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Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs

In Honour of Prof. Gianni Gilardi

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Preface

This special volume is dedicated to Gianni Gilardi on the occasion of his 70th birthday, in tribute to his important achievements in respect of several theoretical and applied problems, especially in the fields of partial differential equations, variational inequalities, optimal control, free boundary problems and phase transition models.

Gianni Gilardi was born in Milan in February 1947. He studied mathematics at the University of Pavia, where he graduated with full marks in October 1970. During that period, he was alumnus of the *Collegio Ghislieri*, a prestigious historical college in Pavia founded by Pope St. Pius V in 1567. After being a teaching assistant at the University of Pavia, Gianni became a full professor of mathematical analysis at the Polytechnic University of Milan in November 1980. He moved back to Pavia in 1985, where he has been appreciated as a teacher and university professor for more than 30 years. He has taught an impressive number of courses in the Schools of Engineering, Physics and Mathematics at both undergraduate and graduate levels, as well as for PhD students. He has been the advisor to a number of master and PhD students, including some of us editors of the present volume. He has never spared himself from helping colleagues, working for the community, or accepting academic responsibilities. Thus he did not hesitate in accepting the invitation to serve as chairman of the Department of Mathematics of the University of Pavia, a position he held for 6 years, or, more recently, as coordinator of the teaching programs in mathematics at the University of Pavia. In both these roles, he has been appreciated not only by colleagues but also by the administrative staff.

Gianni has been an associate fellow in the academy *Istituto Lombardo Accademia di Scienze e Lettere* since 2002. He is the author or coauthor of eight books and of around 100 research papers published in prestigious international journals. He has given numerous talks in Italy and abroad (Canada, Czech Republic, France, Germany, Japan, Portugal, Romania, Spain, Switzerland, USA) and contributed to the organization of a large number of conferences and courses.

Gianni's research activity has been intense and varied, being mainly devoted to the analysis of nonlinear PDEs, but with particular attention to the related applications. He has primarily been interested in the study of free boundary problems and phase transition models. In more detail, we may mention among his scientific interests:

- Well-posedness and regularity theory for second-order abstract evolution equations.
- Monotonicity, speed of propagation and regularity properties of the free boundary for the dam problem, the time-dependent dam problem in a general unbounded domain, and a regularity result for the time derivative of the solution.
- Error estimates for space-time discretizations of parabolic variational inequalities and a class of noncoercive stationary variational inequalities.
- Phase field models with memory and more general nonlinear Volterraintegrodifferential equations.
- Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions (this includes the most cited paper coauthored by Gianni).
- Phase separation and phase segregation models including also mechanical effects.
- General phase field systems: Caginalp and Penrose–Fife models, evolutions based on the entropy balance, shape memory alloys, Cahn–Hilliard systems (also nonlocal), and dynamic boundary conditions.
- Diffuse interface models describing tumor growth dynamics.
- Control problems for phase field systems: distributed and boundary optimal control, sliding mode control, and feedback stabilization.

It is a great pleasure for us five editors of this volume to celebrate the 70th birthday of our friend Gianni. In addition to being a teacher to some of us, he has been a pleasant colleague who could always be approached with questions about mathematics or the proof of a technical lemma, knowing that he would be prepared to discuss and willing to solve problems. Gianni is very generous in providing help to young mathematicians and less young colleagues requiring his advice when checking whether "that solution" could be as regular as necessary.

His webpage contains a number of short notes, lecture notes of courses, and exercises, with examples and counterexamples that he has generously made available to students and colleagues. People who have had the chance to write papers with him experienced his generosity when, during discussions at the blackboard, they somehow began to see how the mathematical results were deduced, with Gianni already declaring his personal willingness to write down the paper.

The appreciation that Gianni always received within the scientific community is reflected in the enthusiasm with which many applied scientists and mathematicians agreed to contribute to this special volume dedicated to him, as announced in the beautiful Palazzone di Cortona during the INdAM conference "Optimal Control for Evolutionary PDEs and Related Topics" in June 2016. We editors of the present volume are warmly grateful to all the authors for their precious contributions,which will surely be appreciated also by Gianni.

The volume gathers original and peer-reviewed research papers in the field of partial differential equations, with special emphasis on mathematical models in phase transitions, complex fluids, and thermomechanics. In particular, the following thematic areas are developed: nonlinear dynamic and stationary equations, well-posedness of initial and boundary value problems for systems of PDEs, regularity properties for the solutions, optimal control problems and optimality conditions, and feedback stabilization and stability results. Most of the papers are presented in a self-contained manner; as a general strategy, the articles describe some new achievements and/or the state of the art in their line of research, providing interested readers with an overview of recent progress and future research items in PDEs.

In conclusion, we would like to join the large family of Gianni, including his wife Ce, his two daughters Carla and Laura and their husbands, his five wonderful grandchildren, his friends and the contributors to the present volume, in celebrating his accomplishments and expressing the wish that he may continue his research activity for many years to come. Let us conclude with a motto that Gianni will surely appreciate: *"Sapientia cum probitate morum coniuncta humanæ mentis perfectio"*.

Pavia, Italy — Pierluigi Colli
Bologna, Italy — Angelo Favini
Pavia, Italy — Elisabetta Rocca
Pavia, Italy — Giulio Schimperna
Berlin, Germany — Jürgen Sprekels
July 2017

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About the Editors

Prof. Pierluigi Colli graduated in Mathematics at the University of Pavia in 1981, before becoming a researcher and associate professor at the same university. He became a professor of mathematical analysis at the University of Torino in 1994, and then he moved back to Pavia in 1998. He is author or coauthor of more than 150 papers, and coeditor of a number of special volumes. His main research area is the mathematical analysis of nonlinear evolution problems, in particular parabolic systems of partial differential equations arising from differential models in physics, thermodynamics, mechanics, and physiology.

Prof. Angelo Favini was an assistant professor from 1971 to 1976, and has been a professor at the University of Bologna since then. He is the author of 230 publications in international journals, mainly devoted to interpolation, differential equations in Banach spaces, partial differential equations, control theory, and ill-posed problems. His main research focus is on degenerate equations, and he has written a monograph on this subject with A. Yagi (Osaka). He is also the author of a monograph on nonlinear equations with G. Marinoschi (Bucharest), Springer-Verlag, 2012.

Prof. Elisabetta Rocca graduated in Mathematics in 1999 at the University of Pavia, where she also obtained her PhD in 2004. She was a researcher at the University of Milan till 2011 when she became associate professor. She moved to the WIAS in Berlin in 2013 where she spent 2 years coordinating a research group with the ERC Starting Grant she received as PI in 2011. She moved to the University of Pavia in 2016 where she is associate professor. She is author of more than 80 papers in mathematical analysis and applications.

Prof. Giulio Schimperna obtained his PhD in Mathematics at Milan University in 2000. Since 2006 he has been a professor of mathematical analysis in Pavia. He has authored more than 70 papers published in international scientific journals. His scientific interests mainly focus on the analysis of nonlinear evolutionary partial differential equations, and, in particular, mathematical models for phase transitions, damaging, thermomechanics, and complex fluids.

Prof. Jürgen Sprekels graduated in 1972 in Mathematics at the University of Hamburg (Germany), where he also received his PhD in 1975. He was a professor at the universities of Augsburg and Essen, before moving to a full professorship at the Humboldt-Universität zu Berlin in 1994. From 1994 to 2015, he was also the director of the Weierstrass Institute (WIAS) in Berlin. He is the coauthor of two monographs, coeditor of several conference proceedings, and coauthor of nearly 200 research papers in various fields of applied mathematics.

Rate of Convergence for Eigenfunctions of Aharonov-Bohm Operators with a Moving Pole

Laura Abatangelo and Veronica Felli

Abstract We study the behavior of eigenfunctions for magnetic Aharonov-Bohm operators with half-integer circulation and Dirichlet boundary conditions in a planar domain. We prove a sharp estimate for the rate of convergence of eigenfunctions as the pole moves in the interior of the domain.

Keywords Aharonov-Bohm potential • Convergence of eigenfunctions • Magnetic Schrödinger operators

2010 AMS Classification 35J10, 35Q40, 35J75

1 Introduction

For every $a = (a_1, a_2) \in \mathbb{R}^2$, we consider the Aharonov-Bohm vector potential with pole a and circulation $1/2$ defined as

$$A_a(x_1, x_2) = A_0(x_1 - a_1, x_2 - a_2), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

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where

$$A_0(x_1, x_2) = \frac{1}{2}\left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}\right), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

The Aharonov-Bohm vector potential A_a generates a δ-type magnetic field, which is called Aharonov–Bohm field: this field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point a, as the radius of the solenoid tends to zero while the flux through the solenoid section remains constantly equal to $1/2$. Neglecting the irrelevant coordinate along the solenoid axis, the problem becomes 2-dimensional.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we consider the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{E_a}$$

in a weak sense, where the magnetic Schrödinger operator with Aharonov-Bohm potential $(i\nabla + A_a)^2$ acts on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

A suitable functional setting for stating a weak formulation of (E_a) can be introduced as follows: for every $a \in \Omega$, the functional space $H^{1,a}(\Omega, \mathbb{C})$ is defined as the completion of

$$\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

In view of the following Hardy type inequality proved in [12]

$$\int_{\mathbb{R}^2} |(i\nabla + A_a)u|^2\, dx \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x-a|^2}\, dx,$$

which holds for all $a \in \mathbb{R}^2$ and $u \in C^\infty_c(\mathbb{R}^2 \setminus \{a\}, \mathbb{C})$, it is easy to verify that

$$H^{1,a}(\Omega, \mathbb{C}) = \left\{u \in H^1(\Omega, \mathbb{C}) : \tfrac{u}{|x-a|} \in L^2(\Omega, \mathbb{C})\right\}.$$

We also denote as $H^{1,a}_0(\Omega, \mathbb{C})$ the space obtained as the completion of

$$C^\infty_c(\Omega \setminus \{a\}, \mathbb{C})$$

with respect to the norm $\|\cdot\|_{H^{1,a}(\Omega,\mathbb{C})}$, so that

$$H_0^{1,a}(\Omega,\mathbb{C}) = \Big\{u \in H_0^1(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\Big\}.$$

For every $a \in \Omega$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) in a weak sense if there exists $u \in H_0^{1,a}(\Omega,\mathbb{C}) \setminus \{0\}$ (called an eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u) \cdot \overline{(i\nabla v + A_a v)}\, dx = \lambda \int_\Omega u\overline{v}\, dx \quad \text{for all } v \in H_0^{1,a}(\Omega,\mathbb{C}).$$

From classical spectral theory, the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues (repeated according to their finite multiplicity)

$$\lambda_1^a \le \lambda_2^a \le \cdots \le \lambda_j^a \le \ldots .$$

The mathematical interest in Aharonov-Bohm operators with half-integer circulation can be motivated by a strong relation between spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity and nodal domains of eigenfunctions of these operators. Indeed, a magnetic characterization of minimal partitions was given in [10] (see also [5–7, 14]): partitions with points of odd multiplicity can be obtained as nodal domains by minimizing a certain eigenvalue of an Aharonov-Bohm Hamiltonian with respect to the number and the position of poles. From this, a natural interest in the study of the properties of the map $a \mapsto \lambda_j^a$ (associating eigenvalues of magnetic operators to the position of poles) arises. In [1, 2, 4, 8, 13, 15] the behaviour of the function $a \mapsto \lambda_j^a$ in a neighborhood of a fixed point $b \in \overline{\Omega}$ has been investigated, both in the cases $b \in \Omega$ and $b \in \partial\Omega$. In particular, the analysis carried out in [1, 2, 4, 8, 15] shows that, as the pole moves towards a fixed limit pole $b \in \overline{\Omega}$, the rate of convergence of λ_j^a to λ_j^b is related to the number of nodal lines of the limit eigenfunction meeting at b. In the present paper we aim at deepening this analysis describing also the behaviour of the corresponding eigenfunctions; in particular, we will derive a sharp estimate for the rate of convergence of eigenfunctions associated to moving poles, in terms of the number of nodal lines of the limit eigenfunction.

Without loss of generality, we can assume that

$$b = 0 \in \Omega.$$

Let us assume that there exists $n_0 \ge 1$ such that

$$\lambda_{n_0}^0 \quad \text{is simple,} \tag{1}$$

and denote $\lambda_0 = \lambda_{n_0}^0$ and, for any $a \in \Omega$, $\lambda_a = \lambda_{n_0}^a$. From [13, Theorem 1.3] it follows that the map $a \mapsto \lambda_a$ is analytic in a neighborhood of 0; in particular we have that

$$\lambda_a \to \lambda_0, \quad \text{as } a \to 0. \tag{2}$$

Let $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C}) \setminus \{0\}$ be a $L^2(\Omega,\mathbb{C})$-normalized eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda^0_{n_0}$, i.e. satisfying

$$\begin{cases} (i\nabla + A_0)^2\varphi_0 = \lambda_0\varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \\ \int_\Omega |\varphi_0(x)|^2\,dx = 1. \end{cases} \tag{3}$$

From [9, Theorem 1.3] (see also [14, Theorem 1.5]) it is known that φ_0 has at 0 a zero of order $\frac{k}{2}$ for some odd $k \in \mathbb{N}$, i.e. there exist $k \in \mathbb{N}$ odd and $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0,0)$ and

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to e^{i\frac{t}{2}}\left(\beta_1 \cos\left(\frac{k}{2}t\right) + \beta_2 \sin\left(\frac{k}{2}t\right)\right) \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C}) \tag{4}$$

as $r \to 0^+$ for any $\tau \in (0,1)$. The asymptotics (4) (together with the fact that the right hand side of (4) is a complex multiple of a real-valued function, see [11]) implies that φ_0 has exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts; such nodal lines are tangent to the k half-lines

$$\left\{\left(t, \tan\left(\alpha_0 + j\frac{2\pi}{k}\right)t\right) : t > 0\right\}, \quad j = 0, 1, \dots, k-1,$$

for some angle $\alpha_0 \in [0, \frac{2\pi}{k})$.

In [1, 2] it has been proved that, under assumption (1) and being k as in (4),

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos\big(k(\alpha - \alpha_0)\big) \qquad \text{as } a \to 0 \text{ with } a = |a|(\cos\alpha, \sin\alpha), \tag{5}$$

where $C_0 > 0$ is a positive constant depending only on k, β_1, and β_2. More precisely, in [1, 2] it has been proved that

$$C_0 = -4(|\beta_1|^2 + |\beta_2|^2)\,\mathfrak{m}_k$$

where

$$\mathfrak{m}_k = \min_{u \in \mathscr{D}^{1,2}_s(\mathbb{R}^2_+)} \left[\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\,dx - \frac{k}{2}\int_0^1 t^{\frac{k}{2}-1}u(t,0)\,dt\right] < 0. \tag{6}$$

In (6), s denotes the half-line $s := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1\}$ and $\mathscr{D}^{1,2}_s(\mathbb{R}^2_+)$ is the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \setminus s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2\,dx)^{1/2}$.

Let us now consider a suitable family of eigenfunctions relative to the approximating eigenvalue λ_a. In order to choose eigenfunctions with a suitably normalized phase, let us introduce the following notations.

For every $\alpha \in [0, 2\pi)$ and $b = (b_1, b_2) = |b|(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \setminus \{0\}$, we define

$$\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\alpha, \alpha + 2\pi) \quad \text{and} \quad \theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\alpha, \alpha + 2\pi)$$

such that

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{and} \quad \theta_0^b(r(\cos t, \sin t)) = t, \\ \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha + 2\pi).$$

We also define

$$\theta_0 : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi)$$

such that

$$\theta_0(r\cos t, r\sin t) = t \quad \text{for all } r > 0 \text{ and } t \in [0, 2\pi).$$

For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a, & \text{in } \Omega, \\ \varphi_a = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

such that its modulus and phase are normalized in such a way that

$$\int_\Omega |\varphi_a(x)|^2\, dx = 1 \quad \text{and} \quad \int_\Omega e^{\frac{i}{2}(\theta_0^a - \theta_a)(x)} \varphi_a(x) \overline{\varphi_0(x)}\, dx \text{ is a positive real number,} \tag{8}$$

where φ_0 is as in (3). From (1), (2), (3), (7), (8), and standard elliptic estimates, it follows that $\varphi_a \to \varphi_0$ in $H^1(\Omega, \mathbb{C})$ and in $C^2_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C})$ and

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}). \tag{9}$$

The main result of the present paper establishes the sharp rate of the convergence (9).

Theorem 1 *For $\alpha \in \mathbb{R}$, $p = (\cos\alpha, \sin\alpha)$ and $a = |a|p \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ solve Eqs. (7)–(8) and $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be a solution to (3) satisfying (1) and (4). Then there exists $\mathfrak{L}_p > 0$ such that*

$$|a|^{-k} \left\| (i\nabla + A_a)\varphi_a - e^{\frac{i}{2}(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0 \right\|^2_{L^2(\Omega, \mathbb{C})} \to (|\beta_1|^2 + |\beta_2|^2)\mathfrak{L}_p \tag{10}$$

as $a = |a|p \to 0$. *Moreover the function* $\alpha \mapsto \mathfrak{L}_{(\cos\alpha,\sin\alpha)}$ *is continuous, even, and periodic with period* $\frac{2\pi}{k}$.

The constant $\mathfrak{L}_p$ in Theorem 1 can be characterized as the energy of the solution of an elliptic problem with cracks (see (22)), where jumping conditions are prescribed on the segment connecting 0 and p and on the tangent to a nodal line of φ_0, see Sect. 3.

For every $\alpha \in \mathbb{R}$, let us denote as $s_\alpha = \{t(\cos\alpha, \sin\alpha) : t \geq 0\}$ the half-line with slope α. We notice that, if $a = |a|(\cos\alpha, \sin\alpha)$, then $\nabla\big(\frac{\theta_a}{2}\big) = A_a$, $\nabla\big(\frac{\theta_0^a}{2}\big) = A_0$, and $e^{-\frac{i}{2}\theta_a}$ and $e^{-\frac{i}{2}\theta_0^a}$ are smooth in $\Omega \setminus s_\alpha$. Thus

$$i\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a) = e^{-\frac{i}{2}\theta_a}(i\nabla + A_a)\varphi_a, \quad i\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_0^a}\varphi_0) = e^{-\frac{i}{2}\theta_0^a}(i\nabla + A_0)\varphi_0,$$

where $\nabla_{\Omega\setminus s_\alpha}$ is the distributional gradient in $\Omega \setminus s_\alpha$. Hence (10) can be rewritten as

$$|a|^{-k}\left\|\nabla_{\Omega\setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a - e^{-\frac{i}{2}\theta_0^a}\varphi_0)\right\|^2_{L^2(\Omega,\mathbb{C})} \to (|\beta_1|^2 + |\beta_2|^2)\mathfrak{L}_p$$

as $a = |a|p \to 0$; thus it can be interpreted as a sharp asymptotics of the rate of convergence of the approximating eigenfunction to the limit eigenfunction in the space $\{u \in H^1(\Omega \setminus s_\alpha) : u = 0 \text{ on } \partial\Omega\}$.

The paper is organized as follows. In Sect. 2 we fix some notation and recall some known facts. In Sect. 3 we give a variational characterization of the limit profile of scaled eigenfunctions, which is used to study the properties (positivity, evenness, periodicity) of the function $p \mapsto \mathfrak{L}_p$. Finally, in Sect. 4 we prove Theorem 1, providing estimates of the energy variation first inside disks with radius $R|a|$ and then outside such disks; this latter outer estimate is performed exploiting the invertibility of an operator associated to the limit eigenvalue problem. We mention that this strategy was first developed in [3] in the context of spectral stability for varying domains, obtained by adding thin handles to a fixed limit domain.

2 Preliminaries and Some Known Facts

Through a rotation, we can easily choose a coordinate system in such a way that one nodal line of φ_0 is tangent to the x_1-axis, i.e. $\alpha_0 = 0$. In this coordinate system, we have that, letting β_1, β_2 be as in (4),

$$\beta_1 = 0. \tag{11}$$

The asymptotics of eigenvalues established in [1, 2], as well as the estimates for eigenfunctions we are going to achieve in the present paper, are based on a blow-up analysis for scaled eigenfunctions performed in [1, 2], whose main results are briefly recalled below for the sake of completeness.

For every $p \in \mathbb{R}^2$ and $r > 0$, we denote as $D_r(p)$ the disk of center p and radius r and as $D_r = D_r(0)$ the disk of center 0 and radius r. Moreover we denote, for every $r > 0$, $D_r^+ = \{(x_1, x_2) \in D_r : x_2 > 0\}$ and $D_r^- = \{(x_1, x_2) \in D_r : x_2 < 0\}$.

First of all, we observe that (4) completely describes the behaviour of φ_0 after scaling; indeed, letting

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{k/2}},$$

from [9, Theorem 1.3 and Lemma 6.1] we have that, under condition (11),

$$W_a \to \beta_2 e^{\frac{i}{2}\theta_0}\psi \quad \text{as } |a| \to 0 \tag{12}$$

in $H^{1,0}(D_R, \mathbb{C})$ for every $R > 1$, where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is the $\frac{k}{2}$-homogeneous function (which is harmonic on $\mathbb{R}^2 \setminus \{(r, 0) : r \geq 0\}$)

$$\psi(r\cos t, r\sin t) = r^{k/2} \sin\left(\frac{k}{2}t\right), \quad r \geq 0, \quad t \in [0, 2\pi]. \tag{13}$$

For every $p \in \mathbb{R}^2$, we denote by $\mathscr{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})$ the completion of $C_c^\infty(\mathbb{R}^N \setminus \{p\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathscr{D}_p^{1,2}(\mathbb{R}^2,\mathbb{C})} := \left(\int_{\mathbb{R}^2} \left|(i\nabla + A_p)u(x)\right|^2 dx\right)^{1/2}. \tag{14}$$

Proposition 1 ([2, Proposition 4]) *Let $\alpha \in [0, 2\pi)$ and $p = (\cos\alpha, \sin\alpha)$. There exists a unique function $\Psi_p \in H^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$ such that*

$$(i\nabla + A_p)^2\Psi_p = 0 \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,p}\text{-sense}, \tag{15}$$

and

$$\int_{\mathbb{R}^2\setminus D_r} \left|(i\nabla + A_p)(\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)}e^{\frac{i}{2}\theta_0}\psi)\right|^2 dx < +\infty, \quad \text{for any } r > 1, \tag{16}$$

where ψ is defined in (13). Furthermore (see [9, Theorem 1.5])

$$\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)}e^{\frac{i}{2}\theta_0}\psi = O(|x|^{-1/2}), \quad \text{as } |x| \to +\infty.$$

Theorem 2 ([2, Theorem 11 and Remark 12]) *For* $\alpha \in [0, 2\pi)$,

$$p = (\cos\alpha, \sin\alpha)$$

and $a = |a|p \in \Omega$, *let* $\varphi_a \in H_0^{1,a}(\Omega,\mathbb{C})$ *solve (7)–(8) and* $\varphi_0 \in H_0^{1,0}(\Omega,\mathbb{C})$ *be a solution to* (3) *satisfying* (1), (4), *and* (11). *Let* Ψ_p *be as in Proposition 1. Then*

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \to \beta_2\Psi_p \quad \text{as } a = |a|p \to 0,$$

in $H^{1,p}(D_R,\mathbb{C})$ *for every* $R > 1$ *and in* $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{p\},\mathbb{C})$.

In the sequel, we will denote

$$\tilde{\varphi}_a(x) = \frac{\varphi_a(|a|x)}{|a|^{k/2}}.$$

Sharp estimates of the energy variation under moving of poles will be derived by approximating the eigenfunction φ_a by $H^{1,0}$-functions in the less expensive way from the energetic point of view. For every $R > 2$ and $|a|$ sufficiently small, we define these approximating functions $v_{R,a}$ as follows:

$$v_{R,a} = \begin{cases} v_{R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{R,a}^{int}, & \text{in } D_{R|a|}, \end{cases}$$

where

$$v_{R,a}^{ext} := e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_a \quad \text{in } \Omega \setminus D_{R|a|}$$

solves

$$\begin{cases} (i\nabla + A_0)^2 v_{R,a}^{ext} = \lambda_a v_{R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{R,a}^{ext} = e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_a & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas $v_{R,a}^{int}$ is the unique solution to the problem

$$\begin{cases} (i\nabla + A_0)^2 v_{R,a}^{int} = 0, & \text{in } D_{R|a|}, \\ v_{R,a}^{int} = e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_a, & \text{on } \partial D_{R|a|}. \end{cases}$$

We notice that $v_{R,a} \in H_0^{1,0}(\Omega,\mathbb{C})$ for all $R > 2$ and a sufficiently small. For all $R > 2$ and $a = |a|p \in \Omega$ with $|a|$ small, we define

$$Z_a^R(x) := \frac{v_{R,a}^{int}(|a|x)}{|a|^{k/2}}. \tag{17}$$

For all $R > 2$ and $p = (\cos\alpha, \sin\alpha)$, we also define $z_{p,R}$ as the unique solution to

$$\begin{cases} (i\nabla + A_0)^2 z_{p,R} = 0, & \text{in } D_R, \\ z_{p,R} = e^{\frac{i}{2}(\theta_0^p - \theta_p)}\Psi_p, & \text{on } \partial D_R, \end{cases} \tag{18}$$

with Ψ_p as in Proposition 1.

Lemma 1 ([2, Remark 12]; [1, Lemma 8.3]) *For $R > 2$, $\alpha \in [0, 2\pi)$,*

$$p = (\cos\alpha, \sin\alpha)$$

and $a = |a|p \in \Omega$ small, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ solve (7)–(8), $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$ be a solution to (3) satisfying (1), (4), and (11), and Z_a^R be as in (17). Then

$$Z_a^R \to \beta_2 z_{p,R} \quad \text{as } a = |a|p \to 0 \text{ in } H^{1,0}(D_R, \mathbb{C}) \text{ for every } R > 2,$$

with $z_{p,R}$ being as in (18).

3 Variational Characterization of the Limit Profile Ψ_p

In [1], the limit profile Ψ_p was constructed by solving a minimization problem in the case $p = (1, 0)$ (i.e. for poles moving tangentially to a nodal line of the limit eigenfunction); in that case the limit profile was null on a half-line. In the spirit of [4] (where poles moving towards the boundary were considered), we extend this variational construction for poles moving along a generic direction $p = (\cos\alpha, \sin\alpha)$ and construct the limit profile by solving an elliptic crack problem prescribing the jump of the solution along the segment joining 0 and p.

Let us fix $\alpha \in (0, 2\pi)$ and $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$. We denote by Γ_p the segment joining 0 to p, that is to say

$$\Gamma_p = \{(r\cos\alpha, r\sin\alpha) : r \in (0, 1)\}.$$

Let $s_0 = \{(x_1, 0) : x_1 \geq 0\}$. We introduce the trace operators

$$\gamma^{\pm} : \bigcap_{R>0} H^1(D_R^{\pm} \setminus \Gamma_p) \longrightarrow H^{1/2}_{\text{loc}}(s_0).$$

We also define $\mathscr{H}$ as the completion of

$$\mathscr{D} = \big\{u \in H^1(\mathbb{R}^2 \setminus s_0) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \text{ and } u = 0$$
$$\text{in neighborhoods of 0 and } \infty\big\}$$

with respect to the Dirichlet norm $\left(\int_{\mathbb{R}^2\setminus s_0} |\nabla u|^2\right)^{1/2}$. In the following lemma we prove that a Hardy-type inequality can be recovered even in dimension 2, under the jump condition $\gamma^+(u) + \gamma^-(u) = 0$ forced for $\mathscr{H}$-functions.

Lemma 2 *The functions in $\mathscr{D}$ satisfy the following Hardy-type inequality:*

$$\int_{\mathbb{R}^2\setminus s_0} |\nabla\varphi(x)|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x|^2}\,dx \quad \textit{for all } u \in \mathscr{D}.$$

Proof This is a consequence of a suitable change of gauge combined with the Hardy-type inequality for magnetic Sobolev spaces proved in [12]. For any $\varphi \in \mathscr{D}$, the function $u := e^{\frac{i}{2}\theta_0}\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})$ according to the definition of the spaces $\mathscr{D}_p^{1,2}(\mathbb{R}^2,\mathbb{C})$ given in Sect. 2 (see (14)). From the Hardy-type inequality proved in [12], it follows that

$$\int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx.$$

Since $\nabla\left(\frac{\theta_0}{2}\right) = A_0$ and $(i\nabla + A_0)u = ie^{\frac{i}{2}\theta_0}\nabla\varphi$ in $\mathbb{R}^2 \setminus s_0$, we have that

$$\int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2\,dx = \int_{\mathbb{R}^2\setminus s_0} |\nabla\varphi(x)|^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx = \int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x|^2}\,dx,$$

thus the proof is complete.

