Keith K. Niall *Editor*

Erwin Schrödinger's Color Theory

Translated with Modern Commentary

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For John Macnamara:

"Some on the leaves of ancient authors prey, / Nor time nor moths e'er spoil so much as they."

Pope, Alexander (1709)
"An essay on criticism," ll. 112 & 113

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Schrödinger, E. Über die subjektiven Sternfarben und die Qualität der Dämmerungsempfindung. *Die Naturwissenschaften*, **13**(18), May, 373–376 (1925).

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Chapter 1
Schrödinger's Color Theory and Its Background

Abstract Translations of Schrödinger's articles on color theory show us the continuing importance of his colorimetry. Schrödinger's color theory develops a tradition which begins with Newton, and which was developed by Helmholtz and by Grassmann. Schrödinger also wrote at a time when Fechner's influence on psychology was much stronger than it is now. Some colorimetric terms have changed since his articles were published: some are more precisely applied than general terms were in the 1920s. There have also been surprises since, such as Wald's discovery of small-field tritanopia, and the discovery of four-cone color systems in some women. Generally Schrödinger's approach to color theory is sophisticated, comprehensive, and usefully didactic. His axiomatic approach to the geometry of color space permits a close examination of current assumptions about the treatment of data from color matching and color comparison.

Keywords Colorimetry • Translation • History of ideas • Newton's Opticks • Grassmann • Helmholtz • Fechner's Law • Color space • Small-field tritanopia • Tetrachromacy • Schrödinger

Section 1: About the Color Theory

The progress of science is not always smooth or easy. Sometimes a great scientist's effort is displaced: Newton spent a great deal of time writing religious tracts. Sometimes a scientists's effort is superseded or forgotten: Helmholtz is known for his work on the physiology of vision, not for his work in kinematics.[a] Then sometimes a scientist's effort in one domain is eclipsed by success in another: Schrödinger's work in theoretical physics is celebrated, but it may come as a surprise that he wrote extensively on color theory – meaning the theory of human color vision. His articles on color theory have been left uncollected, and much of that work has been left untranslated. Translated into English almost a hundred years after they were written, here are his principal articles on the subject.

Schrödinger published his articles on color theory in the 1920s. One article of nearly a hundred pages appeared in three sections, within the journal Annalen der Physik. He sought to interpret his work for the common scientific reader as well, in two popular articles and a textbook chapter. I became aware of

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Schrödinger's writings on color theory when I was a graduate student, leafing through older journals. At that time I believed that the existence and the importance of these articles were matters of common knowledge – at least among researchers in color science. I believe that his approach to colorimetry retains its fundamental importance. If nothing else, his colorimetry licenses a wider discussion of geometry applied to colorimetry. That is to say his colorimetry readies the mathematical foundations of advanced colorimetry. If some modern accounts of colorimetry crawl through masses of unconnected detail, Schrödinger's colorimetry soars in its formal sophistication. There have been translations of one or two of his articles (note the citations at the end of this volume), but this is the most comprehensive collection of his works on colorimetry in English to date.

One may ask the pedigree of Schrödinger's account of color theory, in other words its place in the history of ideas. (His biography has already been written, and larger overviews of the history of color systems are available elsewhere.)[(b)] His color theory has a strong lineage. His account of color space follows on a notion developed from Newton's Opticks, by way of Hermann von Helmholtz and Hermann Grassmann. His account responds directly to Helmholtz's hypothesis of a line element for color space, and it responds to Grassman's formalization of color theory as a vector space. Newton relies on an analogy from music, to arrange colors in a circle whose circumference is divided into seven parts. Though he does not insist on specific arclengths to represent all the different types of color, he considers the arrangement of spectral colors to be "proportional to the seven musical Tones of Intervals of the eight Sounds".[(c)] Though the musical analogy sets an initial arrangement of colors, Newton also introduces another formalism: a 'center of gravity' construction to model combinations of colors. Later authors will abandon the analogy of a musical scale, but the 'center of gravity' construction persists as a feature of color theory. Color mixture for Newton is a domain independent of other properties of physical optics, a domain tractable in a formal way. "And in this respect the Science of Colours becomes a Speculation as truly mathematical as any other part of Optiques."[(d)]

Hermann von Helmholtz sought to carry Newton's legacy forward, though Helmholtz leaves aside the analogy to tonal intervals. Helmholtz (1852) surmises that Newton's musical analogy may have been reinforced by his choice of sunlight as an illuminant, as well as his choices of crown glass or flint glass as prisms.[(e)] Those can distort intervals along the spectrum. Helmholtz promotes Thomas Young's (1802) three-color or **trichromatic** theory, though he considers Wollaston's (1802) work as its basis. Helmholtz also acknowledges limitations on Young's three-color theory, for example in its claim for the objectivity of three fixed primary colors. To extend the body of empirical evidence, Helmholtz (1855) devised a color-wheel apparatus, which he thought could be used to replicate earlier work with prisms – including that of Newton.

Grassmann (1853) responds to Helmholtz in vindication of Newton. Grassmann does seek to show that Helmholtz's results coincide with Newton's for the most part. In the same text Grassmann introduces important new concepts to the study of color. Those concepts inform Schrödinger's later work. Grassmann introduces the notion of a measure of hue, and the axiomatization of operations on colors. Colors can be represented as line lengths with direction, and combinations of colors can be represented as geometric vector sums. Grassmann refers to his own earlier work on vector spaces, but he also introduces Möbius's barycentric coordinates and barycentric calculus.[f] He introduces them as part of the task of revising Newton's arrangement of colors about a circle by a centre-of-mass calculation. Grassmann introduces the formal machinery of affine geometry or projective geometry to the problem of characterizing color space; Helmholtz seems not to appreciate the full import of Grassmann's gambit. Helmholtz's response to Grassmann is once more a defense and re-interpretation of Newton's Opticks. Helmholtz (1855) seeks to explain Newton's color circle results in terms of the spectral sensitivity of the eye, and the optical properties of refractive materials. He attributes some variance to the Purkinje effect, as well. Newton's spacing of colors is said to need revision, and Newton's color circle needs to include purples as mixtures of red and violet (Newton did acknowledge the latter possibility.). Helmholtz claims that Newton's theory of color mixture is expressed by the color circle. Moreover, he claims that Newton's essential contribution to color theory is just the center-of-mass construction for the combination of colors in the color circle. It is noteworthy that Helmholtz interprets Grassmann's 'sum of colors' in the center-of-mass construction in a narrow way rather than as a vector sum. (As a consequence, he believes that Grassman is committed to a circular form as the boundary of color space.)

Helmholtz (1891) continues his formal development of color theory, still aiming to characterize Newton's laws of color combination. Helmholtz's aim is to develop a Riemannian 3-manifold for color space. His methods include the determination of intervals of *just-noticeable difference* or JND, following methods set out by Weber and Fechner earlier in the nineteenth century. (Note that Helmholtz uses 'difference in sensation' – Empfindungsunterschied – almost interchangeably with 'sensation of difference' – Unterschiedsempfindung.) Helmholtz includes comparisons of brightness, not just comparisons of hue among these differences. That is to say, comparisons of heterochromatic brightness also count as intervals or steps of just-noticeable difference. Lines in color space which prove to be lines of smallest color difference are taken as geodesic, in other words as shortest lines between colors as points in the color field. Helmholtz (1892b) generalizes his theory of color further. He assumes that the perception of differences in color originates with the perception of differences in brightness. (Helmholtz does recognize departures from Fechner's Law of just-noticeable differences, for color mixtures which include colors of low saturation.) Consequently differences in brightness and differences in color both contribute to a geometric representation of color. One may continue by characterizing a system of color for dichromats, and then extending the dichromatic system to trichromats. Newton's laws of color mixture are more easily seen to apply to color comparisons by dichromats. Helmholtz's (1892b) color system makes reference to three primary colors as reference points, and it places colors in a frame of positive rectilinear coordinates. Helmholtz's (1892) color system is the principal foil, and the main historical reference for Schrödinger's narrative on color theory. In that system any color can be expressed as a point in terms of three values: three positive rectangular coordinates in x, y, and z. Helmholtz's system is Riemannian in the sense that any distance between two neighbouring points is given by a differential expression of coordinates. That expression for color sensations in coordinates plays the role of an expression for the lengths of line elements. Helmholtz's system is three-dimensional including brightness, though he claims that any plane section of the color system is a color table in the sense given in Newton's Opticks. In contrast Schrödinger's colorimetry leaves us with a programmatic sketch: his colorimetry sets out the geometric framework for color space, but it does not complete the structure. It establishes an affine geometry to suit the basic evidence of color matching in colorimetry, but it stops short of specifying the Riemannian structure of advanced colorimetry for color similarity. The affine geometry is a default structure, sufficient only until a few problems in advanced colorimetry may have been solved. A geometry of advanced colorimetry waits on two things: a fuller corpus of empirical data, and a decision about the validity of the Weber-Fechner law. What is to be done about Fechner? One can – to repeat Stevens's phrase – honor Fechner and repeal his law.[(g)] Otherwise one is faced with the task of reconciling Fechner's Law for color space with legitimate competing interests. In either case Schrödinger's work – equally the work of color theory – remains unfinished.

There is a leitmotiv in psychology, beginning with Weber and continuing with Fechner,[(h)] which maintains there is a determinate relation between changes in physical magnitudes and perceived changes in those magnitudes. Discussion of its psychological validity often dominates discussion in the experimental psychology of the time, as in Meinong's writing and influence.[(i)] That determinate relation has been claimed as a relation between physical magnitudes and qualities perceived in a sense modality. Some would go so far as to call it a relation between physical magnitudes and psychological qualities. The estimation of lifted weights provides an example. Over a large range of weights that can be lifted with one hand, a person may be asked to judge a just-noticeable difference in weight, or else a constant difference in weight over many trials. At least for the estimation of weights, a comparison weight which is noticeably heavier is one that adds a constant positive fraction of weight to the weight which serves as a standard. That constant fraction is maintained across a large range of weights, that is, for a variety of standard weights. Something similar occurs in the brightness of white lights: lights that are seen to exhibit equal steps of brightness will each add a constant fraction of intensity to the previous step. (Note that just-noticeable differences and equal steps are not the same here, though they are related.) That constant fraction for the brightness of white lights need not be the same fraction as the fraction for the heaviness of weights. We may go on to speak of just-noticeable differences or of equal steps for many qualities and modalities. The associated fractions are known as Weber fractions. Fechner surmised that these observations provided evidence for a logarithmic relation between psychological quantities and physical changes for many modalities. Perhaps the elision from the estimation of physical changes to changes in psychological quantities is unwarranted, but it seldom delays anyone in this discussion.

Most often force of will supervenes over logic in discussion of Fechner's Law. Fechner recognized criticisms of his logarithmic relation, but considered them a nuisance.[(j)] Later, Stevens recognized a heuristic value for Fechner's Law, but he did not stop to consider the assumptions which underlie its establishment. Stevens is well-known for supplanting Fechner's Law with a broader power-law relation. Yet there have been close rational and historical considerations of Fechner's Law which illuminate the assumptions involved. That is to say we know what sorts of experimental evidence might be brought in its favour. Still, in our century as previously, we are blithe in our approach to psychological 'laws' based on just-noticeable differences. Blithe, or else simply undisposed to close examination of the reasonable implications of such assumptions. We seem to have accepted the general tenor of the conclusions of such arguments, while we have – for the most part – jettisoned the premises.

One could say that a central question for Schrödinger's colorimetry is whether Weber fractions (i.e., Fechner intervals) remain constant for the brightnesses of many differently colored lights, or not. Better, one can say that Schrödinger realizes the centrality of Fechner's Law to contemporary accounts of color space. By his (Schrödinger's) own deliberations, he was disposed not to accept the assumptions of Fechner's Law. Acceptance of that Law he sees to be incompatible with the specification of a Riemannian line element for color space, in a colorimetry that would unify an account of large differences in color with the account of small differences.

Schrödinger hints at a greater mathematical sophistication in colorimetry.[(k)] In introducing basic colorimetry, he sets out a condition to determine whether or not a definition of brightness is possible in colorimetry – which is his initial motivating question for colorimetry. That stipulation is a condition on a set of partial differential equations, known as a Pfaffian system (see the relevant parts of Chap. 5 here, and also footnote 3 to Chap. 4). Basic colorimetry must satisfy this condition if the notion of heterochromatic brightness is to make sense. Schrödinger also specifies a provisional line element for the Riemannian geometry of the color manifold. He specifies a line element which improves on the one Helmholtz had defined.[(l)] Both line elements are then placed in doubt. Schrödinger leaves the business of a line element for the color manifold unspecified and unfinished. Two aims of his colorimetry collide in the specification of a line element. One aim is to reconcile small-scale differences with large-scale ones, meaning that there should be a meshing of gears between strongly heterochromatic colorimetry and the colorimetry of adjacent colors in the manifold. Another aim is to adhere to Fechnerian proportion for equal increments of intensity across the color manifold. Perhaps the two constraints could even have been reconciled, but for the intrusion of the Bezold-Brücke effect.[(m)] The discriminability of hues changes appreciably across the spectrum as light intensity is varied. Though the Bezold-Brücke pattern of changes is known by experiment, that pattern does not scale with the neat proportions of a colorimetry based on Fechner's Law. In short: specification of the line element remains an open problem. Is there even a line element tractable in mathematical terms, which is subject to all these influences? Subsequent authors quote (and criticize) Schrödinger's line element for color space as if its form had been settled as definite rather than left uncertain, as it is. Here Schrödinger's very hesitation shows the sophistication of his approach to colorimetry. (One might also ask what a suitable form might be, for a line element free of the Fechnerian constraint.)

As with any historical translation, the issue of change in language may be raised for this collection of Schrödinger's papers. Have the technical terms of color science changed so much as to be unrecognizable? I think not. Both in his training (he was Franz Exner's assistant) and in his recognition of Helmholtz's contributions, Schrödinger was in the mainstream of color theory. Some contemporary terms of art have fallen into disuse, such as 'alychne' (it indicates the locus of colors in a color diagram which have an ideal but fictional property of zero brightness). Some other terms have been sharpened over time – for example, 'luminous efficiency' has a more specific role in color theory than does 'brightness'. Yet none seem irretrievably unfamiliar.

I do not wish to say that these texts lack any decent translation at all. David MacAdam's translations provide something of a counterexample, though he published only short fragments of the work in translation.[n] There is a particular problem, though, for which I would like to make my stance clear. There is a cluster of words used in the development of what are known today as color-matching functions and fundamental response curves. The associated terms have changed through time, as can be known even by comparing W.D. Wright's (1947) uses of the terms with Schrödinger's. Of course it might also be that the terms have been made obscure in translation: MacAdam (1970) uses 'calibration' as an adjective to cover terms ranging from chromaticity coordinates to fundamental color stimuli, to an extent that make the modern interpretations of color-matching functions and fundamental stimuli barely recognizable. I take that gloss in translation to pose a greater danger to these texts than change in language; my own glossary for these terms is given as a footnote to this introduction.[o]

There are two more factors which may have clouded other translations. One is appreciation of the subject matter. Translation preserves meaning: a text which lacks meaning is not susceptible to translation. A translator also needs to be comfortable with the meaning of the original text. Schrödinger's writing is clear – evidently – but not everyone will be comfortable with its depth. The second factor concerns a translator's linguistic skill. I mean more by that than education in the technical arts of translation. Style may also cloud the result of translation. Some may consider colloquial style to be offhand for the translation of *Buchwurmsprache*. Others may lament the gap between North American usage and the Queen's English, and so forth. One particular point of style is important here: where translation is less than perfect, translation into one's first language is likely happier – as a matter of familiarity in style – than translation in another direction. Few translations achieve a beautiful balance of meaning and style.

Whatever the faults of the present translation, my earnest wish is that the original meaning may shine through.

There have also been some surprises in color science over the last century, as one might hope. Schrödinger maintained that no authentic cases of tritanopia had been reported – that is, no 'blue-blind' observers had been documented whose condition did not involve severe ocular trauma. Though rare, such individuals are now known to exist. He might have been surprised by discovery of small-field tritanopia – that most of us are tritanopes when the field of view is restricted to a very small angular extent under fixed viewing conditions.[(p)] He might also have been surprised by the efficient manufacture of fluorescent pigments that borrow energy from non-visible regions of the spectrum, to display brighter visible colors. Similarly, he maintained that the color space of his basic colorimetry is three-dimensional, in the sense that triplets of spectral colors are linearly independent. Combinations of four spectral colors – so he supposed – would produce nothing new. He took pains to arbitrate between a color manifold of dimensionality three, and a manifold of dimensionality four. In other words he thought that the manifold of colors is covered completely and exhaustively by pure spectral colors and their binary mixtures. He may also have been surprised in that much: there are indications that a fair proportion of women possess four chromatic systems rather than three, meaning that they possess four distinct populations of cone cells, and that they discriminate color reliably and functionally better as a consequence.[(q)] Then it may well be the case that we need a four-dimensional manifold of color space, to describe the ability these women have to judge differences in color.

Neither discovery – small-field tritanopia or tetrachromacy in some observers – affects Schrödinger's color theory in a fundamental way. On these counts we can tell how his color theory is extensible – how it may be extended to subsume these new findings. His color theory is still fresh in that much. I hope the reader finds a full and reasonable exposition of the theory of colorimetry in these pages, one that ignites our modern imagination about the geometric nature of color space.

Notes

a. This includes [***Helmholtz***, H. L.F. von. The origin and meaning of geometric axioms I. Mind, **1**(3), 301 – 321 (1876). & ***Helmholtz***, H. L.F. von. The origin and meaning of geometric axioms II. Mind, **3**(10), 212 – 225 (1878).]. Sophus ***Lie*** revealed the lacuna in ***Helmholtz***'s account: [***Lie***, S. Bemerkungen zu von Helmholtzs Arbeit: Ueber die Tatsachen, die der Geometrie zu Grunde liegen. In S. Lie, Gesammelte Abhandlungen, 2. Band, 1. Teil. Leipzig: B.G. Teubner, 374 – 379 (1886/1935).]

b. ***Schrödinger***'s biography can be found as: [***Moore***, W. Schrödinger: Life and thought. Cambridge: Cambridge University Press (1989).] A more comprehensive account of color systems, including non-metric systems, is contained in: [***Wyszecki***, G. Farbsysteme. 2d. ed. Göttingen: Musterschmidt-Verlag (1962).] or else [***Wyszecki***, G. and ***Stiles***, W.S. Color science. 2d. ed. New York: John Wiley & Sons (1982).]

c. Their arrangement is "proportional to the seven musical Tones of Intervals of the eight Sounds, *Sol*, *la*, *fa*, *sol*, *la*, *mi*, *fa*, *sol* contained in an Eight, that is, proportional to the numbers $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{16}$, $\frac{1}{9}$ " [***Newton***, p.114, The first book of Opticks, part II (1704)].

d. He continues: "I mean so far as they depend on the nature of Light, and are not produced or altered by the power of imagination, or by striking or pressing the Eyes." [***Newton***, p. 48, The second book of Opticks, part II (1704)].

e. Citations for the articles discussed in this section are:

Grassmann, H.G. Zur Theorie der Farbenmischung. Annalen der Physik und Chemie (J.C. Poggendorff's Annalen), **89** (Dritte Reihe **29**), 69 – 84 (1853).;

Helmholtz, H.L.F. von. IV. Ueber die Theorie der zusammengesetzten Farben. Annalen der Physik und Chemie (J.C. Poggendorff's Annalen), **87** (Dritte Reihe **27**), 45 – 66 (1852).; ***Helmholtz***, H.L.F. von. Ueber die Zusammensetzung von Spectralfarben. Annalen der Physik und Chemie (J.C. Poggendorff's Annalen), **94**(1) (Vierte Reihe **4**), 1 – 28 (1855).; ***Helmholtz***, H.L.F. von. Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem. Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **2**, 1 – 30 (1891).; ***Helmholtz***, H.L.F. von. Kürzeste Linien im Farbensystem: Auszug aus einer Abhandlung gleichen Titels in Sitzungsberichte der Akademie zu Berlin, 17. Dezember 1891. Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **3**, 108 – 122. (1892).; ***Helmholtz***, H.L.F. von. Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden. Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **3**, 1 – 20 (1892b).; ***Wollaston***, W.H. A method of examining refractive and dispersive powers, by prismatic reflection. Philosophical Transactions of the Royal Society, 1 January, **92**, 365 – 380 (1802).; ***Young***, T. The Bakerian lecture: On the theory of light and colors. Philosophical Transactions of the Royal Society of London, January 1, **92**, 12 - 48 (1802).

f. In other words the use of a non-Euclidean coordinate system for color space, as in a system of barycentric coordinates [***Möbius***, A. F. Der barycentrische Calcul 1827. In: Gesammelte Werke, Band 1. Leipzig: Salomon Hirzel, 1 – 389 (1885).], was carried on by [***Grassmann***, H. G. Zur Theorie der Farbenmischung. Annalen der Physik und Chemie (J.C. Poggendorff's Annalen), **89** (Dritte Reihe **29**), 69 – 84 (1853). See page 83.] ***Grassmann***'s earlier work on vector spaces, the Ausdehnungslehre, can be found as [***Lewis***, A.C. (Ed.) Landmark writings in western mathematics 1640 – 1940. Chapter 32 – Hermann G. ***Grassmann***, Ausdehnungslehre, first edition (1844), pp. 431 – 440 (2005).].

g. [***Stevens***, S.S. On the psychophysical law. Psychological Review, **64**(3), 153 – 181 (1957)]. ***Stevens*** proposed a power-law relation whose exponent varied by modality, to supplant ***Fechner***'s 'Law'.

h. Beginning with ***Weber*** [***Weber***, E.H. De pulsu, resorptione, auditu et tactu. Leipzig: C.F. Koehler (1834). & ***Weber***, E.H. Ueber den Raumsinn und die Empfindungskreise in der Haut und im Auge. Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physische Classe, **2**, 85 – 164 (1852).] and continuing with ***Fechner*** [***Fechner***, G. T. Ueber ein psychophysisches Grundgesetz und dessen Beziehung zur Schätzung der Sterngrössen. Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physische Classe, **4**, 457 – 532 (1859). & ***Fechner***, G. T. Revision der Hauptpuncte der Psychophysik. Leipzig: Breitkopf und Härtel (1882). & ***Fechner***, G. T. Über die psychischen Maβprinicpien und das ***Weber***'sche Gesetz. Philosophische Studien, **4**, 161 – 230 (1888).]. Discussion of the 'Law' had a central place in nineteenth-century discussion of experimental psychology (then 'experimental philosophy').

i. ***Meinong*** elaborated ***Fechner***'s arguments at length [eg., ***Meinong***, A. Ueber Sinnesermüdung im Bereiche des Weber'schen Gesetzes. Vierteljahrsschrift für wissenschaftliche Philosophie, **12**(1), 1 – 31 (1888). & ***Meinong***, A. Über die Bedeutung des ***Weber***schen Gesetzes. Beiträge zur Psychologie des Vergleichens und Messens. Erster Abschnitt: Von Grössengedanken und dessen Anwendungsgebiet. Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **11**, 81 – 99 (1896).]. As Chair of Philosophy at Graz, ***Meinong*** had great influence on contemporary psychology. See Bertrand ***Russell*** [***Russell***, B. Review of Alexius ***Meinong***'s Ueber die Bedeutung des Weberschen Gesetzes. Mind (New Series), **8**, 251 – 256 (1899)] for an argument against ***Meinong***'s position.

j. Note especially the 'quality objection' put forward by Johannes von ***Kries*** [***Kries***, J. von. Ueber die Messung intensiver Grössen und über das sogenannte psychophysische Gesetz. Vierteljahrsschrift für wissenschaftliche Philosophie, **4** (3), 257 – 294 (1882b)]. ***Niall*** provides an English translation: [***Kries***, J. von. Conventions of measurement in psychophysics: von ***Kries*** on the so-called psychophysical law. Spatial Vision, **9**(3), 275 – 305 (1882/1995).] See ***Michell*** [Michell, J. Measurement in psychology: critical history of a methodological concept. New York: Cambridge University Press (1999).] for a thorough and critical account of assumptions in Fechnerian psychophysics.

k. This sophistication is evident through his career, of course: cf. ***Schrödinger***, E. Expanding universes. Cambridge at the University Press (1956). Not that such sophistication was always lacking in later color theory: the affine-geometric account of basic colorimetry was taken up by [***Schelling***, H. von. Advanced color geometry. Journal of the Optical Society of America, **45**(12), 1072 – 1079 (1955). & ***Schelling***, H. von. Concept of distance in affine geometry and its applications in theories of vision. Journal of the Optical Society of America, **46** (5), 309 – 315 (1956).]

l. There is a long history of the line-element following Helmholtz and Schrödinger: cf. [***Stiles***, W.S. Line element in colour theory: A historical review. In: J.J. Vos, L.F.C. Friele & P.L. Walraven, Eds. Color metrics: Proceedings of the Helmholtz Memorial Symposium. Soesterberg, Netherlands: AIC/Holland & Institute for Perception TNO, 1 – 25 (1972).; ***Wyszecki***, G. Über die Metrik des visuell homogenen Farbenraumes. In: International discussion of problems in color metrics. Heidelberg: Die Farbe, 100 – 108 (1955).; ***Wyszecki***, G. Recent developments on color-difference evaluations. In: J.J. Vos, L.F.C. Friele & P.L. Walraven, Eds. Color metrics: Proceedings of the Helmholtz Memorial Symposium. Soesterberg, Netherlands: AIC/Holland & Institute for Perception TNO, 339 – 379 (1972).], up to more recent accounts such as [***Raj Pant***, D. & ***Farup***, I. Riemannian formulation and comparison of color-difference formulas. Color Research and Application, **37**(6), 429 – 440 (2012).; ***Raj Pant***, D. & ***Farup***, I. Geodesic calculation of color difference formulas and comparison with the Munsell color order system. Color Research and Application, **38**(4), 259 – 266 (2013).; ***Jain***, A.K. Color distance and geodesics in color 3 space. Journal of the Optical Society of America, 62(11). 1287 – 1291 (1972).]

m. Cf. ***Ejima***, Y. and ***Takahashi***, S. Bezold-Brücke shift and nonlinearity in opponent-color process. Vision Research, **24**, 1897 – 1904 (1984).

n. See the list of Translations at the end of this volume. ***MacAdam*** was also well aware of ***Schrödinger***'s work on the brightness of colored pigments: see [***MacAdam***, D.L. The theory of the maximum visual efficiency of colored materials. Journal of the Optical Society of America, **25**, 249 – 252 (1935). & ***MacAdam***, D.L. Maximum visual efficiency of colored materials. Journal of the Optical Society of America, **25**, 361 – 367 (1935b).]

o. [Trans.] I have tried to maintain the following glossary for ***Schrödinger***'s technical vocabulary of colorimetry:

Aichkurven: color-matching functions (in graphic form)

Eichfarben: calibration colors

Eichfunktionen: color-mixture functions (near in meaning to 'color-matching functions')

Eichkurven: color-mixture curves

Eichlichter: calibration lights

Eichwerte: trichromatic coefficients (near in meaning to 'chromaticity coordinates')

Elementarempfindungskurven: fundamental response curves

Farbkoordinaten: color coordinates

Grundempfindungen: fundamental stimuli

For ***Wright***'s terminology, see [Wright, W.D. Researches on normal and defective colour vision. St. Louis: The C.V. Mosby Company (1947)].

p. ***Wald***, G. Blue-blindness in the normal fovea. Journal of the Optical Society of America, **57**(11), 1289 – 1303 (1967).

q. ***Jameson***, K.A., ***Highnote***, S.M. & ***Wasserman***, L.M. Richer color experience in observers with multiple photopigment opsin genes. Psychonomic Bulletin & Review, **8**(2), June, 244 – 261 (2001).

Chapter 2
Outlines of a Theory of Photopic Colorimetry (Part I): Basic Colorimetry, or Affine Color Properties (First Article)

Abstract A formal system of human color perception is described which is called **basic colorimetry**. Colors may be compared in four ways, and basic colorimetry uses just one of these: color matching. Basic colorimetry forms a coherent system which follows ***Grassmann***'s laws, and this system constitutes a color manifold. The properties of the system are developed in a rigorous way. The color manifold provides a model for affine vector geometry. Properties which are not affine-invariant do not belong to the three-dimensional color space of basic colorimetry. This geometric notion of colorimetry is elaborated in two subsequent chapters.

Keywords Colorimetry • Color vision • Color metric • Heterochromatic brightness • Color matching • Color additivity • Grassmann law • Color dimensionality • Affine geometry • Affine invariant • Color plane • Color coordinates • Alychne • Schrödinger

Preface

The occasion for developing the outline of a color metric – as it is sketched here – was the author's meditation on the concept of **brightness**. Those thoughts are expressed in Sections 3–7 of the second Article of the present work. The questions that are dealt with there, are the following. Suppose a color is given concretely – that is, as a particular mixture of lights. Then can a number be assigned to its **brightness**, in a way that a value may be given to lights of differing 'stimulus qualities',[(a)] to say whether they are **equal in brightness**, or not? How is a definition of equal brightness to be made, posed experimentally? And what would be the consequences of a definition, correctly posed? Brightness could be defined **empirically** as a function of three trichromatic coefficients, or three fundamental stimulus valences, or in terms of any other triple of quantities sufficient to specify color uniquely.

About these three questions: I believe I have established here – for the first time – a rational criterion for answering the **first** question. ('Is there heterochromatic

Schrödinger, E. (1920). Grundlinien einer Theorie der Farbenmetrik im Tagessehen. I. Mitteilung. *Annalen der Physik*, vierte Folge, **63**(21), 397–426. Copyright © 2006, as renewed. Translated with permission from Wiley-VCH Verlag GmbH & Co. KGaA.

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brightness?') In doing so, I have discovered the basic reason that researchers are so divided on this issue. Is a definition of equality of brightness fundamentally possible, or not? That depends whether or not there exists an integrating factor for a certain ***Pfaff***ian expression in three differentials. (Its coefficients are determinable empirically.) Only if that should be the case, would 'plane elements representing equal brightness' of the color manifold be pieced together to form integral surfaces of equal brightness.[(b)]

Under the hypothesis that this experimentally-obtained function may admit integration, a definition of brightness is obtained immediately. Whatever clever experiments may show in future, there will be nothing to be altered in the definition.

The situation is different for the third question, as a consequence of the particular function of color coordinates which stands for brightness. I have run with an idea here, while trying to do justice to the larger structure of our color experience. The idea is that brightness is a **linear** function of the fundamental valences: that it behaves strictly additively when lights are added. A large corpus of observations speaks to this fact, which holds at least to a close approximation. That is why I thought it useful to show how this interpretation may be articulated, without coming into obvious conflict with other observations – such conflict as might have arisen from our **general** definition of brightness. That conflict has not yet been manifest. Yet I do not quite believe that additivity holds in a strict sense; a few isolated but very salient phenomena point to deviations from additivity.

Certainly then, this **particular approach to the line element for color space** – and the **particular brightness function** which follows from it – is to be considered only as an **approximation** which awaits further experimentation for its improvement.

To emphasize the significance of this formal solution to the problem of brightness, it has proved necessary to point out its place in the larger scheme of a color metric. The author has indicated this already in the course of discussion on this question and similar questions. It emerges through experience that all efforts to quantify color fall into two fundamentally distinct categories. Their distinction lies in the criterion used for the adjustment of two adjoining color fields on a device. Either a criterion of **complete identity** is applied exclusively (indistinguishability), or other criteria are applied (just-noticeable difference, maximum similarity, or perhaps maximum contrast). Results of the **first** kind of measurement do form a unified, internally consistent system whose simple axiomatic rules have been formalized by ***Grassmann***, which rules have been substantiated by the work of ***König***. I believe that this system of rules – commonly known as the laws of light-mixture – may reasonably be called **basic colorimetry**, to distinguish it from **advanced colorimetry**, whose laws are much more complex and much less well understood. Advanced colorimetry deals with quantitative results of the **second** kind.

I maintain a strong distinction between these two theoretical domains – abstracted from their immediately apparent meaning in theory – to be extremely important, as a consequence. Otherwise there are a host of unclear concepts and a host of experimental uncertainties that we encounter at every step of our advanced colorimetry, which will also intrude persistently in the domain of basic colorimetry. They threaten to create confusion in the latter domain, though basic colorimetry stands securely and validly on its own merits, being in no way affected by such difficulties.

The deeper reason for such a hermetic, insulated quality is a simple one. Here there is a strong analogy to the situation of a geometer who undertakes to study the **projective properties** of plane figures or spatial forms exclusively. Such a geometer does not lose sleep over ***Euclid***'s parallel postulate; he does not need to know anything about right angles or the lengths of line segments. That is to say, it turns out that basic colorimetry provides a **complete** model for the **affine geometry** of a **pencil of vectors**, taken to represent the color manifold. In the more commonly used plane representation there is an **essentially** complete model for the **projective geometry of the color plane**, in other words. Following an idea from ***Helmholtz***,[(c)] we say by contrast that advanced colorimetry is best taken as a model for the real **metric geometry** of that pencil of vectors normally taken to represent **color space**. That is a metric in the very widest sense established by ***Riemann***, a metric whose line element is not Euclidean. Its line element has variable coefficients which are only to be determined by experiment. (Advanced colorimetry is not so easily represented in a plane diagram, since there the third coordinate is replaced by weights that are both variable and unintuitive.)

The **question of brightness** also plays a role in **advanced** colorimetry, as is immediately apparent, and as will be revealed in the following section. The relation of 'equally bright' is as superfluous and foreign to the strict color **matching** of two fields differing in hue, as are the relations of 'equally long' or 'mutually perpendicular' for two lines in projective geometry. Then I also consider it to be fundamentally misguided if (as in ***Abney***[(d)]) color matches and brightness matches are confounded in their very nature, by the stipulation that fundamental colors should be gauged in terms of shared equal units of brightness. Disparate judgments of equality should not be confounded by the stipulation that fundamental colors can be measured in units of equal brightness. A decision about the mere possibility of such a step can be made only at a much better-developed stage of advanced colorimetry. Whether such a step is useful depends on the successful experimental demonstration of a particular advanced function of brightness, that is, a linear combination of the fundamental stimulus values. Any experimental counterindication or theoretical objection to this very specific assumption will then (seemingly) shake the very foundations of the theoretical edifice.

A link between the two domains is established by the theory of fundamental stimulus colors. These should acquire meaning in basic colorimetry from those colors which are confused by dichromats, which are widely known to have been determined – entirely by **color-matching** ! – by von ***König***.[(e)] On the other hand one might expect that the line element of advanced colorimetry would assume an especially simple form, should one choose the fundamental stimuli as variables. ***Helmholtz***'s expectations were disappointed in that much. He had to assume **new** fundamental stimulus points in the color plane arbitrarily to accommodate the line element which he constructed to suit the ***Weber-Fechner*** law. Only then does his line element correspond to the evidence (the color difference thresholds produced by ***König*** and ***Brodhun***) given the ad-hoc assumption. Consequently ***Helmholtz*** was compelled to adopt much more complicated assumptions about the nature of color blindness in dichromats.[f]

Add to this an error in ***Helmholtz***'s calculations which widens the gap between the experimental evidence and his line element. That may be the reason his profound deliberations towards a Riemannian geometry of color have found little resonance and have not been pressed further, so far as I know. The relevant citations have even been excised from the most recent edition of ***Helmholtz***'s Handbook of physiological optics (Hamburg & Leipzig, published 1909 by Leopold Voss).

On my first encounter with this part of ***Helmholtz***'s work, the incongruity of his position seemed blatant. I seemed to recognize something that appeared to have escaped ***Helmholtz*** – that his line element implies a brightness function which flies in the face of experience (the cube root of the product of fundamental stimulus valences). But then I noticed that another form is possible for the line element, which – though surely capable of being improved – has a simple form I will adopt which establishes an overall regularity, and which alleviates some irritating contradictions.

In this first section dedicated to basic colorimetry, it is obvious that we add nothing to the content provided by ***Helmholtz*** and ***König***. Still in my opinion, one cannot dispense with a full and unified account of the entire subject, since there is no one presentation I can think of that we could have used as a foundation for advanced colorimetry. There is no description I know which emphasizes the purely projective (or purely affine) character of these colorimetric laws as strictly as is necessary to our purposes.[g]

Introduction: Characteristics of the Domain. Delineation of the Basic Metric

The art of quantifying color can be seen as a topic in the physiology of the senses, as it can be seen as a topic in experimental physics. We lend priority to the latter perspective in the present discussion.

The experimental methods of physics that concern us here are proceedings which occupy a particular distinguished position – I would even say a privileged position from an epistemological standpoint. In the final analysis, all other physical measurements always consist in the determination of **superposition in space-time**, whether those measurements be conducted with a balance, a galvanometer, a thermometer, or a telescope. The superposition may be that of a pointer, or of a patch of light; the superimposition may be of a meniscus of mercury with a position on a scale; it may be the superposition of crosshairs in a sight with a graduated scale, or with the image of a star. – Often it consists of the temporal coincidence of two such superimpositions in space, and so on. By contrast, the sensory apparatus of an observer enters into the activity of at least **one** determination for any measurement of color, besides the **kinds** of determinations just listed. In its most basic form, this activity of observation consists of the following: two adjacent color fields are judged to be indistinguishable in color – allowing variation in the values of other instrumental parameters, such as the settings of Nicol prisms, diaphragm settings, and collimator settings. This is a determination where color matches coincide with other coincidences and superpositions of the kind first mentioned.

The distinctive relation of the two halves of the color field does not have to be one of indistinguishability, though. For a certain set of measurements (difference thresholds) one proceeds another way, from a situation of complete identity. A physical parameter of the apparatus is varied until the identical fields become **just-noticeably different**. Further it may be that by a change in one parameter, no setting is discovered for complete identity of the two fields. Rather a setting is found that is distinguished **another way**: the two fields become **less markedly different** for that setting (i.e., that value of the parameter) than for antecedent and succeeding values of the parameter. Following ***Helmholtz***, I call this an adjustment for maximum similarity; I believe that all direct methods for the comparison of heterochromatic brightness – among other methods – fall into this category of measurements. Finally, it is at least conceivable one might proceed in the opposite manner: one could adjust fields for maximum contrast. I know of no case in which such a principle may have been used.

I am of the opinion that all cases are covered by the following exhaustive list. The four **possible** principles of adjustment are:

a) Identity
b) Difference
c) Maximum similarity
d) Maximum contrast

For instance, I believe that one makes an adjustment of the **third** kind if one selects a color of the same hue within a richly-illuminated spectrum of colors to match a color from a dimly-illuminated spectrum. One uses the same principle of adjustment if one compares the photometric properties of two lights of different hue: meaning one may match a color of equal brightness from a first sample, to one of a graded second series of samples. I do not think that 'set' adjustments are possible, as would be indicated by the differing expressions 'identical in hue' or 'identical in brightness'. That is not to deny that one could, in the previous example, at one time choose the **equally bright**, and at another time choose the color of **equal hue** from the dimly illuminated spectrum. One might then choose different positions along the second spectrum, in general. So naturally there can be **two** (or more) colors in a series which are more similar to a given color than their neighbours – just as two (or more) minima can be present along a finite stretch of a curve.

Of course this conception is not subject to logical demonstration. So long as it proves serviceable, it retains the advantage of simplicity against all other schemes – which in current parlance should almost suffice to indicate that it is **correct**.

We have cited four principles for the adjustment of distinct combinations of parameters by means of an apparatus to indicate color judgments. These four principles can be derived from the following two capacities, which are simply our native and intuitive capacities to make sensory judgments.

1. We are able to judge whether adjacent color fields are the **same** or **different**.
2. Let us call a pair of adjacent, non-identical fields presented to an observer, a '**step**'. We can make qualitative decisions about the size of these steps. We can judge simultaneously presented steps, or two steps presented in quick succession (if the temporal interval is not too large). Namely we can judge if a step is 'larger' or if it is 'smaller'. **Corollary**: This relation of larger-and-smaller satisfies the axiom of transitivity: If $A > B$ and $B > C$, then $A > C$; and if $A < B$ and $B < C$, then $A < C$.

I take it that our capacities to make these two color judgments are the only native capacities to be considered in quantitative measurement. They entitle us to frame observers' adjustments according to the four principles just cited. That much is immediately evident when the color fields vary only by **a single variable parameter** of our experimental apparatus. If a one-dimensional scale of steps is in question, we may easily discern the **same** step, the **first different** (by progression in one direction), the **smallest** (by comparison to neighbouring steps), and the **largest** step, if such steps occur in the given scale. If two or more parameters are to be varied, then the task is more difficult, and a solution much less certain. Even for two parameters, a procedure of successive approximation must be adopted. For example, one can maintain the first parameter as it is, and adjust the second parameter for maximum similarity. Then one maintains the second in its altered state and adjusts the first for maximum similarity, and so on. This is repeated either until identity is achieved, or else until maximum attainable similarity is achieved.

In all cases where an adjustment is made to achieve **equality**, the concrete result is not a **single** distinguished value of a parameter (or some combination of parameters), but rather a small region of values. The second principle b) owes its relevance to this, since it comes into play in the investigation of magnitudes in these regions. One cannot rely upon the simultaneous variation of several parameters; rather, one explores the extent of these regions in the direction specified by each parameter in turn. By these means one obtains a somewhat different, but perhaps equally useful measure of its extent, in that one determines an **average error** of the adjustment for equality.

Now at some point we need to substantiate how and in what respect we are justified in posing such a fundamental distinction between the measurements at hand, and all other physical measurements. Other physical measurements are standardized by spatial and temporal coincidences; these other measurements involve a special relation to color. The full import of this distinction is not immediately apparent. It does not depend on the external expression of color judgment in any way. – This counts for so little that I even exclude from colorimetry proper a good number of studies which presuppose this special characteristic; we have to exclude them because they are only peripherally connected to colorimetry proper.

The color fields which are judged to be equivalent in these respects also have **entirely the same physical makeup**. Clearly an observer's eye – and the observer's judgment of color – enters into this business only as a convenient and a sensitive physical instrument. That instrument registers a null-state reading, and it could be replaced by any number of other instruments: a bolometer, or a thermometer column, for example. In that way this kind of measurement could also be traced back to conventions about coincidence in space and time. These measurements are entirely at home with measurements of the usual kind – there is no fundamental difference involved, only an apparent one.

Among those belong for example, strobometric measurements using polarizers, as well as monochromatic photometry in which both lights do have just the same spectral composition. That also holds particularly for studies with spectrophotometers, e.g. determination of the reflectance function of a pigment, or the spectral composition of a light source as compared to standard light sources, and so on. All those measurements could be carried out by a colorblind physicist, and carried out by a set procedure – even by a **blind** physicist. The results would be identical; they would have a well-defined meaning for that physicist – just the same meaning they would have for anyone.

The situation is quite different if, e.g. we combine two complementary spectral lights to produce white, and compare the result to sunlight, or if we combine thallium green and lithium red to obtain a desaturated sodium yellow (in what is called Rayleigh equivalence). The spectral composition of the two fields is entirely different in these cases, as illustrated by Figs. 2.1 and 2.2. Still with a suitable choice of intensities, the two fields appear completely indistinguishable to the eye. In this instance the eye is not to be replaced by an instrument. Further, another observer's eye may distinguish the balance of the two lights differently – in abnormal cases the difference can be **great**. Arguments are all moot here which might say whose judgment might be better or more nearly correct. The two lights which appear the same are completely different entities; though they appear the same, the two lights are entirely different when taken for themselves. They have nothing in common except that they appear the same to a **certain** eye, which in its color judgment is irreplaceable or unsurveyable by another measuring instrument.

This exclusive reliance on the eye of an observer **always** holds for the **other three principles**, i.e., the three **principles** of colorimetry that have been outlined (difference, maximum similarity, and maximum contrast). There is an amount by

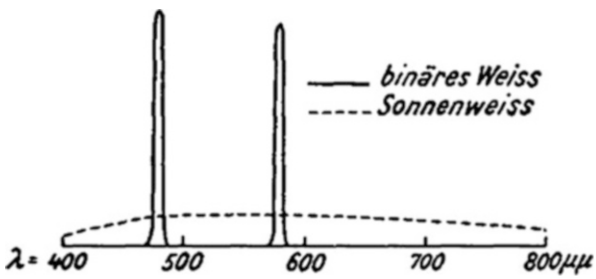

Fig. 2.1 Sunlight, and the combination of two spectral lights. (*binäres Weiss*: white from a pair of spectral lines; *Sonnenweiss*: white sunlight) (Reproduced from Schrödinger 1920)

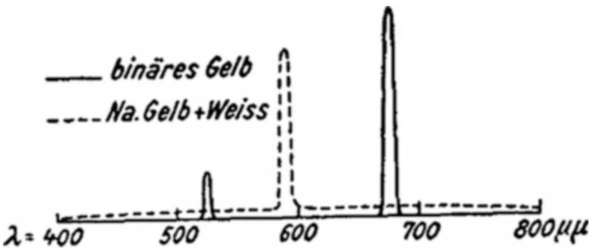

Fig. 2.2 Desaturated sodium yellow, and the combination of thallium green and lithium red. (*binäres Gelb*: yellow from a pair of spectral lines; *Na Gelb* + *Weiss*: sodium yellow plus white) (Reproduced from Schrödinger 1920)

which two otherwise equally constituted lights must be altered in intensity, to be recognized by eye as only just different. Evidently that amount is not determinable by another measuring instrument, but only by eye. Similarly a color that is most similar objectively to a given color cannot be picked from a graded series in the absence of a judgment of color, that is, on a purely physical basis.

By this time and after all this discussion, one might characterize the **special situation** of the domain of experimental physics to which we have turned our attention. One might think it no longer belongs to physics at all. One would then be investigating the subjective properties of sensory organs, and not the objective properties of the physical world. This way of speaking is clear and comprehensible, it is in general use, and it is correct in a certain sense. Nevertheless I have some remarks to make against this way of speaking.

If one says that research on color may not be concerned with the investigation of the properties and lawful regularities of the physical world which surrounds us – if one says rather that it is an investigation of the function of a sensory organ – than a certain form of representation follows easily. The immediate object of investigation – color – is attributed less objective reality than other objects of physics, such as physical bodies, atoms, light, electromagnetic fields, and so on. The barest form of this representation is the view that the only real correlate in the environment is just the anatomic or physiologic constitution and function of the sensory organ. But one could just as well assert that all the rest of physics investigates the functioning of our sensory organs, and the associated processes of the central and peripheral nervous system. One could assert that in the final analysis all knowledge, all science seems subordinate to the all-encompassing discipline of anthropology.

One may or may not cling to such radical Berkeleian views. At any rate, to me it rings false to acknowledge only one part of our experience, and hence to establish nonexistent metaphysical distinctions. Yes, all of physics might be conceived as a categorization of perceptual experience in terms of mental images, which system may allow us to predict the sequence of those images. In that respect one may find it advantageous for many purposes to draw predictions from particular experiential cues or marks. Yet one should not come to believe that such cues lose some of their reality by our abstraction, or that the physical bodies which surround us might possess some cues in themselves known to us, though at the same time sound and color are thought to exist **for us** alone.

The three-dimensional manifold of colors, or **color space** will be the focus of much of the subsequent discussion. My view is that color space is every bit as real as the familiar three-dimensional space of points. Color space should not be conceived as a mathematical representation, such as the state-space diagrams of statistical mechanics by which we illustrate the distribution of speeds of gas molecules. Of course the manner in which we assign, arrange, and survey coordinates in this space is an artificial construct of mathematics. In general we do not consider the construct at all, as objects from the space appear before our eyes. In just the same way we do not consider the familiar space of points when its objects appear to us.

Among the four previously-established principles of colorimetric standardization, the first principle of identity or equality is by far the simplest. There is an advantage to bringing this principle to prominence first of all: i.e., easily accessible data can be collected and examined when the principle is used exclusively. In this way one may achieve a surview and general orientation to the color manifold. We turn now to the definition of that manifold.

Colorimetry, Part I:
Basic Colorimetry, or Affine Color Properties

Section 1: The Notion of Color and Light

Color appears when light strikes the eye, ceteris paribus. Since this is the usual way in which colors arise, and the only one which may be investigated in an exact quantitative manner, we will consider it exclusively in what follows.

By **light** of a certain kind we understand a field of radiation that proceeds from the neighbourhood of a certain point in space. The field has certain properties in the (angular) neighbourhood of a direction which intersects that point in space. Consider a plane element that is placed normally (i.e., perpendicularly) to that direction. The plane element and the direction of a source point subtend a solid angle. For units of area, time, and space, the quantity of energy varies as

$$f(\lambda)d\lambda$$

in the form of waves with lengths between λ and $\lambda + d\,\lambda$. $f(\lambda)$ is a function of λ in the visible spectrum. According to this definition, a quantity of energy can be expressed as

$$f(\lambda)d\lambda\ d\omega\ d\sigma\ dt$$

where $d\omega$ is a measure of angle, $d\sigma$ a measure of plane area, and dt a measure of time. The light in question is a function of wavelength $f(\lambda)$ between $\lambda = 400$ and $\lambda = 800$ μμ, more or less.

In saying that we present this light to the eye, we mean that we place the eye at the locus of our plane element, **so that** the selected rays will be imaged close to the region of most distinct vision on the retina (the macula). Here we presuppose that $f(\lambda)$ does not vary appreciably over the region of the pupil, meaning that the entire pupil is uniformly covered by a homogeneous pencil of rays. Yet as soon as we compare different lights across adjacent fields, the situation becomes entirely different. Then $f(\lambda)$ will exhibit a stepwise dependence on the **direction** that the pupil makes to a flat pencil of rays – representing the clean, straight dividing line between the two fields.

The manifold of lights has a higher **power** than the power of the continuum, that is to say higher than a space of functions, and hence an indefinitely large number of dimensions. A priori it could have been possible this would also hold true for the manifold of color qualities. At least it could have had as large a number of dimensions as does the manifold of combination tones. The ear acts as a harmonic analyzer to some extent. That is not the case here. Rather, lights can be arranged in large sets according to the principle of identical appearance across adjacent fields – each of these sets has the power of a space of functions. For color-normal observers, the manifold of this set of lights which appear identical is a manifold of three dimensions; this is the highest dimensionality yet observed. This fact of its dimensionality is a fundamental proposition of basic colorimetry. We will seek to establish this result in more detail from experience.

We can now elucidate the ordinary-language use of the word **color**, that has been used in absence of a better definition. For quantitative ends, let us designate the **set of identically-appearing lights as a color**. In doing so, we distance ourselves somewhat from the ordinary use of language. That holds particularly since lights of the same color in **our** terminology may produce very different impressions on the eye under different conditions. Occasionally they may even be assigned different **color names**. So it may be for two lights that would be indistinguishable in appearance if presented in adjacent fields. One may be seen as **golden yellow** when say, presented singly in the context of a black background; the other may appear **brown** when it is presented as an object's surface color alongside other relatively intense surface colors. Similarly in the well-known demonstration concerning 'colored shadows': daylight that is reflected from a white patch 'in the candle's shadow' appears now **blue** (when the candle is **lit**), but at another time **gray** (when the candle is **snuffed**).[h]

These differences in appearance depend not at all on a putative difference in the physical composition of lights. Just as in the case given, such difference as does occur, occurs with completely identical lights. Whatever circumstance under which a light may be presented to the eye, the definition given above is maintained in **one** sense: no noticeable change occurs by the substitution of that light with another composite light of the **same color**.

Section 2: The Addition of Lights and Colors

By the **mixture or addition of lights**, we understand a superposition of the pencils of rays in question. If $f(\lambda)$ and $g(\lambda)$ are the spectral functions of two lights, then the function of their mixture or sum is given by

$$f(\lambda) + g(\lambda)$$

As a further basic **fact of experience**, we can cite another basic datum: that one can speak **unequivocally** of **a mixture or an addition of colors**, just as one can speak of a mixture or an addition of **lights**. At the outset this operation is indefinitely equivocal, before one settles on a specific interpretation of experience. From each of the colors to be mixed, one can select any of the infinite number of their representatives for use in a mixture. It might be possible at the outset that the resulting lights would not be all identical. Rather some might appear different, at least in some respect. Then a bare specification of the **colors** that are mixed would not be sufficient to determine a color mixture unequivocally. Yet as we have said, experience teaches us that this is **not** the case. The resulting lights all appear the same. Determination of the summands for their **color** does suffice to determine the **color** of the composite light. ***Grassmann***[(i)] expresses this as follows:

The mixture of lights which are identical in appearance will produce lights identical in appearance. (***Grassmann***'s third proposition)

One may also say: the unconditional and complete equivalence of lights which we have defined as the **same in color**, is retained when they are mixed.

This fact – and only this fact – allows us to operate directly with colors instead of lights, i.e. instead of spectral functions, abstracting from extended consideration of the exact composition of lights by which we produce colors. Throughout we will use a Roman capital letter $A, B, L, M \ldots$ as the symbol for a color, and we will use + as the symbol for a mixture:

$$A + B, \quad A + B + L + M,$$

To indicate that two colors have the same appearance, we use the symbol $=$. So if color L – which we know by other means – turns out to be the mixture of colors A and B, then we may write in symbolic form:

$$L = A + B$$

The equality sign carries an ambiguous meaning in this calculus, that is, it has its common algebraic sense, and another in terms of color matches. So long as we use Roman capitals consistently as symbols for colors only, this will lead to no greater confusion than it does for vector algebra and tensor algebra. This analogy proves to be complete, by the way, since all color matches made by a color-normal observer may be represented by three algebraic equations.

It is evident that the ordinary commutative and associative laws hold for this addition of color. No new data of experience are implied, since these laws are already defined in the course of the ordinary addition of spectral functions.

Once one has defined addition for a category of objects, soon one wants to ask about the possibility of subtraction. Can a color X always be found, which when added to A, produces color B?

$$A + X = B$$

This color would be indicated by

$$X = B - A$$

Such an X is not always realized. **Subtraction is not possible in all cases**, as we show by the following (somewhat premature) example.

We call a light a **spectral light**, whose $f(\lambda)$ is only slightly different in width from zero in a small region of λ; we call the associated color a **spectral** color. Now most spectral colors (from about $\lambda = 475$ to $\lambda = 630$) are special, in that they can be produced in only this **one** manner. The set of equivalent lights has only **one** member, if one ignores the fact that across a sufficiently small λ- region, the distribution of energy **within** the region is arbitrary. Neighbouring wavelengths that are sufficiently close have no different effect on the eye. No doubt there are composite lights of the same **hue**, but they always appear somewhat pale (less saturated) compared to the spectral light.

Now if we choose any two such spectral colors $S_{\lambda'}$ and $S_{\lambda''}$, and if we require that

$$S_{\lambda'} + X = S_{\lambda''}$$

then as a matter of fact this stipulation cannot be satisfied, since the wavelengths λ' on the **left**-hand side do not occur in all circumstances in a light that may represent the spectral color on the **right**.

Subtraction is possible in some cases, for a reliable example by transposition of an equation of addition which has been made manifest already. For the moment let us restrict our use of the $-$ (minus) sign to such cases. We may then transpose elements of any color equation that is recognized as valid, so that a viable instance of a color on one side of the equation can be transposed to the other side, with change in sign. But now we must ask ourselves if the transposition of elements in such an addition is always **unequivocal**, even though certainly it may be **possible** in the present case. For example let us write

$$A + C = B$$

as

$$A = B - C$$

and read the latter as: A is that color, which we must mix with C to obtain B. This only makes sense if A is **unequivocally** determined for given B and C. That is to say then, that the obverse proposition to ***Grassmann***'s proposition about experience (cited above) is also valid:

Only *lights of the same color, when mixed with lights of the same color, produce lights of the same color in turn.*

Otherwise stated: from the color equality of $n - 1$ summands and the sum of the said equation, one may determine the equality of the n^{th} summand.

On that point it may be said this is the case – so far as experience is concerned. That is, insofar as the limits of discrimination thresholds do not play a large role – such limits are the subject of the second part of the present work. If A is a very faint color compared to B, then B and C will be little different from one another. Then a small alteration in A will go unnoticed in the sum $A + C$, though the alteration would be clearly noticed if applied to A alone. This uncertainty in the definition of A as a difference is entirely analogous to the experimental uncertainty present in another case. Suppose for example that one calculates a small **angle** – the angle of deflection for a type of glass – as the difference of two larger angles to determine the index of refraction for the type of glass. No one would want to deny that angles can be subtracted unambiguously, on such a basis. And similarly here we should maintain that the establishment of differences between colors is **univocal** in result when it is possible at all.

One should pay serious attention that only a determinate and manifest color is represented on at least one side of the equation, where it is represented as a sum of values. A value of zero (0) should be admissible too, to represent the absence of all light. Otherwise the application of such an arithmetic model is not assured. It may be added that such equations are innocuous as may be formed as the by-product of algebraic or formal operations. Still, they are to be avoided as a guide to use.

