# New Class of Doubly Nonlinear Evolution Equations Governed by Time-Dependent Subdifferentials

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**Abstract** We discuss a new class of doubly nonlinear evolution equations governed by time-dependent subdifferentials in uniformly convex Banach spaces, and establish an abstract existence result of solutions. Also, we show non-uniqueness of solution, giving some examples. Moreover, we treat a quasi-variational doubly nonlinear evolution equation by applying this result extensively, and give some applications to nonlinear PDEs with gradient constraint for time-derivatives.

**Keywords** Doubly nonlinear • Quasi-variational inequalities • Subdifferential • Time-dependent

## 1 Introduction

This paper is concerned with a new class of doubly nonlinear evolution equations governed by time-dependent subdifferentials. Let $H$ be a real Hilbert space and $V$ be a uniformly convex Banach space such that $V$ is dense in $H$ and the injection from $V$ into $H$ is compact. Also we suppose that the dual space $V^*$ of $V$ is uniformly convex. In this case, identifying $H$ with its dual, we have

$$V \hookrightarrow H \hookrightarrow V^* \text{ with compact embeddings.}$$

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The doubly nonlinear evolution equation, as in the title, is of the following form:

$$(\mathrm{P};f,u_0)\begin{cases}\partial_*\psi^t(u'(t))+\partial_*\varphi^t(u(t))+g(t,u(t))\ni f(t) \text{ in } V^* \text{ for a.e. } t\in(0,T),\\ u(0)=u_0 \ \text{ in } V.\end{cases} \tag{1}$$

Here $0<T<\infty$, $u'=du/dt$ in $V$, $\psi^t : V\to\mathbb{R}\cup\{\infty\}$ and $\varphi^t : V\to\mathbb{R}\cup\{\infty\}$ are time-dependent proper, l.s.c. (lower semi-continuous) and convex functions on $V$ for each $t\in[0,T]$, $\partial_*\psi^t$ and $\partial_*\varphi^t$ are their subdifferentials from $V$ into $V^*$, $g(t,\cdot)$ is a single-valued operator from $V$ into $V^*$, $f$ is a given $V^*$-valued function and $u_0\in V$ is a given initial datum. Suppose that $\partial_*\varphi^t$ is single-valued, linear and continuous from $V$ into $V^*$.

The main aim of this paper is to show the existence of a solution to (P;$f,u_0$) under some additional assumptions. Also, we touch the uniqueness question of solutions to (P;$f,u_0$), together with an example for non-uniqueness of solutions in the general case. We shall show the uniqueness of solutions under the strong monotonicity of $\partial_*\psi^t$.

Similar types of doubly nonlinear evolution equations have been discussed by many mathematicians, for instance, Akagi [1], Arai [2], Aso et al. [3, 4], Colli [8], Colli–Visintin [9] and Senba [14]. Most of them treated the case

$$\partial\psi^t(u'(t))+\partial\varphi(u(t))\ni f(t) \ \text{ in } H \ \text{ for a.e. } t\in(0,T) \tag{2}$$

and it should be noticed that the second term $\partial\varphi$ in (2) is independent of time and there is no perturbation term $g$. There has been no theory on nonlinear evolution equations governed by doubly time-dependent subdifferentials because of lack of energy estimate up to date. In this paper we shall establish an abstract approach to (1), specifying the time-dependence of $\psi^t$ and $\varphi^t$. As to the application of (1), we can treat nonlinear variational inequalities with gradient constraint for time-derivatives (see Sect. 6), which is a new novelty of this paper.

Another aim of this paper is to treat a doubly nonlinear quasi-variational evolution equation of the form:

$$(\mathrm{QP};f,u_0)\begin{cases}\partial_*\psi^t(u'(t))+\partial_*\varphi^t(u;u(t))+g(t,u(t))\ni f(t) \ \text{ in } V^* \ \text{ for a.e. } t\in(0,T),\\ u(0)=u_0 \ \text{ in } V.\end{cases}$$

The solvability will be discussed in the same framework with (P;$f,u_0$) by means of a standard fixed-point argument for compact operators. In this formulation, $\varphi^t(v;z)$ is proper, l.s.c. and convex in $z\in V$, and $(t,v)\in[0,T]\times L^2(0,T;V)$ is a parameter which determines the convex function $\varphi^t(v;\cdot)$ on $V$. The dependence of function $v$ upon $\varphi^t(v;\cdot)$ is allowed to be non-local, in general. Therefore, the expression of (QP;$f,u_0$) includes an extremely wide class of quasi-linear partial differential equations or variational inequalities.

## 1.1 Notations

Throughout this paper, let $H$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. Let $V$ be a uniformly convex (hence reflexive) Banach space with uniformly convex dual space $V^*$. We denote by $|\cdot|_V$, $|\cdot|_{V^*}$ and $\langle\cdot,\cdot\rangle$ the norms in $V$, $V^*$ and duality pairing between $V^*$ and $V$, respectively. Also, suppose that $V$ is dense and embedded compactly in $H$. Then, identifying $H$ with the dual $H^*$, we have $V \hookrightarrow H \hookrightarrow V^*$, where $\hookrightarrow$ stands for the compact embedding. Therefore, $(V, H, V^*)$ is the standard triplet and

$$\langle u, v\rangle = (u, v) \ \text{ for } u \in H \text{ and } v \in V.$$

Also, let $F : V \to V^*$ be the duality mapping, which is single-valued and continuous from $V$ onto $V^*$.

We here prepare some notations and definitions of subdifferential of convex functions. Let $\phi : V \to \mathbb{R} \cup \{\infty\}$ be a proper (i.e., not identically equal to infinity), l.s.c. and convex function. Then, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in V;\ \phi(z) < \infty\}.$$

The subdifferential $\partial_*\phi : V \to V^*$ of $\phi$ is a possibly multi-valued operator and is defined by:

$$z^* \in \partial_*\phi(z) \iff z^* \in V^*,\ z \in D(\phi),\ \langle z^*, y - z\rangle \le \phi(y) - \phi(z),\quad \forall y \in V;$$

and the domain of $\partial_*\phi$ is denoted by $D(\partial_*\phi)$, and set as $D(\partial_*\phi) := \{z \in V\ ;\ \partial_*\phi(z) \ne \emptyset\}$. For basic properties and related notions of proper, l.s.c., convex functions and their subdifferentials, we refer to the monographs of Barbu [6, 7].

Next, we recall a notion of convergence for convex functions, developed by Mosco [12]. Let $\phi$, $\phi_n$ ($n \in \mathbb{N}$) be proper, l.s.c. and convex functions on $V$. Then, we say that $\phi_n$ converges to $\phi$ on $V$ in the sense of Mosco [12] as $n \to \infty$, iff. the following two conditions are satisfied:

1. for any subsequence $\{\phi_{n_k}\} \subset \{\phi_n\}$, if $z_k \to z$ weakly in $V$ as $k \to \infty$, then

$$\liminf_{k\to\infty} \phi_{n_k}(z_k) \ge \phi(z);$$

2. for any $z \in D(\phi)$, there is a sequence $\{z_n\}$ in $V$ such that

$$z_n \to z \text{ in } V \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \phi_n(z_n) = \phi(z).$$

## 2 Main Theorems

We begin with the precise formulation of our problem (P;$f, u_0$).

We suppose that the duality mapping $F : V \to V^*$ is strongly monotone, more precisely there is a positive constant $C_F$ such that

$$\langle Fz_1 - Fz_2, z_1 - z_2 \rangle \geq C_F |z_1 - z_2|_V^2, \quad \forall z_1,\ z_2 \in V. \tag{3}$$

(*Assumption (A)*)

Let $\psi^t(\cdot)$ be a proper l.s.c. and convex function on $V$ for all $t \in [0, T]$. We assume:

(A1) If $\{t_n\} \subset [0, T]$ and $t \in [0, T]$ with $t_n \to t$ as $n \to \infty$, then $\psi^{t_n}(\cdot) \to \psi^t(\cdot)$ in the sense of Mosco [12] as $n \to \infty$.