As a direct consequence of Lemma 2, $\mathscr{H}$ can be characterized as

$$\mathscr{H} = \left\{u \in L^1_{\text{loc}}(\mathbb{R}^2) : \nabla_{\mathbb{R}^2\setminus s_0} u \in L^2(\mathbb{R}^2),\ \tfrac{u}{|x|} \in L^2(\mathbb{R}^2),\ \text{and } \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0\right\},$$

where $\nabla_{\mathbb{R}^2_+\setminus s_0} u$ denotes the distributional gradient of u in $\mathbb{R}^2 \setminus s_0$.

For $p \neq e$ with $e = (1,0)$, we also define the space $\mathscr{H}_p$ as the completion of

$$\mathscr{D}_p = \{u \in H^1(\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \text{ and } u = 0 \\ \text{in neighborhoods of 0 and } \infty\}$$

with respect to the Dirichlet norm

$$\|u\|_{\mathscr{H}_p} := \|\nabla u\|_{L^2(\mathbb{R}^2\setminus(s_0\cup\Gamma_p))}. \tag{19}$$

In order to prove that the space $\mathscr{H}_p$ defined above is a concrete functional space, the argument performed in Lemma 2 is no more suitable, since $\mathscr{H}_p$-functions do not satisfy a Hardy inequality in the whole $\mathbb{R}^2$. We need the following two lemmas, which establish a Hardy inequality in external domains and a Poincaré inequality in D_1 for $\mathscr{H}_p$-functions.

Lemma 3 *The functions in $\mathscr{H}_p$ satisfy the following Hardy inequality in $\mathbb{R}^2 \setminus D_1$:*

$$\|\varphi\|^2_{\mathscr{H}_p} \geq \frac{1}{4}\int_{\mathbb{R}^2\setminus D_1} \frac{|\varphi(x)|^2}{|x|^2}\,dx, \quad \textit{for all } \varphi \in \mathscr{H}_p.$$

Proof The proof follows via a change of gauge as in the proof of Lemma 2. More precisely, we notice that, for any $\varphi \in \mathscr{D}_p$, the function u defined as $u = e^{\frac{i}{2}\theta_0}\varphi$ in $\mathbb{R}^2 \setminus D_1$ and as $u(x) = u(x/|x|^2)$ in D_1 belongs to $\mathscr{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$. From the invariance of Dirichlet magnetic norms and Hardy norms by Kelvin transform and the Hardy-type inequality of [12], it follows that

$$\begin{aligned}\|\varphi\|^2_{\mathscr{H}_p} &\geq \int_{\mathbb{R}^2\setminus(D_1\cup s_0)} |\nabla\varphi(x)|^2\,dx = \frac{1}{2}\int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2\,dx \\ &\geq \frac{1}{8}\int_{\mathbb{R}^2}\frac{|u(x)|^2}{|x|^2}\,dx = \frac{1}{4}\int_{\mathbb{R}^2\setminus D_1}\frac{|\varphi(x)|^2}{|x|^2}\,dx.\end{aligned}$$

The conclusion follows by density of $\mathscr{D}_p$ in $\mathscr{H}_p$.

Lemma 4 *The functions in $\mathscr{H}_p$ satisfy the following Poincaré inequality in D_1:*

$$\|\varphi\|^2_{\mathscr{H}_p} \geq \frac{1}{6}\int_{D_1} |\varphi(x)|^2\,dx, \quad \textit{for all } \varphi \in \mathscr{H}_p.$$

Proof From the Divergence Theorem, the Schwarz inequality and the diamagnetic inequality, it follows that, for every $u \in H^{1,0}(D_1 \setminus \Gamma_p)$,

$$\begin{aligned}2\int_{D_1} |u|^2\,dx &= \int_{D_1\setminus\Gamma_p}\Big(\operatorname{div}(|u|^2 x) - 2|u|\nabla|u|\cdot x\Big)\,dx \\ &\leq \int_{\partial D_1}|u|^2\,ds + \int_{D_1\setminus\Gamma_p}|u|^2\,dx + \int_{D_1\setminus\Gamma_p}|\nabla|u||^2\,dx \\ &\leq \int_{\partial D_1}|u|^2\,ds + \int_{D_1}|u|^2\,dx + \int_{D_1\setminus\Gamma_p}|(i\nabla + A_0)u|^2\,dx\end{aligned}$$

where, when applying the Divergence Theorem, we have use the fact that $x\cdot\nu = 0$ on both sides of Γ_p. If $\varphi \in \mathscr{D}_p$, then $u := e^{\frac{i}{2}\theta_0}\varphi \in H^{1,0}(D_1 \setminus \Gamma_p)$ and

$$(i\nabla + A_0)u = ie^{\frac{i}{2}\theta_0}\nabla\varphi \text{ in } D_1 \setminus (s_0 \cup \Gamma_p),$$

hence the previous inequality yields

$$\int_{D_1}|\varphi|^2\,dx \leq \int_{\partial D_1}|\varphi|^2\,ds + \int_{D_1\setminus(s_0\cup\Gamma_p)}|\nabla\varphi|^2\,dx.$$

On the other hand, via the Divergence Theorem,

$$\int_{\partial D_1} |\varphi|^2 = \int_{\partial D_1} \varphi^2 \frac{x}{|x|^2} \cdot \nu = -\int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \operatorname{div}\left(\varphi^2 \frac{x}{|x|^2}\right)$$
$$+ \int_0^{+\infty} \gamma^+(\varphi^2) \frac{(s,0)}{s^2} \cdot (0,-1)\, ds$$
$$+ \int_0^{+\infty} \gamma^-(\varphi^2) \frac{(s,0)}{s^2} \cdot (0,1)\, ds$$
$$= -\int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \operatorname{div}\left(\varphi^2 \frac{x}{|x|^2}\right) = -2\int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \varphi \nabla\varphi \cdot \frac{x}{|x|^2}$$
$$\leq \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} |\nabla \varphi|^2 + \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi|^2}{|x|^2} \leq 5\|\varphi\|^2_{\mathcal{H}_p},$$

where the last inequality is obtained by Lemma 3. The proof is thus complete.

As a straightforward consequence of Lemmas 3 and 4, we can characterize the space $\mathcal{H}_p$ as

$$\Big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \nabla_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} u \in L^2(\mathbb{R}^2),\ \tfrac{u}{|x|} \in L^2(\mathbb{R}^2 \setminus D_1),\ u \in L^2(D_1),\ \text{and}$$
$$\gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0\Big\}.$$

The functions in $\mathcal{H}_p$ may clearly be discontinuous on Γ_p. For this reason, we introduce two trace operators. Let us consider the sets

$$U_p^+ = \{(x_1,x_2) \in \mathbb{R}^2 : \cos\alpha\, x_2 > \sin\alpha\, x_1\} \cap (D_1 \setminus s_0)$$

and

$$U_p^- = \{(x_1,x_2) \in \mathbb{R}^2 : \cos\alpha\, x_2 < \sin\alpha\, x_1\} \cap (D_1 \setminus s_0).$$

First, for any function u defined in a neighborhood of U_p^+, respectively U_p^-, we define the restriction

$$\mathcal{R}_p^+(u) = u|_{U_p^+}, \quad \text{respectively} \quad \mathcal{R}_p^-(u) = u|_{U_p^-}.$$

We observe that, since $\mathcal{R}_p^\pm$ maps $\mathcal{H}_p$ into $H^1(U_p^\pm)$ continuously, the trace operators

$$\gamma_p^\pm : \mathcal{H}_p \longrightarrow H^{1/2}(\Gamma_p), \quad u \longmapsto \gamma_p^\pm(u) := \mathcal{R}_p^\pm(u)|_{\Gamma_p}$$

are well defined and continuous from $\mathcal{H}_p$ to $H^{1/2}(\Gamma_p)$. Furthermore, by Sobolev trace inequalities and the Poincaré inequality of Lemma 4, it is easy to verify that the operator norm of $\gamma_p^{\pm}$ is bounded uniformly with respect to $p \in \mathbb{S}^1$, in the sense that there exists a constant $L > 0$ independent of p such that, recalling (19),

$$\|\gamma_p^{\pm}(u)\|_{H^{1/2}(\Gamma_p)} \leq L\|u\|_{\mathcal{H}_p} \quad \text{for all } u \in \mathcal{H}_p. \tag{20}$$

Clearly, for a continuous function u, $\gamma_p^{+}(u) = \gamma_p^{-}(u)$.

Furthermore, let $\nu^+ = (0, -1)$ and $\nu^- = (0, 1)$ be the normal unit vectors to s_0, whereas

$$\nu_p^+ = (\sin\alpha, -\cos\alpha) \quad \text{and} \quad \nu_p^- = -\nu_p^+$$

be the normal unit vectors to Γ_p.

For every $u \in C^1(D_1 \setminus (\Gamma_p \cup s_0))$ with

$$\mathcal{R}_p^+(u) \in C^1(\overline{U_p^+} \setminus s_0) \text{ and } \mathcal{R}_p^-(u) \in C^1(\overline{U_p^-} \setminus s_0),$$

we define the normal derivatives $\frac{\partial^{\pm} u}{\partial \nu_p^{\pm}}$ on Γ_p respectively as

$$\frac{\partial^+ u}{\partial \nu_p^+} := \nabla \mathcal{R}_p^+(u) \cdot \nu_p^+ \Big|_{\Gamma_p}, \quad \text{and} \quad \frac{\partial^- u}{\partial \nu_p^-} := \nabla \mathcal{R}_p^-(u) \cdot \nu_p^- \Big|_{\Gamma_p}.$$

Analogous definitions hold for normal derivatives on s_0 (which will be denoted just as $\frac{\partial^{\pm} u}{\partial \nu^{\pm}}$).

For $p \neq e$, where $e = (1, 0)$, we consider the minimization problem for the functional $J_p : \mathcal{H}_p \to \mathbb{R}$ defined as

$$\begin{aligned} J_p(u) &= \frac{1}{2}\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla u|^2\,dx + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}\gamma_p^+(u)\,ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-}\gamma_p^-(u)\,ds \\ &= \frac{1}{2}\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla u|^2\,dx + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(u) - \gamma_p^-(u))\,ds \end{aligned} \tag{21}$$

on the set

$$\mathcal{K}_p := \{u \in \mathcal{H}_p : \ \gamma_p^+(u + \psi) + \gamma_p^-(u + \psi) = 0\}.$$

The set $\mathcal{K}_p$ is nonempty, convex and closed, the functional J_p is coercive (see (34)), so that the problem admits a unique minimum $w_p \in \mathcal{K}_p$ which is a weak solution to the problem

$$\begin{cases} -\Delta w_p = 0, & \text{in } \mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}, \\ \gamma^+(w_p) + \gamma^-(w_p) = 0, & \text{on } s_0, \\ \gamma_p^+(w_p + \psi) + \gamma_p^-(w_p + \psi) = 0, & \text{on } \Gamma_p, \\ \dfrac{\partial^+ w_p}{\partial \nu^+} = \dfrac{\partial^- w_p}{\partial \nu^-}, & \text{on } s_0, \\ \dfrac{\partial^+ (w_p + \psi)}{\partial \nu_p^+} = \dfrac{\partial^- (w_p + \psi)}{\partial \nu_p^-}, & \text{on } \Gamma_p. \end{cases} \tag{22}$$

Remark 1 We note that the trivial function is not a solution to the problem (22), since the two jump conditions for the solution and its normal derivative on Γ_p cannot be satisfied simultaneously by the trivial function if $p \neq e$, hence $w_p \not\equiv 0$ for all $p \neq e$.

One can easily see that the function $e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0}(w_p + \psi)$ satisfies (15) and (16), hence by the uniqueness stated in Proposition 1 we conclude that necessarily

$$\Psi_p = e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0}(w_p + \psi). \tag{23}$$

On the other hand, for $p = e$, we consider the function $w_k \in \mathscr{D}_s^{1,2}(\mathbb{R}^2_+)$ defined as the unique minimizer in (6). The function w_e defined as

$$w_e(x_1, x_2) = \begin{cases} w_k(x_1, x_2), & \text{if } x_2 \geq 0, \\ w_k(x_1, -x_2), & \text{if } x_2 \leq 0, \end{cases} \tag{24}$$

satisfies

$$w_e \in \mathscr{H}_e$$

and

$$\begin{cases} -\Delta(w_e + \psi) = 0, & \text{in } \mathbb{R}^2 \setminus s, \\ \gamma^+(w_e) + \gamma^-(w_e) = 0, & \text{on } s, \\ \dfrac{\partial^+ w_e}{\partial \nu^+} = \dfrac{\partial^- w_e}{\partial \nu^-}, & \text{on } s, \end{cases} \tag{25}$$

where $s = \{(x_1, 0) : \ x_1 \geq 1\}$ and $\mathscr{H}_e$ is defined as the completion of

$$\mathscr{D}_e = \big\{u \in H^1(\mathbb{R}^2 \setminus s) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s \text{ and } u = 0 \\ \text{in neighborhoods of } 0 \text{ and } \infty\big\}$$

with respect to the Dirichlet norm $\|\nabla u\|_{L^2(\mathbb{R}^2\setminus s)}$. One can easily see that the function $e^{\frac{i}{2}\theta_e}(w_e+\psi)$ satisfies (15) and (16) wit $p=e$ (notice that $\theta_0^e=\theta_0$), hence by the uniqueness stated in Proposition 1 we conclude that necessarily

$$\Psi_e = e^{\frac{i}{2}\theta_e}(w_e+\psi). \tag{26}$$

In [2, Proposition 14] it was proved that

$$\lim_{a=|a|p\to 0}\frac{\lambda_0-\lambda_a}{|a|^k} = |\beta_2|^2 k\int_0^{2\pi} w_p(\cos t,\sin t)\sin\left(\frac{k}{2}t\right)dt,$$

which, combined with (5), yields

$$-4\mathfrak{m}_k\cos(k\alpha) = k\int_0^{2\pi} w_p(\cos t,\sin t)\sin\left(\frac{k}{2}t\right)dt. \tag{27}$$

The right hand side of (27) can be related to $J_p(w_p)$ as follows.

Lemma 5 *For every $p\neq e$*

$$\int_0^{2\pi} w_p(\cos t,\sin t)\sin\left(\frac{k}{2}t\right)dt = -\frac{2}{k}J_p(w_p).$$

Proof Throughout this proof, let us denote

$$\omega_p(r) := \int_0^{2\pi} w_p(r\cos t, r\sin t)\sin\left(\frac{k}{2}t\right)dt.$$

Then we have to prove that $k\omega_p(1) = -2J_p(w_p)$. Since $-\Delta w_p = 0$ in $\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}$, $\gamma^+(w_p)+\gamma^-(w_p) = 0$ on s_0, and $\frac{\partial^+ w_p}{\partial\nu^+} = \frac{\partial^- w_p}{\partial\nu^-}$ on s_0, by direct calculations ω_p satisfies

$$-(r^{1+k}(r^{-k/2}\omega_p(r))')' = 0, \quad \text{in } (1,+\infty).$$

Hence there exists a constant $C\in\mathbb{R}$ such that

$$r^{-k/2}\omega_p(r) = \omega_p(1) + \frac{C}{k}\left(1-\frac{1}{r^k}\right), \quad \text{for all } r\geq 1.$$

From (23) and Proposition 1, it follows that $\omega_p(r) = O(r^{-1/2})$ as $r\to+\infty$. Hence, letting $r\to+\infty$ in the previous relation, we find $C=-k\omega_p(1)$, so that

$$\omega_p(r) = \omega_p(1)r^{-k/2}$$

for all $r \geq 1$. By taking the derivative in this relation and in the definition of ω_p, we obtain

$$-\frac{k}{2}\omega_p(1) = \int_{\partial D_1} \frac{\partial w_p}{\partial \nu}\psi \, ds.$$

Multiplying Eq. (22) by ψ and integrating by parts over $D_1 \setminus \{s_0 \cup \Gamma_p\}$, we obtain

$$\begin{aligned}\int_{D_1\setminus\{s_0\cup\Gamma_p\}} \nabla w_p \cdot \nabla\psi \, dx &= \int_{\partial D_1} \frac{\partial w_p}{\partial \nu}\psi \, ds + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi \, ds \\ &= -\frac{k}{2}\omega_p(1) + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi \, ds. \qquad (28)\end{aligned}$$

Testing the equation $-\Delta\psi = 0$ by w_p and integrating by parts in $D_1 \setminus \{s_0 \cup \Gamma_p\}$, we arrive at

$$\begin{aligned}\int_{D_1\setminus\{s_0\cup\Gamma_p\}} \nabla w_p \cdot \nabla\psi \, dx &= \int_{\partial D_1} \frac{\partial \psi}{\partial \nu} w_p \, ds + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds \\ &= \frac{k}{2}\omega_p(1) + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds, \qquad (29)\end{aligned}$$

where in the last step we used the fact that $\frac{\partial\psi}{\partial\nu} = \frac{k}{2}\psi$ on ∂D_1. Combining (28) and (29), we obtain

$$k\omega_p(1) = \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right)\psi \, ds - \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds. \qquad (30)$$

On the other hand, multiplying (22) by w_p and integrating by parts over $\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}$, we obtain

$$\int_{\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}} |\nabla w_p|^2 \, dx = \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+}\gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-}\gamma_p^-(w_p) \, ds.$$

At the same time, recalling the definition of J_p (21) and taking into account the latter equation we have

$$\begin{aligned}2J_p(w_p) &= \int_{\mathbb{R}^2\setminus\{s_0\cup\Gamma_p\}} |\nabla w_p|^2 \, dx + 2\int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+}\gamma_p^+(w_p) \, ds + 2\int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-}\gamma_p^-(w_p) \, ds \\ &= \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+}\gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-}\gamma_p^-(w_p) \, ds\end{aligned}$$

$$
\begin{aligned}
&+ 2\int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p)\, ds + 2\int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p)\, ds \\
&= \int_{\Gamma_p} \frac{\partial^+ (w_p+\psi)}{\partial \nu_p^+} \gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- (w_p+\psi)}{\partial \nu_p^-} \gamma_p^-(w_p)\, ds \\
&\quad + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p)\, ds \\
&= \int_{\Gamma_p} \frac{\partial^+ (w_p+\psi)}{\partial \nu_p^+} \gamma_p^+(w_p+\psi)\, ds + \int_{\Gamma_p} \frac{\partial^- (w_p+\psi)}{\partial \nu_p^-} \gamma_p^-(w_p+\psi)\, ds \\
&\quad + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p)\, ds \\
&\quad - \int_{\Gamma_p} \frac{\partial^+ (w_p+\psi)}{\partial \nu_p^+} \gamma_p^+(\psi)\, ds - \int_{\Gamma_p} \frac{\partial^- (w_p+\psi)}{\partial \nu_p^-} \gamma_p^-(\psi)\, ds
\end{aligned}
$$

from which the thesis follows by comparison with (30) recalling that in the last equivalence the first term is zero by (22) and ψ is regular on Γ_p.
From the fact that w_k attains the minimum in (6) and (24) it follows easily that

$$
\mathfrak{m}_k = \frac{1}{2}\left[\frac{1}{2}\int_{\mathbb{R}^2\setminus s_0} |\nabla w_e|^2\, dx + \int_{\Gamma_e} \frac{\partial^+ \psi}{\partial \nu^+} \gamma^+(w_e)\, ds + \int_{\Gamma_e} \frac{\partial^- \psi}{\partial \nu^-} \gamma^-(w_e)\, ds\right]. \tag{31}
$$

Combining (27), Lemma 5, and (31) we conclude that, for

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_p|^2\, dx + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p)\, ds \\
&= \cos(k\alpha)\left[\frac{1}{2}\int_{\mathbb{R}^2\setminus s_0} |\nabla w_e|^2\, dx + \int_{\Gamma_e} \frac{\partial^+ \psi}{\partial \nu^+} \gamma^+(w_e)\, ds + \int_{\Gamma_e} \frac{\partial^- \psi}{\partial \nu^-} \gamma^-(w_e)\, ds\right].
\end{aligned} \tag{32}
$$

every $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1 \setminus \{e\}$.

Lemma 6

(i) *There exists* $C > 0$ *(independent of* $p \in \mathbb{S}^1$*) such that, for all* $p \in \mathbb{S}^1$,

$$
\int_{\mathbb{R}^2\setminus\Gamma_p} \left|(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi\right|^2 dx \le C. \tag{33}
$$

(ii) *If* $p_n, p \in \mathbb{S}^1$ *and* $p_n \to p$ *in* $\mathbb{S}^1$*, then* $\Psi_{p_n} \to \Psi_p$ *weakly in* $H^1(D_R, \mathbb{C})$ *for every* $R > 1$*, a.e., and in* $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{p\})$.

Proof Let us fix $q > 2$. From the continuity of the embedding $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$ and (20), we have that there exists some const > 0 independent of $p \in \mathbb{S}^1$ such that, for all $u \in \mathcal{H}_p$,

$$\left| \int_{\Gamma_p} \frac{\partial^\pm \psi}{\partial \nu_p^\pm} \gamma_p^\pm(u)\, ds \right| = \left| \frac{k}{2} \cos\left(\frac{k}{2}\alpha\right) \int_{\Gamma_p} |x|^{\frac{k}{2}-1} \gamma_p^\pm(u)\, ds \right|$$
$$\leq \frac{k}{2} \| |x|^{\frac{k}{2}-1} \|_{L^{q'}(\Gamma_p)} \| \gamma_p^\pm(u) \|_{L^q(\Gamma_p)} \leq \text{const}\, \| \gamma_p^\pm(u) \|_{H^{1/2}(\Gamma_p)}$$
$$\leq \text{const}\, L \|u\|_{\mathcal{H}_p}$$

and then, from the elementary inequality $ab \leq \frac{a^2}{4\varepsilon} + \varepsilon b^2$, we deduce that, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ (depending on ε but independent of p) such that, for every $u \in \mathcal{H}_p$,

$$\left| \int_{\Gamma_p} \frac{\partial^\pm \psi}{\partial \nu_p^\pm} \gamma_p^\pm(u)\, ds \right| \leq \varepsilon \|u\|^2_{\mathcal{H}_p} + C_\varepsilon. \tag{34}$$

From (34) and the fact that the right hand side of (32) is bounded uniformly with respect to $p \in \mathbb{S}^1$, we deduce that for any $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$

$$\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2 \leq M \tag{35}$$

for a constant $M > 0$ independent of p. Replacing (23) ((26) for $p = e$) into (35) we obtain (33).

We have that (33) together with the Hardy-type inequality of [12] implies that $\{\Psi_p\}_{p\in\mathbb{S}^1}$ is bounded in $H^1(D_R)$ and $\{A_p\Psi_p\}_{p\in\mathbb{S}^1}$ is bounded in $L^2(D_R)$ for every $R > 1$. Hence, by a diagonal process, for every sequence $p_n \to p$ in $\mathbb{S}^1$, there exist a subsequence (still denoted as p_n) and some $\Psi \in H^1_{\rm loc}(\mathbb{R}^2)$ such that Ψ_{p_n} converges to Ψ weakly in $H^1(D_R)$ and a.e. and $A_{p_n}\Psi_{p_n}$ converges to $A_p\Psi$ weakly in $L^2(D_R)$ for every $R > 1$. In particular this implies that $\Psi \in H^{1,p}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$. Passing to the limit in the equation $(i\nabla + A_{p_n})^2\Psi_{p_n} = 0$, we obtain that $(i\nabla + A_p)^2\Psi = 0$. Furthermore, by weak convergences $\nabla\Psi_{p_n} \rightharpoonup \nabla\Psi$, $A_{p_n}\Psi_{p_n} \rightharpoonup A_p\Psi$ in $L^2(D_R)$ and (33), we have that, for every $R > 1$,

$$\int_{D_R\setminus D_1} \left| (i\nabla + A_p)\Psi - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi \right|^2 dx$$
$$\leq \liminf_{n\to\infty} \int_{D_R\setminus D_1} \left| (i\nabla + A_{p_n})\Psi_{p_n} - e^{\frac{i}{2}(\theta_{p_n} - \theta_0^{p_n})} e^{\frac{i}{2}\theta_0} i\nabla\psi \right|^2 dx \leq C$$

and, since C is independent of R, $\int_{\mathbb{R}^2\setminus D_1} \left|(i\nabla + A_p)\Psi - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0} i\nabla\psi\right|^2 dx \le C$. By the uniqueness stated in Proposition 1 we conclude that necessarily $\Psi = \Psi_p$. Since the limit Ψ depends neither on the sequence p_n nor on the subsequence, we obtain statement (ii). The convergence in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{p\})$ follows by classical elliptic regularity theory.

Lemma 7 *For every $p \in \mathbb{S}^1$, let $f_p : [0,1] \to \mathbb{C}$, $f_p(r) = \Psi_p(rp)$. If $p_n, p \in \mathbb{S}^1$ and $p_n \to p$, then $f_{p_n} \rightharpoonup f_p$ weakly in $L^q(0,1)$ for all $q > 2$.*

Proof If $p_n \to p$ in $\mathbb{S}^1$, then the $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{p\})$-convergence stated in Lemma 6 implies that $f_{p_n} \to f_p$ a.e. in $(0,1)$. Furthermore, from the continuity of the embedding $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$ and boundedness of $\{\Psi_p\}_{p\in\mathbb{S}^1}$ in $H^1(D_1,\mathbb{C})$, we have that

$$\|f_{p_n}\|_{L^q(0,1)} = \left(\int_{\Gamma_{p_n}} |\Psi_{p_n}|^q\, ds\right)^{1/q} \le \text{const}\, \|\Psi_{p_n}\|_{H^{1/2}(\Gamma_{p_n})} \le \text{const}\, \|\Psi_{p_n}\|_{H^1(D_1)} \le \text{const}$$

for positive const > 0 independent of n. Then, along a subsequence, f_{p_n} convergences weakly in $L^q(0,1)$ to some limit which necessarily coincides with f_p by a.e. convergence (then the convergence holds not only along the subsequence).

Proposition 2 *For $\alpha \in \mathbb{R}$, let $p = (\cos\alpha, \sin\alpha)$. Let w_p be the unique solution to problem (22) ((25) if $p = e$). Then the function*

$$\alpha \mapsto \frac{1}{2}\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_{(\cos\alpha,\sin\alpha)}(x)|^2\, dx \tag{36}$$

is continuous, even and periodic with period $\frac{2\pi}{k}$.

Proof In view of (32), to prove the continuity of the map in (36) it is enough to show that the function

$$G : \mathbb{S}^1 \to \mathbb{R}, \quad G(p) = \begin{cases} \int_{\Gamma_p} \frac{\partial^+\psi}{\partial\nu_p^+}\gamma_p^+(w_p)\, ds + \int_{\Gamma_p} \frac{\partial^-\psi}{\partial\nu_p^-}\gamma_p^-(w_p)\, ds, & \text{if } p \ne e, \\ \int_{\Gamma_e} \frac{\partial^+\psi}{\partial\nu^+}\gamma^+(w_e)\, ds + \int_{\Gamma_e} \frac{\partial^-\psi}{\partial\nu^-}\gamma^-(w_e)\, ds, & \text{if } p = e, \end{cases}$$

is continuous. In view of (23) and (26), G can be written also as

$$G(p) = \begin{cases} kie^{-\frac{i}{2}\theta_0(p)}\cos(\frac{k}{2}\theta_0(p)) \int_0^1 r^{\frac{k}{2}-1} f_p(r)\, dr, & \text{if } p \ne e, \\ ki\int_0^1 r^{\frac{k}{2}-1} f_e(r)\, dr, & \text{if } p = e, \end{cases}$$

so that, to prove the continuity of G it is enough to show that the function

$$p \mapsto \int_0^1 r^{\frac{k}{2}-1} f_p(r)\, dr$$

is continuous on $\mathbb{S}^1$ and this follows from Lemma 7 and the fact that $r^{\frac{k}{2}-1}$ is in $L^t(0,1)$ for all $1 < t < 2$.