One more point should be noted. If one knows that a difference $A - B$ in color can be realized,

$$A - B = C$$

then it does not follow **at all** as a consequence that a non-negative difference over all λ,

$$\varphi_A(\lambda) \geq \varphi_B(\lambda)$$

will be produced by **any two** spectral distributions $\varphi_A(\lambda)$ and $\varphi_B(\lambda)$ which represent the colors A and B. Perhaps it can be concluded that there **are** spectral representatives of A and B for which this holds. Since there is a C, there is at least **one** light $\varphi(\lambda)$ which represents the color C. In that case

$$\varphi_B(\lambda) + \varphi_C(\lambda) \text{ and } \varphi_B(\lambda)$$

are two lights which have the desired property.

Section 3: Multiplication by a Scalar

The notion of the **product of a color and an integer** is given directly by a repeated operation of addition.

$$m\,A = A + A + A + \ldots + A$$

The notion of the **division of a color by an integer** is given as the opposite operation. It is clear that such a division can always be **performed**.

To obtain

$$(1/n)A$$

is to find a color which when taken n times produces color A. To that end one need only choose a light that represents A – call it $\varphi_A(\lambda)$ – and divide by n.

$$(1/n)\ \varphi_A(\lambda)$$

What is not clear a priori is that this operation is **unequivocal**; i.e., that it makes no difference which light one chooses for the color A, or that one always arrives at the same fraction of color.

For the moment let us take this univocal result as having been established. Then a notion of the product of a color with a rational number does follow, through a combination of multiplication and division by integers:

$$(m/n)A\ .$$

Therefore it follows that a color equation **remains** valid, if one multiplies it by a **rational** number, without fresh recourse to experience. That is: unless our postponed proof of a univocal result for the division of a color by a positive integer should make necessary such recourse to experience.

An extension to arbitrary non-negative numbers is assured, since we are dealing with physical reality and not just playing a game with mathematical symbols. But if one wants to play, one can begin by relying on the second of ***Grassmann***'s basic propositions. We express the proposition in this form:

If a light changes monotonically the corresponding color changes monotonically. (***Grassmann***'s second proposition)

In our nomenclature, this form has the advantage that it does not mention color perception – only quantitatively tractable things. Expressed more precisely, our proposition of monotonicity reads:

If $\varphi(\lambda)$ and $\varphi(\lambda) + \delta\,\varphi(\lambda)$ are two ***minimally*** *different lights, and $\psi(\lambda)$ is an arbitrary light which appears the same as $\varphi(\lambda)$, then among the lights which appear the same as $\varphi + \delta\,\varphi$, there is at least one – $\psi + \delta\,\psi$ – which appears minimally different from ψ.*

With that step, we gain the notion of a product

$$\mu A\,,$$

where μ is an arbitrary non-negative number. Also, we have the following proposition.

An equivalence in color is retained, if the color is multiplied by a non-negative number.

The validity of the associative and distributive laws of multiplication – a commutative law is not under consideration – emerges from the relation between our operations on **colors** and operations on **spectral functions**. What distinguishes the former from the latter – the essential core of such propositions – is always and only a determination of the **univocal** nature of the result. The result must be unique, no matter which lights one may use to produce the colors that are the subject of these operations.

Let us take up a question once more – the question that we have held in abeyance temporarily – of the **univocal outcome of division by whole numbers**. One may ask whether, from the equality of multiples

$$mA = mB$$

one may conclude the equality of

$$A = B$$

Or is it possible somehow that a **color inequality**

$$A \neq B$$

can be changed into an **equation** just by multiplication by a whole number m?

Where lights are concerned, the question then becomes: can two lights $\varphi(\lambda)$ and $\psi(\lambda)$ which do **not** appear the same, be made to appear the same, as the objective intensity of both lights is increased by a multiplicative factor m?

Firstly, the **facts** are as follows: this is not the case for the fovea, out as far as the blind spot in the paracentral region. There all color equivalence is independent of absolute intensity.[j] Color **equivalences** do change to color inequalities in the paracentral and peripheral regions of the retina, which regions do not concern us at present. But also in these regions, ***Grassmann***'s proposition is invalid for the equality in appearance of mixtures of equal-appearing lights. Two questions concern us here for reasons of principle: do these two kinds of appearance bear a **logical** connection to one another?; or must the equations be endorsed as a distinct datum of experience and as foundational, in **juxtaposition** with ***Grassmann***'s proposition?

Ewald ***Hering***[(k)] affirmed the latter. As a consequence he would have extended the list of ***Grassmann***'s propositions to include the following:

Lights that appear the same remain the same, if one increases or decreases the intensity of each by the same proportion.

I have also tried (in vain) to manage here without having fresh recourse to experience – I think it impossible. Yet surely it would be incorrect to incorporate wholesale all the machinery of multiplication – to erect it as an independent fact of experience. At any rate, multiplication by an **integer** still follows logically from the operation of addition.

The following conclusions may be drawn. If for a value of n there are **different** fractions of colors $(1/n)\,A$, we say:

$$nA_n = nA'_n = A$$
$$A_n \neq A'_n$$

Then neither

$$(n-1)\,A_n = (n-1)\,A'_n$$

nor

$$(n+1)\,A_n = (n+1)\,A'_n$$

because from each of these equations, together with the first part of the assumption (the result of the univocal nature of subtraction, which seems reasonable here), there would follow:

$$A_n = A'_n$$

which contradicts the second part of the assumption. In the same way, it may be concluded that

$$2n\,A_n = 2n\,A'_n$$

but then

$$(2n-1)\,A_n \neq (2n-1)\,A'_n$$

$$(2n+1\ )\,A_n \neq (2n+1)\,A'_n$$

and so on, for arbitrary multiples of n.

It suffices to say one thing for this reasoning to fly in the face of experience, for the assumption consequently to be contradicted, and yet for univocity to be affirmed. It suffices to posit this as a proposition from experience:

There are no two lights that become equal and later unequal ***in a periodic manner*** *as the intensities of these lights increase proportionally.*

Section 4: Dimensionality

There is still **one** piece missing in our construction of an affine geometry of the color manifold. That is the business of dimensionality, as was alluded in passing in the previous section. I have not yet revealed the roots of dimensionality in experience, and have not yet made use of the fact of dimensionality. I have intended to leave this for last, since all those facts of experience remain valid which were mentioned previously. Their consequences hold valid not only for ordinary color-normal observers, but they hold just as well for color-anomalous and color-deficient observers. The differences among them lie simply in the dimensionality of the color manifold.

If a number (n) of colors are given:

$$A, B, C\ldots$$

there may be a linear relation that obtains among them:

$$\alpha\,A + \beta\,B + \gamma\,C + \cdots = 0 \tag{2.1}$$

(naturally, we suppose throughout that not all these magnitudes $\alpha, \beta, \gamma\ldots$ are equal to zero). The existence of such a relation means that from coefficients of certain subsets of the n colors (those which have positive coefficients in the equation), the same color can be produced as a mixture of certain colors of the remaining coefficients (those which have negative coefficients).

But it may also happen that no such relation obtains. In the former case we call the n colors ‘linearly dependent’, in the latter ‘linearly independent’.

Then the basic facts of dimensionality can be stated:

A. For color-normals (trichromats)
There are linearly independent triplets of colors. Any four colours are always linearly dependent.

B. For partially color-blind observers (dichromats)
There exist linearly independent pairs of colors. Any three colors are always linearly dependent.

C. For completely color-blind observers (monochromats)
Any two colors are linearly dependent.

These expressions simply mean that for such people the color manifold has three dimensions, or two, or one, respectively. Indeed let us consider the more complicated case of trichromats. Let us then consider a linearly independent triple of colors

$$A,\ B,\ C.$$

None of these may be obtained as a mixture of the others, under our premise. For an arbitrary color F there will be a color equation

$$\alpha A + \beta B + \gamma C + \zeta F = 0 \tag{2.2}$$

in which ζ is different from zero – otherwise A, B, and C would not be independent. One can also solve for F:

$$\begin{aligned} F &= x_1 A + x_2 B + x_3 C. \\ x_1 &= -\frac{\alpha}{\zeta},\quad x_2 = -\frac{\beta}{\zeta},\quad x_3 = -\frac{\gamma}{\zeta}\ . \end{aligned} \tag{2.3}$$

To each F corresponds **one** and **only** one triplet of values (x_1, x_2, x_3): **only** one, since otherwise a relation could be found by subtraction which would involve A, B, and C alone. The converse does not hold in general, though at least to all **positive** triplets (x_1, x_2, x_3) there corresponds one and only one color F. Then the colors F form a manifold of no more and no less than three dimensions.

Certainly the expression that we have introduced for the fact of dimensionality is simplest in theoretical terms. Yet it doesn't follow that this expression represents the immediate data of experience. One can never be quite certain that one has really evaluated all sets of four colors, or if perhaps a set of linearly independent colors is to be found among them.

In place of our proposition about dimensionality (for trichromats), ***Grassmann*** uses this proposition:

For any light, another light of identical appearance may be formed as the mixture of white light with either a pure spectral color, or with a distinct purple mixture (that is, a mixture of colors from the ends of the visible spectrum). (***Grassmann***'s first proposition)

A **two-dimensional** manifold is formed by the series of spectral colors and the mixtures which give rise to purples, together with variations in intensity. Variations in the intensity of white form a **one-dimensional** manifold. Consequently and empirically, the combination of all these will form a **three-dimensional** manifold.

Yet in my opinion, this expression of ***Grassmann***'s requires too much inductive support – much broader than is necessary – to confirm basic facts about dimensionality. That is, it seems one really must have evaluated all lights, just to be sure there is no mixture which is not representable in the way mentioned.

By contrast, a proof of tri-dimensionality may be adduced systematically, as follows. First one examines the **series of spectral colors**; one recognizes that in general any two of them are distinguished by more than their intensity (apart from the narrow-band spectral 'ends', to be discussed later). Any attempt to match two of them by objective changes in intensity will fail. **Pairs of spectral colors are linearly independent, in general**.

Now if one examines **binary mixtures of spectral colors**, generally one finds that something **new** is produced (if the wavelengths in the mixture do not lie along the dichromatic 'lines of confusion' – more on that later). Namely, what is produced are whitish versions of the pure colors and all the whitish versions of the purples. These cannot be matched to a single spectral color by a simple change in their objective intensity. **Triplets of spectral colors are linearly independent, in general**.

In other words, we are certain the number of dimensions for the manifold is **at least three**.

One recognizes that the number of dimensions is at **most four**, if one examines **ternary mixtures** of **spectral colors**, and finds that they produce **nothing new**. Rather, one finds that each has a **binary equivalent**. One doesn't need to produce an experimental demonstration for mixtures of four spectral colors or more. One can always replace three of those by two others, until a binary equivalent remains. Then since any light can be considered – to a reasonable approximation – as the mixture of n pure spectral lights, it follows that:

The manifold of colors is covered completely and exhaustively by the pure spectral colors and their binary mixtures.

Now the manifold of binary mixtures has a dimension number of 4. The same would be true of the manifold of colors – that is, it would be true if each color could not arise in ∞^1 ways from the mixture of two spectral colors. Similarly, for example, white can arise as the mixture of ∞^1 pairs of complementary colors. We can arbitrate between a dimensionality of *three* and a dimensionality of *four* at **this** time, after having said that the set of interest is not the set of 'all lights' but rather that of binary mixtures. Then we may have recourse to ***Grassmann***'s idea, and show that it holds equally well for desaturated spectral colors and purple mixtures alike. And so finally the dimension number of *three* is vouchsafed.

The procedure is similar, though much simpler, for dichromats and monochromats. To the former, binary mixtures produce nothing new. In general any two spectral colors are linearly independent (different in color). The spectral colors cover the manifold of all colors; the number of dimensions is therefore *two*. For monochromats, simple changes in intensity suffice to match any two spectral colors, and thus any arbitrary mixture. There are then not even linearly independent **pairs** of colors; the number of dimensions is *one*.

Section 5: Mapping onto a Pencil of Vectors

With that, the **general** laws of light mixture are settled. Basic colorimetry – as we have called it – is settled insofar as the three kinds of color system which we have distinguished can be considered independently. An important theoretical complement of a similarly general nature will be discussed later, that draws a relation between the dichromatic and the trichromatic systems of color.

All that has been said to this point can be summarized briefly. Colors form a manifold of things over which several operations are defined unequivocally: equality, addition, in a limited way subtraction, and moreover, multiplication by a non-negative number uniquely. Associative and commutative laws – among others – hold for these operations in the ordinary sense. The dimension number of the manifold is three (in the normal case, to which we always give first attention). All this is grounded in experience, held together by the immediate data of color matching judgments for adjacent color fields.

Then we may compare these laws of **color**, as **validated by experience**, with a set of **axioms**. They are the axioms we must establish to ground an **affine geometry** for a **pencil of vectors**,[(1)] namely **vectors** that radiate from a point. The comparison reveals a complete correspondence. The manifold of colors, or as we would like to say, the **color space**, provides a three-dimensional model for a purely **affine structure**, so far as color matching relations are concerned. In **this** way, all specific relations in the less-than-easily surveyable domain of color can be made splendidly intuitive. One can consider the color domain to have a one-to-one correspondence with a pencil of vectors in space; each color corresponds to **a** vector, and each vector definitely represents no more than a **single** color. But in doing so one should not forget – as ***Grassmann*** stresses – that the domains which provide these models are differentiated only in our immediate intuition, rather than in some elementary reality. It is on the basis of matching judgments that color space comes into its own, and acquires an affine structure. That is entirely distinct from the vector space or the space of points which aids us in making color space perspicuous. Conversely this intuition obscures a certain risk – no matter how comfortable or indispensable the intuition may be. We must be acutely aware that we are wholly accustomed to thinking of the space of points not just as an affine space, but rather as a metric space – specifically as a Euclidean space. We must be careful to interpret color space exclusively in terms of affine relations, rather than imposing metric relations. Those metric relations are completely meaningless here. Color space may have its own metric, but that notion will be developed in the second Article of the present work [Chap. 4].

It may not be too much to remind ourselves of something here. We know the affine properties of spatial forms to be all those properties which remain unaltered under affine transformation. Forms which can be brought into coincidence by such a transformation play the same role in affine geometry as congruent forms of Euclidean geometry. The general affine group takes the place of the group of motions. In ordinary rectilinear coordinate systems, and likewise in arbitrary systems which have oblique, nonindependent axes, the most general affine group can be expressed as the complete group of linear (nonhomogeneous) coordinate transformations. Every point of the new coordinates is mapped onto the original unchanged system of coordinates by the deformation (‘**alias** method’). One can proceed in another way as well: one retains the numeric coordinates of every point, but one frames these as the coordinates of a new skew system whose axes are arbitrarily oriented to one another (‘**alibi** method’). The most general affine transformations can be instantiated in that way too. Speaking in terms of vectors rather than points, one can say: we conceive the original vector components as mapped onto an arbitrarily transformed system of basis vectors.

In this most general characterization, affine transformation subsumes not only its idiothetic basis – as deformation – but also ordinary motions (of translation and rotation). Yet translation (transposition of the origin of coordinates) is meaningless for our purposes, since we are dealing with a pencil of vectors which emanate from a point. The position of that point in space is not determined in advance. Consequently the coordinate transformation can always be termed homogeneous. What remains by way of transformation, apart from rotation, is then so-called linear deformation. In other words there remain three operations of stretch (changes of measure) in three mutually perpendicular directions. (Of course, this does **not** generally mean in the directions of the coordinate axes, even when rectilinear coordinate axes are used.) Only those properties of shape which remain unaltered by the shears and strains in question are affine properties, or have any meaning in color space. Most importantly for us is that in this fashion, lines are mapped onto lines and planes onto planes – and as a consequence of linearity, the order of a curve or the order of a surface remains the same in number. The collinearity and coplanarity of points, as well as the coplanarity of lines are also unchanged as a consequence. Accordingly a curve will not acquire an inflection point as a result of its transformation, and – what will interest us particularly – a cone will not acquire a new generatrix line. – For later reference (in Section 10, on the theory of dichromacy) we note too that the property of **parallelism** of lines and of planes is also affine-invariant. – Of course this body of facts is well-known to physicists, and trusted in the foundations of elasticity and of hydrodynamics. There the application is to infinitesimally small deformations; by contrast our present application is to quite arbitrary finite deformations.

The laws of color mixture deviate from the axioms of affine geometry on **one** point which we have left unmentioned. That is the restricted application of the operation of subtraction, and the connected lack of significance attached to negative colors ($-A$), as well as the whole business of multiplication by negative numbers. The use of color space as a model for a pencil of vectors might allow this operation to be performed unrestrictedly. Yet it may be concluded that at most half of the totality of vectors find a model, meaning that only that many vectors actually represent a color (as we shall see, in reality it is less than half). For any vector which represents a color, the vector of opposite sign remains unused because there are no two lights and hence no two colors which produce darkness when superimposed. That means colors are not represented by the entire vector pencil extending 4π in spatial angle. Rather a more bounded cone with total angular extent less than 2π represents them (the precise value of this spatial angle has no meaning in affine geometry, by the way). This explains the restricted application of the operation of subtraction. Of course a vector difference may be found for any of the representing vectors as a matter of geometry. That operation can always indicate a vector which does not belong to the cone of real-valued colors. Then there exists no color which is represented by the geometric requirement.

By the establishment of this general state of affairs, color space can be considered purely in terms of affine geometry – from the perspective of judgments of color equality. Color space has the affine structure of a cone of vectors whose angular extent is less than 2π. By the establishment of this state of affairs, in my opinion the most important part of basic colorimetry is now complete. The facts have now been established. As the reader is aware, in this section our discussion did not need to touch on such difficult notions such as the definitions of white, saturation, whiteness content, chromatic power, and complementary color, among others. Yet what remains to be added consists only in the search for color vectors which represent particular lights (i.e., especially spectral lights) by simple concatenation, and also in determination of the affine-geometric relations of these vectors to one another. Thus at one stroke we know the nature of the boundary of the spectral cone, the affine-geometric structure of this cone, and its composition in terms of color. To that purpose we will employ a representation of color which corresponds entirely to a vector representation of coordinates. Because of this complete analogy, calculations become simple and perspicuous. Consideration of the meaning of all these operations in color-theoretic terms will yet occupy some space in our discussion, particularly the meaning of coordinate transformations. Then we will be concerned with a single empirically rich proposition of experience – namely the recognition of the set of trichromatic color matches made by dichromats – and the rule-governed relations between dichromatic and trichromatic systems. This leads to the identification of certain (virtual) color vectors as fundamental stimuli. Finally – because of its wide dissemination – we will introduce the representation of colors in a **color plane**. That is simpler in practical application, but it is somewhat more complicated theoretically. We will derive that plane representation from our spatial representation, and go on to describe an 'advanced' color metric in the next article (Chap. 4).

(Submitted March 1920)

Notes

a. Following ***von Kries***, I designate lights as having the **same stimulus quality** when they are made indiscriminable by a simple change in the objective intensity of one of their component lights. [ff.1, p. 398 original].

b. [Trans.] In the original this passage reads: „Ob die Definition der Helligkeitsgleichheit grundsätzlich möglich ist oder nicht, hängt nämlich davon ab, ob ein gewisser ***Pfaff***scher Differenzialausdruck in drei Differenzialen mit experimentell zu ermittelnden Koeffizienten einien integrierenden Faktor besitzt oder nicht. Nur wenn das der Fall ist, lassen sich die „gleichhellen Flächenelemente“ der Mannigfaltigkeit zu Integralflächen gleicher Helligkeit zusammensetzen.“

c. von ***Helmholtz***, H. Versuch einer erweiterten Anwendung des ***Fechner***schen Gesetzes, im Farbensystem. Zeitschrift für die Psychologie und Physiologie der Sinnesorgane, **2** (1891), p. 1.; von ***Helmholtz***, H. Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden, Zeitschrift für die Psychologie und Physiologie der Sinnesorgane, **3** (1892), pp. 1 & 517.; von ***Helmholtz***, H. Kürzeste Linien im Farbensystem. Sitzungsberichte der Akademie zu Berlin, 17 December (1892), p. 1071. See also the relevant sections of his Handbuch der physiologischen Optik (2d. ed.). [ff. 1, p. 400 original].

d. ***Abney***, W. de W. Modified apparatus for the measurement of colour and its application to the determination of the colour sensations. Philosophical Transactions of the Royal Society A (Mathematical, Physical, & Engineering Sciences), January 1st, **205**, 333 – 355 (1906).; ***Abney***, W. de W. Colour blindness and the trichromatic theory of colour vision. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), April 14th, **83** (565), 462 – 473 (1910).; ***Abney***, W. de W. Colour-blindness and the trichromatic theory of vision. Part II. Incomplete red or green blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 15th, **84**(572), 449 – 464 (1910b).; ***Abney***, W. de W. Colour-blindness and the trichromatic theory of colour vision. Part III. Incomplete colour blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 22nd, **86**(583), 42 – 56 (1911).; ***Abney***, W. de W. Colour-blindness and the trichromatic theory of colour vision. Part IV. Incomplete colour-blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), October 2nd, **87**(596), 326 – 330 (1912). [ff. 2, p. 400 original].

e. ***König***, A. & ***Dieterici***, C. Zeitschrift für die Psychologie und Physiologie der Sinnesorgane, **4** (1893) p. 241ff. [ff. 1, p. 401 original].

f. Cf. the second of the above citations, p.15. [ff. 2, p. 401 original].

g. For an initial orientation to this area, I would most like to recommend the illuminating account put forward by Johannes ***von Kries*** in W. ***Nagel***’s Handbuch der Physiologie des Menschen III.1 (Braunschweig, F. Vieweg 1904). [ff. 1, p. 402 original].

h. Cf. ***Helmholtz***, H.L.F. von Handbuch der physiologischen Optik. 3rd ed., Hamburg und Leipzig: Leopold Voβ (1911, second volume published 1910), p. 230ff. [ff. 1, p. 410 original].

i. ***Grassmann***, H. G. Zur Theorie der Farbenmischung. Annalen der Physik und Chemie (J.C. Poggendorff’s Annalen), **89** (Dritte Reihe **29**), 69 – 84 (1853). Also: ***Grassmann***, H. G. & ***Engel***, F. Hermann ***Grassmann***s gesammelte mathematische und physikalische Werke. 2.2 Die Abhandlungen zur Mechanik und zur mathematischen Physik. Leipzig: B.G. Teubner (1902), p. 213ff. Also:

the Annex to Preyer, W. Elementen der reinen Empfindungslehre. Jena: Hermann Dufft (1877). [ff. 1, p. 411 original].

j. 1, p. 417: Cf. ***Nagel***, W. A. Handbuch der Physiologie des Menschen, **3**(1). Braunschweig: Vieweg (1904), p. 181. & Zeitschrift für die Psychologie und Physiologie der Sinnesorgane, **23**, p. 162. [ff. 1, p. 417 original].

k. ***Hering***, E. Ueber Newton's Gesetz der Farbenmischung. Lotos: Jahrbuch für Naturwissenschaft, **35**, N.F. VII (1887), p. 242ff. [ff. 2, p. 417 original].

l. Cf., eg. ***Weyl***, H. K. H. Raum – Zeit – Materie. Vorlesungen über allgemeine Relativitätstheorie. Berlin: Springer (1918), p. 15. [ff. 1, p. 422 original].

Chapter 3
Outlines of a Theory of Photopic Colorimetry (Part II) Basic Colorimetry or Affine Color Properties, Continued (First Article)

Abstract A formal system of human color perception has been described, which is known as **basic colorimetry**. Positions of the associated color space represent spectral colors, purples, and the color white – among others. Actual colors in color space form at least a three-dimensional color envelope. This chapter describes some affine-invariant properties of the color envelope. The role which plane projective geometry plays in the subordinate color plane is the same as the role which affine geometry plays in color space. The derivation of color coordinates is demonstrated for an arbitrary function of wavelength, as is the derivation of color coordinates under affine transformation of color space. A relation is established between the color matches which are made by dichromat observers, and color matches made by trichromats (color-normal observers). Basic colorimetry provides the foundation for advanced colorimetry as developed in a subsequent chapter.

Keywords Colorimetry • Color vision • Color matching • Color space • Affine geometry • Barycentric coordinates • Affine invariant property • Projective geometry • Projective invariant • Color coordinates • Color-matching function • Dichromacy • Trichromacy • Spectral curve • Schrödinger

Section 6: The Spectral Cone. Delineation of the Real-Valued Color Space, and Orientations in Color Space

The mapping of colors onto the cone of vectors – and conversely the catalogue of the vectors which represents objectively given colors – has a theoretical formulation which is extremely simple, though clean experimental demonstrations are difficult. One chooses three objectively given colors as calibration colors among which there is no linear relation, meaning that no one of them is formed by mixture of the other two. One assigns these to three arbitrarily chosen noncoplanar vectors of the pencil. This fixes the nine variable coefficients which are constrained under the general group of affine (homogeneous) transformations. (Any arbitrary triple of

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noncoplanar vectors counts as 'congruent' to any other, in affine geometry.) Then given a fourth objective color, one establishes a linear equation grounded experimentally in the mixture of the four colors. If A, B, and C are the three calibration colors and F is the fourth color, then as previously one obtains following the equation (Eq. 3.1):

$$F = x_1 A + x_2 B + x_3 C. \tag{3.1}$$

The coefficients x_i indicate the color coordinates of F (its trichromatic coefficients) relative to the calibration colors A, B, and C.[(a)] The signs of the x_i may be different, but at least **one** of them must be positive, since the left-hand side stands for an actual color. Either one of the four colors can be mixed from the three others, or else a nicely balanced mixture of two of them is equal to a nicely balanced mixture of the two others. At any rate, from the equation above we may derive the vector that pertains to F. The x_i are simply its components in affine-geometric terms, relative to the basis vectors A, B, and C. If one constitutes the manifold of vectors this way, representing them in homogeneous terms as lights from a concretely-given spectrum, they form a characteristic cone: the spectral cone. The heads of the vectors trace a specific curve around the surface of the spectral cone. The exact shape of this curve depends on the specific distribution of energy in the spectrum at hand, but the locus of the cone does not. That is, an increase in energy at one position along the spectrum pushes the curve out farther from the origin. A decrease in energy for that region of the spectrum draws the curve closer to the origin.

Our main focus of interest is the fully invariant (in the sense of affine geometry, naturally) form of the cone's **envelope**. Its most important property is just this: the envelope has gores which bound two **plane** sections. Those are the spectral intervals – von König called them transitional segments – from $\lambda = 630$ to $\lambda = 655$, and from $\lambda = 430$ to $\lambda = 475$. They appear as the sections *ROG* and *VOI* in Fig. 3.1. (The Figure also shows to good effect the intersection which the spectral cone makes with an arbitrarily inclined plane.)

Along each of these spectral intervals – they include varying degrees of **orange** and of **indigo** – any two intermediate spectral colors can be produced as a mixture of two outlying spectral colors. But in general it is not the case that the three

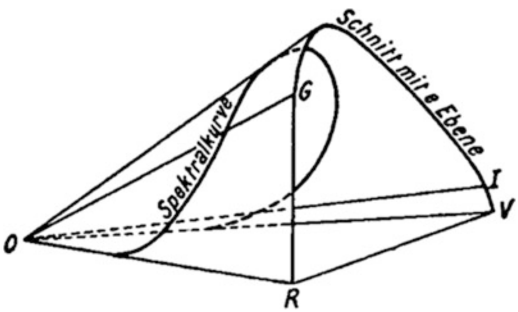


Fig. 3.1 The envelope of the spectral cone (*Spektralkurve*: spectral curve), and its intersection with an arbitrarily inclined plane (*Schnitt mit einer Ebene*). (Reproduced from Schrödinger, 1920)

Fig. 3.2 Sections of the visible spectrum, classified by properties of color mixture. (*Endstrecke*: far region; *Zwischenstrecke*: transitional region; *Mittelstrecke*: middle region; *monochromatisch*: monochromatic; *dichromatisch*: dichromatic; *trichromatisch*: trichromatic; *Rot, Orange, Gelb, Grün, Blau, Indigo, Violett*: ROYGBIV) (Reproduced from Schrödinger, 1920)

generators of a cone lie in a single plane. The plane has no twist: it does have monotonic curvature. Certainly it is not so tightly curved (one may think of the development of a closed envelope) that a triple of three coplanar points may be employed as generators in another way.

The plane gores form the **borders** of the surface in question. Yet if one meanders towards one end of the spectrum, the color vector reaches its boundary, **even before** the end of the visible spectrum is reached. From that point it draws back towards the origin without changing in direction. This implies that within this **end region** (***König***) any color can be produced by any other by a simple change in objective intensity – as we put the matter previously, they are of **equal stimulus quality** (v. ***Kries***). The end regions and transitional regions can also be called monochromatic and dichromatic according to their properties as mixtures of **one** or **two** colors, respectively. Between them lies a **middle region**, along which the spectral cone has convex curvature. Here is an overview of these five regions of the color spectrum, whose boundaries are naturally not determinable precisely, and which may vary slightly from eye to eye (Fig. 3.2).

If one mixes colors from each of the two end segments together in varying proportion, one obtains all possible hues of purple – the most saturated of purple hues which can be produced. The associated color vectors close up the plane segment $R\ O\ V$, which when added to the spectral cone completes it as a closed envelope. Now an arbitrary color must be represented as composed somehow of spectral lights. Then since we know that this mixture must consist of no more than two spectral lights, it follows: only the vectors inside this envelope – but that means all of them – are real-valued color vectors. The three calibration colors with which we began must also lie either in the interior or on the surface of the envelope.

From the rough distribution of colors in the interior we can form an intuitive picture, reasoning that somewhere in the interior there must be a vector direction which corresponds to **white** (say, to sunlight). The sheaf of planes through that direction separates the totality of colors into groups. The groups which lie on the same **half**-plane of the sheaf are related to the same spectral color (or the same saturated hue of purple) by ‘dilution with white’.

They form – following common parlance – less saturated gradations of these spectral colors (or those purples). They are **more** unsaturated, i.e. whiter, the **closer** they lie to the locus of white. Here we mark a clear distinction: this gradation has meaning only for the **same** half-plane, that is, for colors of the **same** hue. Consider the angle that the direction of white makes with pairs of vectors which are **not** coplanar to white. That angle offers no measure of the relative saturation of the associated pair of colors. From our present perspective no such measure can exist, because an affine transformation can eliminate that angle entirely, or change the sign of that relation.

Pairs of colors which are coplanar to white, and which **enclose** white, are complementary colors in the fullest sense: white can be produced from suitably balanced proportions of them.

Pairs of colors which have the **same direction** (vectors which include one another) have – as suggested earlier – the **same stimulus quality** (the same hue **and** the same saturation, so to speak). Colors that are less intense lights are represented by shorter vectors. A similar observation holds, as was made earlier with respect to saturation: A comparison of the lengths of vectors which lie in **different** orientations will not license any inference to a comparison of the proportion of the intensities of lights. From the standpoint of affine geometry, and consequently from the standpoint of basic colorimetry, such vectors count as entirely incommensurate. Up to this point, our system fails to cover any comparison of intensity between colors that have **different stimulus qualities**.

It is important to notice that there is something arbitrary in this approach to color space, in which the direction of white is given prominence. Simply as the color of sunlight, white is not exceptional. It is not distinguished by the elementary makeup of this kind of light, nor is it distinguished by a somehow characteristic place of its assigned vector. Its psychological distinctiveness consists just in this: there is a particular simplicity to the impression, or perhaps better, to the drawing of judgments about this hue independently of sensation. This special situation – in psychological terms – is associated with a composition of light that is in no way exceptional in physical terms, being black-body radiation of about 7000 °C. The situation may be explained readily in terms of genetic psychology, that is, in terms of the evolutionary development of the sense organs. Our eyes have arisen pretty well exclusively under the influence of **this** composition of light, and so they have developed as they have, taking on their present functions. It is hardly amazing that this composition of light plays a special role in color perception as it has developed, mediating between possible extremes. As a footnote, the phylogenetic explanation is also the only natural explanation for the noteworthy coincidence of the **energy** peak of the solar spectrum with the peak of **brightness** across a flat spectral distribution of energy. One might say that the development of better sensitivity to light would be most profitable – so to speak – at the spectral position of the energy peak and on either side of it. There the most distinct perception of objects in dim illumination was inculcated.

But this plausible phylogenetic explanation in no way alleviates a deep ambiguity in the definition of ‘white’. That ambiguity has two sources: a physical source which changes with the time of day and the season of the year, since the composition of sunlight is not rigidly fixed. Secondly, it has a physiological source, since color – meaning the set of lights which appear the same and to which we affix a hue – varies in a significant way with the **adaptation state** of the eye. Even a pre-adapted eye is disposed to change its stance on illumination after only a few minutes of exposure to conditions other than sunlight. After that, either the prevailing light or a light closer to it is recognized as being devoid of hue. Then if a spot of sunlight appears in isolation in that environment, it will be named a color that is complementary to the prevailing illumination. On those grounds we should not declare that what is called 'white' at one time, is always the same in **psychological** terms or not. (I maintain that it **is** the same.) According to our metric definition of color, it is **another color** (cf. Section 1 above). That is the consequence for us presently, since our concept of color is the only tenable account in metric terms. By contrast the psychological concept is indeterminate, and untenable as a metric.

From the previous discussion, it emerges that directions in color space with respect to white do not have a very central role. Rather they are just a guide to intuition. We can make them an exact guide, by precise physical determination of the light mixture that ought to be called 'white'.[(b)] Yet there is an element of convention to this: in purely formal terms we could also assign quite an arbitrary mixture of lights to serve as that signpost, even one that is not white at all. Of course, then descriptions like "a less saturated version of the same hue" would be baseless for mixtures involving the new reference light. But such would also be the case for a properly adapted eye, if we chose sunlight as standard white.

Many observers report that quite apart from the adaptation state of the eye, a color may alter in hue if one simply combines it with a white which does appear free of any tint under the conditions of the adaptation regime.[(c)] This occurs even when the composition of the light is unaltered and only its objective intensity is lowered.[(d)] Then the nomenclature 'same hue' does not even hold for what we have called the same 'stimulus quality' in color, to put the simple facts concisely. At this point I would propose only that even this notable circumstance need not pose any impediment to our deliberations. From the standpoint of the basic metric which is founded on absolute judgments of equality, any such extrapolation in judgments of color – especially to 'equally bright' or 'equal in hue' or the like – is an accessory judgment outside the province of the basic metric. That is not its business, or its competence. At most the accessory judgment can be used for some quick and easy labels, but it should never be applied to delineate exact concepts.

We will return to these phenomena in the final paragraphs of Article 2 [Chap. 4].

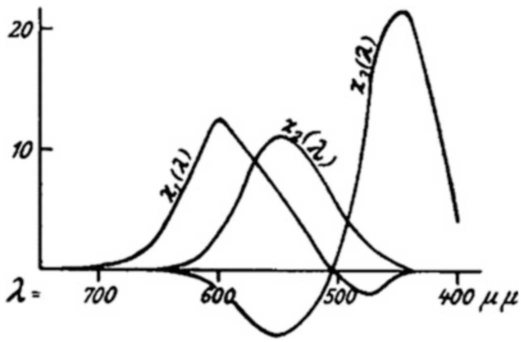


Fig. 3.3 Color-mixture curves of the interference spectrum of sunlight, in relative coordinates across the spectrum. (Reproduced from Schrödinger, 1920)

Section 7: The Theory of Coordinate Assignment

We had thought (in the first part of Section 6, preceding) that we had solved the task of finding coordinates in terms of three calibration colors for an arbitrary color. We considered the task to have been solved in empirical terms by the establishment of relevant color equivalences. For such an equivalence, colors must be given *in concreto*, in a form amenable to experimental manipulation. In comparison, the spectral composition of the light in question – its function of wavelength $f(\lambda)$ – need not be known. Let us now examine the converse situation, in which a light is known only as a numeric function of wavelength. The task is to find, solely by calculation, the color vector to be assigned to the light.

The task can be solved, when a spectrum having a **known energy distribution** has been thoroughly calibrated as a standard, i.e., the ordinates of a continuous sequence of narrow-band lights of that spectrum have been evaluated experimentally by real color matches to produce three continuous functions of wavelength. We call these the color-mixture functions (CMF). In graphical representation color-mixture curves [Eichkurven] are color-matching functions [Aichkurven] for the spectrum. They are formed with reference to three calibration colors which are chosen in advance, as a matter of procedure. Figure 3.3 shows the color-mixture curves of the interference spectrum of sunlight,[(e)] with respect to three spectral colors. The first is at the far red end of the spectrum, the second is green at $\lambda = 505$ μμ, and the third is at the far violet end **depending on the stimulus quality**. The **intensities** of these calibration lights are not to be chosen so that they appear at any arbitrary heights along the spectrum. Instead the **proportion** of these three color-mixture curves is meant above all to produce white, that is to say, a color which has the stimulus quality of sunlight. This has the consequence for the color-mixture curves, as we shall see, that between each one of them and the abscissa there is an area which is subject to algebraic evaluation and which is set to be equal in area for all three. A standard of measure is chosen so that this common area is set to 1000 arbitrarily for each one. This convention supplants the difficult task of determining the **absolute** strength of these calibration lights, that is to say it dispenses with the task altogether.

The **relation of any three trichromatic coefficients** to the color-mixture curve is independent of this particular form of the calibrated spectrum. That is provided the three pertain to the same abscissa (the same wavelengths). An increase in the intensity of light (its energy) at one place in the spectrum will change all three trichromatic coefficients by the same ratio. The form of the spectral cone is given in terms of affine geometry, because of the ensuing changes in those ratios. The precise form of the color-mixture curves is largely dependent on the energy distribution, by comparison. Then by a judicious choice of the energy distribution, **one** of the color-mixture curves might even be given an entirely arbitrary form within the visible region of the spectrum.[(f)] The form of the other two would then be uniquely constrained, however.

The notion of the energy distribution across the said spectrum may lead to a few ambiguities in this discussion or similar discussions. One should by no means take the function of wavelength $\Phi(\lambda)$ – in the sense defined previously – to mean, say, the source of light which serves to create the spectrum. The distribution of energy depends not only on its source, but also on the nature of dispersion across the spectrum.

If we consider a spatial position on the spectrum, a small **band** of wavelengths from λ to $\Delta\lambda$ will be incident on it. How large this band may be, will depend on the strength of dispersion $dx/d\lambda$ (where x is a spatial coordinate along the spectrum) and on the width of the aperture **image**. The wavelength function of **spectral light** $\varphi(\lambda)$ which dominates in the said spatial position, is null everywhere except between λ and $\lambda + \Delta\,\lambda$. There it may count as constant. With a suitable arrangement of prisms it corresponds to the $\Phi(\lambda)$ of the light source, excepting losses due to reflection. **That** $\varphi(\lambda)$ and $\Phi(\lambda)$ agree is not at all so immediately enlightening as is tacitly assumed by most expositions of the topic. Rather it is a consequence of the **cosine law of geometric optics**. The solid angle of a (small) cone of incident rays is transformed in inverse relation to the increase in incident area of each optical image. Then under the law a constant product is formed by the solid angle multiplied by the area of the image incident on the plane.

Then the function $\varphi(\lambda)$ of a spectral source is what we may call a 'rectangular function'. Intensity is given by the **area** of the rectangle, in other words as the product of the height $\Phi(\lambda)$ and the base $\Delta\lambda$. This product is what we had sought to find for the distribution of energy as a function of λ. Only the first of the product's two components – height – is a property of the light source itself; the second depends only on the type of dispersion, by comparison.

It follows from this that $\Delta\lambda$ is the quotient of the image width and the dispersion

$$\Delta\lambda = b : (dx/d\lambda) \tag{3.2}$$

$\Delta\lambda$ is primarily proportional to image width, across the entire spectrum. It depends on wavelength for two reasons: first because the increase varies somewhat, and second because dispersion does vary – **strongly** across the prismatic spectrum.

If one works in the neighbourhood of the diffraction limit, variations in b may be neglected. The rapid increase in dispersion towards violet wavelengths will then provide an accurate picture of the distortion which applies to the function $\Phi(\lambda)$ of the light source under a given type of dispersion. The simplest arrangement, at least **theoretically**, obtains when one uses an optical grating, and projects the spectrum across both sides of the direction normal to the grating. It is simplest because both b and $dx/d\lambda$ are roughly independent of λ. Therefore the distribution of energy is a faithful image of $\Phi(\lambda)$, that is to say, equal to $\Phi(\lambda)$ times a small and constant interval of λ. Yet in the **practical** use of gratings there are idiosyncratic effects of their **sulcate form**, which lead us to suspect considerable, completely unsurveyable complications. In abstraction from such complications, for the sake of simplicity we may consider ourselves justified in calculating results only for the spectrum from an ideal grating. **After all these critical asides, we will still assume that the energy distribution – i.e., the area of our rectangular spectral function – has the form**:

$$C \cdot \phi(\lambda)\ ,$$

where C is a small constant with the dimension of length, and $\Phi(\lambda)$ indicates the λ-function of the light source.

Let us turn once again to the task we set out at the beginning of this section. That was to find by calculation the color vector, or the three color coordinates, for a given function of wavelength $f(\lambda)$. We introduce as known three color-mixture functions $x_1(\lambda)$, $x_2(\lambda)$, and $x_3(\lambda)$ for the spectrum of the energy distribution that we have just considered. Let us divide the visible spectrum of wavelengths into n equal sections, each of about a short width C. Then we may think of the light $f(\lambda)$ as consisting of the superposition of n spectral lights developed from lights given immediately in the spectrum, each by multiplication with the quotient $f(\lambda)/\Phi(\lambda)$. The coordinates of such a spectral light are then:

$$\frac{f(\lambda)x_1(\lambda)}{\Phi(\lambda)}, \frac{f(\lambda)x_2(\lambda)}{\Phi(\lambda)}, \frac{f(\lambda)x_3(\lambda)}{\Phi(\lambda)},$$

and so the color coordinates of $f(\lambda)$ are:

$$\sum_1^n \frac{f\,x_1}{\Phi}, \sum_1^n \frac{f\,x_2}{\Phi}, \sum_1^n \frac{f\,x_3}{\Phi}. \tag{3.3}$$

We ought to be able to replace these sums by definite integrals, and maintain adequate precision. Then for the desired color coordinates of $f(\lambda)$ we find three numbers:

$$\frac{1}{C}\int\frac{f\,x_1}{\Phi}d\lambda,\ \frac{1}{C}\int\frac{f\,x_2}{\Phi}\ d\lambda,\ \frac{1}{C}\int\frac{f\,x_3}{\Phi}d\lambda. \tag{3.4}$$

Concerning the constant C, there is little interest in ascertaining its value by experiment. It becomes superfluous through the normalization of the color-mixture functions, which normalization was alluded to earlier. An especially simple analytic form is obtained for the coordinates of a light drawn from the spectrum. Namely, one obtains for $f = \Phi$:

$$\frac{1}{C}\int x_1\,d\lambda,\ \frac{1}{C}\int x_2\,d\lambda,\ \frac{1}{C}\int x_3\,d\lambda. \tag{3.5}$$

If one then changes the scale of two of the color-mixture functions arbitrarily so that the three integrals become equal, then the incident light will be assigned three equal color coordinates. In doing so, one has chosen three **such** lights of the previously arranged stimulus qualities as calibration lights. They appear the same as the illumination source when they are mixed together. In the main, there is little else to say about the **absolute** intensity of the calibration colors as lights. There is just as little to decide between cases when the illumination source in its actual value is chosen as a norm, and cases when it has some other intensity, say as some multiple of C. To that purpose one ought – one does as a matter of course – to transform the scale of each of the three color-mixture functions proportionally, so that the three integrals produce some round-numbered value like 1000. Ignoring the factor $1/C$ one may then assign the triple

$$1000,\quad 1000,\quad 1000$$

as an approximation to the coordinates of the illumination source. Of course this does not hold for the actual intensity of the source, just for some multiple of C.

Under this arrangement the color coordinates of the light $f(\lambda)$ will become:

$$\int\frac{f(\lambda)x_1(\lambda)}{\Phi(\lambda)}\,d\lambda \tag{3.4$'$}$$

and so on. One may recognize that separate knowledge of $f(\lambda)$ and $\Phi(\lambda)$ is unnecessary to this calculation. Instead it is only the quotient f/Φ as a function of wavelength which is necessary. One might call this the **relative** illumination function of the light in question, taken relative to the illumination source considered as norm. For a light $f(\lambda)$ which we once again consider to be present *in concreto*, the experimental registration of this relative function is a simple business using a spectrophotometer. It is simple if the illumination source – for whose spectrum the color-mixture curves are normalized – is present *in concreto*. This second theoretical method of determining the coordinates can then be applied to a light which is physically present and whose $f(\lambda)$ is unknown beforehand. Certainly it is justified to call the method '**theoretical**' in this case, despite the series of spectrophotometric measurements that is undertaken. The reason is that this method does not constitute color measurement in our sense: it is just a business of the comparison of intensities of pairs of spectral lights, only ones that have a physical congruence. This could be achieved just as well by a colorblind observer, or with a bolometer (cf. our Introduction). A basic judgment of the equality of colors plays only an accessory role here. The overweening part of this work has been achieved once and for all by the mapping of the color-mixture curves.

Section 8: Transformation of Calibration Colors, or of the Coordinate System

It may happen that one **knows** the coordinates of a number of colors – such as the colors of a spectrum, or others still – relative to particular trichromatic coefficients. One would like to derive their coordinates relative to a fresh triple of trichromatic coefficients. Of course the new calibration colors must somehow be expressed uniquely. We may assume that their nine coordinates have been established relative to the original calibration colors following an established method, and that numeric results are available. In computational terms the task comes down to a simple transformation of **one** system of affine vector coordinates to **another** system. The new coordinates will be a homogeneous linear function of the old, in other words. That is an affine transformation, like that which we have encountered already, though in a slightly different sense of the term.[(g)]

Let F be an arbitrary color, and F_1, F_2, F_3 be the original calibration colors, so that

$$F = x_1 F_1 + x_2 F_2 + x_3 F_3, \tag{3.6}$$

and let the new calibration colors be A, B, C so that

$$\begin{cases} A = a_1 F_1 + a_2 F_2 + a_3 F_3, \\ B = b_1 F_1 + b_2 F_2 + b_3 F_3, \\ C = c_1 F_1 + c_2 F_2 + c_3 F_3. \end{cases} \tag{3.7}$$

We wish to find the new coordinates of F, meaning three numbers that we will call y_a, y_b, y_c which satisfy the following color equation

$$F = y_a A + y_b B + y_c C. \tag{3.8}$$

We substitute values for A, B, C from (Eq. 3.7) and collect terms for F_1, F_2, F_3 with the following result:

$$\begin{cases} F = (a_1 y_a + b_1 y_b + c_1 y_c)F_1 \\ \quad +(a_2 y_a + b_2 y_b + c_2 y_c)F_2 \\ \quad + (a_3 y_a + b_3 y_b + c_3 y_c)F_3. \end{cases} \tag{3.9}$$

A comparison with (Eq. 3.6) produces the three ordinary equations:

$$\begin{cases} a_1 y_a + b_1 y_b + c_1 y_c = x_1, \\ a_2 y_a + b_2 y_b + c_2 y_c = x_2, \\ a_3 y_a + b_3 y_b + c_3 y_c = x_3, \end{cases} \tag{3.10}$$

from which the y can be calculated unequivocally, provided that:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0. \tag{3.11}$$

But as is well-known, that is just the condition under which the three vectors A, B, and C are **noncoplanar**. Evidently that is what is required if three colors are to take on the roles of calibration colors.

One attains this end more quickly and with better understanding, if one simply assigns – for **harmonization** of the four color equations (Eqs. 3.6 and 3.7) – the usual condition for the vanishing of a determinant. Those four equations are cast as homogeneous equations in terms of F_1, F_2, F_3, and 1.

$$\begin{vmatrix} F & x_1 & x_2 & x_3 \\ A & a_1 & a_2 & a_3 \\ B & b_1 & b_2 & b_3 \\ C & c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{3.12}$$

In this way one obtains the basic color equations for the new coordinates all at once. The expression can be developed as:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} F - \begin{vmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} A - \ldots. = 0, \tag{3.12$'$}$$

and from that it follows:

$$\left\{ y_a = \begin{vmatrix} x_1 & x_2 & x_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \middle\| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} x_1 + \ldots . \right. \tag{3.13}$$

and so on. One may compare the color equations of (Eq. 3.7) with this system of (as expected) linear homogeneous equations for the conversion of coordinates. One recognizes that color coordinates, indeed colors themselves can be transformed in **contragredient** fashion. This deserves to be recognized in spite of the extreme simplicity of this state of affairs. It should be recognized because the distinction between these two kinds of transformation formulae is not always drawn with sufficient clarity. And with that, our overall task is done.

Two special cases will emerge as important in what follows, and we examine them now. As previously mentioned, it is much easier to delimit the **stimulus quality** of a color in words and symbols than it is to indicate its objective light intensity. Only the two items of data together serve to represent the **absolute values** of color coordinates, while knowledge of the **stimulus quality** can be gotten from knowledge of the **ratios** of coordinates. Previously we saw that, given or having postulated only the stimulus qualities of the original calibration colors, the yet-undetermined task of ascertaining coordinates can be made into a well-formed task. It is sufficiently specified if one stipulates equal areas under the three color-mixture curves. In that way one chooses **such** colors for the three stimulus qualities as primaries, as will sum to produce the undispersed light of the normal spectrum. Since this undispersed light often comes close to meriting the name 'white', let us call this convention – a useful one for normalization – the 'convention about white'. If this convention is satisfied for the **original** calibration colors, i.e., if

$$F_1 + F_2 + F_3 = W, \tag{3.14}$$

where W is the color of the undispersed light at some fixed intensity. Suppose then that the **new** calibration colors are not known exactly, as had been assumed, but are known only by their stimulus qualities. That is, the nine coefficients on the right-hand side of the equations in (Eq. 3.7) are not known exactly. Rather, for each line there is still a factor left undetermined, so that the schema for these coefficients would be replaced by something of the form:

$$\begin{cases} \lambda\, a_1 & \lambda\, a_2 & \lambda\, a_3 \\ \mu\, b_1 & \mu\, b_2 & \mu\, b_3 \\ \nu\, c_1 & \nu\, c_2 & \nu\, c_3 \end{cases} \tag{3.15}$$

An additional stipulation offers a convenient form for the necessary and sufficient supplement to fix this still-incomplete task fully.

$$A + B + C = F_1 + F_2 + F_3 = W \tag{3.14'}$$

i.e., that the 'convention about white' also holds for the new coordinates. If one expresses F_1, F_2, F_3 on the left-hand side, and compares coefficients, the result is:

$$\begin{cases} \lambda a_1 + \mu b_1 + \nu c_1 = 1, \\ \lambda a_2 + \mu b_2 + \nu c_2 = 1, \\ \lambda a_3 + \mu b_3 + \nu c_3 = 1. \end{cases} \tag{3.16}$$

These equations provide unique values for λ, μ, ν if the condition (Eq. 3.11) for the determinant is fulfilled.

At this point we assume that the coefficients $a_1 \ldots c_3$ have already been normalized. Therefore we return to our original system of notation, omitting λ, μ, ν, but then:

$$a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = 1\ . \tag{3.16'}$$

We may then ask ourselves what consequences that has for the coefficients of the transformation in (Eq. 3.13). The following reformulation of the determinants is instructive on that point,

$$\begin{cases} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + b_1 + c_1 & a_2 + b_2 + c_2 & \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ = \begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{cases} \tag{3.17}$$

along with two analogous reformulations.

The 'convention about white' has consequences in both coordinate systems for the schema of coefficients for color equivalence: while there (Eq. 3.7) the three coefficients of each **column** add to one, then too the coefficients of each **row** add to one in the transformation formulae of (Eq. 3.13).

The second special case that we must consider, is one that should inform us of changes that the three **color-mixture curves** suffer, contingent on changes in the calibration colors. The transformation of (Eq. 3.13), valid for the triple of coordinates for arbitrary colors, holds also for the continuous series of the trichromatic coefficients for spectral colors. Namely that is the consequence if we apprehend $x_1 \ldots x_3$ and $y_a \ldots y_c$ as the relevant functions of λ. Each new color-mixture curve then becomes a specific superposition (with constant coefficients) of the three original curves.

Above all one thing is clear, which proceeds immediately from the longhand formulae of (Eq. 3.13). If the **stimulus quality** (**direction**) of the calibration colors remains the same, and only their intensity (length) is modified, then the **form** of the color-mixture curves is unaltered. Each of them is multiplied only by a constant factor. We will consider such a change as inessential, for the moment.

Now let us consider the second case, that only **one** of the calibration colors – say F_3 – is modified.

$$A = F_1 \quad , \quad B = F_2,$$

and of course F_3 should not be translated arbitrarily, but must remain **in the plane** of F_1 and F_3. That means the vectors F_1, F_3, C are coplanar, or in other terms:

$$\begin{vmatrix} 1 & 0 & 0 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 1 \end{vmatrix} = 0 = c_2.$$

The schema of coefficients in (Eq. 3.7) then reads:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & 0 & c_3 \end{matrix}$$

Substituted in the formulae of (Eq. 3.13), that produces:

$$\begin{cases} y_a = x_1 - \dfrac{c_1}{c_3} x_3, \\ y_b = x_2, \\ y_c = \dfrac{1}{c_3} x_3. \end{cases} \tag{3.18}$$

It is **not the same color-mixture curve** whose form is altered in this case, as the one whose dependent calibration **color has been changed**. Rather it is **one** of the two others. Specifically it is the <u>curve whose color vector appears as the intersection of the two planar sides of the fundamental triangle – at the intersection of those two sides which remain unaltered</u>. **That** color-mixture curve can be superimposed on the other color-mixture curve whose color had been changed, using the specific term $\left(-\frac{c_1}{c_3}\right)$.

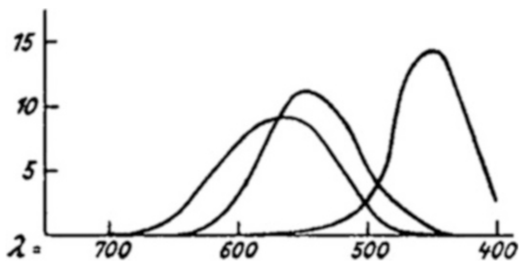
If F_3 had been altered **arbitrarily** instead, then the third color-mixture curve would be superimposable on the first **and** the second, each differing by a respective specific term. This result is important, in that it teaches us how far the spectral distribution of the so-called **fundamental stimuli** may be established through experimental data on dichromats. Healthy eyes exhibit only two kinds of dichromacy: so-called **red-blindness** and **green-blindness**. Two directions in color space can be established from those results: **fundamental red** and **fundamental green**. A third form of dichromacy is to be expected from the theoretical perspective of the ***Young-Helmholtz*** theory: **blue-blindness**. Blue-blindness occurs only with severe pathology of the eye, in cases where experimental results are much less than reliable. The direction attributable to **fundamental blue** is fairly uncertain as a consequence. Our present deliberations will show us that just the distribution of the **blue sensation** along the spectrum is unaffected by this lack of reliability. Instead it is only the distributions of **red** and **green** which are affected, and along each of those a short fragment of the blue distribution may be superimposed without doing violence to the experimental data.

Section 9: Virtual-Valued Calibration Colors. The *Young-Helmholtz* Theory

To begin, we make the simple but important comment that the possibility exists – purely in computational terms, of course – that color coordinates may be recalculated with reference to any three arbitrary basis vectors. Those include triples for which one, two, or all three vectors lie **outside** the envelope described in Section 6. Then there is no color to represent such vectors, meaning no color which can be manifest in light. There is a way – and this is the only way – to ensure the aim that all three calibration colors are positive for the totality of real color. The way is for the chosen fundamental triangle to surround the said envelope entirely, while in turn the triangle that is formed of the real color vectors is surrounded by the envelope. The envelope is not itself a triangle; rather it is curved in places, though it runs convex overall. And so there will always be colors which cannot be obtained by the mixture of real calibration colors. For those colors, the relevant trichromatic coefficients will needs be assigned one or two negative values instead.

Of course for such **virtual calibration colors** as are used in calculation, the relations of their nine coordinates must be known with respect to the original, real-valued calibration colors. The calculation then runs just as before. Having said that much, it is also sufficient to know the virtual **stimulus qualities**, i.e., the **directions** of the new basis vectors. Analytically this means six **proportions** of coordinates are known. The absolute values may be normalized by the 'convention about white' as they were previously. Vector addition of the new basis vectors then produces the same intensity of undispersed light as does addition of the original, real-valued calibration lights.

Fig. 3.4 The elementary-stimulus curves, in relative coordinates across the spectrum. (Reproduced from Schrödinger, 1920)



Let us follow a worked example to show how directions in color space may be determined univocally with color measurements that have been garnered empirically, when the directions fall outside the real-valued color envelope. I cite an example taken from von ***König*** in the evaluation of what he calls **elementary stimuli**. For the fundamental triangle, he chooses the **smallest triangle** which encloses the real-valued colors completely. Two calibration colors are real-valued, namely the colors at the far ends of the spectrum (directions *OR* and *OV* in Fig. 3.1). The third is virtual, and falls on the dihedral formed by the plane acute angles *ROG* and *VOI*. This derivation of the third stimulus quality as a line of intersection between two planes is also uniquely determinate in affine geometry.

