On the Measurement of the Speed of Light in a Cavity

Fabienne Schneiter

Introduction

How precisely do we know the value of the speed of light nowadays? Adopting the current definition of the SI units [1], we would simply say that the speed of light is constant and has the value $c = 299\,792\,458\ \frac{\mathrm{m}}{\mathrm{s}}$. In these units, the second is defined using transition properties of the caesium atom, and the meter is defined by the distance a light pulse travels in a certain amount of time with the speed of light set to the above value.

In this article however, which is based on [2], we work with units for distance and time that are defined independently of the speed of light. We want to measure the speed of light in a certain region of space and for a certain period of time. The measurement is done through the frequency and the wavelength of the light, thus implicitely using the definition of the units for distance and time. Although we need to assume that if the speed of light was not constant its variation would be negligible in the region of space and period of time we consider, we do not assume that the speed of light is constant everywhere and at every time. Performing the measurement at different places or at different times thus allows to verify if the speed of light actually takes the same value everywhere and at every time. If we measured the speed of light assuming that it is the constant parameter c as it appears in modern theories, we could infer it (possibly more precisely) by measuring other quantities—but this is not what we do in this calculation. Our approach can be considered as the measurement an observer does who does not want to rely on any theory and makes his setup in an according way. Looking at his procedure in the framework of quantum mechanics and general relativity, we analyse the errors he makes according to these theories. Assuming that quantum mechanics and general relativity are true, we thus set bounds on the precision of the measurement of an observer who does

F. Schneiter (✉)
Institut für Theoretische Physik, Eberhard-Karls-Universität Tübingen, Tübingen, Germany
e-mail: fabienne.schneiter@uni-tuebingen.de

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his measurement without using these theories, and thus implicitely set bounds on the testability of theories predicting deviations from pure general relativity or pure quantum mechanics, such as some approaches to quantum gravity.

For the measurement, we consider a cubic cavity with reflecting walls containing light. The wavelength of the light is given by the length of the cavity. We measure the frequency of the light at the wall of the cavity and determine its speed according to $c = \frac{\omega\lambda}{2\pi}$. How precisely can we measure this speed? When one wants to measure a quantum mechanical observable, as for example the momentum, the Heisenberg uncertainty relation states that the uncertainty of the observable scales as one over the uncertainty of the conjugate variable, which for the momentum is the position. When we now want to know the uncertainty in the measurement of the speed, we cannot simply use the Heisenberg uncertainty relation, since the speed is not a quantum mechanical observable. What we can do, however, is to estimate its uncertainty using quantum parameter estimation theory. Doing so, we find that it scales as one over the energy inside the cavity. However, when there is a lot of energy inside the cavity, we have to be careful with what we actually measure, since we are not dealing with a vacuum anymore. Determining the speed of light according to $c = \frac{\omega\lambda}{2\pi}$ and believing to be measuring the speed of light in vacuum, one makes a systematic error: Due to the energy inside the cavity, there is a gravitational field, which leads to a change of the frequency of the light, the gravitational redshift. This systematic error is proportional to the energy inside the cavity. Altogether, what we will call in the following the most accurate measurement of the speed of light in vacuum is a measurement for which the uncertainty of the quantum mechanical measurement and the systematic error are of the same order of magnitude. Since the former is inversely proportional and the latter proportional to the amount of energy inside the cavity, there exists a certain amount of energy as a function of the other parameters of the measurement which is optimal to perform the measurement. This optimal amount of energy can be obtained if one takes the light to be in a corresponding quantum state.

Quantum Parameter Estimation Theory and the Quantum Mechanical Uncertainty

Since quantum mechanically, we cannot measure a speed directly, we perform measurements of quantum mechanical observables (actually even more general measurements) and use these to estimate the value of the speed. Optimizing over all measurements leads to the minimal quantum mechanical uncertainty in the estimation procedure of the speed. This is done in quantum parameter estimation theory, which works as follows: Consider a quantum system that depends on a parameter, in our case the speed of light c. We describe the state of this system by the density matrix $\hat{\rho}(c)$. Performing M measurements on the system, we obtain the empirical data $\{x_1, x_2, ..., x_M\}$. Using this data, we find an estimate $c_{\text{est}}(x_1, x_2, ..., x_M)$ of the real value c, depending on the results of the measurement. To know the precision of the measurement, we need to know how close the estimate c_{est} is to the actual value c. Making the reasonable assumption that for many measurements, the expectation value of the estimator c_{est} is equal to the parameter c, the precision of the measurement corresponds to the standard deviation of the estimator c_{est}. A lower bound which is optimized over all estimators and all measurements for this standard deviation is given by the Cramér-Rao Lower Bound (CRLB) [3]