(A2) There exist positive constants $C_1 > 0$ and $C_2 > 0$ such that

$$\psi^t(z) \geq C_1 |z|_V^2 - C_2, \quad \forall t \in [0, T],\ \forall z \in D(\psi^t).$$

(A3) $\partial_* \psi^t(0) \ni 0$ for all $t \in [0, T]$ and $\psi^{(\cdot)}(0) \in L^1(0, T)$.

(*Assumption (B)*)

Let $\varphi^t(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ be a non-negative, finite, continuous and convex function with $D(\varphi^t) = V$ for all $t \in [0, T]$. We assume:

(B1) For each $t \in [0, T]$, the subdifferential $\partial_* \varphi^t : D(\partial_* \varphi^t) = V \to V^*$ is linear and uniformly bounded, i.e., there exists a positive constant $C_3 > 0$ such that

$$|\partial_* \varphi^t(z)|_{V^*} \leq C_3 |z|_V, \quad \forall t \in [0, T],\ \forall z \in V.$$

(B2) $\varphi^t(0) = 0$ for all $t \in [0, T]$ and there exists a positive constant $C_4 > 0$ such that

$$\varphi^t(z) \geq C_4 |z|_V^2, \quad \forall t \in [0, T],\ \forall z \in V.$$

(B3) There is a function $\alpha \in W^{1,1}(0, T)$ such that

$$|\varphi^t(z) - \varphi^s(z)| \leq |\alpha(t) - \alpha(s)| \varphi^s(z), \quad \forall s, t \in [0, T],\ \forall z \in V.$$

*Remark 1* We derive from (B1) and (B2) that the subdifferential $\partial_* \varphi^t$ satisfies that

$$C_3 |z|_V^2 \geq \langle \partial_* \varphi^t(z), z \rangle \geq \varphi^t(z) \geq C_4 |z|_V^2, \quad \forall z \in V,\ \forall t \in [0, T] \tag{4}$$

and from (B3) that the function $t \to \partial_* \varphi^t(z)$ is weakly continuous from $[0, T]$ into $V^*$.

*Remark 2* The assumption (B3) is the standard time-dependence condition of convex functions (cf. [10, 13, 15]).

(*Assumption (C)*)

Let $g$ be a single-valued operator from $[0,T] \times V$ into $V^*$ such that $g(t,z)$ is strongly measurable in $t \in [0,T]$ for each $z \in V$, and assume:

(C1) For each $t \in [0,T]$, the operator $z \to g(t,z)$ is continuous from $V_w$ into $V^*$, i.e., if $z_n \to z$ weakly in $V$ as $n \to \infty$, then $g(t,z_n) \to g(t,z)$ in $V^*$ as $n \to \infty$.
(C2) $g(t,\cdot)$ is uniformly Lipschitz from $V$ into $V^*$, i.e., there is a positive constant $L_g > 0$ such that

$$|g(t,z_1) - g(t,z_2)|_{V^*} \le L_g |z_1 - z_2|_V, \quad \forall t \in [0,T],\ \forall z_i \in V\ (i=1,2).$$

Under the above assumptions we define the solution to (P;$f, u_0$) as follows.

**Definition 1** Given $f \in L^2(0,T;V^*)$ and $u_0 \in V$, a function $u : [0,T] \to V$ is called a solution to (P;$f,u_0$) on $[0,T]$, iff. the following conditions are fulfilled:

(i) $u \in W^{1,2}(0,T;V)$.
(ii) There exists a function $\xi \in L^2(0,T;V^*)$ such that

$$\xi(t) \in \partial_* \psi^t(u'(t)) \ \text{ in } V^* \ \text{ for a.e. } t \in (0,T),$$

$$\xi(t) + \partial_* \varphi^t(u(t)) + g(t,u(t)) = f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0,T).$$

(iii) $u(0) = u_0$ in $V$.

Now, we mention the first main result of this paper, which is concerned with the existence of a solution to problem (P;$f,u_0$).

**Theorem 1** *Suppose that Assumptions (A), (B) and (C) hold. Then, for each $u_0 \in V$ and $f \in L^2(0,T;V^*)$, there exists at least one solution $u$ to* (P;$f,u_0$) *on* $[0,T]$. *Moreover, there exists a positive increasing function $N_0 : \mathbb{R}_+^3 \to \mathbb{R}_+$ with respect to $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ and $|\alpha'|_{L^1(0,T)}$ such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t\in[0,T]} \varphi^t(u(t)) \le N_0\left(\varphi^0(u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)}\right). \tag{5}$$

In Sect. 3, we shall prove Theorem 1, considering the approximate problems of (P;$f,u_0$). It is known that the solution to (P;$f,u_0$) is not unique in general. In Sect. 4, we give an example for non-uniqueness of solutions to (P;$f,u_0$) in the general case, but we can show the uniqueness under strong monotonicity of $\partial_* \psi^t$, as stated below.

**Theorem 2** *Suppose that Assumptions (A), (B) and (C) are fulfilled. In addition, assume that $\partial_* \psi^t$ is strongly monotone in $V^*$, more precisely,* (A4) *There exists a positive constant* $C_5 > 0$ *such that*

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq C_5 |z_1 - z_2|_V^2, \quad \forall [z_i, z_i^*] \in \partial_* \psi^t \ (i = 1, 2), \ \forall t \in [0, T].$$

*Then, the solution to* (P;$f, u_0$) *is unique.*

In Sect. 4, we prove Theorem 2 using the additional assumption (A4) and Gronwall's inequality.

*Remark 3* Colli [8, Theorem 5] and Colli–Visintin [9, Remark 2.5] showed several criteria for the uniqueness of solutions to the following type of doubly nonlinear evolution equations:

$$\partial \psi(u'(t)) + \partial \varphi(u(t)) \ni f(t) \ \text{ in } H \ \text{ for a.e. } t \in (0, T). \tag{6}$$

For instance, if $\partial \varphi$ is linear and positive in $H$ and $\partial \psi$ is strictly monotone in $H$, then the solution to (6) on $[0, T]$ is unique.

## 3 Existence of Solutions to (P;$f, u_0$)

In this section, we discuss the solvability of (P;$f, u_0$) for $f \in L^2(0, T; V^*)$ and $u_0 \in V$.

Throughout this section, we suppose that all the assumptions of Theorem 1 are made. On this basis, we prove Theorem 1 by means of the approximation of (P;$f, u_0$). Indeed, our approximate problem is of the following form with parameter $\varepsilon \in (0, 1]$:

$$(\mathrm{P}; f, u_0)_\varepsilon \begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \ \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0, T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V. \end{cases} \tag{7}$$

We prove the existence-uniqueness of solution to (P;$f, u_0$)$_\varepsilon$ for each $\varepsilon \in (0, 1]$.

**Proposition 1** *Assume (A), (B) and (C) are satisfied. Then, for each* $\varepsilon \in (0, 1]$, $u_0 \in V$ *and* $f \in L^2(0, T; V^*)$, *there exists a unique solution* $u_\varepsilon \in W^{1,2}(0, T; V)$ *to* (P;$f, u_0$)$_\varepsilon$ *on* $[0, T]$ *satisfying* $u_\varepsilon(0) = u_0$ *in* $V$ *and there exists a function* $\xi_\varepsilon \in L^2(0, T; V^*)$ *such that*

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon'(t)) \ \textit{ in } V^* \ \textit{ for a.e. } t \in (0, T),$$

$$\varepsilon F u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \ \textit{ in } V^* \ \textit{ for a.e. } t \in (0, T).$$

*Moreover, there exists a positive increasing function $N_0$ with respect to $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ and $|\alpha'|_{L^1(0,T)}$, independent of $\varepsilon \in (0,1]$, such that*

$$\int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon(t)) \le N_0\left(\varphi^0(u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)}\right). \tag{8}$$

To show (8), we need the following lemma.