To the last part of the proof, following closely [2, Lemma 15], we introduce the two transformations $\mathscr{R}_1, \mathscr{R}_2$ acting on a general point

$$x = (x_1, x_2) = (r\cos t, r\sin t), \quad r > 0,\ t \in [0, 2\pi),$$

as

$$\mathscr{R}_1(x) = \mathscr{R}_1(x_1, x_2) = M_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad M_k = \begin{pmatrix} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}$$

i.e.

$$\mathscr{R}_1(r\cos t, r\sin t) = \Big(r\cos(t+\tfrac{2\pi}{k}), r\sin(t+\tfrac{2\pi}{k})\Big),$$

and

$$\mathscr{R}_2(x) = \mathscr{R}_2(x_1, x_2) = (x_1, -x_2),$$

i.e.

$$\mathscr{R}_2(r\cos t, r\sin t) = (r\cos(2\pi - t), r\sin(2\pi - t)).$$

The transformation $\mathscr{R}_1$ is a rotation of $\frac{2\pi}{k}$ and $\mathscr{R}_2$ is a reflexion through the x_1-axis. We note that

$$\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_p|^2 = \int_{\mathbb{R}^2\setminus\Gamma_p} \left|(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi\right|^2 dx. \tag{37}$$

From the change of variable $x = \mathscr{R}_1(y)$ and [2, Lemma 15, (58) and (66)] we have that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus\Gamma_p} &\left|(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi\right|^2 dx \\ &= \int_{\mathbb{R}^2\setminus\Gamma_{\mathscr{R}_1^{-1}(p)}} \left|(i\nabla + A_{\mathscr{R}_1^{-1}(p)})\Psi_{\mathscr{R}_1^{-1}(p)} - e^{\frac{i}{2}\left(\theta_{\mathscr{R}_1^{-1}(p)} - \theta_0^{\mathscr{R}_1^{-1}(p)} + \theta_0\right)} i\nabla\psi\right|^2 dy\end{aligned}$$

which, in view of (37), yields

$$\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_{\mathscr{R}_1^{-1}(p)})} |\nabla w_{\mathscr{R}_1^{-1}(p)}|^2 = \int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_p|^2$$

and hence $\frac{2\pi}{k}$-periodicity of the map (36). On the other hand, from the change of variable $x = \mathscr{R}_2(y)$ and [2, Lemma 15, (72)] we have that

$$\int_{\mathbb{R}^2\setminus\Gamma_p} \left|(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}i\nabla\psi\right|^2 dx$$

$$= \int_{\mathbb{R}^2\setminus\Gamma_{\mathscr{R}_2(p)}} \left|(i\nabla + A_{\mathscr{R}_2(p)})\Psi_{\mathscr{R}_2(p)} - e^{\frac{i}{2}\left(\theta_{\mathscr{R}_2(p)}-\theta_0^{\mathscr{R}_2(p)}+\theta_0\right)}i\nabla\psi\right|^2 dy$$

which, in view of (37), yields

$$\int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_{\mathscr{R}_2(p)})} |\nabla w_{\mathscr{R}_2(p)}|^2 = \int_{\mathbb{R}^2\setminus(s_0\cup\Gamma_p)} |\nabla w_p|^2$$

and hence evenness of the map (36).

4 Rate of Convergence for Eigenfunctions

In this section we prove a sharp estimate for the rate of convergence of eigenfunctions. The estimate of the energy variation will be derived first inside disks with radius of order $|a|$ and later outside such disks.

4.1 Energy Variation Inside Disks with Radius of Order $|a|$

As a straightforward corollary of the blow-up results described in Sect. 2, we obtain the following result.

Lemma 8 *Under the same assumptions as in Theorem 2, we have that*

$$\lim_{a=|a|p\to 0} \frac{1}{|a|^k}\int_{D_{R|a|}} \left|(i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a-\theta_a)(x)}(i\nabla + A_0)\varphi_0(x)\right|^2 dx = |\beta_2|^2\mathscr{F}_p(R)$$

for all $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1$ and $R > 2$, where

$$\mathscr{F}_p(R) = \int_{D_R} \left|(i\nabla + A_p)\Psi_p(x) - e^{-\frac{i}{2}(\theta_0^p-\theta_p)(x)}(i\nabla + A_0)(e^{\frac{i}{2}\theta_0}\psi)(x)\right|^2 dx.$$

Proof By a change of variable we obtain that

$$\int_{D_{R|a|}} \left|(i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a - \theta_a)(x)}(i\nabla + A_0)\varphi_0(x)\right|^2 dx$$
$$= |a|^k \int_{D_R} \left|(i\nabla + A_p)\tilde{\varphi}_a(x) - e^{-\frac{i}{2}(\theta_0^p - \theta_p)(x)}(i\nabla + A_0)W_a(x)\right|^2 dx$$

so that the conclusion follows from convergence (12) and Theorem 2.

Lemma 9 *Let $\mathscr{F}_p(R)$ be as in Lemma 8. Then*

$$\lim_{R\to+\infty} \mathscr{F}_p(R) = \mathfrak{L}_p > 0$$

where

$$\mathfrak{L}_p = \int_{\mathbb{R}^2\setminus(\Gamma_p\cup s_0)} |\nabla w_p|^2$$

and w_p is the weak solution to the problem (22).

Proof Via a change of gauge, we can write

$$\mathscr{F}_p(R) = \int_{D_R\setminus(\Gamma_p\cup s_0)} \left|e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \left(i\nabla(w_p + \psi) - i\nabla\psi\right)\right|^2$$
$$= \int_{D_R\setminus(\Gamma_p\cup s_0)} \left|\nabla w_p\right|^2 \to \int_{\mathbb{R}^2\setminus(\Gamma_p\cup s_0)} \left|\nabla w_p\right|^2$$

as $R \to +\infty$. Thanks to Remark 1, we stress that the limit is non zero. This concludes the proof.

4.2 Energy Variation Outside Disks with Radius of Order |a|

In order to estimate the energy variation outside disks with radius $R|a|$, we consider the following operator:

$$F : \mathbb{C} \times H_0^{1,0}(\Omega,\mathbb{C}) \longrightarrow \mathbb{R}\times\mathbb{R}\times(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$$
$$(\lambda,\varphi) \longmapsto \Big(\|\varphi\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_0,\ \mathfrak{Im}\big(\textstyle\int_\Omega \varphi\overline{\varphi_0}\,dx\big),\ (i\nabla + A_0)^2\varphi - \lambda\varphi\Big).$$

In the above definition, $(H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}))^\star$ is the real dual space of

$$H_{0,\mathbb{R}}^{1,0}(\Omega,\mathbb{C}) = H_0^{1,0}(\Omega,\mathbb{C}),$$

which is here meant as a vector space over $\mathbb{R}$ endowed with the norm

$$\|u\|_{H_0^{1,0}(\Omega,\mathbb{C})} = \left(\int_\Omega \left|(i\nabla + A_0)u\right|^2 dx \right)^{1/2},$$

and $(i\nabla + A_0)^2\varphi - \lambda\varphi \in (H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star$ acts as

$$\begin{aligned} &{}_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}\Big\langle (i\nabla + A_0)^2\varphi - \lambda\varphi, u \Big\rangle_{H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})} \\ &= \mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)\varphi \cdot \overline{(i\nabla + A_0)u}\, dx - \lambda \int_\Omega \varphi\overline{u}\, dx \right) \end{aligned}$$

for all $\varphi \in H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})$.

Lemma 10 *For $\alpha \in [0, 2\pi)$, $p = (\cos\alpha, \sin\alpha)$ and $a = |a|p \in \Omega$, let*

$$\varphi_a \in H_0^{1,a}(\Omega,\mathbb{C})$$

solve (7)–(8) and $\varphi_0 \in H_0^{1,0}(\Omega,\mathbb{C})$ be a solution to (3) satisfying (1), (4), and (11). Then, for all $R > 2$,

$$\|e^{\frac{i}{2}(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\|^2_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} \le |a|^k g(a, R)$$

where, for all $R > 2$,

$$\lim_{a=|a|p\to 0} g(a, R) = g(R) \tag{38}$$

and

$$\lim_{R\to+\infty} g(R) = 0. \tag{39}$$

Proof From [1, Lemma 7.1] we know that the function F is Fréchet-differentiable at (λ_0, φ_0) and its Fréchet-differential $dF(\lambda_0, \varphi_0)$ is invertible. From the invertibility of $dF(\lambda_0, \varphi_0)$ it follows that

$$\begin{aligned} &\left\| e^{\frac{i}{2}(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0 \right\|_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} \\ &= \left\| (i\nabla + A_0)(e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_a - \varphi_0) \right\|_{L^2(\Omega\setminus D_{R|a|},\mathbb{C})} \\ &\le |\lambda_a - \lambda_0| + \|v_{R,a} - \varphi_0\|_{H_0^{1,0}(\Omega,\mathbb{C})} \\ &\le \|(dF(\lambda_0,\varphi_0))^{-1}\|_{\mathscr{L}(\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star,\mathbb{C}\times H_0^{1,0}(\Omega,\mathbb{C}))} \\ &\quad \times \|F(\lambda_a, v_{R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega))^\star}(1 + o(1)) \end{aligned}$$

as $|a| \to 0^+$. We have that

$$F(\lambda_a, v_{R,a}) = (\alpha_a, \beta_a, w_a)$$

where

$$\begin{aligned}
\alpha_a &= \|v_{R,a}\|^2_{H^{1,0}_0(\Omega,\mathbb{C})} - \lambda_0 \in \mathbb{R},\\
\beta_a &= \mathfrak{Im}\left(\int_\Omega v_{R,a}\overline{\varphi_0}\,dx\right) \in \mathbb{R},\\
w_a &= (i\nabla + A_0)^2 v_{R,a} - \lambda_a v_{R,a} \in (H^{1,0}_{0,\mathbb{R}}(\Omega))^\star.
\end{aligned}$$

We mention that in [1, 2], the norm of $\|F(\lambda_a, v_{R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega))^\star}$ was estimated before proving the blow-up results recalled in Theorem 2 and Lemma 1 (actually some preliminary estimates of $F(\lambda_a, v_{R,a})$ were carried out to obtain an energy control in terms of an implicit normalization needed to prove the blow-up results). Here we are going to exploit the sharp blow-up results Theorem 2 and Lemma 1 to improve the preliminary estimates in [1, 2]. From (5), Theorem 2 and Lemma 1 we have that

$$\begin{aligned}
\alpha_a &= \left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{int}_{R,a}|^2\,dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\,dx\right) + (\lambda_a - \lambda_0)\\
&= |a|^k\left(\int_{D_R} |(i\nabla + A_0)Z^R_a|^2\,dx - \int_{D_R} |(i\nabla + A_p)\tilde{\varphi}_a|^2\,dx\right) + (\lambda_a - \lambda_0) = O(|a|^k),\\
&\quad \text{as } |a| \to 0^+.
\end{aligned}$$

The normalization condition for the phase in (8) together with the blow-up results (12), Theorem 2 and Lemma 1 yield

$$\begin{aligned}
\beta_a &= \Im\left(\int_{D_{R|a|}} v^{int}_{R,a}\overline{\varphi_0}\,dx - \int_{D_{R|a|}} e^{\frac{i}{2}(\theta^a_0-\theta_a)}\varphi_a\overline{\varphi_0}\,dx + \int_\Omega e^{\frac{i}{2}(\theta^a_0-\theta_a)}\varphi_a\overline{\varphi_0}\,dx\right)\\
&= \Im\left(|a|^{k+2}\int_{D_R} Z^R_a\overline{W_a}\,dx - |a|^{k+2}\int_{D_R} e^{\frac{i}{2}(\theta^p_0-\theta_p)}\tilde{\varphi}_a\overline{W_a}\,dx\right) = O(|a|^{k+2})\\
&\quad \text{as } |a| \to 0^+.
\end{aligned}$$

Let $\mathscr{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$ be the functional space defined in (14). For every $a \in \Omega$, we define the map

$$\mathscr{T}_a : \mathscr{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C}) \to \mathscr{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C}), \quad \mathscr{T}_a\varphi(x) = \varphi(|a|x).$$

It is easy to verify that $\mathscr{T}_a$ is an isometry of $\mathscr{D}^{1,2}_0(\mathbb{R}^2, \mathbb{C})$.

Since $H_0^{1,0}(\Omega,\mathbb{C})$ can be thought as continuously embedded into $\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})$ by trivial extension outside Ω and $\|u\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = \|u\|_{H_0^{1,0}(\Omega,\mathbb{C})}$ for every function $u \in H_0^{1,0}(\Omega,\mathbb{C})$, we have that

$$\begin{aligned}
&\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} \\
&= \sup_{\substack{\varphi\in H_0^{1,0}(\Omega,\mathbb{C})\\ \|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}=1}} \left|\Re\left(\int_\Omega (i\nabla + A_0)v_{R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{R,a}\overline{\varphi}\,dx\right)\right| \\
&\le \sup_{\substack{\varphi\in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})}=1}} \left|\Re\left(\int_\Omega (i\nabla + A_0)v_{R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{R,a}\overline{\varphi}\,dx\right)\right|.
\end{aligned} \tag{40}$$

For every $\varphi \in H_0^{1,0}(\Omega,\mathbb{C})$ we have that

$$\begin{aligned}
&\int_\Omega (i\nabla + A_0)v_{R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_\Omega v_{R,a}\overline{\varphi}\,dx \\
&= \int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_a\overline{\varphi}\,dx \\
&\quad + \int_{D_{R|a|}} (i\nabla + A_0)v_{R,a}\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_{D_{R|a|}} v_{R,a}\overline{\varphi}\,dx.
\end{aligned} \tag{41}$$

From scaling and integration by parts

$$\begin{aligned}
&\int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_a\cdot\overline{(i\nabla + A_0)\varphi}\,dx - \lambda_a\int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_a\overline{\varphi}\,dx \\
&\quad = |a|^{\frac{k}{2}}\Bigg(\int_{\frac{\Omega}{|a|}\setminus D_R} e^{\frac{i}{2}(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde{\varphi}_a\cdot\overline{(i\nabla + A_0)(\mathscr{T}_a\varphi)}\,dx \\
&\qquad -\lambda_a|a|^2\int_{\frac{\Omega}{|a|}\setminus D_R} e^{\frac{i}{2}(\theta_0^p-\theta_p)}\tilde{\varphi}_a\overline{\mathscr{T}_a\varphi}\,dx\Bigg) \\
&\quad = |a|^{\frac{k}{2}}\Bigg(\int_{\frac{\Omega}{|a|}\setminus D_R} (i\nabla + A_p)\tilde{\varphi}_a\cdot\overline{(i\nabla + A_p)(e^{-\frac{i}{2}(\theta_0^p-\theta_p)}\mathscr{T}_a\varphi)}\,dx \\
&\qquad -\lambda_a|a|^2\int_{\frac{\Omega}{|a|}\setminus D_R} \tilde{\varphi}_a\overline{e^{-\frac{i}{2}(\theta_0^p-\theta_p)}\mathscr{T}_a\varphi}\,dx\Bigg)
\end{aligned}$$

$$= |a|^{\frac{k}{2}} i \int_{\partial D_R} \overline{\mathscr{T}_a \varphi} e^{\frac{i}{2}(\theta_0^p - \theta_p)} (i\nabla + A_p)\tilde{\varphi}_a \cdot \nu \, d\sigma$$

$$= |a|^{\frac{k}{2}} i \int_{\partial D_R} \overline{\mathscr{T}_a \varphi} (i\nabla + A_0)(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a) \cdot \nu \, d\sigma \tag{42}$$

being $\nu = \frac{x}{|x|}$ the outer unit vector. In a similar way we have that

$$\int_{D_{R|a|}} (i\nabla + A_0) v_{R,a} \cdot \overline{(i\nabla + A_0)\varphi} \, dx - \lambda_a \int_{D_{R|a|}} v_{R,a} \overline{\varphi} \, dx$$

$$= |a|^{\frac{k}{2}} \left(\int_{D_R} (i\nabla + A_0) Z_a^R \cdot \overline{(i\nabla + A_0)(\mathscr{T}_a \varphi)} \, dx - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathscr{T}_a \varphi} \, dx \right)$$

$$= |a|^{\frac{k}{2}} \left(-i \int_{\partial D_R} (i\nabla + A_0) Z_a^R \cdot \nu \overline{\mathscr{T}_a \varphi} \, d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathscr{T}_a \varphi} \, dx \right), \tag{43}$$

with Z_a^R being as in (17). Combining (40)–(43), and recalling that $\mathscr{T}_a$ is an isometry of $\mathscr{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$, we obtain that

$$|a|^{-\frac{k}{2}} \|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}$$

$$\leq \sup_{\substack{\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = 1}} \left| i \int_{\partial D_R} (i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \, \overline{\mathscr{T}_a \varphi} \, d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathscr{T}_a \varphi} \, dx \right|$$

$$= \sup_{\substack{\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = 1}} \left| i \int_{\partial D_R} (i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \, \overline{\varphi} \, d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\varphi} \, dx \right|$$

$$\leq \sup_{\substack{\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = 1}} \left| \int_{\partial D_R} (i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \, \overline{\varphi} \, d\sigma \right|$$

$$+ \lambda_a |a|^2 \sup_{\substack{\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = 1}} \left| \int_{D_R} Z_a^R \overline{\varphi} \, dx \right|$$

$$= \sup_{\substack{\varphi \in \mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C}) \\ \|\varphi\|_{\mathscr{D}_0^{1,2}(\mathbb{R}^2,\mathbb{C})} = 1}} \left| \int_{\partial D_R} (i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \, \overline{\varphi} \, d\sigma \right|$$

$$+ |a|^2 O\left(\|Z_a^R\|_{L^2(D_R,\mathbb{C})} \right).$$

From Theorem 2 and Lemma 1 it follows that

$$(i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \to \beta_2 (i\nabla + A_0) \left(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p - z_{p,R} \right) \cdot \nu$$

in $H^{-1/2}(\partial D_R)$ as $a = |a|p \to 0$ and

$$|a|^2 O\left(\|Z_a^R\|_{L^2(D_R,\mathbb{C})}\right) \to 0 \quad \text{as } a = |a|p \to 0.$$

Hence we conclude that

$$|a|^{-\frac{k}{2}} \|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} \le h(a,R)$$

with

$$\lim_{a=|a|p\to 0} h(a,R) = |\beta_2| h(R)$$

being

$$h(R) = \sup_{\substack{\varphi\in\mathscr{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})\\ \|\varphi\|_{\mathscr{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}=1}} \left| \int_{\partial D_R} (i\nabla + A_0)\left(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - z_{p,R}\right)\cdot \nu\, \overline{\varphi}\, d\sigma\right|.$$

We observe that, for every $\varphi \in \mathscr{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})$,

$$\begin{aligned}
&\left| \int_{\partial D_R} (i\nabla + A_0)\left(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - z_{p,R}\right)\cdot \nu\, \overline{\varphi}\, d\sigma\right| \\
&= \left| \int_{\partial D_R} e^{\frac{i}{2}(\theta_0^p-\theta_p)}(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\right)\cdot \nu\, \overline{\varphi}\, d\sigma \right.\\
&\qquad \left. + \int_{\partial D_R} (i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\right)\cdot \nu\, \overline{\varphi}\, d\sigma\right| \\
&= \left| -i\int_{\mathbb{R}^2\setminus D_R} (i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\right)\cdot \overline{(i\nabla + A_0)\varphi}\, e^{\frac{i}{2}(\theta_0^p-\theta_p)}\, dx \right.\\
&\qquad \left. + i\int_{D_R} (i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\right)\cdot \overline{(i\nabla + A_0)\varphi}\, dx\right| \\
&\le \left(\sqrt{\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\right)\right|^2 dx} \right.\\
&\qquad \left. + \sqrt{\int_{D_R} \left|(i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\right)\right|^2 dx}\right) \|\varphi\|_{\mathscr{D}^{1,2}_0(\mathbb{R}^2,\mathbb{C})}
\end{aligned}$$

and hence

$$h(R) \le \sqrt{\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\right)\right|^2 dx} + \sqrt{\int_{D_R} \left|(i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\right)\right|^2 dx}.$$

From Proposition 1 it follows that

$$\lim_{R\to+\infty}\int_{\mathbb{R}^2\setminus D_R} \left|(i\nabla + A_p)\big(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\big)\right|^2 dx = 0.$$

Since $(i\nabla + A_0)^2\big(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\big) = 0$ in D_R and

$$(e^{\frac{i}{2}\theta_0}\psi - z_{p,R})\big|_{\partial D_R} = e^{\frac{i}{2}\theta_0}\psi - e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p,$$

if η_R is a smooth cut-off function satisfying

$$\eta_R \equiv 0 \text{ in } D_{R/2}, \quad \eta \equiv 1 \text{ in } \mathbb{R}^2\setminus D_R, \quad 0 \le \eta_R \le 1, \quad |\nabla\eta_R| \le \frac{4}{R} \quad \text{in } D_R\setminus D_{R/2},$$

from the Dirichlet Principle we can estimate

$$\begin{aligned}
&\int_{D_R} \left|(i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\right)\right|^2 dx \\
&\quad\le \int_{D_R} \left|(i\nabla + A_0)\left(\eta_R(e^{\frac{i}{2}\theta_0}\psi - e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p)\right)\right|^2 dx \\
&\quad\le 2\int_{D_R} |\nabla\eta_R|^2 |e^{\frac{i}{2}\theta_0}\psi - e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p|^2 dx \\
&\qquad + 2\int_{\mathbb{R}^2\setminus D_{R/2}} \left|(i\nabla + A_0)\left(e^{\frac{i}{2}\theta_0}\psi - e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p\right)\right|^2 dx \\
&\quad\le \frac{32}{R^2}\int_{D_R\setminus D_{R/2}} |\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi|^2 dx \\
&\qquad + 2\int_{\mathbb{R}^2\setminus D_{R/2}} \left|(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\theta_0}\psi\right)\right|^2 dx
\end{aligned}$$

which, in view of Proposition 1, implies that

$$\lim_{R\to+\infty}\int_{D_R} \left|(i\nabla + A_0)\big(e^{\frac{i}{2}\theta_0}\psi - z_{p,R}\big)\right|^2 dx = 0.$$

Therefore we can conclude that $h(R) \to 0$ as $R \to +\infty$. The proof is thereby complete.

4.3 Proof of Theorem 1

As observed in Sect. 2, it is not restrictive to assume $\beta_1 = 0$. Let $\varepsilon > 0$. From Lemma 9 and (39) there exists $R_0 > 0$ sufficiently large such that

$$|\mathscr{F}_p(R_0) - \mathfrak{L}_p| < \varepsilon \quad \text{and} \quad |g(R_0)| < \varepsilon.$$

From (38) and Lemma 8 there exists $\delta > 0$ (depending on ε and R_0) such that, if $|a| < \delta$, then

$$|g(a, R_0) - g(R_0)| < \varepsilon$$

and

$$\left| \frac{1}{|a|^k} \int_{D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a - \theta_a)(x)} (i\nabla + A_0)\varphi_0(x) \right|^2 dx - |\beta_2|^2 \mathscr{F}_p(R_0) \right| < \varepsilon.$$

Therefore, taking into account also Lemma 10, we have that, for all $a = |a|p$ with $|a| < \delta$,

$$\begin{aligned}
&\left| |a|^{-k} \int_\Omega \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx - |\beta_2|^2 \mathfrak{L}_p \right| \\
&\quad \leq \left| |a|^{-k} \int_{D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx - |\beta_2|^2 \mathscr{F}_p(R_0) \right| \\
&\quad + |a|^{-k} \int_{\Omega \setminus D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx + |\beta_2|^2 |\mathfrak{L}_p - \mathscr{F}_p(R_0)| \\
&\quad < \varepsilon + g(a, R_0) + |\beta_2|^2 \varepsilon \leq \varepsilon + |g(a, R_0) - g(R_0)| + |g(R_0)| + |\beta_2|^2 \varepsilon = (3 + |\beta_2|^2)\varepsilon,
\end{aligned}$$

thus concluding the proof of Theorem 1. □

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Nondecreasing Solutions to Doubly Nonlinear Equations

Goro Akagi and Ulisse Stefanelli

Abstract Nondecreasing evolution is described via the coupling of an abstract doubly nonlinear differential inclusion and a constraint on the rate. The latter is formulated by imposing the monotonicity in time of the solution with respect to a given preorder in a Hilbert space. We discuss a solution notion for this problem and prove existence and long-time behavior.

Keywords Existence • Long-time behavior • Nondecreasing dynamics • Nondecreasing solution • Preorders in Hilbert spaces

AMS (MOS) Subject Classification 35K90

1 Introduction

Doubly nonlinear evolution problems arise in connection with the modelization of a variety of physical systems, from heat conduction and phase change, to viscous dynamics. Indeed, at low-frequency regimes inertial effects can be often neglected and systems are driven by the balance of conservative, dissipative, and external actions. This can be modeled in abstract terms by describing the state of the system

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as a point in a Hilbert space $\mathcal{H}$ and by letting its evolution follow a trajectory $t \in [0,T] \mapsto u(t) \in \mathcal{H}$ up to some final reference time $T > 0$. In particular, such a trajectory can be asked to solve the differential-inclusion problem

$$A(u)' + \partial\varphi(u) \ni f, \quad u(0) = u^0, \tag{1.1}$$

starting from the given initial state $u^0 \in \mathcal{H}$. The convex potential $\varphi : \mathcal{H} \to [0,\infty]$ models *conservative* effects, ∂ denotes the subdifferential (see below), and $t \mapsto f(t) \in \mathcal{H}$ is a given external forcing. The maximal monotone operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ describes dissipation. Indeed, the term $A(u)'$ (the prime denotes time differentiation) corresponds to a system of dissipative effects, so that relation (1.1) ensures as a balance law.

Problem (1.1) classically arises in connection with nonlinear diffusion phenomena. In relation with different choices of the operators A and $\partial\varphi$, Eq. (1.1) may arise in a variety of different dissipative situations. Letting A be linear and symmetric, relation (1.1) is a classical gradient flow. Nonlinear operators A arise in connection with the Stefan problem, the porous media equation, or the Hele-Shaw model, just to mention a few, see [35]. Classical references for the analysis of (1.1) are Grange and Mignot [20], Barbu [11], Di Benedetto and Showalter [17], Alt and Luckhaus [7], and Bernis [12]. For a collection of further developments, the Reader can check [1, 2, 4, 6, 18, 19, 22–24, 27, 31–33], among many others.

The evolution of the dissipative system (1.1) is irreversible with respect to time, for the complementary energy $u \mapsto \varphi(u) - (f,u)$ is nonincreasing along time-evolution of solutions (here $(\cdot,\cdot)$ stands for the scalar product in $\mathcal{H}$). In addition to this, some physical systems show *unidirectional* evolution, in the sense that trajectories are monotone in time with respect to some suitable order. Glue hardening, food cooking, and material damage are examples of such unidirectionality in real applications. Once the glue has hardened, the food is cooked, or the specimen is cracked, no external forcing can drive the system back to its original condition. With the aim of dealing with different unidirectional situations in a unified fashion, we address here unidirectional evolution in abstract terms, by additionally constraining the system (1.1) to

$$u' \in \mathcal{P}. \tag{1.2}$$

Here, $\mathcal{P} \subset \mathcal{H}$ models the *cone of nonnegative directions* in the Hilbert space $\mathcal{H}$. In particular, $\mathcal{P}$ is assumed to be a closed convex cone with vertex at 0, thus entailing a *preorder* on $\mathcal{H}$ via the position $u \preceq v$ iff $v - u \in \mathcal{P}$. The constraint (1.2) forces the trajectory to be *nondecreasing* with respect to the preorder, namely $s \le t \Rightarrow u(s) \preceq u(t)$.

Problem (1.1)–(1.2) admits no strong solutions in general, namely trajectories fulfilling the inclusions for almost every time. A counterexample to strong existence is given by $\mathcal{H} = \mathbb{R}$, $\mathcal{P} = [0, \infty)$, $A = \mathrm{id}$, $\varphi = 0$, and $f < 0$. This forces us to look for weaker solution notions. A first possibility is that of replacing (1.1)–(1.2) with

$$A(u)' + \partial I_{\mathcal{P}}(u') + \partial\varphi(u) \ni f, \quad u(0) = u^0. \tag{1.3}$$

Here $I_{\mathcal{P}}$ stands for the indicator function of the set $\mathcal{P}$, namely $I_{\mathcal{P}}(u') = 0$ if $u' \in \mathcal{P}$ and $I_{\mathcal{P}}(u') = \infty$ otherwise. Equation (1.3) explicitly includes the Lagrange multiplier $\partial I_{\mathcal{P}}(u')$ of the constraint (1.2). Nonetheless, an existence theory for strong solutions to (1.3) is still not available. The presence of three nonlinear operators, two of which are unbounded, does not allow the application of classical monotonicity and compactness techniques, see [5] where some triply nonlinear problem subject to a nonnegative-cone constraint is treated. In case $A = \mathrm{id}$, $f \in H^1(0, T; \mathcal{H})$, and some restrictions in the choice of $\mathcal{P}$ and u^0 (see (3.10) later on), the existence of a strong solution has been obtained by Barbu [11] and Arai [8]. Moreover, again for $A = \mathrm{id}$ and $\mathcal{P} = \mathcal{H}$ the equation falls within the class studied by Colli and Visintin [16] and is strongly solvable.

In order to introduce our weak notion of solution to (1.1)–(1.2) let us start by equivalently rewriting (1.3) in the form of an almost-everywhere-in-time variational inequality as

$$u \in D(\varphi), \;\; u' \in \mathcal{P}, \;\; \text{and} \;\; (v' - f, u - w) + (y, u - w) + \varphi(u) \le \varphi(w) \quad \forall w \in \mathcal{H}, \tag{1.4}$$

where $y \in \partial I_{\mathcal{P}}(u')$ and $v \in A(u)$ almost everywhere in time.