$$\delta c_{\text{est}} \geq \frac{1}{\sqrt{M F_Q(c)}} , \tag{1}$$

where $F_Q(c)$ is the Quantum Fisher Information (QFI). The QFI is a measure for the sensitivity of the quantum state on the parameter: If a small change of the parameter results in a big change of the state, the QFI is high, and if it induces only a small change of the state, the QFI is low (see Fig. 1). Intuitively this explains the statement of the CRLB, as when the state is very sensitive on the parameter (big $F_Q(c)$), the parameter is more easily measurable (small standard deviation of the estimator).

Let us now find the CRLB for our measurement of the speed of light. Our system is described by the Hamiltonian

$$\hat{H} = \sum_{m=0}^{\infty} \hbar \omega_m \hat{n}_m , \tag{2}$$

where ω_m is the frequency and $\hat{n}_\omega$ the number operator. We assume that the Hamiltonian is bounded, which is equivalent to claim that the total energy in the system is finite. It turns out that the CRLB depends only on the possible minimal amount of energy inside the cavity, which is zero, and the possible maximal amount of energy inside the cavity. Therefore we can choose that all photons have the same energy, i.e. the same frequency ω. We call the number of photons that gives the maximal amount of energy $n_{\max}$. The QRLB for this system leads to the minimal standard deviation

$$\frac{\delta c_{\text{CRLB}}}{c} \sim \frac{1}{t n_{\max} \omega \sqrt{M}} \quad \sim \quad \frac{1}{tc\sqrt{M}\frac{n_{\max}}{\lambda}} , \tag{3}$$

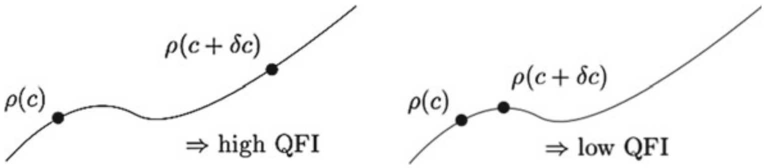

Fig. 1 The QFI is a measure for the sensitivity of the quantum state on the parameter. If the state is very sensitive on the parameter, it changes a lot when the parameter is changed by only a little, and the QFI is high (*left image*). When the state is barely sensitive on changes of the parameter, the QFI is low (*right image*)

where t is the duration of the measurement. The state for which this minimal standard deviation is achieved turns out to be the superposition of the states with minimal and maximal energy [4],

$$\left|\psi_{\text{opt}}\right\rangle = \frac{|0\rangle_\omega + |n_{\max}\rangle_\omega}{\sqrt{2}} \ . \tag{4}$$

The Gravitational Field of a Light Field Inside the Cavity and the Systematic Error Due to Gravitational Redshift

Once there is light inside the cavity, we are not in vacuum anymore. There is energy inside the cavity, and this energy leads to a gravitational field. We use the semi-classical approximation of general relativity [5], since we treat the light quantum mechanically and the gravitational field classically. To make the Einstein equations in this formalism meaningful, one takes the quantum mechanical expectation value of the energy-momentum tensor of the light, $\hat{T}_{\alpha\beta}$. Then the Einstein equations read

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi G \left\langle \hat{T}_{\alpha\beta} \right\rangle \ , \tag{5}$$

where $g_{\alpha\beta}$ is the metric, $R_{\alpha\beta}$ the Ricci tensor, R the Ricci scalar and G Newtons constant. On the right-hand side of these equations stands the energy, and on the left-hand side terms describing the curvature of the spacetime and thus the gravitational field. Altogether, this equation tells us how energy induces a gravitational field. Since we deal with very small energies, we use the linearized theory of gravity [6]: We make the ansatz that the metric $g_{\alpha\beta}$ equals the Minkowski metric $\eta_{\alpha\beta}$ for the flat spacetime plus a small perturbation $h_{\alpha\beta}$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ , \tag{6}$$

where $|h_{\alpha\beta}| \ll 1$ ensures that the deviation from the flat spacetime is small. In other words, this equation is valid if the gravitational field is very weak. The Einstein equations lead to (in transverse-traceless gauge)

$$h_{\alpha\beta}(\vec{y}) = \frac{4G}{c^4} \int_0^L d^3y \, \frac{\left\langle \hat{T}_{\alpha\beta}(\vec{y}) \right\rangle}{|\vec{x} - \vec{y}|} \ . \tag{7}$$