Figure 3.4 shows the corresponding primary curves, the so-called elementary stimulus curves. Here the intensities of the calibration lights (lengths of the basis vectors) are determined by the ‘convention about white’.

The basic fact about color dimensionality for normal trichromats, already known to ***Newton***, was originally expressed – since the exact form of the spectral cone was still unknown – in the form that undoubtedly all colors may be shown to be mixtures of three fundamental colors. Mostly red, green, and blue were cited, but sometimes also red, green, and violet. Thomas ***Young*** was the first to tie this to the hypothesis that three distinct processes or excitations are present simultaneously without mutual interference as the physical correlate of color sensation in the eye. Only three fundamental colors were thought each to activate one of these three processes; all other processes were thought to do so in varying proportion. The totality of mixture data would then have been explained easily, on one assumption. The assumption was that the intensity with which any mixed light excites the **first** fundamental process – let us suppose – is composed in an additive way from the differential excitations that are produced by the light’s monochromatic components. If one were to choose actual fundamental colors as calibration colors, then the trichromatic coefficients of a color would be an accurate measure of the excitation strength of the three fundamental processes in the presence of that color. Clearly here the calibration of unit excitation strength for each fundamental process is established in arbitrary fashion, as in the convention about white.

We should refrain from a thoroughgoing critique of this hypothesis here, though many serious objections could be raised. Among the most daring of its corollaries, it appears that under the hypothesis any **color in our metric** would be assigned very specific states of the affected part of the retina. They would be three definite quantitative excitation states. Then how is it, one may ask, that quite different **sensations** – such as brown and golden yellow – can be occasioned by the same condition of a single retinal location? Adherents of the theory may easily be misled into some attempt to argue away these differences in sensation, by some explanation in terms of ‘unconscious inference’. Now it is certainly possible that the physiological correlate of this difference in sensation is due at least in part to a central influence rather than to a retinal process (cf. von ***Kries***’s ‘zone theory’). But in no sense does any of this correspond to any physiological correlate of **logical inference** (as indicated by the word ‘conclusion’). Rather, at most it indicates a capacity for **ratiomorphic intuition**. We must presuppose **at least** one thing, to express the hypothesis in the elementary form it has in the preceding text. That is: the eye must be in the same state of accommodation throughout. It would surely be nonsensical to claim that under the same illumination – and, say, different **pupillary diameter** – the same strength of excitation would be evinced. We need not dwell on this point for our present purpose, not least since **color equivalences** on the fovea are independent of accommodation states.

Ewald ***Hering*** has drawn special attention to such difficulties. Despite them, ***Young***'s hypothesis has proven to be indispensable as a **working hypothesis**. It is the only hypothesis to date which enables a simple consolidation of the results of measurement.

The remarks at the beginning of this section tell us directly that there is no way three real stimulus qualities could stand as fundamental colors in the roles supposed by ***Young***'s theory or in the theory subsequently elaborated by von ***Helmholtz*** and ***König***. No three real stimulus qualities can actually be combined to form all colors. One might however adopt the received idea that one may assume a **two-valued** excitatory process, which is set into action in one direction by a positive trichromatic coefficient, and in the other direction by a negative trichromatic coefficient. If one were to believe that much, one would still have free choice among **all possible noncoplanar vector triples** in the interior of the envelope. The facts of color mixture themselves do not serve to distinguish any of those as special. In principle any three linearly independent colors can be combined to form all others, as soon as one admits the validity of 'improper' mixtures.

On closer consideration it does not appear outright absurd that the fundamental colors may not stand just for directions within the envelope, but also for directions outside the envelope. Those would be virtual stimulus qualities, meaning that the envelope would be surrounded entirely by the fundamental color triangle. That means only: no light elicits **one** fundamental process exclusively. Rather each acts on all three, but in varying proportion. On the evidence of photochemical data, and equally on the evidence of magnetic resonance phenomena (should one wish to elevate such evidence to the level of theory formation), such a state of affairs appears even more probable than another: that there should always be two of the perhaps similarly-constituted processes which are robust to changes in a particular mixture of lights that activates the third process strongly.

There would then be no objection, eg. to calling ***König***'s three elementary colors 'fundamental colors' (as cited above. Only one – green – is a virtual color.). There would be no objection to interpreting König's 'elementary stimulus curves' (cf. Fig. 3.6) as the distribution of excitation values of the three fundamental processes in sunlight. The only thing that is still lacking is a **sufficient reason** to distinguish this triangle from all the others. Once again, as soon as one has decided to admit virtual stimulus qualities as fundamental colors, then every triangle serves the purpose so long as it encloses the envelope of colors. Every such triangle serves the same purpose, at least from the standpoint of the states of affairs which that have been introduced to this point. None is pre-eminent or distinctive.

Section 10: The Relation of Dichromacy to Trichromacy

Let us recall the fact that there are people whose color space has only two dimensions rather than three. They are dichromats. The argument of Sections 5–9 can be developed for the dichromatic eye as well as for the normal trichromatic eye. That development will then be so much simpler, since it would have to deal with a planar vector pencil rather than a spatial pencil. We do not need to develop that argument separately. As we shall see in a moment, the color space of dichromats can be derived directly from the color space of trichromats in a much simpler – hence much more meaningful – way, on the basis of a simple but far-reaching proposition of experience.

A priori it might be possible there would be no relation between a two-dimensional color space for vision and a three-dimensional color space for vision. Mixtures of light which appear the same to color-normal observers might also be distinguished by color-blind observers, and vice versa. Overall it could still be the case that the manifold of distinguishable lights would be smaller for the color-blind.

Yet the facts are different: at the conclusion of an extended comparison of colors, a color-blind observer will be found to have made a majority of errors – from the standpoint of a normal observer. On the other hand, the normal observer will never make errors from the standpoint of a color-blind observer. If one allows the form of expression (which does not prejudice us in favor of any theory) that the trichromat distinguishes **three features** and the dichromat only **two** of any light, then one is struck by the thought that the color-blind observer does not distinguish **other** features of the light, but rather **two of the** features which are apparent to the normal observer. The color-blind observer fails to perceive the third at all. He will recognize two lights as equal which correspond in all three features. If on the other hand he should adjust two lights one against another, as a rule they will still be distinct in the third feature. He will not have attended to that feature, and the lights will appear unequal to the color-normal observer as a consequence.

It hardly needs to be added that adherents of the ***Young-Helmholtz*** theory will try to identify these ‘three features’ with the excitation strengths of three fundamental processes. They will expect there to be three kinds of color blindness, namely kinds in which the first, second, or third of the basic processes is lacking. The theory would receive strong support if that were the case.

In our three-dimensional color space, we may visualize the dichromat’s color vision best in the following way. We connect all the points (heads of vectors) that appear (entirely) equal to him, by a system of curves. Immediately it emerges that this system of curves forms a pencil of parallel lines.

Then let A and B be two distinct colors that are confused by a particular dichromat.

$$A \overset{d}{=} B \,. \tag{3.19}$$

To indicate that this is equality for a **dichromat**, we place a *d* over the equal sign. Moreover, let *C* indicate an arbitrary third color. Then for the dichromat at least, an identity holds:

$$C \overset{d}{=} C \,. \tag{3.20}$$

On each side of this equation we add the terms of another equation (Eq. 3.19) multiplied by $\lambda \geq 1$.

$$C + \lambda A \overset{d}{=} C + \lambda B \,.$$

Then if the trichromatic color $C + \lambda\,(A - B)$ **does** exist, i.e., if the pertinent vector is part of the real-valued color space, then it follows for dichromats:

$$C + \lambda(A - B) \overset{d}{=} C \,. \tag{3.21}$$

And if:

$$C + \lambda(B - A)$$

then there exists:

$$C + \lambda(B - A) \overset{d}{=} C \,. \tag{3.21$'$}$$

That is, (Eq. 3.21) holds for any **real** λ for which the trichromatic color on the left-hand side does exist. Yet in real color space, these colors will cover the real-valued segments of those lines drawn through the color point (the vector head) of an arbitrary color *C*, parallel to the lines which connect points *A* and *B*. That is, their color vector protrudes beyond color vector *C* by the addition of a multiplicandum of the vector difference $(A - B)$. That vector difference is not required to represent a real-valued color, but nevertheless it does exist as a vector.

As soon as **one** definite pair of confusion colors is known for a dichromat – meaning two colors which are indistinguishable to him but which are distinguished by color-normals – then one knows that all colors will be confused that lie on a parallel to the lines which connect those two colors.

Since the color manifold is actually reduced to two dimensions by the identification of such a pencil of parallels, then it may be assumed also that all the confusions have been exhausted for the dichromat in question. The claim is that the converse of the proposition just proven also holds: **only** such pairs of colors will appear equal to the dichromat, as lie on the same parallel. That proposition admits an exact demonstration.

If there were only a single pair of confusion colors A' and B' such that the vectors $A - B$ and $A' - B'$ were **not** (directly or inversely) parallel, then it would follow that **the entire plane** of colors

$$C + \lambda(A - B) + \mu(A' - B')$$

would appear identical to C for dichromats – insofar as this expression represents real colors at all, meaning any part of the plane which lies in real-valued color space. In that eventuality the sheaf of parallel planes would reduce the color manifold to only **one** dimension. That contradicts the supposition that a **dichromat** is still under discussion.

One may recognize immediately that if one of the fundamental vectors is given for a color coordinate system – say that the vector F_1 specifies a direction for the pencil of parallels – then the first coordinate of the system does not matter at all for the color-blind individual in question. Those and **only** those color pairs appear equal to him, as correspond in their second and third coordinates. If F_1 should be a real-valued color vector, then colors of this stimulus quality would not be perceived by him at all, at any arbitrary objective intensity. That is because any color of the coordinates

$$(x_1,\ \ 0,\ \ 0)$$

appears the same as total darkness, at

$$(0, 0, 0)$$

One might call these **faux colors** for dichromats.

Actually, this never happens: the faux colors for dichromats are always **virtual** colors.

Suppose there were just three kinds of dichromats – that is, suppose one were always to be led to the same three distinct directions for the pencils of parallels in trichromatic color space, through investigation of a large number of color-blind individuals. That would be very solid support for the ***Young-Helmholtz*** theory. Uncertainty in the choice of fundamental colors – alluded to at the end of the previous section – would be alleviated by the irrepressible force of conviction that in each of these three conditions **one** of ***Young***'s fundamental processes is **lacking**. One might then transform these three directions, to interpret them as coordinate axes. The color coordinates so derived would then be ascribed a deeper meaning than any others. They would be a measure of the intensity with which the incident color excites the three fundamental processes. Independently of any hypothesis, at least **these** three numbers would have the advantage that one could tell at a glance not only if two colors are equal (or of equal stimulus quality) for color-normals, but also if they may be equal for one of the three groups of color-blind individuals.

There would be a contradiction – or at least a very great complication – in our theory, if one found **more than three** distinct groups of color-blind individuals.

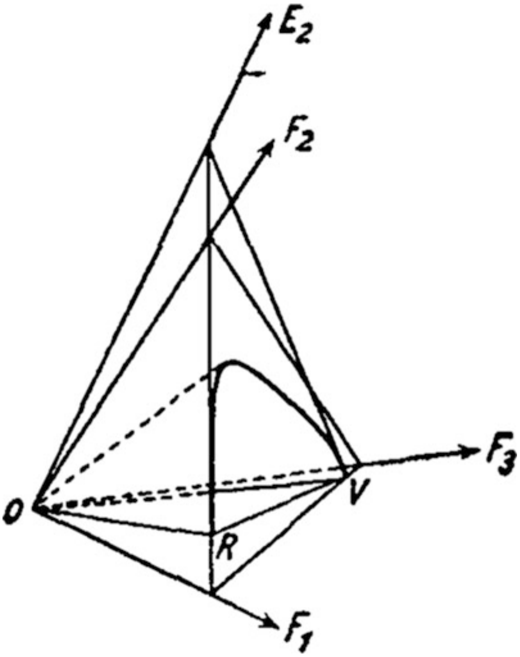


Fig. 3.5 The basic color metric, represented as intersecting a plane. $\boldsymbol{F_1}$ $\boldsymbol{F_2}$ $\boldsymbol{F_3}$ are three directions of confusion. (Reproduced from Schrödinger, 1920)

As the investigations pursued by ***König*** and others have shown, only two sorts of dichromacy occur as characteristic physiological variants of healthy eyes. The lines of confusion (those directions of pencils of parallels) fell along one of two affine-geometric directions within the bounds of experimental error, for all eyes which are dichromatic but which prove to be healthy otherwise. By the way – as had been mentioned – those affine-geometric directions of normal trichromatic color space do not represent any real color drawn from the origin. Those two directions drawn from the origin by parallel vectors will both lie outside the envelope of real-valued colors.[h]

One assumes that these two directions represent two of the three virtual fundamental colors, to be called **fundamental red** and **fundamental green**, for reasons to be elaborated shortly. Correspondingly the two kinds of color blindness will be called **red-blindness** and **green-blindness**.

Certainly **blue-blindness** – the third kind postulated by theory – has been observed,[i] but only in areas of retinal pathology in severely damaged eyes (as in retinitis or ablatio retinae). Those areas were also functionally damaged in other respects, especially in acuity. Then the determination of lines of confusion is made very difficult for the adapted eye on the basis of adjustments for color equality. Nevertheless it does appear that these blue-blind observers judge a faux color with close concordance among themselves.

Figure 3.5 illustrates the rough situation of these three directions of confusion F_1 F_2 F_3 in relation to the spectral cone. For the sake of clarity the entire vector pencil is represented as intersecting a plane – meaningless in itself – as is the trace of its projection onto the plane.

In passing I would like to suggest that this two-dimensional projection can be used on its own as a representation of the entire basic color metric, in place of the spatial vector pencil. It has found much more frequent application than the spatial construction. One calls this the color plane or the color triangle. A substitution is made in this way for the third coordinate: colors of the same stimulus quality may be projected onto the same point. One can distinguish those projected points by a measure or a quantity where stimulus quality is assumed to be proportional to the length of the vector in question. By judicious choice of the relative proportions of the three coordinate directions (which will fix proportions in every other direction) one can arrive at a color plane diagram for which ***Newton***'s well-known centre-of-mass construction holds. In that construction, the location of a mixture color is the centre of mass of its components: its mass is the sum of their measures of mass. Curves of the **same color** for color-blind observers are not intuitively evident in this color plane, because of the missing third coordinate. Yet **curves of the same stimulus quality** are evident. They are straight lines through the **point of confusion** (vector head of the direction of confusion). The role that plane projective geometry plays in the color plane is the same as the role that affine geometry plays in color space. Vector coordinates become projective coordinates whose reference triangle is outlined by the coordinate triangle.

However, the relationship of the two geometries is not exact. One has to make the projective coordinates **barycentric** (i.e., one must translate their 'unit point' to the centre of mass of the coordinate triangle) if a simple version of ***Newton***'s centre-of-mass construction is to hold.

After this digression, let us turn once more to the spatial representation. We bring this to the fore not only for its comparative simplicity, but also because it alone will find application to the next part of the present work.

As was intimated, the directions F_1 and F_2 are more or less fixed by equations for red-blind and green-blind observers. There is one peculiarity: the plane $F_1\ F_2$ coincides with the plane ROG in an intermediate segment (cf. Fig. 3.1). The implication is that some colors appear to be 'mixtures' of only fundamental red and fundamental green: not only at the far red end of the spectrum, but also for scarlet and orange colors up to about $\lambda = 630$. The blue process is not at all stimulated by lights at these wavelengths.

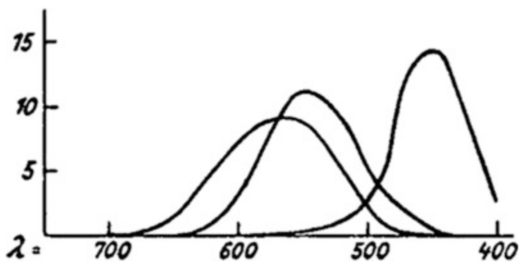


Fig. 3.6 ***König***'s fundamental stimulus curves, in relative coordinates across the spectrum. (Reproduced from Schrödinger, 1920)

It must be noted 1) that ***König*** **took pains** to simplify the representation **in order to** arrive at **consistency** between the directions of confusion as they are given by experiment, and to arrive at the result just mentioned. In reality the directions of confusion are scattered across two small spatial angles. It must also be noted 2) that the eye is quite insensitive to small additions of spectral blue or violet in this domain of wavelength. Yet it may still be possible that exclusion of the third fundamental process from this domain of wavelength is still only approximate.

The fundamental green F_2 which we used in Section 9 lies **nearer to red** than ***König***'s second **elementary** color (E_2 in Fig. 3.5).

As mentioned, the location of F_3 is quite uncertain. Indeed the locus that we have given here in connection with ***König*** (coplanar with F_1 and far violet) represents ***König***'s own investigations on blue-blind individuals very badly. As long as a more definite localization is not possible, we may not want to shake our conviction in this locus of record from the literature. ***König***'s so-called fundamental stimulus curves are shown in Fig. 3.6.

Those are the color-mixture curves of the diffraction spectrum of sunlight, drawn along fundamental vectors with the directions F_1 F_2 F_3, and of such lengths that they produce the color of sunlight once combined. The ordinates of the red curve show the spectral distribution of the excitation strength of the red process, as one allows the eye to be stimulated by the colors of the said spectrum in unbroken succession. To that purpose the units of excitation strength are chosen arbitrarily, provided that all three are represented for sunlight (or a light which appears the same) in **equal** measure.

We still have to justify why we speak of red, green, and blue processes and so on, not just of processes numbered 1, 2, and 3. After all this is not about real colors, but rather only about the sensation which is represented by the excitation of a **single** process, which process is unknown to us.

If one connects the direction of a fundamental color and the direction of white by a plane, then that plane will intersect the color envelope in two lines. Of the two, one lies in the acute angle between white and the fundamental color. Vectors of this direction may be composed additively from white and the fundamental color. Yet the relevant colors do not excite the fundamental process in isolation. Rather that excitation is joined by the excitation strengths of all the processes as would produce white. Now we do have an intuitive idea how such an addition of white would alter a **real** color. We are accustomed to say that hue is unchanged or little changed in that instance; rather only that color saturation is changed. Perhaps we ought to assume that the colors which lie between white and the fundamental color (and which are coplanar with them in direction) behave in the same way as the transformation of a whitish version of a spectral color to the spectral color itself. It might be thought possible that colors between white and the fundamental color would elicit sole excitation of a particular fundamental process. Then we may believe that those partial colors provide some qualitative image of the fundamental color, which may be thought of as the more saturated result of a farther transformation.

The whitish exemplars of the fundamental colors are then a reddish purple (roughly complementary to the hue at $\lambda = 494$), a green of about $\lambda = 505$, and a blue of about $\lambda = 470$ (complementary to $\lambda = 573$). This supports our system of notation.

By the way, it may be recognized immediately that these spectral lights should be attributed a straightforward meaning in experiment – which proceeds from the representation of the relation between dichromatic and trichromatic vision by the pencil of parallels. This does work for the first two lights. For the red-blind individual, $\lambda = 494$ appears indistinguishable from trichromatic white (sunlight) of a particular intensity, as does $\lambda = 505$ for the green-blind. The two 'neutral points' lie so near to one another that sometimes they have been confounded or even reversed in order. They have been confounded by inexact procedures, as well as by the coloration of ocular media. That is one of the reasons why the fundamental difference between red-blind and green-blind individuals had been denied for so long.

Two neutral points around $\lambda = 470$ and $\lambda = 573$ were to be expected for a blue-blind individual. In fact for the diseased eyes that have been mentioned, only one neutral point has ever been found – and that was between $\lambda = 560$ and $\lambda = 570$. These and yet another irregularity leave the position of F_3 in doubt, insofar as one might like to ascribe a meaning to these results for establishment of the fundamental colors. If one wishes not to make the interpretation, the choice of F_3 is **entirely** arbitrary, of course.

We had already mentioned that a possible change in this position would not occasion a change in the form of the blue curve, but in the forms of the red and green curves. Each of those curves could be superimposed along a bounded segment of the blue curve, which segment would depend on the new position of F_3.

Unfortunately the visual function of **monochromats** cannot be used to better the calibration of the fundamental colors. No such simple relation obtains between monochromats on the one hand and dichromats and trichromats on the other, as obtains between dichromats and trichromats. In the rod-free region of the central retina, which region alone concerns us in the present work, it is very likely that monochromats are completely blind. The sensitivity curve for their paracentral and peripheral regions does not correspond, say, to one of the three normal fundamental stimulus curves. Rather it corresponds to what is a normal sensitivity curve for achromatic scotopic vision, which is just absent from rod-free regions.

We will not elaborate on the interesting cases of **anomalous trichromacy** at this point.

Notes

a. Occasionally one designates other things as vector components: namely **orthogonal projections** onto the coordinate directions. Components of that kind are not considered here, because the property of perpendicularity is meaningless here. Clearly in metric geometry the two types of components stand in the relation covariant $\leftrightarrow$ contravariant. They coincide numerically if the basis vectors are pairwise orthogonal and equal in length. [ff. 1, p. 428 original].

b. Cf. ***von Kries***, in: ***Nagel***, W. A. Handbuch der Physiologie des Menschen, 3(1). Braunschweig: Vieweg (1904), p. 116. [ff. 1, p. 432 original].
c. Cf. the third of ***Helmholtz***'s articles cited above, and: ***Abney***, W. de W. On the change in hue of spectrum colours by dilution with white light. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 10th, **83**(560), 120-127 (1909). The "equivalences of hue" that ***Abney*** establishes there by experimental means belong wholly to advanced colorimetry. [ff. 2, p. 432 original].
d. Cf. especially ***Exner***, F. Über die Grundempfindungen im ***Young-Helmholtz***'schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **111**, 857 – 877 (1902). [ff. 3, p. 432 original].
e. Adapted from ***Dieterici***s elementary stimulus curves (see below) in ***König***, A. & ***Dieterici***, C. Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsverteilung im Spektrum. [Fundamental stimuli of normal and anomalous color systems, and their intensity distributions across the spectrum] Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **4**, 241 – 347 (1893). [ff. 1, p. 433 original].
f. Only the **signs** of the trichromatic coefficients may not be altered ! [ff. 1, p. 434 original].
g. In § 5 above, our theme was that ordinary Euclidean-metric space must be considered subject to all possible affine transformations. The purpose is to liberate Euclidean-metric space from all those relational properties which are meaningless for color space, though they may be properties in common intuition. The change in a coordinate system that may have been established is unconnected to those transformations; each transformation **affects** this system, together with all the color vectors the system contains. Here all the color vectors **maintain** their locations, and three new vectors are sought as basis vectors; **that** is the reason why coordinates change value here. [ff. 1, p. 438 original].
h. Mainly one should note: ***König***, A. & ***Dieterici***, C. Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsverteilung im Spektrum. [Fundamental stimuli of normal and anomalous color systems, and their intensity distributions across the spectrum] Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **4**, 241 – 347 (1893). [ff. 1, p. 452 original].
i. Note especially: ***König***, A. Über « Blaublindheit ». Sitzungsberichte der Königlich Preuβische Akademie der Wissenschaften zu Berlin, **34**(2), 8. Juli, 718 – 731 (1897). [ff. 2, p. 452].

Chapter 4
Outlines of a Theory of Photopic Colorimetry (Part III): Advanced Colorimetry, or Full-Blooded Colorimetry (Second Article)

Abstract There are familiar relations among colors which remain unrepresented in the affine geometry of basic colorimetry. An **advanced colorimetry** will incorporate maximum color similarity. Advanced colorimetry applies Riemannian geometry, not Euclidean geometry. A line element is proposed for color space; it preserves additivity of brightness. The line element bridges heterochromatic differences and just-noticeable differences in color. One problem is that the ***Bezold-Brücke*** phenomenon pervades this range of conditions; a correction factor must then be applied to the line element. Then two constraints conflict: the constraint that both large and small differences are accounted for, and the constraint that Fechner's law holds uniformly across color space. This tension changes the theory of advanced colorimetry to a transitional theory or heuristic account.

Keywords Colorimetry • Advanced colorimetry • Color vision • Color manifold • Color space • Line element • Fechner's law • Color similarity • Just-noticeable difference • Affine geometry • Riemannian metric • Helmholtz line-element • Pfaffian • Isolychne • Color wheel • MacAdam ellipse • Heterochromatic brightness • Geodesic line • Projective transformation • Bezold-Brücke • Schrödinger

Section 1: The Need to Transcend the Basic Metric

In the first part of the present work, we were concerned with relations among colors for which experimental data can be garnered exclusively by the use of **judgments of equality**. By that we mean the adjustment of two adjacent color fields to achieve complete indistinguishability. The line of separation between them disappears if it is not marked by anything but their contingent difference in color. These relations – commonly known as the laws of color mixture and which we have called basic colorimetry or the affine geometry of color – found a complete and adequate representation in the modelling by the color manifold of a spatial pencil of vectors in affine geometry. We have shown there is an exact correspondence of raw

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color experience with the axioms of affine geometry. That is the reason we have indicated repeatedly that such – and only such – relations between candidate vectors have a correlate in colorimetry: those relations which remain invariant under affine transformation. There are a multitude of simple geometric relations for which that condition does not hold. So: the angle between two vectors, the relative lengths of two vectors which differ in direction, and the separation between the heads of two vectors (color points): all must remain without any correlate in color space. That is because these magnitudes can be varied arbitrarily across a wide range of values by a choice of representations among equally-well justified formalisms (or what comes to the same thing, by the exercise of an affine transformation applied to the original representation).

On the other hand, both our intuition and the judgments about color that we make in daily life tell us that our present formulation is far from complete. There are still many **relations among colors** that have found no **geometric correlate** in our vector space. Rather we have ignored them completely. We cannot tell at all from our representation which of two colors of different stimulus quality may be brighter, and which darker. The relative length of vectors has almost nothing to tell us here. We may assign three quite arbitrary vectors to colors in the role of calibration colors. For example a very bright color may be assigned quite a short vector; at the same time a much darker color may be assigned a very long one. On similar grounds we ought never to draw conclusions about the degree of difference between particular colors from the separation of two color points. And yet we know that judgments about the relative size of differences between distinct pairs of color ('this step in color is very much larger than that one') are carried out decisively and with conviction, at least for extreme examples.

It is far from astonishing that all of this lies outside the scope of the argument we have developed to this point. Until now, two colors were either equal or unequal for our purposes. We did not concern ourselves either with degrees of dissimilarity, nor with any particular form of equality ('in hue', 'in saturation', 'in brightness'). <u>The proposition which will be developed in this next part is that all conceptual schemes of this type can be formulated and quantitatively substantiated solely by the notion of adjustment for maximum similarity as it was described in the introduction to the first part</u>. The notion of adjustment b) which was mentioned there – concerning **just-noticeable difference** – serves as an important anchor to hypotheses in experiment, though they be very general.

One can imagine a closely allied attempt to specify concepts such as relative brightness, color difference, and others quantitatively. One might think that among the many equipotent representations in affine geometry, there might possibly be one that is distinctive and suitable. It would be distinctive **just in that** relative lengths, the (normally Euclidean) separation of color points or other similar properties would actually retain the meaning that we automatically feel tempted to impart to them. Such a significant interpretation would emerge as a very specific case of the general proposition we might put forward. Such an impulse can be shown to be impossible at once. To be clear, this impossibility proceeds from ***Fechner***'s Law for simple changes in intensity – which law does hold most of the time. Let us consider two colors

$$A(1+\varepsilon)A\ ,$$

for which ε is chosen to be large enough that the colors are fully and clearly distinguishable. At the same time the two are not distinguishable at all if ε is 10 times smaller. So then, the colors will also be indistinguishable.

$$10A\,(10+\varepsilon)A$$

Yet the Euclidean separation of the vectors is the same for both sets of color pairs, however one would like to elaborate the vector representation otherwise. It would not be a congenial conclusion for us to say that two color intervals are the same, when one is clearly distinguishable and the other lies below the threshold of discriminability.

Section 2: A Measure of Difference. The Line-Element of Color Space

Suppose that we did possess a measure of the difference between any two colors X (with coordinates x_1, x_2, x_3) and Y (with coordinates y_1, y_2, y_3), that is, a function

$$s(x_1, x_2, x_3, y_1, y_2, y_3)$$

which specifies the difference. Then it would be clear that **an adjustment for maximum similarity** would need to be a way to minimize the function s under certain accessory conditions in experiment for colors X and Y (e.g., setting X constant and varying Y in a predetermined manner). That much is implicit in the concepts, and requires no special assumption.

The general hypothesis – that we introduce as ***Helmholtz*** did – deals with the measure of difference: more precisely, the measure of difference for colors in a small neighbourhood.

We surmise:

Assumption 1: for $y_1 = x_1 + d\,x_1, y_2 = x_2 + d\,x_2, y_3 = x_3 + d\,x_3$ the square of s, for which we can reasonably write ds^2, the quadratic form of the differential coordinate equation is:

$$ds^2 = \sum_1^3 \sum_{k=1}^3 a_{ik}\, dx_i\, d\,x_k \qquad a_{ik} = a_{ki}, \tag{4.1}$$

for which the a_{ik} indicate definite functions of x_1, x_2, x_3.

Assumption 2: for any two **just distinguishable** colors the $d\,x_i$ (hence also ds) should be considered differentials (meaning the a_{ik} do not vary markedly) and **that *ds* has the same value for every such pair of colors**.

The exact value which one bestows on *ds* for such a pair of colors is unimportant, and indicates only a means of standardization. Primarily it is important that one thinks of a standard color as having been chosen, and *ds* defined for increments in just-noticeable intensity of that standard color – rather than for the **relative increase in intensity** which is required here. Because of ***Fechner***'s Law (as commonly understood) – which holds true for a large part of color space[(a)] – if the standard is chosen only for a local region it proves to be the case that exact specification of the standard color is unnecessary. (I mention the standard color only to indicate that the **general** validity of ***Fechner***'s Law is quite an independent matter. If it were invalid, then one would need to specify the standard color **precisely**. Come to that, one would also have to fix *ds* to a value of one for just-noticeably different colors. That would mean investing all subsequent formulae with a constant term for proportionality.)

Naturally our second assumption is the **essential** one. The first assumption must be derived faithfully from experiment, that is, strictly derived from measurement of the discrimination of differences. I find it hard to believe that such measurements would ever prove inconsistent. It is impossible to derive anything but an integer-valued root from the relevant integer series, for obvious reasons. (A change in sign across all the differentials must leave *ds* unchanged in all instances.)

It is well known that the formulation (Eq. 4.1) completely specifies a **metric** – under the general classification set out by Bernhard ***Riemann*** – for the manifold of number triples (x_1 x_2 x_3) if one conceives *ds* as the **line element** of the manifold. Now – and only now – we may speak of an actual metric for color space. Its general form is invariant for this formulation, not only under the linear transformations considered earlier, but also under quite arbitrary transformations of the x_i. Still of course the a_{ik} alter their form in the process in a familiar manner. The **general** proposition then holds for every system of trichromatic coefficients. As soon as we want to try a new system of particular functions a_{ik}, we must employ an entirely different system, meaning an entirely different triad of fundamental colors. Of course we would hope and expect that the line element would take on its simplest form for the 'true' fundamental colors – the ones we arrived at in the first section by entirely different methods.

For many purposes, that is to say when the comparison of **very similar colors** is in question, we find that ***Helmholtz***'s differential formula is a sufficient statement. For other purposes, especially in 'strongly heterochromatic' photometry, we require a measure of difference *s* which also holds for definite and finite differences of color. Attention should be paid to the fact that according to (Eq. 4.1), *ds* is a homogeneous function of degree one for $d\,x_i$, but it is not the differential expression of a specific coordinate function. Nevertheless there is only **one** significant generalization – in fact one can say there is only one generalization possible at all – of ***Helmholtz***'s measure of difference, if one does not regard the establishment of such a measure as altogether impossible for very different colors. That assumption follows.

Assumption 3: The difference of arbitrary colors is to be judged by the magnitude of:

$$\int ds$$

taken as the shortest connecting line (geodesic line) between two color points in the manifold whose metric is established by (Eq. 4.1).

It need hardly be mentioned that what is meant as the shortest line or geodesic line may not be a straight line of vector space. Rather this is to be understood as the curve for which the integral takes on its smallest value.

With reference to Assumption 2, our Assumption 3 amounts to this much in the intuitive language of colorimetry: Two colors will be declared greater in similarity, the fewer just-noticeable steps that can be established in a continuous progression from one to the other – **if one chooses the intervals for this procedure as skillfully as possible**.

To offer an easily visualizable representation, we ought to retain the vector space that we used in the first section. We ought to add even more specialized content. Yet we should keep in mind that concepts like the length of a curve, perpendicularity (or any concept of angle), direction outwards from a point, or any such concepts are **no longer** to be judged in terms of immediate intuition, but rather in terms of ***Riemann***ian geometry. Then we may proceed to apply **entirely arbitrary** transformations to the coordinates, which will prove useful in the following section.

Here is a description to provide an intuitive grasp of our three assumptions, should we choose to identify the terms 'equally distant' and 'equal magnitude of difference' a priori:

Color points count as equally far from a color point F, that are equally discriminable from F (***Assumption 2***). Those points lie about a small ellipsoid, whose centroid is F (***Assumption 1***). Every diameter of all these small ellipsoids counts as the same in length (***Assumption 2***). One can derive surfaces of equal separation distance around F, for **small** separations: that is, one can expand all diameters of the first ellipsoid – which we will call an 'scatter ellipsoid' – by the same proportion, meaning that similar and similarly situated ellipsoids are formed by expansion. The proportion of expansion is the proportion of separation distance (***Assumption 1***). Strictly speaking, this Euclidean construction can only be performed over an infinitely small neighbourhood. In order to construct surfaces that lie at equal and **finite** distances from F, one does not expand the diameter of the scatter ellipsoid proportionally **in the same direction** (in the Euclidean sense). Instead one expands **along the straightest direction** in the Riemannian sense. That is, it follows along the geodesic by equal sections of the line integral

$$\int ds,$$

which is the measure of separation. (***Assumption 3***)

At this point we will begin by drawing some general conclusions from these assumptions, still not specifying the coefficients a_{ik}. Along the way we will propose a particular line element and investigate that further. Although certain not to be **exactly** what is needed, this will prove to be the best approximation to reality that maintains maximum simplicity.

Section 3: The Notion of Brightness

Many people deny the very possibility of a photometry of **strongly** different colors. Against that opinion, there is no doubt that one can adjust two colors that differ only a **little** in stimulus quality (colloquially, in hue and saturation) so that they attain the same brightness.

One can adjust these to the same brightness by suitably altering the objective intensity of one of the colors. We will tie our concept of brightness to these 'nearly monochrome' comparisons of brightness, which also seem to be gaining dominance as an experimental manipulation. Clearly our notion here is of an adjustment for maximum similarity.

Let OF and OF' (Fig. 4.1) stand for the directions of two adjacent color vectors in a neighbourhood. Let F be a **fixed** point on the first vector and F' a **variable** point along the other. In order to find the position of F' which lies closest to point F, we have drawn an ellipsoid of equal separation of such size that it that just touches the line OF'. The point of contact F'' is the point that is selected by the process of adjustment. That is, in the colloquial language we would adopt: F'' and F are **equally bright**. If we proceed with this same construction for all possible directions

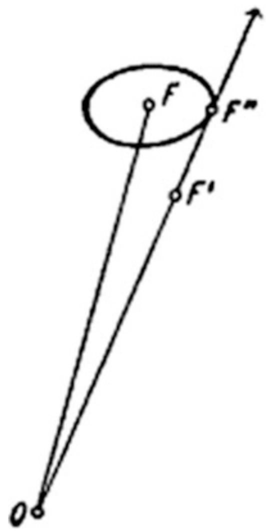

Fig. 4.1 ***OF*** and ***OF′*** stand for directions of adjacent color vectors. Points ***F*** and ***F″*** are equally bright; the ellipse (projection of an ellipsoid) joins points of equal brightness about ***F***. (Reproduced from Schrödinger, 1920c)

which lie in the neighbourhood of OF, then the points of equal brightness F'' subtend a surface element. We can assign a short name to this surface element: the 'surface of equal brightness' for F. Since one can construe the tangent cone from O to one of these small ellipsoids as a cylinder with the axis OF, then the surface element of equal brightness is simply the plane section of the ellipsoid with that cone's diameter. Then the equation of the ellipsoid[(b)] is:

$$a_{ik}\, d\, x_i\, d\, x_k = \text{constant} \tag{4.2}$$

If the coordinates of F are $x_1\ x_2\ x_3$, and those of F'' are $x_1 + d\, x_1$, $x_2 + d\, x_2$, $x_3 + d\, x_3$, then the direction of progression along OF towards the point F – call that $\delta x_1\ \delta x_2\ \delta x_3$ – (a pure change in intensity) is governed by

$$\delta x_1 : \delta x_2 : \delta x_3 = x_1 : x_2 : x_3 \ .$$

The dx_2 [Trans.: dx_i] satisfy the equation of the plane with the diameter which runs perpendicular to the direction that defines the ellipsoid of (Eq. 4.2). Then:

$$a_{ik}\, d\, x_i\, d\, x_k = 0; \tag{4.3}$$

which is the differential equation of the surface element of equal brightness for the point x_1, x_2, x_3. In the sense given by Riemannian geometry, it lies in the direction OF perpendicular to point F.

Now if the notion of brightness is to make any sense, one must require that two colors which are equally bright to a third color, must also be equally bright to one another. One may use that fact in the following construction. One takes a step from F along the surface element of equal brightness in any direction – a small step out to a point G. From there one takes a step along the surface of equal brightness **through *G***, then again a step in an arbitrary direction until H is reached, and so on. In that way one obtains a **stretch of equal brightness** along a curve. Incidentally, such a curve may have arbitrary inflections. One can now arrange matters that after covering a certain path, one again encounters a point of OF: call that K. One must require that K corresponds to F, if the notion of brightness is to make sense. If this holds for any chosen path, then all the surface elements of (Eq. 4.3) are capable of being consolidated into integral surfaces. The stretches of equal brightness that have been described must traverse those surfaces. The ***Pfaff***ian differential expression on the left-hand side must then be integrable; there must exist a multiplier μ (x_1, x_2, x_3) such that:

$$\frac{\partial\, \mu\, a_{ik}\, x_i}{\partial x_l} = \frac{\partial\, \mu\, a_{il}\, x_i}{\partial x_k} \quad \text{for} \qquad \begin{array}{c} k, l = 1, 2, 3 \\ k \neq l. \end{array}$$

It is well-known that this is by no means the case for entirely arbitrary functions of a_{ik}. It obtains only if the a_{ik} satisfy a specific condition. Namely it must be the case that:

$$\mu \cdot \left(\frac{\partial \, a_{ik} \, x_i}{\partial x_l} - \frac{\partial \, a_{il} \, x_i}{\partial x_k} \right) \quad = \quad - \left(a_{ik} \, x_i \;\; \frac{\partial \mu}{\partial x_l} - a_{il} \, x_i \;\; \frac{\partial \mu}{\partial x_k} \right).$$

If one multiplies each of the three equations of this type (which are found for the three combinations of $(k\,,l)$) by $a_{pm}\,x_p$, where m is the third index, and then these are summed, then the right-hand sides are zero identically. So it must be the case that:

$$\underset{(k\;l\;m)}{\mathfrak{S}} \; a_{pm} \, x_p \left(\frac{\partial \, a_{ik} \, x_{\mathrm{i}}}{\partial x_l} - \frac{\partial \, a_{il} \, x_i}{\partial x_k} \right) = 0 \;\; .$$

The unusual summation sign is meant to indicate that apart from summations over the indices i and p (not shown explicitly, according to convention), the triple index of $(k\,,l\,,m)$ must iterate over these combinations of values: (1, 2, 3), (2, 3, 1), and (3, 1, 2). The relation can be somewhat further simplified. Since $a_{\;i\;k} = a_{\;k\;i}$, it holds that:

$$\frac{\partial \, a_{ik} \, x_i}{\partial x_l} - \frac{\partial \, a_{il} \, x_{\mathrm{i}}}{\partial x_k} = x_i \left(\frac{\partial a_{ik}}{\partial x_l} - \frac{\partial a_{il}}{\partial x_k} \right) \;\; .$$

And therefore:

$$\underset{(k\;l\;m)}{\mathfrak{S}} \; a_{pm} \left(\frac{\partial \, a_{ik}}{\partial x_l} - \frac{\partial \, a_{il}}{\partial x_k} \right) \; x_i \, x_p = 0 \;\; . \tag{4.4}$$

This condition is **necessary** for (Eq. 4.3) to be integrable. Moreover it can also be shown that this condition is also **sufficient**.[(c)]

Under our assumption, in the first place the a_{ik} are empirically determinable coordinate functions, measured through the discrimination of differences. It is entirely imaginable they do **not** satisfy the differential equation (Eq. 4.4). In that case the notion of brightness would have no comprehensible quantitative meaning. Whether it has such meaning is a matter for experiment. Perhaps it would be best to apply methods of photometry through a progression of small steps over closed cycles of colors. Then one may examine whether the product of all the successive ratios of brightness results in a value of one. Or else perhaps the frequent replication of this experiment will uncover a systematic departure from unity. In the latter case the concept of brightness would have to be abandoned as meaningless: that is itself illuminating aside from other theoretical considerations.

A pronounced lack of clarity dominates this notion of brightness in the scientific literature. Such lack of clarity might arouse the suspicion that the latter situation does hold in nature, after all. Fortunately at my present institution, such investigations were conducted while the present article was being written. Brightness measurements of the kind described were conducted for cycles on

Ostwald's tableaux of colors. The results will be published in a multipage report very soon. The integrability of (Eq. 4.3) was documented to a very good approximation. In the following we will consider the condition as having been fulfilled precisely. We still hold its determination to be important, since the result is purely empirical, and since it could possibly turn out differently under other conditions (for example, in parafoveal vision or by intrusion of the Purkinje phenomenon).

Then for a suitable choice of the multiplier μ (x_1, x_2, x_3) there is a function h (x_1, x_2, x_3) for which:

$$\frac{\partial h}{\partial x_k} = \mu \, a_{ik} \, x_i \qquad k = 1, 2, 3 \, . \tag{4.5}$$

For progression across the surface:

$$h = \text{const.},$$

which pares color space into planes nested like the layers of an onion along which (Eq. 4.3) is satisfied. The planes are orthogonal planes (in the Riemannian geometric sense) to color vectors drawn from the origin. We will call them planes of equal brightness, or ***isolychnes***. The provisional role of a measure of brightness will be assigned to h.

Yet h is in no way constrained by (Eq. 4.5). Each monotone function of h represents a single bundle of surfaces. Each is equally justified a priori as a measure of brightness – and so an operation of normalization is called for. The normalization can be prefigured as simple only if we may assume that the isolychnes are similar (in the sense of elementary geometry) in vector space and similarly situated in vector space with respect to the origin. In experimental terms that means despite proportional changes in objective intensity, equality of brightness (and also color equivalence) is preserved for pairs of light mixtures. We are aware that this constancy is **in no way a consequence** of the constancy of color equivalence. Nonetheless we believe the assumption of constancy to be justified at least **as an approximation**, on the basis of general experience. That runs counter to the data presented by A. ***König***,[(d)] which data may be based on the ***Purkinje*** phenomenon at least in part (see the end of Section 10, however).

One can then normalize the brightness parameter h, so that it becomes proportional to the intersection of the isolychne with any color vector (just because this intersection varies proportionally from isolychne to isolychne, for all colors). In experimental terms this means one agrees to designate a surface illuminated by two, three, or four lightbulbs to be two, three, or four times as bright.[(e)] For the purposes of calculation, it means that h should be a homogeneous function of degree one in the coordinates. Then by ***Euler***'s theorem for homogeneous functions:

$$h = x_1 \frac{\partial h}{\partial x_1} + x_2 \frac{\partial h}{\partial x_2} + x_3 \frac{\partial h}{\partial x_3} = x_\mathrm{k} \frac{\partial h}{\partial x_\mathrm{k}} \quad ,$$

therefore from (Eq. 4.5):

$$h = \mu \cdot a_{ik}\, x_i\, x_k \;; \qquad \mu = \frac{h}{a_{ik}\, x_i\, x_k} \quad .$$

Substituting in (Eq. 4.5):

$$\frac{\partial \log h}{\partial\, x_l} = \frac{\sum_i a_{il}\, x_i}{\sum_i \sum_k a_{ik}\, x_i\, x_k} \qquad l = 1, 2, 3\,. \tag{4.6}$$

These three equations determine *log h* up to an additive constant. In other words they determine *h* up to a ratio which is a negligible proportional factor.

Section 4: A First Estimate of the Line-Element

In order not to complicate the subsequent calculations too far, we will derive them presently for the specific line element I am recommending, which I maintain represents a fair approximation over a large part of color space. Of course the most conservative tack would be to determine a_{ik} over the entire color space by experiment. Such an effort would consist of the measurement of difference thresholds for every color, in all possible directions of excursion. Only then with experimentally determined coefficients could the theory of maximally similar colors be demonstrated strictly for the most varied excursions. The theory of brightness that we have put forward is just a special case of that larger theory. Unfortunately such a complete line of investigation would be hopelessly difficult if it were to be carried out by experiment. The line element has only been measured in experiment along a few color vectors,[(f)] and along certain curves of the spectral cone.[(g)] Measurements along the spectral cone to white (sensitivity to ‘changes in saturation’) are almost entirely lacking. For that direction there are only a few measurements made by M. ***Gottlieb***[(h)] (and those were for pigments, not for spectral colors). Good information may well be generated with the help of ***Ostwald***’s splendid color atlas, but for the fact that color coordinates for these colors have not been defined. The exact determination of those coordinates will require substantial effort.

There is nothing left to do but to attempt to characterize the line element by hypothesis in a way which satisfies the principal constraints. The hypothesis can then be examined rigorously, on one hand by the measurement of difference thresholds, and on the other by the adjustment of maximally similar colors.

H. von ***Helmholtz*** tried the following proposition, based on considerations that emerged from ***Fechner***'s Law. His proposition was meant to apply to x_i that have been linked to actual fundamental colors.

$$a_{ik} = 0 \quad for \quad i \neq k \; ; \quad a_{ii} = \frac{1}{3\, x_i^2} \quad , \tag{4.7}$$

which means:

$$ds^2 = \frac{1}{3}\left(\frac{d\, x_1^2}{x_1^2} + \frac{d\, x_2^2}{x_2^2} + \frac{d\, x_3^2}{x_3^2}\right) \quad . \tag{4.8}$$

This proposition has not held up, insofar as its predictions did not correspond to ***König***'s measurements of color difference thresholds across the spectrum. They do not correspond as the x_i are interpreted as fiducial coordinates for the faux colors of dichromats. There would have to be new fundamental colors calculated ad hoc, in order to make the line element somewhat constant for just-noticeably different pairs of colors. The simple notion of dichromacy that we detailed in Section 10 of the first part [Chap. 3] would be thrown overboard.

The line element (Eq. 4.8) leads to something else – which ***Helmholtz*** seems not to have noticed – that is, to an absurd brightness function. In itself it contradicts experience, quite apart from its relation to the theory of dichromacy. One is easily convinced that the integrability condition and the assumption of homogeneity (from the previous paragraphs) are satisfied. Then formula (Eq. 4.6) taken together with the specific values (Eq. 4.7) of a_{ik} produces:

$$\frac{\partial \log h}{\partial\, x_l} = \frac{1}{3\, x_l} \qquad\qquad l = \; 1, 2, 3 \, ,$$

and therefore:

$$h = \text{constant} \;\bullet\; \sqrt[3]{x_1\, x_2\, x_3} \quad .$$

Yet this brightness function is absurd, since e.g. it produces a hideous dromedary-like curve with two pronounced maxima as a brightness distribution for the spectrum of sunlight. It produces that result both for ***König***'s fundamental stimulus valences – derived from the elementary theory of dichromacy – as well as for von ***Helmholtz***'s ad hoc conversion values for the fundamental stimulus valences. And this brightness function – together with its line element – is unacceptable for a more general reason still.

In recent times – namely through the researches of W. ***Abney***[(i)] and subsequently those of F. ***Exner***,[(j)] as well as K.W.F. ***Kohlrausch***[(k)] – it has been demonstrated that brightness is an additive property of color, at least to a good approximation. That is, when equally bright lights are combined, equally bright lights are produced. This additivity property of brightness is not a consequence of the additivity of color

matches. ***Helmholtz*** even doubted that additivity holds for brightness, though he steadfastly asserted the additivity of color matches. His assertion was based on a few very interesting trials with color wheels; we will come back to examine his evidence more closely in Section 6. There we shall show how ***Helmholtz*** misinterpreted these results; in reality they do **not** contradict the additivity of brightness. We will use the latter claim as a guide in our search for the right line element. Our method of presentation has the advantage that we do not commit ourselves to a fixed solution. At worst we will arrive only at a rough approximation to the real situation, which approximation may always be improved.

Now if

$$\begin{cases} h\,(x_1, x_2, x_3) \;+\; h\,(x'_1, x'_2, x'_3) \\ \quad = \; h\,(x_1 + x'_1,\; x_2 + x'_2,\; x_3 + x'_3) \end{cases} \tag{4.9}$$

counts as an identity for any two triples of values x_i and x'_i, then by differentiation for x_l it follows that:

$$\frac{\partial h}{\partial x_l}\;(x_1, x_2, x_3) = \frac{\partial h}{\partial x_l}\;(x_1 + x'_1,\; x_2 + x'_2,\; x_3 + x'_3) \qquad l = \; 1, 2, 3$$

and since once again x_i and $x_i + x'_i$ are two arbitrary triples of values, then the three partial differential quotients $\frac{\partial h}{\partial x_l}$ must be constants. Then h must be of the form

$$h = a_1\,x_1 \;+\; a_2\,x_2 \;+\; a_3\,x_3\;, \tag{4.10}$$

i.e. **homogeneous** and linear, so that the requirement of integrability (Eq. 4.9) is still satisfied. In passing we note that this form is not altered by linear transformation, so it holds for arbitrary choices of calibration colors.

In order to be consonant with experience, our line element must produce

$$a_1\,x_1 \;+\; a_2\,x_2 \;+\; a_3\,x_3 = \text{constant} \tag{4.11}$$

as the **isolychnes of a bundle of planes**. I.e., according to Section 3, it must be constituted so that the planes of the bundle always stand perpendicular (that is, perpendicular in the sense given by Riemannian geometry) to the direction of the radial vectors ($d\,x_1$: $d\,x_2$: $d\,x_3 = x_1$: x_2: x_3)Now the transformation

$$\xi_1 = \sqrt{\alpha_1\,x_1}\;, \qquad \xi_2 = \sqrt{\alpha_2\,x_2}\;, \qquad \xi_3 = \sqrt{\alpha_3\,x_3}$$

maps radial vectors to radial vectors, but it maps the bundle of planes (Eq. 4.11) onto a collection of concentric spheres.

$$\xi_1^2 \;+\; \xi_2^2 \;+\; \xi_3^2 = \text{constant}$$

Then in ξ_i space, the Euclidean line element

$$4\,(d\xi_1^2 + d\xi_2^2 + d\xi_3^2) = \alpha_1 \frac{d\,x_1^2}{x_1} + \alpha_2 \frac{d\,x_2^2}{x_2} + \alpha_3 \frac{d\,x_3^2}{x_3} = ds^2$$

would also satisfy our requirements.

This simplest of premises, which was brought to my attention as an insightful observation made by W. ***Pauli*** Jr., is blocked by ***Fechner***'s Law. Actually if one varies colors of the same stimulus quality but of different intensities, and if those colors vary only in intensity,

$$d\,x_1 = \varepsilon\,x_1, \quad d\,x_2 = \varepsilon\,x_2, \quad d\,x_3 = \varepsilon\,x_3 \quad (\varepsilon \angle\angle 1),$$

then the outcome would be

$$d\,s = \varepsilon\,\sqrt{\alpha_1\,x_1 + \alpha_2\,x_2 + \alpha_3\,x_3}$$

which is not constant for **constant** ε as required by ***Fechner***'s Law. Rather it is constant when ε is inversely proportional to the square root of brightness. That runs counter to experience.

Yet we do not need to abandon the advantages of the real isolychnes we have established. That is because the isolychnes change – as one may recognize from equation (Eq. 4.3) – if the factor of an arbitrary function of the coordinates is added to the line element. If we choose the reciprocal of brightness as that factor,

$$\frac{1}{a_1 x_1 + a_2 x_2 + a_3 x_3}\ ,$$

then we obtain constant intervals following ***Fechner***'s Law not only for the same stimulus quality but also over the whole gamut of color, as experience appears to require over a large region of color space.

On the basis of these considerations, we should like to propose the following line element as the one likely to hew most closely to the facts:

$$\begin{cases} a_{ik} = 0\,, & i \neq k\,, \\ a_{ii} = \dfrac{\alpha_i}{x_i\,(\alpha_1\,x_1 + \alpha_2\,x_2 + \alpha_3\,x_3)}\,, & i\,e., \\ ds^2 = \dfrac{1}{\alpha_1\,x_1 + \alpha_2\,x_2 + \alpha_3\,x_3} \left(\dfrac{\alpha_1\,dx_1^2}{x_1} + \dfrac{\alpha_2\,dx_2^2}{x_2} + \dfrac{a_3\,dx_3^2}{x_3} \right)\,, & \end{cases} \tag{4.12}$$

where the a_i are certain constants determined by experiment. They are selected so that the following expression

$$h = a_1\,x_1 + a_2\,x_2 + a_3\,x_3$$

measures the **brightness** of the color $(x_1\,x_2\,x_3)$.

The consequent application of this line element delivers the brightness function just cited, which is consonant with experience. It is consonant with ***Fechner***'s Law for intensity, and produces the same constants for all colors.

This is not the only possible line element which can satisfy the said constraints, since it is **not invariant** under linear transformations of the x_i. Any form to which it may be linearly transformed would serve equally well as a possible line element.

We assume that the simple form given by (Eq. 4.12) **holds for the actual fundamental colors** – which we consider to have the form derived by ***König*** and ***Dieterici***, at least provisionally.

Section 5: Noticeable Differences in Color Across the Spectrum

As an additional reality check, we look to derive from our initial proposition the noticeable differences in color across the spectrum. These are measured in rational fashion, always by equalizing the difference in brightness of the spectral colors to be compared. Otherwise the result depends on the particular distribution of intensity or of brightness in the spectrum which is employed. One progresses forward from a position λ to a neighbouring position $d + d\,\lambda$ which is distinguishable from λ **even when** and just discernably when one matches differences in brightness as well as possible.

Let

$$x_i, \quad x_i + d\,x_i$$

be the coordinates of the just-noticeably different hues λ, $\lambda + d\,\lambda$, as they occur in spectral order. The line element is not gauged between these, but between

$$(1 + \varepsilon)\,x_i\;, \qquad x_i + d\,x_i$$

The requisite value of ε for the comparison of brightness is easily established as $d\log h$.And therefore:

$$\left\{\begin{aligned} ds^2 &= a_{ik}\,(x_i\,d\log h - d\,x_i)\,(x_k\,d\log h - d\,x_k)\\ &= a_{ik}\,x_i\,x_k\,d\log\frac{x_i}{h}\,d\log\frac{x_k}{h} \qquad . \end{aligned}\right. \tag{4.13}$$

Then for our line element (Eq. 4.12)

$$ds^2 = \frac{a_1\,x_1}{h}\left(d\,\log\frac{x_1}{h}\right)^2 + \frac{a_2\,x_2}{h}\left(d\,\log\frac{x_2}{h}\right)^2 + \frac{a_3\,x_3}{h}\left(d\,\log\frac{x_3}{h}\right)^2 \tag{4.14}$$

with

$$h = a_1\, x_1 \;+\; a_2\, x_2 \;+\; a_3\, x_3 \;.$$

Here we treat h and the x_i as functions of λ. Deviations from λ are marked by a prime mark. A form suited to numeric methods proves to be:

$$ds = \sqrt{\frac{\alpha_1\, x_1}{h}\left(\frac{x_1'}{x_1} - \frac{h'}{h}\right)^2 + \frac{\alpha_2\, x_2}{h}\left(\frac{x_2'}{x_2} - \frac{h'}{h}\right)^2 + \frac{\alpha_3\, x_3}{h}\left(\frac{x_3'}{x_3} - \frac{h'}{h}\right)^2} \cdot d\lambda \tag{4.15}$$

It is a ticklish business to determine these numeric results from empirical data, because of the appearance of differential quotients in the expression. The numeric data that we use were given by ***Helmholtz***[l] for the same purpose, and based on careful adjustment values. But ***Helmholtz*** gives the values r, g, v for ***König***'s **elementary colors**. We transform them instead to fundamental colors using the coefficients provided by ***König***.[m] It helps to incorporate multiplication by the constants a_i in this conversion. That is, we include the x_i from here on (in the sections to follow, too) not in such units of the fundamental colors as mix to produce white, but rather in such units as appear **equally bright** – those which have the same brightness values in mixtures. The brightness function becomes:

$$h = x_1 + x_2 + x_3 \;, \tag{4.10$'$}$$

Then in the general case the line element is:

$$ds^2 = \frac{1}{h}\left(\frac{d\,x_1^2}{x_1} + \frac{d\,x_2^2}{x_2} + \frac{d\,x_3^2}{x_3}\right) \tag{4.12$'$}$$

and in the case we are considering presently:

$$d\,s = \sqrt{\frac{x_1}{h}\left(\frac{x_1'}{x_1} - \frac{h'}{h}\right)^2 + \frac{x_2}{h}\left(\frac{x_2'}{x_2} - \frac{h'}{h}\right)^2 + \frac{x_3}{h}\left(\frac{x_3'}{x_3} - \frac{h'}{h}\right)^2} \cdot d\lambda \;. \tag{4.15$'$}$$

I would like to thank F. ***Exner*** for his kindness in supplying the a_i from his recent experimental trials. (Our focus is on the proportions of a_i .)

$$a_1 = 43 \cdot 33, \quad a_2 = 32 \cdot 76, \quad a_3 = 1. \tag{4.16}$$

In the derivation of fresh x_i from the elementary valences r, g, v we find that:

$$\left\{\begin{array}{l} x_1 = 45 \cdot 6\; r \;-\; 6 \cdot 84\, g \;+\; 4 \cdot 56\; v, \\ x_2 = 6 \cdot 57\; r \;+\; 26 \cdot 3\, g\;, \\ x_3 = v. \end{array}\right. \tag{4.17}$$

The same formulae hold for the tacit quantities.