We now check that the term $(y, u - w)$ is nonnegative if $w - u \in \mathcal{P}$, namely for $u \preceq w$. As $\mathcal{P}$ is a convex cone with vertex in 0 and both $w - u$ and u' belong to $\mathcal{P}$ we have that $w - u + u' \in \mathcal{P}$ as well. Then, $y \in \partial I_{\mathcal{P}}(u')$ implies that $(y, w - u) = (y, (w-u+u')-u') \le 0$. Hence, by dropping the nonnegative term $(y, u-w)$ in (1.4) we obtain the weaker

$$\begin{gathered} u \in D(\varphi), \; u' \in \mathcal{P}, \; v \in A(u), \\ \text{and } (v' - f, u - w) + \varphi(u) \le \varphi(w) \;\; \forall w \in \mathcal{H} \text{ with } u \preceq w. \end{gathered} \tag{1.5}$$

A trajectory u solving (1.5) almost everywhere is called a *nondecreasing solution* of the doubly nonlinear problem (1.1)–(1.2). Clearly, all strong solutions of (1.4) are nondecreasing solutions.

This note deals with the existence and long-time behavior of nondecreasing solutions. We start by setting some preliminary material on orders in Hilbert spaces and convex differentiation in Sect. 2. We believe some of these to be of independent interest. The main results are stated in Sect. 3 where we also discuss the relation between nondecreasing solutions and strong solutions of (1.3) and supersolutions of (1.1)–(1.2). By means of an approximation and passage to the limit procedure, we prove in Sect. 4 that nondecreasing solutions exist. Section 5 addresses the characterization of the long-time behavior of nondecreasing solutions: due to their monotonicity with respect to time, a nondecreasing solution converges to a single equilibrium point. We conclude by mentioning some directions of possible future research in Sect. 6.

2 Mathematical Preliminaries

We collect in this section some definitions and properties of our abstract framework. In all of the following $\mathcal{H}$ is a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot| = \sqrt{(\cdot,\cdot)}$. We use the symbols $u_n \to u$ and $u_n \rightharpoonup u$ for strong and weak convergence, respectively.

2.1 Orders in Hilbert Spaces

Let $\mathcal{P} \subset \mathcal{H}$ be a nonempty, convex, and closed cone and let $u \preceq v$ iff $v - u \in \mathcal{P}$. Relation $\preceq$ defines a *preorder* [29, Prop. 3.38, p. 95]. We will also use the equivalent notations $v \succeq u$ and $u \preceq v$. Indeed, for all $p, q \in \mathcal{P}$, one readily checks that $p+q \in \mathcal{P}$ as well, whence the transitivity of $\preceq$ follows. If $\mathcal{P}$ is strict (i.e. $\mathcal{P} \cap -\mathcal{P} = \{0\}$) then relation $\preceq$ is actually an *order*. In all cases, the (pre)order $\preceq$ is *partial* as two elements in $\mathcal{H}$ are not necessarily comparable.

Given the nonempty set $\mathcal{Q} \subset \mathcal{H}$ we define its *polar cone* as $\mathcal{Q}^* = \{u \in \mathcal{H} : (u, v) \le 0 \ \ \forall v \in \mathcal{Q}\}$. This is a closed convex cone with vertex at 0. All $u \in \mathcal{H}$ can be decomposed as $u = u_1 + u_2$ where $u_1 \in \mathcal{P}$, $u_2 \in \mathcal{P}^*$, and $(u_1, u_2) = 0$ [28]. Indeed, u_1 and u_2 are the projections of u on $\mathcal{P}$ and $\mathcal{P}^*$, respectively. We will use the notation $u_1 = u^+ = \pi_{\mathcal{P}}(u)$ and $u_2 = -u^- = \pi_{\mathcal{P}^*}(u)$ (here π stands for the projection). The choice of the symbol $(\cdot)^+$ is suggested by the fact that we shall regard $\mathcal{P}$ as the cone of *nonnegative elements* of $\mathcal{H}$. In particular, in case of a *self-polar cone* $\mathcal{P}^* = -\mathcal{P}$ one indeed has that $u^- = \pi_{\mathcal{P}}(-u)$.

For all $u, v \in \mathcal{H}$, we introduce the notations

$$u \max v = u + (v - u)^+, \quad u \min v = u - (u - v)^+.$$

The symbols min and max are chosen just for the sake of notational simplicity. In particular, if $\mathcal{P}$ is self-polar one can check that $u \min v = \inf\{u, v\}$, $u \max v = \sup\{u, v\}$, $u \max v = v \max u$, and $u \min v = v \min u$. The pair $(\mathcal{H}, \mathcal{P})$ is called *Hilbert pseudolattice* if $\mathcal{P}$ is self-polar. Moreover, $(\mathcal{H}, \mathcal{P})$ is called *Hilbert lattice* if it is a lattice with respect to the preorder $\preceq$, namely if $\inf\{u, v\}$ and $\sup\{u, v\}$ exist for all $u,\ v \in \mathcal{H}$ [10]. Here are some examples.

Example 1 (Orthant) Our first example is the finite-dimensional non-negative *orthant* $\mathcal{P} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \ge 0 \ \text{for} \ i = 1, \ldots, n\}$. The cone $\mathcal{P}$ is self-polar and $(\mathbb{R}^n, \mathcal{P})$ is actually a lattice.

Example 2 (Nonnegative Functions) Let Ω be a measure space, μ be a positive measure on Ω and denote by $L^2(\Omega, \mu)$ the Hilbert space of all square μ-integrable functions on Ω endowed with the standard inner product. Letting

$$\mathcal{P} = \{u \in L^2(\Omega, \mu) \ : \ u \ge 0 \ \ \mu - \text{a.e. in} \ \ \Omega\}$$

relation $u \preceq v$ reads $u \le v$ μ-a.e. in Ω, which defines a lattice.

Example 3 (Projections) Let $\mathcal{H}$ be given and define $\mathcal{P} = \mathcal{Q}^*$ for some non-empty set $\mathcal{Q} \subset \mathcal{H}$. Hence the order relation reads $u \succeq v$ iff $(v, q) \geq (u, q)$ for all $q \in \mathcal{Q}$. In case $\mathcal{Q} = \{q\}$ is a singleton with $q \neq 0$ we readily check that $u^+ = u + (u, q)^- q/|q|^2$ for all $u \in \mathcal{H}$ where of course $(\cdot)^-$ stands for the standard negative part in $\mathbb{R}$. This is a pseudolattice iff $\mathcal{Q}^{**} = -\mathcal{Q}^*$.

Example 4 (Positive Semidefinite Matrices) Let $\mathcal{H}$ be the space of symmetric $n \times n$ real matrices endowed with the standard contraction product $(A, B) = \sum_{i,j=1}^{n} A_{ij} B_{ij}$ for all $A, B \in \mathcal{H}$. We define $\mathcal{P} = \{A \in \mathcal{H} \ : \ A \geq 0\}$ and check that $\mathcal{P}$ is self-polar [25, Cor. 7.5.4, p. 459]. Relation $A \preceq B$ corresponds to $B - A$ being positive semidefinite and $(\mathcal{H}, \mathcal{P})$ is a lattice. In particular, for all $A \in \mathcal{H}$, U orthogonal matrix, and $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ diagonal matrix such that $A = U^T D U$, the projection operator turns out to be

$$A^+ = U^T D^+ U \quad \text{where} \quad D^+ = \operatorname{diag}\{d_1^+, \ldots, d_n^+\},$$

where $d_i^+ = 0 \max d_i$ is the positive part in $\mathbb{R}$.

Example 5 (Second-Order Cone) Given the Hilbert space $\mathcal{H}$, we consider the convex cone in $\mathbb{R} \times \mathcal{H}$ defined by $\mathcal{P} = \{(t, u) \in \mathbb{R} \times \mathcal{H} \ : \ t \geq |u|\}$. It is a standard matter to verify that $\mathcal{P}$ is self-polar in $\mathbb{R} \times \mathcal{H}$. This example in particular shows that there exist self-polar cones in $\mathbb{R}^n$ that are not isometric to the nonnegative orthant of Example 1. Relation $(s, v) \preceq (t, u)$ reads in this case $t - s \geq |u - v|$, and the projection operator is defined as follows

$$\pi_{\mathcal{P}}(t, u) = \left(t + \frac{(|u| - t)^+}{2}, \frac{u}{|u|} \left(|u| - \frac{(|u| - t)^+}{2} \right) \right) \qquad \forall (t, u) \in \mathbb{R} \times \mathcal{H} \setminus \{0\}$$

where the positive part in the above right-hand side stands for that of $\mathbb{R}$. Note that $(\mathbb{R} \times \mathcal{H}, \mathcal{P})$ is in general not a lattice.

Example 6 (Infinite-Dimensional Orthant) Let $(\mathcal{H}_i, \mathcal{P}_i)$ be Hilbert (pseudo)lattices for $i = 1, \ldots, n$. Then $(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n, \mathcal{P}_1 \times \cdots \times \mathcal{P}_n)$ is a Hilbert (pseudo)lattice.

Example 7 (Conic Combination) Let $\{u_n\}$ denote a countable orthonormal basis of the separable Hilbert space $\mathcal{H}$. We denote by $\mathcal{P}$ the range of the mapping $u \mapsto \sum_{n \in \mathbb{N}} (u, u_n)^+ u_n$. Namely, $\mathcal{P}$ is the set of linear combinations of $\{u_n\}$ with non-negative coefficients (*conic combination*). Then $\mathcal{P}$ is self-polar and $(\mathcal{H}, \mathcal{P})$ is a lattice.

Example 8 (Vector-Valued Nonnegative Functions) Let (H, P) be a Hilbert (pseudo)lattice, Ω be a measure space, μ be a positive measure on Ω and denote by $\mathcal{H} = L^2(\Omega, \mu; H)$ the Hilbert space of all square μ-integrable functions on Ω with values in H. By letting $\mathcal{P} = \{u \in L^2(\Omega, \mu; H) \ : \ u \in P \ \ \mu - \text{a.e. in} \ \ \Omega\}$ relation $u \preceq v$ reads $v - u \in P$, μ-a.e. in Ω, which defines a (pseudo)lattice. Example 2 corresponds to the choice $(H, P) = (\mathbb{R}, [0, \infty))$.

We record here a characterization of Hilbert lattices. As we could not find a precise reference in the literature, we include its proof.

Proposition 1 (Characterization) *Let* $(\mathcal{H}, \mathcal{P})$ *be a Hilbert pseudolattice. Then, the following properties are equivalent:*

i) $(\mathcal{H}, \mathcal{P})$ *is a Hilbert lattice,*
ii) $x \preceq y \wedge (x \vee z)$ *for every* $x, y, z \in \mathcal{H}$ *with* $x \preceq y$,
iii) the map $x \mapsto x^+$ *is nondecreasing, i.e. for all* $x, y \in \mathcal{H}$ *such that* $x \preceq y$ *then* $x^+ \preceq y^+$.

Proof Let us first of all prove the implication *i*) $\Rightarrow$ *ii*). Indeed, it suffices to note that since $x \preceq x \vee z$ and $x \preceq y$, the lattice structure entails that $x \preceq \inf\{y, x \vee z\} = y \wedge (x \vee z)$.

As for the implication *ii*) $\Rightarrow$ *iii*), let us first point out that

$$x \preceq y \wedge (x \vee z) \;\Leftrightarrow\; (y - x - (z - x)^+)^+ = (y - (x \vee z))^+ \preceq y - x. \tag{2.1}$$

On the other hand, *iii)* is implied by the following

$$q \preceq p,\ p \in \mathcal{P} \;\Rightarrow q^+ \preceq p,$$

which ensues from (2.1) by choosing $y = p$, $x = 0$ and $z = p - q$.

To conclude the equivalence, it remains to prove *iii*) $\Rightarrow$ *i*). Indeed, we have to show that, given any pair of elements $w, v \in \mathcal{H}$, the set $\{w, v\}$ has a supremum, which is in fact $w \vee v$. Clearly, $w \vee v$ is an upper bound for $\{w, v\}$, as $w \preceq w + (v - w)^+$ and $v \preceq v + (w - v)^+$. On the other hand, let u be *any* upper bound for $\{w, v\}$. Then $u = v + p$, with $p \in \mathcal{P}$, and the fact $w \preceq u = v + p$, as well as assumption *iii)*, yield that $(w - v)^+ \preceq p^+ = p$. Then $v \vee w = v + (w - v)^+ \preceq v + p = u$, hence $w \vee v$ is the minimum of the set of the upper bounds of $\{w, v\}$.

2.2 Subdifferentials

Recall that, given $\varphi : \mathcal{H} \to (-\infty, \infty]$ convex, proper, and lower semicontinuous, the *subdifferential* $\partial\varphi : \mathcal{H} \to 2^{\mathcal{H}}$ is defined as

$$v \in \partial\varphi(u) \ \text{ iff } \ u \in D(\varphi) \ \text{ and } \ (v, w - u) + \varphi(u) \le \varphi(w) \ \ \forall w \in \mathcal{H},$$

where $D(\varphi) = \{u \in \mathcal{H} : \varphi(u) < \infty\}$ stands for the *effective domain* of φ. The latter operator with domain $D(\partial\varphi) = \{u \in D(\varphi) : \partial\varphi(u) \neq \emptyset\}$ turns out to be maximal monotone and strongly-weakly closed. Namely, its graph is closed with respect to the strong-weak topology of $\mathcal{H} \times \mathcal{H}$ [13].

We define the *nonnegative subdifferential* $\partial^+\varphi : \mathcal{H} \to 2^{\mathcal{H}}$ as the set-valued operator

$$v \in \partial^+\varphi(u) \ \text{ iff } \ u \in D(\varphi) \ \text{ and } \ (v, w - u) + \varphi(u) \le \varphi(w) \ \ \forall w \succeq u$$

with domain $D(\partial^+\varphi) = \{u \in D(\varphi) : \partial^+\varphi(u) \neq \emptyset\}$. This definition corresponds to the same variational inequality from the definition of $\partial\varphi(u)$ but restricted to tests w such that $u \preceq w$. It is of course straightforward to check that $\partial\varphi(u) \subset \partial^+\varphi(u)$ for all $u \in \mathcal{H}$ and, in particular, $D(\partial\varphi) \subset D(\partial^+\varphi) \subset D(\varphi)$.

Let now φ fulfill

$$\varphi(u \min v) + \varphi(v \max u) \leq \varphi(u) + \varphi(v) \quad \forall u, v \in \mathcal{H}. \tag{2.2}$$

In this case it is a standard matter to deduce that, for all $v_1 \in \partial\varphi(u_1)$ and $v_2 \in \partial\varphi(u_2)$, one has that

$$(v_1 - v_2, (u_1 - u_2)^+) \geq 0. \tag{2.3}$$

The latter is nothing but the well-known notion of *T-monotonicity*, originally introduced in [14]. In the case of a Hilbert pseudolattice the two relations (2.2) and (2.3) are actually equivalent. Moreover, the same properties are preserved by Moreau-Yosida approximations of φ. Furthermore, one readily checks from (2.3) that

$$(v_1 - v_2, -(u_1 - u_2)^-) = (v_2 - v_1, (u_2 - u_1)^+) \geq 0. \tag{2.4}$$

The reader is referred to [26] and [34, Lemma 4.1] for some detail in this direction.

The T-monotonicity condition (2.2) implies the weaker *conditional upper-semi-continuity*

$$\begin{aligned} &\forall u \in \mathcal{H},\ \forall w \in D(\varphi),\ w \succeq u,\ \forall u_n \to u: \quad \exists w_n \in D(\varphi) \ \text{ such that} \\ &w_n \to w,\ u_n \preceq w_n,\ \text{ and } \liminf_{n\to\infty} (\varphi(w_n) - \varphi(u_n)) \leq \varphi(w) - \varphi(u). \end{aligned} \tag{2.5}$$

Indeed, in order to check (2.5) starting from (2.2) one simply chooses $w_n = w \max u_n$, so that $u_n \preceq w_n$, $w_n = w + (u_n - w)^+ \to w + (u - w)^+ = w$, and one computes

$$\varphi(w_n) - \varphi(u_n) = \varphi(w \max u_n) - \varphi(u_n) \overset{(2.2)}{\leq} \varphi(w) - \varphi(w \min u_n).$$

The inequality in (2.5) follows then by passing to the lim inf as $n \to \infty$ and using the fact that $w \min u_n = w - (w - u_n)^+ \to w - (w - u)^+ = u$ and the lower semicontinuity of φ.

Condition (2.5) implies that the graph of $\partial^+\varphi$ is strongly-weakly closed. We have the following.

Lemma 1 (Strong-Weak Closure) *Let* $(\mathcal{H}, \mathcal{P})$ *be a pseudolattice and* $\varphi : \mathcal{H} \to (-\infty, \infty]$ *be convex, proper, and lower semicontinuous and fulfill* (2.5)*. Then,* $\partial^+\varphi$ *is strongly-weakly closed.*

Proof Let $z_n \in \partial^+\varphi(u_n)$ be such that $u_n \to u$ and $z_n \rightharpoonup z$. In particular, one has that

$$(z_n, w - u_n) + \varphi(u_n) \le \varphi(w) \quad \forall w \succeq u_n. \tag{2.6}$$

Fix any $w \in \mathcal{H}$ with $u \preceq w$ and find w_n according to (2.5). Then, one passes to the lim inf in (2.6) with $w = w_n$ getting

$$(z, w - u) = \liminf_{n\to\infty}(z_n, w_n - u_n) \le \liminf_{n\to\infty}(\varphi(w_n) - \varphi(u_n)) \overset{(2.5)}{\le} \varphi(w) - \varphi(u)$$

which entails $z \in \partial^+\varphi(u)$.

Note that $\partial^+\varphi(u) - \mathcal{P} \subset \partial^+\varphi(u)$ for all $u \in D(\partial^+\varphi)$. On the other hand we have already observed that $\partial\varphi(u) \subset \partial^+\varphi(u)$ for all $u \in D(\partial\varphi)$. This implies that $\partial\varphi(u) - \mathcal{P} \subset \partial^+\varphi(u)$ for all $u \in D(\partial\varphi)$. We present some sufficient condition in order to prove the converse inequality (see also [30, Thm. 2.1]).

Proposition 2 (Decomposition) *Let $\varphi : \mathcal{H} \to (-\infty, \infty]$ be convex, proper, and lower semicontinuous. Then, $\partial\varphi(u) - \mathcal{P} = \partial^+\varphi(u)$ for all $u \in D(\partial\varphi)$.*

Proof Let $u \in D(\partial\varphi) \subset D(\partial^+\varphi)$. We have to prove that $\partial^+\varphi(u) \subset \partial\varphi(u) - \mathcal{P}$. To this aim let $z \in \partial^+\varphi(u)$ and define, for all $\lambda > 0$, the element u_λ as a solution to the problem

$$u_\lambda + z_\lambda + y_\lambda = u + z, \tag{2.7}$$

$$z_\lambda \in \partial\varphi(u_\lambda), \tag{2.8}$$

$$y_\lambda = \big((u_\lambda - u) - (u_\lambda - u)^+\big)/\lambda = -(u_\lambda - u)^-/\lambda = \partial I_{\mathcal{P},\lambda}(u_\lambda - u), \tag{2.9}$$

where $\partial I_{\mathcal{P},\lambda}$ is the *Yosida* approximation of $\partial I_{\mathcal{P}}$. The latter is uniquely solvable since the operator

$$w \mapsto \partial\varphi(w) + \frac{1}{\lambda}\big((w - u) - (w - u)^+\big)$$

is maximal monotone (see, e.g., [13, Lemme 2.4, p. 34]).

Let now $\zeta \in \partial\varphi(u)$. By adding $-\zeta$ to both sides of (2.7) and testing it by y_λ we get that

$$(u_\lambda - u, y_\lambda) + (z_\lambda - \zeta, y_\lambda) + |y_\lambda|^2 = (z - \zeta, y_\lambda).$$

As for the first term in the above left-hand side one has that

$$(u_\lambda - u, y_\lambda) = \lambda|y_\lambda|^2 + \frac{1}{\lambda}((u_\lambda - u)^+, -(u_\lambda - u)^-) = \lambda|y_\lambda|^2.$$

On the other hand, owing to (2.4), one obtains that $(z_\lambda - \zeta, y_\lambda) \ge 0$. Thus y_λ is bounded in $\mathcal{H}$. Moreover, testing (2.7) by $u_\lambda - u$, one can derive the boundedness in $\mathcal{H}$ of u_λ in a standard way. Finally, by comparison in (2.7), we obtain the boundedness of z_λ in $\mathcal{H}$ as well. Moreover, we readily check that u_λ minimizes

$$w \mapsto \Phi_\lambda(w) = \varphi(w) + I_{\mathcal{P},\lambda}(w-u) + \frac{1}{2}|w|^2 - (u+z,w),$$

where $I_{\mathcal{P},\lambda}(w) = \inf_{h\in\mathcal{H}}\{|w-h|^2/(2\lambda)+I_{\mathcal{P}}(h)\}$ stands for the well known *Moreau-Yosida* approximation of $I_{\mathcal{P}}$ and $\partial(I_{\mathcal{P},\lambda}) = \partial I_{\mathcal{P},\lambda}$ (see [9]). In particular, we have the bound

$$\varphi(u_\lambda) + \frac{1}{2}|u_\lambda|^2 - (u+z,u_\lambda) \le \varphi(u) + \frac{1}{2}|u|^2 - (u+z,u),$$

so that $\varphi(u_\lambda)+|u_\lambda|$ is uniformly bounded as well. We next claim that u_λ is a Cauchy sequence in $\mathcal{H}$. Indeed, subtract Eq. (2.7) with λ and μ and test it by $u_\lambda - u_\mu$. Then it follows that

$$|u_\lambda - u_\mu|^2 + (y_\lambda - y_\mu, (u_\lambda - u) - (u_\mu - u)) \le 0.$$

Here the second term of the left-hand side can be handled by Kōmura's trick,

$$\begin{aligned}(y_\lambda - y_\mu, (u_\lambda - u) - (u_\mu - u)) &\ge \lambda|y_\lambda|^2 + \mu|y_\mu|^2 - (\lambda+\mu)(y_\lambda, y_\mu)\\ &\ge -\frac{\lambda}{4}|y_\mu|^2 - \frac{\mu}{4}|y_\lambda|^2 \to 0\end{aligned}$$

as $\lambda, \mu \to 0$, since y_λ and y_μ are bounded in $\mathcal{H}$. Thus, u_λ is a Cauchy sequence in $\mathcal{H}$.

Hence one can extract not relabeled subsequences and find three functions $\bar{u}, \bar{z}$, and $\bar{y}$ such that $u_\lambda \to \overline{u}$, $z_\lambda \rightharpoonup \overline{z}$, and $y_\lambda \rightharpoonup \overline{y}$. One can hence pass to the limit in (2.7) as λ goes to 0 and get that

$$\overline{u} + \overline{z} + \overline{y} = u + z, \quad \overline{z} \in \partial\varphi(\overline{u}), \quad \text{and} \quad \overline{y} \in \partial I_{\mathcal{P}}(\overline{u} - u).$$

As for the last inclusion above we have exploited the convergence of the Moreau-Yosida approximation [9].

It is now a standard matter to check that the functional Φ_λ converges to

$$w \mapsto \Phi(w) = \varphi(w) + I_{\mathcal{P}}(w-u) + \frac{1}{2}|w|^2 - (u+z,w)$$

in the *Mosco sense* [9], namely it Γ-converges with respect to both the weak and the strong topology. In particular, since u_λ minimizes Φ_λ and is precompact, one has that the limit $\overline{u}$ is a minimizer for Φ. On the other hand, we readily observe that u

minimizes Φ as well. Indeed, by $z \in \partial^+\varphi(u)$, one has that, for all $w \in \mathcal{P} + u$,

$$\begin{aligned}\Phi(u) &= \varphi(u) + \frac{1}{2}|u|^2 - (u+z,u) \\ &= (\varphi(u) - (z,u)) - \frac{1}{2}|u|^2 \le (\varphi(w) - (z,w)) - \frac{1}{2}|u|^2 \\ &= \varphi(w) + \frac{1}{2}|w|^2 - (u+z,w) + \left((u,w) - \frac{1}{2}|u|^2 - \frac{1}{2}|w|^2\right) \le \Phi(w).\end{aligned}$$

Since Φ is strictly convex, one has that $\overline{u} = u$, hence $\overline{y} \in \partial I_{\mathcal{P}}(0) = \mathcal{P}^* = -\mathcal{P}$, and $z = \overline{z} + \overline{y}$.

3 Main Results

This section is devoted to the statement of our main results on existence and long-time behavior, Theorems 1 and 2 respectively. We start by listing assumptions:

(A1) $(\mathcal{H}, \mathcal{P})$ is a Hilbert pseudolattice.
(A2) $A : \mathcal{H} \to \mathcal{H}$ is maximal strongly monotone and Lipschitz continuous (hence single-valued). In particular, there exists a constant $\alpha > 0$ such that

$$\alpha\left(|u-v|^2 + |A(u) - A(v)|^2\right) \le (A(u) - A(v), u - v) \quad \forall u, v \in \mathcal{H}. \tag{3.1}$$

(A3) $\varphi : H \to [0,\infty]$ is a convex, proper, and lower semicontinuous function with compact sublevels and fulfills the following time-dependent version of condition (2.5)

$$\begin{aligned}&\forall u \in L^2(0,T;\mathcal{H}) \text{ with } \varphi \circ u \in L^1(0,T) \text{ and} \\ &\forall w \in L^2(0,T;\mathcal{H}) \text{ with } w \succeq u \text{ a.e.},\ \forall u_n \to u \text{ strongly in } L^\infty(0,T;\mathcal{H}) : \\ &\exists w_n \in L^2(0,T;\mathcal{H}) \text{ with } \varphi \circ w_n \in L^1(0,T) \text{ such that} \\ &w_n \to w \text{ strongly in } L^2(0,T;\mathcal{H}),\ \ u_n \preceq w_n \ \text{ a.e., and} \\ &\liminf_{n\to\infty} \int_0^T (\varphi(w_n) - \varphi(u_n)) \le \int_0^T (\varphi(w) - \varphi(u)).\end{aligned} \tag{3.2}$$

(A4) $u^0 \in D(\varphi),\ v^0 = A(u^0),\ f \in L^2(0,T;\mathcal{H})$.

Before moving on, let us mention that the conditional upper semicontinuity assumption (3.2) follows whenever $\partial\varphi$ is T-monotone, namely when (2.2) or (2.3) holds. On the other hand, by arguing as in Lemma 1 one can prove that condition (3.2) yields that $\partial^+\varphi$ is strongly-weakly closed as operator from $L^2(0,T;\mathcal{H})$ to

$2^{L^2(0,T;\mathcal{H})}$. Note that the above hypotheses on φ may be somewhat weakened. The nonnegativity assumption $\varphi \geq 0$ can be replaced by $0 \in D(\varphi)$ and it suffices to ask $u \mapsto |u|^2 + \varphi(u)$ to have compact sublevels. We prefer to stick to (A3) for the sake of notational simplicity. Our existence result reads as follows.

Theorem 1 (Existence) *Assume* (A1)–(A4). *Then there exist* $u, v \in H^1(0,T;\mathcal{H})$ *such that* $v(0) = v^0$, $u(0) = u^0$ *and the following relations*

$$u' \in \mathcal{P}, \quad u \in D(\partial^+\varphi), \tag{3.3}$$

$$v' + \partial^+\varphi(u) \ni f, \tag{3.4}$$

$$v \in A(u), \tag{3.5}$$

hold almost everywhere in $(0,T)$. *Moreover, u and v fulfill the bound*

$$\int_0^t (v', u') + \varphi(u(t)) \leq \varphi(u^0) + \int_0^t (f, u') \quad \forall t \in [0,T]. \tag{3.6}$$

The proof of Theorem 1 is based on a classical approximation—a priori estimates—passage to the limit argument and is detailed in Sect. 4.

Recall that each trajectory u solving (3.3)–(3.5) in the sense of Theorem 1 is called a *nondecreasing* solution of the doubly nonlinear problem. This class includes all strong solutions to (1.3) and, a fortiori, of (1.1)–(1.2) (whenever existing).

Let us shed more light on the concept of nondecreasing solution by remaking that, in case the nondecreasing solution u belongs to $D(\partial\varphi)$ almost everywhere (a property which however seems not to follow directly from our analysis) and the stronger (2.2) (equivalently, (2.3) or (2.4)) holds, problem (3.3)–(3.5) can be equivalently recasted by means of Proposition 2 as that of finding $u \in H^1(0,T;\mathcal{H})$ and $v \in H^1(0,T;\mathcal{H})$, such that $u(0) = u^0$ one has that

$$u' \in \mathcal{P}, \tag{3.7}$$

$$v' + \partial\varphi(u) - f \in \mathcal{P}, \tag{3.8}$$

$$v \in A(u), \tag{3.9}$$

almost everywhere in time. Nondecreasing solutions solve hence problem (1.1)–(1.2), where however the equation in (1.1) is replaced by an inequality in (3.8).

Let us now turn to the special case already addressed in [11], namely

$$A = \text{id}, \ \varphi \text{ is T-monotone}, \ u^0 \in D(\partial\varphi), \ f \in H^1(0,T;\mathcal{H}), \ \text{and} \, f(0) - \partial\varphi(u^0) \subset \mathcal{P}. \tag{3.10}$$

In this case, relation (1.4) is strongly solvable. As a by-product of our existence proof we will observe that, whenever (3.10) holds, the approximation method indeed leads us to obtain a solution of the stronger (1.4). In this case, relation (3.8) turns to be an equality, namely $u' + z - f = 0$ for some $z \in L^2(0,T;\mathcal{H})$ such that $z \in \partial\varphi(u)$ almost everywhere.