Using this formalism, we calculate the frequency an observer measures at the wall of the cavity. Because of the gravitational redshift [6], i.e. the different frequencies an observer in the gravitational field and an observer in a space without a gravitational field measure, the observer at the wall of the cavity will measure a frequency which deviates from the frequency an observer in vacuum would measure. This deviation turns out to be

$$\delta\omega = \frac{h_{00}}{2}\omega \, . \tag{8}$$

Since the observer wants to measure the speed of light in vacuum, i.e. without any gravitational field, this deviation is a systematic error in his measurement. In terms of the measurement of the speed of light, it is found to be

$$\frac{\delta c_{\text{err}}}{c} \sim \frac{\hbar G}{c^3 L}\frac{n_{\max}}{\lambda} \, . \tag{9}$$

Minimizing the Quantum Mechanical Uncertainty Plus the Systematic Error

We found that the minimal quantum mechanical uncertainty scales as

$$\frac{\delta c_{\text{CRLB}}}{c} \sim \frac{1}{\sqrt{M t c \frac{n_{\max}}{\lambda}}} \, . \tag{10}$$

δc_{CRLB} can thus be lowered by

- increasing the number of measurements M
- increasing the measurement duration t
- increasing the energy (increasing the ratio $\frac{n_{\max}}{\lambda}$)

On the other hand, we found that the systematic error due to the gravitational redshift scales as

$$\frac{\delta c_{\text{err}}}{c} \sim \frac{\hbar G}{c^3 L}\frac{n_{\max}}{\lambda} \, , \tag{11}$$

and we see that $\delta c_{\text{err.}}$ can be lowered by

- increasing the size of the cavity L
- decreasing the energy (decreasing the ratio $\frac{n_{\max}}{\lambda}$)

By increasing the number of measurements or the measurement duration (and keeping the other parameters constant), we can make the quantum mechanical uncertainty of the measurement arbitrarily small, but without affecting the systematic error, which corresponds to a shift of the measured value. One can thus think of the measurement outcomes in this case as being close to a value which deviates from the actual value. On the other hand, increasing the size of the cavity (and keeping the other parameters constant), we can make the systematic error arbitrarily small, but not the quantum mechanical uncertainty. This case thus corresponds to measurement outcomes that are centered around the actual value of c, but possibly spread widely. Altogether, increasing at the same time the number of measurements or the measurement duration as well as the size of the cavity, one can make the quantum mechanical uncertainty of the measurement as well as the systematic error arbitrarily small.

Contrarily, since δc_{CRLB} is inversely proportional and δc_{err} is proportional to the energy inside the cavity, there must exist a certain amount of energy that minimizes the sum of both errors for given values of the length of the cavity, the number of measurements and the measurement duration (Fig. 2).

Equating the minimal uncertainty δc_{CRLB} and the systematic error δc_{err}, we find that the optimal amount of energy corresponds to the optimal ratio of number of photons per wavelength

$$\left(\frac{n_{\max}}{\lambda}\right)_{\text{opt}} \sim c\sqrt{\frac{L}{\hbar G t\sqrt{M}}}\,. \tag{12}$$

Inserting this into Eqs. (10) or (11) leads to the minimal measurement uncertainty, and thus best precision

$$\frac{\delta c_{\min}}{c} \sim \frac{1}{c^2}\sqrt{\frac{\hbar G}{L t\sqrt{M}}}\,. \tag{13}$$