**Lemma 1 (cf. [10, Lemma 2.1.1])** *Assume* $(B)$. *Let* $v \in W^{1,1}(0,T;V)$. *Then, we have:*

$$\frac{d}{dt}\varphi^t(v(t)) - \langle \partial_*\varphi^t(v(t)), v'(t)\rangle \le |\alpha'(t)|\varphi^t(v(t)), \quad a.e.\ t \in (0,T). \tag{9}$$

*Proof* We observe from (B3) that $\varphi^t(v(t))$ is absolutely continuous on $[0,T]$ and also observe from the definition of subdifferential that

$$\begin{aligned}
&\varphi^t(v(t)) - \varphi^s(v(s)) - \langle \partial_*\varphi^t(v(t)), v(t)-v(s)\rangle \\
\le&\varphi^t(v(s)) - \varphi^s(v(s)) \\
\le&|\alpha(t)-\alpha(s)|\varphi^s(v(s)) \quad \text{for all } s,t \in [0,T].
\end{aligned}$$

Then, we get (9) by dividing the above inequalities by $t-s$ and letting $s \uparrow t$. □

*Proof (Proof of Proposition 1)* Note that the approximate problem $(\mathrm{P};f,u_0)_\varepsilon$ can be reformulated in the following form:

$$\begin{cases} u_\varepsilon'(t) = (\varepsilon F + \partial_*\psi^t)^{-1}\left(f(t) - \partial_*\varphi^t(u_\varepsilon(t)) - g(t,u_\varepsilon(t))\right) \text{ in } V \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \text{ in } V. \end{cases} \tag{10}$$

Here, we put

$$\mathscr{B}(t)z^* := (\varepsilon F + \partial_*\psi^t)^{-1}z^* \text{ for all } z^* \in V^*,\ t \in (0,T)$$

and

$$\mathscr{F}(t,z) := f(t) - \partial_*\varphi^t(z) - g(t,z) \text{ for all } z \in V,\ t \in (0,T).$$

Now we show that the operator $\mathscr{B}(t)z^* : [0,T]\times V^* \to V$ is Lipschitz in $z^* \in V^*$ and is continuous in $t \in [0,T]$. We first fix any $t \in [0,T]$ to show that $z^* \in V^* \mapsto \mathscr{B}(t)z^* \in V$ is Lipschitz continuous. To this end, put $z_i = \mathscr{B}(t)z_i^*$ $(i=1,2)$. Then,

$$z_i^* = \varepsilon F z_i + z_{i,*} \text{ for some } z_{i,*} \in \partial_*\psi^t(z_i).$$

Hence, we infer from (3) and the monotonicity of $\partial_* \psi^t(\cdot)$ that

$$\begin{aligned}\langle z_1^* - z_2^*, z_1 - z_2\rangle =&\langle \varepsilon F z_1 + z_{1,*} - \varepsilon F z_2 - z_{2,*}, z_1 - z_2\rangle \\ \geq&\varepsilon\langle F z_1 - F z_2, z_1 - z_2\rangle \\ \geq&\varepsilon C_F |z_1 - z_2|_V^2,\end{aligned}$$

which implies that

$$|\mathscr{B}(t)z_1^* - \mathscr{B}(t)z_2^*|_V = |z_1 - z_2|_V \leq \frac{1}{\varepsilon C_F}|z_1^* - z_2^*|_{V^*}.$$

Thus, the operator $\mathscr{B}(t)z^*$ is Lipschitz in $z^* \in V^*$ for all $t \in [0, T]$ with a uniform constant $1/\varepsilon C_F$.

Next, we fix any $z^* \in V^*$ to show that $t \in [0, T] \mapsto \mathscr{B}(t)z^* \in V$ is continuous. Let $z^* \in V^*$ be an arbitrary element and put $z^t := \mathscr{B}(t)z^*$, hence $\varepsilon F z^t + \partial_* \psi^t(z^t) \ni z^*$. Let $\{s_n\} \subset [0, T]$ with $s_n \to t$ (as $n \to \infty$). Note that

$$z^* = \varepsilon F z^{s_n} + z_*^{s_n} \ \text{ for some } z_*^{s_n} \in \partial_* \psi^{s_n}(z^{s_n}). \tag{11}$$

Also, we observe from (A1) that $\partial_* \psi^{s_n}$ converges to $\partial_* \psi^t$ in the sense of graph as $n \to \infty$ (cf. [5, 11]). Therefore, for $[z^t, z^* - \varepsilon F z^t] \in \partial_* \psi^t$, there exists a sequence $\{[z_n, z_n^*]\} \subset V \times V^*$ such that $[z_n, z_n^*] \in \partial_* \psi^{s_n}$ in $V \times V^*$ for all $n \in \mathbb{N}$,

$$z_n \to z^t \ \text{ in } V \ \text{ and } \ z_n^* \to z^* - \varepsilon F z^t \ \text{ in } V^* \ \text{ as } n \to \infty. \tag{12}$$

Since the dual space $V^*$ is uniformly convex, the duality mapping $F$ is uniformly continuous on every bounded subset of $V$. Therefore, we observe from (12) that

$$z_n^* + \varepsilon F z_n \to z^* - \varepsilon F z^t + \varepsilon F z^t = z^* \ \text{ in } V^* \ \text{ as } n \to \infty. \tag{13}$$

Hence, we infer from (11), (13) and the monotonicity of $\partial_* \psi^{s_n}$ that

$$\begin{aligned}0 =& \lim_{n\to\infty} \langle z^* - z_n^* - \varepsilon F z_n, z^{s_n} - z_n\rangle \\ =& \lim_{n\to\infty} \langle \varepsilon F z^{s_n} + z_*^{s_n} - z_n^* - \varepsilon F z_n, z^{s_n} - z_n\rangle \\ \geq& \limsup_{n\to\infty} \varepsilon\langle F z^{s_n} - F z_n, z^{s_n} - z_n\rangle \\ \geq&\varepsilon C_F \limsup_{n\to\infty} |z^{s_n} - z_n|_V^2,\end{aligned}$$

which implies from (12) that

$$z^{s_n} = \mathscr{B}(s_n)z^* \to z^t = \mathscr{B}(t)z^* \ \text{ as } s_n \to t.$$

Thus, the operator $\mathscr{B}(t)z^*$ is continuous in $t \in [0, T]$ for all $z^* \in V^*$.

Furthermore, it follows from (B1), (B3), (C2) and $f \in L^2(0, T; V^*)$ that the operator $\mathscr{F}(t, z) : [0, T] \times V \to V^*$ is (strongly) measurable in $t \in [0, T]$ and Lipschitz in $z \in V$.

Now we show the existence-uniqueness of a solution to (10), i.e., $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0, T]$. To this end, for given $u \in C([0, T]; V)$, we define the operator $S : C([0, T]; V) \to C([0, T]; V)$ by:

$$S(u)(t) := u_0 + \int_0^t \mathscr{B}(s)[\mathscr{F}(s, u(s))]ds \quad \text{for all } t \in [0, T].$$

Note that the operator $\mathscr{B}(\cdot)[\mathscr{F}(\cdot, \cdot)] : [0, T] \times V \to V$ satisfies the Carathéodory condition, $\mathscr{B}(\cdot)[\mathscr{F}(\cdot, z)]$ is Lipschitz in $z \in V$ and $\mathscr{B}(\cdot)[\mathscr{F}(\cdot, u)] \in L^1(0, T; V)$ for all $u \in C([0, T]; V)$. Therefore, by Cauchy–Lipschitz–Picard's existence theorem, we can prove that $S$ has the fixed point $u \in C([0, T_0]; V)$ for some small $T_0 \in (0, T]$, which is a unique solution to $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0, T_0]$. By repeating the above argument, we can construct a unique solution $u_\varepsilon$ to $(\mathrm{P}; f, u_0)_\varepsilon$ on the whole time interval $[0, T]$.