In the case of a nonlinear operator A, by still assuming the T-monotonicity of φ, if $(\mathcal{H},\mathcal{P})$ is a Hilbert *lattice* and A is such that $u' \in \mathcal{P}$ and $v \in A(u) \Rightarrow v' \in \mathcal{P}$ (this is for instance the case of nondecreasing operators, namely $u \preceq w \Rightarrow A(u) \preceq A(w)$), we can conclude by means of the characterization Proposition 1 that

$$v' - (f - \partial\varphi(u))^+ \in \mathcal{P} \tag{3.11}$$

almost everywhere in time. Indeed, (3.8) entails, that $f - \partial\varphi(u) \preceq v'$. By taking the positive part on both sides (recall Proposition 1.iii) we have $(f - \partial\varphi(u))^+ \preceq (v')^+ = v'$ so that (3.11) follows. In particular, in the special case of (3.10) we get that

$$u' - (f - \partial\varphi(u))^+ \in \mathcal{P}$$

almost everywhere as well.

As far as uniqueness is concerned, we just remark that the problem (3.3)–(3.5) admits, in general, multiple solutions. Indeed, this is also the case of the stronger (1.3). First of all, note that the operators A and $\partial\varphi$ may be multivalued and it is straightforward to find examples of nonuniqueness for the respective selections. Furthermore u is nonunique as well [17, Sec. 5].

In addition to existence we provide an asymptotic result for the long-time behavior of nondecreasing solutions. In fact, the reader should notice that the above stated existence result is actually independent of the choice of the reference time T. In particular, by assuming

(A5) $f - f_\infty \in L^2(0,\infty;H)$ for some $f_\infty \in H$, and $u \mapsto \varphi(u) - (f_\infty, u)$ has compact sublevels

one can find $u \in H^1_{\mathrm{loc}}([0,\infty);\mathcal{H})$ and $v \in L^2_{\mathrm{loc}}([0,\infty);\mathcal{H})$ solving (3.3)–(3.5) on $(0,T)$ for all $T < \infty$. Indeed, since u turns out to be continuous with valued in the closed set $D(\varphi)$, one simply exploits a continuation argument. For any such solutions, let now the ω-limit set be defined as

$$\omega(u) = \{u_\infty \in \mathcal{H} \ : \ \exists t_n \in [0,T], \ t_n \to \infty, \ \text{ and } \ u(t_n) \to u_\infty\}.$$

We prove the following statement.

Theorem 2 (Long-Time Behavior) *Assume* (A1)–(A5) *and let* $u \in H^1_{\mathrm{loc}}(0,\infty;\mathcal{H})$ *be a nondecreasing solution on* $(0,T)$ *for all* $T < \infty$. *Then, the* ω*-limit set* $\omega(u)$ *is nonempty, compact, and connected. Moreover, all* $u_\infty \in \omega(u)$ *fulfill* $\partial^+\varphi(u_\infty) \ni f_\infty$ *and* $\omega(u)$ *reduces to a single point so that the whole trajectory converges.*

The proof of Theorem 2 is detailed in Sect. 5. We just remark here that, despite the set $\partial^+\varphi$ being quite large, the monotonicity in time of the nondecreasing solution implies that the ω-limit reduces to a point, a property that is often not available even in the context of classical gradient flows.

4 Existence

This section is focused on the proof of Theorem 1, which is based on a combined regularization and time-discretization procedure. Let $n \in \mathbb{N}$ and

$$\partial I_{\mathcal{P},\lambda}(w) = \frac{1}{\lambda}(w - w^+) \quad \forall w \in \mathcal{H}$$

be the Moreau-Yosida approximation of $\partial I_{\mathcal{P}}$ at level $\lambda = 1/n$. The operator $\partial I_{\mathcal{P},\lambda}$ is everywhere defined, monotone, and Lipschitz continuous. Let the time step τ be defined as $\tau = T/n$. For all vectors $\{x_i\}_{i=0}^n \in \mathcal{H}^{n+1}$ we define the piecewise constant in time and the piecewise affine interpolants $\overline{x}_n$, $\widehat{x}_n : [0,T] \to \mathcal{H}$ as

$$\overline{x}_n(0) = \widehat{x}_n(0) = 0, \quad \overline{x}_n(t) = x_i, \quad \widehat{x}_n(t) = \alpha_i(t)x_i + (1-\alpha_i(t))x_{i-1}$$
$$\text{with } \alpha_i(t) = (t-(i-1)\tau)/\tau \text{ for all } t \in ((i-1)\tau, i\tau] \text{ and } i = 1,\dots,n.$$

Moreover, indicate with $M_n : L^1(0,T;\mathcal{H}) \to L^\infty(0,T;\mathcal{H})$ the *mean operator* defined by

$$M_n(x) = \overline{x}_n \quad \text{where} \quad x_i = \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} x(t)\,\mathrm{d}t \qquad \text{for } i = 1,\dots,n,$$

for all $x \in L^1(0,T;\mathcal{H})$.

We will also use the shorthand notation δx_i for the *discrete time derivative* $\delta x_i = (x_i - x_{i-1})/\tau$.

By letting $u_0 = u^0$, $v_0 = v^0$, we consider the regularized and time-discretized problem consisting in finding $\{(u_i, v_i, z_i)\}_{i=1}^n \in \mathcal{H}^{3n}$ such that

$$v_i \in A(u_i), \quad z_i \in \partial\varphi(u_i), \quad \delta v_i + \partial I_{\mathcal{P},\lambda}(\delta u_i) + z_i = f_i \quad \text{for } i = 1,\dots,n \tag{4.1}$$

where $f_i = \tau^{-1}\int_{(i-1)\tau}^{i\tau} f(t)\,\mathrm{d}t$. At each level i, the latter can be rewritten as

$$A(u_i) + \tau\partial I_{\mathcal{P},\lambda}\left(\frac{u_i - u_{i-1}}{\tau}\right) + \tau\partial\varphi(u_i) \ni \tau f_i + A(u_{i-1}). \tag{4.2}$$

Given u_{i-1}, relation (4.2) has a unique solution u_i as the operator

$$u \in D(\varphi) \mapsto A(u) + \tau\partial I_{\mathcal{P},\lambda}\left(\frac{u - u_{i-1}}{\tau}\right) + \tau\partial\varphi(u)$$

is maximal monotone and coercive, hence onto [13, Cor. 2.3, p. 31]. The maximal monotonicity follows from the fact that $D(A) = D(\partial I_{\mathcal{P},\lambda}((\cdot - u_{i-1})/\tau)) = \mathcal{H}$ (the regularization of $\partial I_{\mathcal{P}}$ is here needed) by the classical [13, Cor. 2.7, p. 36]. The coercivity is a direct consequence of (A2).

By testing the last equation in (4.1) by $\tau\delta u_i$ and summing for $i = 1, \ldots, m$, $m \leq n$ one gets

$$\int_0^{m\tau} (\widehat{v}_n', \widehat{u}_n') + \int_0^{m\tau} (\partial I_{\mathcal{P},\lambda}(\widehat{u}_n'), \widehat{u}_n') + \varphi(\overline{u}_n(m\tau)) \leq \varphi(u^0) + \int_0^{m\tau} (\overline{f}_n, \widehat{u}_n'). \tag{4.3}$$

The second term in the above left-hand side reads

$$\int_0^{m\tau} (\partial I_{\mathcal{P},\lambda}(\widehat{u}_n'), \widehat{u}_n') = \frac{1}{\lambda}\int_0^{m\tau} |\widehat{u}_n' - (\widehat{u}_n')^+|^2$$

so that relation (4.3) and assumption (A2) entail that

$$\alpha \int_0^{m\tau} |\widehat{u}_n'|^2 + \alpha \int_0^{m\tau} |\widehat{v}_n'|^2 + \frac{1}{\lambda}\int_0^{m\tau} |\widehat{u}_n' - (\widehat{u}_n')^+|^2 + \varphi(\overline{u}_n(m\tau))$$
$$\leq \varphi(u^0) + \int_0^{m\tau} (\overline{f}_n, \widehat{u}_n').$$

The nonnegativity of φ from (A3) allows us to estimate

$$\int_0^{T} |\widehat{u}_n'|^2 + \int_0^{T} |\widehat{v}_n'|^2 + \frac{1}{\lambda}\int_0^{T} |\widehat{u}_n' - (\widehat{u}_n')^+|^2 + \sup_{t\in[0,T]} \varphi(\overline{u}_n(t)) \leq c \tag{4.4}$$

where c depends on α, $\varphi(u^0)$, and $\|f\|_{L^2(0,T;\mathcal{H})}$. By convexity, it also follows that

$$\sup_{t\in[0,T]} \varphi(\widehat{u}_n) \leq c.$$

As the sublevels of φ are precompact, see (A3), we can extract not relabeled subsequences such that

$$\widehat{u}_n \to u \quad \text{weakly in } H^1(0,T;\mathcal{H}) \text{ and strongly in } C([0,T];\mathcal{H}), \tag{4.5}$$
$$\overline{u}_n \to u \quad \text{strongly in } L^\infty(0,T;\mathcal{H}), \tag{4.6}$$
$$\widehat{v}_n \to v \quad \text{weakly in } H^1(0,T;\mathcal{H}) \tag{4.7}$$

as $n \to \infty$. Here we also note that $\widehat{u}_n - \overline{u}_n \to 0$ strongly in $L^2(0,T;\mathcal{H})$. It further implies that $\varphi \circ u \in L^\infty(0,T)$ by lower-semicontinuity of φ. Taking bound (4.4) into account, convergence (4.5) implies that

$$(\widehat{u}_n')^+ \to u' \quad \text{weakly in } L^2(0,T;\mathcal{H}).$$

Hence, the closedness of $\mathcal{P}$ ensures that $u' \in \mathcal{P}$ almost everywhere in time and (3.3) follows. On the other hand, since A is Lipschitz continuous and $\overline{v}_n = A(\overline{u}_n)$,

$$\overline{v}_n \to \overline{v} = A(u) \quad \text{strongly in } L^\infty(0,T;\mathcal{H}).$$

Moreover, $\overline{v}$ turns out to coincide with v, since $\widehat{v}_n - \overline{v}_n \to 0$ in $L^2(0,T;\mathcal{H})$. This ensures $v = A(u)$ almost everywhere in time, namely (3.5).

We now rewrite (4.1) as

$$\overline{u}_n \in D(\varphi) \text{ and } (\overline{f}_n - \widehat{v}'_n, \overline{w}_n - \overline{u}_n) - (\partial I_{\mathcal{P},\lambda}(\widehat{u}'_n), \overline{w}_n - \overline{u}_n) + \varphi(\overline{u}_n) \le \varphi(\overline{w}_n)$$
$$\forall \overline{w}_n \in L^2(0,T;\mathcal{H}), \text{ a.e. in } (0,T). \tag{4.8}$$

Fix $w \in L^2(0,T;\mathcal{H})$ such that $u \preceq w$ almost everywhere and let w_n be defined according to condition (3.2) with $u_n = \overline{u}_n$. One can check that the piecewise constant sequence $M_n(w_n)$ can be used in (3.2) instead of w_n. In fact,

$$\overline{u}_n \preceq w_n \quad \Rightarrow \quad \overline{u}_n = M_n(\overline{u}_n) \preceq M_n(w_n)$$

almost everywhere, as $\overline{u}_n$ is piecewise constant on the partition and $\mathcal{P}$ is convex. Moreover, $M_n(w_n) \to w$ strongly in $L^2(0,T;\mathcal{H})$ as we have that

$$\begin{aligned}\|w - M_n(w_n)\|_{L^2(0,T;\mathcal{H})} &\le \|w - M_n(w)\|_{L^2(0,T;\mathcal{H})} + \|M_n(w) - M_n(w_n)\|_{L^2(0,T;\mathcal{H})} \\ &\le \|w - M_n(w)\|_{L^2(0,T;\mathcal{H})} + \|w - w_n\|_{L^2(0,T;\mathcal{H})} \to 0\end{aligned}$$

where we have used that $M_n(w) \to w$ strongly in $L^2(0,T;\mathcal{H})$ and that M_n is a contraction in $L^2(0,T;\mathcal{H})$. Finally, the convexity of φ entails that $\varphi(M_n(w_n)) \le M_n(\varphi(w_n))$ almost everywhere (recall that $\varphi \circ w_n \in L^1(0,T)$), hence

$$\begin{aligned}&\int_0^T \big(\varphi(M_n(w_n)) - \varphi(\overline{u}_n)\big) \le \int_0^T \big(M_n(\varphi(w_n)) - \varphi(\overline{u}_n)\big) \\ &= \int_0^T \big(M_n(\varphi(w_n)) - M_n(\varphi(\overline{u}_n))\big) = \int_0^T \big(\varphi(w_n) - \varphi(\overline{u}_n)\big).\end{aligned}$$

Note that the first equality above holds as $\varphi(\overline{u}_n)$ is piecewise constant on the partition and the last equality follows from a direct computation.

Choose now $\overline{w}_n = M_n(w_n)$ in (4.8). By dropping the nonnegative term

$$-(\partial I_{\mathcal{P},\lambda}(\widehat{u}'_n), \overline{w}_n - \overline{u}_n)$$

relation (4.8) can be integrated on $(0,T)$ giving

$$\int_0^T (\overline{f}_n - \widehat{v}'_n, \overline{w}_n - \overline{u}_n) \le \int_0^T \big(\varphi(\overline{w}_n) - \varphi(\overline{u}_n)\big).$$

It suffices to take the lim inf as $n \to \infty$ and use (A3) in order to obtain that

$$\int_0^T (f - v', w - u) \le \int_0^T \big(\varphi(w) - \varphi(u)\big)$$

so that relation (3.4) holds. This concludes the proof of Theorem (1).

Before closing this section let us show that, in the case $A = \text{id}$ and φ is T-monotone (2.2), the nondecreasing solution u is a supersolution to the original unconstrained problem (1.1) in the sense that it bounds almost everywhere from above any strong solution to (1.1) with the same initial datum. Indeed, let $\tilde{u}$ solve (1.1), namely

$$\tilde{u} \in D(\varphi) \ \text{ and } \ (\tilde{u}' - f, \tilde{u} - \tilde{w}) + \varphi(\tilde{u}) \le \varphi(\tilde{w}) \quad \forall \tilde{w} \in L^2(0,T;\mathcal{H}), \ \text{a.e. in } \ (0,T).$$

By choosing $\tilde{w} = \tilde{u} \min u$ above and $w = u \max \tilde{u}$ in (1.5) (note that this choice is admissible since of course $u \max \tilde{u} - u \in \mathcal{P}$), taking the sum of the corresponding relations, and using (2.2) we readily get that

$$((\tilde{u} - u)', (\tilde{u} - u)^+) \le 0 \quad \text{a.e. in } \ (0,T).$$

Hence, also taking the initial conditions into account and using the Gronwall Lemma, it follows from and integration in time that $\tilde{u} \le u$ a.e. Note nonetheless that nondecreasing-in-time functions bounding $\tilde{u}$ from above are not necessarily nondecreasing solutions.

If the stronger conditions (3.10) hold, one could pass to the limit as $\tau \to 0$ first, for $\lambda > 0$ fixed. This is doable as one can identify the limit of $\partial I_{\mathcal{P},\lambda}(\widehat{u}_n')$ by a classical semicontinuity argument, see, e.g., [16]. For all $\lambda > 0$ we hence find $u_\lambda,\ v_\lambda \in H^1(0,T;\mathcal{H})$ such that

$$u_\lambda' + \partial I_{\mathcal{P},\lambda}(u_\lambda') + z_\lambda \ni f, \quad z_\lambda \in \partial\varphi(u_\lambda) \quad \text{a.e. in } (0,T). \tag{4.9}$$

By formally taking the time derivative above and testing on $y_\lambda = \partial I_{\mathcal{P},\lambda}(u_\lambda')$ one gets

$$\int_0^t (u_\lambda'', y_\lambda) + \int_0^t (y_\lambda', y_\lambda) + \int_0^t (z_\lambda', y_\lambda) = \int_0^t (f', y_\lambda). \tag{4.10}$$

By evaluating (4.9) at the initial time one finds

$$u_\lambda'(0) + y_\lambda(0) = f(0) - z_\lambda(0) \in f(0) - \partial\varphi(u^0) \overset{(3.10)}{\subset} \mathcal{P}$$

so that $u_\lambda'(0) \in \mathcal{P}$ follows by adding $-y_\lambda(0) \in \mathcal{P}$ to both sides. Note however that this argument is presently just formal, as (4.9) is valid almost everywhere and cannot be evaluated at the initial time. The argument could however be made rigorous by resorting to an additional approximation, namely by replacing φ by its Moreau-Yosida regularization, see [8] for a similar approach. The first term in the right-hand side of (4.10) is nonnegative as

$$\int_0^t (u_\lambda'', y_\lambda) = I_{\mathcal{P},\lambda}(u_\lambda'(t)) - I_{\mathcal{P},\lambda}(u_\lambda'(0)) \geq -I_{\mathcal{P},\lambda}(u_\lambda'(0)) = -\frac{1}{2\lambda}|(u_\lambda'(0))^-|^2 = 0.$$

The second term in the left-hand side of (4.10) yields

$$\int_0^t (y_\lambda', y_\lambda) = \frac{1}{2}|y_\lambda(t)|^2 - \frac{1}{2}|y_\lambda(0)|^2 = \frac{1}{2}|y_\lambda(t)|^2,$$

since $y_\lambda(0) = \partial I_{\mathcal{P},\lambda}(u_\lambda'(0)) = -(u_\lambda'(0))^-/\lambda = 0$. The third term in the left-hand side of (4.10) is nonnegative due to T-monotonicity and the right-hand side can be controlled via the Gronwall Lemma. An additional comparison in (4.9) yields the estimates

$$\sup_{[0,T]} |y_\lambda'|^2 + \int_0^T |z_\lambda|^2 \leq c$$

where now c depends on $\|f'\|_{L^2(0,T;\mathcal{H})}$, $|(\partial\varphi(u^0))^\circ|$, and T. These estimates suffice in order to pass to the limit in each term in (4.9) and obtain a strong solution to (1.3).

5 Long-Time Behavior

We now turn to the proof of Theorem 2. Let $u \in H^1_{\mathrm{loc}}(0,\infty;\mathcal{H})$ be a nondecreasing solution on $(0,T)$, for all $T < \infty$. Rewrite bound (3.6) as

$$\int_0^t (v', u') + \varphi(u(t)) - (f_\infty, u(t)) \leq \varphi(u^0) - (f_\infty, u^0) + \int_0^t (f - f_\infty, u') \quad \forall t \geq 0.$$

Then, assumptions (A2) and (A3) allow us to deduce that

$$\int_0^\infty |u'|^2 + \int_0^\infty |v'|^2 + \sup_{t\geq 0} \big(\varphi(u(t)) - (f_\infty, u(t))\big) \leq c \tag{5.1}$$

where $c > 0$ depends on α, $\varphi(u^0)$, $|f_\infty|$, $\|f - f_\infty\|_{L^2(0,T;\mathcal{H})}$, and $\inf(\varphi(u) - (f_\infty, u))$ and the latter infimum is not $-\infty$ as $u \mapsto \varphi(u) - (f_\infty, u)$ has compact sublevels due to (A5). Hence, the trajectory $t \mapsto u(t)$ for $t > 0$ is relatively compact in $\mathcal{H}$. Therefore $\omega(u)$ is a nonempty and compact subset of $\mathcal{H}$. In addition, since $u \in C^0([0,\infty), \mathcal{H})$, a classical argument from the theory of dynamical systems ensures that $\omega(u)$ is connected in $\mathcal{H}$ (see e.g. [21, p. 12]).

Let now $u_\infty \in \omega(u)$. Namely, there exists a sequence $t_n > 0$ of strictly increasing time such that $t_n \to \infty$ and $u(t_n) \to u_\infty$. For $n \in \mathbb{N}$ and $t \in [0, T)$, we define $u_n(t) = u(t_n + t)$ and $f_n(t) = f(t + t_n)$. It is straightforward to check that $u_n, v_n \in H^1(0, T; \mathcal{H})$ solve

$$\begin{aligned}
&u_n(0) = u(t_n), \quad u_n \in D(\varphi), \quad u_n' \in \mathcal{P}, \\
&(f_n - v_n', w - u_n) + \varphi(u_n) \le \varphi(w) \quad \forall w \in L^2(0, T; \mathcal{H}) \\
&\text{such that } \ u_n \preceq w, \ \ v_n \in A(u_n'), \ \text{ a.e. in } \ (0, T).
\end{aligned} \tag{5.2}$$

We readily identify the limits of u_n, v_n, and f_n as $n \to \infty$. Namely, we have that

$$\begin{aligned}
&u_n \to u_\infty, \quad u_n' \to 0 \quad \text{strongly in } \ L^2(0, T; \mathcal{H}), \\
&v_n \to v_\infty, \quad v_n' \to 0 \quad \text{strongly in } \ L^2(0, T; \mathcal{H}), \\
&f_n \to f_\infty \quad \text{strongly in } \ L^2(0, T; \mathcal{H}).
\end{aligned}$$

Indeed, taking (5.1) into account we easily check that

$$\int_0^T |u_n'|^2 = \int_{t_n}^{t_n+T} |u'|^2 \to 0, \quad \int_0^T |v_n'|^2 = \int_{t_n}^{t_n+T} |v'|^2 \to 0 \quad \text{as} \quad n \to +\infty.$$

This computation entails in particular that

$$\begin{aligned}
|u_n(t) - u_\infty| &\le |u_n(t) - u_n(0)| + |u(t_n) - u_\infty| \\
&\le T^{1/2} \left(\int_0^T |u_n'|^2 \right)^{1/2} + |u(t_n) - u_\infty| \to 0,
\end{aligned}$$

an $v_n \to v_\infty = A(u_\infty)$ by Lipschitz continuity of A (see (A2)).

Let now $w \in \mathcal{H}$ be given such that $u_\infty \preceq w$ and use w_n given by condition (3.2) in relation (5.2). One has that

$$\begin{aligned}
\int_0^T (f_\infty, w - u_\infty) &= \liminf_{n\to\infty} \int_0^T (f_n - v_n', w_n - u_n) \\
&\le \liminf_{n\to\infty} \int_0^T \big(\varphi(w_n) - \varphi(u_n)\big) \overset{(3.2)}{\le} \int_0^T \big(\varphi(w) - \varphi(u_\infty)\big)
\end{aligned}$$

which proves that $\partial^+\varphi(u_\infty) \ni f_\infty$.

The monotonicity of the trajectory with respect to the order entails that $\omega(u)$ reduces to a single point. Indeed, for all $t > 0$ by letting t_n be such that $t_n \le t < t_{n+1}$ one has

$$|u_\infty - u(t)|^2 = (u_\infty - u(t), u_\infty - u(t)) \le (u_\infty - u(t_n), u_\infty - u^0) \to 0.$$

Here we used that

$$u(t) - u(s) = \int_s^t u' \in \mathcal{P} \;\text{ for }\; 0 \leq s \leq t \leq \infty \quad \text{and} \quad (w, z) \geq 0 \;\text{ for all }\; w, z \in \mathcal{P}.$$

Hence, $u(t) \to u_\infty$ and the proof is completed.

6 Concluding Remarks

We conclude this note by highlighting some possible ideas for further investigations.

6.1 Ordering of Nondecreasing Solutions

As already mentioned, nondecreasing solutions are likely to be nonunique. It would be interesting to identify setting where the set of nondecreasing solutions has itself some order structure, possibly including a minimal element. This seems to be the case in some concrete situations and would offer a variational selection procedure towards uniqueness.

More generally, it would be worth investigating the relation between nondecreasing solutions and supersolutions of (1.3), namely trajectories $t \mapsto \tilde{u}(t)$ such that $u \preceq \tilde{u}$ almost everywhere, for all solutions u of (1.3). It is clear that supersolutions of (1.3) are not nondecreasing solutions in general. On the other hand, in some cases (see the end of Sect. 4) nondecreasing solutions are supersolutions of (1.3). It would be interesting to understand if nondecreasing solutions are minimal within the class of supersolutions of (1.3). This would again provide a selection criterion for uniqueness.

6.2 Other Approximations

An alternative strategy to prove Theorem 1 is that of regularizing the problem by letting φ be replaced by its Moreau-Yosida approximation φ_λ. In case A is nondecreasing one has $u' \in \mathcal{P}$ and $v \in A(u)$ implies $v' \in \mathcal{P}$. This is for instance the case if $\mathcal{H} = L^2(\Omega)$ and $A(u)(x) = \alpha(u(x))$ for $\alpha : \mathbb{R} \to \mathbb{R}$ strongly monotone and Lipschitz continuous with $\alpha(0) = 0$. By assuming additionally that $\partial I_{\mathcal{P}}(u') \subset \partial I_{\mathcal{P}}(v')$ one can find a solution to

$$A(u_\lambda)' + \partial I_{\mathcal{P}}(u_\lambda') + \partial \varphi_\lambda(u_\lambda) \ni f$$

by looking to relation

$$A(u_\lambda)' + \partial I_{\mathcal{P}}(A(u_\lambda)') + \partial \varphi_\lambda(u_\lambda) \ni f,$$

instead. The latter can be rewritten as

$$v_\lambda' = (I + \partial I_{\mathcal{P}})^{-1} \left(f - \partial \varphi_\lambda(A^{-1}(v_\lambda))\right).$$

Due to (A2), A^{-1} is Lipschitz continuous in H, and therefore, one can prove the existence of solutions $v_\lambda \in C^1([0,T];H)$, which also implies $u_\lambda \in C^1([0,T];H)$.

Approximating in space, for instance by a Galerkin method, namely by reducing to finite-dimensional subspaces $\mathcal{H}_h$ invading $\mathcal{H}$, may also be amenable. This would correspond to a conformal finite element method. The corresponding discretization consists of a system of ODEs. It would probably turn out sensible to coordinate the discretization with the order structure. In particular, one could assume the finite-dimensional subspaces $\mathcal{H}_h$ to be preordered via the cones $\mathcal{P}_h = \mathcal{P} \cap \mathcal{H}_h$.

6.3 Stability

Assume to be given a sequence of data $(A_n, \varphi_n, u_n^0, f_n)$ fulfilling assumptions (A1)–(A4) uniformly with respect to n. Correspondingly, for all n the problem (3.3)–(3.5) with initial datum u_n^0 admits a nondecreasing solution u_n by extending the results in [3]. One could try to identify a suitable convergence frame for data $(A_n, \varphi_n, u_n^0, f_n) \to (A, \varphi, u^0, f)$ ensuring that $u_n \to u$ where u is a decreasing solution of the limiting problem (3.3)–(3.5).

A relevant assumption in this direction seems to be the *conditional Mosco-convergence*

$$\begin{aligned}
&\forall u \in \mathcal{H},\ \forall w \in D(\varphi) \text{ with } u \preceq w,\ \forall u_n \in D(\varphi_n) \text{ with } u_n \to u :\\
&\exists w_n \in D(\varphi_n) \text{ with } u_n \preceq w_n,\ w_n \to w, \text{ and}\\
&\liminf_{n\to\infty} \left(\varphi_n(w_n) - \varphi_n(u_n)\right) \leq \varphi(w) - \varphi(u),
\end{aligned} \tag{6.1}$$

$$\varphi(u) \leq \inf\{\liminf_{n\to\infty} \varphi(u_n) \ : \ u_n \rightharpoonup u\}. \tag{6.2}$$

Note that the two conditions (6.1)–(6.2) are weaker than (2.2) and $\varphi_n \to \varphi$ in the sense of Mosco (choose $w_n = w \max u_n$).

6.4 Dynamic Problems

It would be interesting to investigate the existence and approximation of nondecreasing solutions for classes of hyperbolic problems of the form

$$u'' + A(u)' + \partial\varphi(u) \ni f, \quad u' \in \mathcal{P}.$$

This could arise as a variational version of balance equations, with potential applications to Mechanics.

The basic estimate for the problem is again a test on u', which directly entails the possibility of considering nondecreasing solutions by using $\partial^+\varphi$ instead of $\partial\varphi$.

6.5 Refined Assumptions

We have developed the analysis in the case of a Hilbert pseudolattice, namely by assuming the cone $\mathcal{P}$ of positive elements to be self-polar. Some arguments seem however to be valid even for not self-polar cones $\mathcal{P}$, i.e. without assuming the pseudolattice structure.

The existence theory relies on convexity and compactness, as all doubly nonlinear theories do. T-monotonicity plays however a role, even if limited to the conditional upper semicontinuity assumptions (2.5) and (3.2). It would be probably possible, likely at expense of a lesser generality, to present cases in which the compactness assumptions can be weakened in favor of a stronger ordering framework, see [15].