Finally, for $d\,\lambda$ I use ***Uhthoff***'s[(n)] recent experimental results.

The accompanying Table 4.1 is self-evident.

The first nine columns show ***Helmholtz***'s[(o)] data, recalculated by the formula given above. The last four columns are: 1) 100 times the derived square root, 2) ***Uhthoff***'s observations, 3) the product of those two, which should be a constant value, and 4) the mean of column 3 divided by column 1, **computed as *d λ***.

The requirement for the constancy of *ds* is not at all well-satisfied, just as it was not well-satisfied for ***Helmholtz***. Yet if we plot the observed and calculated values of $d\,\lambda$ (Fig. 4.2), one notes that the general trend of detectable difference is captured by theoretical values across the spectrum. Those values are just a little exaggerated, in that extreme values are shifted towards red.

In no way do I wish to claim that one can mount a strong argument in favor of the proposed line element from this rough correspondence. Yet I do believe that one could not have expected a better correspondence from an entirely accurate theory. One may consider that none of these numbers was adopted ad hoc to be used in the calculation. Rather the numbers arose from very heterogeneous sets of observations (from ***König***, ***Uhthoff***, and ***Exner***). Moreover the true positions of the fundamental colors are by no means certain yet, as was shown by our analysis at the end of the first part of the present work. In particular, interpolation of differential quotients (whose values are crucial and decisive here) is always an uncertain and risky business from curves of purely empirical, altogether irregular form – as for ***König***'s curves. One may recognize the errors which can creep in, from the notable position of the **theoretical** point at $\lambda = 530$. ***Helmholtz***'s calculations produce an outlier, as do mine; both fall outside the **theoretical** trend.

The mean of the *ds* for just-noticeably different colors proved to be very close to 0.01 (cf. Table 4.1, next-to-last column). According to theory, this number should correspond to ***Fechner***'s step – his increment of intensity. Unfortunately the reckoning is not exact, since different observers produced very different values for that increment. The values vary with the method used, but they may also be subject to individual differences. The largest value is given by ***König*** as 1/57. Other results listed by von ***Kries*** (in ***Nagel***'s Handbuch der Physiologie, **3**(1), p. 250) are: ***Arago*** 1/130, ***Masson*** 1/120, ***Volkmann*** 1/100, ***Helmholtz*** 1/167, ***Aubert*** 1/186. The number 1/94 which we have calculated here sits very comfortably among those values.

Section 6: *Helmholtz*'s Color-Wheel Experiments Which Seem to Contradict Additivity of Brightness

Now we turn to ***Helmholtz***'s experiments with a color wheel, which were mentioned earlier. On the basis of those experiments he concluded wrongly that an additive rule could not be formulated for the brightness of mixtures. His experiments provide an illustration of the need for a adjustment for maximally similar

Table 4.1 Reproduced from Schrödinger, 1920c

λ	x_1	x_2	x_3	h	x_1'	x_2'	x_3'	h'	*1.*	*2.*	*3.*	*4.*
640	119	23.3	–	142	−5.12	−1.37	–	−6.49	0.586	2.97	1.74	1.81
630	176	40.2	–	216	−5.58	−2.01	–	−7.59	0.712	1.68	1.20	1.49
620	236	64.5	–	301	−6.77	−3.10	–	−9.87	0.793	1.24	0.98	1.34
610	286	100.5	–	387	−4.03	−3.93	–	−7.96	1.100	1.08	1.11	0.96
600	317	144.1	–	461	−2.56	−4.87	–	−7.43	1.194	1.02	1.20	0.89
590	340	193	–	538	−1.63	−5.91	–	−7.54	1.207	0.91	1.10	0.88
580	354	259	–	613	−1.14	−5.62	–	−6.76	0.915	0.88	0.81	1.16
570	359	324	–	683	−0.42	−5.50	–	−5.92	0.789	1.10	0.87	1.34
560	359	364	0.22	723	+0.68	−2.63	–	−1.95	0.456	–	–	2.33
550	339	375	0.30	714	+3.04	+0.45	−0.0138	+3.48	0.401	1.66	0.67	2.64
540	301	358	0.49	659	+4.83	+2.98	−0.0233	+7.79	0.420	–	–	2.53
530	230	315	0.75	546	+8.10	+4.98	−0.0326	+13.05	0.991	1.88	1.86*	1.07
520	158	255	1.10	414	+6.06	+7.12	−0.0400	+13.14	0.618	–	–	1.72
510	105	171	1.55	278	+5.28	+8.18	−0.0536	+13.41	0.632	1.29	0.83	1.68
500	55.7	97.2	2.2	155	+3.65	+5.17	−0.0887	+8.73	1.493	–	–	0.71
490	36.7	64.0	3.6	104	+0.97	+1.89	−0.208	+2.65	1.58	0.72	1.14	0.67
480	41.3	52.0	7.9	101	−0.60	+1.02	−0.52	−0.10	–	0.95	–	–

Average of Column ***3****. :* 1.06
Column ***1.***: 100 ($ds/d\,\lambda$); Column ***2.***: $d\,\lambda$ observed; Column ***3.***: 100 ds calculated; Column ***4.***: $d\,\lambda$ calculated

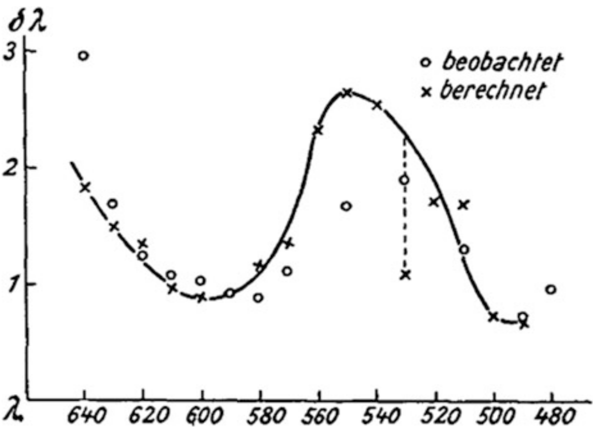

Fig. 4.2 Observed and calculated values of $d\lambda$ are plotted along the spectrum. The observed values are from Uhthoff. (*beobachtet*: observed; *berechnet*: calculated) (Reproduced from Schrödinger, 1920c)

colors, which is **not** the same as a ordinary comparison of brightness. ***Helmholtz*** did not recognize this, and that is where his error lay.

The experiment runs as follows. One covers a color wheel with a mixture of two pigment colors F and V, with the exclusion of a small angular extent ε' (about 1/50 to 1/100th of the circle) which remains dark at first. We call this mixture **background** $\boldsymbol{G}$. More precisely, F or V should be the colors which appear when one covers the entire extent $2\pi - \varepsilon'$ with the first (or second, respectively) pigment as the wheel rotates. G is its color if λ' is covered by the first pigment and μ' by the second, so that $\lambda' + \mu' = 2\pi - \varepsilon'$. Then:

$$G = \lambda F + \mu V ; \quad \lambda + \mu = 1$$
$$\left[\lambda = \frac{\lambda'}{2\pi - \varepsilon'} \quad , \quad \mu = \frac{\mu'}{2\pi - \varepsilon'}\right] \quad . \tag{4.18}$$

F **should be the darker pigment**. – Say the angle ε' around an inner ring is also covered by F. Along an adjacent outer ring, only a section of angle ε' is covered by V – the angle ζ'. At the same time $\varepsilon' - \zeta'$ is covered in black.[(p)] Then ζ' is varied in extent, so that the border between the two rings becomes its most indistinguishable. In other words the two colors become least distinguishable. ***Helmholtz*** identifies this operation as an adjustment for equal brightness. According to him, increments in color

$$\frac{\varepsilon'}{2\pi - \varepsilon'} F = \varepsilon F \; ; \qquad \frac{\zeta'}{2\pi - \varepsilon'} V = \zeta V$$
$$\left[\varepsilon = \frac{\varepsilon'}{2\pi - \varepsilon'} \quad , \quad \zeta = \frac{\zeta'}{2\pi - \varepsilon'}\right] \tag{4.19}$$

relative to the background G will have changed the brightness in equal strength. Then if brightness were an additive property, it would also be the case that the brightnesses of F and V would vary inversely with the increments ε and ζ, or with the small angles ε' and ζ'.

$$\frac{h\,(F)}{h\,(V)} = \frac{\zeta}{\varepsilon} = \frac{\zeta'}{\varepsilon'} \tag{4.20}$$

This proportion would necessarily prove to be constant, **independent of the background color**. That is just not the case. The ratio of angles varies markedly – up to ½ in extreme cases. The ratio varies in **this** direction: it appears to favor the color that is **less** well-represented (because of its 'brightening power') in the background. ***Helmholtz*** attributed this to a generalization of ***Fechner***'s Law.

Now we should like to show that the ratio of angles ζ' / ε' does **not** simply coincide with a brightness ratio, and that this proportion should **not** be expected to be independent of the background color. Certainly the observed changes in the ratio of angle with the background color may be predicted from our line element (Eq. 4.12, or else Eq. 4.12′) which implies the brightness function of (Eq. 4.10, or Eq. 4.10′). That allows us to predict the observed variation in angular ratio with background color.

If OF and OV (Fig. 4.3) are the color vectors of pure pigments (in the sense given previously) then the head of the vector for the background

$$G = \lambda F \;+\; \mu V \qquad\qquad [\lambda + \mu = 1]$$

lies along the connecting line FV. A particular position is indicated for G. One obtains the color of the **inner** ring, when one adjoins to OG a small vector GG' that has direction $\overrightarrow{OF}$ and length $\varepsilon \bullet \overline{OF}$. (This and the following constructions are

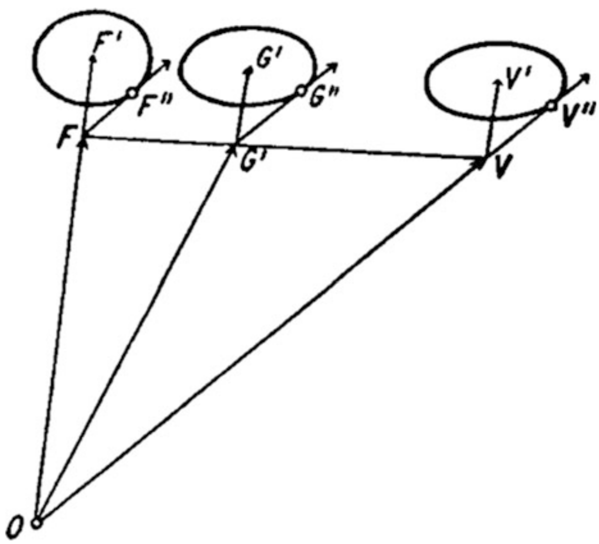

Fig. 4.3 This construction illustrates the calculation of a color equal in brightness to two others. ***OF*** and ***OV*** are the color vectors of pure pigments. (Reproduced from Schrödinger, 1920c)

developed in the Figure for an intermediate position, and for the two end positions where G coincides with F or V.) For the **outer** ring, what is added to OG is a vector GG'' of direction $\overrightarrow{OV}$ and a **requisite** length so that G'' is situated closest to G'. As before, one inscribes a scatter ellipse about G' (Eq. 4.2) of such size that it just touches the line GG'' (the parallel to OV through G). G'' is its point of contact. That point lies on the plane of the diameter conjugate to GG'' (analogously for the extrema of FF'' and VV''), not on the plane of the diameter conjugate to OG (similarly for OF and OV). Therefore in general it is not equally bright to G' (similarly for F', V'). Only in the borderline case in which G coincides with V are the two directions – hence the two conjugate planes – identical. An ordinary comparison of brightness is available only in that case.

The brightness ratio of the two pigments may be calculated directly from the ratio of the small supplementary angles only if the color which is manipulated in experiment is the same as the background color.

Having said that, one recognizes immediately from the Figure that GG'' must be **lengthened** in all other cases, in order to intersect the plane conjugate to GO. A **shorter** addition to the variable color V (**shorter** than would be necessary for equality of brightness) suffices to establish maximum similarity. That means, (if one commits ***Helmholtz***'s error) the variable color appears to have a relatively stronger brightening effect against a background of a different color than against a background of its own color: that is what ***Helmholtz*** found. One may also conjecture that the discrepancy becomes even greater the more the directions OG and GG'' diverge from one another (i.e., the nearer G draws to F).

This last conclusion is uncertain, however. The form of the scatter ellipsoid changes too, if its centroid slides along the line $V'\,F'$.

Here is the quantitative situation. If the coordinates of

$$\begin{array}{lll} F & \cdots\cdots & y_i \\ V & \cdots\cdots & z_i \\ G & \cdots\cdots & x_i \;= \lambda\, y_i - \mu\, z_i \end{array}$$

and the coordinates of

$$\begin{array}{llll} G' & \text{are} & x_i & +\;\varepsilon\, y_i \\ G'' & \text{are} & x_i & +\;\zeta\, z_i . \end{array}$$

then the vector GG'' has as its components

$$\zeta\, z_i - \varepsilon\, y_i \quad .$$

The vector must be conjugate to the direction GG'' (relative to the scatter ellipsoid of G'), whose components behave as:

$$z_i$$

That is, it is necessary that either

$$a_{ik}\ (\ \zeta\, z_i - \varepsilon\, y_i\ )\ \ z_k = 0$$

or

$$\frac{\zeta}{\varepsilon} = \frac{\sum\limits_i \sum_k a_{ik}\, y_i\, z_k}{\sum\limits_i \sum_k a_{ik}\, z_i\, z_k} \quad . \tag{4.21}$$

According to (Eq. 4.20)**, this is the *apparent* ratio of brightness of the fixed color *F* to the variable color *V*.** Here the a_{ik} are meant – strictly speaking their values at point G'. Naturally G may be substituted with adequate precision for those values, and therefore they may be substituted for $x_i = \lambda\, y_i + \mu\, z_i$ as well.

As we have seen, then, the *real* ratio of brightness must be given by the same formula as if one set out to assign a_{ik} their value at point *V*.

Let us refine this step for our line element (Eq. 4.12′), and stipulate that:

$$a_{ik} = 0 \qquad [\ i\ \ \neq\ \ k\ ]\ \ . \qquad\qquad a_{ii} = \frac{1}{(x_1 + x_2 + x_3)\, x_i} \quad ,$$

This produces:

$$\frac{\zeta}{\delta} = \frac{\frac{z_1}{x_1} \cdot y_1 \ + \ \frac{z_2}{x_2} \cdot y_2 \ + \ \frac{z_3}{x_3} \cdot y_3}{\frac{z_1}{x_1} \cdot z_1 \ + \ \frac{z_2}{x_2} \cdot z_2 \ + \ \frac{z_3}{x_3} \cdot z_3} \quad , \tag{4.22}$$

where $x_i = \lambda\, y_i + \mu\, z_i$. One can recognize that the **correct** brightness ratio results, if the x_i coincide with z_i .

Just **how** the apparent ratio of brightness varies with the background color (x_i), may be clarified in the following way. Three **weighting factors** z_i / x_i are added 'under hypothesis' to the fundamental stimulus valences of formula (Eq. 4.22). A fundamental valence receives a stronger weight in formation of the ratio, the more it predominates in the variable color (z_i) over the background color (x_i). The background color will not coincide with the variable color because of an admixture of the fixed color (y_i). The weighting factors will favor just those fundamental valences in which the variable color is amply represented – also with respect to the fixed color. The effect is stronger, as the background diverges from the variable color. That is, weighting favors the variable color more and more, the more strongly the background diverges – just as ***Helmholtz*** had found.

A quantitative comparison is excluded, since coordinates are unknown for the colors which von ***Helmholtz*** used. Yet there is something one may insist upon. Since ***Helmholtz*** had used three different pigments in three combinations, one can examine whether the results of the three extreme cases for $G = V$, are **consistent** results. In other words we can see if the product of the brightness ratios

$$\frac{\text{Red}}{\text{Blue}} \times \frac{\text{Blue}}{\text{Green}} \times \frac{\text{Green}}{\text{Red}}$$

is equal to unity, as theory demands in this case (since a pedestrian measure of brightness applies just in this case). If one constructs these ratios from the data found in the place cited, unfortunately we find:

$$1.33 \times 0.41 \times 1.20 = 0.66$$

When combined in the same way, the other extreme cases produce 1.55. For a background mixture of half-and-half, the result is 1.19.

In these respects the experiments do not correspond to theory – yet also not to the collective experience of ***Abney***, ***Exner***, and ***Kohlrausch***, who apply methods which enable much greater precision. If the variable color does coincide with that of the background, this business is really only one of a simple change in intensity for the outer ring (given a constant inner ring). That may be carried out much more conveniently and exactly using a polarimeter. Then one need not stop the disk to change the color wheel with each alteration. Rather the various adjustments can be lined up continuously, and compared in value. In just this way ***Exner*** has determined with great precision a constancy in brightness value for a pigment color in the most diverse of mixtures.

Section 7: Geodesics. The Measurement of Finite Differences. Heterochromatic Photometry

Until now we have always employed a propositional expression for the line element in differential form – as a measure of difference between pairs of colors which differ but **little**. We have made no use of the third assumption in Section 2, which sets out to measure differences in the line integral of ds along a shortest connecting line for **starkly** different colors. We will use it now in order to evaluate what theory has to say on the subject of **strongly** heterochromatic photometry. Does it coincide or not with the ‘nearly monochrome’ study of photometry **in small increments**, which we have given as sole support to the definition of brightness?

Once again let us consider a fixed color point F, plus another point F' which is variable in its radial vector. Here the radial vector does not pass close to F; rather it may take an arbitrary direction. We seek the position F'' from F' which lies closest to F. **Then F'' is the foot of the perpendicular geodesic which extends from F to the radius vector OF'**. One may ask if this point of the foot lies along the same isolychne as F:

$$x_1 + x_2 + x_3 = \text{constant}$$

In order to calculate the geodesic line, we apply a transformation to line element (Eq. 4.12′)

$$\xi_1 = \sqrt{x_1}\ , \quad \xi_2 = \sqrt{x_2}\ , \quad \xi_3 = \sqrt{x_3} \tag{4.23}$$

to bring the line element into the form:

$$ds^2 = 4 \ \cdot \ \frac{d\,\xi_1^2 + d\,\xi_2^2 + d\,\xi_3^2}{\xi_1^2 + \xi_2^2 + \xi_3^2} \tag{4.23a}$$

We note that if one interprets the ξ_i provisionally as rectilinear coordinates, then complete spherical symmetry about the origin holds over the space of ξ_i. Hence in this ξ_i space, the geodesics between two color points must be **plane curves**, such as a plane curve of Y and Z in the plane $Y\,Z\,O$. Let us then introduce polar coordinates r and ψ to this plane. The expression of ds^2 in terms of $d\,r$ and $d\,\psi$ must remain the same, independent of the pose of the plane. By specialization (such as for $\xi_3 = 0$) we see that that expression must read:

$$ds^2 = 4\ \left(\frac{d\,r^2}{r^2} + d\,\psi^2\right) = 4\ \left[\ (d\,\log r)^2 + d\,\psi^2\ \right] \quad . \tag{4.24}$$

From this we recognize that the geodesics – which are **straight lines in terms of the variables ψ and log r** – are **plane logarithmic spirals** in ξ_i space, whose asymptotic point lies at 0. We may write their equation in ψ and r [which in reality contains only **two** independent constants; two other constants pertain to the arbitrary **pose of the plane**] in this form:

$$\frac{\psi - \psi'}{\psi'' - \psi'} = \log \frac{\frac{r}{r'}}{\log \frac{r''}{r'}} \quad , \tag{4.25}$$

where ψ', r', ψ'', r'' are constants. Namely they are polar coordinates of the points Y and Z through which the geodesic lines pass.

Under the following limited conditions:

$$a)\quad \psi' = \ \psi''\ .\ \ r' \neq \ r'' \quad ; \quad b)\quad \psi' \neq \ \psi''\ .\ \ r' = \ r''$$

this result is obtained:

$$a)\quad \psi = \text{constant} \ \ ; \quad b)\quad r = \text{constant} \tag{4.26}$$

Those a) are lines through the origin, not only in $\bar{\xi}_i$ but also in x_i - space, since it follows from (Eq. 4.23) that fixed ratios of ξ_i will imply fixed ratios of x_i as well. Then we note:

According to our theory, in the transition from an intense illuminant color to a weak illuminant color of the same stimulus quality *(that is, under a simple change* *in absolute intensity without change in the composition of the light mixture)* ***one moves along a shortest series of color points***.

We will return to this subject in Section 10.

b) gives us circles centred on the origin – or otherwise expressed, great circles of a sphere circumscribing the origin.

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \text{const.}$$

These spheres in ξ - space are nothing else (following Eq. 4.23) than our isolychnes in x- space (cf. Eq. 4.10′). Therefore the geodesic between two equally-bright points is one such great circle in ξ - space, and it is a plane curve in x- space **on the isolychne** (namely an ellipse, as we shall see). Since the isolychnes – as we know already – are orthogonal (in the sense given by Riemannian geometry) to any radial vector, then those plane curves form the **set of geodesic normals between any two of their radial vectors**.

In that way, the question we posed in the preamble has been answered. The foot and the head of the perpendicular (i.e., the normal) always lie on the same isolychne. Heterochromatic photometry – insofar as it is feasible – should produce the same result as incremental, nearly monochrome photometry, according to our theory.

For later use we note something which is immediately evident from the form of the line element given in (Eq. 4.23a). **On the isolychnic spheres of $\boldsymbol{\xi}$ - space, our metric coincides with the ordinary Euclidean metric**. The standard of measure changes, however, from sphere to sphere; in fact the sphere's radius serves as the unit of length. The metric relations, and with them all the geometry **within an isolychne** have become entirely perspicuous by means of our ξ - transformation. Namely, that geometry is reduced to ordinary spherical trigonometry. It is in those terms that the **angle** between two line elements can be judged, or that **perpendicularity** and so forth can be judged.

In the next section we will discuss the **general** path of geodesics in x- space. Here by way of an appendix – since it is so easily developed – is an exposition of **calibration**. In our theory such calibration gives the **measure of difference between two finitely different colors, meaning any two arbitrary colors**. We began with that subject in Section 2, along the geodesic following (Eqs. 4.24 and 4.25):

$$\left\{\begin{aligned} \int_Y^Z ds &= 2\int_Y^Z \sqrt{(d\log r)^2 + d\psi^2} \\ &= 2\sqrt{\left(\frac{\log\frac{r''}{r'}}{\psi'' - \psi'}\right)^2 + 1}\cdot\int_Y^Z d\psi \\ &= \sqrt{\left(\log\frac{r''^2}{r'^2}\right)^2 + 4\,(\psi'' - \psi')^2} \quad . \end{aligned}\right. \tag{4.27}$$

where of course r', ψ', r'', ψ'' now have reference to the endpoints Y and Z. It is therefore plausible to say that the measure of difference brings together two components: **difference in brightness** – indicating the ratio of lengths of radial vectors **in ξ - space** – and **difference in color** – in a narrow sense for which the angle between these radial vectors is the measure. In the intrinsic coordinates y_i and z_i of the color points Y and Z, and their associated brightnesses, say:

$$h_y = \sum y_i \ , \qquad h_z = \sum z_i$$

one obtains, obviously since:

$$h_y = r'^2 \ , \qquad h_z = r''^2 \quad ,$$

$$\arccos \ \frac{\sqrt{y_1 \, z_1} + \sqrt{y_2 \, z_2} + \sqrt{y_3 \, z_3}}{\sqrt{h_y \, h_z}} \quad = \quad \psi'' - \psi'$$

the following expression for the measure of difference:

$$\int_Y^Z ds \ = \ \sqrt{\left(\log \frac{h_z}{h_y}\right)^2 + \ 4 \ \left(\arccos \ \frac{\sqrt{y_1 \, z_1} + \sqrt{y_2 \, z_2} + \sqrt{y_3 \, z_3}}{\sqrt{h_y \, h_z}}\right)^2} \quad . \tag{4.28}$$

As an example I have evaluated this expression using the data of Table 4.1 for red at $\lambda = 640$ and cyan blue at $\lambda = 480$, but **maintaining equal brightness**. One obtains a value very close to 1. This indicates that for viewing conditions under which ***Fechner***'s intervals have – say, a value of 0.01 – that a progression of at least 100 intermediate just-noticeable steps should be established if one selects the intermediate steps as skillfully as possible.

Section 8: Description of the Paths of Geodesics in *x*-Space

It is of some interest to present a clearer representation of the paths of geodesic lines in our **vector space of** x_i, so meaningful in basic colorimetry. One awkwardness of transformation (Eq. 4.23) is that it is not a one-to-one correspondence.[(q)] The **positive** octant of x- space, to which it is restricted in domain, is mapped onto **each** of the 8 octants of ξ - space. Consider the consequence: for example if one of our logarithmic spirals continues on from the first ξ - octant, it intersects a coordinate plane and continues to another octant. Then the corresponding curve in x- space doubles back to the first octant, and steadily approaches the origin (or better stated: it reduces the sum Σx_i). From there it continues to the second coordinate plane, from there to the third, and so on. Thus the curve approaches the origin asymptotically by repeated turns on the three positive quadrants of the coordinate planes, always remaining in the first octant.

How is the **plane through the origin of ξ - space** transformed in x- space? Each such plane – along which every geodesic runs – such as:

$$\gamma_1\,\xi_1 \;+\; \gamma_2\,\xi_2 \;+\; \gamma_3\,\xi_3 = 0\ . \tag{4.29}$$

represents

$$\gamma_1\sqrt{x_1} \;+\; \gamma_2\sqrt{x_2} \;+\; \gamma_3\sqrt{x_3} = 0$$

or

$$\gamma_1^2\,x_1 \;=\; \gamma_2^2\,x_2 \;+\; \gamma_3^2\,x_3 - 2\,\gamma_2\,\gamma_3\,\sqrt{x_2\,x_3}$$

or

$$(\gamma_1^2\,x_1 \;-\; \gamma_2^2\,x_2 \;-\; \gamma_3^2\,x_3)^2 = \;4\,\gamma_2^2\,\gamma_3^2\,x_2\,x_3 \tag{4.30}$$

That is a second-degree conic with its apex at the origin. If one traces its intersection with the coordinate planes, then in every case a **doubly-counted** line is formed (e.g., for $x_1 = 0$):

$$(y_2^2\,x_2 \;-\; y_3^2\,x_3)^2 = 0$$

The cone has a point of contact with all three coordinate planes.

Let us now consider (as we did in "Sect. 10" of the first part) the **intersection** of our vector pencil **with a plane**, say with the isolychne

$$x_1 + x_2 + x_3 = \text{const.}$$

Its projection onto the coordinate planes is the well-known color triangle. With the conventions that have been adopted for our representation, this proves to be equilateral. The projections of our conic are conic sections which are themselves geodesic, since they lie in the isolychne. Since they lie at least **in part** in the interior of the triangle, and since they never pass beyond it – a conic section has no point of inflection – they lie **entirely** in its interior, meaning they must be ellipses.

And so almost without benefit of calculation, we have gained a **unified** intuitive grasp of the **complete set of geodesic lines on an isolychne**: that is the **complete set of ellipses inscribed in the color triangle**. That is because there must be a geodesic line in any prescribed direction which proceeds through any point. If we then add tangents to the points of the isolychne as well, then those ellipses are **uniquely** determined by the additional constraint as having a point of contact with the sides of the color triangle.

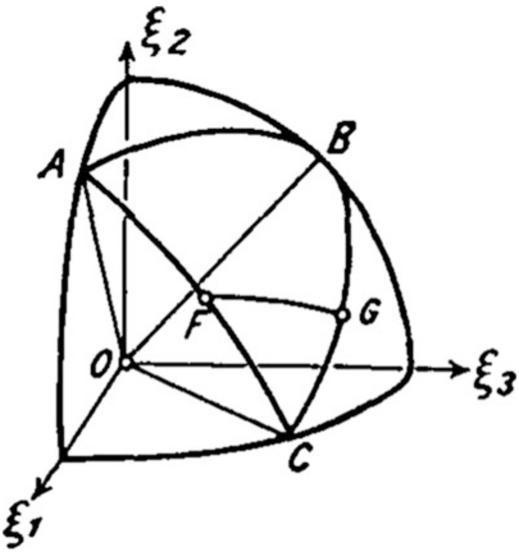

Fig. 4.4 One of the octants of $\boldsymbol{\xi}$ – space, showing the transformed triangle ***ABC*** as well as the geodesic arc ***FG***. (Reproduced from Schrödinger, 1920c)

We should still dispel some residual **ambiguity** concerning the **shortest** line to connect two points on the color triangle of the isolychne. In general given two points there are **two** ellipses which can be inscribed including them. That means **four** geodesic elliptical arcs: which of them is shortest?[(r)]

As we have intimated, **one** octant in ξ - space – say the first – suffices to map the first x- octant, which is the only one of interest to us. Now **that** part of the plane (Eq. 4.29) which lies in the first octant obviously represents **that** part of the cone (Eq. 4.30) that lies between two lines which touch the coordinate planes. Suppose one seeks those points which correspond to the two other portions of the envelope **in the first $\boldsymbol{\xi}$ - octant**. Then one needs to reflect the remaining portions of the plane (Eq. 4.29) through the coordinate planes into the first ξ - octant. By that means one finds one of the triangles inscribed in the first ξ - octant (( ) *ABC* in Fig. 4.4).

This is just the triangle which is deformed into the positive x- cone by transformation (Eq. 4.23). Under the transformation, corners become rounded, and ( ) *A*, ( ) *B*, and ( ) *C* transform into the lines of contact. Then we see that **the shortest connecting lines between two points are just those elliptical arcs which do not contain points of contact**. Of the four possible arcs, only one ever satisfies this condition. Naturally in ξ - space the arc segment *FG* will be shortest between *F* and *G*, and not say the bent segment *GCF* or even *GBAF* for example.

Something similar holds for the logarithmic spirals, of course. They project onto triangle *ABC* as the path of a buckled curve, which converges to *O* as it winds around the triangle. The path of this curve is smoothed in x- space, and it winds around the cone in the same way as the threads of a conical screw. The screw's pitch decreases without bound as it leads to the origin as asymptote. Yet what serves as **shortest** is still just the segment between two successive points of contact on the coordinate planes.

Previously we spoke of the isolychne color triangle, that happened to be equilateral by virtue of our special assumptions. Clearly these two constraints are not essential, since the property of **contact** is invariant under the linear transformations in question. The second-order conics inscribed in a triangle will become – under affine transformation – once again second-order conics inscribed in a triangle [Trans.: technically, one may use the less familiar 'trihedral' throughout]. A similar situation holds for the **projective** transformation of the ellipses inscribed in a triangle. And so we find the following:

1. **Consider an arbitrary color triangle drawn on the true fundamental colors** (but with arbitrary units!) and that all the colors on it are positioned to have equal brightness. *Then according to our theory, the inscribed ellipses indicate shortest transitions between two equally bright colors, where all the tints in the transition series can be assumed to be equally bright.*
2. *The elliptical arcs fix the succession of the stimulus qualities by which a quickest transition may be arranged, for pairs of colors which differ in brightness too.* This means brightness and stimulus quality can be varied simultaneously in a determinate and lawful way.

The nature of this lawful relation is given in ξ - space by the logarithmic spirals (Eq. 4.25). If one introduced x_i into this equation, then one would not gain an intuitive picture by one's effort. The differential form of the law is perfectly intuitive:

$$d\,\psi = c\,d\ \log r \qquad \left[c \ = \ \frac{\psi'' - \psi'}{\log \frac{r''}{r'}}\right] \ .$$

since from (Eq. 4.24) it follows that:

$$ds = 2\sqrt{(d\log r)^2 \ + \ d\,\psi^2} \ .$$

The whole increment ds then consists of two parts: the increment of brightness $d\ \log\ r$ (= ½ $d\ \log\ h$) and the increment of stimulus quality $d\ \psi$. These are proportional to one another along a geodesic path. That is, if one needs to **apportion equal** changes in brightness at the **pace** of changes in stimulus quality, then one has available the change in brightness necessary to assign **increments of stimulus quality** in a specific way. One can choose them as **equally-noticeable differences for fixed brightness**. One varies the **brightness** by a **fraction** (or by a fixed multiple) that is constant along the geodesic line. That fraction is the **equally-noticeable increment of brightness when stimulus quality is fixed**.

Section 9: Special Cases. Changes in Hue With the Addition of White

The geodesic connection is a straight line only in special cases even in *x*- space: those cases for which the shortest series of colors coincides with the mixture colors of the endpoints. We foreshadowed **one** such special case in Section 7 (progress along a radial vector, that is, for unchanged stimulus quality). The only other possibility occurs when an elliptical arc of equally-bright colors degenerates into a straight line. That is the case if the line connecting the two colors passes through a corner of the color triangle (or in terms of the vector space: if it intersects a coordinate axis). In the doubly-counted portion of this line, the ellipse is degenerate insofar as it lies within the color triangle. The colorimetric peculiarity of this case is that far spectral endpoint colors **coincide in *one* ratio of coordinates** (e.g., x_2/x_3).

This theoretical statement coincides remarkably with a result by W. ***Abney***.[(s)] He took care to select the most similar pure (undesaturated) spectral color to a spectral color diluted with white, the latter from the long-wave portion of the spectrum. For ease of comparison brightness was uncontrolled, meaning it was made equal only as far as possible. ***Abney*** found – and proved quantitatively – that in such comparisons it is always **those** pure colors that are chosen, which match the red / green ratio of the spectral colors diluted with white. The hue appears to be shifted somewhat by the dilution with white – towards that yellow (at $\lambda = 577$) which has the same red / green ratio as white. This is just as predicted by theory, insofar as the spectral curve along the part of the spectrum in question can be considered to align closely with the outer border $F_1 F_2$ (cf. Fig. 4.5) of the color triangle. Then that spectral color which has the same red/green ratio as the dilute color V is the color A, which can be

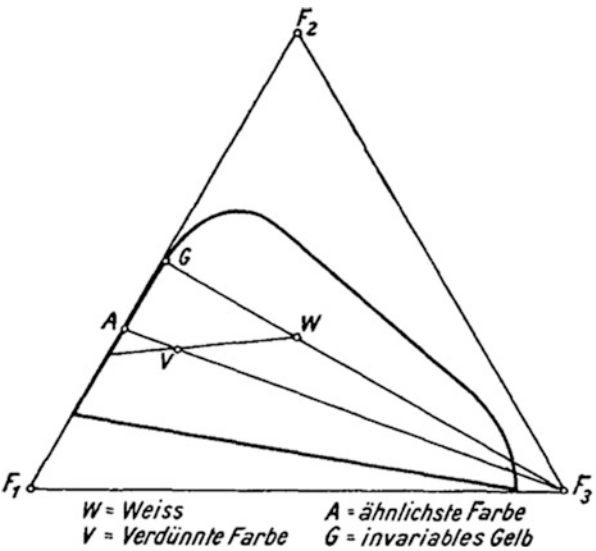

Fig. 4.5 The color triangle $\boldsymbol{F_1F_2F_3}$, showing the spectrum locus and the geodesic normal of $\boldsymbol{AF_3}$ to $\boldsymbol{F_1F_2}$. (*Weiss*: white; *verdünnte Farbe*: dilute color; *ähnlichste Farbe*: most similar color; *invariables Gelb*: unique yellow) (Reproduced from Schrödinger, 1920c)

found on the line VF_3 which connects to the third corner of the triangle. It is perspicuous from our ξ - transformation that not only is this connecting line a geodesic line, but also that it stands perpendicular to F_1 F_2 at A, the point of intersection. It really does constitute the **geodesic normal** of V to F_1 F_2. That ξ - transformation maps F_1 F_2 and F_3 A to orthogonal great circles in the pertinent isolychne sphere.

The result is somewhat muted in value, since ***Abney*** did not employ just the right fundamental colors (those of ***König***). The differences are not great, however.

I would also like to put the issue forward, whether ***Abney***'s expression is entirely fitting when he says that **colors of the same hue** were sought. My intuition is that spectral red is very saturated, while the yellow is much less saturated and much closer to white. It does not surprise me then, that with the addition of white, one looks nearer to yellow for the most similar color. Yet that is, I believe, better attributed to the sharp decline of saturation in this spectral direction than perhaps to say that red becomes yellowish by the addition of white. In trials I have run, it appeared to me quite the opposite: red acquired a purple tinge.

By contrast, the theory of shortest color paths brings another definition of equality in hue close to hand. Let me emphasize explicitly that what is meant here is **not** a 'consequence of theory'. Theory can make no pronouncement at all on **sensation**. Clearly theory only enters into the attempt to wield a concept differently or more profoundly, whose quantitative employment has proved superficial to date, and when a concept does **not** jibe with sensation. Theory enters in so that the concept may correspond better to sensation, if possible.

In order to put the matter shortly, it was common practise until now to call all mixtures of white and a spectral light of wavelength λ as 'equal in hue'. The hue was then characterized by giving the wavelength. On the other hand it is known that hue – subjectively evaluated – is changed by the addition of white to a mixture.

Now I should like to make the link clearer (as ***Helmholtz*** did in a similar situation, by the way) that the series of color mixtures is **not** the shortest path to bridge spectral colors and white. It seems likely to me that **the fastest progression from any distinctive color to colorlessness proceeds without change in hue**. That is because any new mark of difference between adjacent colors in a series would be superfluous, and would needs lengthen the path.

That a given color is 'equal in hue' to another would then imply a definition of all the colors as equal in hue which lie on its shortest line to white.

At least this definition is consistent, and it has consequences. Whether it really represents sensation in a better way than the definition given previously, is clearly a matter for subjective judgment.

What specifically would this definition tell us? Figure 4.6 illustrates the pencil of geodesic lines through the white point of a color triangle. We would like to think this captures the loci of equally bright colors. Only the connections between fundamental colors and their complements are straight lines. Only in those cases would their mixtures with white be constant in hue. For all intermediate colors, their mixtures with white appear pushed towards the dominant fundamental color. That means towards the fundamental color most strongly represented – as compared to

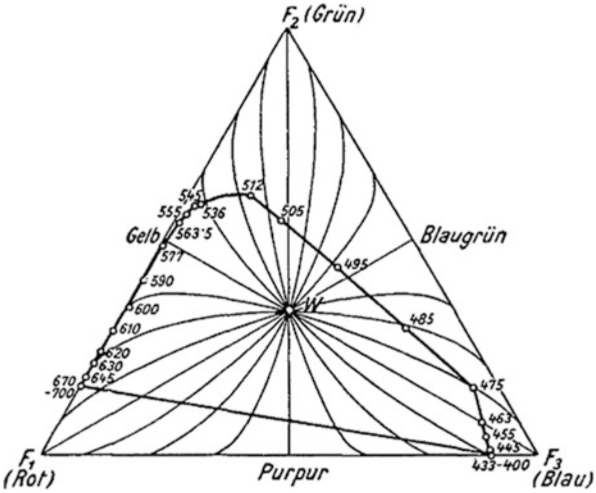

Fig. 4.6 The curves that form a pencil of geodesic lines emanating from the white point of a color triangle. (*Rot*: red; *Gelb*: yellow; *Grün*: green; *Blaugrün*: blue-green; *Blau*: blue; *Purpur*: purple). (Reproduced from Schrödinger, 1920c)

the mixture ratio of the three fundamental colors which appears to be colorless. Then following our assumption for a color in close proximity to the white point, one finds its equivalent in hue along the edge of the triangle, by proceeding along one curve of the bushel. The bushel draws together a three-pointed star from the locations of the fundamental colors.

Colors which can actually be produced are found along these curved segments, that constitute the projection of the amply-described envelope from the first part of this article. One may recognize that the appearance of a color will hardly be distinctive, when the color is from a part of the spectrum in which the arcs of the ellipse depart but little from straight lines. The colors that will be most strongly distinctive are in red and orange, and then in the indigo and violet regions.

One should expect a number of changes when a spectrum is flooded by white in increasing strength. One change is that all the colors draw towards the nearest fundamental color. Another is that, as in the ***Bezold-Brücke*** phenomenon, the three fundamental components of hue are preserved, but with fairly abrupt transitions towards nearby yellow and blue-green. As complementary colors to fundamental blue and fundamental red, these colors are likewise persistent, but will not be very salient since neighbouring colors draw away from them.

I have carried out a rudimentary trial with the diffraction spectrum of sunlight from a grating about a meter in length. I did this at a time when the theoretical stance just outlined was as yet unclear. These expectations were fulfilled insofar as three hues actually did persist – with abrupt transitions – under conditions of strong desaturation. It appeared to me that red acquired a distinct rosy tinge, which brought it closer to fundamental red in hue. The persistent color at the long-wave end was not blue, however. As many observers have determined already, it was a purplish violet (quite a whitish one, of course). The prominence of red was more distinct than in the pure undilute spectrum.

I would not hazard to base a proposal for the constitution of the third fundamental color on the theory of changes in hue which has been proposed. Such a theory remains uncertain, and it may depart from the views of other, more experienced investigators. But our result that the third fundamental color must be reddish-violet rather than blue – if the theory makes sense – is consonant with the proposition that has been expressed by ***König*** himself[(t)] in his later work. It is so plainly consonant that we seem to glimpse an error in the commonly-agreed position of the third fundamental, rather than an error in the theory.

One may adjust the point F_3 – holding the location of the spectral curve constant – downwards and to the left, so that the line WF_3 no longer intersects the spectral curve, but rather intersects the violet end of the line of purples. By that operation the geodesics are to be redrawn as inscribed ellipses of the **transformed** triangle, as should be clear. In that way the shift in position towards reddish-violet seems reasonable, as do the results from blue-blind observers which ***König*** describes in the same text.[(u)]

Section 10: The *Bezold-Brücke* Phenomenon. Prospects for Future Improvement to the Line Element

A near relative to the question of change in hue by the addition of white is the question of change in hue induced by mere diminution of the objective intensity of a light mixture (without change of percentage in its composition). One might call this: difference in hue among colors of the same stimulus quality. In like manner, the hue of these colors moves towards that of the closest fundamental color. Under very low-light conditions, only three hues remain in the spectrum, with quite abrupt transitions just as for a spectrum diluted with white (the ***Bezold-Brücke*** phenomenon).

We may understand how this appearance is **not** subsumed by our theory of the line element, given the simple qualitative explanation of the effect provided by ***Brücke***.

First: the phenomenon is really not subsumed by the theory. It has been said that one may demand no immediate proposition of experience from theory. Then let us examine the compelling assumption that to a **given** color the **most similar** color of a **given brightness** (higher or minimal) is **equal in hue**. The assumption is that the quickest transition can be arranged by proceeding to salient colors of the same hue. Yet the geodesic normals to the isolychnes are the radial vectors. As a consequence, colors of the same stimulus quality should also be equal in hue; change in hue should not be manifest on the mere basis of change in objective intensity.

This failure of theory is unremarkable **for the reason** that it is based wholly on the assumption that ***Fechner***'s Law holds for increments of intensity. Surely it only represents an approximation to reality, as does the Law itself. According to ***Brücke*** the appearance in question depends on a subliminal threshold effect – sublimation in whole or in part of the affected color in relation to white. The effect works on the fundamental valence that is most weakly represented. (Here it is assumed that the threshold values of fundamental valences behave in the same way as does the fundamental valence of white – in other words: white disappears colorlessly below threshold.)

Even for reasons of mathematics, ***Fechner***'s Law must lose its validity around the absolute zero of sensation. At that point an indefinitely large sensitivity to difference arises. (Our line integral $\int ds$ **diverges** as one attempts to approach the origin.) At the detection threshold, this stands in blatant contradiction to experience, since there changes in stimuli stop being perceived entirely. Instead, the Law would have it that sensitivity becomes ever greater as the strength of the stimulus diminishes. Appreciable deviations from this Law should then also be present for much larger stimulus values; that much is also validated by experience.

Then we cannot expect that a phenomenon will be accounted for by our theory, when the effect depends primarily on approximation to a detection threshold. At least we shall not expect that, so long as we depend on ***Fechner***'s Law as a rough guide.

Helmholtz also tried to produce a rough sort of correction, though of course it was for **his** line element, which we have seen is unusable.

One might have – as a second approximation – not identified the x_i in (Eq. 4.12′) with the fundamental valences. One could have identified the x_i with the fundamental valences augmented by a small but definite constant ε_i. (***Helmholtz*** relates the ε_i to entoptic light, "the intrinsic light of the retina".) Keeping to the notation of the first part, one would have to make the following substitution:

$$x_i + \varepsilon_i \quad \text{for} \quad x_i$$

And so the disappearance of the denominator is avoided as a fundamental valence disappears. In the present article I have distanced myself from introducing such a correction, so not to complicate the material further – since it is not simple to begin with. Clearly the metric of color space is unchanged, but it is then **deferred** to the affine space we considered in the first part [Chaps. 2 and 3]. Using such a correction, the isolychnes would be slightly deformed, if only because in actual experiment we adjust the **objective** properties of the fundamental valences x_i, not "entoptically" adjusted terms $x_i + \varepsilon_i$ which are simpler for representation of the metric. I reckon the shape of the deformed isolychnes as:

$$x_1 \ + \ x_2 \ + \ x_3 \ - \ \log\left(x_1^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_3}\right) = \text{const.}$$

Then there should be deviations from additivity of brightness, particularly in the case of trifling brightness (for **large** x_i the logarithmic term recedes). Then there should also be deviations in the brightness ratio for clearly distinct colors, as ***König*** found in experiment.[(v)]

Overall one can say: either this correction or a similar correction factor may be applied to our line element. <u>The need to apply them is beyond all doubt, because of</u>

the approximate nature of ***Fechner***'s Law. These corrections debase all the 'laws' which we have established for our 'advanced colorimetry' – for example the precise correspondence between strongly heterochromatic photometry and the photometry of small differences. Such 'laws' only have the status of simple approximations. Therefore we would have them known as such – and only as such – from the very outset. The value of theory seems to us to lie in this: theory shows us the intrinsic connection of all these heuristics. At the same time it shows us the intrinsic connection of **deviations of the actual color manifold from our idealized, "purely *Fechner*ian" color manifold**. Assays for the greater number of these deviations wait for the thorough and precise experimental research which has yet to be conducted.

(Submitted March 1920)

Notes

a. ***König***, A. & ***Brodhun***, E. Experimentelle Untersuchungen über die psychologische Fundamentalformel in Bezug auf den Gesichtssinn. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, **37**(2), 26 Juli, 917 – 931 (1888).; ***König***, A. & ***Brodhun***, E. Experimentelle Untersuchungen über die psychophysische Fundamentalformel in Bezug auf den Gesichtssinn. Zweite Mittheilung. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, **32**(2), 27. Juni, 641 – 644 (1889). [ff. 1, p. 484 original].

b. From this point on we will omit the summation signs, as has become customary. Wherever general indices appear in double summation, we always sum from 1 to 3 ! [ff. 1, p. 487 original].

c. ***Forsyth***, A. R. Theorie der Differentialgleichungen. Deutsche Ausgabe. [Theory of differential equations, German edition] part 1. Leipzig: B.G. Teubner (1893), p. 7 ff. [ff. 1, p. 489 original]

[Trans.] In the original this passage reads:

» Wenn das nun für jeden beliebigen Weg gelten soll, so müssen sich die Flächenelemente (4.3) zu Integralflächen zusammenfassen lassen, auf denen dann die früher betrachteten gleichhellen Streifen verlaufen. Der ***Pfaff***sche Differentialausdruck auf der linken Seite muβ einen Multiplikator μ (x_1, x_2, x_3) zulassen, derart, daβ

$$\frac{\partial\, \mu\, a_{ik}\, x_i}{\partial x_l} = \frac{\partial\, \mu\, a_{il}\, x_i}{\partial x_k} \quad \text{für} \quad \begin{array}{l} k, l = 1, 2, 3 \\ k \neq l. \end{array}$$

Das ist nun für ganz beliebige Funktionen $a_{i\,k}$ bekanntlich keineswegs der Fall, sondern nur, wenn die $a_{i\,k}$ einer gewissen Bedingung genügen. Es muβ nämlich

$$\mu \cdot \left(\frac{\partial\, a_{ik}\, x_i}{\partial x_l} - \frac{\partial\, a_{il}\, x_i}{\partial x_k} \right) = - \left(a_{ik}\, x_i\, \frac{\partial \mu}{\partial x_l} - a_{il}\, x_i\, \frac{\partial \mu}{\partial x_k} \right) .$$

Multipliziert man jede der drei Gleichungen dieser Art, die sich für die drei Kombinationen $(k\,,l)$ ergeben, mit $a_{p\,m}\,x_p$, wo m der dritte Index, und addiert, so kommt rechts identisch Null. Es muβ also

$$\underset{(k\;l\;m)}{\mathfrak{S}}\; a_{pm}\,x_p\left(\frac{\partial\;a_{ik}\,x_{\mathrm{i}}}{\partial x_l}-\frac{\partial\;a_{il}\,x_i}{\partial x_k}\right)\;=\;0\;\;.$$

Das eigenartige Summenzeichen soll andeuten, daβ auβer den beiden, nach Übereinkunft nicht angezeigten Summationen über die Indizes i und p, das Indextripel $(k\,,l\,,m)$ die drei Wertekombinationen (1, 2, 3), (2, 3, 1), (3, 1, 2) zu durchlaufen hat. Die Relation läβt sich noch etwas vereinfachen, da wegen $a_{\,i\,k}=a_{\,k\,i}$, gilt

$$\frac{\partial\;a_{ik}\,x_i}{\partial x_l}-\frac{\partial\;a_{il}\,x_i}{\partial x_k}=x_i\left(\frac{\partial a_{ik}}{\partial x_l}-\frac{\partial a_{il}}{\partial x_k}\right)\;\;.$$

Daher

$$\underset{(k\;l\;m)}{\mathfrak{S}}\;a_{pm}\left(\frac{\partial a_{ik}}{\partial x_l}-\frac{\partial a_{il}}{\partial x_k}\right)\;x_i\,x_p=0\;\;. \tag{4.4}$$

Diese Bedingung ist **notwendig** für die Integrabilität von (4.3), es kann übrigens gezeigt werden, daβ sie dafür auch **hinreicht**. «

d. ***König***, A. Über den Helligkeitswert der Spektralfarben bei verschiedener absoluter Intensität. Aus: Beiträge zur Psychologie und Physiologie der Sinnesorgane (Festschrift zur Feier des 70. Geburtstages von Hermann v. ***Helmholtz***). Hamburg: Leopold Voss, 309 – 392 (1891).; cf. also ***Allen***, F. The persistence of vision of colours of varying intensity. Philosophical Magazine (Philosophical Journal) S.6, **38**(223), July, 81 – 89 (1919). [ff. 1, p. 490 original]
e. In common comparisons of brightness with the ***Nicol*** prism, the possibility of using a like concept of brightness is assumed tacitly. [ff. 1, p. 491 original].
f. ***König***, A. & ***Brodhun***, E. Experimentelle Untersuchungen über die psychologische Fundamentalformel in Bezug auf den Gesichtssinn. Sitzungsberichte der Königlich Preuβischen Akademie der Wissenschaften zu Berlin, **37**(2), 26 Juli, 917 – 931 (1888).; & ***König***, A. & ***Brodhun***, E. Experimentelle Untersuchungen über die psychophysische Fundamentalformel in Bezug auf den Gesichtssinn. Zweite Mittheilung. Sitzungsberichte der Königlich Preuβischen Akademie der Wissenschaften zu Berlin, **32**(2), 27. Juni, 641 – 644 (1889). [ff. 1, p. 492 original]
g. ***König***, A. & ***Dieterici***, C. Ueber die Empfindlichkeit des normalen Auges für Wellenlängeunterschiede des Lichtes. Annalen der Physik, **22**(8), (Neue Folge **22**, Ganze Folge **258**), 579 – 589 (1884).; ***Uhthoff***, W. Ueber die Unterschiedsempfindlichkeit des normalen Auges gegen Farbentöne im Spektrum. Gräfes Archiv für Ophthalmologie [Gräfe's Archive for Clinical

and Experimental Ophthalmology], **34**(4), December, 1 – 15 (1888).; ***Exner***, F. Über die Grundempfindungen im ***Young-Helmholtz***'schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **111**, 857 – 877 (1902).; ***Steindler***, O. Die Farbenempfindlichkeit des normalen und farbenblinden Auges. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **115**, 39 – 62 (1906). [ff. 2, p. 492 original].

h. ***Gottlieb***, M. Über die Empfindlichkeit des Auges gegenüber Sättigungsänderungen von Farben. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **126**, 1299 – 1316 (1917). [ff. 3, p. 492 original].

i. ***Abney***, W. de W. Philosophical Transactions of the Royal Society A (Mathematical, Physical, & Engineering Sciences), January 1st, **193**, 259 – 287 (1900); Ibid.: ***Abney***, W. de W. Modified apparatus for the measurement of colour and its application to the determination of the colour sensations. Philosophical Transactions of the Royal Society A (Mathematical, Physical, & Engineering Sciences), January 1st, **205**, 333 – 355 (1906). Since ***Abney***'s photometry is strongly heterochromatic (see Section 7 below) – in contrast to that of ***Exner*** or of ***Kohlrausch*** – ***Abney***'s results so not have the same central importance to our theoretical standpoint (presented in Section 3) as the results of the other two researchers. I would also like to take this opportunity to justify why – when the clarification of basic concepts is in question – I never make use of the many elegant results that use the inspired method of **flicker photometry**. That is because in my opinion one must first establish **what brightness is**. **Then** one may determine by empirical methods, **if** and **how** brightness may be reckoned from tests on afterimages which are quite complicated in theoretical terms. **Then** one may use such methods for theoretical purposes in an unprejudiced way. [ff. 1, p. 493 original].

j. ***Exner***, F. Einige Versuche und Bemerkungen zur Farbenlehre. [Some experiments and discussion on color theory] Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **127**, 1829 – 1864 (1918).; Ibid. ***Exner***, F. Zur Kenntnis der Grundempfindungen im ***Helmholtz***'schen Farbensystem. [Towards a characterization of the fundamental stimuli in ***Helmholtz***'s color system] Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **129**, 27 – 46 (1920, in press). [ff. 2, p. 493 original]

k. ***Kohlrausch***, K.W.F. Mitteilungen des techn. Versuchsamtes in Wien, 9. Jahrg. 1920 (in press); See: ***Kohlrausch***, F.W.F. (1920). Beiträge zur Farbenlehre

II. Die Hellkigkeit der Pigmentfarben. [Articles on color theory II. Brightness of pigment colors.] Physikalische Zeitschrift, **21**, 423 – 440.

l. ***Helmholtz***, H.L.F. von Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem. [Proposal for an extended application of Fechner's Law to the color system.] Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **2**, 1 – 30 (1891) (Table 1). [ff. 1, p. 497 original]

m. ***König***, A. & ***Dieterici***, C. Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsverteilung im Spektrum. [Fundamental sensations in normal and anomalous color systems, and their intensity distributions across the spectrum] Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **4**, 241 – 347 (1893) & § **23**. (1892). [ff. 2, p. 497 original].

n. ***Uhthoff***, W. Ueber die Unterschiedsempfindlichkeit des normalen Auges gegen Farbentöne im Spektrum. [On the ability of normal eyes to distinguish sensations of color tint across the spectrum.] Gräfes Archiv für Ophthalmologie [Gräfe's Archive for Clinical and Experimental Ophthalmology], **34**(4), December, 1 – 15 (1888).; cited in ***Nagel***, W. A. Handbuch der Physiologie des Menschen, 3 vol. III(1), Braunschweig: Viehweg (1904), p. 251. [ff. 1, p. 498 original].

o. ***Helmholtz***, H. L.F. von Versuch einer erweiterten Anwendung des ***Fechner***schen Gesetzes im Farbensystem. Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **2**, 1 – 30 (1891). [ff. 1, p. 500 original]

p. For 'F' and 'V' one may think of '**fixed**' and '**variable**' colors ! [ff. 1, p. 501 original].

q. To make this representation vivid, let us consider the x_i as **orthogonal** coordinates ! [ff. 1, p. 509 original].

r. The reader is invited to sketch this simple Figure, which I omit here for reasons of space. [ff. 1, p. 510 original].

s. ***Abney***, W. de W. On the change in hue of spectrum colours by dilution with white light. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 10th, **83**(560), 120-127 (1909). [ff. 1, p. 513 original].

t. ***König***, A. Über « Blaublindheit ». Sitzungsberichte der Königlich Preuβische Akademie der Wissenschaften zu Berlin, **34**(2), 8. Juli, 718 – 731 (1897). The expository note is found on p. 406 of ***König***, A. Gesammelte Abhandlungen. [Collected works.] Leipzig: Johann Ambrosius Barth (1903). [ff. 1, p. 517 original].

u. ***König***'s experiments produced the following results for pathological observers who were blue-blind: 1) lights at the spectrum ends could be mixed to match any spectral color, and 2) there was only **one** neutral (color-free) point along the spectrum – and that was in the yellowish-green. Both results belie the location of the faux color F_3 along the spectral curve, as it is indicated by Figure 4.5. [ff. 2, p. 517 original].

v. Compare Section 3 above, after Eqn. 4.5. [ff. 1, p. 519 original].