Next we show a priori estimate (8). To this end, multiply (7) by $u'_\varepsilon$ to obtain:

$$\begin{aligned} &\langle \varepsilon F u'_\varepsilon(t), u'_\varepsilon(t)\rangle + \langle \xi_\varepsilon(t), u'_\varepsilon(t)\rangle + \langle \partial_*\varphi^t(u_\varepsilon(t)), u'_\varepsilon(t)\rangle \\ &\quad + \langle g(t, u_\varepsilon(t)), u'_\varepsilon(t)\rangle \\ &= \langle f(t), u'_\varepsilon(t)\rangle \quad \text{for a.e. } t \in (0, T), \end{aligned} \tag{14}$$

with $\xi_\varepsilon \in L^2(0, T; V^*)$ satisfying $\xi_\varepsilon(t) \in \partial_*\psi^t(u'_\varepsilon(t))$ in $V^*$ for a.e. $t \in (0, T)$. It follows from the definition of $F$ and $\partial_*\psi^t$, and Lemma 1 that:

$$\langle \varepsilon F u'_\varepsilon(t), u'_\varepsilon(t)\rangle = \varepsilon |u'_\varepsilon(t)|_V^2, \tag{15}$$

$$\langle \xi_\varepsilon(t), u'_\varepsilon(t)\rangle \geq \psi^t(u'_\varepsilon(t)) - \psi^t(0), \tag{16}$$

$$\langle \partial_*\varphi^t(u_\varepsilon(t)), u'_\varepsilon(t)\rangle \geq \frac{d}{dt}\varphi^t(u_\varepsilon(t)) - |\alpha'(t)|\varphi^t(u_\varepsilon(t)) \tag{17}$$

for a.e. $t \in (0, T)$. Also, from (A2), (B2), (C2) and Schwarz's inequality, we observe that

$$\begin{aligned} &|\langle g(t, u_\varepsilon(t)), u'_\varepsilon(t)\rangle| \leq |g(t, u_\varepsilon(t))|_{V^*} |u'_\varepsilon(t)|_V \\ \leq & \frac{C_1}{4}|u'_\varepsilon(t)|_V^2 + \frac{1}{C_1}|g(t, u_\varepsilon(t))|_{V^*}^2 \\ \leq & \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{1}{C_1}\left(|g(t, 0)|_{V^*} + L_g|u_\varepsilon(t)|_V\right)^2 \\ \leq & \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{2|g(t, 0)|_{V^*}^2}{C_1} + \frac{2L_g^2}{C_1 C_4}\varphi^t(u_\varepsilon(t)) \end{aligned} \tag{18}$$

and

$$|\langle f(t), u'_\varepsilon(t)\rangle| \le \frac{C_1}{4}|u'_\varepsilon(t)|_V^2 + \frac{1}{C_1}|f(t)|_{V^*}^2 \le \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{1}{C_1}|f(t)|_{V^*}^2 \quad (19)$$

for a.e. $t \in (0, T)$. Thus, using (15)–(19), it follows from (14) that:

$$\begin{aligned} &\varepsilon|u'_\varepsilon(t)|_V^2 + \frac{1}{2}\psi^t(u'_\varepsilon(t)) + \frac{d}{dt}\varphi^t(u_\varepsilon(t)) \\ &\le M_1\left(|\alpha'(t)| + 1)\varphi^t(u_\varepsilon(t)) + M_2(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t, 0)|_{V^*}^2 + 1\right) \\ &\qquad \text{for a.e. } t \in (0, T), \end{aligned} \quad (20)$$

where $M_1 > 0$ and $M_2 > 0$ are constants independent of $\varepsilon \in (0, 1]$; for instance, $M_1 = \frac{2L_g^2}{C_1C_4} + 1$ and $M_2 = \frac{2}{C_1} + \frac{C_2}{2} + 1$. Multiplying (20) by $e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}$, we get

$$\begin{aligned} &\varepsilon e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}|u'_\varepsilon(t)|_V^2 + \frac{1}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(\psi^t(u'_\varepsilon(t)) + C_2) \\ &\qquad + \frac{d}{dt}\left\{e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}\varphi^t(u_\varepsilon(t))\right\} \quad (21) \\ &\le \frac{C_2}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau} + M_2 e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t, 0)|_{V^*}^2 + 1) \\ &=: M_3(t). \end{aligned}$$

Integrating (21) in time, we obtain

$$\begin{aligned} &\int_0^T \psi^t(u'_\varepsilon(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon(t)) \\ &\le 3e^{\int_0^T M_1(|\alpha'(\tau)|+1)d\tau}\left\{\varphi^0(u_0) + \int_0^T M_3(\tau)d\tau\right\} =: N_0. \end{aligned}$$

It is easy to see from the above construction of $N_0$ that $N_0$ is a positive increasing function with respect to $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ and $|\alpha'|_{L^1(0,T)}$, and is independent of $\varepsilon \in (0, 1]$. Thus, the proof of Proposition 1 has been completed. □

Now, let us prove the main Theorem 1.

*Proof (Proof of Theorem 1)* Let $u_\varepsilon$ be a solution to $(\mathrm{P}; f, u_0)_\varepsilon$ with initial datum $u_0$, which is obtained by Proposition 1, and let $\xi_\varepsilon$ be a function in $L^2(0, T; V^*)$ such that

$$\xi_\varepsilon(t) \in \partial_*\psi^t(u'_\varepsilon(t)) \text{ in } V^* \text{ for a.e. } t \in (0, T) \quad (22)$$

and

$$\varepsilon F u'_\varepsilon(t) + \xi_\varepsilon(t) + \partial_*\varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \text{ in } V^* \text{ for a.e. } t \in (0, T). \quad (23)$$

From (B2), (8) and the Ascoli–Arzelà theorem, we see that there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ (as $n \to \infty$) and a function $u \in W^{1,2}(0,T;V)$ such that

$$\left.\begin{aligned} u_{\varepsilon_n} \to u \ \ &\text{weakly in } W^{1,2}(0,T;V), \ \ \text{in } C([0,T];H) \\ &\text{and weakly-}* \ \text{ in } L^\infty(0,T;V) \text{ as } n \to \infty, \end{aligned}\right\} \tag{24}$$

$$u_{\varepsilon_n}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty, \tag{25}$$

$$\int_0^t \psi^\tau(u'(\tau))d\tau \le \liminf_{n\to\infty} \int_0^t \psi^\tau(u'_{\varepsilon_n}(\tau))d\tau \le N_0 \ \text{ for all } t \in [0,T].$$

Next, we show that $u_{\varepsilon_n} \to u$ in $L^2(0,T;V)$. To this end, we multiply (23) by $u'_{\varepsilon_n} - u'$ to get:

$$\begin{aligned} &\langle \varepsilon_n F u'_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\quad + \langle \partial_* \varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &= \langle f(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \qquad \text{for a.e. } t \in (0,T). \end{aligned} \tag{26}$$

Here, we have by the definition of $\partial_* \psi^t$ (cf. (22)) that

$$\langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \ge \psi^t(u'_{\varepsilon_n}(t)) - \psi^t(u'(t)) \qquad \text{for a.e. } t \in (0,T), \tag{27}$$

and by Lemma 1 that

$$\begin{aligned} &\langle \partial_* \varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &= \langle \partial_* \varphi^t(u_{\varepsilon_n}(t) - u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \partial_* \varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\ge \frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) - |\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\qquad + \langle \partial_* \varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \qquad \text{for a.e. } t \in (0,T). \end{aligned} \tag{28}$$

Therefore, from (26)–(28) we obtain that:

$$\begin{aligned} &\frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\le |\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) + \tilde{L}_{\varepsilon_n}(t) + \psi^t(u'(t)) - \psi^t(u'_{\varepsilon_n}(t)), \end{aligned} \tag{29}$$

for a.e. $t \in (0,T)$, where $\tilde{L}_{\varepsilon_n}(\cdot)$ is a function defined by:

$$\begin{aligned} \tilde{L}_{\varepsilon_n}(t) :=& \langle f(t) - \partial_* \varphi^t(u(t)) - g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &+ \varepsilon_n |F u'_{\varepsilon_n}(t)|_{V^*} |u'_{\varepsilon_n}(t) - u'(t)|_V \qquad \text{for a.e. } t \in (0,T). \end{aligned}$$