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Identification Problems for Degenerate Integro-Differential Equations

Mohammed Al Horani, Mauro Fabrizio, Angelo Favini, and Hiroki Tanabe

Abstract We are concerned with two different inverse problems for degenerate integro-differential equations in Banach spaces. In the first, we handle a strongly degenerate problem on a finite interval, while in the second we consider a related inverse problem for integro-differential equations studied by G. Da Prato and A. Lunardi in the regular case. All these results can be applied to inverse problems for equations from mathematical physics.

Keywords Integrodifferential equations • Inverse problems • Periodic solutions

AMS (MOS) Subject Classification 35R09, 35R30, 35A01, 35A02, 35K20

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1 Introduction

In previous years many researchers studied inverse problems for differential equations in Banach spaces, cfr. [15], which consists of recovering the unknowns $(y, f_1, \ldots, f_n)$ such that

$$\frac{dy}{dt} = Ay + \int_0^t k(t-s)A_1 y(s)\, ds + \sum_{i=1}^n f_i(t) z_i + h(t), \quad 0 \le t \le \tau\,,$$
$$y(0) = y_0 \in D(A)\,,$$
$$\Phi_i[\, y(t)] = g_i(t), \quad i = 1, \ldots, n, \quad t \in [0, \tau]\,,$$

where A generates a c_0-semigroup in a complex Banach space X, A_1 is a closed linear operator on X with $D(A) \subseteq D(A_1)$, $k \in C([0,\tau];\mathbb{C})$, $z_i \in X$, $h \in C([0,\tau];X)$, $\Phi \in X^*$, $g_i \in C([0,\tau];\mathbb{C})$, $\Phi_i[\,y_0] = g_i(0)$, $f_i \in C([0,\tau];\mathbb{C})$, $y \in C([0,\tau];D(A))$.

This type of problems has been discussed using different methods. The first approach, devoted to a simpler case where the linear operators are assumed to be bounded, was described in [28]. Of course, Alfredo Lorenzi, in collaboration with many co-authors, studied the inverse problem where the involved operators are unbounded using the fixed point theorem approach, see [20, 21]. Similar results on nonsingular systems are obtained in [30] under different additional conditions. Recently, both degenerate and non-degenerate inverse differential problems were discussed by some authors using perturbation method, see [2, 3] and [5, 6].

More recently, see [22] and [23], some results appeared on the degenerate case, i.e.,

$$\frac{d}{dt}(My(t)) = Ly(t) + \int_0^t k(t-s)L_1 y(s)\, ds + \sum_{i=1}^n f_i(t) z_i + h(t), \quad 0 \le t \le \tau\,,$$
$$(My)(0) = My_0, \quad y_0 \in D(L)\,,$$
$$\Phi_i[My(t)] = g_i(t), \quad i = 1, \ldots, n, \quad t \in [0,\tau], \quad \Phi_i[My_0] = g_i(0).$$

Here L, M, L_1 are closed linear operators on X, L is invertible, $D(L) \subseteq D(M) \cap D(L_1)$. In particular, we point out the paper of Favini and Tanabe [17], where it is shown that under suitable assumptions, the inverse problem is reduced to a related direct problem to which the previous arguments in [22] and [23] apply. Let us illustrate this, for sake of brevity, in the case $n = 1$.

Lemma 1 *Let $\alpha + \beta > 1$, $2 - \alpha - \beta < \theta < 1$, $D(L) \subseteq D(M) \cap D(L_1)$,*

$$\|(\xi + L_1)^{-1}\|_{\mathscr{L}(X)} \le \frac{c}{(1+|\xi|)^{\beta_1}}\,, \quad \operatorname{Re}\xi \ge -c(1+|\operatorname{Im}\xi|)^{\alpha_1}, \quad 0 < \beta_1 < \alpha_1 \le 1\,.$$

If A is the multivalued operator LM^{-1} and $T = ML^{-1}$ is its inverse, $z \in D(A) \cap X^{\delta}_{L_1}$, $\delta > 1-\beta_1$, $h \in C^{\theta}([0,\tau];X)$, $g \in C^{\theta+1}([0,\tau];\mathbb{C})$, $k \in C^{\theta}([0,\tau];\mathbb{C})$, $h(0) - Ly_0 \in R(T) = D(A)$, $\Phi[z] \neq 0$, then problem

$$\begin{aligned}
&\frac{d}{dt}(My)(t) = Ly(t) + \int_0^t k(t-s)L_1 y(s)\,ds + f(t)z + h(t)\,,\\
&My(0) = My_0\,,\\
&\Phi[My(t)] = g(t), \quad \Phi[My_0] = g(0)
\end{aligned}$$

admits a unique strict solution

$$(y,f) \in C^{\theta-2+\alpha+\beta}([0,\tau];D(L)) \times C^{\theta-2+\alpha+\beta}([0,\tau];\mathbb{C}).$$

Indeed, the above inverse problem is reduced to a direct problem by noting that the additional information $\Phi[My(t)] = g(t)$ leads to

$$g'(t) = \Phi[Ly(t)] + \int_0^t k(t-s)\Phi[L_1 y(s)]\,ds + f(t)\Phi[z] + \Phi[h(t)]\,,$$

so that, using the assumption $\Phi[z] \neq 0$, $y(\cdot)$ necessarily satisfies

$$\begin{aligned}
&\frac{d}{dt}(My(t)) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}z + \int_0^t k(t-s)\left[L_1 y(s) - \frac{\Phi[L_1 y(s)]}{\Phi[z]}z\right]ds\\
&\qquad\qquad - \frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)\\
&(My)(0) = My_0\,.
\end{aligned}$$

Hence, one can use the results of existence, uniqueness and regularity of solutions which are applied to such direct problem, see Lemma 5. This method is new at all and improves the usual fixed point theorem approach. For this purpose, we recall the monographs [7, 13] and [30]. In order to improve Theorem 6.1 in [17], a basic help comes from a new step in [15], see also [16, Lemma 1], precisely

Lemma 2 *Suppose $k > 0$ sufficiently large so that $0 \in \rho(kM + L)$, $0 \in \rho(kM + L + L_1)$, where $L_1 \in \mathscr{L}\left(D(L), X^{\bar\theta}_A\right)$, $\bar\theta > 1-\beta$, $A = LM^{-1}$, $A_1 = (L+L_1)M^{-1}$. Then, for all $\delta \in (0,1)$,*

$$X^{\delta}_A = X^{\delta}_{A_1} = X^{\delta}_{(kM+L+L_1)M^{-1}}\,.$$

In Sect. 2, we shall consider a more complicated inverse problem concerning

$$M\frac{dy}{dt} = Ly(t) + \int_0^t k(t-s)L_1 y(s)\,ds + \sum_{i=1}^n f_i(t)z_i + h(t), \quad 0 \le t \le \tau\,,$$

with initial condition

$$y(0) = y_0$$

and the additional information

$$\Phi_i[My(t)] = g_i(t), \quad i = 1, \dots, n\,, \quad t \in [0, \tau]\,.$$

It is clear that more regularity for the inputs must be required, because the solution y of the problem considered above is not necessarily differentiable, together with compatibility relations. The main step is noting that if ξ satisfies

$$\frac{d}{dt}(M\xi(t)) = L\xi(t) + \int_0^t k(t-s)L_1\xi(s)\,ds + k(t)L_1y_0 + h'(t),$$
$$(M\xi)(0) = My_1 = Ly_0 + h(0),$$

then $y(t) = \displaystyle\int_0^t \xi(s)\,ds + y_0$ is such that

$$My'(t) = Ly(t) + \int_0^t k(t-s)L_1y(s)\,ds + h(t), \quad 0 \le t \le \tau\,,$$
$$y(0) = y_0\,.$$

Thus, the second problem can be solved by means of a suitable reduction to the first problem. We want to recall that some related inverse problems for possibly degenerate equations are discussed in [7] using a fixed point theorem argument. Section 3 is devoted to a special case of Sect. 2, where $z = 0$ is a simple pole for $(z - T)^{-1}$, $T = ML^{-1}$. In Sect. 4, we shall consider an inverse problem of the type

$$\frac{d}{dt}(A+1)y(t) = Ay(t) + \int_{-\infty}^t k(t-s)Ay(s)\,ds + f(t)z + h(t), \quad t \in \mathbb{R}\,,$$
$$(A+1)y(0) = (A+1)y(1)\,,$$
$$\Phi\big[(A+1)y(t)\big] = g(t), \quad t \in [0,1]\,,$$

where -1 is a simple pole for the resolvent $(\lambda - A)^{-1}$. If instead of $(A + 1)$, we consider the identity operator, then a problem like

$$\frac{dy}{dt} = Ay(t) + \int_{-\infty}^t k(t-s)Ay(s)\,ds + F(t)$$

was considered (even in more general form) by Da Prato and Lunardi, see [12]. Notice that the argument in [12] can not be applied directly, since the operator $(A + 1)$ has no bounded inverse by assumption.

2 An Inverse Problem for an Integro-Differential Equation

As a first step, we begin this section by considering the simplest case, i.e., only one term f has to be recovered. Let L, M be two closed linear operators on the complex Banach space X, such that for all $\lambda \in \Sigma_\alpha$ where

$$\Sigma_\alpha := \left\{\lambda \in \mathbb{C} : Re\,\lambda \geq -c(1 + |Im\,\lambda|)^\alpha\right\}, \tag{2.1}$$

$\lambda M - L$ has a bounded inverse $(\lambda M - L)^{-1}$ and

$$\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \leq \frac{c}{(|\lambda| + 1)^\beta}, \quad \forall\, \lambda \in \Sigma_\alpha \tag{2.2}$$

with, $c > 0$, $0 < \beta \leq \alpha \leq 1$ (degenerate parabolic case).

Of concern is to find a unique strict solution

$$(y, f) \in C([0, \tau]; D(L)) \times C([0, \tau]; \mathbb{C})$$

(i.e., there exists $y'(t) \in C([0, \tau]; X)$, $My'(t)$ being continuous on $[0, \tau]$) to the inverse problem

$$My'(t) = Ly(t) + \int_0^t k(t-s)L_1 y(s)\,ds + f(t)z + h(t), \quad 0 \leq t \leq \tau\,, \tag{2.3}$$

$$y(0) = y_0 \in D(L), \tag{2.4}$$

$$\Phi[My(t)] = g(t), \quad 0 \leq t \leq \tau\,, \tag{2.5}$$

where L_1 is a closed linear operator on X with $D(L) \subseteq D(L_1)$, $k \in C([0, \tau]; \mathbb{C})$, z is a fixed element in X, $y_0 \in D(L)$, $h \in C([0, \tau]; X)$, $\Phi \in X^*$, $g \in C^1([0, \tau]; \mathbb{C})$, $\Phi[My_0] = g(0)$. Recall that we always suppose $D(L) \subseteq D(M)$. Applying Φ to both sides of (2.3) and taking into account (2.5), we get

$$g'(t) = \Phi[Ly(t)] + \int_0^t k(t-s)\Phi[L_1 y(s)]\,ds + f(t)\Phi[z] + \Phi[h(t)]\,.$$

Assuming that $\Phi[z] \neq 0$, we necessarily obtain

$$f(t) = \frac{g'(t) - \Phi[Ly(t)] - \int_0^t k(t-s)\Phi[L_1 y(s)]\,ds - \Phi[h(t)]}{\Phi[z]}\,,$$

so that our inverse problem is reduced to the direct problem

$$My'(t) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}z + \int_0^t k(t-s)\left[L_1 y(s) - \frac{\Phi[L_1 y(s)]}{\Phi[z]}z\right] ds$$

$$-\frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)\,, \quad 0 \le t \le \tau\,, \tag{2.6}$$

$$y(0) = y_0\,.$$

Let L_2 be the operator defined by

$$D(L_2) = D(L), \quad L_2 y = -\frac{\Phi[Ly]}{\Phi[z]}z\,,$$

Eq. (2.6) reads

$$My'(t) = (L + L_2)y(t) + \int_0^t k(t-s)L_3 y(s)\,ds + \frac{g'(t)}{\Phi[z]}z - \frac{\Phi[h(t)]}{\Phi[z]}z + h(t)\,, \tag{2.7}$$

where L_3 denotes the linear operator

$$D(L_3) = D(L_1), \quad L_3 x = L_1 x - \frac{\Phi[L_1 x]}{\Phi[z]}z\,.$$

Of course, such an operator may not be closed. To this end, it is important to introduce the multivalued operator $A = LM^{-1}$ and the Banach spaces

$$X_A^{\gamma,p} = \Big\{x \in X,\ [x]_{X_A^{\gamma,p}} = \|\xi^\gamma A^0(\xi I - A)^{-1}x\|_{L_*^p(X)} < \infty, \quad \gamma \in (0,1),\ p \in [1,\infty)\,,\Big\}$$

$$\|x\|_{X_A^{\gamma,p}} = \|x\|_X + [x]_{X_A^{\gamma,p}}, \quad \gamma \in (0,1), \quad p \in [1,\infty)\,,$$

where

$$A^0(\xi I - A)^{-1} := -I + \xi(\xi I - A)^{-1}\,,$$

$$L_*^p(X) = \Big\{f : (0,\infty) \to X \ \text{strongly measurable, such that} \quad \|f\|^p_{L_*^p(X)} = \int_0^\infty \|f(t)\|_X^p \frac{dt}{t} < \infty\Big\}\,,$$

if $1 \le p < \infty$,

$$L_*^\infty(X) = \Big\{f : (0,\infty) \to X \ \text{strongly measurable on } R^+ \quad \text{and } \ \|f\|_{L_*^\infty(X)} = \operatorname*{ess\,sup}_{R^+} \|f(t)\|_X < \infty\Big\}\,.$$

It is well known that the embedding, see [15]; see also [14] and [25]

$$(X, D(A))_{\gamma_1,p_1} \hookrightarrow (X, D(A))_{\gamma_2,p_2}$$

holds for any $0 < \gamma_2 < \gamma_1 \leq 1$ and $p_1, p_2 \in [1, \infty]$. Finally, in view of formula (3.4) in [19, p. 49],

$$\|A^0(\lambda I - A)^{-1}f\|_X \leq c|\lambda|^{1-\beta-\theta}\|f\|_{X_A^{\theta,\infty}}, \quad \lambda \in \Sigma_\alpha, \quad f \in X_A^{\theta,\infty}.$$

Therefore, if $z \in X_A^{\bar{\theta},\infty}$, where $\bar{\theta} > 1 - \beta$,

$$L^{-1}A^0(h + A)^{-1}L_1 \in \mathscr{L}(D(L))$$

and

$$\begin{aligned}\left\|L^{-1}A^0(h + A)^{-1}L_1\right\|_{\mathscr{L}(D(L))} &\leq c\left\|A^0(h + A)^{-1}\right\|_{\mathscr{L}(X_A^{\bar{\theta},\infty},X)}\|L_1\|_{\mathscr{L}\left(D(L),X_A^{\bar{\theta},\infty}\right)}\\ &\leq c'|h|^{1-\beta-\bar{\theta}}\|L_1\|_{\mathscr{L}\left(D(L),X_A^{\bar{\theta},\infty}\right)}.\end{aligned}$$

Since $\bar{\theta} > 1 - \beta$, the result follows from [22, Theorem 1]. In order to ensure that L_3 is a closed operator, suppose for all $\lambda \in \Sigma_\alpha$ with sufficiently large $|\lambda|$, there exists $(\lambda \pm L_1)^{-1}$ such that

$$\|(\lambda \pm L_1)^{-1}\|_{\mathscr{L}(X)} \leq c(1 + |\lambda|)^{-\beta_1}, \tag{2.8}$$

and

$$z \in X_{L_1}^{\delta,\infty}, \quad \delta > 1 - \beta_1. \tag{2.9}$$

Write

$$\lambda x \pm L_1 x - \varepsilon x = (\lambda \pm L_1)x - \varepsilon x = (\lambda \pm L_1)\left(1 - (\lambda \pm L_1)^{-1}\varepsilon\right)x$$

with

$$\varepsilon : D(L_1) \to X_{L_1}^{\delta,\infty}, \quad \varepsilon x = -\frac{\Phi[L_1 x]}{\Phi[z]}z.$$

We know that

$$\left\|L_1(\lambda \pm L_1)^{-1}z\right\|_X \leq c\,|\lambda|^{1-\beta_1-\delta}\,\|z\|_{X_{L_1}^{\delta,\infty}}.$$

It follows that $(\lambda \pm L_1)^{-1}\varepsilon \in \mathscr{L}(D(L_1))$.

In order to obtain the resolvent estimates, we add $-hMy$ to both sides of Eq. (2.7), obtaining

$$M(y' - hy) = -hMy + (L + L_2)y + \int_0^t k(t-s)L_3 y(s)\,ds + \frac{g'(t)}{\Phi[z]}z - \frac{\Phi[h(t)]}{\Phi[z]}z + h(t).$$

Take $z \in X_A^{\bar\theta,\infty} \cap X_{L_1}^{\delta,\infty}$, $\delta > 1-\beta_1$, $\bar\theta > 1-\beta$. Then it is proved in [4]

$$\|M(\lambda M + hM - L - L_2)^{-1}\|_{\mathscr{L}(X)} \le c(1+|\lambda|)^{-\beta}, \quad \lambda \in \Sigma_\alpha$$

and h large enough. Indeed, the main step consists in observing that

$$\begin{aligned}(\lambda+h)M - L - L_2 &= ((\lambda+h)M - L)\big(1 - ((\lambda+h)M-L)^{-1}L_2\big)\\ &= ((\lambda+h)M - L)\left(1 - L^{-1}A^0((\lambda+h)-A)^{-1}L_2\right).\end{aligned}$$

We know by assumption that $L_2 \in \mathscr{L}\left(D(L), X_A^{\bar\theta,\infty}\right)$ and by [19, p. 49] (see above)

$$\|A^0(\lambda I - A)^{-1}\|_{\mathscr{L}\left(X_A^{\bar\theta,\infty};X\right)} \le c\,|\lambda|^{1-\beta-\bar\theta}.$$

Therefore, repeating the same argument, also,

$$\lambda \pm L_1 - \varepsilon = (\lambda \pm L_1)\left[1 - (\lambda \pm L_1)^{-1}\varepsilon\right]$$

is invertible, so that $\pm L_1 - \varepsilon$ is closed. Therefore, we write (2.6) as

$$M(y' - hy) = -hMy + (L+L_2)y(t) + \int_0^t k(t-s)L_3 y(s)\,ds + F(t)$$

where

$$F(t) = -\frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t).$$

In order to solve such a problem, we prove the following lemma

Lemma 3 *If $k_1 \in C([0,\tau];\mathbb{C})$ and $\xi(t)$ is a strict solution to*

$$\frac{d}{dt}M\xi(t) = L\xi(t) + \int_0^t k_1(t-s)L_3\xi(s)\,ds + k_1(t)L_3y_0 + F'(t) \tag{2.10}$$

$$M\xi(0) = My_1 = Ly_0 + F(0), \tag{2.11}$$

(recall that ξ is a strict solution of (2.10)–(2.11) means that $\xi \in C([0,\tau];D(L))$, $M\xi \in C^1([0,\tau];X)$ and (2.10)–(2.11) holds) then

$$y(t) = \int_0^t \xi(s)\,ds + y_0$$

satisfies

$$My'(t) = Ly(t) + \int_0^t k_1(t-s)L_3 y(s)\,ds + F(t)\,, \tag{2.12}$$

$$y(0) = y_0\,. \tag{2.13}$$

Proof First of all, observe that

$$\int_0^t k_1(t-s)L_3 y(s)\,ds = \int_0^t k_1(s)L_3 y(t-s)\,ds$$

so that

$$\begin{aligned}\frac{d}{dt}\int_0^t k_1(t-s)L_3 y(s)\,ds &= k_1(t)L_3 y_0 + \int_0^t k_1(s)L_3 \frac{\partial}{\partial t} y(t-s)\,ds \\ &= k_1(t)L_3 y_0 + \int_0^t k_1(t-s)L_3 y'(s)\,ds\,.\end{aligned}$$

Integrating (2.10) on $(0, t)$ we get

$$M\xi(t) - M\xi(0) = \int_0^t L\xi(s)\,ds + \int_0^t k_1(t-s)L_3\left[\int_0^s \xi(\tau)\,d\tau + y_0\right] ds + F(t) - F(0)\,.$$

Using $M\xi(0) = Ly_0 + F(0)$, we obtain

$$M\xi(t) = L\left[\int_0^t \xi(\tau)\,d\tau + y_0\right] + \int_0^t k_1(t-s)L_3\left[\int_0^s \xi(\tau)\,d\tau + y_0\right] ds + F(t)\,.$$

Hence

$$My'(t) = Ly(t) + \int_0^t k_1(t-s)L_3 y(s)\,ds + F(t)\,.$$

Therefore, all is reduced to study existence, uniqueness, and regularity of the problem (2.10)–(2.11). To this end, we can use both results from [21] and [15]; see [18] and [20], too. Remember that the additional information

$$\Phi[My(t)] = g(t)$$

transforms the inverse problem (2.3)–(2.4) into

$$My'(t) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}z + \int_0^t k(t-s)\left[L_1 y(s) - \frac{\Phi[L_1 y(s)]}{\Phi[z]}z\right] ds$$

$$-\frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)$$
$$= (L+L_2)y(t) + \int_0^t k(t-s)L_3 y(s)\,ds + \frac{g'(t)}{\Phi[z]}z - \frac{\Phi[h(t)]}{\Phi[z]}z + h(t)\,,$$

i.e.,

$$M\left(\frac{d}{dt}-h\right)y = (-hM+L+L_2)y(t) + \int_0^t k(t-s)L_3 y(s)\,ds$$
$$+\frac{g'(t)}{\Phi[z]}z - \frac{\Phi[h(t)]}{\Phi[z]}z + h(t)\,,$$

$h > 0$ large, where it can be supposed

$$\|M(\lambda M + hM - L - L_2)^{-1}\|_{\mathscr{L}(X)} \le \frac{c}{|\lambda|^{\beta}}\,, \quad \lambda \in \Sigma_\alpha\,.$$

In view of Lemma 3, if $\xi(t)$ solves

$$\frac{d}{dt}M\xi(t) = (L+L_2)\xi(t) + \int_0^t k(t-s)L_3\xi(s)\,ds$$
$$+ k(t)L_3 y_0 + \frac{g''(t)}{\Phi[z]}z - \frac{\Phi[h'(t)]}{\Phi[z]}z + h'(t)\,, \tag{2.14}$$

$$(M\xi)(0) = My_1 = (L+L_2)y_0 + \frac{g'(0)}{\Phi[z]}z - \frac{\Phi[h(0)]}{\Phi[z]}z + h(0)\,, \tag{2.15}$$

then

$$y(t) = \int_0^t \xi(s)\,ds + y_0$$

satisfies indeed

$$M\frac{dy}{dt} = (L+L_2)y(t) + \int_0^t k(t-s)L_3 y(s)\,ds + \frac{g'(t)}{\Phi[z]}z - \frac{\Phi[h(t)]}{\Phi[z]}z + h(t)\,, \tag{2.16}$$

$$y(0) = y_0\,. \tag{2.17}$$

Therefore, all is reduced to solve (2.14)–(2.15). Then we shall use two basic theorems due to [21] and [15] respectively. We report them as lemmas (cfr. Lemma 5).

Lemma 4 (Theorem 2.1, [21, p. 469]) *Consider the problem*

$$\frac{d}{dt}(My) = Ly(t) + \int_0^t k(t-s)L_1 y(s)\, ds + f(t)\,, \tag{2.18}$$

$$My(0) = My_0\,, \tag{2.19}$$

with $D(L) \subseteq D(M) \cap D(L_1)$, $0 \in \rho(L)$

$$\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \le \frac{c}{|\lambda|^{\beta}}\,, \quad \lambda \in \Sigma_\alpha\,, \quad 0 < \beta \le \alpha \le 1, \quad \alpha + \beta > 1\,,$$

$y_0 \in D(L)$, $f \in C^{\theta}([0,\tau];X)$, $k \in C^{\theta}([0,\tau];\mathbb{C})$. *Moreover, assume* $f(0) + Ly_0 \in R(T) = R(ML^{-1})$, $\omega = 2 - \alpha - \beta < \theta < 1$. *Then problem (2.18)–(2.19) admits a unique global strict solution* $y \in C^{\theta-\omega}([0,\tau];D(L)) = C^{\theta-2+\alpha+\beta}([0,\tau];D(L))$ *and* $My \in C^{1+\theta-\omega}([0,\tau];X)$.

Theorem 1 *Suppose* $z \in X_A^{\bar{\theta},\infty}$, $\bar{\theta} > 1 - \beta$, $k \in C^{\theta}([0,\tau];\mathbb{C})$, $g \in C^{2+\theta}([0,\tau];\mathbb{C})$, $h \in C^{1+\theta}([0,\tau];X)$, $0 < \beta \le \alpha \le 1$, $\alpha + \beta > 1$, $2 - \alpha - \beta < \theta < 1$, $y_0 \in D(L)$,

$$k(0)L_3 y_0 + \frac{g''(0)}{\Phi[z]}z - \frac{\Phi[h'(0)]}{\Phi[z]}z + h'(0) + (L + L_2)y_1 \in R(ML^{-1})$$

$$\left(\text{where } My_1 = (L + L_2)y_0 + \frac{g'(0)}{\Phi[z]}z - \frac{\Phi[h(0)]}{\Phi[z]}z + h(0)\right).$$

Suppose $(\lambda \pm L_1)$ *has a bounded inverse such that (2.8)–(2.9) hold. Then (2.14)–(2.15) admits a unique strict global solution* $\xi(t)$ *such that*

$$\xi \in C^{\theta-2+\alpha+\beta}([0,\tau];D(L)) \quad \text{and} \quad M\xi \in C^{1+\theta-\omega}([0,\tau];X).$$

Therefore, problem (2.16)–(2.17) admits a unique solution y satisfying the property $y \in C^{\theta-1+\alpha+\beta}([0,\tau];D(L))$.

A more detailed result is given by Favaron and Favini, see [15, Theorem 5.9], that we recall as a lemma.

Lemma 5 *Suppose that L, M are closed linear operators in the complex Banach space X*, $0 \in \rho(L)$, $D(L) \subseteq D(M)$, *such that*

$$\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \le \frac{c}{(|\lambda| + 1)^{\beta}}$$

for any $\lambda \in \Sigma_\alpha := \big\{z \in \mathbb{C} :\ Re\, z \ge -c(1 + |Im\, z|)^{\alpha},\ \ c > 0,\ \ 0 < \beta \le \alpha \le 1\big\}$.