Chapter 5
A Metric of Color

Abstract Color space is a generalization of the color diagram; it embodies the laws of color mixture for human color vision. If it is based on color-matching judgments alone, the color metric is an affine geometry of a spatial pencil of vectors having a common origin. (Its projective representation is less clear and less useful as confined to a color plane or a color triangle.) A metric of color implies a line element for differences of color. The line element should reconcile the results of strongly heterochromatic photometry with those of photometry which proceeds by just-noticeable differences. There is an assumption implicit in the some versions of the line element: that the Weber–Fechner law holds precisely true over the entire color space. This summary article presages Schrödinger's detailed development of colorimetry, published shortly afterward.

Keywords Color vision • Color metric • Color space • Color manifold • Color mixture • Affine geometry • Color triangle • Color coordinates • Trichromatic • Dichromatic • Heterochromatic photometry • Riemannian metric • Line element • Pfaffian • Helmholtz line-element • Fechner's law • Just-noticeable difference • Geodesic line • Isolychne • Chromatic brightness • Bezold-Brücke • Schrödinger

The task of physiological optics is the determination of colors and their relations through precise experimental measurement.[(a)] In this domain of physiological optics, the **basic observation** is this: As opposed to the enormous manifold composed of **lights** that have the most diverse spectral composition – as **stimuli** distinguished purely in physical terms – there stands a very much smaller manifold of **colors** – as **responses** to those stimuli. As is familiar, a **light** is characterized by its wavelength function $f(\lambda)$. All possible lights, i.e. $f(\lambda)$, fall into broad categories such that any two lights of the same group – when presented to neighbouring areas of the eye (meaning always in the **centre of the retina**, the fovea centralis!) – are completely indistinguishable (for example white sunlight, and white from the combination of complementary colors). We call such a category a **color**, and the associated lights are **the same in color**. <u>Among normal, color-capable individuals, the color manifold is of dimension three</u>.

Schrödinger, E. (1920). Farbenmetrik. *Zeitschrift für Physik*, **1**(5), 1 April, 459–466.

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The second basic observation is this: Lights of the same color can replace one another fully in mixtures. The color of a mixture is completely determined by its constituent **colors**. That justifies us in speaking not only of an **addition** (and later of the **subtraction**) of **lights** (as the concatenation of the respective wavelength functions), but also in speaking of the **addition of colors**. Another notion emerges from this: the multiplication of a color by a number (a scalar). On those two notions rests the classical edifice of color theory, in the tradition of ***Newton***, ***Maxwell***, ***Helmholtz*** and ***König***, commonly known as "the laws of color mixture".

A representation may be given for this consistent and complete state of affairs. The representation maps these facts onto the affine geometry of a spatial pencil of vectors which emerge from a single point. (A representation that is confined to a **color plane** or a **color triangle** may seem more functional, but it is less clear, less transparent, and so less suited to our purposes.) Each color is assigned a point in space, or the corresponding radial vector. The addition of colors corresponds to vector addition. The customary 'vocabulary' of the representation can be derived immediately from this convention. For example, colors which are related to one another by a simple change in light intensity (colors "of the same **stimulus quality**" as von ***Kries*** puts it) lie along the same radial vector. In that much, vector lengths behave like intensities. Similarly the vector for a two-color mixture is coplanar with the vectors of the constituent colors, and so forth.

Just as in affine vector geometry, a **numeric** representation of colors may be obtained. One may specify three numbers (x_1, x_2, x_3) by which three basis vectors F_1, F_2, F_3 must be multiplied and then summed, to produce a given color:

$$F = x_1 F_1 + x_2 F_2 + x_3 F_3 \,. \tag{5.1}$$

However, the three basis vectors $F_1\ F_2\ F_3$ cannot be chosen so that they represent real colors at the same time that any arbitrary real color F can be produced by their mixture. That is, they cannot all be represented as purely positive values of the x_i in the manner just outlined. The reason is that colors do not fill the entire space of our construction. Rather only the interior and the surface of a cone ('the color envelope') are filled, and those approximate the outlined form of our Figure. The pure spectral colors lie along the curved part of the surface ($R\ G\ V$). Along the plane angle $R\ O\ V$ lie saturated purple mixtures that are produced by light from the far ends of the visible spectrum. All the more complicated mixtures of light, including white, lie somewhere in the interior. In passing, one recognizes right away that the color of any arbitrarily complicated mixture of light – white included – can be the combination of two spectral colors. One of them can even be chosen arbitrarily, within set limits. – Now due to its convex form, our color envelope will jut out partly over the sides of any triangle whose edges lie along the envelope's surface. It follows that in fact the totality of all colors may not be mixable from **any** three real fundamental colors (Fig. 5.1).

At this point one can either admit 'virtual color vectors' (which lie outside the 'envelope') as basis vectors, or else admit negative values of the x_i. Both expressions are permitted. The latter – say for a negative x_1 with positive x_2 and x_3 – means

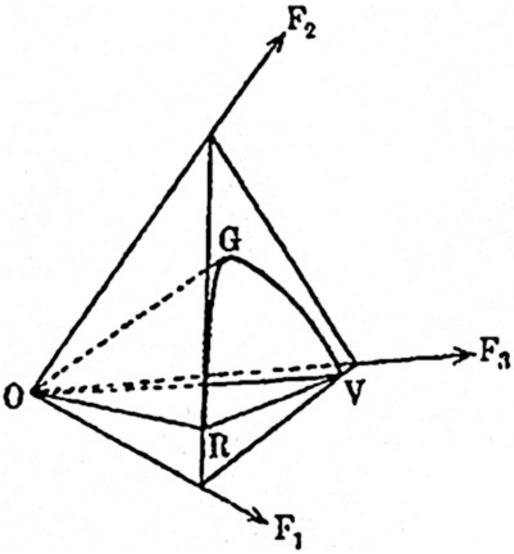

Fig. 5.1 A coordinate representation of color space with three basis vectors $\boldsymbol{F_1}$ $\boldsymbol{F_2}$ $\boldsymbol{F_3}$, and a spectral locus ***RGV***

that to the appointed color F one must mix a precisely determined amount of F_1. That mixture will appear identical to a precisely determined mix of F_2 and F_3. In this way F is unequivocally determined, just as it would be by wholly positive values of x_i. In no case – by the rules of mixture found for normal, color-capable eyes – does any triple (either real or virtual) of fundamental colors appear to be distinctive over and above other triples. After the successful assignment of colors to positions, in principle any (noncoplanar!) vector triple is equally well suited to representation of the coordinate frame.

There are some cases when a definite and certainly virtual triple of fundamental colors can be designated for which the x_i are always positive. This occurs in the state of affairs known as partial color-blindness, or 'dichromacy'. For such color-blind individuals ('dichromats'), the color manifold has only two dimensions. There are three types of dichromacy; among them the third type ('blue-blindness') arises seldom, and only as a result of pathology. At the same time, red-blindness and green-blindness occur in otherwise healthy eyes, as an anomaly of physiology. Each type can be characterized by specifying a particular virtual direction of color vector, which has the following significance: For the dichromat in question, an arbitrary color does not change its appearance at all, if the point representing a color is displaced **parallel to that direction**, or if one adds a vector which has that direction. If one chooses vectors of **these three directions** as basis vectors, then only two of the three x_{i-} values will have import for any dichromat. The third escapes his perception, so to speak. Under this formulation it could be admitted (***Helmholtz***, ***König***) that color vision arises from the action of three fundamental processes in the eye. These act in concert, and change in proportion for different colors. Their (the fundamental processes) activation strengths are a measure of the x_i, which can be specified in that way. **One** of each of them is missing in the dichromat eye, and this explains the triple of colorblindness types. The nomenclature of 'red'-, 'green'-, and 'blue'-blindness has its origin in the real-valued 'characteristic colors' which are coplanar to both a dichromatic basis vector and the vector for white. Those characteristic colors may be conceived as a whitish transformation of a fundamental stimulus which is not expressible on its own.

2. As has been stated, so long as an experiment only takes into account the complete equality (indistinguishability) of two colors to be compared, and so long as this counts as the distinctive relation between them, then color provides a meaningful model only for the affine geometry of our vector pencil. It follows from the nature of affine invariance, that consequently from our representation we are able to derive nothing about greater or lesser degrees of **similarity**, and nothing about the **ratio of brightness** between colors of different stimulus type (vector direction). According to a foundational idea that also can be traced back to ***Helmholtz***,[(b)] it ought to be possible to understand all the other relations between colors – equality of brightness, equality of hue, and so on – by the ability of judgment to select the most similar from among many color pairs along an ordered and continuous series of such pairs. That pair is most similar which shows a smaller difference than the pair immediately preceding it and the pair immediately following. Of course this assumes that such a color pair is present in the series. For example, in heterochromatic photometry a color point may be held constant, and another point displaced along a radial vector and adjusted for 'sameness of brightness'. Then that heterochromatic photometry is simply an approach to photometry by a method of adjustment for maximum similarity. To pursue this notion further, one needs a **measure** of similarity – or equivalently, a measure of difference. To that end we make the very broad assumption that this measure is given by the length of the geodesic line between the two color points under a general ***Riemann***ian metric which we establish for our vector space. Specifically, for small differences this is given as the line element of the metric:

$$ds^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} a_{ik}\, dx_i\, dx_k\,. \qquad [\, a_{ik} = a_{ki}] \tag{5.2}$$

According to ***Helmholtz***, this line element (i.e., the a_{ik} as functions of the x_i) is to be specified so that the ds is assigned the same value for **every just-noticeably different pair of colors**. It is crucially important that the a_{ik} are **determinable empirically**. One expects that they would assume a particularly simple form, should the determination of these coordinates happen upon actual fundamental colors.

Here is a first general corollary for color pairs that are not much different (x_i and $x_i + dx_i$) as a **condition of equal brightness** (i.e., maximum similarity, given **radial** displacement of one color).

$$\sum_i \sum_k a_{ik}\, x_i\, dx_k = 0. \tag{5.3}$$

Given a fixed x_i, one may call this an equation for **equally-bright surface elements** in terms of the variable (relative) coordinates dx_k. It is of some interest that for **arbitrary** a_{ik}, such surface elements do **not** assemble themselves into integral surfaces of equal brightness (or 'isolychnes'). That is because – in general – a ***Pfaff***ian expression in three differentials has no integrating factor. Yet experience speaks to the issue, to the effect that this is the case here. If one assumes that much, and then one requires the normalization of the brightness function h – that it should be a homogeneous first-degree function in x_i – one obtains the following differential equations as the form of the expression.

$$\frac{\partial \log h}{\partial x_l} = \frac{\sum_i a_{il}\, x_i}{\sum_i \sum_k \ a_{ik}\, x_i x_k} \qquad l = 1, 2, 3. \tag{5.4}$$

Neither the available experimental data – nor any which could be hoped for – about difference thresholds suffice for the direct and complete experimental determination of the a_{ik}. One is consigned to evaluating various rough formulations from experiment. ***Helmholtz*** was led to the following formulation by the outcome of certain badly-calibrated experiments with colored disks (see below):

$$ds^2 = \sum_i \frac{dx_i^2}{x_i^2} \tag{5.5}$$

but this **fails** in each and every direction. It is not only that it returns merely middlingly correct values for difference thresholds across the spectrum, if it is applied to fundamental colors chosen freshly ad hoc – which depart radically from the dichromatic primaries. It is also that – according to (Eq. 5.4) – it produces quite the impossible function

$$h = \sqrt[3]{x_1\, x_2\, x_3}$$

for brightness. To satisfy that function, the spectrum of sunlight would need to exhibit **two** prominent maxima of brightness. They would be as prominent whether one chose to employ dichromatic primaries, or else the fundamental colors calculated ad hoc.

Empirical results (***Abney***, F. ***Exner***, K.W.F. ***Kohlrausch***) indicate that brightness behaves **additively** in mixtures, to a fine approximation.

$$h\,(x_1, x_2, x_3) + h\,(y_1, y_2, y_3) = h\,(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3). \tag{5.6}$$

From this functional equation, it follows straightforwardly that:

$$h = \alpha\, x_1 + \beta\, x_2 + \gamma\, x_3\ , \tag{5.7}$$

where α, β, and γ are constants. If one incorporates these in specifying the coordinates – by altering the scales of units – then one obtains the yet simpler form

$$h = x_1 + x_2 + x_3. \tag{5.8}$$

The simplest line element produced by **this** brightness function, which at the same time does justice to the ***Weber-Fechner*** law (valid within wide bounds) for pure changes in intensity, is:

$$ds^2 = \sum_{i=1}^{3} \frac{dx_i^2}{x_i(x_1 + x_2 + x_3)} \tag{5.9}$$

This attempt at calculation leads to two propositions: 1) that **this** line element follows the empirical difference threshold for spectral colors, at least in broad outline without use of any assumption ad hoc, and 2) that **this** threshold also replicates a quantitative property of the Fechner interval with a degree of approximation that might be expected, given the exigencies of using very heterogeneous materials in experiment. (The desired property is a constant ds for any noticeably different pair of colors.)

Even ***Helmholtz***'s above-mentioned experiments with a color wheel find an explanation. Those results were obtained with a mixture of two very different pigments – such as red and blue, where a small extra portion of red must be added in order to attain the same increase in brightness as occurs with a small but **definite** addition of blue. The extra red must be added in **larger** quantities, the **greater** the original quantity of red. Naturally the converse holds too: a small addition of blue is more strongly 'knocked down' in its brightening effect by the amount of blue present at first. More strongly, that is, than by the amount of red already present. This **appears** to contradict the additivity of brightness (which was ***Helmholtz***'s express opinion). As a matter of fact this does not contradict additivity at all.

In reality what was adjusted for was not equal brightness, but rather maximum similarity – that is, adjustment to the fullest possible blurring of the dividing line between the two shades of color. ***Helmholtz*** overlooked the fact that the minimum condition for this situation is not the aforementioned equation (Eq. 5.3). Rather another different equation results, if the variable color is changed not along its radial vector as in Eq. (5.3), but in another direction, namely by small additions of one of these two pure pigments. An exact computation with **our** line element (Eq. 5.9) – which **strictly** fulfils the additivity constraint – does in fact produce the observed variation of the **apparent** brightness values of the two colors, in just the direction given above – the direction observed by ***Helmholtz***!

In the above, we have sought fundamental support for our notion of brightness in the comparison of **minimally different** color pairs. The general brightness function was found by integration over those. This procedure does not emerge from a purely theoretical construction. Rather the procedure is entered into by predilection in experiment (as in the intercalation of intermediate colors). Further experiment teaches us that direct comparison of the brightness of **strongly different** hues may be difficult. Yet insofar as it may be accomplished, it leads to nearly the same result as the intercalation of intermediate steps. One may ask if theory produces this same result, i.e., if the geodesic normal to a color point F runs along any radial vector in the isolychne that intersects F.

The geodesic lines of our metric arise in a space where one considers the $\sqrt{x_i}$ to be arrayed as independent coordinate axes, but they are pictured as plane logarithmic spirals centred on the origin. In the x_i vector space,[(c)] they wind around cones whose bases are elliptical, which have their peaks at the origin, and which have points of contact with all three coordinate planes. This is a similar arrangement to the threads of a conical screw, but with threads whose turns diminish infinitely and asymptotically towards the origin. There are a couple of special cases: a) the radial vectors themselves, and b) the elliptical conic-sections which are the intersection of the isolychnes and the cone itself. The geodesic connection of two equally-bright points then runs **along the isolychne** (for case (b)). Otherwise the connection has a normal direction to all its radial vectors. In terms of Riemannian geometry, **that** is the force of the general equation (Eq. 5.3) which holds for any line element along the isolychne. (The isolychnes are plane segments orthogonal to the the radial vectors.)

Then the question posed earlier is answered in the affirmative: our theory does ensure an exact correspondence of strongly heterochromatic photometry, with photometry that proceeds by gentle increments.

A closed-form equation results, for the **length of the geodesic** between two points with the color coordinates designated by y_i and z_i .

$$\int_{(y)}^{(z)} ds = \sqrt{\left(\log \frac{z_1+z_2+z_3}{y_1+y_2+y_3}\right)^2 + 4\left(\arccos \frac{\sqrt{y_1\ z_1}+\sqrt{y_2\ z_2}+\sqrt{y_3\ z_3}}{\sqrt{(y_1+y_2+y_3)\ (z_1+z_2+z_3)}}\right)^2} \tag{5.10}$$

This provides a measure of the difference between two colors, that is, it gives the minimum number of just-noticeable intervals which serve to bridge one color to another. The first item under the radical sign vanishes when the points are **equally bright**; the second vanishes when they are colors of the **same stimulus quality** (i.e., colors that fall along the same radial vector).

As has been intimated, the geodesics lie along the surfaces of **cones**, not on **planes through the origin**. Therefore except in special cases, the “shortest series of colors” does not coincide with a series produced by elementary color-mixture. Consider the familiar **color-triangle** diagram – which we have not made any use of, but which just turns out to be an arbitrary **plane section** of our vector pencil. In the color triangle, the fastest transition is not achieved along straight lines. (Elementary mixtures fall along straight segments.) Rather the fastest transition falls along the curved arc of an ellipse which intersects both points, and that has points of contact with all three sides of the color triangle. In general there will be **two** such ellipses, and consequently **four** elliptical arcs which join the points. From those four we choose the arc which contains no point of contact. If the colors are the **same in stimulus quality**, then the arc of fastest transition proceeds from one to the other maintaining the same stimulus quality (pure change in intensity). If the colors are the **same in brightness**, the arc maintains constant brightness. If the two points are different in both, then the law of geodesic connection may be expressed as follows: the whole of the necessary **change in brightness** is **equally distributed** across **changes in stimulus quality**, such that the ratio of the two is constant for each small increment along the shortest series of colors. (One may compare our earlier remarks about the two terms contained underneath the square root sign in expression Eq. (5.10).)

There is ample room for speculation about deviation of the shortest series of colors from the series that represents color mixture – especially on the subject of **changes in hue which follow from the addition of white**. Let us omit such speculation here, for the sake of space. Changes in hue which accompany simple variations in intensity have certainly been observed (cf. the '***Bezold-Brücke*** phenomenon'). Such changes in hue have **not yet been incorporated** in this theory in its present formulation.

The fact that this theoretical standpoint is only transitional, follows from the assumption implicit in our line-element (Eq. 5.9): that the ***Weber-Fechner*** law (for simple changes in intensity) holds precisely true over the entire color space. That stands not only in contradiction to experience – it stands in contradiction to **all** possible experience (as when it predicts continuing increase in sensitivity as one moves towards intensity **zero**!). At any rate we still await the form of a correction under which **all** the propositions put forward here as "laws" are more plausibly downgraded to the status of approximations. **From this point onwards they should only be considered as such** – as approximations. To me it seems that the value of theory lies in this: that it brings to light the internal connections of all these heuristics. At the same time it brings to light the internal connections among all the discrepancies between our idealized '**purely *Fechner***ian' color manifold, and the **actual** color manifold. In most cases those discrepancies still await a painstaking and exact empirical investigation. I do not believe that the "riddle of color is solved once and for all". I do think I have cleared a fertile path to its understanding once again. Thus we may catch sight of the fortunate intuitions of the old Master. He may have stumbled a little on this path, and as a consequence he may have thought it should be avoided afterwards as impassable to future research.

Notes

a. Lectures delivered at the meetings of the Vienna Gauverein on February 26, March 4, and March 11 of 1920. A more comprehensive publication will appear shortly, in Annalen der Physik.

b. ***Helmholtz***, H.L.F. von. Handbuch der physiologischen Optik. 2e Auflage. Hamburg und Leipzig: Leopold Voβ , pp. 434 – 472 (1896) (passage suppressed in the 3rd edition of 1909-11). [ff. 1, p. 462 original]

c. Disregarding the abstraction, at this point one ought to think of the x_i as **orthogonal** coordinates too, since the metric is determined by (Eq. 5.9) independently anyways. [ff. 1, p. 465 original]

Kapitel 5
Farbenmetrik

1. Auf dem Gebiete der physiologischen Optik, das sich die Festlegung der Farben und ihrer Beziehungen untereinander durch das exakte, messende Experiment zur Aufgabe macht, ist die **Grunderfahrung** diese, daß der ungeheuren Mannigfaltigkeit von **Lichtern** der verschiedensten spektralen Zusammensetzung, als lauter physikalisch verschiedenen **Reizen**, eine sehr viel kleinere Mannigfaltigkeit von **Farben**, als den **Reaktionen** auf diese Reize, gegenübersteht.[1] Ein **Licht** wird in bekannter Weise durch Angabe einer Wellenlängenfunktion $f\,(\lambda)$ gekennzeichnet. Alle möglichen Lichter *i.e.* $f\,(\lambda)$ ordnen sich in große Gruppen derart, daß irgend zwei Lichter derselben Gruppe, auf angrenzenden Feldern dem Auge (wir meinen immer die **Netzhautmitte**, die fovea centralis!) dargeboten, völlig ununterscheidbar sind (Beispiel: Sonnenweiß – Weiß aus Komplementärfarben). Eine solche Gruppe nennen wir eine **Farbe**, bzw. die Lichter **gleichfarbig**. Für **normale, farbentüchtige Individuen ist die Mannigfaltigkeit der Farben von der Dimensionszahl drei**.

Die zweite Grunderfahrung ist die, daß **gleichfarbige Lichter einander auch in Mischungen vollkommen vertreten können**. Die Farbe einer Mischung ist vollkommen festgelegt durch die **Farben** der Konstituenten. Das berechtigt uns, nicht nur von einer **Addition** (und eventuell Subtraktion) von **Lichtern** (Superposition der betreffenden Wellenlängenfunktionen), sondern auch von einer **Addition von Farben** zu sprechen. Auf diesem Begriff und dem daraus abgeleiteten: Multiplikation einer Farbe mit einer Zahl, beruht das klassische Gebäude der Farbenlehre von ***Newton***, ***Maxwell***, ***Helmholtz***, ***König***, gewöhnlich als „Gesetzte der Lichtmischung" bezeichnet.

Es läßt sich eine Darstellung dieses in sich vollkommen abgeschlossenen Tatsachenkomplexes geben, wobei derselbe abgebildet wird auf die **affine**

von Erwin Schrödinger. *Zeitschrift für Physik*, 1(5), 1920, S. 459–466. Mit einer Abbildung (Eingegangen am 1. April 1920.)

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Geometrie des von einem Punkte entspringenden räumlichen Vektorbüschels. (Gebräuchlicher, aber weniger klar und durchsichtig, darum für unsere Zwecke weniger geeignet, ist die Darstellung in der **Farbenebene** oder dem **Farbendreieck**). Jeder Farbe ist ein Raumpunkt oder der betreffende Radiusvektor zugeordnet. Der Addition von Farben entspricht die Vektoraddition. Aus dieser Festsetzung ist das übrige „Vokabular" der Abbildung sofort abzuleiten, z. B. daß Farben, die durch bloße Intensitätsänderung des Lichtes auseinander hervorgehen (Farben **gleicher Reizart**, wie v. ***Kries*** sagt) auf demselben Radiusvektor liegen, daß dabei die Vektorlängen sich wie die Intensitäten verhalten, daß der Vektor einer binären Mischfarbe mit den Konstituenten komplanar ist usw.

Wie in der affinen Vektorgeometrie läßt sich eine **zahlenmäßige** Darstellung der Farben gewinnen, indem man drei Zahlen angibt (x_1, x_2, x_3) mit denen drei Grundvektoren $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$ multipliziert und addiert werden müssen, um die vorgelegte Farbe $\boldsymbol{F}$ zu ergeben:

$$F \;=\; x_1 F_1 \;+\; x_2 F_2 \;+\; x_3 F_3 \,. \tag{1}$$

Die drei Grundvektoren $\boldsymbol{F_1}$ $\boldsymbol{F_2}$ $\boldsymbol{F_3}$ können aber nicht so gewählt werden, daß sie wirklichen Farben entsprechen und zugleich jede beliebige wirkliche Farbe $\boldsymbol{F}$ aus ihnen mischbar ist, d.h. mit lauter positiven Werten der $\boldsymbol{x_i}$ auf die obige Weise dargestellt werden kann. Der Grund ist der, daß sich bei unserer Abbildung nicht der ganze Raum mit Farben füllt, sondern nur das Innere und die Oberfläche eines Kegels („Farbdüte") von etwa der nebenstehend gezeichneten Gestalt. Auf dem gewölbten Teil der Oberfläche (***R G V***) liegen die reinen Spektralfarben, auf dem ebenen Winkel ***R O V*** die aus den Endlichtern des Spektrums herstellbaren gesättigten Purpurgemische. Alle komplizierteren Lichtgemische, so auch das Weiß, liegen irgendwo im Innern, übrigens erkennt man sofort, daß die Farbe jedes beliebig komplizierten Lichtgemisches, ebenso wie das Weiß, auch aus zwei Spektralfarben gemischt werden kann, von denen eine sogar noch, innerhalb gewisser Grenzen, beliebig wählbar ist. – Da nun unsere Farbdüte wegen ihrer konvexen Gestalt über jedes Dreikant, dessen Kanten auf ihrer Oberfläche liegen, teilweise hinausragt, so folgt, daß sich in der Tat aus **keinem** reellen Grundfarbentripel die Gesamtheit aller Farben ermischen läßt (Abb. 5.1).

Man kann nun entweder auch „irreelle Farbvektoren" (außerhalb der „Düte" gelegene) als Grundvektoren zulassen, oder negative Werte für die x_i. Beides ist zulässig. Das Letztere, z. B. ein negatives x_1, bei positiven x_2, x_3 bedeutet, daß man der so gekennzeichneten Farbe $\boldsymbol{F}$ erst eine genau bestimmte Menge von $\boldsymbol{F_1}$ zuzumischen hat, welches Gemisch dann einem genau bestimmten Gemisch von $\boldsymbol{F_2}$ und $\boldsymbol{F_3}$ gleich erscheint. Hierdurch ist $\boldsymbol{F}$ ebenso eindeutig festgelegt, wie bei durchaus positiven x_i. **In keinem Falle erscheint durch die Mischungstatsachen des normalen farbentüchtigen Auges allein ein bestimmtes** (reelles oder irreelles) **Grundfarbentripel vor den übrigen ausgezeichnet**; nach erfolgter Lokalisation der Farben eignet sich jedes (nicht komplanare!) Vektortripel prinzipell gleich gut zur Koordinatendarstellung.

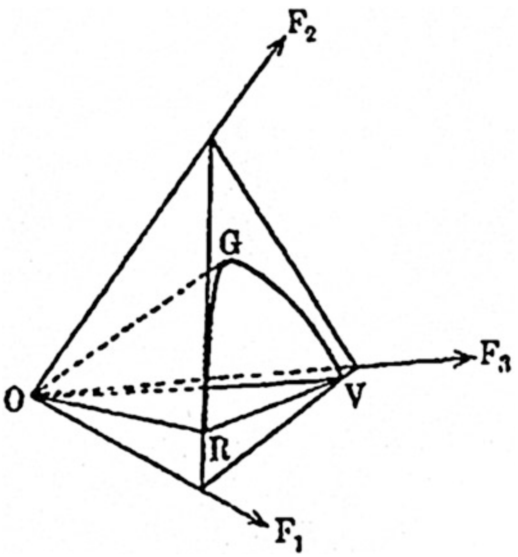

Abb. 5.1

Die Auszeichnung eines bestimmten und zwar irreellen Grundfarbentripels, für welches die x_i stets positiv sind, erfolgt durch die Tatsachen der partiellen Farbenblindheit („Dichromasie“). Für diese Farbenblinden („Dichromaten“) hat die Farbmannigfaltigkeit nur zwei Dimensionen. Es gibt davon drei Typen, wovon allerdings der dritte („Blaublindheit“) nur selten und nur pathologisch vorkommt, während Rotblindheit und Grünblindheit als physiologische Anomalie an sonst gesunden Augen auftreten. Jeder Typus läßt sich durch Angabe einer bestimmten irreellen Farbvektorrichtung kennzeichnen, welche für ihn folgende Bedeutung hat: für den betreffenden Dichromaten ändert eine beliebige Farbe ihr Aussehen gar nicht, wenn sich der Farbpunkt **parallel zu dieser Richtung** verschiebt oder wenn man einen Vektor dieser Richtung addiert. Wählt man Vektoren **dieser drei Richtungen** zu Grundvektoren, so kommt es für jeden Dichromaten nur auf zwei von den drei x_i_ Werten an, der dritte entgeht sozusagen seiner Wahrnehmung. Es ist die Auffassung zulässig (***Helmholtz***, ***König***), daß das Farbensehen durch drei physiologische Grundprozesse im Auge zustande kommt, die bei den verschiedenen Farben in wechselnden Verhältnissen zusammenwirken, für deren (der Grundprozesse) Erregungsstärken die in dieser speziellen Weise bestimmten x_i ein Maß sind und von denen je **einer** dem dichromatischen Auge fehlt, woraus sich eben die Dreizahl der Typen erklärt. Die Bezeichnung als „Rot“-, „Grün“-, „Blau“ – Blindheit hat ihren Grund in der Färbung desjenigen reellen „Farbenfächers“, der mit je einem dichromatischen Grundvektor und mit dem Weißvektor komplanar ist, und deshalb als weißliche Abwandlung der isoliert nicht auslösbaren Grunderregung aufzufassen ist.

2. Solange das Experiment nur die völlige Gleichheit (Ununterscheidbarkeit) zweier verglichener Farben in Betracht zieht und als ausgezeichnete Beziehung zwischen ihnen gelten läßt, hat, wie gesagt, nur die affine Geometrie unseres Vektorbüschels für die Farben Bedeutung. Aus dem Charakter der affinen Invarianz folgt, daß wir deshalb aus unserer Darstellung nichts über den größeren oder geringeren Grad von **Ähnlichkeit**, nichts über das **Helligkeitsverhältnis** von Farben verschiedener Reizart (Vektorrichtung) ablesen können. Nach einem Grundgedanken, der gleichfalls auf ***Helmholtz***[2] zurückgeht, dürfte es möglich sein, alle übrigen Beziehungen zwischen Farben, z. B. Gleichheit der Helligkeit, Gleichheit des Farbtons, usw. zu verstehen aus der Fähigkeit des Urteils, aus einer vorgelegten kontinuierlichen Folge von Farbenpaaren das ähnlichste herauszusuchen, d.h. dasjenige, das einen kleineren Unterschied aufweist, als die unmittelbar voraufgehenden und nachfolgenden; vorausgesetzt natürlich, daß ein solches Farbenpaar sich in der Reihe vorfindet. Die heterochrome Photometrie z.B., bei der etwa ein Farbpunkt festgehalten, der andere auf einem Radiusvektor verschoben und auf „gleiche Helligkeit" eingestellt wird, soll nichts anderes als eine Einstellung auf größte Ähnlichkeit sein. Um diesen Gedanken durchzuführen, benötigt man ein **Maß** der Ähnlichkeit bzw. des Unterschiedes. Wir machen darüber die sehr allgemeine Annahme, daß es geliefert wird durch die Länge der geodätischen Linie zwischen den beiden Farbpunkten in einer allgemeinen **Riemann**schen Metrik, die wir in unserem Vektorraum etablieren: bzw. bei kleinen Unterschieden, durch das Linienelement dieser Metrik:

$$ds^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} a_{ik}\, dx_i\, dx_k\,. \qquad [\, a_{ik} = a_{ki}] \tag{2}$$

Dieses Linienelement (d.h. die a_{ik} als Funktionen der x_i) ist, nach ***Helmholtz***, so zu bestimmen, daß ds denselben Zahlenwert bekommt **für jedes eben unterscheidbare Farbenpaar**; prinzipell sind also die a_{ik} **empirisch zugänglich**. Man wird erwarten, daß sie **dann** besonders einfache Gestalt annehmen, wenn die Koordinatenbestimmung auf die wahren Grundfarben erfolgt ist.

Zunächst folgt allgemein für wenig verschiedene Farbenpaare (x_i und $x_i + dx_i$) als **Bedingung gleicher Helligkeit** (d.i. größter Ähnlichkeit bei **radialer** Verlagerung der einen Farbe):

$$\sum_i \sum_k a_{ik} x_i\, dx_k = \; 0. \tag{3}$$

Mann kann das, bei festgehaltenen x_i, die Gleichung des **gleichhellen Flächenelements** in den laufenden (relativen) Koordinaten dx_k nennen. Es ist von Interesse, daß bei **beliebigen** a_{ik} diese Flächenelemente sich **nicht** zu Integralflächen gleicher Helligkeit („Isolychnen") zusammenfügen werden, da ein ***Pfaff***scher Ausdruck in drei Differentialen im allgemeinen keinen Multiplikator hat. Die Erfahrung spricht aber dafür, daß dies hier doch der Fall ist. Nimmt man das an und verlangt zur Normierung der Helligkeitsfunktion h, daß sie homogen vom 1. Grade in den x_i sein soll, so erhält man für sie die Differentialgleichungen

$$\frac{\partial \log h}{\partial x_l} = \frac{\sum_i a_{il}\, x_i}{\sum_i \sum_k a_{ik}\, x_i x_k} \qquad l = 1, 2, 3. \tag{4}$$

Zur unmittelbaren und vollständigen empirischen Festlegung der a_{ik} reicht weder das vorhandene noch das irgend zu erhoffende Versuchsmaterial über Unterschiedsschwellen aus. Man ist darauf angewiesen, verschiedene Ansätze an der Erfahrung zu prüfen. Der Ansatz, auf den ***Helmholtz*** durch gewisse, unrichtig gedeutete Farbscheibenversuche (s. unten) geführt wurde, nämlich

$$ds^2 = \sum_{\mathrm{i}} \frac{dx_i^2}{x_i^2} \tag{5}$$

versagt in aller und jeder Richtung. Nicht nur gibt er die Unterschiedsempfindlichkeit im Spektrum bloß dann einigermaßen richtig wieder, wenn man ihn auf neue, von den dichromatischen gänzlich abweichende, ad hoc gewählte Grundfarben bezieht, sondern er liefert auch nach (4) die ganz unmögliche Funktion

$$h = \sqrt[3]{x_1\, x_2\, x_3}$$

für die Helligkeit, wonach das Sonnenspektrum **zwei** ausgeprägte Helligkeitsmaxima aufweisen müßte, und zwar gleichviel ob man die dichromatischen oder die ad hoc berechneten Grundfarben benutzt.

Die Erfahrung (***Abney***, F. ***Exner***, K.W.F. ***Kohlrausch***) spricht dafür, daß die Helligkeit sich in Mischungen mit großer Annäherung **additiv** verhält:

$$h\,(x_1, x_2, x_3) \;+\; h\,(y_1, y_2, y_3) \;=\; h\,(x_1 + y_1,\; x_2 + y_2,\; x_3 + y_3) \tag{6}$$

Aus dieser Funktionalgleichung folgt leicht

$$h \;=\; \alpha\, x_1 \;+\; \beta\, x_2 \;+\; \gamma\, x_3\;, \tag{7}$$

wo α, β, γ Konstante; zieht man sie, durch Änderung der Einheiten, in die Koordinatenbestimmung, so hat man noch einfacher

$$h \;=\; x_1 + x_2 + x_3. \tag{8}$$

Das einfachste Linienelement, das **diese** Helligkeitsfunktion liefert und gleichzeitig dem innerhalb weiter Grenzen gültigen ***Weber-Fechner***schen Gesetz für reine Intensitätsänderungen Rechnung trägt, ist

$$ds^2 = \sum_{i=1}^{3} \frac{dx_i^2}{x_i(x_1 + x_2 + x_3)} \tag{9}$$

Der Rechenversuch ergibt, daß **dieses** Linienelement **ohne jede ad hoc-Annahme auch dem empirischen Verlauf der Farbunterschiedsschwelle für Spektralfarben mindestens in groben Zügen gerecht wird** und auch die quantitative Beziehung **dieser** Schwelle zur Fechnerstufe mit dem Grade der Annäherung wieder gibt, der bei der notgedrungenen Verwendung sehr heterogenen Versuchsmaterials erwartet werden darf (die zu fordende Beziehung ist: Konstanz von ***ds*** für jedes eben unterscheidbare Farbenpaar).

Auch die oben erwähnten ***Helmholtz***schen Farbscheibenversuche finden ihre Aufklärung. Diese hatten ergeben, daß in einem Gemisch zweier stark verschiedener Pigmente, etwa Rot und Blau, ein weiterer kleiner Rotzusatz, um damit dieselbe Helligkeitsvermehrung zu erzielen, wie mit einem **bestimmten** kleinen Blauzusatz, um so **größer** bemessen werden muß, je **mehr** Rot das Gemisch von Haus aus schon enthält; und natürlich vice versa: ein kleiner Blauzusatz wird durch schon vorhandenes Blau in seiner aufhellenden Wirkung stärker „geschlagen" als durch schon vorhandenes Rot. Das **scheint** der Additivität der Helligkeit zu widersprechen (wie ***Helmholtz*** ausdrücklich betont), widerspricht ihr aber in Wahrheit nicht.

In Wahrheit wurde nämlich, bewußt, nicht auf gleiche Helligkeit, sondern auf größte Ähnlichkeit, d.h. auf möglichstes Vorschwimmen der Grenzlinie zwischen den beiden Nuancen eingestellt. ***Helmholtz*** hat übersehen, daß sich hierfür als Minimalbedingung durchaus nicht die frühere Gleichung (3), sondern eine abweichende ergibt, wenn die veränderliche Farbe nicht, wie dort, auf ihrem Radiusvektor, sondern in anderer Richtung, nämlich durch kleine Zusätze eines der beiden reinen Pigmente variiert wird. Die genaue Durchrechnung mit **unserem** Linienelement (9), das die Additivität **strenge** erfüllt, ergibt in der Tat eine Variation des **scheinbaren** Helligkeitswertes der beiden Farben genau in dem oben angegebenen, vom ***Helmholtz*** beobachteten Sinn!

Wir haben oben unseren Helligkeitsbegriff grundsätzlich gestützt auf den Vergleich **wenig verschiedener** Farbenpaare [Gl. (3)], und die allgemeine Helligkeitsfunktion daraus durch Integration abgeleitet. Dieser Weg ist nicht eine rein theoretische Konstruktion, sondern wird auch vom Experiment mit Vorliebe betreten (Einschaltung von Zwischenfarben). Das Experiment lehrt weiter, daß der direkte Helligkeitsvergleich **stark verschiedener** Farbtöne zwar schwieriger ist, aber, soweit ausführbar, zu nahe demselben Ergebnis führt, wie die Einschaltung von Zwischenstufen. Es fragt sich, ob die Theorie dasselbe ergibt, d.h. ob das geodätische Lot von einem Farbpunkt F auf einem beliebigen Radiusvektor in der durch F gelegten Isolychne verläuft?

Die geodätischen Linien unserer Metrik ergeben sich in einem Bildraum, in welchem man die $\sqrt{x_i}$ als rechtwinkelige Koordinaten aufgetragen denkt, als die ebenen logarithmischen Spiralen um den Ursprung. Im Vektorraum der x_i [3] winden sie sich auf den elliptischen Kegeln, die ihre Spitze im Ursprung haben und alle drei

Koordinatenebenen berühren, ähnlich den Zügen eines konischen Bohrers aber mit unbegrenzt abnehmender Ganghöhe asymptotisch in den Ursprung hinein. Grenzfälle bilden a) die Radienvektoren selbst, b) die Schnittellipsen der genannten Kegel mit den Isolychnen. Die geodätische Verbindung zweier gleichheller Punkte verläuft also [nach b)] **auf der Isolychne**; sie steht außerdem auf allen ihren Radienvektoren senkrecht; **das** ist nämlich, im Riemanngeometrischen Sinne, der Inhalt der allgemeinen Gleichung (3), die für jedes Linienelement der Isolychne gilt (die Isolychnen sind Orthogonalflächen der Radienvektoren).

Die oben gestellte Frage ist also zu bejahen: unsere Theorie fordert exakte Übereinstimmung der stark heterochromen Photometrie mit der Photometrie durch sanft abgestufte Zwischenfarben.

Für die **Länge der Geodätischen** zwischen zwei Punkten mit den Farbkoordinaten y_i bzw. z_i ergibt sich der geschlossene Ausdruck

$$\int_{(y)}^{(z)} ds = \sqrt{\left(\log \frac{z_1+z_2+z_3}{y_1+y_2+y_3}\right)^2 + 4\left(\arccos \frac{\sqrt{y_1\ z_1}+\sqrt{y_2\ z_2}+\sqrt{y_3\ z_3}}{\sqrt{(y_1+y_2+y_3)\ (z_1+z_2+z_3)}}\right)^2} \qquad (10)$$

Er bildet ein Maß für die Verschiedenheit der beiden Farben, d.h. für die minimale Zahl ebenmerklicher Zwischenstufen, durch die sich der Übergang von der einen zur anderen bahnen läßt. Das erste Glied unter der Wurzel verschwindet für **gleichhelle**, das zweite für Farben **gleicher Reizart** (d.i. Farben auf demselben Radiusvektor).

Da die Geodätischen, wie erwähnt, auf **Kegeln**, nicht auf **Ebenen durch den Ursprung** verlaufen, so fällt, von speziellen Fällen abgesehen, die „kürzeste Farbreihe" nicht mit der Reihe der Mischfarben zusammen. In der bekannten **Dreiecksdarstellung**, von der wir hier sonst nicht Gebrauch gemacht haben, die sich aber einfach als beliebiger **ebener Schnitt** unseres Vektorenbündels ergibt, stellt sich heraus: der rascheste Übergang vollzieht sich nicht auf der Geraden (auf der die Mischfarben liegen), sondern auf dem Bogen einer Ellipse, welche durch die beiden Punkte geht und alle drei Seiten des Farbendreiecks berührt. Es gibt im allgemeinen **zwei** solche Ellipsen, daher **vier** verbindende Ellipsenbogen: von ihnen ist derjenige auszuwählen, der keinen Berührungspunkt enthält. Sind die Farben **reizartgleich**, so vollzieht sich der rascheste Übergang bei konstanter Reizart (reine Intensitätsänderung); sind sie **gleichhell**, bei konstanter Helligkeit. Sind sie in beiden verschieden, so läßt sich das Gesetz der Geodätischen auch so aussprechen: Die ganze notwendige **Helligkeitsänderung** ist **gleichmäßig** auf die **Reizartänderungen** aufzuteilen, derart, daß das Verhältnis beider auf jedem kleinen Schritt entlang der kürzesten Farbreihe konstant ist [man vergleiche die Bemerkung hinsichtlich der beiden Terme unter der Quadratwurzel in dem Ausdruck (10)].

An die Abweichung der kürzesten Farbreihen von der Reihe der Mischfarben lassen sich noch Spekulationen knüpfen betreffend die **Änderung des Farbtons beim Zumischen von Weiß**, die wir hier um der Raumersparnis willen unterdrücken. Die Farbtonänderung bei bloßer Intensitätsvariation, die gleichfalls mit Sicherheit beobachtet ist (vgl. das „***Bezold – Brücke***sche Phänomen“), wird von der Theorie auf ihrem gegenwärtigen Standpunkt **noch nicht erfaßt**.

Daß dieser Standpunkt nur ein vorläufiger ist, folgt schon daraus, daß unser Linienelement (9) die exakte Gültigkeit des ***Weber – Fechner***schen Gesetzes (für bloße Intensitätsänderung) für den ganzen Farbenraum fordert, was nicht nur mit der tatsächlichen, sondern sogar mit jeder möglichen Erfahrung im Widerspruch steht (unendliches Anwachsen der Empfindlichkeit bei der Intensität **Null**!) Es wird sich darum jedenfalls eine Korrektur gefallen lassen müssen, durch welche wahrscheinlich **alle** hier aufgestellten „Gesetze“ auf den Rang von Näherungen herabgedrückt werden. **Nur als solche wollen sie darum von vornherein aufgefaßt sein**. Der Wert der Theorie scheint uns darin zu liegen, daß sie den inneren Zusammenhang all dieser Näherungsgesetze in Evidenz setzt, zugleich aber auch den inneren Zusammenhang aller Abweichungen der **realen** von unserer idealisierten „rein ***Fechner***schen“ Farbenmannigfaltigkeit; Abweichungen, die in den meisten Fällen erst der genauen experimentellen Durchforschung harren. – Wir glauben nicht, das „Rätsel der Farbe endlich und endgültig gelöst“, wohl aber einen fruchtbaren Weg wieder freigelegt zu haben, den des Altmeisters glückliche Intuition erspäht, sein Fuß unter leichtem Straucheln betreten hatte, weshalb ihn die weitere Forschung seither für ungangbar gehalten und gemieden hat.

1. Vorgetragen in den Sitzungen des Gauvereins Wien am 26. Februar, 4. und 11. März 1920; die ausführliche Veröffentlichung erfolgt demnächst in den Annalen der Physik.
2. H. v. Helmholtz, Handbuch der physiologischen Optik, 2. Aufl., S. 434 – 472 (in der 3. Auflage unterdrückt).
3. Unbeschadet der Allgemeinheit darf man sich jetzt auch die x_i als **rechtwinkelige** Koordinaten denken, da die Metrik ohnedies separat durch (9) festgelegt ist.

Chapter 6
A Theory of Pigments of Maximum Luminous Efficiency

Abstract Non-fluorescent pigments capable of reflecting saturated light close to narrow-band spectral lights may only be manufactured with vanishingly low reflectance. Schrödinger asks what maximum intensity of light across the visible spectrum can be reflected from a pigment color, and what the reflectance function of a pigment must be to attain such a maximum. He describes a two-dimensional manifold of optimal pigments. The manifold represents those pigments under arbitrary illumination conditions, provided the illuminant is present in an unbroken way across the visible spectrum. Under this scheme all possible optimal pigment colors are represented, each by a single exemplar.

Keywords Color vision • Luminous efficiency • Pigment color • Spectral curve • Illumination conditions • Pigment manifold • Reflectance function • Bivalent pigment • Pigment appearance • Absorptive pigment • Spectral reflectance • Optimal pigment • Reflectance coefficient • Monochromatic region • Barycentric • Projective coordinates • Schrödinger

Section 1: Problem Statement

It is known that the color of light which is reflected from a patch of pigment never attains that degree of **saturation** which is characteristic of pure spectral lights. The reflected light always appears more or less whitish beside narrow-band light of the same hue. The reflected light may be reproduced as a mixture of the narrow-band light plus a certain amount of white light. It is not a technical barrier which prevents us making a pigment color of the saturation possessed by spectral light. That impossibility is one of principle, to some extent. Here is one constraint: the mixture of two spectral lights which are not too far apart from one another will match a light which lies between them in hue, but the result will be whiter. To attain the full saturation of a spectral light, the pigment would actually have to reflect back only an infinitesimal band of wavelengths, while absorbing all others completely. But

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Schrödinger, E. (1920d). Theorie der Pigmente von größter Leuchtkraft. *Annalen der Physik*, vierte Folge, **62**(15), 603–622. Copyright © 2006, as renewed. Translated with permission from Wiley-VCH Verlag GmbH & Co. KGaA.

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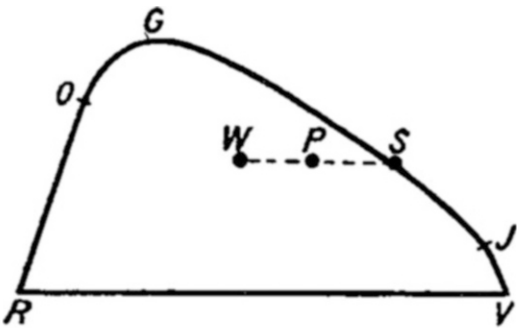

Fig. 6.1 The ***Newton–König*** diagram, with spectral curve ***RGV***. *VJ* and *RO* approximate straight-line segments (Reproduced from Schrödinger 1920d)

then it would surely appear extraordinarily dark – as ***Helmholtz*** remarked – and in the limit it would appear black.

In general, pigments capable of narrow-band spectral saturation may only be manufactured with vanishingly low reflectance (we will address the necessary constraints shortly).

The reason for the whitish cast of pigment colors becomes more intuitive by inspection of the ***Newton–König*** color diagram. The reason is the convexity of the spectral curve *R G V* (cf. Fig. 6.1). The pigment color is represented as the centre of a given linear distribution of mass along that spectral curve. The distribution of mass is determined by the reflectance function (coefficient of reflectance as a function of wavelength) and by the incident illumination. Generally the centre of mass *P* falls in the interior of real color segments (bounded by the spectral curve and the 'line of purples' *RV*). This means the pigment color may be produced as a certain mixture of a spectral light *S* and a white light *W*, or else as some mixture of purple with *W*.

This situation admits an exception only if the reflectance is confined wholly to the short-wave, or else wholly to the long-wave end of the spectrum: specifically if the distribution of mass covers from *V* to *J* only (λ about 475 μμ) or from *R* to *O* only (λ about 630 μμ). According to ***König***, these ends of the spectral curve – whose boundaries are naturally less sharp than just cited, by the way – fall strictly along straight lines. The centres of mass for such pigments would then fall **onto** the spectral curve; they would not be inferior in saturation to corresponding spectral lights.

Orangy-red and indigo-violet pigments may be manufactured to reflect light of appreciable intensity (if not very elevated intensity) and with full saturation in spectral terms.

It should be recalled that the points *R* and *V* each represent a **bounded** region of wavelength, namely the two **monochromatic** ends of the spectrum (up to $\lambda = 655$ and $\lambda = 430$). Then as the reflectance pertains to these ends of the spectrum, each will integrate a **finite** point-mass (and not only the width of a spectral line).

One may pose the question: **how large** a maximum strength of light can be realized with a pigment, for any point of the red-orange region, or of the indigo-violet region, or just as well for a point along the line of purples? One may also pose the same question for points which **approach** the borders of these regions – if one dispenses with the requirement that they reach saturations attainable with pure lights of high intensity. And so finally one is led to ask of an arbitrary point on the real color surface:

A central question: What is the maximum intensity of light in which this point may be manifest as a pigment color? And how must those pigments be constituted – in their reflectance functions – to provide that maximum intensity? The answer to this question is the object of the small article which follows.

Consider any three reference points in the color plane which do not lie in a straight line. These may be points that represent ***König***'s fundamental sensations, points that represent three real-valued calibration lights, or any other three points inside or outside the region of real-valued colors. Suppose then one considers values of units for these points somehow to have been established. Then as is well known, we are used to define any color by a set of three numbers – that is, by the number of units of the three fundamental colors which have been selected, and from which any color can be mixed – in an unusual if not in the ordinary way.[(a)] Considered as proportional values, the three trichromatic coefficients form a triangle of origin, as the projective coordinates of a triangular barycentric coordinate system.[(b)] The colors of representative points may be gauged by the coordinate triangle, namely by this triangle of fundamental colors. The sum of these trichromatic coefficients is the mass of the points represented, and their sum is designated a standard quantity of color. Among representative colors of distinct **position** (colors of different "**stimulus qualities**" as von ***Kries*** says so aptly), the sum is by no means a measure of their proportional **brightness**. That is, except perhaps by careful selection of the units of the fundamental colors chosen. That much is still an unanswered question, one which need not be re-opened here.

For colors of the **same** stimulus quality (those which fall on the same point, and which can be made equal merely by changing their objective intensity) the quantitative value of their objective intensity will be proportional, and accordingly it is surely a monotonic measure of brightness – ceteris paribus. Then let us expand our two-dimensional spatial representation of the color domain to a three-dimensional one. We do so by establishing a unit based on the ordinate for each real-valued color point, at a normal to the color diagram. In that way a **surface** is formed of the pigment colors of largest value, or of the maximal intensity of light, or else – ceteris paribus – of greatest brightness, as discussed. This surface follows trajectories along real-valued color segments over the middle of the diagram, and it has ordinate values of zero at the boundary region of the **curved** part of the diagram. Along the three **straight-line** segments, it declines to small ordinate values which connect to form a continuous curve. This surface should be considered together with the usual real-valued color surface (represented in Fig. 6.1) and also with three vertical 'walls' formed by the ordinates at the border of the diagram. These surfaces demarcate the region of colors which can be represented in pigments within our three-dimensional model. Incidentally we have selected this three-dimensional model as an aid to visualization just for the moment. We will not emphasize the model in what follows. (It becomes impractical, because an absence of objective intensity should not be represented as a **single** point, but should be represented as holding across the entire base plane.)

All that which has been said will hold for arbitrary **illumination** of the pigments. One qualification is that their exact **physical** composition and intensity must be determined in advance, and maintained despite other considerations. Of course the locus of the color surface changes with different illuminants, which surface represents an upper bound for our representation of the domain of pigments. All its ordinates increase in proportion to the **intensity** of an illuminant. Yet the **form** of the surface also varies with the **composition** of the illuminant. For example, under bluish illumination blue hues may be produced more easily with appreciable saturation and intensity, relative to white or reddish illumination – the latter facilitates the production of saturated, intense red pigments, and so on. We aim to obtain a certain result for pigments of maximum luminous efficiency – for a given stimulus quality. I will call these **optimal** pigments for short. In a sense this result will prove to be entirely independent of illumination. It will prove to be the case that the **same** pigments are always optimal under arbitrary illumination. At least, a two-dimensional manifold of pigments – i.e. of reflectance functions – can be given, which are illumination-independent. They are even defined independently of the precise form of the spectral curve given in Fig. 6.1. Under any arbitrary illumination, the manifold represents the pigments which are optimal under that illuminant.

Section 2: The Search for a Pigment Manifold Which Captures the Outer Surface

In order to make this representation definite, let us adopt the following specific coordinate convention for pigments. We choose ***König***'s fundamental sensation points as reference points. The illuminant may be freely determined, given that it should be known in advance to include all visible wavelengths. In this context we may still think of sunlight. We translate the color of ideal white and all neutral gray pigments under this illuminant to the centre of mass of the reference triangle. That color is the color of all pigments which share a **constant** reflectance function; clearly an ideal white pigment of reflectance 1 will be assigned the coordinates (1, 1, 1). If then

$$x_1(\lambda) \quad x_2(\lambda) \quad x_3(\lambda)$$

are ***König***'s fundamental stimulus curves for the interference spectrum of the illuminant under such a measure as sets:

$$\int x_1(\lambda)\, d\lambda \;=\; \int x_2(\lambda)\, d\lambda \;=\; \int x_3(\lambda)\, d\lambda \;=\; 1\ , \tag{6.1}$$

then the coordinates of a pigment with the reflectance function $r(\lambda)$ are:

$$p_1 = \int x_1(\lambda)\, r(\lambda)\, d\lambda, \quad p_2 = \int x_2(\lambda)\, r(\lambda)\, d\lambda, \quad p_3 = \int x_3(\lambda)\, r(\lambda)\, d\lambda, \tag{6.2}$$

and its magnitude becomes:

$$q \;=\; p_1 + p_2 + p_3 \;=\; \int (x_1 + x_2 + x_3)\, r\, d\lambda \ . \tag{6.3}$$

Naturally for r we admit only values between 0 and 1, borders included. The values of p have the same range, while q falls between 0 and 3. This coordinate representation is independent of the **intensity** of the illuminant. As a consequence of our convention about the coordinates of white pigment, the 'unit quanta of the fundamental sensations' will vary automatically in step.