Now, just as (20)–(21) in the proof of Proposition 1, by multiplying (29) by $e^{-\int_0^t |\alpha'(\tau)|d\tau}$ and integrating it in time, we get

$$\begin{aligned} &e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\le \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \tilde{L}_{\varepsilon_n}(s)ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \{\psi^s(u'(s)) - \psi^s(u'_{\varepsilon_n}(s))\}ds. \end{aligned}$$

By (24) and (25) the first integral of the right hand side goes to 0 as $n \to \infty$ and by the weak lower semicontinuity of the functional $v \to \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \psi^s(v(s))ds$ on $L^2(0,t;V)$ the limit supremum of the second integral is bounded by 0 as $n \to \infty$. Hence we conclude that

$$\limsup_{n\to\infty} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \le 0, \quad \text{hence } u_{\varepsilon_n}(t) \to u(t) \text{ in } V, \quad \forall t \in [0,T], \tag{30}$$

so that by the Lebesgue dominated convergence theorem,

$$u_{\varepsilon_n} \to u \ \text{ in } L^2(0,T;V) \ \text{ as } n \to \infty. \tag{31}$$

Now we show that $u$ is a solution of (P;$f, u_0$) with initial datum $u_0$. We first note from (B1), (30) and the Lebesgue dominated convergence theorem that

$$\partial_* \varphi^{(\cdot)}(u_{\varepsilon_n}(\cdot)) \to \partial_* \varphi^{(\cdot)}(u(\cdot)) \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty \tag{32}$$

and by (8) that

$$\varepsilon_n F u'_{\varepsilon_n} \to 0 \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{33}$$

By (31)–(33) and (C2),

$$\xi_{\varepsilon_n} = f - \partial_* \varphi^t(u_{\varepsilon_n}) - g(t,u_{\varepsilon_n}) - \varepsilon_n F u'_{\varepsilon_n} \to f - \partial_* \varphi^t(u) - g(t,u) =: \xi \text{ in } L^2(0,T;V^*).$$

Therefore, from the demi-closedness of $\partial_* \psi^t$ in $L^2(0,T;V) \times L^2(0,T;V^*)$ it follows that $\xi(t) \in \partial_* \psi^t(u'(t))$ in $V^*$ for a.e. $t \in (0,T)$ and

$$\xi(t) + \partial_* \varphi^t(u(t)) + g(t,u(t)) = f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0,T).$$

Therefore, we conclude that $u$ is a solution of (P;$f, u_0$) and from a priori estimate (8) that (5) holds for the same function $N_0$ as in Proposition 1.

Thus, the proof of Theorem 1 has been completed. □

## 4 Uniqueness of Solutions to (P;$f, u_0$)

In this section, we discuss the uniqueness of solutions to (P;$f, u_0$).

We begin with showing a counterexample for uniqueness of solutions to (P;$f, u_0$).

*Example 4.1 (cf. [8, Section 2])* Let $\Omega = (0, 1)$. Also, let $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Define a closed convex subset $K$ of $V$ by

$$K := \{z \in V\,;\ |z(x)| \leq 1,\ |z_x(x)| \leq 1,\ \text{a.e. } x \in \Omega\}.$$

Then, we consider the following variational problem with constraint:

$$\begin{cases} u_t(t) \in K,\ \text{a.e. } t \in (0, T), \\ \displaystyle\int_\Omega u_x(t, x)(u_{xt}(t, x) - v_x(x))dx \leq 0, \quad \forall v \in K,\ \text{a.e. } t \in (0, T), \\ u(0, x) = 0, \quad x \in \Omega, \end{cases} \tag{34}$$

where $0 < T < +\infty$.

Here, for each $t \in [0, T]$ we consider the following convex functions:

$$\psi^t(z) = I_K(z), \quad \varphi^t(z) = \frac{1}{2}|z|_V^2, \quad \forall z \in V.$$

Then we have:

1. $z^* \in \partial_* \psi^t(z)$ if and only if $z^* \in V^*,\ z \in K$ and $\langle z^*, v - z\rangle \leq 0$ for all $v \in K$,
2. $\langle \partial_* \varphi^t(z), v\rangle = \int_\Omega z(x)v(x)dx + \int_\Omega z_x(x)v_x(x)dx$ for all $v,\ z \in V$,

and problem (34) is reformulated as (P;0, 0) with $g(t, z) = -z$. Therefore, applying Theorem 1, problem (34) has at least one solution $u$.

Moreover, for each constant $c \in (0, 1)$ the function $u_c$ defined by

$$u_c(t, x) := c(1 - \exp(-t)) \ \text{ for all } (t, x) \in (0, T) \times \Omega$$

is a solution to (34). Indeed, we observe that

$$(u_c)_t(t, x) = c\exp(-t) \in K, \quad (u_c)_x(t, x) = 0, \quad (u_c)_{xt}(t, x) = 0$$

for all $(t, x) \in (0, T) \times \Omega$. Therefore, for each $c \in (0, 1)$, (34) is satisfied. Hence $\{u_c\}_{c\in(0,1)}$ provides with an infinite family of solutions to (34).

Now, we prove Theorem 2 concerning the uniqueness of solutions to (P;$f, u_0$) under the additional condition (A4) of strict monotonicity of $\partial_* \psi^t$.

*Proof (Proof of Theorem 2)* Let $u_i$, $i = 1, 2$, be two solutions to (P;$f, u_0$) on $[0, T]$. Subtract (P;$f, u_0$) for $i = 2$ from the one for $i = 1$, and multiply it by $u_1' - u_2'$. Then:

$$\begin{aligned} &\langle \xi_1(t) - \xi_2(t), u_1'(t) - u_2'(t)\rangle + \langle \partial_* \varphi^t(u_1(t)) - \partial_* \varphi^t(u_2(t)), u_1'(t) - u_2'(t)\rangle \\ &\quad + \langle g(t, u_1(t)) - g(t, u_2(t)), u_1'(t) - u_2'(t)\rangle = 0 \quad \text{ for a.e. } t \in (0, T), \end{aligned} \tag{35}$$

where $\xi_i(t) \in \partial_* \psi^t(u_i'(t))$ for a.e. $t \in (0, T)$ $(i = 1, 2)$. From (A4) we observe that

$$\langle \xi_1(t) - \xi_2(t), u_1'(t) - u_2'(t)\rangle \geq C_5 |u_1'(t) - u_2'(t)|_V^2 \ \text{ for a.e. } t \in (0, T) \tag{36}$$

and by Lemma 1 that

$$\begin{aligned} &\langle \partial_* \varphi^t(u_1(t)) - \partial_* \varphi^t(u_2(t)), u_1'(t) - u_2'(t)\rangle \\ = &\langle \partial_* \varphi^t(u_1(t) - u_2(t)), u_1'(t) - u_2'(t)\rangle \\ \geq &\frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) - |\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) \quad \text{ for a.e. } t \in (0, T). \end{aligned} \tag{37}$$

Therefore, we observe from (35)–(37) and (C2) with the help of the Schwarz inequality that

$$\begin{aligned} &C_5 |u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) \\ \leq &|\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + |g(t, u_1(t)) - g(t, u_2(t))|_{V^*} |u_1'(t) - u_2'(t)|_V \\ \leq &|\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + \frac{1}{2C_5}|g(t, u_1(t)) - g(t, u_2(t))|_{V^*}^2 + \frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 \\ \leq &|\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + \frac{L_g^2}{2C_5}|u_1(t) - u_2(t)|_V^2 + \frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 \end{aligned}$$

for a.e. $t \in (0, T)$. From the above inequality we infer that

$$\begin{aligned} &\frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) \\ &\leq K_1(|\alpha'(t)| + 1)\varphi^t(u_1(t) - u_2(t)) \quad \text{ for a.e. } t \in (0, T), \end{aligned} \tag{38}$$

for some constant $K_1 > 0$ being independent of $u_i$ $(i = 1, 2)$. Hence, applying the Gronwall inequality to (38), we conclude that

$$u_1(t) - u_2(t) = 0 \ \text{ in } V \text{ for all } t \in [0, T].$$

Thus the proof of Theorem 2 has been completed. □

## 5 Doubly Nonlinear Quasi-Variational Inequality

In this section we discuss a doubly nonlinear quasi-variational inequality of the form:

$$(\mathrm{QP};f,u_0)\begin{cases}\partial_*\psi^t(u'(t))+\partial_*\varphi^t(u;u(t))+g(t,u(t))\ni f(t) \text{ in } V^* \text{ for a.e. } t\in(0,T),\\ u(0)=u_0 \ \text{ in } V,\end{cases}$$

where $\psi^t(z)$ and $g(t,z)$ are the same ones as before, and $\varphi^t(v;z)$ is precisely formulated below.