Consider the initial value problem

$$\frac{d}{dt}(My) = (\lambda_0 M + L)y(t) + \sum_{i_1=1}^{n_1} \int_0^t k_{i_1}(t-s)L_{i_1}y(s)\,ds$$

$$+ \sum_{i_2=1}^{n_2} h_{i_2}(t)z_{i_2} + f(t), \quad t \in I_\tau = [0,\tau]\,, \tag{2.20}$$

$$(My)(0) = My_0\,, \tag{2.21}$$

where $y_0 \in D(L)$, k_{i_1} *is a continuous function from* I_τ *in* $\mathbb{C}$, L_{i_1}, $i_1 = 1, 2, \ldots, n_1$, *are closed linear operators on* X, *such that* $D(L) \subseteq D(L_{i_1})$ *for each* i_1, h_{i_2} *are complex valued continuous functions on* I_τ, z_{i_2} *are fixed elements in* X, $f \in C(I_\tau; X)$. *Suppose* $5\alpha + 2\beta > 6$, $k_{i_1} \in C^{\eta_{i_1}}(I_\tau; \mathbb{C})$, $h_{i_2} \in C^{\delta_{i_2}}(I_\tau; \mathbb{C})$, $z_{i_2} \in Y^r_{\gamma_{i_2}} \in \left\{(X, D(A))_{\gamma_{i_2},r}, X_A^{\gamma_{i_2},r}\right\}$, $r \in [1,\infty]$, $y_1 = (\lambda_0 M + L)y_0$, $y_1 + f(0) \in Y^r_\varphi$, *where* η_{i_1}, $\delta_{i_2} \in \big((3-2\alpha-\beta)/\alpha, 1\big)$, γ_{i_2}, $\varphi \in \big(5-3\alpha-2\beta, 1\big)$, $i_j = 1, 2, \ldots, n_j$, $j = 1, 2$. *Let* $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ *and* $\tau = \min_{\substack{i_j=1,\ldots,n_j\\ j=1,2}}\{\eta_{i_1}, \delta_{i_2}, X_{\alpha,\beta,\gamma}\}$, *where* $X_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. *Then for every* $\delta \in I_{\alpha,\beta,\tau}$, *problem (2.20)–(2.21) admits a unique strict solution* $y \in C^\delta(I_\tau; D(L))$, *such that* $\frac{d}{dt}My$, $Ly \in C^\delta(I_\tau; X)$, *provided that* $f \in C^\mu(I_\tau; X)$, $\mu \in \left[\delta + \frac{3-2\alpha-\beta}{\alpha}, 1\right)$ *where*

$$I_{\alpha,\beta,\gamma} = \begin{cases} \left(\frac{2-\alpha-\beta}{\alpha}, \gamma\right) & \text{if } \gamma \in \left(\frac{2-\alpha-\beta}{\alpha}, 1/2\right), \\ \left(\frac{2-\alpha-\beta}{\alpha}, 1/2\right) & \text{if } \gamma \in [1/2, 1)\,. \end{cases}$$

In our case, $n_{i_1} = 1$, $n_{i_j} = 0$, $j \neq 1$, and the problem reads

$$\left(\frac{d}{dt} - k\right)M\xi(t) = (-kM + L + L_2)\xi(t) + \int_0^t k(t-s)L_3\xi(s)\,ds + k(t)L_3y_0$$

$$+\frac{g''(t)}{\Phi[z]}z - \frac{\Phi[h'(t)]}{\Phi[z]}z + h'(t)\,, \tag{2.22}$$

$$(M\xi)(0) = My_1 = (-kM + L + L_2)y_0 + \frac{g'(0)}{\Phi[z]}z - \frac{\Phi[h(0)]}{\Phi[z]}z + h(0)\,, \tag{2.23}$$

$k > 0$ large, $(-kM + L + L_2)y_1 + h'(0) \in Y^r_\varphi \in \left\{(X, D(A))_{\varphi,r}, X_A^{\varphi,r}\right\}$, $r \in [1,\infty]$, $\varphi \in \big(5-3\alpha-2\beta, 1\big)$, $5\alpha + 2\beta > 6$, $k \in C^\eta(I_\tau; \mathbb{C})$, $\eta \in (3-2\alpha-\beta, 1)$, $z \in X_A^{\bar\theta,\infty}$, $\bar\theta > 1-\beta$, $A = LM^{-1}$, $L_3y_0 \in Y^r_{\gamma_0} \in \left\{(X, D(A))_{\gamma_0,r}, X_A^{\gamma_0,r}\right\}$, $\gamma_0 \in \big(5-3\alpha-2\beta, 1\big)$, $z \in Y^r_{\gamma_1} \in \left\{(X, D(A))_{\gamma_1,r}, X_A^{\gamma_1,r}\right\}$, $\gamma_1 \in \big(5-3\alpha-2\beta, 1\big)$, $g \in C^{2+\delta}(I_\tau; \mathbb{C})$, $\delta \in$

$(3-2\alpha-\beta,1)$. Let $\gamma = \min\{\gamma_0, \gamma_1, \varphi\}$, $\tau = \min\{\eta, \delta, (\alpha+\beta+\delta-2)/\alpha\}$. Then, for all fixed $\sigma \in I_{\alpha,\beta,\tau}$, problem (2.22)–(2.23) admits a unique strict solution $\xi \in C^{\sigma}(I_\tau, D(L))$ such that $(-kM + L + L_2)\xi$, $\left(\frac{d}{dt} - k\right)M\xi \in C^{\sigma}(I_\tau, X)$, provided that $h \in C^{1+\mu}(I_\tau; X)$, $\mu \in \left[\sigma + \frac{3-2\alpha-\beta}{\alpha}, 1\right)$.

Theorem 2 *Let L, M be two closed linear operators on the complex Banach space X satisfying (2.2) with $D(L) \subseteq D(M)$. Suppose $(\lambda \pm L_1)$ has a bounded inverse such that (2.8)–(2.9) hold with $D(L) \subseteq D(L_1)$. If $y_1 \in D(L)$, where y_1 is described in (2.23), $L_3 y = L_1 y - \frac{\Phi[L_1 y]}{\Phi[z]} z$, $L_2 y = -\frac{\Phi[Ly]}{\Phi[z]} z$, $(-kM + L + L_2)y_1 + h'(0) \in Y^r_{\varphi} \in \left\{(X, D(A))_{\varphi,r}, X_A^{\varphi,r}\right\}$, $r \in [1,\infty]$, $\varphi \in \left(5-3\alpha-2\beta, 1\right)$, $5\alpha + 2\beta > 6$, $k \in C^{\eta}(I_\tau; \mathbb{C})$, $\eta \in (3-2\alpha-\beta, 1)$, $z \in X_A^{\bar{\theta},\infty}$, $\bar{\theta} > 1-\beta$, $A = LM^{-1}$, $L_3 y_0 \in Y^r_{\gamma_0} \in \left\{(X, D(A))_{\gamma_0,r}, X_A^{\gamma_0,r}\right\}$, $\gamma_0 \in \left(5-3\alpha-2\beta, 1\right)$, $z \in Y^r_{\gamma_1} \in \left\{(X, D(A))_{\gamma_1,r}, X_A^{\gamma_1,r}\right\}$, $\gamma_1 \in \left(5-3\alpha-2\beta, 1\right)$, $g''(t) \in C^{\delta}(I_\tau; \mathbb{C})$, $\delta \in (3-2\alpha-\beta, 1)$, $\gamma = \min\{\gamma_0, \gamma_1, \varphi\}$, $\tau = \min\{\eta, \delta, (\alpha+\beta+\delta-2)/\alpha\}$, then, for all fixed $\sigma \in I_{\alpha,\beta,\tau}$, problem (2.22)–(2.23) admits a unique strict solution $\xi \in C^{\sigma}(I_\tau, D(L))$ such that $(-kM + L + L_2)\xi$, $\left(\frac{d}{dt} - k\right)M\xi \in C^{\sigma}(I_\tau, X)$, provided that $h'(t) \in C^{\mu}(I_\tau; X)$, $\mu \in \left[\sigma + \frac{3-2\alpha-\beta}{\alpha}, 1\right)$.*

Proof It is an easy consequence of Lemma 5.

Corollary 1 *Under the assumptions described in Theorem 2, the identification problem*

$$My'(t) = (L + L_1)y(t) + \int_0^t k(t-s)L_1 y(s)\, ds + f(t)z + h(t), \quad 0 \le t \le \tau,$$

$$y(0) = y_0,$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le \tau$$

admits a unique strict solution (y, f) such that $y \in C^{1+\sigma}(I_\tau; X)$, $f \in C^{1+\sigma}(I_\tau; \mathbb{C})$.

Proof It follows by observing that problem (2.16)–(2.17) is solved by reduction to a problem like (2.14)–(2.15), whose solution is described in Theorem 2.

3 Simple Pole Case

Suppose $T = ML^{-1} = A^{-1}$ has a closed range, then all previous assumptions can be weakened.

Consider

$$My'(t) = Ly(t) + \int_0^t k(t-s)L_1 y(s)\, ds + h(t), \quad 0 \le t \le \tau, \tag{3.1}$$

$$y(0) = y_0. \tag{3.2}$$

Call $Ly = \xi$. Then (3.1)–(3.2) can be written as

$$ML^{-1}\frac{d\xi}{dt} = \xi + \int_0^t k(t-s)L_1L^{-1}\xi(s)\,ds + h(t), \quad 0 \le t \le \tau\,, \tag{3.3}$$

$$\xi(0) = Ly_0\,. \tag{3.4}$$

If $\lambda = 0$ is a simple pole of $(\lambda - T)^{-1}$, $T = ML^{-1}$, then $X = N(T) \oplus R(T)$ and Eq. (3.3) is written by

$$T\frac{d\xi}{dt} = \xi + \int_0^t k(t-s)L_1L^{-1}\xi(s)\,ds + h(t), \quad 0 \le t \le \tau\,. \tag{3.5}$$

If P is a projection onto $N(T)$ along $R(T)$, then one can split (3.5) into

$$\tilde{T}\frac{d}{dt}(I-P)\xi = (I-P)\xi + \int_0^t k(t-s)(I-P)L_1L^{-1}\xi(s)\,ds + (I-P)h(t)\,, \tag{3.6}$$

$$0 = P\xi(t) + \int_0^t k(t-s)PL_1L^{-1}\xi(s)\,ds + Ph(t)\,. \tag{3.7}$$

If we write (3.7) in the form

$$0 = P\xi(t) + \int_0^t k(t-s)PL_1L^{-1}P\xi(s)\,ds + \int_0^t k(t-s)PL_1L^{-1}(I-P)\xi(s)\,ds + Ph(t)\,,$$

then we can find $P\xi(t)$ in terms of $(I-P)\xi(t)$ where $(I-P)\xi(t)$ is known from (3.6), since $\tilde{T}$ has a bounded inverse in $R(T)$ for which we obtain a solvable integro-differential equation.

Example 1 Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix}' = \begin{bmatrix} y \\ x \end{bmatrix} + f(t)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \qquad \forall\, t \in [0,\tau]\,,$$

i.e.,

$$y'(t) = y(t) + f(t)z_1 + h_1(t), \tag{3.8}$$

$$0 = x(t) + f(t)z_2 + h_2(t)\,. \tag{3.9}$$

Let $\Phi = \begin{bmatrix} 1 & 0 \end{bmatrix}$, then $y(t) = g(t)$. If $z_1 \neq 0$,

$$g'(t) = g(t) + f(t)z_1 + h_1(t)$$

is solvable and

$$f(t) = \frac{g'(t) - g(t) - h_1(t)}{z_1},$$

and this is determined by Eq. (3.8). However, if we want the same regularity for $x(t)$, we need to assume more regularity on g and h_1, precisely, what we required in our last theorem (Theorem 2).

4 Applications to Integro-Differential Equations

In the paper [12], the authors considered the problem

$$u'(t) = Au(t) + \int_{-\infty}^{t} K(t-s)u(s)\,ds + f(t), \quad 0 < t < 2\pi, \tag{4.1}$$

$$u(0) = u(2\pi), \tag{4.2}$$

where A and $K(t)$ are linear operators in a real Banach space and f is a periodic X-valued function. Using Fourier transform, it is easy to see that if X is a Hilbert space, $f \in L^2(0, 2\pi; X)$, then

$$u(t) = \sum_{k=-\infty}^{\infty} F(ik) f_k e^{ikt}, \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-iks} f(s)\,ds$$

where $F(\lambda) = (\lambda - A - \hat{K}(\lambda))^{-1}$ is the resolvent for the initial value problem. Thus, if $F(\lambda)$ is defined for all $\lambda = ik$, $k \in \mathbb{Z}$, the problem admits a unique strict solution. Moreover, $u',\ Au \in L^2(0, 2\pi; X)$. Moreover, for applications to nonlinear problems, it is not convenient to deal with general Banach spaces. The treatment in [12] starts with real Banach space X, A, $K(t) : D \to X$. $\tilde{X}$ and $\tilde{D}$ denote the usual complexification of X and D, respectively. One sets $\tilde{A} : \tilde{D} \to \tilde{X}$, $\tilde{A}(x + iy) = Ax + iAy$, $\forall\, x, y \in D$.

The main assumptions are as follows

(A) $\rho(\tilde{A})$ contains a sector $S = \{z \in \mathbb{C};\ z \neq 0,\ \arg(z - \omega) < \theta_0\}$ with $\omega \in \mathbb{R}$, $\theta_0 \in (\pi/2, \pi)$ and there exists $\mu > 0$, such that

$$\left\|(z-\omega)(z-\tilde{A})^{-1}\right\|_{\mathscr{L}(\tilde{X})} \le \mu, \text{ for } z \in S$$

(B) For $s \ge 0$, $K(s) \in \mathscr{L}(D; X)$, $\forall\, x \in D$, the function $K(\cdot)x$ is absolutely Laplace transformable. The Laplace transform $\hat{K}(\cdot)x$ is extendable to S and there exist $\theta, N > 0$ such that

$$\left|(z-\omega)^{\theta}\right| \, \|\hat{K}(z)\|_{\mathscr{L}(D;X)} \le N, \quad z \in S.$$

Under **(A)** and **(B)** it is shown, see [12], that the resolvent operator $R(t)$ is given by

$$R(t) = \frac{1}{2\pi i}\int_\gamma e^{zt}F(z)\,dz$$

where γ is a suitable path in $\mathbb{C}$ joining $\infty e^{-i\theta_1}$ with $\infty e^{i\theta_1}$, $\theta_1 \in (\frac{\pi}{2},\, \theta_0)$, and

$$F(z) = \left(z - \tilde{A} - \tilde{K}(z)\right)^{-1}. \tag{4.3}$$

It is easily seen

$$\sup_{z\in S,\,|z|\geq k} \|zF(z)\|_{\mathscr{L}(\tilde{X})} < \infty\,,$$

$$\sup_{z\in S,\,|z|\geq k} \|\tilde{A}F(z)\|_{\mathscr{L}(\tilde{X})} < \infty\,.$$

In order to handle the problem

$$u'(t) = Au(t) + \int_{-\infty}^t K(t-s)u(s)\,ds + f(t)\,,$$

one assumes, besides **(A)** and **(B)**

(C) $iz \in \rho_F = \left\{z \in \mathbb{C} \text{ such that } \left(z - \tilde{A} - \tilde{K}(z)\right)^{-1} \in \mathscr{L}(\tilde{X}, \tilde{D})\right\}$

so that

$$\|kF(ik)\|_{\mathscr{L}(\tilde{X})} + \|\tilde{A}F(ik)\|_{\mathscr{L}(\tilde{X})} < \mu, \quad k \in \mathbb{Z}\,.$$

An existence and regularity result for problem (4.1)–(4.2) is as follows (see [12, Theorem 2.3, p. 54]).

Theorem 3 *Let* ***(A)****,* ***(B)****, and* ***(C)*** *be satisfied. Assume moreover, that there exists* $\theta \in (0,1)$ *such that for any* $v \in C_\sharp(D)$*, the function* Φ *given by*

$$\Phi(t) = \int_{-\infty}^0 K(t-s)v(s)\,ds, \quad t \geq 0\,,$$

belongs to $C^{1,\theta}([0,T] : X)$ *for any* $T > 0$*. Then, for any* $f \in C_\sharp^{1,\theta}(X)$ *there exists a unique* 2π*-periodic solution* u *of (4.1)–(4.2) such that* $u', Au \in C_\sharp^{1,\infty}(X)$*.*

Here, $C_\sharp(X)$ represents the Banach space of all continuous and 2π-periodic functions $\mathbb{R} \to X$, $C_\sharp^\theta(X)$ denotes the space of all θ-Hölder continuous and 2π-

periodic functions $f : \mathbb{R} \to X$ endowed with the norm

$$\|f\|_{C^\theta_\sharp(X)} = \sup_{t\in[0,2\pi]} \|f(t)\|_X + \sup_{0\le s\le t\le 2\pi} \frac{\|f(t) - f(s)\|_X}{|t-s|^\theta}.$$

$C^k_\sharp(X)$, $C^{k,\theta}_\sharp(X)$ are analogously defined.

Let $k(t) = be^{-ct}$, $K(t) = k(t)A$, $b \in \mathbb{R}$, $c > 0$, A satisfies **(A)** and the spectrum of A consists of the sequence $(-\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $\mu_n \neq \infty$. Then

$$\hat{K}(\lambda) = \frac{b}{\lambda + c}\tilde{A}, \quad Re\,\lambda > -c\,.$$

One easily recognizes that all assumptions in Theorem 3 are satisfied.

Let $z = -1$ be a simple pole for the resolvent $(z - A)^{-1}$, where A is a closed linear operator on X, $0 \in \rho(A)$. Of concern is the equation

$$\frac{d}{dt}(1+\tilde{A})y(t) = \tilde{A}y(t) + \int_{-\infty}^t be^{-c(t-s)}\tilde{A}y(s)\,ds + f(t)\,. \tag{4.4}$$

Introducing the change of variable $Ay = x$, we get

$$\frac{d}{dt}(1+\tilde{A})A^{-1}x(t) = x(t) + \int_{-\infty}^t be^{-c(t-s)}x(s)\,ds + f(t)\,. \tag{4.5}$$

Since $(1+\tilde{A})\tilde{A}^{-1} = \tilde{A}^{-1} + I$, $N((1+\tilde{A})\tilde{A}^{-1}) = N(1+\tilde{A})$, $R((1+\tilde{A})\tilde{A}^{-1}) = R(1+\tilde{A})$, thus, if $T = 1 + \tilde{A}^{-1}$, then $\tilde{X} = N(T) \oplus R(T)$, $R(T)$ is closed and the restriction $\tilde{T}$ of T to $R(T)$ has an inverse $\tilde{T}^{-1}$ bounded from $R(T)$ to itself. Denote by P the projection onto $N(T)$ along $R(T)$. Our system reduces to

$$\frac{d}{dt}\tilde{T}(I-P)x(t) = (I-P)x(t) + \int_{-\infty}^t be^{-c(t-s)}(I-P)x(s)\,ds + (I-P)f(t)\,, \tag{4.6}$$

$$0 = Px(t) + \int_{-\infty}^t be^{-c(t-s)}Px(s)\,ds + Pf(t)\,. \tag{4.7}$$

Concerning (4.6), the change of variable $\tilde{T}(I-P)x(t) = \xi(t)$ furnishes the equivalent equation

$$\frac{d}{dt}\xi(t) = \tilde{T}^{-1}\xi(t) + \int_{-\infty}^t be^{-c(t-s)}\tilde{T}^{-1}\xi(s)\,ds + (I-P)f(t)\,. \tag{4.8}$$

If $f \in C^{1,\theta}_\sharp(X)$, see [12], (4.8) admits a unique solution ξ such that ξ is a 2π-periodic solution to (4.8) such that ξ', $\tilde{T}^{-1}\xi \in C^{1,\theta}_\sharp\big((1-P)\tilde{X}\big)$, i.e., $(\tilde{T}(I-P)x)'$, $(I-P)x$ belong to $C^{1,\theta}_\sharp(R(T))$. Equation (4.7) is an integral equation and no periodicity property of its solution can be prescribed of the equation

$$Px(t) = -\int_{-\infty}^{t} k(t-s)Px(s)\,ds - Pf(t)\,. \tag{4.9}$$

Notice that if $\int_0^\infty |k(t)|\,dt = c_0 < 1$,

$$\left|\int_{-\infty}^{t} k(t-s)Px(s)\,ds\right| = \left|\int_0^\infty k(\tau)Px(t-\tau)\,d\tau\right| \le c_0\,\|P_x\|_{C(\mathbb{R};PX)}\,.$$

Analogously, since

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{t} k(t-s)Px(s)\,ds &= \frac{d}{dt}\int_0^\infty k(s)Px(t-s)\,ds \\
&= \int_0^\infty k(s)\frac{d}{dt}Px(t-s)\,ds \\
&= \int_0^\infty k(s)\frac{d}{d(t-s)}Px(t-s)\,ds \\
&= \int_{-\infty}^{t} k(t-s)\frac{d}{ds}Px(s)\,ds\,,
\end{aligned}$$

$$\frac{d^2}{dt^2}\int_{-\infty}^{t} k(t-s)Px(s)\,ds = \int_{-\infty}^{t} k(t-s)\frac{d^2}{ds^2}Px(s)\,ds\,.$$

We conclude that

$$\left\|\int_{-\infty}^{t} k(t-s)Px(s)\,ds\right\|_{C^1(\mathbb{R};PX)} \le c_0\|Px\|_{C^1(\mathbb{R};PX)}\,,$$

$$\left\|\int_{-\infty}^{t} k(t-s)Px(s)\,ds\right\|_{C^2(\mathbb{R};PX)} \le c_0\|Px\|_{C^2(\mathbb{R};PX)}\,.$$

By interpolation, see [29],

$$\left\|\int_{-\infty}^{t} k(t-s)Px(s)\,ds\right\|_{C^{1+\theta}(\mathbb{R};PX)} \le \bar{c}\|Px\|_{C^{1+\theta}(\mathbb{R};PX)}\,.$$

Hence, Eq. (4.9) admits a unique solution for any $f \in C^{1+\theta}(\mathbb{R};PX)$. Therefore,

Theorem 4 *Suppose the kernel* $k \in C(\mathbb{R}^+;\mathbb{C}) \cap L^1(\mathbb{R}^+;\mathbb{C})$ *and* $\int_0^\infty |k(t)|\,dt < 1$. *Then for all* $f \in C^{1,\theta}_\sharp(X)$, *problem*

$$\frac{d}{dt}(1+\tilde{A})y(t) = \tilde{A}y(t) + \int_{-\infty}^{t} k(t-s)\tilde{A}y(s)\,ds + f(t)\,,$$
$$(1+\tilde{A})y(0) = (1+\tilde{A})y(2\pi)$$

admits a unique solution y such that

$$(1+\tilde{A})y \in C_{\sharp}^{1+\theta}([0,2\pi];X) \quad \textit{and} \quad \tilde{A}y \in C^{1+\theta}([0,2\pi];X).$$

We want to recall that integral equations over unbounded intervals are considered, for example, in the monographs [1, 10] and [11]. Periodic problems for degenerate differential equations are considered in [8, 9], [24] and [26, 27].

Consider

$$\frac{d}{dt}(My(t)) = Ly(t) + \int_{-\infty}^{t} k(t-s)L_1 y(s)\,ds + f(t)z + h(t), \quad 0 < t < 2\pi, \tag{4.10}$$
$$(My)(0) = (My)(2\pi)\,, \tag{4.11}$$
$$\Phi[My(t)] = g(t)\,. \tag{4.12}$$

Applying Φ to both sides of Eq. (4.10). Then, taking account of (4.12), we get

$$g'(t) = \Phi[Ly(t)] + \int_{-\infty}^{t} k(t-s)\Phi[L_1 y(s)]\,ds + f(t)\Phi[z] + \Phi[h(t)]\,.$$

If $\Phi[z] \neq 0$, then we obtain

$$f(t) = \frac{g'(t) - \Phi[Ly(t)] - \int_{-\infty}^{t} k(t-s)\Phi[L_1 y(s)]\,ds - \Phi[h(t)]}{\Phi[z]}\,. \tag{4.13}$$

Substitute (4.13) in (4.10), we deduce

$$\frac{d}{dt}(My(t)) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}z + \int_{-\infty}^{t} k(t-s)\left[L_1 y(s) - \frac{\Phi[L_1 y(s)]}{\Phi[z]}z\right]ds$$
$$-\frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)\,.$$

In particular, if $L = L_1$, we get

$$\frac{d}{dt}(My(t)) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}z + \int_{-\infty}^{t} k(t-s)\left[Ly(s) - \frac{\Phi[Ly(s)]}{\Phi[z]}z\right]ds$$
$$-\frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)\,. \tag{4.14}$$

Let $\mathscr{L}y = Ly(s) - \dfrac{\Phi[Ly]}{\Phi[z]}z$. Then we can write (4.14) as

$$\frac{d}{dt}(My(t)) = \mathscr{L}y(t) + \int_{-\infty}^{t} k(t-s)\mathscr{L}y(s)\,ds - \frac{\Phi[h(t)]}{\Phi[z]}z + \frac{g'(t)}{\Phi[z]}z + h(t)\,.$$

Here we have the same situation with $\mathscr{L}$ instead of L. That is, a direct problem that we discussed previously. Therefore, we can easily handle the inverse problem, too.

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A Phase Transition Model Describing Auxetic Materials

Elena Bonetti, Mauro Fabrizio, and Michel Frémond

Abstract In this paper we introduce a new model describing the behavior of auxetic materials in terms of a phase-field PDE system. More precisely, the evolution equations are recovered by a generalization of the principle of virtual power in which microscopic motions and forces, responsible for the phase transitions, are included. The momentum balance is written in the setting of a second gradient theory, and it presents nonlinear contributions depending on the phases. The evolution of the phases is governed by variational inclusions with non-linear coupling terms. By use of a fixed point theorem and monotonicity arguments, we are able to show that the resulting initial and boundary value problem admits a weak solution.

Keywords Auxetic materials • Nonlinear PDE system • Phase transitions • Second gradient theory

AMS (MOS) Subject Classification 35K87, 74N25

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1 Introduction

Auxetic materials are smart materials showing shape memory effects combined with negative Poisson's coefficient. As it is introduced in [8] and [20] they have the property of expanding in all directions when pulled in one and vice versa. These phenomena can be observed, for example in sandwich components, in compliant structures, in structural integrity, and in passive smart devices [21] and [10].

From experimental results it has been proved that these materials show important mechanical features compared with classical materials, as indentation resistance, enhanced bending stiffness in structural systems, shear resistance, high dissipation energy per unit volume under compressive cyclic loading, and optimal passive tuning of structural vibration [3].

Transverse isotropy and rhombic materials also have been the object of an intensive study with respect to the hypothesis of strong ellipticity of the elasticity tensor, closely related to the auxetic behavior of materials: at this regard, a detailed review (also concerning Mindlin-type plates) can be found in [17] and [18].

We underline that the Poisson's ratio is not constant during the deformation processes, then it is not obvious how to obtain the thermodynamic coherence of a model with a Poisson's ratio depending on the process. To address this problem, we need to represent the mesoscopic process of the internal structure of the material, which is actually characterized by a phase transition (in the sequel called auxetic transition). Finally, in a recent work by Li et al. [14], a bi-material structure is studied for which through control of temperature it is possible to observe mechanical deformations implying a transition from positive to negative Poisson's ratio or vice versa. Concerning the shape memory behavior, i.e. the possibility of recovering by heating the original shape (after permanent deformations), we recall that it is related to a solid-solid phase transition (see, e.g., to [9] which is one of the first paper introducing a modeling approach via phase-transitions for shape memory alloys).

In this paper, we address the model for auxetic materials introducing two phase parameters (φ_1, φ_2) representing the different configurations of the microscopic lattice. The reaction of the material to external loads (applied in different directions, as tension and compression loads) is related to the presence of these two configurations. In Sect. 2 we introduce the model which is recovered as a balance laws PDE system by virtue of a generalization of the principle of virtual power (see [12, 13]). More precisely, we include the power of forces acting at a microscopic level and responsible for the phase transition, i.e. the change of the microscopic configuration of lattice. In Sect. 3 we state the main existence result (for a suitable variational formulation of the resulting initial and boundary value problem), which is proved mainly applying a fixed point argument. Finally, in Appendix, we suggest an extension of the model presented in [8], which is related to the model we are investigating, in order to describe a lower limit of the auxetic phase.

2 The Model

The microscopic structure of the material is described at the macroscopic level, by the two phase parameters φ_1 and φ_2, together with the volume variation $tr\boldsymbol{\varepsilon}$ (where $\boldsymbol{\varepsilon}$ is the symmetric linearized strain tensor in a small deformation regime). The evolution of the mixture of the two phases is governed by volume variation $tr\boldsymbol{\varepsilon}$. More precisely the phases evolution depends on the sign of the volume variation.

Assuming that the evolution at a point depends on its spatial neighborhood we introduce the gradients of the phase parameters and the gradient of the volume variation as state variables

$$\nabla\varphi_1, \nabla\varphi_2, \nabla tr\boldsymbol{\varepsilon}.$$

The physical meaning is that phase quantities and volume variation evolve smoothly with respect to space. Such a point of view has been used, e.g., to describe the evolution of shape memory alloys and smart materials (see, e.g., [1, 4, 5, 7, 9, 19]). As already said, the variation of the phases is mainly governed by the volume spatial variation. Because the spatial variations of the phases are assumed to be smooth, introducing their gradients, the spatial variations of their governing quantities should also be assumed to be smooth, thus introducing the gradient of $tr\boldsymbol{\varepsilon}$.

Then the state quantities are

$$(\varphi_1, \varphi_2, \nabla\varphi_1, \nabla\varphi_2, \boldsymbol{\varepsilon}, \nabla tr\boldsymbol{\varepsilon}, \theta),$$

where θ is the absolute temperature. Actually, we assume that the temperature θ is a known function.

2.1 The Equations of Motion

The PDE system we are going to investigate results from the principle of virtual power, where the power of the internal forces is

$$\mathscr{P}^{int}(\gamma_1, \gamma_2, \nabla\gamma_1, \nabla\gamma_2, \boldsymbol{\varepsilon}(\boldsymbol{V}), \nabla tr\boldsymbol{\varepsilon}(\boldsymbol{V}))$$
$$= -\int_\Omega B_1\gamma_1 + B_2\gamma_2 + \nabla\gamma_1 \cdot \boldsymbol{H}_1 + \nabla\gamma_2 \cdot \boldsymbol{H}_2 + \sigma : \boldsymbol{\varepsilon}(\boldsymbol{V}) + \nabla tr\boldsymbol{\varepsilon}(\boldsymbol{V}) \cdot \boldsymbol{F} d\Omega.$$

Here, γ_1, γ_2 are the virtual phase velocities and $\boldsymbol{V}$ the virtual macroscopic velocity. Indeed, we are following [13] generalizing the principle of virtual power including the effects of microscopic forces and motions related to the phase transition process. More precisely, the internal forces are works B_i and work flux vectors $\boldsymbol{H}_i$, which are responsible for the evolution of the phase transition and they account at the macroscopic level of the power of the motions occurring at the microscopic level during the evolution. The other internal forces are the classical stress σ and the force $\boldsymbol{F}$ working with the spatial variation of the volume of the material.