We constrain the possible forms of the reflectance function for optimal pigments, first by the following proposition – which is fundamental to our present little investigation:

<u>Suppose that a pigment lies in the neighbourhood of three positions along the spectrum which do not lie along a straight line of the color triangle. Suppose too, that one of the three has a reflectance different from 0 or 1: that it has a reflectance between 0 and 1. Then the reflectances at these three positions can be transformed so that a strongly reflective pigment of the same stimulus quality can be found</u>.

From any three narrow-band lights, there may be combined a **positive** unit of the color possessed by the pigment. That may still be an operation in some unusual sense, meaning perhaps with one or two negative coefficients in the mixture. Now if reflectances at the three positions lie **between** 0 and 1, then I can transform them by small amounts (increasing them or decreasing them) so that the assigned units of color stand in the proper proportion, and so that a strongly reflective pigment of the same stimulus quality is produced. In symbolic form: let

$$\lambda = a, \quad \lambda = b, \quad \lambda = c$$

be the three positions in the spectrum. Then the assumption of noncollinearity implies:

$$\begin{vmatrix} x_1(a) & x_2(a) & x_3(a) \\ x_1(b) & x_2(b) & x_3(b) \\ x_1(c) & x_2(c) & x_3(c) \end{vmatrix} \neq 0 \ . \tag{6.4}$$

The equations

$$\begin{cases} x_1(a)\,\delta_a + x_1(b)\,\delta_b + x_1(c)\,\delta_c = p_1\,\delta \\ x_2(a)\,\delta_a + x_2(b)\,\delta_b + x_2(c)\,\delta_c = p_2\,\delta \\ x_3(a)\,\delta_a + x_3(b)\,\delta_b + x_3(c)\,\delta_c = p_3\,\delta \end{cases} \tag{6.5}$$

then have solutions in δ_a, δ_b, δ_c, for given small $\delta > 0$. If one then transforms, along short segments ($\varepsilon > 0$)

$$a \leq \lambda \leq a+\varepsilon, \quad b \leq \lambda \leq b+\varepsilon, \quad c \leq \lambda \leq c+\varepsilon, \tag{6.6}$$

and transforms the reflectance $r(\lambda)$ in relative fashion as:

$$r(a)+\delta_a, \quad r(b)+\delta_b, \quad r(c)+\delta_c \tag{6.7}$$

then the pigment coordinates of (Eq. 6.2) are altered by the left-hand side of (Eq. 6.5) as multiplied by ε – which indicates stimulus quality. This changes the unit or quantum of (Eq. 6.3) to a positive magnitude (but **not because** of Eq. 6.5), which is:

$$\varepsilon\,\delta\,(p_1+p_2+p_3) = \varepsilon\,\delta\,q,$$

which was the proposition to be proven.

It follows that the reflectance of optimal pigments cannot be different from zero or one along a **finite** portion of the bent part of the spectral curve. In the same way the reflectance of optimal pigments cannot be different from zero or one along finite portions of **both** straight-line end segments. It is clear that for the bent portion and one of the straight portions that the reflectance can take only one of these limiting values.

To be concise, I will call a pigment '**bivalent**' for a region of wavelengths along which the pigment's reflectance is always zero or one. I will call a pigment 'bivalent' altogether whose reflectance always takes on one of these two values across the spectrum.

The proof given above fails for one of the two straight-line (or 'dichromatic') spectral regions, because of the vanishing of the determinant (Eq. 6.4). Moreover we may consider that the endpoints R and V of the spectral curve each represent a finite ('monochromatic') region of wavelengths. Yet our proof does not exclude **the** case in which deviations from bivalence occur in **both** monochromatic regions simultaneously – but then nowhere else. Then such deviations still appear possible, either

1) in **one** dichromatic region including the bordering monochromatic region, or
2) in both monochromatic regions simultaneously.

Nevertheless we ought to **confine** our observations **to bivalent pigments**, if we **only** put value on knowing each optimal pigment color by at least **one** of its representatives: that is, if we choose not to count **physiological duplicates**. It is easy to see how the bivalence of cases 1) and 2) can be made to coincide **without change in the appearance of the pigments** by successive transformations of the reflectance function.

So, if the three points a, b, c lie along a straight line of the color diagram, then the determinant (Eq. 6.4) disappears and the equations in Eq. (6.5) have nonvanishing solutions in δ_a, δ_b, and δ_c for $\delta = 0$. The relevant transformation (Eq. 6.7) does **not** change the appearance of pigments. It can be applied repeatedly in the same sense (or else it may be given a **specific** appreciable size at the outset), until one of the three numbers $r\ (a)$, $r\ (b)$, or $r\ (c)$ is brought to zero or one. This procedure can be applied repeatedly, so long as $r\ (\lambda)$ still contravenes the bivalence property at three locations. That is to say, the pigment may be replaced by a bivalent pigment, without change in its appearance – which was the point to be proven.

So now we restrict our considerations to bivalent pigments; we would like to find the optimal ones among them. The reflectance of a bivalent pigment is a non-monotonic function of λ. At one place or at several places the function switches from zero to one or from one to zero; I call such a position a **switch point** $(1 \to 0)$ or $(0 \to 1)$. The arrow is to be taken to indicate increasing wavelength. One realizes quickly that **the total number of switch points** for optimal pigments **cannot be greater than two**. Indeed this shows how duplicates are excluded once more. Now the following proposition holds, which is analogous to the first proposition on many points:

Suppose a pigment has three switch points that do not lie on a straight line in the color diagram. Then the pigment's reflectance can be transformed by translation of its switch points, so that a luminously efficient pigment of the same stimulus quality is produced. This implies the original pigment cannot be optimal.

The proof of this proposition runs just the same way as before; therefore it may not need to be conducted *in extenso*. Its basis is that a pigment color can be produced as a mixture of the three switchpoint colors, at least in the unusual sense if not otherwise. Then a translation of the switchpoints that is suitably chosen for its direction and magnitude will lead to the desired reinforcement of luminous efficiency without change in stimulus quality.

Even three or more **collinear** switchpoints appear possible. Yet so long as three of them are present they may be translated without change in appearance of the pigment, so that they coincide. After that, at most two will still remain. This may become perspicuous in the following manner – e.g. for the long-wavelength segment RO – more clearly than by a logical argument analogous to that pursued earlier.

Let a pigment be given which is bivalent over all the spectrum except over RO, but whose reflectance may be arbitrarily formed along the segment RO of the spectrum – including its limits. (Compare Fig. 6.2, which is somewhat peculiar but quite readily understandable. There the reflectance coefficient is erected as a perpendicular outside the spectral curve.) Our task is to replace the given pigment by one which is bivalent over RO, one which has as few switchpoints over RO as possible, and which is physiologically equivalent to the original.

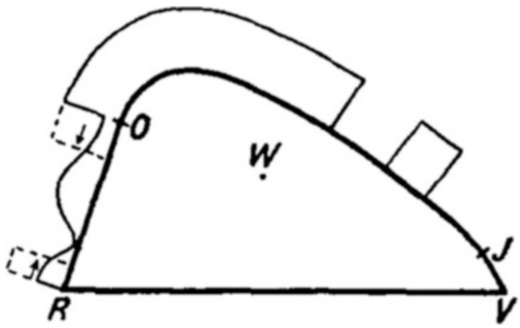

Fig. 6.2 The relative spectral reflectance of a pigment which is bivalent across the spectrum – except over segment ***RO***, where it is arbitrarily formed. (Reproduced from Schrödinger 1920d)

For the latter condition to hold, it must be the case – according to ***Grassmann***'s proposition that mixtures of equal-appearing lights will produce equal-appearing lights – that "component colors" which are produced just by the reflectance of *RO* will be the same for the transformed color as for the originally given color. The point which represents that component color lies somewhere along *RO*, at least. In order to end up with as few switchpoints as possible, we 'generate' the component color in different ways. Our choice of method depends whether the given pigment reflects entirely or absorbs entirely the section that lies adjacent to *O* in the direction of violet.

In **both** cases we begin with complete absorption along *RO*. Then in the **first** case (as indicated by the sign) we displace a switchpoint ($0 \rightarrow 1$) from the red end of the spectrum towards violet – and simultaneously displace the switchpoint ($1 \rightarrow 0$) from *O* towards red. We do this with a relative pace such that the reflectance which is produced in this way across region *RO* will always match the **stimulus quality** of the component color. Clearly this can be achieved, and it may be advanced so far as to produce a **quantum** of the component color. The progress of this process may only be thwarted if one of the switchpoints crosses over the locus of the component color. Yet that may not occur before the quantum of the component color has been reached. Otherwise only spectral lights would remain "unused" between the switchpoints; those spectral lights would either be redder without exception or yellower without exception than the component color itself. From those no quantum of the component color could ever be mixed. The latter could not be mixed from the light that would then be available – which simply contradicts the assumption.

In the second case under which movement from *O* towards violet connects to a **region of absorption**, we produce component colors of increasing strength by a single region of reflectances which contains the position of the component color in its interior. Once again proceeding from the *r* in *RO* which vanish identically, we displace a switchpoint ($1 \rightarrow 0$) out from the component color in the direction of red. At the same time we displace a switchpoint ($0 \rightarrow 1$) out from the component color in the direction of violet. We do this with a relative pace such that the stimulus quality of the component color is maintained. In this way its quantum must be reached, on similar grounds as have been argued above. This happens before – or at worst at just the moment when – the progress of this process is thwarted because the first switchpoint reaches the red end of the spectrum, or else the second switchpoint reaches point *O*.

This completes our task.

If one is led to a **single** switchpoint by the procedures that have been sketched – which can occur in exceptional cases – then a second is possible in the spectrum somehow. If there are two, however, they must be the only ones – if the pigment is indeed optimal.

Another thing may occur: the two switchpoints to which one is led, may both lie along the monochromatic ends of the spectrum. The two then coincide in a single point *R* of the color diagram, and a third switchpoint might be possible somewhere along the spectrum. But then one can recognize something right away about the isolated monochromatic region of reflectance or region of absorption in question. That region can be displaced to the far end of the spectrum – so that once again there is only **one** monochrome switchpoint.

If we survey all that has been said, **we obtain by way of conclusion** – after excluding many duplicate cases – a **manifold of optimal pigments**. That proves to be a **two-dimensional manifold of bivalent pigments which have only one or two switchpoints**. Every optimal pigment color must have at least one representative among these pigments.

We have now only to exclude a few small groups of candidates which are certainly not optimal. Of the remainder it will be shown that no other two pigments of the same stimulus quality arise among them. Thus it will be shown that these pigments are really all optimal. They simply represent the outer manifold that was alluded to at the beginning; they represent it without duplicates, by **single** exemplars.

Next we will categorize the bivalent pigments of one switchpoint and two switchpoints in groups, as follows. These categories are clearly evident on inspection of the adjacent Figure (Fig. 6.3). The baseline represents the visible spectrum, while reflectance is plotted on the ordinate.

We exclude the following as being clearly non-optimal:

a) from the long-wave pigments, those pigments whose reflectance does not reach to the short-wave border of monochromatic red;
b) from the middle pigments, those whose switchpoints both fall either **between** *O* and the red end of the spectrum, or **both between** *J* and the short-wave end. Clearly for those an increase in the intensity of light is possible, by extension of the domain of reflectance;
c) from the short-wave pigments, those whose reflectance does not reach the long-wave border of monochromatic violet;
d) from the middle-absorptive pigments, those which have **one** switchpoint **inside** monochromatic red, and the **other**, **inside** of monochromatic violet – since the purple in question may be intensified by the extension of both domains of reflectance until one of the two switchpoints reaches a border of the monochromatic domain. (Those with **two identically-colored** monochromatic switchpoints were excluded before – as duplicates – and were replaced by end-spectrum pigments.)

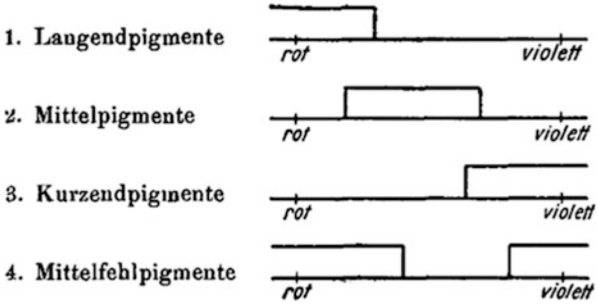

Fig. 6.3 The relative spectral reflectance of bivalent pigments of one switchpoint and of two switchpoints, as four groups. (*Langendpigmente*: long-wave pigments; *Mittelpigmente*: middle-wave pigments; *Kurzendpigmente*: short-wave pigments; *Mittelfehlpigmente*: pigments which absorb middle frequencies; *rot*: red; *violett*: violet) (Reproduced from Schrödinger 1920d)

At this point a demonstration must still be conducted to show that – after the exclusions a) to d) – all the remaining pigments are distinct from one another in stimulus quality.

In this deliberation we anticipate those three subgroups, whose points are represented to lie along one of the three straight-line segments which border the diagram. They enter into competition only among themselves. The accuracy of this claim is clear for them without further explanation, as the exclusion conditions a) to d) hold decisively for them.

For the rest, an ungainly proof must be conducted piecemeal. That is, it must begin by considering each of the four groups on their own, then comparing them to one another, then taking into consideration all possible combinations of switchpoint positions. Nonetheless we should cover those cases, neglecting end-spectrum pigments, since they can be considered after all as degenerate cases of middle pigments, or else of middle-absorptive pigments.

Let us now compare:

A. **Middle pigments only**

For widely separate locations of reflectance regions, a superposition of centres of mass is not possible. That much holds just as well for more broadly circumscribed locations. Reflectance regions (after excision of the saturated spectral pigments described above) must also contain bent parts of the spectral curve. Then the centre of mass of the unshared outer parts of the broader reflectance regions will fall outside the segments which belong to the smaller reflectance regions. The former must shift the centre of mass of those segments. The same thing holds if the reflectance regions overlap. The shared middle portion – with which each unshared outer part is combined – cannot lead to the same point twice.

B. **Middle-absorption pigments only**

Here we always begin with a pigment that covers both absorption regions at once. Always consider that the centre of mass of this "difference pigment" may be displaced differently by the relevant complementaries.

C. **Middle pigments with middle-absorptive pigments**

α) The reflectance region of the first and the absorption region of the second pigment are far separate. Then the middle-absorptive pigment arises from the middle pigment by the addition of reflectances which may not possibly leave its centre of mass unchanged.

β) The reflectance region falls within the absorption region. This case requires no explanation.

γ) The absorption region falls within the reflectance region (see Fig. 6.4). Then the entire color region comes apart into three stripes. If the pigments are to be of **the same** stimulus quality, their centre of mass (i.e., that of the sections of spectral curve which border them) would have to lie along a single line **in such a way** that the middle stripe's centre of mass does **not** lie in the centre. Clearly, that is impossible.

δ) The absorption region and the reflectance region overlap (see Fig. 6.5). There would have to be a substantive point in the interior of each of the hatched areas I, II, and III, so that the three points lie along a straight line. Of course point III must **not** lie in the middle between I and II. That is because the

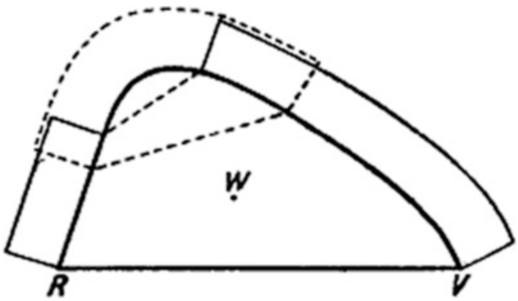

Fig. 6.4 An aid to visualization of the condition in which the absorption region falls within the reflectance region. (Reproduced from Schrödinger 1920d)

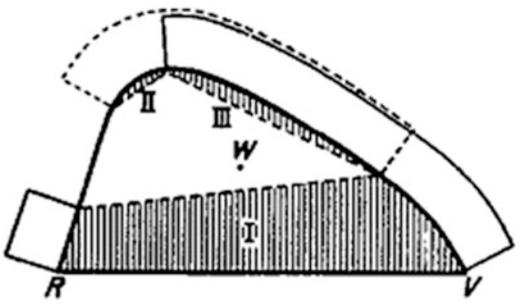

Fig. 6.5 The condition in which the absorption region and the reflectance region overlap. (Reproduced from Schrödinger 1920d)

> addition of the curve II to III would have to make the centre of mass of III coincide in one point, as does the addition of the pair of curves at I. Clearly that is impossible.

We had arrived at a two-dimensional manifold of pigments. That manifold represents all the optimal pigment colors completely, each by a **single** example. And surely – as had been foreshadowed – it represents them **under entirely arbitrary conditions of illumination**, so long as they subsume all wavelengths. Two things have not happened here: 1) the definition of our pigments has not made any reference to illumination conditions, and 2) in this investigation we have not used any other property, than those which tacitly eschew any kind of homogeneous illumination.

One may recognize that our pigments do not lose their optimal character, even if the incident illumination exhibits spectral gaps. One can recognize this directly in the transition between illumination conditions that are little different, in which small ordinate values bridge gaps which one may dampen to zero in a regular manner. Certainly a pigment will generally change its position in the color diagram as a consequence, but not its physical composition. So it remains optimal even in the borderline case. Naturally then an unambiguous mapping from the manifold of pigments to the optimal colors is not maintained. It is not maintained because large sets of pigments become identical in color, namely all those whose switchpoints fall into a spectral gap of the incident illumination. In the same way for this case, the trend of reflectance within such a gap has no influence at all on the appearance of the pigment.

I would direct the reader to the end of the present article for a short summary of the main results which have been obtained so far – a summary free of such terms of art as have been introduced.

Section 3: Concerning the Answers to Questions About Maximum Attainable Intensity of Light, About Highest Attainable Saturation, and About Necessary Conditions for Maximally Luminous Pigments

It is now clear how one should proceed in calculation to answer the **first part of the *central question* posed in the first section**, for any particular case. That first part was: what is the maximum intensity of light which may be manifested by a pigment, for a particular point of the color diagram?

This question makes sense only for a given illumination condition, of course. In preparation one needs to derive a Table of the three integrals of the fundamental stimulus curves for the interference spectrum of the illuminant in question:

$$\int_{\lambda_0}^{\lambda} x_1(\lambda)\, d\lambda, \qquad \int_{\lambda_0}^{\lambda} x_2(\lambda)\, d\lambda, \qquad \int_{\lambda_0}^{\lambda} x_3(\lambda)\, d\lambda,$$

where λ_0 is the short-wave limit of the visible spectrum, while the upper bound of λ_0 extends to the long-wave end of the spectrum. The coordinates of the optimal colors may be read from such a Table with little effort. For more frequent use, one may unite these in a Table with double-entry columns. By a reasonable process of trial-and-error – I do not see a simpler method – or perhaps best by graphical means, one can ascertain switchpoints. By graphical means, one can always trace **one** integral as a function of the others, and construct chords of the correct slope. That is a way one can ascertain **those** switchpoints (limits of integration) for which the three coordinates of the pigment fix the ratio given by the point listed in the Table. In turn these coordinates indicate which fraction of the red-, green-, and blue-sensations are stimulated under the given illuminant for the given stimulus quality, when compared to the red-, green-, and blue-sensations stimulated under the same illuminant but for an ideal white pigment, three coordinates indicate which fractions will be maximally excited by a pigment.

Of course the same deliberation and calculations may be used with other curves, say color-mixture curves drawn from three real-valued calibration lights, in place of the fundamental stimulus curves. In that case optimal coordinates acquire a more concrete meaning, one even superficially free of any hypothesis about the origins of the color sense.

I would almost hold the actual commission of the preliminary calculations – as they have been portrayed – to constitute useless labor, **because** it is so very uncertain if the preliminary results that are delivered will prove at all worthwhile in some practical way. It is quite uncertain that they will prove to be suited to daylight, or to ***König***'s color-mixture curves.

In passing I would like to mention a related question which has just been touched upon, and which may be dearer to a practitioner's heart. Namely: what is the maximum **saturation** of a given **hue** that can be produced by a pigment, supposing that **brightness** will be maintained above a given threshold?

Doubtless the said pigment is to be found in our outer manifold – certainly along a half-ray through the position of white.[(c)] The pigment will be found as far out as possible – **so** far out, that the desired brightness lies at least above threshold. One would need to construct a "*brightness surface*" here, just as before a "*quantitative surface*" was constructed along the line of colors. The ordinates of the brightness surface would represent the **brightness** of the optimal colors. Brightness should be susceptible to a calculation which results in unique values. If the operationalization of this concept is to be determinate, it must be defined from the three given coordinates – which of course are calculated uniquely from pigment color. Yet opinions do vary widely **how** this is to be accomplished. Some hold brightness to be a linear function of coordinates with constant coefficients, namely the "specific brightnesses" of fundamental stimuli. Others – among them ***Helmholtz*** in his treatise on the application of ***Fechner***'s Law to color systems – believe that other associations must be considered, which are more than strictly additive in nature.

The problem of a correct understanding of the concept of heterochromatic brightness is an extraordinarily important one. It has a much more far-reaching importance than does our preceding investigation. Before long I will offer a much better exposition of this subject in another forum. Clearly for us this has only an accessory role. If one really did compound optimal mixtures of light – of the desired hues in increasing saturations according to the procedure outlined – then it might be discoverable from them just what would satisfy the demand. That would indicate just how much these constructs are established on reasonable grounds, and are not just verbal recipes.

Then consider **the second part of the *central question* that has been posed**, which was: how must optimal pigments be constituted – meaning in their reflectance functions? – This will not be answered in a thoroughgoing way by our two-dimensional manifold of pigments. Note that all the excluded duplicates are equally optimal. An optimal pigment does not entirely need to be bivalent; it may possess more than two switchpoints; and so forth...

The uncertainties that arise all have their basis in the existence of dichromatic and monochromatic regions of the spectrum. In those regions the reflectance function is in the widest sense arbitrary, under certain conditions. What is licensed there and what is not, is a matter that may be surveyed easily for any eventuality, given the results of our earlier deliberations. I maintain that an airtight enumeration of all possible cases would be both uninteresting and superfluous. The question here is always whether the reflectance function may or may not take on one of the described forms by a suitable transformation.

By way of example, a pigment which is reflective continuously from the short-wave end to point *O* (at the border of orange) will be an optimal pigment, however its reflectance function may run in the long-wave portion. That is because this pigment may be transformed into a middle-absorption pigment.

By contrast, deviations from bivalence in the dichromatic region are inadmissible for pigments whose reflectance is confined to the section from the long-wave end up to point *O*. Such deviations are admissible in the monochromatic portion only if the dichromatic portion is completely reflective. Otherwise the pigment would be mapped to a prohibited middle pigment by transformation (see above, exclusion b)).

Of course, very similar scenarios hold for analogous cases at the short-wave end of the spectrum.

Section 4: Comparison to Experience

Wilhelm ***Ostwald*** has drawn some consequences from purely empirical work, drawing on his broad experimental investigations of pigments. He finds that to achieve the greatest **purity of color** for pigments, it is advisable that only reflectances of one or zero should occur, with their step transitions being steep. Further, there should be only either **one** compact region of reflectance, or else **one** compact region of absorption.[(d)]

In that much, ***Ostwald***'s empirical findings coincide exactly with the major conditions for optimal pigments which we have established theoretically.

Ostwald puts forward other requirements for the achievement of maximum purity. One is that the absorption or else the reflectance should encompass exactly a 'half-color', meaning a region that extends just from a spectral color to its complementary color. In order to gain at least a qualitative understanding of this requirement, let us recall what ***Ostwald*** would have us understand by **purity**. That is the **fraction of a pure color** that is contained in the global impression which is evoked by a mixture, and which may be abstracted from it conceptually. Certain fractions of **white** and **black** may still enter into this fraction of pure color as an impurity or cloudiness.

According to ***Ostwald***, a color's black content may be permitted to increase e.g. if one mixes it with an ideal black pigment on a color wheel – i.e., with a non-reflective pigment, or better, with a black hole. With recourse to ***Talbot***'s Law it may be concluded – whatever one may judge black to be, in terms of sensation – that at any rate **a relatively negligible intensity of light** is **the objective correlate** of what ***Ostwald*** calls black content.

Now it is clear that those optimal pigments will have very low intensities of light, if their reflectance is restricted to a much too **small** region of the spectrum. According to ***Ostwald*** in other words, they will have an elevated black content, and therefore will exhibit little purity. On the other hand, pigments whose reflectance encompasses too **large** a portion of the spectrum will be intense; they will contain little black, but will contain a lot of white as a consequence. The latter follows from well-known general laws of light mixture. ***Ostwald***'s pigments of maximum purity are to be found only among our optimal pigments. That is a natural consequence of the proposition that a pigment of little light intensity and the same stimulus quality as another will possess the same white content, but a higher black content – meaning lesser purity.

The restriction of a region of reflectance to the region between two complementary colors is evidently a practically demonstrated compromise between the Scylla of whitish impurity and the Charybdis of blackish cloudiness. Or to express the matter as ***Helmholtz*** did, using a form of expression which is geared to the objective composition of a mixture of light rays: a means to attain maximal color saturation without too great a loss of light through absorption.

Because of the nature of this compromise, it might be expected that the best of the pigment colors produced by ***Ostwald*** would have a residual quality of cloudiness, that is, a middling gray which is produced by a mixture of not-so-different proportions of white and black. ***Ostwald***'s account of purity corresponds quite nearly to that for some pigments (loc. cit. p. 560). For many others however, namely for the blue and green pigments, the proportion of black is substantially greater than the proportion of white.

Nonetheless I cannot refrain from remarking that I have used ***Ostwald***'s terminology here, only to be able to compare – to some extent – my theoretical results with the **facts** which he has shown. It is not as if I might have been convinced that the same level of quantitative determinacy is a feature of constructs such as 'purity', 'black content', or 'gray', when compared to the physiological constructs of the ***Helmholtz-König*** color metric. With all the respect due to ***Ostwald***'s valuable and painstakingly forged results, I consider e.g. the absolute values he has determined for 'purity' and 'gray' from reflectance values to hold for at most **two** positions of the spectrum, even if they are special positions (maximum and minimum). All that provides for a good rule-of-thumb, at best; it is in no way suitable to a precise definition of these concepts.

Summary

1. The pigments – of a given stimulus quality – which reflect the greatest intensity of light will have the following constitution:
   a) At no place in the spectrum do they have another reflectance coefficient other than zero or one.
   b) Their reflectance shows at most two points of inflection ('switchpoints' from 0 to 1 or from 1 to 0). Their reflectance is not zero throughout.
   c) If their reflectance is confined to one of the two dichromatic regions of the spectrum – including the adjacent monochromatic region – then they reach at least to one end of that region.
   d) If the reflectance is confined to the monochromatic regions, then the reflectance covers at least one of the two entirely.
   e) If the **absorption** is confined to **one** monochromatic region, then it begins at the end of the spectrum.
2. The pigments as described possess the specified property under arbitrary illumination; i.e. under any arbitrary illumination, none of them will be exceeded in light intensity by any pigment that reflects light of the same stimulus quality with it **under just that illuminant**.
3. If the illuminant shows no spectral gaps, then the pigments are all distinct from one another in physiological terms. They overlap the line of real-valued colors simply and exactly, including its boundaries.
4. Apart from the pigments cited in 1., there are other maximally luminous representatives of their stimulus quality. Incidentally this property holds for **all** those pigments whose **absorption** is confined to **one** of the monochromatic or dichromatic ends of the spectrum. The reflectance functions of those pigments may vary quite arbitrarily within those regions otherwise. Overall, admissible deviations from the properties listed in 1. all pertain to the monochromatic and dichromatic regions. Of course any such pigment which is composed as an exception will match **one** of the pigments of the manifold cited at the beginning.

Which of those pigments the exception will match, depends in general on illumination conditions.

5. These theoretical results match several of Wilhelm ***Ostwald***'s empirical findings.

Vienna, December 1919, II. Institute of Physics at the University

(Submitted December 22, 1919)

Notes

a. In an unaccustomed sense, I deem a color as capable of being mixed, even if it lies outside the triangle of fundamental colors. Then one or two of the trichromatic coefficients will be negative. The practical meaning of this kind of mixture is by now familiar. [ff. 1, p. 605 original]

b. By '**barycentric**' I mean a system of triangular coordinates whose 'origin' lies at the centre of mass of the coordinate triangle. This convention should not be confused with the arbitrary – but convenient and therefore frequently used – translation of the **position of white** to the centre of mass. [ff. 2, p. 605 original]

c. One should note that in the color diagram we have established for our pigments, it is not the white of sunlight which is located at the centre, but rather the color of an ideally reflective surface. In other words the color of incident light lies at that centre. **Here** the 'position of white' does not lie at the said central location; rather it lies at the position assigned to sunlight. [ff. 1, p. 618 original]

d. ***Ostwald***, Wilhelm 50.: Das Fechnersche Gesetz., pp. 417 – 419. Beiträge zur Farbenlehre (erstes bis fünftes Stück). Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, **34**(3), 365 – 571, p. 471 ff. (1917). ; ***Ostwald***, W. Neue Forschungen zur Farbenlehre. Physikalische Zeitschrift, **17**, 322 – 332, p. 328 ff. (1916). [ff. 1, p. 619 original]

Chapter 7
On the Origin of the Eye's Sensitivity Curves

Abstract The spectral brightness sensitivity function of the ordinary human eye may be considered in terms of the availability of natural sources of illumination. The human eye possesses both a cone system of vision and a rod system of vision. Here it is speculated that their peak brightness sensitivities arose in phylogenetic development, with the cone system being relatively recent in appearance. The older rod system became adapted to the role of low-illumination, or 'twilight' vision. The comparative physiology of vision offers support to these conjectures, based on differences in illumination for terrestrial animals and aquatic animals.

Keywords Color vision • Visual sensitivity • Cone response curve • Cone sensitivity • Rod response curve • Rod sensitivity • Phylogeny of vision • Photopic vision • Scotopic vision • Resonance curve • Development of vision • Aquatic vision • Evolution of vision • Schrödinger

It is well-known that our eyes are sensitive to only a relatively small portion of the radiation that is emitted by a glowing object. The **visible** region of the electromagnetic spectrum extends from about $\lambda = 800$ μμ to $\lambda = 400$ μμ. If one asks **why** our sensitivity to light has developed in just this region and not in another of longer or shorter wavelengths, there can be no doubt about the answer. Namely the visible region extends **across either side of the peak intensity of sunlight**. It seems that the eye has developed to make best possible use of that light source which was almost the only one available to us before the advent of civilization.[a] One may consider certain constraints on the possibility of organic assembly in the biological task of capturing this peak energy; one might call them 'accessory conditions' to the task. The addition of some conditions – such as a broader spectral extent for the visible region – would be difficult to balance with the trivial advantage they might bring.

In the adjacent Figure, *E* represents the energy distribution of the sun, following ***Abbot***'s measurements.[b] This is the distribution for such ordinary conditions (at sea level, in Washington, for a solar elevation of 45° from the horizon) as should be considered in the development of the visual system. Under these conditions the

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distribution peak is in the blue-green at about $\lambda = 515$ μμ. One may note the much steeper descent of the curve towards shorter wavelengths than towards longer wavelengths. This has its origin in the overall effect of absorption within the sun itself,[c] and with absorption through the earth's atmosphere. (The strong Frauenhofer line A at $\lambda = 760$ μμ of absorption by oxygen is roughly interpolated from ***Langley***'s historical measurements, since for us this is otherwise of no consequence.) Z and Z' are **cone response curves** for data from two different observers.[d] They represent the distribution of brightness – **derived** from measurements of brightness across the spectrum – for a light-adapted eye, and for an **ideal** spectrum of a **flat** energy distribution whose curve E would be given as a horizontal line, and along which an energy-measuring device (bolometer or thermocouple) would display a **constant** reading. Then the ordinates of those Z-curves are a measure of the brightness which **equal** amounts of energy **of different wavelengths** produce in the eye, or as one may call it: the spectral sensitivity distribution of the eye. The maximum sensitivity is found at **yellowish-green** between $\lambda = 550$ μμ and 560 μμ, that is, noticeably towards **red** when compared to the energy maximum of sunlight on earth. Under our biological hypothesis, this might be sufficiently explained by the strongly asymmetric trend of the curve E. Certainly if a more optimal employment is the goal, the maximum of sensitivity would be shifted a little towards the more gently descending curve E.

What has been said above relates to brightness sensitivity for the ordinary process of vision in a normally light-adapted eye. In physiological terms this means it relates to the activity of **cones** in the retina. As is now known, we possess a second vehicle for vision: the so-called **rods**. The rods, sometimes mixed with cones, form a palisade-like surround to the retina. Rods are especially numerous in the periphery of the retina. There cones occur less and less frequently, while only very closely-packed cones are present in a small neighbourhood that is the locus of highest acuity (about 1½° of visual angle). In contrast to the cones, an almost colorless (perhaps a little bluish) sensation is mediated by the rods, and this quality is independent of wavelength. They are **completely colorblind**. A second predominate quality of the rod process is its extraordinary capacity to adapt to low intensities of light. The more the ambient illumination of our environment decreases, the more that '**rod vision**' mixes with the so-called **photopic** vision of the cones. At lower light levels, finally only rod vision remains (so-called 'twilight vision'[e]). This allows the spectral sensitivity curve of the rod process to be determined independently. It turns out (cf. curve *St* of our Fig. 7.1[f]) that this is strongly displaced towards shorter wavelengths, when compared to the curves for cones. It has its maximum around $\lambda = 517$ μμ,[g] in the **blue-green**. Anyone can see an obvious consequence of this displacement. An example is given at the onset of twilight in a picture gallery. As a consequence of the increasing intrusion of rod vision, the brightness of red hues decreases much more quickly than that of blue hues. By contrast, the latter acquire a particularly strong luminous quality (the ***Purkinje*** phenomenon). By a sudden switch to a strong artificial light source, one can reverse this relation of brightness once more. One gains quite a strong

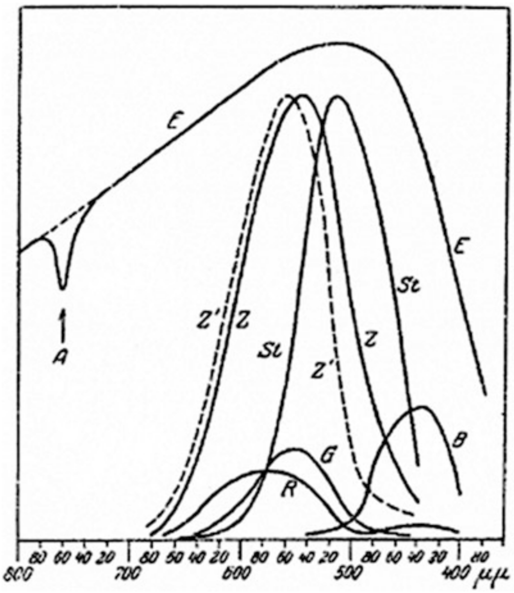

Fig. 7.1 The energy distribution ***E*** of the sun, with cone response curves ***Z*** (H. Bender) and ***Z′*** (F. Exner), Frauenhofer line ***A***, and a spectral sensitivity curve ***St*** for the rod process. ***R***, ***G***, and ***B*** are the *red*, *green*, and *blue* curves. (Reproduced from Schrödinger, 1924)

impression of the state of affairs which can also be expressed exactly by the measurement of the said displacement of the distribution peaks.

What is the origin of this remarkable displacement of the rod curve towards short wavelengths? I do not recall having seen any attempt at an explanation for this. I am fully aware that I have only a small part of the knowledge necessary to evaluate this question. Nonetheless I would like here to put forward several possible explanations, for the sake of discussion. This may at least draw the attention of others who may be better suited to passing judgment on the matter.

In the first place, naturally it would be possible that the inner mechanism of the rod apparatus has led to a somewhat different solution. The organic conditions of the rod makeup deviate so much from those of the cone apparatus. Then for its maximum (biological) function, or the best-possible exploitation of the available light source, 'altered accessory conditions' may have led to a different solution. Of course one would be throwing in the towel by adopting such an explanation outright, meaning one would be renouncing a real explanation. Moreover, according to the view advanced recently by F. ***Exner***[(h)] and F. ***Aigner***[(i)] on the nature of retinal excitation, it is likely that the mechanisms of rod vision and cone vision are qualitatively identical. Their view is that all these cases are a matter of **electromagnetic resonators** in nerve endings. Those resonators would operate within a bounded domain of wavelengths, following a **resonance curve** similar to those familiar from other physical phenomena. The breadth and intensity of the resonance domain are determined by two physical constants of the resonator: its eigen- (or natural) frequency, and its damping coefficient. These resonance coefficients are that which – in biological terms – would have adapted to external conditions. On that point, **three** different kinds of resonators must be taken into consideration for color vision with cones. Their resonance domains are represented in the previous Figure by curves *R* (for red), *G* (for green) and *B* (for blue). The *Z*-curve is seen to be composed of these by the addition of certain equations.[j] For the moment we will not dive into these details any more closely. By contrast, rods should be represented as a **single** kind of resonator, whose domain of resonance is represented directly by the *St*-curve of the figure. Why then – speaking in the vocabulary of the theory at hand – is it that the resonators for rods cover a domain that is shifted so much farther towards blue than the resonators for cones (that is, the resonators for red and for green), which are tuned primarily to the brightness of daylight?

A real explanation would be one which does not appeal to unknown constraints on organic assembly. In my opinion, a real explanation will only be found if the rod system has developed differently than the cone system, under the influence of **another illuminant** with a different energy distribution curve. At this point the following possibilities seem to be open.

1. The special adaptability of the rod apparatus to residual light levels raises the proposition that this concerns the visual organ of a **nocturnal animal**. What then is the energy distribution curve of light at night? For what concerns starlight, we know the spectra of an extraordinarily large number of individual stars. We know from their color index as well, that many are redder and others are bluer than our Sun. From a table[k] of the brightest stars visible in our region, I find among 42 stars that there are 25 bluer and only 16 redder than our Sun. By contrast, light reflected from the moon, which perhaps has greater meaning for the visual capacities of a nocturnal animal, is somewhat redder than sunlight[l] (by some 0.5 of a magnitude in the color index). So this hypothesis seems to offer no satisfactory explanation.
2. Secondly one may consider that the phylogenetic development of the rod system extends so far back in the past, that its peak sensitivity indicates a **higher solar temperature** than the current temperature. Actually the Sun is one of the so-called dwarf stars, according to the findings of recent astrophysical research. It is already in the declining arc of its development, meaning that it is in the process of **cooling**. While its current surface temperature lies between 5900 and 6000° K, its maximum temperature once attained 6600°, according to ***Eddington***'s calculations.[m] Following Wilhelm ***Wien***'s displacement law, this change in temperature towards a **higher** temperature would represent a displacement towards **shorter** wavelengths in the ratio:

$$6000/6600 = 0.91$$

The wavelengths of the two sensitivity peaks stand in the ratio:

$$517/550 = 0.94$$

Then indeed one could say there has been a development in the first stages of the sun's cooling, in which the rod curve was established in something like its present composition in relation to the cone curve. Given the uncertainty of **dating** such intervals of time, I would not dismiss this hypothesis unqualifiedly. Nevertheless it loses its plausibility if we consider that ***Eddington*** estimates the **entire** time for the developmental life of a star (from dark red embers to its maximum temperature and back to dark red) as some tens of billions of years. Even the oldest granites are estimated to be at most 1–1½ billion years old[(n)] by reliable methods using radioactive decay. The development of the visual capacity of our forebears must be of far more recent date.

3. The third, most likely explanation seems to me to lie in the **blue-green color** which **water** exhibits in thick layers. For an **aquatic animal** that lives under the surface at some depth, the composition of sunlight must be transformed in just about the way we need for our explanation. Such an animal would have particular need of robust **adaptation** to different levels of brightness, too, if it explored changes in depth under the water's surface. Under this hypothesis, the rod system would be an **older** system of vision which emerged during the age of aquatic life. These two organs – the rods and cones – serve the same purpose; they bear a certain parallel to the famous case of gills and lungs. Under the hypothesis one would need to assume that the cone system had achieved its full fruition in animals which sought daylight, **while** rods were still urgently needed for use underwater. Moreover one needs to assume that cones assumed the principal function over time. The rods – relegated to the function of an accessory organ – no longer played a sufficiently important biological role to occasion their full adaptation to the altered conditions of illumination after the animals proceeded from aquatic to terrestrial life.

Following this notion most cases of **complete colorblindness**, which amount to a return to pure rod vision, actually would represent an **atavistic** state.

This subject tempts one to further speculation on the gradual development of daylight vision into **color** vision. Of course the grounds for speculation in this matter are appreciably less certain. One may hold it probable that the origin of cone vision lay – like that of rod vision – in an undifferentiated type of vision without color. In fact isolated cases of complete colorblindness are known, which appear **not** to be rod vision at all.[(o)] The hypersensitivity to light is absent which is observed otherwise in totally colorblind observers. The peak of brightness sensitivity lies at just the same place as in the cone curves of young observers, namely at $\lambda = 550\ \mu\mu$. Further, this kind of colorless **cone** vision can also be found in the far periphery of the retinae of **normal** eyes. One can imagine that these border areas – biologically less significant – have not participated in the continuing evolution. The next stage of evolution would have been towards **dichromacy**, that is, yellow-blue vision. On the normal retina there is a region **between** the color-capable centre and the completely color-blind border just mentioned. This inbetween region is also found in insects (in bees: von ***Hess*** and von ***Frisch***, and in hummingbird hawk-moths: ***Knoll***). This forms by far the most common kind of **partial colorblindness**. Yet a complication emerges at this point: there are **two** types of colorblindness. The most common is that for which the 'yellow curve' takes the normal place of the red curve (*R* in the Figure). There is also that for which the yellow curve takes the normal place of the green curve (*G*). The relatively high prevalence of just these kinds of anomalies (in some four percent of men!) seems to me to indicate that the decomposition of a long-wave excitation curve into a red curve and a green curve represents the **most recent** stage in the evolution of our visual system. Consequently it is the least firmly established, and the most susceptible to disturbance or regression. – Let me emphasize once more that the remarks of this last section are simply vague intimations.

As I see the matter, **the** import of the location of the rod peak – that which I held to be most likely – coincides materially with the view held by C. von ***Hess***. This finds support in the rich data of his research. Here are several citations of passages from his 'Color theory' (Farbenlehre)[(p)]:

P. 81: "The measurements I have made on over 100 types of animals… lead unanimously to the unexpected result that fish and invertebrates are in accord in their reactions to diverse spectral radiations. They show the same behavior as a dark-adapted normal observer under low-light conditions, or **as a completely colorblind person** at any light level."

P. 103: " … that I was able to demonstrate a strong attenuation at the long-wave end of the spectrum for all the aquatic animals previously investigated, i.e., I could demonstrate one of the defining characteristics of complete color-blindness. The strong absorption of long wavelengths in water – and their consequently trivial biological importance – renders this behavior comprehensible."

P. 83: "… what a large importance must be assigned to the ability to adapt to different light conditions… by fish, who swim from the shallows to the depths."

P. 82: "The transition by vertebrates to life in air led … to a fundamental development of the sensory neural system. This comes to expression in the development of a color sense, and in the thorough exploitation of long-wave radiation."

P. 80: "In visual systems that have developed along the same principle as the vertebrate eye… the transition to life in air dictated a neural reconstruction, evidently due to the influence of very much larger amounts of long-wave radiation which arrive at the eye. Among other changes, this is expressed as a significant extension of the visible spectrum at the long-wave end."

P. 47: "Consequently this is tantamount to saying that complete colorblindness in humans can be considered as arrested development at an early stage of development, one which we now encounter among vertebrate species only in fishes."

P. 29: " … that in our current state of knowledge, the white-black sensation is to be considered ancient and ancestral; it seems an accordingly well-established capability. At the same time we may consider the dimension of color sensation as a relatively new phylogenetic acquisition, which entered into the development of vertebrates only with the transition from aquatic life to life in air."

Now admittedly ***Hess***'s prediction of total colorblindness has proven to be wrong in a few cases, namely for **invertebrates which live in air**. And it would be too odd, if all our **insect-pollinated flowers** had donned their fine-colored raiment for nothing at all! So in fact von ***Frisch*** can furnish certain proof of a color sense in the honey bee,[(q)] as ***Knoll*** does for a variety of hawkmoth. For example, a bee can easily pick out a **blue**-colored paper from a randomly arranged array of gray papers **of the most widely disparate levels of brightness**. **–** ***Hess***'s error lies mainly in the following: that in the absence of other criteria, he believes he ought to be able to draw conclusions about total colorblindness just from the correspondence of a sensitivity curve with that of completely colorblind persons. Therefore it is of particular value to note here, that von ***Hess*** could adduce the proof of total colorblindness by something like an infallible method for **one** aquatic species, namely the cephalopods, by a particularly sensitive **pupillary response**. If one stimulates the human eye by a moderately quick alternation of two colorless lights of **different brightness**, then the alternation results in a sudden and successive **dilation** and **contraction** of the pupil. This is an **objective procedure for demonstrating differences in brightness**. Yet if **one** of those lights is colorless and the **other** colored (or if the two are differently colored), then a pupillary response occurs also **for two lights of approximately equal brightness**, to be more precise a **contraction** with **each** alternation. This form of response is **unchanged**, if one alters the brightness of one of the two lights within a certain **finite range of their brightness ratio**. The magnitude of this range of 'attenuation to change' is a measure of the **qualitative/color difference** of the two light stimuli. Doubtless the phenomenon is to be interpreted so that as a consequence of the **individually** specific habituation effects of the two lights, **each** of them acts at its onset as the brighter in pupillary-motor terms. Accordingly, attenuation to change is completely **lacking** in completely colorblind observers. It is also lacking in cephalopods, as von ***Hess*** has shown, although they demonstrate a distinct reflexive pupil action otherwise.

I have to point out a crucial difference between ***Hess***'s conception and mine. ***Hess*** is an opponent of von ***Kries***'s 'zone theory', which attributes twilight (scotopic) vision and daylight vision to the two anatomically established neural systems: **rods** and **cones**, respectively. ***Hess*** speaks only of a "transformation of the neural receptive system" through the transition to terrestrial life. Yet it is just in our phylogenetic deliberations that I glimpse strong support for the zone theory. Stated quite generally, suppose it had been the case that a reconstruction of the visual system did take place. And suppose it took place in the sense that there was a development of the color sense, with displacement of the sensitivity curves towards longer wavelengths. Then it would hardly be expected that our greatly transformed eye would exhibit noticeable traces of 'archaic' vision. One such archaïsm is the manner in which the eye regresses **completely** at minimal light levels, so far as the lack of a color sense and so far as the spectral sensitivity distribution are concerned. These characteristics support much better the notion that **alongside the essential retention of the old visual system** – which took on other, less biologically-important functions – **a new system was added**. The new system adapted to the new requirements, and it developed divergent properties as a consequence. The old rod system adopted the role of a twilight system. It was especially suited to that in advance by the great breadth of its capacity for adaptation.

Notes

a. As far as I know, this notion was first expressed clearly by ***Otto Lummer*** in: Lummer, O. Grundlagen, Ziele und Grenzen der Leuchttechnik (Auge und Lichterzeugung). München und Berlin: Oldenbourg Verlag (§ 86) (1918). ***Lummer*** shows the converse: that our eyes – constituted **just as they are** – will best exploit the light of a glowing body that radiates energy of any temperature, just when the color temperature of that body is the same **as the Sun's**. That is why – for the sake of economy – we must take care that our artificial light sources approach the color temperature of sunlight ! [ff. 1, p. 925 col. 1 original]

b. ***Abbot***, C. G. The sun's energy-spectrum and temperature. Astrophysical Journal, **34**, 197-208 (1911). – The area which lies between any two ordinates of the E- curve provides a measure of the energy falling within the spectral interval of wavelengths demarcated by the ordinates (for the solar spectrum on earth). [ff. 2, p. 925 col. 1 original]

c. Cf. ***Milne***, E. A. Radiative equilibrium in the outer layers of a star: the temperature distribution and the law of darkening. Monthly Notices of the Royal Astronomical Society. **81**, 361-375 (1921). [ff. 3, p. 926 col. 1 original]

d. We should like to illustrate the range of individual differences by tracing **two** curves. ***Z*** was obtained from the young ***Hedwig Bender*** by flicker photometry (see ***Lummer*** above, loc. cit. p. 61). ***Z′*** was obtained from the seventy-year old ***Franz Exner*** by the direct method: ***Exner***, F. Zur Kenntnis der Grundempfindungen im ***Helmholtz***'schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **129**, 27 – 46 (1920), cf. p. 41, as calculated by ***Aigner***, F. Ibid. **131**, p. 305 (1922). An explanation of the displacement between these two curves is due chiefly to coloration of the **macula lutea**, which color is stronger in older eyes. The stronger color absorbs short-wave light more readily. [ff. 1, p. 925 col. 2 original]

e. Compare for example, the text by O. ***Lummer*** in: ***Müller-Pouillet***, Lehrbuch der Physik, 10. Auflage, Band II, 3, p. 399ff. Von ***Kries*** is the originator of the zone theory, or 'Duplizitätstheorie'. [ff. 1, p.926 col. 1 original]

f. From: ***Lummer***, O. Grundlagen, Ziele und Grenzen der Leuchttechnik (Auge und Lichterzeugung). München und Berlin: Oldenbourg Verlag, p. 61 (1918). Actually the curves used here were derived from measurements on completely color-blind individuals, the majority of whom have only rod vision. [ff. 2, p. 926 col. 1 original]

g. Also note ***Exner***, F. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **131**, p. 622 (1922). Data on the monochromat ***Beysell***. [ff. 3, p. 926 col. 1 original]
h. ***Exner***, F. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **131**, p. 615 (1922). [ff. 1, p. 926 col. 2 original]
i. Ibid. ***Aigner***, F. p. 299. [ff. 2, p. 926 col. 2 original]
j. In this the 'blue curve' has almost no role at all. The sensation of brightness depends almost exclusively on the sensations of red and green. See ***Exner***, F. Zur Kenntnis der Grundempfindungen im ***Helmholtz***'schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **129**, 27 – 46 (1920). [ff. 3, p. 926, col. 2 original]
k. ***Scheiner***, J. & ***Graff***, K. Astrophysik. Leipzig, Berlin: B.G. Teubner, p. 325 (1922). [ff. 1, p. 927, col. 1 original]
l. Ibid. p. 256. [ff. 2, p. 927 col. 1 original]
m. ***Eddington***, A.S. Applications of the theory of the stellar absorption coefficient**.** Monthly Notices of the Royal Astronomical Society, **83**, 98-109 (1922). I take these data from the exemplary report by ***Jean Bosler*** L'évolution des étoiles. Paris (1923). [ff. 3, p. 927, col. 1 original]
n. ***Lawson***, R.W., this Journal: Über absolute Zeitmessung in der Geologie auf Grund der radioaktiven Erscheinungen. Die Naturwissenschaften. Two parts: **5** (26), 429-435 & **5**(27), 452-459 (both 1917). [ff. 1, p. 927, col. 2 original]
o. ***Exner***, F. (1922). Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **131**, p. 636 (1922). [ff. 2, p. 927 col. 2 original]
p. ***Hess***, C. von. Farbenlehre. In L. ***Asher*** und K. ***Spiro***, Ergebnisse der Physiologie, **20**. Jahrgang, München und Wiesbaden: J.F. Bergmann, 1 – 51 (1922). [ff. 1, p. 928, col. 1 original]
q. Note for example the lectures given this year by these researchers to scientific conferences. [ff. 1, p. 928 col. 2 original]

Kapitel 7
Über den Ursprung der Empfindlichkeitskurven des Auges

Bekanntlich ist unser Auge nur für einen verhältnismäβig kleinen Teil der Strahlung, die ein glühender Körper aussendet, empfänglich; das **sichtbare** Gebiet des Wärmespektrums erstreckt sich von etwa $\lambda = 800$ μμ bis $\lambda = 400$ μμ. Fragt man sich, **warum** wir unseren Lichtsinn gerade in diesem Bereich ausgebildet haben und nicht in einem anderen, bei gröβeren oder kleineren Wellenlängen, so kann die Antwort nicht zweifelhaft sein. Das sichtbare Gebiet liegt nämlich **zu beiden Seiten des Intensitätsmaximums der Sonnenstrahlung**. Es scheint, daβ sich das Auge auf bestmögliche Ausnützung derjenigen Lichtquelle eingestellt hat, die vor menschlicher Kultur fast die einzige in Betracht kommende war.[1] Sozusagen als „Nebenbedingungen“ bei dieser biologischen Maximumsaufgabe wird man sich gewisse Beschränkungen der organischen Konstruktionsmöglichkeit zu denken haben, welche z.B. eine gröβere spektrale Ausdehnung des Sehbereiches, im Vergleich zu dem geringen Vorteil den sie gebracht hätte, zu sehr erschwert haben mögen.

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von Erwin Schrödinger, Zürich. *Die Naturwissenschaften* Zwölfter Jahrgang 7. November 1924 Heft 45

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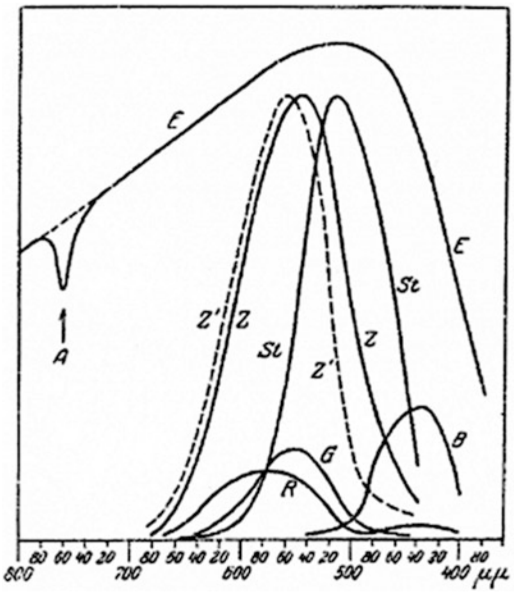

E: Energieverteilung der Sonne; ***Z***: Zapfenkurve nach H. Bender; ***Z′***: Zapfenkurve nach F. Exner; ***St***: Stäbchenkurve; **R G B**: Rot- , Grün- , Blaukurve.

In beistehender Figur ist ***E*** die Energiekurve der Sonne, nach den Messungen ***Abbott***s[2] berechnet, für solche mittlere Verhältnisse, wie sie bei Bildung des Sehorgans durchschnittlich in Betracht gekommen sein mögen: Meeresniveau (Washington), 45° Sonnenhöhe. Das Maximum der Strahlung findet sich unter diesen Verhältnissen bei etwa $\lambda = 515$ μμ im Blaugrün. Bemerkenswert ist das viel steilere Absinken der Kurve gegen kurze Wellen als gegen lange Wellen hin, was von der vereinigten Wirkung der Absorption in der Sonne selbst[3] und in der Erdatmosphäre herrührt. (Die starke Sauerstoffabsorption bei der Fraunhoferschen Linie ***A***, $\lambda = 760$ μμ, ist nach älteren Messungen ***Langley***s nur schätzungsweise hineinkorrigiert, für uns übrigens ohne Belang.) **Z** bzw. ***Z′*** ist die sog. **Zapfenkurve** nach Messungen zweier verschiedener Beobachter.[4] Es ist das die aus Helligkeitsmessungen im Spektrum **errechnete** Helligkeitsverteilung für ein helladaptiertes Auge in einem **idealen** Spektrum von **konstanter** Energie, dessen ***E***-Kurve durch eine horizontale Gerade gegeben wäre, dem entlang geführt ein Energiemesser (Bolometer oder Thermosäule) **konstanten** Ausschlag zeigen würde. Die Ordinaten der **Z**-Kurve sind also ein Maβ für die Helligkeit, welche **gleiche** Energiemengen **von verschiedener Wellenlänge** im Auge hervorbringen, man kann sagen: für die spektrale Empfindlichkeitsverteilung des Auges. Das Empfindlichkeitsmaximum liegt **im Gelbgrün** bei $\lambda = 550$ μμ bis 560 μμ, also merklich **rotwärts** vom Energiemaximum der irdischen Sonnenstrahlung. Das erklärt sich nach unserer biologischen Hypothese wohl hinlänglich aus dem stark unsymmetrischen Verlauf der ***E***-Kurve. Es ist klar, daβ eine etwas bessere

Ausnützung erzielt wird, wenn das Empfindlichkeitsmaximum ein wenig nach der Seite des sanfteren Abfalles der ***E***-Kurve verschoben ist.