*(Assumption (B'))*

Putting

$$D_0 := \left\{ v\in W^{1,2}(0,T;V) \;\middle|\; \int_0^T \psi^t(v'(t))dt<\infty \right\},$$

we define a functional $\varphi^t : [0,T]\times D_0\times V\to\mathbb{R}$ such that $\varphi^t(v;z)$ is non-negative, finite, continuous and convex in $z\in V$ for any $t\in[0,T]$ and any $v\in D_0$, and

$$\varphi^t(v_1;z)=\varphi^t(v_2;z),\ \forall z\in V,\ \text{if } v_1=v_2 \text{ on } [0,t],$$

for $v_i\in D_0,\ i=1,2$, and assume:

(B1') The subdifferential $\partial_*\varphi^t(v;z)$ of $\varphi^t(v;z)$ with respect to $z\in V$ is linear and bounded from $D(\partial_*\varphi^t(v;\cdot))=V$ into $V^*$ for each $t\in[0,T]$ and $v\in D_0$, and there is a positive constant $C_3'$ such that

$$|\partial_*\varphi^t(v;z)|_{V^*}\le C_3'|z|_V,\ \ \forall z\in V,\ \forall v\in D_0,\ \forall t\in[0,T].$$

(B2') If $\{v_n\}\subset D_0$, $\sup_{n\in\mathbb{N}}\int_0^T\psi^t(v_n'(t))dt<\infty$ and $v_n\to v\in C([0,T];H)$ (as $n\to\infty$), then

$$\partial_*\varphi^t(v_n;z)\to\partial_*\varphi^t(v;z)\ \text{ in } V^*,\ \forall z\in V,\ \forall t\in[0,T]\ \text{ as } n\to\infty.$$

(B3') $\varphi^t(v;0)=0$ for all $v\in D_0$ and $t\in[0,T]$. There is a positive constant $C_4'$ such that

$$\varphi^t(v;z)\ge C_4'|z|_V^2,\ \ \forall z\in V,\ \forall v\in D_0,\ \forall t\in[0,T].$$

(B4') There is a function $\alpha\in W^{1,1}(0,T)$ such that

$$|\varphi^t(v;z)-\varphi^s(v;z)|\le|\alpha(t)-\alpha(s)|\varphi^s(v;z)$$
$$\forall z\in V,\ \ \forall v\in D_0,\ \ \forall s,\ t\in[0,T].$$

We now state the final main theorem of this paper.

**Theorem 3** *Suppose that Assumptions (A), (B') and (C) are fulfilled. Let $f$ be any function in $L^2(0,T;V^*)$ and $u_0$ be any element in $V$ such that*

$$u_0 \in D(\varphi^0(\tilde{v};\cdot)) \text{ for some } \tilde{v} \in D_0 \text{ with } \tilde{v}(0) = u_0.$$

*Then* (QP;$f, u_0$) *admits at least one solution $u : [0,T] \to V$ in the sense that:*

*(i) $u \in D_0$ with $u(0) = u_0$ in $V$,*
*(ii) there is $\xi \in L^2(0,T;V)$ such that $\xi(t) \in \partial_* \psi^t(u'(t))$ in $V^*$ for a.e. $t \in (0,T)$ and*

$$\xi(t) + \partial_* \varphi^t(u;u(t)) + g(t,u(t)) = f(t) \text{ in } V^* \text{ for a.e. } t \in (0,T).$$

*Proof* Let $\varepsilon$ be a fixed positive constant in $(0,1]$ and consider the Cauchy problem for any given $v \in D_0$:

$$\begin{cases} \varepsilon F u'(t) + \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(v;u(t)) + g(t,u(t)) \ni f(t) \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u(0) = u_0 \text{ in } V. \end{cases} \tag{39}$$

Then, by virtue of Theorems 1 and 2, problem (39) possesses one and only one solution $u$ in the same sense of Definition 1, enjoying the estimate

$$\begin{aligned} &\int_0^T \{\varepsilon |u'(t)|_V^2 + \psi^t(u'(t))\} dt + \sup_{t\in[0,T]} \varphi^t(v;u(t)) \\ &\qquad \le N_0 := N_0(\varphi^0(v;u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)}). \end{aligned} \tag{40}$$

Now, putting

$$X(u_0) := \left\{ v \in W^{1,2}(0,T;V) \;\middle|\; v(0) = u_0, \; \int_0^T \psi^t(v'(t))dt \le N_0 \right\},$$

we define a mapping $\mathscr{S} : X(u_0) \to X(u_0)$ which maps each $v \in X(u_0) \subset D_0$ to the unique solution $u$ of (39), namely $\mathscr{S} v = u$; note from (40) that $u \in X(u_0)$. Clearly $X(u_0)$ is non-empty, convex and compact in $C([0,T];H)$.

Next we show that $\mathscr{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0,T];H)$. Let $v \in C([0,T];H)$, and let $\{v_n\}$ be a sequence in $X(u_0)$ such that $v_n \to v$ in $C([0,T];H)$ (as $n \to \infty$), and put $u_n = \mathscr{S} v_n$. Then we see that $v \in X(u_0)$, $v_n \to v$ weakly in $W^{1,2}(0,T;V)$ and $\sup_{n\in\mathbb{N}} \int_0^T \psi^t(v_n'(t))dt \le N_0$. From (40) it follows that there is a subsequence of $\{u_n\}$ (not relabeled) and a function $u \in W^{1,2}(0,T;V)$ such that

$$u_n \to u \text{ in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;V) \text{ as } n \to \infty$$

and

$$u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0, T] \text{ as } n \to \infty.$$

Also, we have

$$\varepsilon F u_n'(t) + \partial_* \psi^t(u_n'(t)) + \partial_* \varphi^t(v_n; u_n(t)) + g(t, u_n(t)) \ni f(t) \text{ in } V^* \quad \text{for a.e. } t \in (0, T). \tag{41}$$

Just as (30) in the proof of Proposition 1, we obtain by multiplying (41) for $t = s$ by $u_n'(s) - u'(s)$ and using (3) that

$$\begin{aligned} &\varepsilon C_F |u_n'(s) - u'(s)|_V^2 + \frac{d}{ds}\varphi^s(v_n; u_n(s) - u(s)) \\ &\le |\alpha'(s)|\varphi^s(v_n; u_n(s) - u(s)) + \bar{L}_n(s) \quad \text{for a.e. } s \in (0, T), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \bar{L}_n(s) = &\langle f(s) - g(s, u_n(s)) - \partial_* \varphi^s(v_n; u(s)), u_n'(s) - u'(s)\rangle \\ &- \varepsilon\langle F u'(s), u_n'(s) - u'(s)\rangle + \psi^s(u'(s)) - \psi^s(u_n'(s)) \quad \text{for a.e. } s \in (0, T). \end{aligned}$$