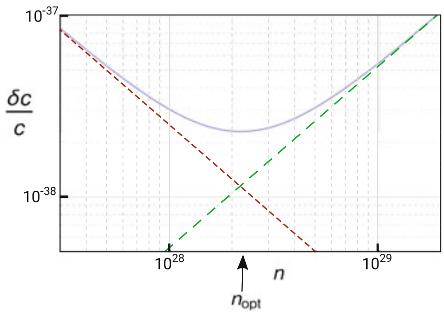

Fig. 2 The minimal uncertainty $\frac{\delta c_{\text{CRLB}}}{c}$ (*short-dashed red line*, Eq. (3)) and the systematic error $\frac{\delta c_{\text{err}}}{c}$ (*long-dashed green line*, Eq. (9)) as a function of the number of photons n: The sum of both of them is shown by the *plain grey line*, and one sees that the number of photons minimizing it lies at the intersection of the curves for the minimal uncertainty and the systematic error. For the plot we chose the wavelength $\lambda = 5 \cdot 10^{-7}$m, the measurement duration $t = \frac{L}{c}$, the length of the cavity $L = 1$m and the number of measurements $M = 10^6$

Conclusion

We consider an observer who has units for time and length that are defined independently of the speed of light. He determines the speed of light in these units by measuring the frequency of the light inside a cubic cavity and calculating the speed of light through $c = \frac{\omega\lambda}{2\pi}$. The minimal uncertainty in his measurement scales as $\frac{\delta c_{\min}}{c} \sim \frac{1}{c^2}\sqrt{\frac{\hbar G}{Lt\sqrt{M}}}$. For a cavity of sidelength $L = 1$m, finesse $F = 10^4$, for a measurement duration of $t \sim \frac{LF}{c}$, and $M = 10^6$ repetitions of the measurement, the minimal uncertainty scales as $\frac{\delta c_{\min}}{c} \sim 10^{-38}$. In an experiment, any additional noise or error taken into consideration will lead to a bigger uncertainty of the measurement, but not invalidate the lower bound we found.

Typically, the light used in an experiment will be in a coherent state, which is defined as $|\psi_{\text{coh}}\rangle_\omega = \exp\left(\alpha\hat{a}_\omega^\dagger - \alpha^*\hat{a}_\omega\right)|0\rangle_\omega$. Calculating the minimal uncertainty given by the the CRLB and the systematic error for a coherent state of a given average excitation number and comparing them, we find that the minimal uncertainty scales as $\frac{\delta c_{\min}}{c} \sim \left(\frac{\hbar G\lambda}{Lc^5t^2M}\right)^{\frac{1}{3}}$. For the same parameters as we used in the numerical example for the optimal state and with $\lambda = 10^{-6}$m, one obtains $\frac{\delta c_{\min}}{c} \sim 10^{-30}$.

Instead of assuming that we have units for time and length that are defined independently of the speed of light and use them to determine the speed of light, we can also proceed in the more modern way, consider the speed of light to have a fixed value and use it to define the unit for distances. Then we obtain in the same way a minimal uncertainty for a measurement of a distance, $\frac{\delta L_{\min}}{L} \sim \frac{\delta c_{\min}}{c}$.

The order of magnitude of $\delta c_{\min}$ or $\delta L_{\min}$ can be used to estimate whether a quantum effect will be measurable or not with this setup and certain values for the size of the cavity, the number of measurements and the measurement duration. For example, if a theory predicts values for quantum fluctuations of a length well below $\delta L_{\min}$, it can, from a purely theoretical point of view, never be detected.

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References

1. Bureau International des Poids et Mesures (2016). http://www.bipm.org/en/measurement-units/
2. Braun, D., Schneiter, F., Fischer, U.R.: How precisely can the speed of light in vacuum be determined in principle? Class. Quantum Grav. **34**, 175009 (2017)
3. Braunstein, S.L., Caves, C.M.: Statistical Distance and the Geometry of quantum States. Phys. Rev. Lett. **72**, 3439–3443 (1994)
4. Braun, D.: Ultimate quantum bounds on mass measurements with a nano-mechanical resonator. Europhys. Lett. **94**, 68007 (2011) (Erratum. Europhys. Lett. **99**, 49901 (2012))
5. Ford, L.H.: Gravitational radiation by quantum systems. Ann. Phys. **144**, 238–248 (1982)
6. Charles Misner, W., Kip Thorne, S.: John Archibald Wheeler. Gravitation. (Freemann 1973)