Das bisher Gesagte bezieht sich auf die Helligkeitsempfindung bei der gewöhnlichen Art des Sehens mit normal-helladaptiertem Auge, physiologisch gesprochen auf die Tätigkeit der Netzhaut**zapfen**. Nun besitzen wir, wie man heute weiβ, noch einen zweiten Sehapparat, die sog. **Stäbchen**, die, mit den Zapfen vermischt, palisadenförmig die Netzhaut bedecken, und zwar besonders zahlreich die peripheren Teile, wo die Zapfen immer seltener werden, während in einer kleinen Umgebung der Stelle des deutlichsten Sehens (ca. 1½° Winkeldurchmesser) nur die hier besonders dicht stehenden Zapfen vorhanden sind. Im Gegensatz zu den Zapfen vermitteln die Stäbchen eine fast farblose (vielleicht ein wenig bläuliche) Empfindung, deren Qualität von der Wellenlänge unabhängig ist – sie sind **total farbenblind**. Die zweite hervorstechende Eigenschaft des Stäbchenapparates ist seine auβerordentlich groβe Anpassungsfähigkeit an geringe Lichtstärken. Je mehr die allgemeine Erhellung unserer Umgebung abnimmt, um so mehr mischt sich in das sog. **Tagessehen** der Zapfen das „**Stäbchensehen**" und bleibt schlieβlich bei niederen Lichtstärken allein zurück [sog. „Dämmerungssehen"[5]]. Das macht die getrennte Bestimmung der spektralen Empfindlichkeitskurve des Stäbchenapparates möglich, und es zeigt sich [Kurve ***St*** unserer Figur[6]], daβ sie gegenüber der Zapfenkurven stark gegen kurze Wellenlängen verlagert ist, sie hat ihr Maximum bei etwa $\lambda = 517$ µµ[7]) im **Blaugrün**. Eine stark in die Augen fallende Folge dieser Verschiebung ist für jedermann leicht zu beobachten, z. B. in einer Bildergalerie bei einbrechender Dämmerung. Infolge wachsender Beteiligung des Stäbchensehens nimmt die Helligkeit der roten Farbtöne viel stärker ab als die der blauen, welch letztere im Vergleich eine eigentümlich starke Leuchtkraft gewinnen (Purkinjesches Phänomen); durch plötzliches Einschalten einer starken künstlichen Lichtquelle kann man das Helligkeitsverhältnis wieder umkehren und gewinnt dann einen sehr starken Eindruck von der Tatsache, die sich für die exakte Messung in der besprochenen Verlagerung des Kurvengipfels ausspricht.

Woher rührt nun diese merkwürdige Verlagerung der Stäbchenkurve nach kurzen Wellenlängen? Ich erinnere mich nicht, irgendwo den Versuch einer Erklärung dafür gefunden zu haben. Im vollen Bewuβtsein, daβ ich nur über einen Teil der zur Beurteilung dieser Frage nötigen Kenntnis verfüge, möchte ich gleichwohl hier einige Erklärungsmöglichkeiten zur Diskussion stellen, sei es auch nur, um die Aufmerksamkeit anderer darauf zu lenken, die zu ihrer Beurteilung berufener sind.

Erstens wäre es natürlich möglich, daβ der innere Mechanismus des Stäbchenapparates, seine organischen Konstruktionsbedingungen, von denen des Zapfenapparates so stark abweichen, daβ bei der "biologischen Maximumsaufgabe" bestmöglicher Ausnützung der zur Verfügung stehenden Lichtquelle die "veränderten Nebenbedingungen" zu einer etwas verschiedenen Lösung geführt haben. Eine solche Erklärung annehmen, hieβe natürlich die Flinte ins Korn werfen und auf eine eigentliche Erklärung verzichten. Nach den neuerdings von F. ***Exner***[8] und F. ***Aigner***[9] vertretenen Ansichten über die Natur der Netzhauterregung ist es überdies wahrscheinlich, daβ der Mechanismus des Stäbchensehens und des Zapfensehens qualitativ der nämliche ist. Danach soll es sich in allen Fällen um **elektromagnetische Resonnatoren** in den Nervenenden handeln, die innerhalb eines gewissen Resonanzbereiches auf die verschiedenen Wellenlängen nach einer von anderen physikalischen Erscheinungen her wohlbekannten **Resonanzkurve** ansprechen, wobei Lage und Breite des Resonanzgebietes durch zwei physikalische Konstanten des Resonators (Eigenschwingungszahl und Dämpfung) bestimmt sind. Diese Resonatorenkonstanten wären also dasjenige, was sich den äußeren Bedingungen biologisch angepaßt hat. Dabei müssen – worauf wir im Augenblick nicht näher eingehen wollen – für das farbige Zapfensehen **drei** verschiedene Resonatorenarten in Anspruch genommen werden; ihre Resonanzgebiete werden durch die Kurven ***R*** („Rot"), ***G*** („Grün"), ***B*** („Blau") der Figur dargestellt, aus denen sich durch gewisse additive Verknüpfung die **Z**-Kurve zusammensetzt.[10] Dagegen sollen die Stäbchen nur **eine** Resonatorenart enthalten, deren Resonanzgebiet direkt durch die ***St***-Kurve dargestellt wird. Warum haben nun – so würden wir in der Sprache dieser speziellen Theorie fragen – die Stäbchenresonatoren ihr Ansprechgebiet so viel weiter nach Blau verschoben als diejenigen Zapfenresonatoren, die hauptsächlich die Helligkeit im Tagessehen vermitteln? (D. i. die Rot- und Grünresonatoren.)

Eine wirkliche Erklärung, welche sich nicht auf unbekannte organische Konstruktionsbedingungen beruft, kann m. E. nur darin gefunden werden, daß der Stäbchenapparat sich unter der Einwirkung eines **anderen Beleuchtungslichtes** mit anderer Energieverteilungskurve ausgebildet hat als der Zapfenapparat. Hier scheinen mir nun folgende Möglichkeiten sich darzubieten.

1. Die besondere Anpassungsfähigkeit des Stäbchenapparates an geringe Lichtstärken läßt daran denken, daß es dabei um das Sehorgan eines **Nachttieres** sich handelt. Wie steht es nun mit der Energieverteilungskurve des nächtlichen Lichtes? Was das Sternenlicht betrifft, so kennen wir die Spektren einer außerordentlich großen Zahl einzelner Sterne und wissen, auch schon aus dem Farbenindex, daß viele röter, andere blauer sind als die Sonne. In einer Tabelle der hellsten in unseren Gegenden sichtbaren Sterne[11] finde ich unter 42 Sternen 25 blauer, nur 16 röter als die Sonne. Dagegen ist das Licht des Mondes, dem für die Sehleistungen eines Nachttieres doch wohl eine erhebliche Bedeutung zukommt, ein wenig röter als das der Sonne[12] (etwa 0,5 Größenklassen im Farbenindex). Diese Hypothese liefert also wohl keine befriedigende Erklärung.
2. Man kann zweitens daran denken, daß die Entstehung des Stäbchenapparates phylogenetisch so weit zurückliegt, daß sein Empfindlichkeitsmaximum noch auf eine **höhere Sonnentemperatur** zurückweist als die jetzt herrschende. Tatsächlich gehört nach den Ergebnissen der neueren astrophysikalischen Forschung die Sonne zu den sog. Zwergsternen und befindet sich schon auf dem absteigenden Ast ihrer Entwicklung, d.h. sie ist im **Abkühlung** begriffen. Während ihre gegenwärtige Temperatur zwischen 5900 und 6000° absolut liegt, hat ihre Maximaltemperatur nach ***Eddingtons*** Berechnung 6600° betragen.[13] Nach dem Wienschen Verschiebungsgesetz würde dieser Temperaturänderung eine Verschiebung des Energiemaximums zu einer für die **höhere** Temperatur im Verhältnis

$$6000/6600 = 0,91$$

kürzeren Wellenlänge entsprechen. Die Wellenlängen der beiden Empfindlichkeitsmaxima stehen im Verhältnis

$$517/550 = 0,94$$

Es hat also tatsächlich in den ersten Stadien der Abkühlung das Sonnenlicht eine Zusammensetzung gehabt, die zur Stäbchenkurve in etwa demselben Verhältnis steht, wie seine gegenwärtige Zusammensetzung zur Zapfenkurve. Bei der Unsicherheit aller **Zeitschätzungen** auf diesem Gebiet möchte ich die Hypothese nicht unbedingt verwerfen. Immerhin verliert sie an Wahrscheinlichkeit, wenn wir bedenken, daß ***Eddington*** die **gesamte** Entwicklungsdauer eines Sternes (von Dunkelrotglut über die Maximaltemperatur zur Dunkelrotglut) auf einige Zehnmilliarden Jahre schätzt, während selbst die ältesten Granite nach der ziemlich zuverlässigen radioaktiven Methode auf höchstens 1–1½ Milliarden Jahre zu schätzen sind,[14] die Entwicklung des Sehvermögens unserer Ahnen also doch wohl sehr viel jüngeren Datums sein muß.

3. Die dritte und wahrscheinlichste Erklärung scheint mir in der **grünblauen Farbe** zu liegen, die das **Wasser** in dickeren Schichten zeigt. Für ein **Wassertier**, das in einiger Tiefe unter der Oberfläche lebt, muß die Zusammensetzung des Sonnenlichtes tatsächlich in ungefähr dem Sinne geändert werden, den wir zur Erklärung nötig haben. Auch die starke **Anpassungsfähigkeit** an verschiedene Helligkeiten würde ein solches Tier besonders nötig haben, wenn es wechselnde Tiefen unter dem Wasserspiegel aufsucht. Der Stäbchenapparat würde also nach dieser Hypothese ein **älteres** Sehorgan sein, das zur Zeit des Wasserlebens entstanden ist. Die zwei demselben Zweck dienenden Organe: Stäbchen, Zapfen würden eine gewisse Parallele bilden zu dem wohlbekannten Fall: Kiemen, Lunge. Dabei müßte man annehmen, daß der Zapfenapparat bei den das Tageslicht aufsuchenden Tieren zur vollen Ausbildung gelangte, **während** die Stäbchen für den Gebrauch unter Wasser immer noch dringend benötigt wurden; ferner, daß die Zäpfchen mit der Zeit die Hauptfunktion übernahmen und die zu Hilfsorganen herabgedrückten Stäbchen keine genügende biologische Wichtigkeit mehr besaßen, um ihre genaue Anpassung an die veränderten Beleuchtungsverhältnisse herbeizuführen, nachdem die Tiere vom Wasserleben ganz zum Landleben übergegangen waren.

Die meisten Fälle von **totaler Farbenblindheit**, die in einer Rückkehr zum reinen Stäbchensehen bestehen, wären nach dieser Auffassung ein eigentlicher **Atavismus**.

Der Gegenstand verlockt zu weiteren Spekulationen über die allmähliche Ausbildung des Tagessehens zum **Farben**sehen. Freilich wird der Boden damit zusehends unsicherer. Man wird es für wahrscheinlich halten, daß das erste Zapfensehen ein undifferenziertes farbloses Sehen war wie das Stäbchensehen. In der Tat sind seltene Fälle totaler Farbenblindheit bekannt, die augenscheinlich **kein**

Stäbchensehen sind.[15] Es fehlt die sonst bei Totalfarbenblinden beobachtete Lichtscheu, und das Maximum der Helligkeitsempfindung liegt genau an derselben Stelle wie bei der Zapfenkurve junger Augen, nämlich bei $\lambda = 550\ \mu\mu$. Ferner findet sich diese Art des farblosen **Zapfen**sehens auch in den periphersten Teilen der Netzhaut **normaler** Augen; man kann sich denken, daβ diese biologisch minder wichtigen Randpartien die weitere Entwicklung nicht mitgemacht haben. Das nächste Entwicklungsstadium dürfte das der **Dichromasie**, des Gelb-Blausehens, gewesen sein. Es findet sich auf der normalen Netzhaut **zwischen** der farbentüchtigen Mitte und der eben erwähnten total farbenblinden Randzone, ferner bei Insekten (Bienen nach v. ***Hess*** und v. ***Frisch***, Taubenschwänzen nach ***Knoll***), endlich bildet es die weitaus häufigste Art der **partiellen Farbenblindheit**. Hier zeigt sich allerdings die Komplikation, daβ es **zwei** Typen dieser Farbenblinden gibt, solche, bei denen die „Gelbkurve" die Lage der normalen Rotkurve (***R*** in der Figur) hat, diese sind die häufigsten, und solche, bei denen sie die Lage der normalen Grünkurve (***G***) hat. Die relativ groβe Häufigkeit gerade dieser Art von Anomalien (etwa 4% aller Männer!) scheint mir darauf hinzudeuten, daβ die Zerfällung der langwelligen Erregungskurve in eine Rot- und Grünkurve das **letzte** Stadium der Entwicklung unseres Sehorgans ist, daher noch am schlechtesten fixiert, Rückfällen und Störungen am meisten ausgesetzt ist. – Ich betone aber nochmals, daβ es bei den Bemerkungen dieses letzten Absatzes nur um vage Vermutungen sich handelt.

Wie ich sehe, trifft **die** Deutung der Lage des Stäbchenmaximums, die ich für die wahrscheinlichste halte, sachlich vollkommen zusammen mit der Ansicht von C. von ***Hess*** und erfährt durch dessen reiches Versuchsmaterial eine Stütze. Ich führe einige Stellen aus der „Farbenlehre" dieses Forschers[16] hier an:

S. 81: „Meine Messungen an über 100 Tierarten … führen übereinstimmend zu dem unerwarteten Ergebnisse, daβ hinsichtlich der Reaktionen gegenüber verschiedenen spektralen Strahlungen Fische und Wirbellose übereinstimmendes und das gleiche Verhalten zeigen wie der dunkeladaptierte, bei herabgesetzter Lichtstärke sehende normale und **wie der total farbenblinde Mensch** bei jeder Lichtstärke."

S. 103: „… daβ ich für alle bisher untersuchten Wassertiere starke Verkürzung des Spektrums am langwelligen Ende, d.h. eines der charakteristischen Merkmale der totalen Farbenblindheit nachweisen konnte; die starke Absorption jener langwelligen Strahlen im Wasser und ihre entsprechend geringe biologische Bedeutung macht dieses Verhalten verständlich."

S. 83: „… von wie groβer Bedeutung die Fähigkeit der Anpassung an verschiedene Lichtstärken … sein muβ … für Fische, die von der Oberfläche zur Tiefe schwimmen …"

S. 82: „Bei den Wirbeltieren führte der Übergang zum Luftleben … zu einer wesentlichen Weiterbildung des nervösen Empfangapparates, die in der Entwicklung eines Farbensinnes und der ausgiebigeren Verwertung langwelliger Strahlen zum Ausdruck kommt."

S. 80: „Bei den nach dem Prinzip des Wirbeltierauges gebauten Sehorganen ... hat sich ... mit dem Übergange zum Luftleben, offenbar unter dem Einflusse der jetzt in viel größeren Mengen zum Auge gelangenden langwelligen Strahlen, eine Umbildung der nervösen Substanz vollzogen, die unter anderem in einer wesentlichen Ausdehnung des Spektrums nach der langwelligen Seite zum Ausdrucke kommt.“

S. 47: „Danach liegt es nahe, die totale Farbenblindheit beim Menschen als Stehenbleiben auf einer niederen Entwicklungsstufe aufzufassen, der wir in der Wirbeltierreihe nur noch bei Fischen begegnen.“

S. 29: „ ... daß nach dem heutigen Stande unserer Kenntnisse die Schwarz-Weißempfindung als ein stammesgeschichtlich uralter und wohl entsprechend gefestigter Besitz zu betrachten ist, während wir in den farbigen Empfindungsreihen einen phylogenetisch verhältnismaßig jungen Erwerb zu sehen haben, der in der Wirbeltierreihe erst mit dem Übergange vom Wasser- zum Luftleben zur Entwicklung gekommen ist.“ – – –

Nun hat sich allerdings in einigen Fällen, namentlich für die **luftlebenden Wirbellosen**, die Hesssche Diagnose auf totale Farbenblindheit als irrtümlich herausgestellt. Es wäre ja auch gar zu merkwürdig, wenn alle unsere **insektenbefruchteten Blüten** ihr herrliches Farbenkleid für nichts und wieder nichts sollten angelegt haben! So konnten denn in der Tat v. ***Frisch*** für die Hönigbiene, ***Knoll*** für eine Schwärmerart den Nachweis des Farbensinnes mit Sicherheit erbringen,[17] indem z.B. die Biene ein **blaues** Farbpapier aus einer großen Anzahl regellos angeordneter Graupapiere **von den verschiedensten Helligkeitsstufen** leicht herausfindet. – Der Hesssche Irrtum bestand hauptsächlich darin, daß er beim Fehlen anderer Kriterien allein schon aus der Übereinstimmung der Empfindlichkeitskurve mit derjenigen des totalfarbenblinden Menschen auf totale Farbenblindheit schließen zu dürfen glaubte. Da ist es denn von besonderem Wert, daß v. ***Hess*** wenigstens für **eine** wasserbewohnende Tiergattung, nämlich die Cephalopoden, den Nachweis der totalen Farbenblindheit noch auf einem ziemlich untrüglichen Wege erbringen konnte, nämlich durch eine besonders sinnreiche **Pupillenreaktion**. Bietet man dem menschlichen Auge in mäßig raschem Wechsel zwei **verschieden helle** farblose Lichter dar, so erfolgt beim Lichtwechsel abwechselnd plötzliche **Kontraktion** und **Dilatation** der Pupille, als ein **objektivier Nachweis des Helligkeitsunterschiedes**. Ist aber das **eine** Licht farblos, das **andere** farbig (oder auch beide verschieden farbig), so tritt **auch bei annähernd gleicher Helligkeit der beiden Lichter** eine Pupillenreaktion auf, und zwar plötzliche **Kontraktion** bei **jedem** Lichtwechsel. Diese Reaktionsweise **bleibt**, wenn man die Helligkeit des einen der beiden Lichter abändert, innerhalb eines gewissen **endlichen Bereiches des Helligkeitsverhältnisses bestehen**, und die Größe dieses Bereiches der „Wechselverengerung“ ist ein Maß für die **qualitativ-farbliche Verschiedenheit** der beiden Lichteindrücke. Die Erscheinung ist zweifellos so zu deuten, daß infolge des **spezifisch** verschiedenen Ermüdungseffektes der beiden Lichter **jedes** von ihnen bei seinem Auftauchen als das pupillomotorisch hellere wirkt. Dementsprechend **fehlt** die

Wechselverengerung völlig beim totalfarbenblinden Menschen, und sie fehlt, wie v. ***Hess*** zeigt, auch bei den Cephalopoden, die im übrigen ein deutliches reflektorisches Pupillenspiel zeigen.

Auf einen wesentlichen Unterschied der Hessschen Auffassung und der meinen muß ich noch hinweisen. ***Hess*** ist ein Gegner der Kriesschen „Duplizitätstheorie", die das Dämmerungssehen, bzw. das Sehen am hellen Tage, den zwei anatomisch festgestellten Nervenendorganen, den **Stäbchen** bzw. den **Zapfen**, zuweist. ***Hess*** spricht daher einfach nur von einer „Umbildung des nervösen Empfangsapparates" beim Übergang zum Landleben. Ich erblicke aber gerade in unseren phylogenetischen Betrachtungen eine starke Stütze der Duplizitätstheorie. Hätte, ganz allgemein gesprochen, eine Umbildung des Sehorgans stattgefunden im Sinne der Entwicklung eines Farbensinnes und der Verschiebung der Empfindlichkeitskurven nach längeren Wellen, dann wäre doch kaum zu erwarten, daß unser stark umgebildetes Auge noch derart merkliche Spuren jener „altertümlichen" Art des Sehens aufweist, in die es bei geringen Lichtstärken sogar **gänzlich** zurückverfällt, sowohl was den Mangel des Farbensinnes als auch was die spektrale Empfindlichkeitsverteilung anlangt. Diesem Verhalten entspricht viel besser die Auffassung, daß **unter wesentlicher Erhaltung des alten Sehapparates**, welcher andere, biologisch weniger wichtige Funktionen übernimmt, **ein neuer Apparat hinzugebildet wurde**, der sich den neuen Anforderungen angepaßt und daher wesentlich abweichende Eigenschaften erhalten hat. Der alte Stäbchenapparat dagegen übernahm die Rolle eines Dämmerungsorganes, wofür er durch seine große Adaptationsbreite von vornherein besonders geeignet war.

1. So viel mir bekannt, hat diesen Gedanken zum ersten Mal klar ausgesprochen Otto ***Lummer***, Ziele der Leuchttechnik (§ 86). München und Berlin 1918. ***Lummer*** zeigt umgekehrt, daß unser Auge **so**, **wie es ist**, unter allen Temperaturstrahlern einen glühenden Körper **von Sonnentemperatur** am besten ausnützt. Das ist der Grund, weshalb wir aus Ökonomiegründen uns bemühen müssen, unsere künstlichen Lichtquellen der Sonnentemperatur zu nähern!
2. C.G. ***Abbott***, Astrophysical Journal, **34**, 197. 1911. Die Fläche zwischen irgend zwei Ordinaten der $\boldsymbol{E}$ – Kurve ist ein Maß der Energie, die auf das betreffende Wellenlängenintervall im Spektrum der irdischen Sonnenstrahlung entfällt.
3. Siehe E.A. ***Milne***, Monthly Notices of the Royal Astronomical Society **81**, 375. 1921.
4. Durch Anführung **zweier** Kurven wollen wir die individuelle Variationsbreite illustrieren. $\boldsymbol{Z}$ ist von der **jungen *Hedwig Bender*** mit dem Flimmerphotometer gewonnen (s.O. ***Lummer***, 1. c. S. 61), $\boldsymbol{Z'}$ von dem **siebzigjährigen *Franz Exner*** nach direkter Methode (Sitzungsber. d. Akad. Wien, Mathem.-naturw. Kl. IIa **129**, 41. 1920; berechnet von F. ***Aigner***, ibid. **131**, 305. 1922). Die Verlagerung der Kurven gegeneinander wird zum größten Teil auf der

stärkeren Färbung des **gelben Flecks** in älteren Augen beruhen, wodurch das kurzwellige Licht stärker absorbiert wird.

5. Man vgl. z.B. ***Müller – Pouillet***, Lehrbuch der Physik, 10. Aufl., Bd. II, 3 (O. ***Lummer***) S. 399 ff. – Der Begründer der "Duplizitätstheorie" ist ***v. Kries***.
6. Nach ***Lummer***, Ziele der Leuchttechnik S. 61. In Wahrheit ist die hier benützte Kurve durch Messungen an Totalfarbenblinden gewonnen, die in der Mehrzahl reine Stäbchenseher sind.
7. Siehe auch F. ***Exner***, Sitzungsber. d. Akad. Wien, Mathem.-naturw. Kl. **131**, 622. 1922. Angabe über ***Monochromat Beyssell***.
8. F. ***Exner***, Sitzungsber. d. Akad. Wien, Mathem.-naturw. Kl. IIa **131**, 615. 1922.
9. F. ***Aigner***, ibid. S. 299.
10. Die „Blaukurve" spielt dabei fast keine Rolle. Die Helligkeitsempfindung hängt fast ausschließlich an der Rot- und Grünempfindung. Siehe F. ***Exner***, Sitzungsber. d. Akad. Wien, Mathem.-naturw. Kl. **129**, 27. 1920.
11. ***Scheiner-Graff***, Astrophysik S. 325., Teubner 1922.
12. Ibid. S. 256.
13. A.S. ***Eddington***, Monthly Notices **83**, 98. 1922. Ich entnehme die Angaben dem vortrefflichen Bericht von ***Jean Bosler***, L'évolution des étoiles. Paris 1923.
14. R.W. ***Lawson***, Diese Zeitschr. 1917, H. 26/27.
15. F. ***Exner***, Sitzungsber. d. Akad. Wien, Mathem.-naturw. Kl. IIa **131**, 636. 1922.
16. In „Ergebnisse der Physiologie" **20**, ***1***, 1922; bei J.F. Bergmann, München u. Wiesbaden.
17. Vgl. z. B. die Vorträge dieser beiden Forscher auf der diesjährigen Naturforscherversammlung

Chapter 8
On the Subjective Color of Starlight and the Quality of Twilight Sensation

Abstract A popular account is given to explain the subjective color of starlight. One's predominant sensation while observing a star in the visual field is given by scotopic or 'twilight' vision, mediated by the rod cells of the eye. The hue of the characteristic rod sensation can be assessed. There is yet another influence on starlight color, however. Conditions for emergence of the ***Bezold-Brücke*** phenomenon are ideally fulfilled in the observation of starlight. A complete explanation of the color of starlight is provided by the effects of contrast on rod 'blue', together with the Bezold-Brücke phenomenon.

Keywords Color vision • Starlight • Nocturnal vision • Cone cells • Cone sensitivity • Rod cells • Rod sensitivity • Tritanopia • Anomalous trichromat • Bezold-Brücke • Purkinje • Schrödinger

Recently in this Journal Mr. ***Bottlinger***[(a)] has been very convincing in pointing out the contradiction between the temperature of fixed stars and the color names that we give them, in comparison to the names given to terrestrial light sources of known temperature. Likewise this contradiction occurred to me some time ago. An explanation presented itself to me, which I sought to reinforce with some experimental trials on the quality of sensation in twilight. Although these experiments have not yet come to full conclusion, I would like to report briefly on the train of my thoughts, now that the problem has been put forward for discussion.

It has often been stressed, as it was by ***Bottlinger*** in the note under discussion, that 'the notion of white is quite a variable one'. White is displaced away from the hue that **prevails** across the field of view. If we don a pair of not-too-strongly colored glasses, after a while we hardly notice that colors are transformed with respect to their normal appearance. Certainly in part this may be attributed to the "specific adaptation" which emerges in simple aftereffects as complementary color. Partly – and perhaps to a large extent – this may be a matter of a purely **psychological** circumstance, that is, a displacement in **judgment** about color. Under that condition, it follows in simultaneous color contrast that strongly saturated colors may not be more strongly inducing of an effect, but rather more weakly inducing

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than a less saturated color. One may be reminded of a well-known demonstration involving flowers.

The predominant sensation while observing a star in the visual field is normally one of twilight vision – or theoretically put, that of rod cells. As we term it, the sensation is of ‘rod white’ or ‘rod gray’. Actually the sensation is of a **single** color, but it is in no sense **colorless**. That is, it is **not** identical with the gray of daylight vision; rather it is **bluish**. It is a matter of a shift in judgment – of the kind discussed above – that we are most often unconscious of this situation. Yet with that, it is clear that judgment of the color of a star seen in strictly foveal view will also be shifted – although as ***Bottlinger*** notes correctly, the twilight system of vision is not **directly** involved. Then of course the color will be judged such that the star’s color will draw closer to the color sensation **complementary** to the rod sensation.

What do we know about the hue of this twilight sensation? Let us delve into the question – interesting in and of itself – more thoroughly than may inevitably be necessary for the purpose at hand. Qualitative research by ***Nagel*** and von ***Kries***[(b)] has shown at any rate that the ‘rod color’ is quite noticeably **blue**. A quantitative determination for ***Nagel*** as a **deuteranope** (so-called ‘green-blind’) showed equivalence to this rod color at $\lambda = 480$–485 μμ in daylight vision. It should be emphasized that this is **not** a matter of determining the **hue**, but rather the **saturation** of ***Nagel***’s rod sensation. As is well-known, that is because to the partially colorblind there are only **two** hues. For him the spectrum presents a range of saturation, from yellow through white to blue. Then for ***Nagel*** those values $\lambda < 480$ must also match the twilight color, with the addition of a suitable amount of white.

By an original and inventive procedure, von ***Hauer***[(c)] has attempted to determine these wavelengths for ordinary people. Suppose one illuminates a larger (parafoveal) area of the retina with bright white light, and then decreases the light suddenly. Then to achieve a color equivalence with a neighbouring area that has **not** been illuminated before, one requires two things. Understandably one requires **less** white in the previously unilluminated field, but one must also add some blue to it. To be precise, Hauer finds that one must add $\lambda = 457$, 460, or 465 μμ for each of three normal experimental subjects. He reasons as follows: in the early application of strong illumination, the rods are excluded. **Only the cones are adapted**. After the light intensity is decreased, **rods** are at an advantage in the **adapted** field during the immediate rise of dark adaptation. They are at an advantage as they compete with the already-**adapted** cones there. In comparison they vie with unadapted cones in the comparison field. Therefore the blue is the rod color. Even if this interpretation holds, yet ***Hauer***’s experiment is restricted to a very special case: incomplete dark adaptation of short duration with a strong participation of daylight vision. (No more time is given in the trial than is supposed for the white-light adaptation of cones.) No definite conclusions may be drawn from this about sensations in the near-**fully** dark-adapted eye, as is most often employed for star-gazing.

To begin purely qualitatively, one can convince oneself easily of the blueness of twilight color using the simplest of means. To this end I use two abutted brass tubes at right angles, each of about 2 cm. in diameter and about 20 cm. in length. The assembly contains a white surface angled to 45° at the tube's bend. Otherwise they are lined with black velvet and outfitted with multiple shutters. At one open end of the tube assembly, one eye is attached by a completely light-tight connection, using a camera condenser of black paper and a stiff eye patch with an aperture. The other end of the tube serves to regulate the attenuated daylight illumination that is visible on the white surface. That end has a second white surface which seals at 45°; there is a small variable aperture at that end of the tube. Then if one places one's eye on this apparatus, after a few minutes of dark adaptation one sees a patch with **both** eyes open, in the middle of the field provided by the brightly-lit eye. That field can be the room itself; one sees a small floating patch of a twilight (scotopic) field. The best effect occurs when one projects that onto a dark corner of the room; then one may compare it easily to arbitrary daylight colors. The relatively strong subjective brightness of this patch is indeed striking, and it may be compared throughout with the field that is seen by the brightly-lit eye. An evident increase in brightness occurs when one's gaze is suddenly averted, and this effect is beautifully demonstrated. Another effect is easily demonstrated too: the complete colorblindness of the rods may be shown by putting colored glass in front of the illumination window. The **hue** of this patch is called a flat **reddish** blue by normal observers, something like pale lilac. The color phenomenon is still more striking if one performs the demonstration in the evening under artificial illumination. For the darkened eye, naturally the quality of illumination is irrelevant. Still the displacement in judgment for the brightly-lit eye due to the yellow or reddish-yellow of the artificial light source does increase the disparity of 'white' from the unaltered twilight color.

In order to determine hue quantitatively, the darkened tube of the apparatus was mounted in a binocular viewing arrangement next to a spectrophotometer telescope, [d] which – instead of a plain oculus – had a divided ocular field which contained a Nicol prism. Then the brightly-lit eye could mix: 1) an arbitrary spectral color on the face of a second Nicol prism placed between the collimator and a first prism, with 2) daylight entering into the apparatus sideways by reflection. This mixture can be adjusted for different ratios and different overall intensity. And so it can be used to match the color of the twilight patch – given that that does not fall within the spectral 'gaps'.

Trials were run with four normal trichromats. Almost always they adjusted wavelength to values smaller than 430 μμ in the narrow-band violet end of the spectrum. Within this region adjustments were strongly variable – which makes sense, because along there hue does not change across the spectrum. An observer well-acquainted with use of the spectroscope reported that he would rather have still a little **more red** in the comparison color than is present in the far violet. Nonetheless adjustments occurred for all the observers separately in which $\lambda > 430$ μμ – some up to about $\lambda = 445$ μμ (indigo), but never to **greenish**-blue, however. Values greater than $\lambda = 430$ μμ occurred after more than a half-hour of adaptation for three of the four observers; still one cannot really speak of a **trend** here. Neither does there seem to be much evidence of a clear influence of the subjective brightness of the twilight patch.

My own result was a complete outlier – I am an anomalous trichromat: to be precise, deuteranomalous or 'red-sighted'. Subjectively I judge the patch to be greenish-blue. Accordingly my estimates lie around cyan, close to Fraunhofer's *F* line. Since hue varies strongly in this region, these estimates have much better replicability than estimates by normal observers. A very pronounced influence is exerted by the subjective brightness of the twilight patch. For a subjectively dark patch I adjust the comparison field towards longer wavelengths; for a subjectively bright patch more towards the short-wave. For all that it is the same whether brightness is varied as a function of the intensity of illumination, or the adaptation state, or through more or less paracentral observation. The outer limits of the twilight color – as it is intentionally manipulated in these ways – lay at $\lambda = 484$ μμ (for an extremely bright patch) and $\lambda = 495$ μμ (for an extremely dark one).

Of course the responses of color-anomalous and color-blind observers are of secondary interest to our present purpose. I think it certain that in general the twilight color for normal observers is an unsaturated red-violet, perhaps somewhat redder than violet at the end of the spectrum. Under particular circumstances which have not yet been wholly specified, perhaps this shifts to indigo but it never moves to green.

Does the predominance of this rod-blue or rod-violet in the visual field allow us to understand the subjective color of starlight? To a large extent yes, but not completely. It stands to reason that white sun-like stars must appear yellow 'by contrast'. Similarly, only stars that are considerably bluer than the sun will appear to be white. An observation made by ***Bottlinger*** also fits our explanation exceptionally well. He observes that α-Lyrae (color temperature of about 10,000°) actually appeared **blue** when near a yellowish-red flame. In that situation the otherwise predominant rod color was replaced by the color of the flame. We know from a famous experiment with colored shadows that in such a context even ordinary daylight appears blue.

What about red stars from 2000° to 3000°? Here the mere hypothesis of contrast is insufficient. A subjective displacement towards **red** on the basis of contrast with the twilight color is ruled out; we have established that such a displacement in judgment **never** moves **towards green** in normal trichromats. Even an explanation in objective terms seems to be ruled out, though the light of these stars does **not** correspond to a terrestrial light source of the same temperature, because of their strong absorption band in the short-wave region of the spectrum. One can easily overlook the fact that the resulting color cannot really move back into 'deep red'. ***Bottlinger***'s most interesting demonstration with the artificial star and an incandescent lamp also shows it is not the absorption bands which cause reddening, but rather the conditions of observation.

Yet now this subjective reddening can be explained very simply in another way. We have only to free ourselves from the admittedly widespread conception that the photopic color of a white-hot body – say a metal filament lamp – is **actually white**. If that were the case, then one would not have to fit it with a fairly strong blue or blue-green filter in order to convert it to a 'daylight lamp'. For example, even to untrained judgment a 'Philips Argenta' lamp appears to glow a warm golden-yellow in daylight. That is, it appears yellow with a distinct tinge of reddish color – and I have had this confirmed by normal trichromats. In what follows, it is important to note that a **greenish**-yellow never occurs in grey-body radiation. The sequence of colors that emerges with rising temperature proceeds from reddish-yellow through yellow to white. This occurs without crossing over the border from pure yellow to green. It also seems that when this sequence is extended beyond the temperature of our sun, no hue emerges with a preponderance of green – in the sense of the **three**-component (trichromatic) theory. Rather, only greenish-blue to bluish hues occur – naturally less saturated colors.

Having said that much, let us recall the long-known ***Bezold-Brücke*** **phenomenon**.[(e)] This consists of a characteristic transformation of the spectral color sequence when the intensity of light is lowered substantially. The spectrum becomes divided into three almost monochrome regions: red, green, and violet. Two abrupt transitions emerge between red and green, and green and violet. The most striking part is the radical compression of the yellow region, in which all reddish-yellow hues transition towards red, and all greenish-yellows transition towards green. This is **not** some business or phenomenon of twilight (scotopic) vision. When **that** is involved, then the entire spectrum **loses its color** (that is, it assumes the twilight color throughout). Von ***Kries*** proposes that the phenomenon is strongest in a **small** visual field with maximum **exclusion** of the process of dark adaptation. According to the trichromatic theory it is based on the following: with the lowering of light intensity, the two weaker contributions of fundamental sensation **dip below threshold**, and only the strongest component remains active. Consequently any color will approach that fundamental color, which is most strongly represented in it. In the process the color will **gain in saturation**. Under the trichromatic theory, that is because a lack of saturation (or an 'admixture of white') depends on the three fundamental components acting in equal measure to produce a common effect. To clarify: the **weakest** component is acted upon as well, while the advantage of the two stronger components over the weaker one will determine the color characteristic of the result. Based on this idea, F. ***Exner***[(f)] was able to locate **three** of the four intersection points of the so-called fundamental response curves. Confirming what A. ***König*** had found by completely different methods, ***Exner*** was able to locate the three with heightened precision by means of the Bezold-Brücke effect.

A physiological explanation is now at hand for the fact that cooler stars appear to have such a pronounced red color. Conditions for the emergence of the Bezold-Brücke phenomenon are ideally fulfilled by the minimal light level of the tiny field subtended by the star, and by its extremely small size. The star is seen **foveally** insofar as it is seen as colored at all. Then there must be a strong approximation of the color, towards the fundamental color which is dominant in it. For reddish-yellow that is **fundamental red**. (According to ***König*** and ***Exner***, even sodium yellow – which we hardly experience as reddish – has something like 33% more fundamental red than fundamental green.)

The red which arises this way is a fairly saturated red, aligned more or less with fundamental red. It is no longer very strongly changed by contrast with twilight color, as is otherwise to be expected from experience. Also, fundamental red is known not to be pure red in **psychological** terms. Rather it is somewhat bluish. In the context of rod stimulus it ought to be experienced as pure red.

Following on that explanation, yellowish-green, greenish-yellow or bluish-green stars ought to appear saturated green. There may not be such stars however, at least not along the normal scale of black-body temperatures. Such a claim could easily be substantiated using artificial starlight. It should be demonstrable as well that a **truly** white star – for example one produced by a good daylight lamp – appears yellow (by its contrast with rod blue) rather than red as ***Bottlinger***'s incandescent-lamp star had appeared.

Also, the **pure** blue of α – Lyrae seen beside a nocturnal flame is **fully** understandable only with reference to ***Brücke***'s phenomenon. Objectively, the color of an A – star still must be a fairly unsaturated, somewhat greenish blue.

If one could raise the brightness of a red star significantly, its saturation of red would needs be diminished. Its color would necessarily move towards yellow. I made such an observation on the occasion of the last opposition of the planet ***Mars***; I do not know if the observation would be confirmed by normal trichromats. And now some light is cast, too, on the paradoxical state of affairs that red stars appear to be so little red to us 'red-sighted' people. Namely our anomaly consists in this: that our 'green sensitivity curve' approaches our 'red curve'. That is, the 'green curve' is shifted towards longer wavelengths. As a consequence, to us all reddish-yellow hues contain relatively **more** fundamental green and **less** fundamental red. The relation of the two components is nearer to identity than for normal observers. Since the Bezold-Brücke shift affects the difference of these two components, it is clear that the shift will apply to this region of the spectrum less easily and less distinctly for color-anomalous observers than for normal observers.

In summary it seems to me that a complete explanation of the subjective color of starlight is provided by the effects of contrast with rod blue, together with the Bezold-Brücke phenomenon.

By way of a postscript, I would like to add the following note to the contrast theory. At one point in his Physiological optics, ***Helmholtz*** remarks that one might free oneself of the shift in color judgment under artificial illumination. By means of a black-lined tube, that might be achieved by fading a small patch to black within an illuminated 'white' field. The 'entoptic light of the retina' would then serve as a comparison, superimposed on the dark background of the tube wall; that comparison would allow the reddish-yellow cast of the area to be recognized. Now I would not like to judge whether the intrinsic activity of the retina really plays a central role in this, or better, if the stimulus is just light weakly reflected from the tube wall. At any rate I think it extremely likely that the quality of color for real entoptic light coincides substantially with our twilight color. Then the objection can hardly be raised to the explanation given above for the color of starlight, that the light of the night sky is altogether too weak to elicit a noticeable activation of the rods.

I would like to return shortly in another work to the remarkable difference in twilight color that was alluded to above, between normal and anomalous trichromats. I believe this may be explained **solely** by a difference in **photopic systems**. The rod color is ‘really’ one and the same for both at the same time – and perhaps for all eyes as well. It is known that the spectral excitation curve of the rod system is not altered in the least by any kind of disturbance of the color sense. Neither does the variation with brightness need to be real, as it was above for anomalous observers; that is, it does not really need to be a variation in the twilight color. Rather, it may be a transformation of the comparison field due to the Bezold-Brücke effect. The unchanging nature of the rod apparatus ought to be closely tied to its phylogenetic development – as should be the particular color characteristic of the sensation it delivers.[g]

Notes

a. ***Bottlinger***, K.F. Astronomische Mitteilungen. Die Naturwissenschaften, **13**(9), 27 Feb., 179–180 (1925). [ff. 1, p. 373 col. 1 original]

b. ***Kries***, J. von & ***Nagel***, W.A. Über den Einfluß von Lichtstärke und Adaptation auf das Sehen des Dichromaten (Grünblinden). Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **12**, p. 28 (1896)., or ***Helmholtz***, H. L.F. von. Handbuch der physiologischen Optik. 3e Auflage. Hamburg und Leipzig: Leopold Voβ, p. 295 (1909). [ff. 1, p. 373, col. 2 original]

c. v. ***Hauer***, F. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abt. 2a, **123**, p. 647 (1914). [ff. 2, p. 373 col. 2]

d. I would like to offer my heartfelt thanks to Mr. ***Debye*** at this point. It was at his Institute that this research was conducted. [ff. 1, p. 374 col. 1 original]

e. ***Brücke***, M.E. Über einige Empfindungen im Gebiet der Sehnerven. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **3**, 39 – 77 (1878). ; ***Exner***, F. Über die Grundempfindungen im Young-Helmholtz’schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **111**, 857 – 877 (1902). ; see for example ***Nagel***, W. A. Handbuch der Physiologie des Menschen, vol. 3. Braunschweig: Vieweg, p.261 (1905). [ff. 1, p. 375, col. 1 original]

f. Loc. cit. F. ***Exner***; See also ***Steindler***, O. Die Farbenempfindlichkeit des normalen und farbenblinden Auges. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **115**, 39–62 (1906).; L. ***Richtera***, Ibid. **122**, p. 1913 (1915).; F. von ***Hauer***, Ibid. **123**, p. 654 (1914). [ff. 1, p. 375 col. 2 original]

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Kapitel 8
Über die subjektiven Sternfarben und die Qualität der Dämmerungsempfindung

Neulich hat an dieser Stelle Herr BOTTLINGER[1] in sehr treffender Weise auf den Widerspruch hingewiesen, welcher besteht zwischen der Temperatur der Fixsterne und der Farbbezeichnung, die wir ihnen geben, im Vergleich mit der Farbbezeichnung irdischer Lichtquellen von bekannter Temperatur. Mir ist vor einiger Zeit der Widerspruch gleichfalls aufgefallen, und es hat sich mir eine Erklärung aufgedrängt, die ich durch einige Versuche, die Qualität der Dämmerungsempfindung betreffend, zu erhärten suchte. Obwohl dieselben noch nicht völlig abgeschlossen sind, möchte ich doch kurz über den Gedankengang berichten, da das Problem einmal zur Diskussion gestellt ist.

Es ist schon oft, so auch von BOTTLINGER in der erwähnten Note, betont worden, daβ „der Begriff Weiβ sehr variabel ist“. Er verschiebt sich gegen den im gesamten Gesichtsfeld **vorherrschenden** Farbton; wenn wir eine nicht allzu stark gefärbte Brille aufsetzen, bemerken wir nach einiger Zeit kaum mehr etwas davon, daβ alle Farben gegenüber ihrem normalen Aussehen verändert sind. Das ist zu einem Teil gewiβ auf dieselbe „spezifische Ermüdung“ zurückzuführen, die sich in den negativ komplementären Nachbildern äuβert; teilweise – und wahrscheinlich zum gröβeren Teil – handelt es sich aber wohl um eine rein **psychologische** Angelegenheit, eine Verschiebung des Farb**urteils**. Es folgt dies daraus, daβ stark gesättigte Farben im Simultankontrast nicht etwa stärker sondern schwächer induzierend wirken als wenig gesättigte – man denke an den bekannten Florversuch.

Die beim Betrachten eines Sternes im Gesichtsfeld vorherrschende Empfindung ist nun normalerweise die des Dämmerungssehens oder, theoretisch gesprochen, die der Stäbchen. Man pflegt sie als „Stäbchenweiβ“ oder „Stäbchengrau“ zu bezeichnen. In Wahrheit ist sie zwar **ein**färbig, aber keineswegs **farblos**, d.h. **nicht** identisch mit einem Grau des Tagessehens, sondern **bläulich**. Daβ wir

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uns dieses Umstandes meist nicht bewuβt sind, liegt offenbar an einer Urteilsverschiebung von der oben besprochenen Art. Dann ist aber klar, daβ dabei auch das Urteil über die Farbe eines streng foveal gesehenen Sterns – obwohl hier, wie BOTTLINGER richtig bemerkt, der Dämmerungsapparat nicht **direkt** beteiligt ist – sich gleichwohl mit verschieben wird, und zwar ungefähr in dem Sinne, daβ die Sternfarbe der zur Stäbchenempfindung **komplementären** Farbempfindung näherrückt.

Was wissen wir nun über den Farbton der Dämmerungsempfindung? Es sei gestattet, uns über diese an und für sich interessante Frage etwas eingehender zu verbreiten, als für den vorliegenden Zweck gerade unumgänglich nötig wäre. Qualitative Versuche von ***Nagel*** und v. ***Kries***[2] haben gezeigt, daβ die „Stäbchenfarbe" jedenfalls sehr merklich **blau** ist. Eine quantitative Bestimmung ergab für den **deuteranopen** (sog. grünblinden) ***Nagel*** Gleichheit der Dämmerungsfarbe mit $\lambda = 480–485$ μμ des Tagessehens. Es muβ aber betont werden, daβ es sich dabei eigentlich **nicht** um die Bestimmung des **Farbtons**, sondern der **Sättigung** von ***Nagel***s Stäbchenempfindung handelt, weil es bekanntlich für den partiell Farbenblinden überhaupt nur **zwei** Farbtöne gibt; das Spektrum ist ihm eine reine Sättigungsreihe von Gelb über Weiβ nach Blau. Es muβ also für ***Nagel*** auch jedes $\lambda < 480$, mit passendem Weiβzusatz die Dämmerungsfarbe kopieren.

Die Wellenlänge für normale Personen festzulegen, hat v. ***Hauer***[3] nach einer eigenartigen und geistvollen Methode versucht. Belichtet man ein gröβeres (überfoveales) Feld der Netzhaut mit starkem Weiβ und setzt die Lichtstärke dann plötzlich herab, so benötigt man zur Erzielung einer Farbengleichung mit einem benachbarten **nicht** vorbelichteten Feld, auf dem letzteren nicht nur – selbstverständlich – **weniger** Weiβ, sondern man muβ diesem Weiβ etwas Blau zusetzen, und zwar findet ***Hauer*** $\lambda = 457$, 460, 465 μμ für drei normale Versuchspersonen. Er deutet dies so: während der starken Vorbelichtung sind die Stäbchen ausgeschaltet, und es werden **nur die Zapfen ermüdet**. Nach dem Herabsetzen der Lichtstärke sind bei der sogleich beginnenden Dunkeladaptation die Stäbchen auf dem **ermüdeten** Feld im Vorteil, weil sie hier mit **ermüdeten** Zapfen konkurrieren, auf dem Vergleichsfeld hingegen mit nicht ermüdeten. Der Farbton des Blau ist also die Stäbchenfarbe. Auch wenn diese Deutung zutrifft, bezieht sich der Hauersche Versuch doch auf einen sehr speziellen Fall: kurze und unvollkommene Dunkeladaptation – es steht nicht mehr Zeit zur Verfügung als die Weiβermüdung der Zapfen andauert – bei noch starker Beteiligung des Tagesapparates. Auf die Empfindung des **stark** dunkeladaptierten Auges, wie es beim Sternesehen meistens vorliegt, lassen sich daraus wohl keine sicheren Schlüsse ziehen.

Sich von der Bläue der Dämmerungsfarbe, zunächst rein qualitativ, zu überzeugen, gelingt leicht mit den einfachsten Mitteln. Ich bediene mich dazu zweier rechtwinkelig aneinandergesetzter Messingrohre von etwa 2 cm Durchmesser und je 20 cm Länge, die an der Knickstelle unter 45° eine weiβe Fläche enthalten, im übrigen aber mit schwarzem Sammet ausgekleidet und mit zahlreichen Blenden versehen sind. An das eine offene Ende des Röhrensystems wird das Auge mittels einer durchlochten steifen Augenbinde und eines Cameraauszuges aus schwarzem Papier vollkommen lichtdicht angesetzt, das andere Ende dient zur regulierbaren Beleuchtung der weißen Sichtfläche mit geschwächtem Tageslicht, dieses Ende wird durch eine zweite weiße Fläche unter 45° abgeschlossen, der gegenüber das Rohr ein kleines regulierbares Loch hat. Setzt man nun das eine Auge an diese Vorrichtung an, so sieht man nach einigen Minuten Dunkeladaptation bei geöffneten **beiden** Augen inmitten des von dem Hellauge gelieferten Gesichtsfeldes, z. B. des Zimmers, ein Dämmerungsfeldchen schweben, das man am besten in eine dunkle Zimmerecke projiziert und dann bequem mit jeder beliebigen Tagesfarbe vergleichen kann. Sehr frappant ist die relativ große subjektive Helligkeit des Feldes, die durchaus mit den vom Hellauge gesehenen vergleichbar ist. Sehr schön zeigt sich das bekannte Aufleuchten beim Abwenden des Blickes, auch ist die totale Farbenblindheit der Stäbchen durch Vorhalten farbiger Gläser vor das Beleuchtungsfenster leicht zu demonstrieren. Der **Farbton** dieses Feldchens wird von Normalen als ein mattes **rötliches** Blau bezeichnet, etwa wie blasser Flieder. Der Farbeffekt ist noch viel auffallender, wenn man den Versuch des Abends bei künstlicher Beleuchtung anstellt. Für das Dunkelauge ist die Qualität der Beleuchtung natürlich irrelevant, aber die Urteilsverschiebung des Hellauges durch das Gelb oder Rotgelb der künstlichen Lichtquelle vergrößert den Abstand des „Weiß" von der ungeänderten Dämmerungsfarbe.

Um den Farbton quantitativ festzulegen, wurde das Dämmerungsrohr zur binokularen Durchsicht neben ein Spektrometerfernrohr montiert,[4] das statt des Okulars einen Okularspalt mit vorgesetztem Nicol trug. Das Hellauge konnte auf der Stirnfläche eines zwischen Kollimator und Prisma eingebauten zweiten Nicols eine beliebige Spektralfarbe mit seitlich in den Apparat reflektiertem Tageslicht in variablem Verhältnis und variabler Gesamtintensität mischen und so die Farbe des Dämmerungsfeldes – vorausgesetzt, daß sie nicht in die spektrale „Lücke" fällt – kopieren.

Es wurden vier normale Trichromaten untersucht. Diese stellten fast immer eine Wellenlänge kleiner als 430 μμ in der monochromvioletten Endstrecke des Spektrums ein. Innerhalb dieses Gebietes schwankte die Einstellung stark, was selbstverständlich ist, weil sich hier der Farbton im Spektrum nicht mehr ändert. Ein spektroskopisch gut geschulter Beobachter gab an, daß er eher noch ein wenig **mehr rot** in der Vergleichsfarbe wünschen würde, als im Endviolett enthalten ist. Immerhin kamen bei allen Beobachtern vereinzelt auch Einstellungen $\lambda > 430$ μμ, bis etwa $\lambda = 445$ μμ (Indigo) vor, niemals jedoch bis zu einem **grünlichen** Blau. Bei drei von den vier Beobachtern ereigneten sich die Überschreitungen von $\lambda = 430$ nach mehr als halbstündiger Adaptation, doch kann man nicht von einem deutlichen **Gang** sprechen. Ebensowenig war ein deutlicher Einfluß der subjektiven Helligkeit des Dämmerungsfeldes nachweisbar.

Gänzlich abweichend war mein eigener Befund – ich bin anomaler Trichromat, und zwar deuteranomal – „rotsichtig". Subjektiv beurteile ich das Feld grünblau, und dementsprechend liegen auch meine Einstellungen im Cyan, nahe der Fraunhoferschen Linie ***F***. Da in dieser Gegend der Farbton sehr stark variiert, sind sie viel besser reproduzierbar als die der Normalen; dabei zeigt sich ein sehr ausgesprochener Einfluß der subjektiven Helligkeit des Dämmerungsfeldes, bei subjektiv dunklem Feld stelle ich das Vergleichsfeld langwelliger, bei subjektiv hellem Feld kurzwelliger ein, einerlei ob die Helligkeit durch die Beleuchtungsstärke, den Adaptationszustand oder durch mehr oder weniger parazentrale Beobachtung variiert wird. Die äußersten Grenzen der auf diese Weise absichtlich variierten Dämmerungsfarbe waren $\lambda = 484$ μμ (bei extrem hellem Feld) und $\lambda = 495$ μμ (bei extrem dunklem Feld).

Für unsere gegenwärtige Absicht ist das Verhalten der Anomalen und Anopen natürlich von untergeordnetem Interesse. Für die Normalen sehe ich als gesichert an, daß ihre Dämmerungsfarbe im allgemeinen ein ungesättigtes Rotviolett, vielleicht noch etwas röter als das Endviolett des Spektrums ist, unter gewissen Umständen, die noch nicht völlig geklärt sind, vielleicht gegen Indigo geht, niemals aber nach Grün zieht.

Läßt nun das Vorherrschen dieses Stäbchenblau oder –violett im Gesichtsfeld die beobachteten subjektiven Sternfarben verstehen? Zum großen Teil ja, aber nicht restlos. Daß die weißen Sonnensterne „durch Kontrast“ gelb erscheinen müssen, leuchtet ein; ebenso, daß erst Sterne, die erheblich blauer sind, als die Sonne, weiß erscheinen werden. Auch stimmt zu unserer Erklärung ausgezeichnet die Beobachtung ***Bottlinger***s, dem α-Lyrae (Temperatur etwa 10,000°) in der Umgebung eines gelbroten Feuers tatsächlich **blau** erschien. Dabei war eben die sonst vorherrschende Stäbchenfarbe durch die Farbe des Feuers ersetzt, und daß gegen diese schon gewöhnliches Tageslicht blau wirkt, wissen wir aus dem bekannten Versuch der farbigen Schatten.

Wie steht es aber mit den Rotsternen von 2000 bis 3000°? Hier reicht die Kontrasttheorie allein offenbar nicht aus. Eine subjektive Verschiebung gegen **Rot** auf Grund des Kontrastes gegen die Dämmerungsfarbe ist ausgeschlossen, da wir festgestellt haben, daß diese für normale Trichromaten **keinesfalls gegen Grün** zieht. Auch eine objektive Erklärung erscheint ausgeschlossen, obwohl ja das Licht dieser Sterne wegen der starken Bandenabsorption im kurzwelligen Teil **nicht** genau mit dem einer irdischen Lichtquelle von gleicher Temperatur übereinstimmt. Man überblickt aber leicht, daß die resultierende Farbe dadurch nicht wirklich ins „Tiefrot“ rücken kann, auch zeigt der höchst interessante Bottlingersche Versuch mit dem künstlichen Glühlampenstern, daß nicht die Absorptionsbanden die Rötung bewirken, sondern die Art der Betrachtung.

Diese subjektive Rötung erklärt sich nun aber sehr einfach auf andere Weise, nur müssen wir uns von der freilich sehr weitverbreiteten Vorstellung losmachen, es sei die Tagesfarbe eines weißglühenden Körpers, z.B. einer Metallfadenlampe, **wirklich weiß**. Wäre das der Fall, so brauchte man eine solche Lampe nicht erst mit einem ziemlich starken blauen bis grünblauen Filter zu versehen, um sie in eine „Tageslichtlampe“ zu verwandeln. Auch nach dem unmittelbaren Urteil erscheint z.B. eine „Philips Argenta“, am hellen Tage gebrannt, in einem warmen Goldgelb, d.h. Gelb mit einem deutlichen Zug ins Rötliche – was ich mir von normalen Trichromaten habe bestätigen lassen. Es ist für das Folgende wichtig zu bemerken, daß ein **grünliches** Gelb bei grauer Temperaturstrahlung überhaupt nicht auftritt, die Farbe geht mit steigender Temperatur von Rotgelb über Gelb nach Weiß, ohne daß die Grenze des reinen Gelb gegen Grün zu überschritten wird. Auch scheint es, daß bei Fortsetzung der Reihe über die Sonnentemperatur hinaus wieder keine Farbtöne mit dem Hauptanteil Grün – im Sinne der **Drei**komponententheorie – auftreten, sondern nur grünblaue bis blaue, natürlich wenig gesättigte Töne.