Since $g(\cdot, u_n) \to g(\cdot, u)$ and $\partial_* \varphi^{(\cdot)}(v_n; u) \to \partial_* \varphi^{(\cdot)}(v; u)$ (strongly) in $L^2(0, T; V^*)$ by conditions (C1), (B2') and the functional $w \to \int_0^t \psi^s(w(s))ds$ is lower semicontinuous on $L^2(0, T; V)$, it follows that

$$\limsup_{n\to\infty} \int_0^t \bar{L}_n(s)ds \le 0, \quad \forall t \in [0, T],$$

so that applying the Gronwall inequality to (42) yields that

$$\limsup_{n\to\infty} \varphi^t(v_n; u_n(t) - u(t)) \le 0, \text{ i.e. } u_n(t) \to u(t) \text{ in } V, \ \forall t \in [0, T]$$

and $u_n' \to u'$ in $L^2(0, T; V)$ as $n \to \infty$. This implies from (B1') and (B2') that $\partial_* \varphi^t(v_n; u_n(t)) \to \partial_* \varphi^t(v; u(t))$ in $V^*$ for all $t \in [0, T]$, whence

$$\begin{aligned} \varepsilon F u_n'(t) + \partial_* \psi^t(u_n'(t)) \ni \xi_n(t) &:= f(t) - \partial_* \varphi^t(v_n; u_n(t)) - g(t, u_n(t)) \\ &\to f(t) - \partial_* \varphi^t(v; u(t)) - g(t, u(t)) =: \xi(t) \text{ in } V^* \end{aligned}$$

for a.e. $t \in [0, T]$ as $n \to \infty$. Accordingly, by the demi-closedness of maximal monotone mappings, we have $\xi(t) \in \varepsilon F u'(t) + \partial_* \psi^t(u'(t))$ for a.e. $t \in [0, T]$. As a consequence, $u$ satisfies (39), namely $u = \mathscr{S} v$. By the uniqueness of solution to (39) we conclude that $\mathscr{S} v_n = u_n \to u = \mathscr{S} v$ in $C([0, T]; H)$ without extracting any subsequence from $\{u_n\}$. Thus $\mathscr{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0,T];H)$. Therefore, by the Schauder fixed point theorem, $\mathscr{S}$ has at least one fixed point $u$ in $X(u_0)$. This is a solution of (39) with $v = u$.

We showed above that for every small $\varepsilon > 0$ the Cauchy problem

$$\begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V \end{cases}$$

admits at least one solution $u_\varepsilon \in W^{1,2}(0,T;V)$ enjoying estimate

$$\varepsilon \int_0^T |u_\varepsilon'(t)|_V^2 dt + \int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon; u_\varepsilon(t)) \le N_0, \quad \forall \varepsilon \in (0,1].$$

Therefore, we can choose a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ (as $n \to \infty$) and a function $u \in D_0$ so that

$$\begin{gathered} u_n := u_{\varepsilon_n} \to u \text{ in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;V) \text{ as } n \to \infty, \\ u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty, \\ \varepsilon_n u_n' \to 0 \text{ in } L^2(0,T;V) \text{ as } n \to \infty, \\ \sup_{n\in\mathbb{N}} \int_0^T \psi^t(u_n'(t))dt \le N_0. \end{gathered}$$

Now, in the same way just as in the convergence proof of Theorem 1 again, we can infer from (B2') and (C1) that the limit $u$ satisfies

$$\begin{cases} \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V. \end{cases}$$

Thus $u$ is a required solution to (QP;$f, u_0$). □

## 6 Applications

In this section, we consider two applications of the general results (Theorems 1 and 3).

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($1 \le N < \infty$) with a smooth boundary $\Gamma := \partial\Omega$, and let us set

$$V := H_0^1(\Omega), \quad H := L^2(\Omega);$$

note that condition (3) is satisfied with $C_F = 1$.

**(Application 1)**

Let $T > 0$ be a fixed real number, and let $Q := (0,T) \times \Omega$. Also, let $\rho$ be a prescribed obstacle function in $C(\overline{Q})$ such that

$$(0 <)\rho_* \le \rho(t,x) \le \rho^*, \quad \forall (t,x) \in \overline{Q}, \tag{43}$$

where $\rho_*$ and $\rho^*$ are positive constants.

Now, for each $t \in [0,T]$ define a closed convex set $K(t)$ in $V$ by

$$K(t) := \{z \in V\ ;\ |\nabla z(x)| \le \rho(t,x) \text{ for a.e. } x \in \Omega\}.$$

Then, our variational inequality with constraint is of the form:

$$\left.\begin{array}{c} u_t(t) \in K(t) \ \text{ for a.e. } t \in (0,T), \\ \displaystyle\int_\Omega a(t,x)\nabla u(t,x)\cdot\nabla(u_t(t,x)-v(x))dx + \int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \\ \displaystyle\le \int_\Omega f(t)(u_t(t,x)-v(x))dx \quad \text{for all } v \in K(t) \text{ and a.e. } t \in (0,T), \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{array}\right\} \tag{44}$$

where $g(\cdot,\cdot)$ is a Lipschitz continuous function on $[0,T]\times\mathbb{R}$, $f$ is a function given in $L^2(0,T;H)$, $u_0$ is an initial datum in $V$, and $a(\cdot,\cdot)$ is a prescribed function on $Q$ such that

$$(0 <)a_* \le a(t,x) \le a^*,\ \forall (t,x) \in \overline{Q}, \qquad a = a(t) \in W^{1,1}(0,T;C(\overline{\Omega})),$$

where $a_*$ and $a^*$ are positive constants.

Now we show the existence of a solution to (44) on $[0,T]$ by applying the abstract result Theorem 1. To this end, for each $t \in [0,T]$ define proper l.s.c. and convex functions $\psi^t$, $\varphi^t$ on $V$ and $\alpha(t)$ by

$$\psi^t(z) := I_{K(t)}(z) = \begin{cases} 0, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise}, \end{cases}, \quad \forall z \in V,\ \forall t \in [0,T], \tag{45}$$

$$\varphi^t(z) := \frac{1}{2}\int_\Omega a(t,x)|\nabla z(x)|^2 dx, \quad \forall z \in V,\ \forall t \in [0,T] \tag{46}$$

and

$$\alpha(t) := \frac{1}{a_*}\int_0^t \left|\frac{\partial}{\partial\tau}a(\tau,x)\right| d\tau, \quad \forall t \in [0,T]. \tag{47}$$

We see easily that

$$z^* \in \partial_* \psi^t(z) \Longleftrightarrow z^* \in V^*,\ z \in K(t) \text{ and } \langle z^*, v - z\rangle \le 0, \quad \forall v \in K(t) \tag{48}$$

and

$$\langle \partial_* \varphi^t(z), v\rangle = \int_\Omega a(t,x)\nabla z(x)\cdot \nabla v(x)dx, \quad \forall z,\ v \in V \tag{49}$$

for all $t \in [0, T]$. In our present case it is easy to check *Assumptions* $(A)$–$(C)$, except for (A1). We prove (A1) in the following lemma.

**Lemma 2 (cf. [11, Lemma 10.1])** *For any sequence* $\{t_n\} \subset [0, T]$ *with* $t_n \to t$ *(as* $n \to \infty$*),* $\psi^{t_n}$ *converges to* $\psi^t$ *on* $V$ *in the sense of Mosco as* $n \to \infty$*.*

*Proof* Assume that

$$\{z_n\} \subset V, z_n \to z \text{ weakly in } V \text{ and } \liminf_{n\to\infty} \psi^{t_n}(z_n) < \infty. \tag{50}$$

We may assume that $z_n \in K(t_n)$ for all $n$. By definition

$$|\nabla z_n(x)| \le \rho(t_n, x), \quad \text{a.e. } x \in \Omega. \tag{51}$$

Also, by $\rho \in C(\overline{Q})$, given $\varepsilon > 0$, there exists a positive integer $n_\varepsilon$ such that

$$\rho(t_n, x) \le \rho(t, x) + \varepsilon \ \text{ for all } x \in \Omega \text{ and all } n \ge n_\varepsilon. \tag{52}$$

Therefore, it follows from (51) and (52) that

$$|\nabla z_n(x)| \le \rho(t, x) + \varepsilon, \quad \text{a.e. } x \in \Omega \text{ and all } n \ge n_\varepsilon,$$

which implies that

$$z_n \in K_\varepsilon(t) := \{z \in V\ ;\ |\nabla z(x)| \le \rho(t, x) + \varepsilon,\ \text{a.e. } x \in \Omega\} \ \text{ for all } n \ge n_\varepsilon. \tag{53}$$

Note that $K_\varepsilon(t)$ is weakly compact in $V$, since the set $K_\varepsilon(t)$ is bounded, closed and convex in $V$. Therefore, it follows from (50) and (53) that

$$z \in K_\varepsilon(t).$$

Since $\varepsilon$ is arbitrary, we have $z \in K(t)$. Hence, we observe that

$$\liminf_{n\to\infty} \psi^{t_n}(z_n) = 0 = \psi^t(z).$$

Next, we verify another condition of the Mosco convergence. To this end, assume $z \in K(t)$. Note from $\rho \in C(\overline{Q})$ that for each $k$, choose a positive integer $N_k$ so that $N_k \geq k$ and

$$\rho(t,x) \leq \rho(t_n,x) + \frac{\rho_*}{k} \quad \text{for all } x \in \Omega \text{ and all } n \geq N_k. \tag{54}$$

Then, we observe from $z \in K(t)$, (43) and (54) that

$$|\nabla z(x)| \leq \rho(t,x) \leq \rho(t_n,x) + \frac{\rho_*}{k} \leq \left(1 + \frac{1}{k}\right)\rho(t_n,x),$$

for a.e. $x \in \Omega$ and all $n \geq N_k$, which implies that

$$\left|\nabla\left(\frac{1}{1+\frac{1}{k}} z(x)\right)\right| \leq \rho(t_n,x), \quad \text{a.e. } x \in \Omega \text{ and all } n \geq N_k. \tag{55}$$

Putting

$$z_n := \begin{cases} \dfrac{1}{1+\frac{1}{k}} z, & \text{if } n \geq N_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{if } 1 \leq n < N_1, \end{cases}$$

we observe from (55) and $z \in K(t)$ that $t_n \to t$ as $n \to \infty$,

$$K(t_n) \ni z_n \to z \ \text{ in } V \ \text{ as } n \to \infty$$

and

$$\lim_{n\to\infty} \psi^{t_n}(z_n) = 0 = \psi^t(z).$$

Thus, $\psi^{t_n}$ converges to $\psi^t$ on $V$ in the sense of Mosco. □

Taking account of (45)–(49), problem (44) can be reformulated in the abstract form (P;$f, u_0$). Therefore, by Theorem 1, problem (44) admits a solution $u \in W^{1,2}(0,T;V)$.

**(Application 2)**

Let us consider problem (44) with the diffusion coefficient $a(t,x)$ replaced by $a(t,x,u)$, namely

$$\left.\begin{array}{c} u_t(t) \in K(t) \ \text{ for a.e. } t \in (0,T), \\ \displaystyle\int_\Omega a(t,x,u(t,x))\nabla u(t,x)\cdot\nabla(u_t(t,x)-v(x))dx \\ \displaystyle+\int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \le \int_\Omega f(t)(u_t(t,x)-v(x))dx \\ \text{for all } v\in K(t) \text{ and a.e. } t\in(0,T), \\ u(0,x)=u_0(x), \quad x\in\Omega, \end{array}\right\} \tag{56}$$

where $K(t)$, $f$ and $u_0$ are the same as in Application 1; the obstacle function $\rho$ satisfies (43) as well. As to the function $a(t,x,r)$ we suppose that

$$\begin{cases} (0<)a_* \le a(t,x,r) \le a^*, \quad \forall (t,x)\in\overline{Q},\ \forall r\in\mathbb{R}, \\ |a(t_1,x,r_1)-a(t_2,x,r_2)| \le L_a(|t_1-t_2|+|r_1-r_2|), \\ \qquad \forall t_i\in[0,T],\ r_i\in\mathbb{R},\ i=1,2,\ \forall x\in\overline{\Omega}, \end{cases} \tag{57}$$

where $a_*,\ a^*$ and $L_a$ are positive constants. Also, condition (43) is assumed and $\psi^t$ is defined by (45) as well. Furthermore the $(t,v)$-dependent functional $\varphi^t(v;z)$ is given by

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(t,x,v(t,x))|\nabla z(x)|^2dx, \ \ \forall t\in[0,T],\ \forall v\in D_0,\ \forall z\in V, \tag{58}$$

where

$$D_0 = \{v\in W^{1,2}(0,T;V) \mid v'(t)\in K(t) \text{ for a.e. } t\in[0,T]\}.$$

The subdifferential $\partial_*\varphi^t(v;\cdot)$ of $\varphi^t(v;\cdot)$ is given by

$$\langle \partial_*\varphi^t(v;z), w\rangle = \int_\Omega a(t,x,v(t,x))\nabla z(x)\cdot\nabla w(x)dx \tag{59}$$

for all $t\in[0,T]$, $v\in D_0$ and $z,\ w\in V$. Note from (43) that

$$|\nabla v'(t,x)| \le \rho^* \text{ for a.e. } (t,x)\in Q,$$

which implies that

$$\sup_{t\in[0,T]} |v'(t)|_{L^\infty(\Omega)} \le \bar\rho^*,\ \forall v\in D_0,\ \text{ for some constant } \bar\rho^*>0. \tag{60}$$

Therefore, it is easy to check by (57) that *Assumption (B')* holds with

$$C_3' := a^*,\ C_4' := \frac{1}{2}a_*,\ \alpha(t) := \frac{1}{a_*}L_a(1+\bar{\rho}^*)t.$$

In fact, (B1') and (B3') are immediately seen from the definition of $\varphi^t(v,z)$. Also, if $v_n \in D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi^t(v_n'(t))dt < \infty$ and $v_n \to v$ in $C([0,T];H)$, then we have

$$\begin{aligned}
&|\langle \partial_*\varphi^t(v_n;z) - \partial_*\varphi^t(v;z), w\rangle| \\
&\le \int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))||\nabla z(x)||\nabla w(x)|dx \\
&\le \left(\int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))|^2|\nabla z(x)|^2dx\right)^{\frac{1}{2}} |w|_V
\end{aligned}$$

and the last integral converges to 0 by the Lebesgue dominated convergence theorem, so that $\partial_*\varphi^t(v_n;z) \to \partial_*\varphi^t(v;z)$ (strongly) in $V^*$. Thus (B2') holds. Condition (B4') is verified by using (43), (57) and (60) as follows:

$$\begin{aligned}
&|\varphi^t(v;z) - \varphi^s(v;z)| \\
&\le \frac{1}{2}\int_\Omega |a(t,x,v(t,x)) - a(s,x,v(s,x))||\nabla z(x)|^2dx \\
&\le \frac{1}{2}\int_\Omega \int_s^t |a_\tau(\tau,x,v(\tau,x)) + a_v(\tau,x,v(\tau,x))v_\tau(\tau,x)||\nabla z(x)|^2 d\tau dx \\
&\le \frac{1}{a_*}(L_a + L_a\bar{\rho}^*)|t-s| \cdot \frac{1}{2}\int_\Omega a(s,x,v(s,x))|\nabla z(x)|^2dx \\
&= \frac{1}{a_*}L_a(1+\bar{\rho}^*)|t-s|\varphi^s(v;z),
\end{aligned}$$

where $a_\tau := \frac{\partial}{\partial\tau}a(\tau,x,v)$ and $a_v := \frac{\partial}{\partial v}a(\tau,x,v)$.

By (58)–(59) problem (56) can be described as

$$\begin{cases} \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u;u(t)) + g(t,u(t)) \ni f(t) \text{ in } V^*, \\ u(0) = u_0 \ \text{ in } V. \end{cases}$$

By virtue of Theorem 3, this Cauchy problem admits a solution $u \in D_0$, so does problem (56).

*Remark 4* (44) is the variational formulation of (P;$f, u_0$). It seems similar to hyperbolic variational problems and our abstract result might be evolved to the hyperbolic case. However, in this paper, we do not touch it, since the mathematical structure is essentially of parabolic or pseudo-parabolic type.

*Remark 5* Problems (P;$f, u_0$) and (QP;$f, u_0$) have a wide class of real world applications, for instance, reaction-diffusion systems for multi-species bacteria and solid-liquid phase transition systems with partial irreversibility (cf. [3, 4]). Moreover, when such phenomena are considered in fluid flows, they are coupled with various variational inequalities of the Navier-Stokes type which can be described by our doubly nonlinear evolution equations, too.

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