Dies vorausgeschickt, erinnern wir an das seit langem bekannte ***Bezold-Brücke*sche Phänomen**.[5] Es besteht darin, daß die Farbenfolge des Spektrums bei starker Herabsetzung der Lichtstärke eine eigentümliche Veränderung erfährt, indem das Spektrum in drei fast monochrome Bezirke: Rot, Grün, und Violett, zerfällt mit zwei sehr schroffen Übergangsstellen zwischen Rot und Grün, Grün und Violett. Am auffallendsten ist **das völlige Zusammenschrumpfen des gelben Bereiches, indem alle rötlichgelben Töne gegen Rot, die grünlichgelben gegen Grün wandern**. Dabei handelt es sich **nicht** etwa um eine Erscheinung des Dämmerungssehens. Tritt **dieses** ein, so **entfärbt** sich ja das ganze Spektrum (bzw. nimmt die Dämmerungsfarbe an). Von ***Kries*** hebt hervor, daß besonders auf **kleinem** Feld und bei möglichstem **Ausschluß** von Dunkeladaptation das Phänomen deutlich ist. Es beruht nach der Dreikomponententheorie darauf, daß bei Herabsetzung der Lichtstärke die beiden schwächeren Grundempfindungskomponenten **unterschwellig** werden und die stärkste allein übrig bleibt, wodurch **jede Farbe derjenigen Grundfarbe sich nähert, die in ihr am stärksten vertreten ist**; dabei muß sie zugleich **an Sättigung zunehmen**, da ja der Mangel an Sättigung oder das „beigemischte Weiß" nach der Dreikomponententheorie auf dem Zusammenwirken der drei Komponenten in gleicher Stärke beruht, und zwar natürlich in der Stärke der **schwächsten** Komponente, während der Überschuß der beiden stärkeren über die schwächste den bunten Charakter der Farbe bestimmt. Auf Grund dieser Vorstellung konnte F. ***Exner***[6] von den vier Schnittpunkten der sog. Grundempfindungskurven, die A. ***König*** auf ganz anderem Wege gefunden hatte, mittels des Bezold-Brückeschen Phänomens **drei** mit erheblicher Genauigkeit bestätigen.

Die physiologische Erklärung dafür, daß die kühleren Sterne so ausgesprochen rot erscheinen, liegt nun auf der Hand. Durch die äußerste Kleinheit und immerhin recht geringe Lichtstärke des Sternscheibchens, das gleichwohl, sofern es überhaupt farbig erscheint, **foveal** gesehen wird, sind die Bedingungen für das Auftreten des Bezold-Brückeschen Phänomens in idealer Weise erfüllt. Es muß daher eine weitgehende Annäherung an diejenige Grundfarbe stattfinden, die in der Farbe vorherrscht, und das ist bei rötlichem Gelb das **Grundrot**. (Selbst Na-Gelb, das wir kaum noch rötlich empfinden, enthält nach ***König*** und ***Exner*** noch etwa 33% mehr Grundrot als Grundgrün!)

Daß das so zustande kommende, dem Grundrot mehr oder weniger nahestehende, ziemlich gesättigte Rot durch den Kontrast mit der Dämmerungsfarbe nicht mehr sehr stark verändert wird, ist nach sonstiger Erfahrung zu erwarten; übrigens ist das Grundrot **psychologisch** bekanntlich kein reines Rot, sondern etwas bläulich. Bei „Stäbchenstimmung" dürfte es gerade als reines Rot wirken.

Gelbgrüne, grüngelbe oder blaugrüne Sterne sollten nach dieser Erklärung gesättigt grün erscheinen. Es gibt sie aber wohl nicht, jedenfalls nicht in der normalen Temperaturreihe. An künstlichen Sternen dürfte sich die Behauptung leicht bestätigen lassen. Auch müßte sich zeigen, daß ein **wirklich** weißer Stern, z.B. mit einer guten Tageslichtlampe hergestellt, nicht wie der Bottlingersche Glühlampenstern rot, sondern (durch Kontrast mit Stäbchenblau) gelb aussieht.

Auch das **reine** Blau von α-Lyrae neben dem nächtlichen Feuer ist **ganz** verständlich nur mit Berücksichtigung des Brückeschen Phänomens. Objektiv muß die Farbe eines **A**-Sternes noch ein ziemlich ungesättigtes, etwas grünliches Blau sein.

Könnte man die Helligkeit eines Rotsternes stark erhöhen, so müßte die Sättigung des Rot abnehmen, und es müßte eine Annäherung an Gelb stattfinden. Ich habe diese Bemerkung am **Mars** bei seiner letzten Opposition gemacht, weiß aber nicht, ob sie von normalen Trichromaten bestätigt wird. Übrigens fällt auch auf die paradoxe Tatsache, daß wir „Rotsichtigen" die Rotsterne überhaupt nur so wenig deutlich rot sehen, jetzt einiges Licht. Unsere Anomalie besteht nämlich darin, daß unsere „Grünkurve" der „Rotkurve" angenähert, d.h. gegen lange Wellen verschoben ist. Infolgedessen enthalten alle rotgelben Farbtöne für uns relativ **mehr** Grundgrün und **weniger** Grundrot, das Verhältnis der beiden Komponenten ist der Einheit näher gerückt, als für den Normalen. Da die Bezold-Brückesche Verschiebung auf der Verschiedenheit der beiden Komponenten beruht, ist es klar, daß sie für den Anomalen in diesem Spektralgebiet weniger leicht eintreten und weniger ausgesprochen sein wird, als für den Normalen.

Zusammenfassend scheint es mir, daß die subjektiven Sternfarben durch den Kontrast mit dem Stäbchenblau in Verbindung mit dem Bezold-Brückeschen Phänomen ihre vollkommene Aufklärung finden.

Nachtragsweise möchte ich zur Kontrasttheorie noch folgendes erwähnen. ***Helmholtz*** bemerkt einmal in der „Physiologischen Optik", daß man von der Verschiebung des Farburteils bei künstlicher Beleuchtung sich befreien könne, indem man mittels einer innen geschwärzten Röhre ein kleines Feld einer beleuchteten „weißen" Fläche sich ausblendet. Das „Eigenlicht der Netzhaut" auf dem dunklen Hintergrund der Röhrenwand diene alsdann zum Vergleich und lasse die rotgelbe Färbung des Feldes erkennen. Ob nun bei diesem Versuch wirklich schon die Selbsterregung der Netzhaut die Hauptrolle spielt oder vielmehr eine Erregung durch das schwache von der Röhrenwand kommende Licht, möchte ich nicht entscheiden. Jedenfalls halte ich für äußerst wahrscheinlich, daß auch die Farbqualität der wirklichen Selbsterregung mit der Dämmerungsfarbe merklich übereinstimmt, so daß gegen die oben gegebene Erklärung der Sternfarben kaum der Einwand zu erheben ist, das Licht des Himmelsgrundes sei überhaupt zu schwach, um eine merkliche Erregung der Stäbchen hervorzubringen.

Auf die oben erwähnte merkwürdige Verschiedenheit der Dämmerungsfarbe für normale und anomale Trichromaten möchte ich demnächst in anderem Zusammenhang zurückkommen. Sie läßt sich, glaube ich, aus der Verschiedenheit **des Tagesapparates allein** erklären, während die Stäbchenfarbe selbst „in Wirklichkeit" für beide – und wahrscheinlich für alle – Augen die nämliche ist;

wird doch auch die spektrale Anregungskurve des Stäbchenapparates bekanntlich durch Farbensinnstörungen irgendwelcher Art nicht im geringsten beeinflußt. Auch die oben am Anomalen gefundene Variation mit der Helligkeit braucht nicht echt, d.h. nicht wirklich eine Variation der Dämmerungsfarbe zu sein, sondern liegt wahrscheinlich an einer Bezold-Brückeschen Veränderung des Vergleichsfeldes. Die Unveränderlichkeit des Stäbchenapparates sowie auch der spezielle Farbcharakter der von ihm vermittelten Empfindung dürften eng mit seiner phylogenetischen Entstehung zusammenhängen.[7]

1. Die Naturwissenschaften 27. Febr. 1925, S. 180.
2. ***Nagel*** und v. ***Kries***, Zeitschr. f. Psychol. u. Physiol. d. Sinnesorg. **12**, 28 oder v. ***Helmholtz***, Physiol. Optik III. Aufl., S. 295 .
3. F. v. ***Hauer***, Sitzungsber. d. Wien. Akad. (2a) **123**, 647. 1914.
4. Ich möchte Herrn ***Debye***, an dessen Institut die Versuche ausgeführt wurden, auch an dieser Stelle herzlich danken.
5. ***Brücke***, Sitzungsber. d. Wien Akad. (3) **77**, 1878; F. ***Exner***, Sitzungsber. d. Wien. Akad. (2a) **111**, 857. 1902; s. z.B. ***Nagel***s Handb. d. Physiol. **3**, S. 261. Braunschweig: Vieweg 1905.
6. F. ***Exner*** l. c.; s. a.O. ***Steindler***, Sitzungsber. d. Wien. Akad. (2a) **115**, 39, 1906 ; L. ***Richtera***, ibid. **122**, 1915, 1913 ; F. v. ***Hauer***, ibid. **123**, 654, 1914.
7. C. v. ***Hess***, Ergebn. d. Physiol. **20**, I. 1922 München u. Wiesbaden: J.F. Bergmann. A. ***Vogt***, Züricher Antrittsrede vom 1. Dezember 1923, S. 14ff. (Zürich: Seldwyla Verlag); E. ***Schrödinger***, Die Naturwissenschaften **12**, 925. 1924. Ich ergreife gern die Gelegenheit, um hinsichtlich der Phylogenie des Stäbchenapparates die **Priorität von *Vogt*** mir gegenüber festzustellen, die mir bei Abfassung der zitierten Note leider entgangen war.

Chapter 9
On the Relation of the Tetrachromatic Theory to the Trichromatic Theory

Abstract The trichromatic and opponent-process theories are two very different conceptions of the color continuum – perhaps. The formal relation between these two theories of human color vision (i.e., the three-color and the four-color theories) may be understood as a mere transformation of variables in color space. A projective transformation of the color plane unifies the two representations. A simple diagram can be constructed to illustrate relations of color for both the Young-Helmholtz trichromatic theory and the Hering opponent-process theory at once. Schrödinger adds some remarks on the likely phylogenetic development of color vision.

Keywords Color vision • Color theory • Color diagram • Trichromatic • Opponent-process • NCS system • Young-Helmholtz • Fundamental colors • Unique colors • Opponent-color • Projective transformation • Barycentric • Chromaticity coordinates • Alychne • Spectral curve • Line of purples • Coordinate transformation • Spectral distribution function • Color triangle • Twin coordinates • Dichromacy • Phylogeny of vision • Schrödinger

There are two very different conceptions of the color continuum. At least, it is plain that they have appeared to contrast until now. The **trichromatic** theory is usually associated with the name of ***Helmholtz***, while the **tetrachromatic** theory (which is more frequently called the opponent-process theory) has been staunchly defended by ***Hering***, though its origins can be traced to ***Aubert***. ***Helmholtz***'s theory maintains, in keeping with the undisputed three-dimensionality of the color continuum, that every color may be thought of as a mixture of three fundamental colors: fundamental red, fundamental green, and fundamental blue (or violet). Hence the trichromatic theory stands in clear opposition to immediate intuition, since pure yellow affords naïve observers a psychologically homogeneous sensation of color just as do red, green and blue. The naïve observer, try as he or she might, does not apperceive a mixture of equal redness and greenness, as should be the case on the trichromatic theory. Likewise the sensation of pure white has nothing to do with any

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of the colors mentioned. Rather it appears homogeneous, not to be analysed further in psychological terms, though on the trichromatic theory white arises from a mixture of equal proportions of all three fundamental colors.

In contrast the tetrachromatic theory bears a close connection to the psychological color manifold. ***Hering*** states that every particular color has, besides a 'valence of whiteness', another **two chromatic** valences. The first of these chromatic valences is called a 'red-green' valence – which is **either** red or green for any specific color – while the second is called a 'blue-yellow' valence, which is either blue or yellow. This addresses two psychological findings: firstly, that sensations of redness and greenness are incompatible, as are sensations of blueness and yellowness, and secondly, that every sensation of color can be classified by its hue as lying **between** one of the two colors of the first pair and one of the two colours of the second 'opponent-color' pair. An increase in the 'valence for whiteness' corresponds to an increase in brightness, and to desaturation (i.e., a tendency to white) when other chromatic valences are unchanged. A decrease in the valence for whiteness corresponds to shading and darkening. Yet if chromatic valence supervenes on a pure valence for whiteness, then **brightness** ought not to remain unchanged; rather a certain 'brightening' effect is attributable to redness and yellowness, and a certain 'darkening' effect is attributable to greenness and blueness.

As is well known, this theory provides the impulse for – or more likely just the occasion for – ***Hering***'s distinctive biochemical interpretation of visual function, which still has many supporters today. Surely it seems natural that many psychologists are among these supporters. Opponents of the theory retort that their claims may not be parsimonious – in other words, that they have multiplied the variables of the theory needlessly, at variance with the empirically determined **tri**dimensionality of the color continuum. Hence their theoretical terms may not allow the results of **quantitative** color research to be expressed adequately or sufficiently. Von ***Kries*** has clearly taken a conciliatory position in what he calls his '**zone theory**'.[(a)] In this theory, the trichromatic theory finds a model in the physiological processes of the retina; in counterpoint, the tetrachromatic theory would hold true for a more centripetally located 'zone' of the visual system, whereby the latter theory's closer connection to the **psychologically**-defined color continuum would be made comprehensible.

Von ***Kries***'s twist on the theory is quite plausible, in my opinion. Nonetheless, what I want to show in this article is completely independent of a deeper explication of the physiological substrate of visual processing. Simply, it is a matter of ascertaining that the purely formal relation between the two theories – the trichromatic and tetrachromatic theories – can be comprehended as an exceptionally simple relation, namely, as a mere **transformation of variables**. This material is not especially profound from a purely mathematical point of view, but all the same it has never been expressed with full clarity until now, so far as I know. Certainly this has not been recognised by many authors; otherwise debate would have proceeded along different lines. The apparent **numerical** contrast between the

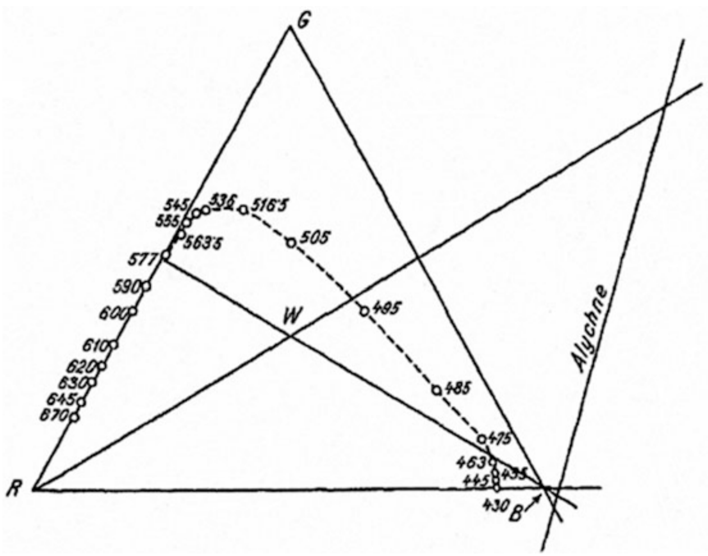

Fig. 9.1 ***König***'s color triangle. (Reproduced from Schrödinger 1925)

two – which through a choice of notation was settled upon as something essential – may have helped obscure this point.

Consider the standard color triangle of ***Helmholtz***'s theory (Fig. 9.1). The proportions of fundamental colors in a color – which we will call its coordinates, designated by x_1, x_2, x_3 – represent, in geometric terms, the projective or barycentric coordinates of a point that denotes the color in question. We choose the 'centre of mass' or barycentre (a geometric convention) of the triangle as the origin of coordinates (in the sense given by projective geometry). Whereas the coordinates of projective geometry only have meaning as relational quantities, chromaticity coordinates have the absolute significance that their sum $x_1 + x_2 + x_3$ specifies the **measure** that one must apportion to a color point, in order to derive the outcome of a color mixture, consistent with the familiar 'centre of mass' construction. A further – and by the way, not entirely necessary – convention which concerns the independently-chosen units of the calibration lights, is that **white** is translated to the centre of mass of the triangle (call this the 'physiophysical' convention).

The three chromaticity coordinates of spectral lights in the diffraction spectrum of terrestrial daylight, distributed as a function of wavelength, are called 'fundamental stimulus curves' or FSC.[(b)] By the way, the relevant color points for an **arbitrary** spectral distribution of intensity lie on the dashed curve of Fig. 9.1.

One can relate the spectral colors, and then naturally, all other colors at the same time, not only to the triangle of reference that has been established (which indeed should be accorded special significance in ***Helmholtz***'s theory, because of investigations carried out with dichromats), but also one can relate them to arbitrarily chosen triangles. As is well known, this is simply a linear homogeneous transformation of coordinates. Of course the form of the FSC is changed, though each new FSC can be superposed on the original curve (with certain constant coefficients). One may ask whether or not the triangle of reference can be given such a form that the obtained FSCs can be pronounced to be ***Hering***'s valence curves. To this end they would need only to satisfy certain **qualitative** constraints, as I have suggested, since **quantitative** findings for the valences of ***Hering***'s theory have not been established.

Hering's valence measures follow this qualitative trend across the spectrum: homogeneous long-wave lights, from the red end of the spectrum up to $\lambda = 575$ μμ (unique yellow) have first an increasing, then a diminishing red valence, and increasing yellow valence besides. Just at unique yellow, red valence crosses over to become green valence. Up to $\lambda = 495$ μμ (unique green) yellow valence diminishes as green valence increases, and from that point up to $\lambda = 472$ μμ (unique blue) green valence diminishes as blue valence increases. From there, red valence again takes the place of green valence, it traverses a secondary peak, and then disappears at the same time as blue valence, at the violet end of the spectrum. Unique yellow and unique blue are complementary colors, that is, they lie on a line on the color triangle through the point assigned to white. There is no physically homogeneous unique red; unique red is represented by extreme spectral red with a small portion of blue added, which makes the mixture complementary to unique green. The '**whiteness**' valence should have the following property across the spectrum: it should yield the empirically observable spectral distribution function for brightness, with regard for a certain'brightening' influence of the valences for red and yellow, and a 'darkening' influence of the valences for green and blue.

One may now look for corresponding wavelengths in ***König***'s color triangle, or better yet in ***König***'s FSCs, in which they are represented accurately as the abscissae of the points of intersection of each pair of FSCs: first of all, that which is complementary to ***König***'s fundamental blue (the long-wavelength intersection of the red and green curves, at approximately $\lambda = 577$ μμ); second, the wavelength that is complementary to fundamental red (the intersection of the green and blue curves, at approximately $\lambda = 497$ μμ); third, the wavelength of fundamental blue (the short-wavelength intersection of the red and green curves, at approximately $\lambda = 469$ μμ). These three wavelengths correspond to the wavelengths specified above for ***Hering***'s unique yellow, unique green, and unique blue. These intersections of the FSCs are fairly precise, as determined by several methods.[(c)] (On the other hand, it is well known that the **psychological** results for a normally functioning eye were of first consideration in establishing ***Hering***'s unique colors – where **yellow** is **pure** yellow, without a tinge of red or green, for example.) – In this way the nearly exact correspondence of ***Hering***'s unique red with ***König***'s fundamental red is obtained, so we may say the following: ***Hering***'s unique colors correspond precisely in hue with fundamental red, fundamental blue, and their complements, while fundamental green (whose wavelength lies very close to that of the complement of fundamental red, by the way) does not have any special role in ***Hering***'s theory.[(d)]

The lines that intersect the two complementary color pairs just named, and which intersect at the position of white, are the lines *RW* and *BW* of Fig. 9.1. If one constructs these on the sides of a new color triangle, there is a direct and unequivocal consequence of the geometrical meaning of projective coordinates: the two new 'FSCs' that are assigned to the new coordinates will exhibit the correct change of sign for the positive ordinate of one (assigned to *BW*) and the negative ordinate of the other. The change of sign is correct in that the FSC will now indicate red and green valences, respectively, in exactly ***Hering***'s sense. Likewise the positive ordinate of the other new FSC (assigned to RW) will indicate yellow valences and its negative ordinate will indicate blue valences. This results from the fact that each projective coordinate of every point is proportional to the perpendiculars drawn from that point to one side of the color triangle, and the sign of that coordinate changes if the point traverses the relevant side of the triangle.

For the moment the choice of a third side for the triangle has not been made. The question is how this line can be drawn so that the new third FSC can be considered a valence for whiteness in ***Hering***'s sense.

Since one would like to have a surveyable procedure by which to choose this third side, let us interpose some steps. We choose the side such that the relevant spectral distribution function for the brightness of the interference spectrum of daylight acts as the new third FSC. According to the comprehensive investigation made by Franz ***Exner***[(e)] into the estimation of the brightness of arbitrary colors from their fundamental stimulus weights (in ***König***'s system), this is both **possible** and exceedingly simple. ***Exner*** says that brightness is expressed through the fundamental stimulus weights $x_1, \ x_2, \ x_3$ by the following homogeneous linear equation:

$$h = \alpha x_1 + \beta x_2 + \gamma x_3 \ ,$$

Exner found that the values of these three coefficients, which are to be interpreted only as ratio values, were:

$$\alpha = 1 \quad \beta = 0.756 \quad \gamma = 0.024 \tag{9.1}$$

Now one only needs to choose for the third side of the new triangle that line which satisfies the following equation relative to ***König***'s triangle:

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0. \tag{9.2}$$

Consequently the third of the new coordinates of this linear expression is indeed proportional to brightness. The line (Eq. 9.2) is constructed in Fig. 9.1, and, for reasons to be discussed shortly, is called the 'alychne' (which means 'lightless').

In ***König***'s triangle, the lines RW and BW satisfy the equations

$$\begin{aligned} x_3 - x_2 &= 0 \\ x_2 - x_1 &= 0. \end{aligned} \tag{9.3}$$

The conversion formulae, which transform the triangle which is constructed from the lines just mentioned, have the following form:

$$\begin{aligned} x'_1 &= a\,(x_3 - x_2) \\ x'_2 &= b\,(x_2 - x_1) \\ x'_3 &= c\,(\alpha x_1 + \beta x_2 + \gamma x_3) \end{aligned} \tag{9.4}$$

Here the choice of a, b, and c is arbitrary at first. It is convenient to fix them by the constraint that

$$x'_1 + x'_2 + x'_3 \equiv x_1 + x_2 + x_3 \tag{9.5}$$

should obtain identically. This leaves unchanged the '**mass**' with which each color enters into the 'centre of mass' construction. The barycentric coordinates x'_i can be applied to the color continuum as depicted in Fig. 9.1, and interpreted in relation to the new color diagram, without need or benefit of distortion of the Figure. The constraint (Eq. 9.5) leads to:

$$\begin{aligned} a &= \frac{\alpha + \beta - 2\gamma}{\alpha + \beta + \gamma} \\ b &= \frac{2\alpha - \beta - \gamma}{\alpha + \beta + \gamma} \\ c &= \frac{3}{\alpha + \beta + \gamma} \end{aligned} \tag{9.6}$$

There obtains, by substitution of Eq. (9.1) into Eq. (9.4) the numeric formulae

$$\begin{aligned} x'_1 &= 0.960\,(x_3 - x_2) \\ x'_2 &= 0.685\,(x_2 - x_1) \\ x'_3 &= 1.685\,x_1 + 1.274\,x_2 + 0.040\,x_3. \end{aligned} \tag{9.4$'$}$$

Before we examine this derived representation more closely, let me append a few remarks about the intriguing line of Eq. (9.2) discussed by ***Exner***, which we call the alychne.

This line is – as may seem somewhat mystical at first reading – the geometrical locus of colors of **vanishing brightness**. Naturally these colors of vanishing brightness are all virtual; the line does not intersect the surface that represents real colors. Two such colors have been chosen as fundamental colors in this representation (Eq. 9.4′); the third fundamental color is white. Actually there is nothing at all mystical about these colors. What it means to add such a color to a mixture cannot only be specified, it can easily be accomplished – for example, as on a color-mixing rotor – more easily than, say, the addition of one of ***König***'s fundamental colors. It simply means a change in hue (or possibly saturation) with brightness held constant. So, for example, if one chooses two colored papers of the same brightness, and combines them in a rotor in various proportions, then all these mixtures will be distinguished solely by their varying content of a particular color situated on the line (Eq. 9.2). The position of this color will be maintained while one brings the line (Eq. 9.2) into intersection with the line which connects the positions of the two colored papers.

As can be seen, the alychne passes quite close to ***König***'s fundamental blue. That is a consequence of the small values for brightness γ of fundamental blue. Now it is common knowledge that the choice of this fundamental blue is pretty much conventional, since the accuracy of spectral calibration allows for a much broader range of error than for fundamental red and fundamental green, and since tritanopia occurs nearly always in severely disordered visual organs, with which sustained and precise experimentation cannot be carried out. In any case, a very small translation of the position of blue, which suffices to translate that point on to the alychne, seems to be in the admissible range. ***König***'s FSCs would then be quite unnoticeably altered, and one can even see to it that the abscissae of their intersections are preserved (which positions, of course, have been confirmed by other methods[(f)]), as one translates the position of blue outwards along the line BW. This modification would have the practical advantage, anyhow, that γ would have a value of zero, that is, one could calculate the brightness of any color from its content of red and green without consideration of its blue content, which as things stand, contributes almost nothing to brightness. Also the determination of this small coefficient of blue has been so inconsistent and unreliable among different observers [***Kohlrausch***[(g)] reports 0.047; ***Ives***[(h)] reports 0.011 both expressed in terms of $\alpha = 1$], that it would not be too risky to appraise these values as error residuals, which indeed could be found to be negative from time to time. From the standpoint of the ***Young-Helmholtz*** theory, the interpretation that the blue sensation can be attributed a brightness that is strictly zero – an interpretation which might as well count as having been demonstrated by experiment – carries considerable significance. The 'blue process' of this theory would then be something to be considered essentially different from the other two processes, insofar as it modifies only the quality of sensation of light and leaves its intensity unchanged.

I do not want to make use of this observation in what follows, so that it should not appear that we must, or want to, do violence to the observed data.

Let us return to formula (Eq. 9.4′). Figure 9.2 depicts the new 'fundamental stimulus curves' that represent these formulae. The FSCs are the loci of the three quantities x'_i as a function of wavelength for the interference spectrum of daylight. Here we use ***König***'s original values for the x'_i. As predicted, the initial two curves

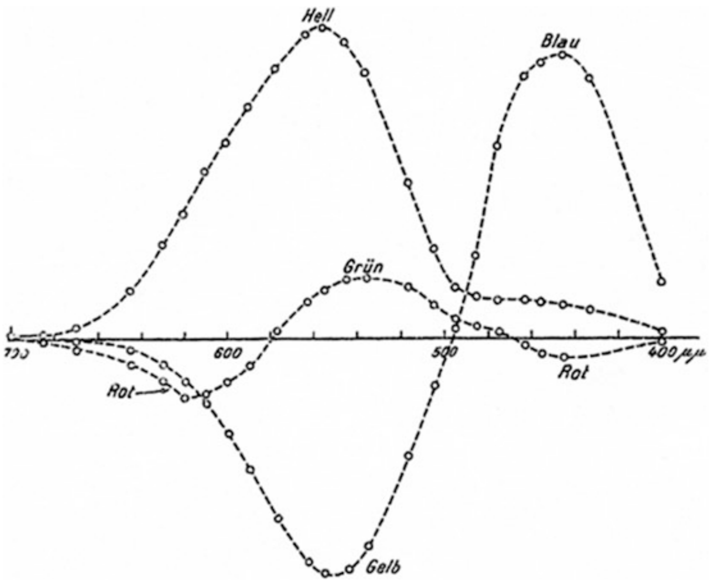

Fig. 9.2 ***Hering***'s valence curves, derived from ***König***'s fundamental stimulus curves. (*Rot*: red; *Gelb*: yellow; *Grün*: green; *Blau*: blue; *Hell*: bright) (Reproduced from Schrödinger 1925)

are an exact portrayal of ***Hering***'s color valences, if one takes the negative ordinate of x_1' to indicate red valence, and the positive ordinate of the same curve to indicate green valence. Further, one may take the negative ordinate of x_2' to indicate yellow valence and its positive ordinate to indicate blue valence. The intersections on the γ axis are unique yellow, unique green and unique blue, while naturally, unique red cannot appear, since it does not occur in the spectrum. The third curve (whose ordinate is presented in the Figure with the ordinate to twice its proper scale, for reasons of space) simply gives the proportion of brightness, and is numerically identical to Franz ***Exner***'s distribution (loc. cit. p. 40, his Table 4 & Fig. 2) which is derived from ***König***'s original data by means of his own calculations[(i)] of the values (which we have used) of proportions of **brightness** (α, β, γ). Still, the values of x_3' **cannot yet** be identified as ***Hering***'s valence for white, because these – according to ***Hering***'s followers – should not determine brightness by themselves. Rather, color valences should have specific brightening or shading effects when they are added in. It would be simple, and possible at least in at least ∞^2 ways, to modify the curves by a second linear transformation, so that the initial two curves could be interpreted as ***Hering***'s color valences, as before, while the third could be interpreted as ***Hering***'s valence for white. It could be interpreted that way with **due consideration** of the putative specific brightness characteristics of the color valences. In other words one might **incorporate** the specific brightness effects into the color valences in a purely formal way. To that end, one needs only to posit

$$\begin{aligned} x_1'' &= -A\,x_1' \\ x_2'' &= -B\,x_2' \\ x_3'' &= x_3' + A\,x_1' + B\,x_2' \\ \text{Brightness} &= x_1'' + x_2'' + x_3'' \ (= x_3') \end{aligned} \tag{9.7}$$

A and B are positive coefficients that are almost entirely arbitrary. Naturally the positive x_1'' are to be interpreted as red valences here – exactly opposite to what we obtained before, and the negative x_1'' are to be interpreted as green valences; similarly with x_2''. Brightness is the sum of trichromatic coefficients; red and yellow have a brightening effect, while blue and green have a darkening effect. (Of course, one could just as easily bring about the opposite by applying a negative sign to A and B. This bizarre interpretation does not lack for proponents in recent literature, either.[(j)])

According to the interpretation originally propounded by ***Hering***'s followers, white valences should be given by scotopic values. Consequently A and B should be determined such that x_3'' coincides with the spectral luminous efficiency function for scotopic vision, which has its maximum at approximately $\lambda = 505$ μμ for the solar interference spectrum. As von ***Kries*** has emphasized strongly,[(k)] **this** conception of the scotopic brightness values, as pure white valence independent of the influence of color valences, is no longer accepted; it is contradicted by the fact that deuteranopes show a Purkinje phenomenon in the large magnitude of 1:100 for certain mixtures that are **photopically matched** and **colorless** to the deuteranope, but without there being a noticeable difference in hue when intensity is diminished. This phenomenon prevents one from describing the strong difference between photopic and scotopic brightness in trichromats merely through the exclusion of color valences. Our present deliberations add something to this argument. On closer inspection it proves to be **impossible** to choose the above-mentioned quantities A and B so that x_3'' represents scotopic brightness. Otherwise it should be possible to construct the scotopic brightness function from linear combinations of ***König***'s original FSCs (with the appropriate positive or negative coefficients). One recognizes almost at first glance that this is impossible, since the maximum of the scotopic brightness function lies at a point at which all three FSCs have relatively small values. However, I have tried to fit the curves, in that I have tried to establish a correspondence at three points: at the peak of the scotopic luminous efficiency curve and at two positions of medium height. In this way, one arrives at completely nonsensical values of the coefficients, which confer very large negative values on other points of the fitted curve.

The only reasonable interpretation of the white valence in the compass of the tetrachromatic theory may be that proposed by von ***Kries***, namely that they are values for peripheral vision.[(l)] It turns out that the disappearance of the quality of hue in eccentric vision hardly involves any change at all in brightness. A clear difference is not found between central or parafoveal chromatic brightness, and peripheral achromatic brightness, provided one light-adapts fully in the determination of the latter, and if one takes into account the coloration of the macula in the case of strict central view.[(m)] That is just the interpretation of our representation (Eq. 9.4′) in which, besides white, two 'zero-brightness' colors on the alychne are chosen as fundamental colors. At this point, there remains no reason to implement further the transformation (Eq. 9.7); the x'_1 that are introduced in (Eq. 9.4′) appear as **the simplest quantitative expression of the tetrachromatic theory**.

This expression has several interesting and serendipitous features. Recently, it has been emphasized by many authors[(n)] how much more expedient it is for the execution of real constructions on the color triangle to use a **right-angled** triangle instead of an equilateral one, perhaps with separate scales for the ordinate and abscissa. Independent Cartesian coordinates for this right-angled triangle can be plotted simply as:

$$\begin{aligned} x &= \frac{x'_1}{x'_1 + x'_2 + x'_3} \\ y &= \frac{x'_2}{x'_1 + x'_2 + x'_3}, \end{aligned} \tag{9.8}$$

while the quotient corresponding to the **third** trichromatic coefficient is given by the sum of trichromatic coefficients, either algebraically as $1 - x - y$, or, geometrically, as the deviation of the color point from the **hypotenuse** [measured in some suitable unit; the hypotenuse is the line segment that extends between the points (1,0) and (0,1) on the coordinate axes].

If we perform these operations on the color coordinates x'_i, we will arrive at Fig. 9.3, in which ***König***'s original fundamental colors are incorporated, as well as the spectral curve. White lies at the origin. The x and y coordinates express content of green, red, blue, and yellow, each according to their sign. The distance of a point from the hypotenuse – which hypotenuse we recognize readily as the alychne – calibrates the brightness of a **unit amount** of the relevant color. As may be recalled, we left this **unit** (i.e. the sum of coordinates) invariant in the transformation (Eq. 9.4′). Therefore it has the same value as in ***König***'s system, and these are the **masses** depicted in Fig. 9.3, which are associated with a color point in the execution of the 'centre-of-mass' construction. – Hence there is a familiar question that can be answered on inspection of the figure: how do the brightnesses of two (simple or compound) lights behave in a complementary mixture? One only has to **divide** the distance of each of the lights from the alychne by its distance from the position of white. If one wants to find two complementary colors of **equal brightness**, one will find that they lie on a **conic section**, which has the position of white as its focus and the alychne as its directrix.

These are relatively trivial consequences, which by the way, already hold true mutatis mutandis for ***König***'s color triangle, once the alychne has been constructed in accord with ***Exner***'s data. The change of geometrical notation is merely meant to achieve this simplification: that the two altitudes of ***König***'s triangle are now perpendicular to one another, and that the alychne, whose construction was comparatively troublesome before, since it required the use of special values of α, β, and

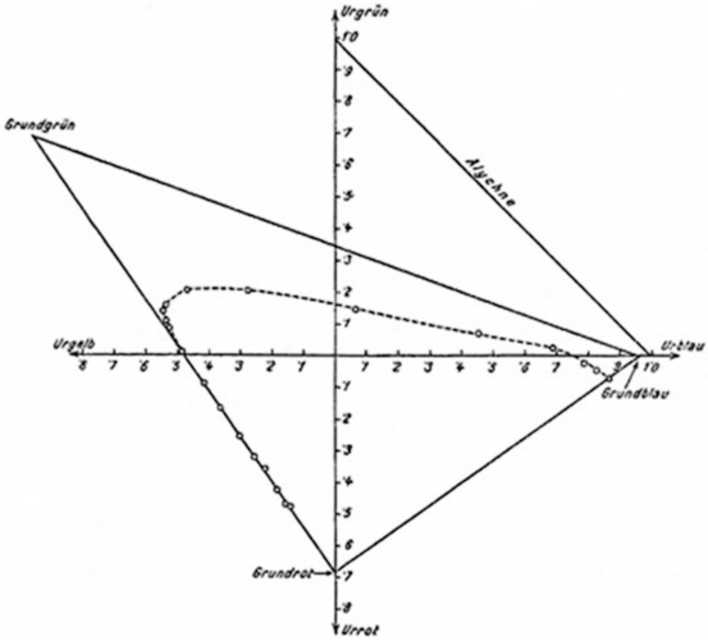

Fig. 9.3 ***Hering***'s color values in terms of units plotted as independent Cartesian coordinates. (*Urgrün*: unique green, *Gründgrün*: fundamental green; *Urgelb*: unique yellow; *Urblau*: unique blue; *Gründblau*: fundamental blue; *Gründrot*: fundamental red; *Urrot*: unique red) (Reproduced from Schrödinger 1925)

γ, now just becomes a 45° line. One might expect that **now *König***'s triangle has lost its simple character as a consequence, since it can only be transposed to the Figure by recourse to the numeric constants α, β, and γ.

Quite remarkably, that is **not** the case. ***König***'s triangle of primaries red, green, and blue (Fig. 9.3) whose vertices lie (according to formula (Eq. 9.4′)) on the points (1,0,0), (0,1,0), and (0,0,1) of the untransformed coordinates, proves to be right-angled. The vertex of its right angle is at fundamental red. This happens purely by chance, one might say, or in consequence of happenstance numeric values of the coefficients of (Eq. 9.4′). These are dependent on ***Exner***'s **empirically**-derived coefficients of brightness α, β, and γ, respectively. In the same way, the legs of this triangle prove to stand in the relation $1{:}\sqrt{2}$ to one another. That is not simply a new-found chance occurrence, however. Rather it is a necessary consequence of the former state of affairs, when one reflects that the x and y coordinates of the vertices of ***König***'s triangle must have the following form (according to Eqs. (9.4), (9.5), and (9.8)):

$$\left(\ 0,\quad -b\ \right)$$

$$\begin{matrix} (-a, & b\ ) \\ (\ \ a, & 0\ ) \end{matrix}$$

And one may add that the **particular** values of α, β, and γ only specify these quantities:

$$a = 0.960$$
$$b = 0.685$$

Once this is recognized, one may consider that if a right angle should occur at fundamental red, then the two right triangles that lie under the x-axis, which are components of the larger right triangle whose hypotenuse lies on the x-axis, must be **similar** both to the larger triangle and to **each other**. (Conversely, if these two triangles are similar to **each other**, then a right angle occurs at fundamental red, and the two triangles are similar to the larger triangle of which they are the components.) Then the legs of these two smaller triangles stand, in absolute value, at

$$\begin{matrix} a/2, & b & \text{on the left} \\ b\quad, & a & \text{on the right} \end{matrix}$$

This similarity then implies:

$$a = b\sqrt{2}$$

This relation obtains in the above-mentioned numeric values to an accuracy of less than one percent. Then the smaller (right-hand) leg of ***König***'s triangle is:

$$\sqrt{a^2 + b^2} = b\sqrt{3}$$

and its larger (left-hand) leg is

$$2\sqrt{b^2 + (a^2/4)} = b\sqrt{6}$$

König's triangle is therefore similar to all the aforementioned triangles; likewise it has the proportion of sides $1{:}\sqrt{2}$, as was claimed.

If one identifies the legs of ***König***'s triangle with unit scales of mass on the abscissa and ordinate of a right-angled coordinate system that is formed by those legs, then the Cartesian coordinates of color points have the following interpretation relative to that coordinate system:

$$\frac{x_3}{x_1 + x_2 + x_3},$$
$$\frac{x_2}{x_1 + x_2 + x_3},$$

which are ***König***'s values of blue and his value of green expressed in terms of the sum of coordinates, respectively. The corresponding quotient for ***König***'s value of red is easily recovered as the sum of coordinates by subtracting both these quantities from 1 (or alternatively, by distance from the hypotenuse of ***König***'s triangle).

Our representation achieves the following, then: it depicts color valence according to the ***Helmholtz-König*** trichromatic system and color valence according to ***Hering***'s tetrachromatic system in **one** diagram. It even depicts **both** in their simplest form, the form most convenient for practical geometric constructions. In this form, ***Hering***'s coordinate axes are simply the altitudes of ***König***'s right-angled triangle. At the same time, this presentation brings the brightnesses of all (simple and compound) lights – according to ***Exner***'s measurements – to an immediately perspicuous form. The very possibility of this theoretical consolidation depends upon the empirical values of the specific brightness function for the fundamental colors, as they have been determined by ***Exner*** and, above all, on the relationship between the brightnesses of green and red (as we shall see in a moment).

In order to construct this twin system of coordinates, still **another** particular datum is required besides those already ascertained – for example, that fundamental blue must have its position at $x = 0.960$. This necessity would have been transcended, had ***Exner***'s value for γ been exactly zero, instead of 0.024, as it is. For in that case fundamental blue would lie on the alychne, and consequently be found at $x = 1$. I have already alluded to the fact that no experimental observation bars such a small translation of fundamental blue (which is accompanied by a small conversion of ***König***'s red and green curves). I have attributed almost no consequence to this remark, in order not to create the suspicion that the desired simplification depends on this step. I will not use **this** method even now, but I will use a still simpler method. It appears to me that the real but small value of γ, which has been derived from ***Exner***'s experiments with color rotors, is still not a sufficiently secure reference point for any conjecture how fundamental blue might be translated in order to produce a better approximation to a brightness-free color, as we have determined it to be already. In practise the insignificant brightness of blue, which has been used conscientiously in calculations until now, proves to be so unimportant that there is hardly an experimental result that could be used to claim definitively that it would be a mistake simply to omit it. The daylight brightness function that has been calculated by ***Exner*** is modified extremely little by exclusion of the brightness of blue, much less than the not wholly insignificant deviation of the theoretical from the obtained brightness function that is found at the short-wave end of the spectrum (in addition, this latter deviation is diminished by the exclusion of the brightness of blue!). If with these most saturated of blue hues, contribution of the blue content to brightness cannot be proved with certainty, it can also be observed how much less this is the case with other colors. We have already intimated above that ***Ives*** finds a value for blue that is less than half as large as that which ***Exner*** finds; in contrast we may add that they are completely in accord in reporting the important value for green – ***Ives*** has this as 0.750 and ***Exner*** has it as 0.756 (the value of red is constant and set exactly to one, as above).

Shortly put, we think ourselves justified for the moment in assigning $\gamma = 0$, even **without** compensating for this by modification of the FSC. The numeric coefficients in equation (Eq. 9.4′) become:

$$\begin{aligned} x_1' &= x_3 - x_2 \\ x_2' &= 0.708\ (x_2 - x_1) = \frac{1}{\sqrt{2}}\ (x_2 - x_1) \\ x_3' &= 1.708\, x_1 + 1.292\, x_2 = \left(1 + \frac{1}{\sqrt{2}}\right) x_1 - \left(2 - \frac{1}{\sqrt{2}}\right) x_2 \quad , \end{aligned} \tag{9.9}$$

The mutual perpendicularity of the two coordinate axis pairs of Fig. 9.3 is not only maintained, but becomes still more exact, in that the $\sqrt{2}$ ratio holds with greater accuracy for the new coefficients.

$$1 : 0.708 = 1.41\ldots = \sqrt{2}$$

If one poses the question more closely how this relation might come about, with the aid of Eqs. (9.4) and (9.6), one finds for ($\alpha = 1$, $\gamma = 0$) that:

$$\frac{2 - \beta}{1 + \beta} = \frac{1}{\sqrt{2}}$$

or, numerically,

$$\beta = 5 - 3\sqrt{2} = 0.75736\ldots \quad (\text{observed} : 0.756).$$

We may attribute the slightly errant value for the observed brightness of green to rounding error.

All previously established conventions **now become completely superfluous** to the production of this new diagram (the old ones are troublesome, despite their apparent simplicity, since they must always be established numerically for some 20 points along the spectrum). The procedure is as follows: one plots color coordinates for the spectral colors, or any colors of interest, in a right triangle scaled to the dimensions 1:$\sqrt{2}$, and puts the relative **value of blue** (that is, the value divided by the sum of coordinates) as the abscissa, and the relative **value of green** as the ordinate. Then one erects a pair of mutually perpendicular and intersecting altitudes on this triangle; for both of these the distance between the positions of white and blue is taken to be unity. Finally one draws ***Exner***'s line through the

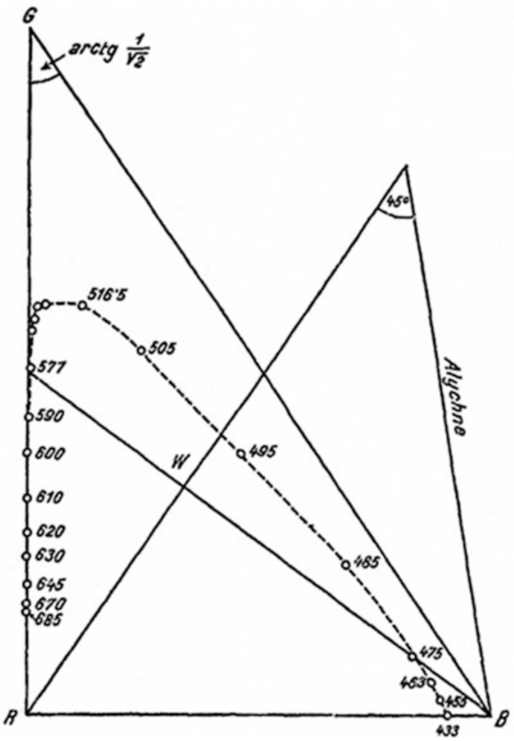

Fig. 9.4 The direct construction of Fig. 9.3 (largely unsimplified) from ***König***'s untransformed data. (Reproduced from Schrödinger 1925)

position of blue, inclined at 45° to the two altitude lines. This Fig. 9.4 expresses the ***König-Helmholtz*** valences, ***Hering***'s valences, **and** the brightnesses of colors all in one, in the same way as in Fig. 9.3. The difference of the two figures should lie in the degree of precision that is attained with the data which have been employed to date.

I would like to leave the matter at that provisionally, and not deliberate on whether the **quite special** simplification of Fig. 9.4 – through the inclusion of the particular ***Exner-Ives*** numeric value for the relative brightness of green – bears deeper meaning for the understanding of the visual process. It ought to be mentioned that ***Kohlrausch*** (loc. cit.) has found a fairly deviant numerical value for the brightness of green, namely 0.618 (compare 0.756) and too, that ***Abney***[(o)] has claimed large individual differences in the coefficients, though these latter, to be sure, are not derived from wholly unobjectionable methods. These methods have led him to the paradoxical claim that **anomalous trichromacy** may consist merely in the abnormal relation of the brightnesses of red and green. (The truth is that it is necessary on a logical basis that coefficients of brightness can have no influence on **precise matches** of colors, that is, if it is experimentally demonstrated that two individuals fail to recognize **each other's** color matches,[(p)] then the adequate theoretical expression of this fact can **never** consist **merely** in a difference of those brightness coefficients.)

With **another** value of the coefficient for green, that is, another than that given by ***Exner*** and ***Ives***, one is always able to arrange the construction of Fig. 9.4 such that the altitudes of the right-angled ***König*** triangle cross each other at a right angle. One can easily assign them the same proportion of $\sqrt{2}$:1 but then ***Hering***'s two axes do not have the same scale, and in consequence the alychne is no longer a 45° line with respect to ***Hering***'s axes.

With the same reserve respecting an interpretation that might run deeper, I would like to allude to a further remarkable circumstance, namely that in Fig. 9.4 the descending branch of the spectral curve nestles extremely closely to the hypotenuse of a right **isosceles** triangle, which has *RB* as its horizontal base.

To return to the argument, it appears to me of undoubted importance for a deeper understanding of the visual system to know that **fundamental** blue is a **brightness-free color** – be it as it has been assumed here, or be it replaced by another that has as much justification for its position. If this is right, then the three 'components' of the visual system may **not be all of the same kind**, and the representation suggested by theory, that the three components play a **symmetric role**, is invalidated. This could be noteworthy in the search for a physiological substratum for these visual functions.

II.

Let us now consider briefly another perspective on the relation of the trichromatic theory to the opponent-process (tetrachromatic) theory. For the moment I will forebear from drawing direct reference to the graphical construction which has just been developed.[(q)]

The principal contrast between these two conceptions had initially been presented as follows: according to **one**, **white** and **yellow** are presented as psychological elements that are almost equipotent (in the case of white) or completely equivalent (as for yellow) to the three other colors (namely red, green and blue). By contrast, according to the **other**, white and yellow represent only the simplest of mixtures: respectively, 1:1:1 (by definition) and 1:1:0 (empirically).

Now if one considers the phylogenetic origins of a light-perceiving organ, one arrives at an almost self-evident conjecture: that in its earliest origins its function would be restricted to detecting **some** sort of electromagnetic radiation, but naturally only radiation of a **restricted** range of frequencies. That much is clear from the enormous differences in physical effects of radiation ranging from radio waves to gamma rays. A **second** stage of adaptation could be considered, should the organ start to react differently – in a qualitative way – to different frequencies **within** this range. In this respect a course of development like that of the ear is ruled out in advance – in which 'separate' or distinct responses develop as finely graded steps for each small range of frequencies. That is ruled out for a **light**-sensitive organ, because it is never exposed to anything like pure frequencies under natural conditions. (If somehow thin layers of self-luminous gases were to play an important biological role, the situation might be different.) Then since preconditions are lacking for the development of a capacity to discriminate individual frequencies by separate mechanisms, another course is more likely. That is the development of a summary capacity to discriminate, where any **preponderance** of either short-wave or long-wave components (compared to the 'normal' makeup of light, i.e., sunlight) becomes a cue, a distinctive feature of sensation. This cue forms the blue-yellow scale, whose balance point is neutral white. The elemental simplicity of this scale is not lost as **deviations** from the normal gradually acquire the character of sensations themselves, in turn. The end state is **dichromacy**, which we observe in people with partial colorblindness, in the peripheral retinas of color-normals, and apparently also in many animals (e.g. insects).

In full analogy to this step, another step leads to the state of **trichromacy**. The **division** of scale that had been based on the preponderance of short wavelengths or long wavelengths, and which had been applied to the entire visible spectrum, is replicated so that the division is applied solely to the range of long-wavelength lights. Then yellow is sundered into red and green, as white had sundered into blue and yellow. Yellow does not stand to lose its essential chromatic quality in this new differentiation, just as white did not surrender its quality as a color before. Yellow is to the pair red and green what white is to the color pair blue and yellow: that is, the neutral point of their transition one to another.

This representation of a succession in development of the color sense is not something to be proven by strictly quantitative methods. Yet to me it seems to reveal the roots of the controversy surrounding the 'elemental' nature of white and yellow, and their role as fundamental sensations. White and yellow really **are** fundamental as sensations – sensations not of recent but of ancient origin. One emerged from a monochromatic stage, and the other from a dichromatic stage. Among the fundamental sensations which remain 'undifferentiated', one (blue) emerged from the stage of dichromacy. The other two (red and green) are the most recently acquired. This explains why the latter are more subject to impairments and 'regression'. It also explains why a disturbance of the 'blue-sense' does not occur in isolation as an anomaly of physiology; under the trichromatic theory this should be as likely as any other. The line of descent or phylogeny evinces no such stage of development. The failure of a 'blue-yellow' process is regularly accompanied by failure of the 'red-green' process; this presents itself as complete congenital colorblindness. In fact cases have been discovered for which this visual condition is **not** based on rod cells, but on a degeneracy or as one might say, on an atavism (a reversion to earlier type) of the photopic mechanism.[r]

If this interpretation is right, the peripheral region of the retina can almost be said to present a map of ancestral modes of vision, with the oldest forms lying furthest out in the periphery. Given the biological importance of foveal vision, it seems entirely reasonable that more recent differentiation would originate in the fovea, and then radiate gradually to the periphery.

Although the present work is of a purely theoretical nature, I would like to take this opportunity to offer my deepest thanks to the **Stiftung für wissenschaftliche Forschung at the University of Zürich**. Through its generosity, it has been possible for me to engage in the experimental study of several problems in the domain of color. This circumstance has been the stimulus for the present theoretical investigation as well.

Notes

a. Compare, for instance ***Kries***, J. von. 3. Gesichtsempfindungen, XII. Übersicht der Tatsachen, Ergebnisse für die theoretische Auffassung des Sehorgans: Zonentheorie. In: Wilibald ***Nagel***, Ed. Handbuch der Physiologie des Menschen: Physiologie der Sinne, **3**(1). Braunschweig: Vieweg und Sohn, 269 – 274 (1905). Also as ***Kries***, J. von. Die Gesichts-Empfindungen und ihre Analyse. Leipzig: Viet & Comp., 178 Seite (1882). [Gelangt gleichzeitig als Supplementband – Band 2 – zur physiologischen Abtheilung des Jahrganges 1882 des Archives für Anatomie und Physiologie zur Ausgabe.] This foundational article will be cited as "J. v. ***Kries***, GE" in what follows. [ff. 1, p. 472 original]

b. ***König***, A. & ***Dieterici***, C. Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsverteilung im Spektrum. [Fundamental stimuli of normal and anomalous color systems, and their intensity distributions across the spectrum] Zeitschrift für Psychologie und Physiologie der Sinnesorgane, **4**, 241 – 347 (1893). ***König***, A. Gesammelte Abhandlungen. Leipzig: Johann Ambrosius Barth (1903). [ff. 1, p.473, original]

c. Cf. Reports such as: ***Exner***, F. Über die Grundempfindungen im ***Young-Helmholtz***'schen Farbensystem. Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **111**, 857 – 877 (1902). ; ***Steindler***, O. Die Farbenempfindlichkeit des normalen und farbenblinden Auges. Op. cit., **115**, 39 – 62 (1906). Op. cit. ***Richtera***, L. **122**, p. 1915 (1913) & ***Hauer***, F. **123**, p. 624 (1914). [ff. 1, p. 474 original]

d. Also see: A. ***König*** u. C. ***Dieterici***, in ***König***, A. Gesammelte Abhandlungen. Leipzig: Johann Ambrosius Barth. footnote to p. 317 (1903). [ff. 1 , p. 475 original]

e. ***Exner***, F., these **Reports**: Einige Versuche und Bemerkungen zur Farbenlehre. [Some experiments and remarks on color theory] Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **127**, 1829 – 1864 (1918). ; ***Exner***, F. Zur Kenntnis der Grundempfindungen im Helmholtz'schen Farbensystem. [Towards a characterization of the fundamental stimuli in Helmholtz's color system] Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **129**, 27 – 46 (1920). [ff. 2, p. 475 original]

f. See the citations above (F. ***Exner***, O. ***Steindler***, etc.). [ff. 1, p.477 original]

g. ***Kohlrausch***, F.W.F. Beiträge zur Farbenlehre II. Die Helligkeit der Pigmentfarben. [Articles on color theory II. Brightness of pigment colors] Physikalische Zeitschrift, **21**, 423 – 440 (1920). [ff. 1, p. 478 original]

h. ***Ives***, H.E. The transformation of color mixture equations from one system to another. II. Graphical aids. Journal of the Franklin Institute, **195**(1), 23 – 44 (1923). [ff. 2, p. 478 original]

i. ***Exner***, F., these Reports: Zur Kenntnis der Grundempfindungen im ***Helmholtz***'schen Farbensystem. [Towards a characterization of the fundamental stimuli in Helmholtz's color system] Sitzungsberichte der Kaiserliche Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse, Abteilung 2a: Mathematik, Astronomie, Physik, Meteorologie, und Mechanik, **129**, p. 40ff., the inscribed curves of Table IV and Figure 2 (1920). Concerning the **additional** corrections to ***König***'s FSC which ***Exner*** applies: I am not able to make use of them. The **reason** is that some flaws are apparent in the resulting spectral curve of the color diagram (such as the appearance of concavities in the wrong direction, meaning outwards). [ff. 1, p. 479 original]
j. ***Fröhlich***, F.W. Grundzüge einer Lehre vom Licht- und Farbensinn. Jena: Gustav Fischer, pp. 46, 51 & 82 (1921). [ff. 1, p. 480 original]
k. J. v. ***Kries***, GE, p. 192ff. [ff. 2, p. 480 original]
l. J. v. ***Kries***, GE, p. 203. [ff. 3, p. 480 original]
m. Cf. the discussion in v. ***Kries***, J. Zeitschrift für technische Physik, **5**, 327 & 340 (1924). [ff. 1, p. 481 original]
n. Journal of the Optical Society of America and Review of Scientific Instruments, **6**, p. 527, August 1922. – ***Guild***, J. The transformation of trichromatic mixture data: algebraic methods. Transactions of the Optical Society, **26**(2), 95 – 108 (1924). ; ***Guild***, J. The geometrical solution of colour mixture problems. Transactions of the Optical Society, **26**(2), 139 – 174 (1924). [ff. 2, p. 481 original]
o. ***Abney***, W. de W. Colour blindness and the trichromatic theory of colour vision. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), April 14th, **83**(565), 462 – 473 (1910). ; ***Abney***, W. de W. - Colour-blindness and the trichromatic theory of vision. Part II. Incomplete red or green blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 15th, **84**(572), 449 – 464 (1910). ; ***Abney***, W. de W. Colour-blindness and the trichromatic theory of colour vision. Part III. Incomplete colour blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), December 22nd, **86**(583), 42 – 56 (1911). ; ***Abney***, W. de W. Colour-blindness and the trichromatic theory of colour vision. Part IV. Incomplete colour-blindness. Proceedings of the Royal Society London A (Mathematical, Physical, & Engineering Sciences), October 2nd, **87**(596), 326 – 330 (1912). [ff. 1, p. 486 original]
p. As is well-known, this is the case for normal and anomalous trichromatic observers. [ff. 1, p. 487 original]
q. See also: ***Schrödinger***, E. Über den Ursprung der Empfindlichkeit des Auges. Die Naturwissenschaften, **12**(45), November, 925 – 929 (1924). [ff. 1, p. 488 original]
r. ***Exner***, F., these **Reports** (2a), **131**, p. 636 (1922). [ff. 1, p. 490 original]

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