The principle of virtual power is the weak formulation of the equations of motion. Formal integration by parts in the principle gives the equations of motion. Concerning the second gradient contribution, we have

$$-\int_{\Omega}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}(\boldsymbol{V})+\mathrm{grad}tr\boldsymbol{\varepsilon}(\boldsymbol{V})\cdot\boldsymbol{F}d\Omega=\int_{\Omega}(\mathrm{div}\boldsymbol{\sigma}-\nabla\mathrm{div}\boldsymbol{F})\cdot\boldsymbol{V}d\Omega$$
$$-\int_{\partial\Omega}\boldsymbol{\sigma}\boldsymbol{N}\cdot\boldsymbol{V}+\boldsymbol{F}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{F})d\Gamma$$

with

$$\int_{\partial\Omega}(\boldsymbol{F}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{F})d\Gamma$$
$$=\int_{\partial\Omega}\mathrm{div}((\boldsymbol{F}\cdot\boldsymbol{N})\boldsymbol{V})-\nabla(\boldsymbol{F}\cdot\boldsymbol{N})\cdot\boldsymbol{V}-\boldsymbol{V}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{F}d\Gamma$$
$$=\int_{\partial\Omega}2C(\boldsymbol{F}\cdot\boldsymbol{N})\boldsymbol{N}\cdot\boldsymbol{V}+\frac{\partial(\boldsymbol{F}\cdot\boldsymbol{N})}{\partial N}\boldsymbol{V}\cdot\boldsymbol{N}+(\boldsymbol{F}\cdot\boldsymbol{N})\boldsymbol{N}\cdot\frac{\partial\boldsymbol{V}}{\partial N}-\boldsymbol{V}\cdot\boldsymbol{N}\mathrm{div}\boldsymbol{F}d\Gamma$$
$$=\int_{\partial\Omega}\left\{2C(\boldsymbol{F}\cdot\boldsymbol{N})+\frac{\partial(\boldsymbol{F}\cdot\boldsymbol{N})}{\partial N}-\mathrm{div}\boldsymbol{F}\right\}\boldsymbol{N}\cdot\boldsymbol{V}-\nabla(\boldsymbol{F}\cdot\boldsymbol{N})\cdot\boldsymbol{V}+(\boldsymbol{F}\cdot\boldsymbol{N})\boldsymbol{N}\cdot\frac{\partial\boldsymbol{V}}{\partial N}d\Gamma$$
$$=\int_{\partial\Omega}\mathscr{L}_1(\boldsymbol{F})\cdot\boldsymbol{V}+\mathscr{L}_2(\boldsymbol{F})\cdot\frac{\partial\boldsymbol{V}}{\partial N}d\Gamma,$$

where C is the mean curvature of the boundary $\partial\Omega$ (which is assumed to be smooth). The linear functions $\mathscr{L}_1(\boldsymbol{F})$ and $\mathscr{L}_2(\boldsymbol{F})$ are

$$\mathscr{L}_1(\boldsymbol{F})=\left\{2C(\boldsymbol{F}\cdot\boldsymbol{N})+\frac{\partial(\boldsymbol{F}\cdot\boldsymbol{N})}{\partial N}-\mathrm{div}\boldsymbol{F}\right\}\boldsymbol{N}-\nabla(\boldsymbol{F}\cdot\boldsymbol{N}),$$
$$\mathscr{L}_2(\boldsymbol{F})=(\boldsymbol{F}\cdot\boldsymbol{N})\boldsymbol{N}.$$

Assuming that no body external power acts at a microscopic level, we recover the equation written in Ω

$$\mathrm{div}\boldsymbol{\sigma}-\nabla\mathrm{div}\boldsymbol{F}+\boldsymbol{f}=0, \tag{2.1}$$

$$B_1-\mathrm{div}\boldsymbol{H}_1=0,\ B_2-\mathrm{div}\boldsymbol{H}_2=0,\ in\ \Omega, \tag{2.2}$$

where $\boldsymbol{f}$ is the macroscopic body force. On part Γ_1 of boundary $\partial\Omega$, we assume neither external forces nor surface external work producing phase transition without macroscopic motion, so that the surface equations of motion or the boundary conditions are defined in Γ_1 by

$$\boldsymbol{H}_1 \cdot \boldsymbol{N} = 0,\ \boldsymbol{H}_2 \cdot \boldsymbol{N} = 0, \tag{2.3}$$

$$\boldsymbol{\sigma N} + \mathscr{L}_1(\boldsymbol{F}) = 0,\ \mathscr{L}_2(\boldsymbol{F}) = 0. \tag{2.4}$$

2.2 The Free Energy and the Pseudo-Potential of Dissipation

We choose the Lamé parameter λ_i of each phase as the phase parameters φ_i. They are clearly related to the Poisson's coefficients which are the interesting quantities. We assume the other Lamé's parameter $\mu > 0$, is the same for the two phases. The phase parameters satisfy the usual property

$$(\varphi_1, \varphi_2) \in K,$$

where convex set K is defined by

$$K = \{(\lambda_1, \lambda_2); 3\lambda_1 + 2\mu \geq \alpha, 3\lambda_2 + 2\mu \geq \alpha\},\ \text{with}\ \alpha > 0. \tag{2.5}$$

Note that the definition of K forces the Poisson coefficients to stay between $1/2$ and -1. It may be modified to take into account interactions between φ_1 and φ_2 (including a possible constraint like $\varphi_1 \geq \varphi_2$).

The free energy depends on the state quantities $(\varphi_1, \varphi_2, \nabla\varphi_1, \nabla\varphi_2, \boldsymbol{\varepsilon}, \nabla tr\boldsymbol{\varepsilon}, \theta)$

$$\Psi(\varphi_1, \varphi_2, \nabla\varphi_1, \nabla\varphi_2, \boldsymbol{\varepsilon}, \nabla tr\boldsymbol{\varepsilon}, \theta) \tag{2.6}$$

$$= \frac{1}{2}\left\{\varphi_1\left((tr\boldsymbol{\varepsilon})^+\right)^2 + \varphi_2\left((tr\boldsymbol{\varepsilon})^-\right)^2 + 2\mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \frac{l_1}{\theta_1}(\theta - \theta_1)\varphi_1 - \frac{l_2}{\theta_2}(\theta - \theta_2)\varphi_2\right\}$$

$$+ \frac{k_1}{2}(\nabla\varphi_1)^2 + \frac{k_2}{2}(\nabla\varphi_2)^2 + \frac{k}{2}(\nabla tr\boldsymbol{\varepsilon})^2 - C\theta\ln\theta + I_K(\varphi_1, \varphi_2), \tag{2.7}$$

for C, l_1, l_2, k_1, k_2, k positive physical constants and I_K is the indicator function of convex set K and it forces the phase variables (φ_1, φ_2) to take only admissible physical values (in K).

Quantities l_i are latent heats of the two phases, k_i and k quantify the interaction of the neighborhood of a point on this point: if it is large the interaction domain is large, if it is small the interaction domain is small, i.e., only the very neighborhood intervenes. The heat capacity is C.

We assume the phase evolutions are viscous (so that we have a rate dependent evolution) and have the pseudo-potential of

$$\Phi = \frac{c_1}{2}\dot{\varphi}_1^2 + \frac{c_2}{2}\dot{\varphi}_2^2, \tag{2.8}$$

where the c_i are the phase viscosities. Let us recall that a pseudo-potential of dissipation is a non-negative convex l.s.c. function vanishing for null velocities (cf. [16]).

2.3 Constitutive Laws

To recover the equation we specify the physical quantities in the balance laws (2.1)–(2.2) in terms of the functionals (2.6) and (2.8), by virtue of the following constitutive equations

$$
\begin{aligned}
\boldsymbol{\sigma} &= \frac{\partial \Psi}{\partial \varepsilon} = \varphi_1 \, (tr\boldsymbol{\varepsilon})^+ \, \mathbf{1} - \varphi_2 \, (tr\boldsymbol{\varepsilon})^- \, \mathbf{1} + 2\mu\boldsymbol{\varepsilon}, \\
\boldsymbol{F} &= \frac{\partial \Psi}{\partial \nabla tr\varepsilon} = k \nabla tr\boldsymbol{\varepsilon}, \\
B_1 &= \frac{\partial \Psi}{\partial \varphi_1} + \frac{\partial \Phi}{\partial \dot{\varphi}_1} = c_1 \dot{\varphi}_1 + \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^+\right)^2 - \frac{l_1}{\theta_1}(\theta - \theta_1) + R_1, \\
B_2 &= \frac{\partial \Psi}{\partial \varphi_2} + \frac{\partial \Phi}{\partial \dot{\varphi}_2} = c_2 \dot{\varphi}_2 + \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^-\right)^2 - \frac{l_2}{\theta_2}(\theta - \theta_2) + R_2, \\
&(R_1, R_2) \in \partial I_K(\varphi_1, \varphi_2), \\
\boldsymbol{H}_1 &= \frac{\partial \Psi}{\partial \nabla \varphi_1} = k_1 \nabla \varphi_1, \; \boldsymbol{H}_2 = \frac{\partial \Psi}{\partial \nabla \varphi_2} = k_2 \nabla \varphi_2.
\end{aligned}
$$

Remark 2.1 It is natural to write the free energy (2.6) for ε components in $L^2(\Omega)$. Hence, we observe that the trace of the stress depends on positive volume variation $(tr\boldsymbol{\varepsilon})^+$ and negative volume variation $-(tr\boldsymbol{\varepsilon})^+$ through the relation

$$
tr\boldsymbol{\sigma} = (\varphi_1 + 2\mu) \, (tr\boldsymbol{\varepsilon})^+ - (\varphi_2 + 2\mu) \, (tr\boldsymbol{\varepsilon})^- .
$$

On the other hand it is known that there exists sequences $\boldsymbol{\varepsilon}_n$ such that their traces converge weakly in $L^2(\Omega)$ to 0

$$
\lim_{n \to \infty} (tr\boldsymbol{\varepsilon}_n) = 0,
$$

but their positive and negative part satisfy

$$
\lim_{n \to \infty} (tr\boldsymbol{\varepsilon}_n)^+ = \lim_{n \to \infty} (tr\boldsymbol{\varepsilon}_n)^- = \eta > 0.
$$

Thus, we could introduce as constitutive equation ($\mathbf{1}$ is the identity matrix)

$$
\boldsymbol{\sigma} = \varphi_1((tr\boldsymbol{\varepsilon})^+ + \eta)\mathbf{1} - \varphi_{2(}\,(tr\boldsymbol{\varepsilon})^- + \eta)\mathbf{1} + 2\mu\boldsymbol{\varepsilon}, \; with \; \eta > 0.
$$

It results that when the phase quantities are known the stress is not a function of the strain $\boldsymbol{\varepsilon}$. In order to avoid this difficulty, it is wise to control the spatial variation of $tr\boldsymbol{\varepsilon}$. This is achieved by choosing $\nabla tr\boldsymbol{\varepsilon}$ as a state quantity, which is related to the local interactions between the phases.

2.4 The Resulting PDE System

We are now in the position of introducing the resulting initial and boundary value problem we are dealing with (here **u** stand for small displacement)

$$\mathrm{div}(\varphi_1 \left(tr\boldsymbol{\varepsilon}(\boldsymbol{u})\right)^+ \mathbf{1} - \varphi_2 \left(tr\boldsymbol{\varepsilon}(\boldsymbol{u})\right)^- \mathbf{1} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - k\Delta tr\boldsymbol{\varepsilon}(\boldsymbol{u})\mathbf{1}) + \boldsymbol{f} = 0, \tag{2.9}$$

$$c_1\dot{\varphi}_1 - k_1\Delta\varphi_1 + R_1 = \frac{l_1}{\theta_1}(\theta - \theta_1) - \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^+\right)^2, \tag{2.10}$$

$$c_2\dot{\varphi}_2 - k_2\Delta\varphi_2 + R_2 = \frac{l_2}{\theta_2}(\theta - \theta_2) - \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^-\right)^2, \tag{2.11}$$

with suitably associated boundary and initial conditions. More precisely, let $\Gamma_0,\ \Gamma_1$ be a partition of the boundary $\partial\Omega$. Finally, for Eq. (2.9) we assume sthenic boundary conditions (2.4) on Γ_1 and kinematic boundary conditions on Γ_0 for **u**. We have, in particular,

$$(\varphi_1 \left(tr\boldsymbol{\varepsilon}(\boldsymbol{u})\right)^+ \mathbf{1} - \varphi_2 \left(tr\boldsymbol{\varepsilon}(\boldsymbol{u})\right)^- \mathbf{1} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - k\Delta tr\boldsymbol{\varepsilon}(\boldsymbol{u})\mathbf{1}) \cdot N = 0 \quad \text{on } \Gamma_1$$

and

$$\boldsymbol{u} = 0,\ \mathrm{div}\boldsymbol{u} = 0 \ on\ \Gamma_0.$$

For Eqs. (2.10) and (2.11), we assume sthenic boundary conditions (2.3) and initial conditions

$$\varphi_1(x,0) = \varphi_1^0(x),\ \varphi_2(x,0) = \varphi_2^0(x),$$
$$(\varphi_1^0, \varphi_2^0) \in K.$$

3 Variational Formulation and Main Existence Result

In this section, we state the abstract setting of the problem and the existence theorem for corresponding weak solutions. We consider the evolution of a body located in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ during a finite time interval $[0,T]$. The boundary of Ω is splitted into two parts $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where the measure of Γ_0 is strictly positive. Hence, we consider the Hilbert triplet (V, H, V'), with $V := H^1(\Omega)$ and $H := L^2(\Omega)$. As usual H is identified with its dual space; we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V. We also introduce the functional space

$$W = \left\{\boldsymbol{v} \in V^3 : \operatorname{div}\boldsymbol{v}(= tr\boldsymbol{\varepsilon}(\boldsymbol{v})) \in H^1(\Omega),\ \boldsymbol{v} = 0 \text{ and } \operatorname{div}\boldsymbol{v} = 0 \text{ on } \Gamma_0\right\}.$$

Note that on the boundary, even if we do not specify it by using a proper notation, we mean the trace of a function as it is defined for Sobolev spaces. Due to the Poincaré inequality, we can endow W with the following norm, which is equivalent to the natural one,

$$\|\boldsymbol{v}\|_W := \left(\int_\Omega |\nabla \operatorname{div} \boldsymbol{v}|^2 + \sum_{i=1}^{3}\int_\Omega |\nabla v_i|^2\right)^{1/2} \tag{3.1}$$

where $\boldsymbol{v} = (v_1, v_2, v_3)$. Hence, $\langle\langle\cdot,\cdot\rangle\rangle$ stands for the duality pairing between W' and W. To simplify notation we will use the same symbol $\|\cdot\|_X$ for the norm in a Banach space X and in any power of it. In addition, in the sequel we will possibly denote by the same symbol c different positive constants depending only on the data of the problem.

In order to write the variational formulation of (2.9), we consider test functions $\boldsymbol{v} \in W$, so that for a.e. t the equation is written as follows

$$\int_\Omega (\varphi_1 (\operatorname{div} \boldsymbol{u})^+ - \varphi_2 (\operatorname{div} \boldsymbol{u})^-)\operatorname{div} \boldsymbol{v} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})$$
$$+ \int_\Omega \nu\nabla(\operatorname{div} \boldsymbol{u}) \cdot \nabla(\operatorname{div} \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in W, \tag{3.2}$$

For any fixed (φ_1, φ_2), we introduce the abstract operator $\mathscr{A}_{(\varphi_1,\varphi_2)} : W \to W'$ defined by

$$\langle\langle \mathscr{A}_{\varphi_1,\varphi_2)}(\boldsymbol{u}), \boldsymbol{v}\rangle\rangle = \int_\Omega (\varphi_1 (\operatorname{div} \boldsymbol{u})^+ - \varphi_2 (\operatorname{div} \boldsymbol{u})^-)\operatorname{div} \boldsymbol{v}, \quad \boldsymbol{u}, \boldsymbol{v} \in W. \tag{3.3}$$

Note that considering the free energy (2.6) for $(\varphi_1, \varphi_2) \in K$ fixed the term

$$\psi_{(\varphi_1,\varphi_2)}(u) = \frac{1}{2}\int_\Omega \varphi_1((\operatorname{div} \boldsymbol{u})^+)^2 + \varphi_2((\operatorname{div} \boldsymbol{u})^-)^2$$

is a convex function, so that its subdifferential (in the sense of convex analysis) results

$$\partial\psi_{(\varphi_1,\varphi_2)} = \mathscr{A}_{(\varphi_1,\varphi_2)}$$

and it is a monotone operator, see [2] and [15, Prop. 1.1 p. 158]. Then, we introduce the following linear abstract operator $\mathscr{B} : W \to W'$

$$\langle\langle \mathscr{B}(\boldsymbol{u}), \boldsymbol{v} \rangle\rangle = \int_\Omega 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \int_\Omega \nu \nabla(\operatorname{div} \boldsymbol{u}) \cdot \nabla(\operatorname{div} \boldsymbol{v}). \tag{3.4}$$

In particular, it results (by Poincaré inequality and (3.1))

$$\langle\langle \mathscr{B}(\boldsymbol{u}), \boldsymbol{u} \rangle\rangle \geq c\|\boldsymbol{u}\|_W^2, \quad c > 0. \tag{3.5}$$

Note that, by definition of K (see (2.5)), (3.3)–(3.4), and by monotonicity and (3.5), we have also that for any $\boldsymbol{u} \in W$

$$\langle\langle \mathscr{A}_{(\varphi_1,\varphi_2)}(\boldsymbol{u}) + \mathscr{B}(\boldsymbol{u}), \boldsymbol{u} \rangle\rangle \geq c\|\boldsymbol{u}\|_W^2. \tag{3.6}$$

Analogously, we introduce the operator $A : V \to V'$ defined by

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V,$$

which corresponds to the realization of the Laplace operator, with homogeneous boundary conditions, in the duality of V, V'.

Thus the problem can be eventually rewritten in the abstract setting of W' and V' as follows.

Problem Find $(\boldsymbol{u}, \varphi_1, \varphi_2)$ such that for a.e. $t \in [0, T]$, there holds

$$\mathscr{A}_{(\varphi_1,\varphi_2)}(\boldsymbol{u}) + \mathscr{B}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } W', \tag{3.7}$$

$$c_1\dot{\varphi}_1 + A\varphi_1 + \xi_1 = h_1(\theta) - \frac{1}{2}((\operatorname{div} \boldsymbol{u})^+)^2 \quad \text{in } V' \tag{3.8}$$

$$c_2\dot{\varphi}_2 + A\varphi_2 + \xi_2 = h_2(\theta) - \frac{1}{2}((\operatorname{div} \boldsymbol{u})^-)^2 \quad \text{in } V' \tag{3.9}$$

for some

$$(\xi_1, \xi_2) \in \partial I_K(\varphi_1, \varphi_2) \text{ a.e. in } \Omega. \tag{3.10}$$

Here, the subdifferential ∂I_K is defined for $(\varphi_1, \varphi_2) \in K$: it is null if (φ_1, φ_2) belong to the interior of K, while it corresponds to the normal cone to K if (φ_1, φ_2) belong to the boundary ∂K. Hence, we have set $h_i(\theta) = \frac{1}{\theta_i}(\theta - \theta_i)$, $i = 1, 2$. Indeed, we are assuming that the temperature θ it is a known datum of the problem and it given in such a way that

$$h_i(\theta) \in L^2(0, T; H), \quad i = 1, 2.$$

Here, we recall that θ_1, θ_2 stand for phase transition temperatures. The equations are combined with the initial conditions

$$(\varphi_1(0), \varphi_2(0)) = (\varphi_1^0, \varphi_2^0). \tag{3.11}$$

Now, we are in the position of stating the main existence result of the paper.

Theorem 1 *Let* $\boldsymbol{f} \in L^\infty(0,T;W')$, $h_i(\theta) \in L^2(0,T;H)$ *for* $i = 1, 2$, *and* $(\varphi_1^0, \varphi_2^0)$ *in* $V^2 \cap K$ *(in the sense that* φ_i^0 *in* (3.11) *are V-functions such that the couple belongs to K a.e.). Then, there exist* $(\boldsymbol{u}, \varphi_1, \varphi_2)$ *fulfilling* (3.7), (3.8)–(3.9), *and* (3.11), *for some* (ξ_1, ξ_2) *satisfying* (3.10). *In addition* $(\boldsymbol{u}, \varphi_1, \varphi_2)$ *and* (ξ_1, ξ_2) *have the following regularity*

$$\boldsymbol{u} \in L^\infty(0,T;W), \tag{3.12}$$

$$\varphi_i \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)),\ i = 1, 2, \tag{3.13}$$

$$\xi_i \in L^2(0,T;H),\ i = 1, 2. \tag{3.14}$$

Remark 3.1 Note that (3.8)–(3.9) can be actually solved a.e., due to the regularity of the solutions (3.12)–(3.14). In particular, (3.12) implies (we are in dimension 3) that div $\boldsymbol{u} \in L^\infty(0,T;L^4(\Omega))$ and the quadratic terms on the right hand side of (3.8)–(3.9) are in $L^\infty(0,T;H)$.

4 Proof of the Existence Result

In this Section we prove Theorem 1 mainly exploiting the Schauder fixed point theorem. To this aim, let us consider the space

$$\mathscr{X} := \{(\gamma_1, \gamma_2) \in L^2(0,T;V)^2,\ (\gamma_1, \gamma_2) \in K \text{ a.e.}, \|(\gamma_1, \gamma_2)\|_{L^2(0,T;V)^2} \le R\},$$

where R will be chosen later. We fix $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in \mathscr{X}$ and consider Eq. (3.7) written with $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ in place of (φ_1, φ_2). Let us point out that, by construction and owing to (3.5) (cf. (3.3) and (3.4)), due to the convexity of the free energy, we can apply fairly classical monotonicity arguments to show that the resulting equation admits a solution (e.g. the reader can refer to the arguments exploited [15] in the proof of Theorem 1.1 p. 156). More precisely, there exists a solution $\boldsymbol{u} = \mathscr{T}_1(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in L^\infty(0,T;W)$ (see also [6] and references therein for a proof of the existence of a solution for a similar equation related to the helium supercooling phenomenon). Hence, we can show that the solution is unique by contracting arguments. Indeed, let us write the difference of the equations solved by two solutions, say $\boldsymbol{u}_1$and $\boldsymbol{u}_2$. We point out that, by monotonicity we have

$$\langle\langle \mathscr{A}_{(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)}(\boldsymbol{u}_1) - \mathscr{A}_{(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle\rangle \ge 0$$

Thus, after testing the difference of the resulting equations by $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, and exploiting (3.5) we deduce

$$\langle\langle \mathscr{B}(\boldsymbol{u}_1 - \boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2 \rangle\rangle \geq c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_W^2,$$

so that at the end we have

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_W^2 \leq 0,$$

and it follows $\boldsymbol{u}_1 = \boldsymbol{u}_2$ a.e. Now, we aim to prove some estimates on the solution $\boldsymbol{u}$, which eventually do not depend on the choice of $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ in $\mathscr{X}$. To this aim, we test (3.7) (with fixed $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$) by $\boldsymbol{u}$. Due to (3.6) and exploiting the Young inequality to estimate the right hand side, it is a standard matter to deduce for a.e. t

$$c\|\boldsymbol{u}(t)\|_W^2 \leq \|\boldsymbol{f}(t)\|_{W'}\|\boldsymbol{u}(t)\|_W \leq \frac{c}{2}\|\boldsymbol{u}(t)\|_W^2 + c\|\boldsymbol{f}\|_{L^\infty(0,T;W')}^2, \tag{4.1}$$

so that

$$\|\boldsymbol{u}\|_{L^\infty(0,T;W)} \leq c. \tag{4.2}$$

Note that in particular the constant c here does not depend on the choice of $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \in \mathscr{X}$, but only on the data of the problem.

Secondly, we fix $\boldsymbol{u} = \mathscr{T}_1(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$ in (3.8)–(3.9) and look for a corresponding (unique) solution $(\varphi_1, \varphi_2) = \mathscr{T}_2(\boldsymbol{u})$. Note that the right hand sides of the two equations are, at least, functions in $L^2(0, T; H)$. This regularity follows by the definition of W and Sobolev embedding. Thus, well known results on parabolic equations combined with maximal monotone operators (see [2]) ensure that there exists a unique couple $(\varphi_1, \varphi_2) \in (H^1(0, T; H) \cap L^\infty(0, T; V)^2$, with $(\varphi_1, \varphi_2) \in K$ a.e. solving the inclusion

$$\begin{aligned}(\dot{\varphi}_1, \dot{\varphi}_2) + (A\varphi_1, A\varphi_2) + \partial I_K(\varphi_1, \varphi_2) \\ \ni (h_1(\theta) - \frac{1}{2}((\operatorname{div} \boldsymbol{u})^+)^2, h_2(\theta) - \frac{1}{2}((\operatorname{div} \boldsymbol{u})^-)^2).\end{aligned} \tag{4.3}$$

Here, we have exploited the regularity on the initial data and the functions h_i. Hence, we aim to find some estimates on the solutions. Let us test (4.3) by $(\dot{\varphi}_1, \dot{\varphi}_2)$ and integrate over $(0, t)$. This estimate is formal at the moment, but the following argument shows that it can be made rigorous (through an approximating procedure). Exploiting (4.2), the assumptions on h_i, the definition of the sub-differential and the chain rule, after integrating by parts in time and by virtue of the Young inequality, we get

$$\sum_{i=1}^{2}\left(\int_0^t \|\dot{\varphi}_i\|_H^2 + \frac{1}{2}\|\nabla\varphi_i(t)\|_H^2\right) + \int_\Omega I_K(\varphi_1(t),\varphi_2(t)) \tag{4.4}$$
$$\leq \frac{1}{2}\sum_{i=1}^{2}\int_0^t \|\dot{\varphi}_i\|_H^2 + c\left(1 + \sum_{i=1}^{2}\|h_i(\theta)\|^2_{L^2(0,T;H)} + \int_0^t \|\boldsymbol{u}\|_W^2\right),$$

so that (cf. (4.2))

$$\sum_{i=1}^{2}\|\varphi_i\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c, \tag{4.5}$$

where the constant c in (4.5) depends on the data of the problem and (4.2), and thus it does not depend on the choice of $(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)$. By continuous embedding theorems and choosing R sufficiently large, we can infer that the operator $\mathscr{T}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2) = \mathscr{T}_2(\mathscr{T}_1(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2))$, turns out to be well-defined $\mathscr{X} \to \mathscr{X}$.

The next step consists in showing that $\mathscr{T}$ is a compact operator and for this we need to prove a further estimate. We test Eq. (4.3) by (ξ_1,ξ_2) and integrate over $(0,t)$. Using the chain rule and exploiting monotonicity properties to infer that

$$\sum_{i=1}^{2}\int_0^t \langle A\varphi_i,\xi_i\rangle \geq 0,$$

we can deduce, once more using the Young inequality and (4.2)

$$\int_\Omega I_K(\varphi_1(t),\varphi_2(t)) + \sum_{i=1}^{2}\int_0^t \|\xi_i\|_H^2 \tag{4.6}$$
$$\leq \frac{1}{2}\sum_{i=1}^{2}\int_0^t \|\xi_i\|_H^2 + c\left(\sum_{i=1}^{2}\|h_i(\theta)\|^2_{L^2(0,T;H)} + \sum_{i=1}^{2}\int_0^t \|\boldsymbol{u}\|_H^2\right),$$

so that

$$\sum_{i=1}^{2}\|\xi_i\|_{L^2(0,T;H)} \leq c \tag{4.7}$$

and by a further comparison in (4.3)

$$\sum_{i=1}^{2}\|A\varphi_i\|_{L^2(0,T;H)} \leq c. \tag{4.8}$$

Combining (4.5) and (4.8) leads to

$$\sum_{i=1}^{2} \|\varphi_i\|_{L^2(0,T;H^2(\Omega))} \leq c. \tag{4.9}$$

Also in this case the constant c depends just on the data of the problem (through the norm $\|\boldsymbol{u}\|_{L^\infty(0,T;W)}$).

Now, (4.5) and (4.9) ensure in particular that $\mathscr{T}$ is actually a compact operator in $\mathscr{X}$. In order to show that it admits a fixed point, it remains to prove that it is continuous w.r.t. the topology of $\mathscr{X}$. With this aim, we consider a sequence $(\boldsymbol{\varphi}_{1n}, \boldsymbol{\varphi}_{2n})$ strongly converging in $\mathscr{X}$

$$(\boldsymbol{\varphi}_{1n}, \boldsymbol{\varphi}_{2n}) \to (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \quad \text{in } L^2(0,T;V)^2. \tag{4.10}$$

Then, we consider the corresponding sequences $\boldsymbol{u}_n = \mathscr{T}_1(\boldsymbol{\varphi}_{1n}, \boldsymbol{\varphi}_{2n})$ and $(\varphi_{1n}, \varphi_{2n}) = \mathscr{T}_2(\boldsymbol{u}_n)$ (and the corresponding $(\xi_{1n}, \xi_{2n}) \in \partial I_K(\varphi_{1n}, \varphi_{2n})$). We can deduce from (4.2), (4.5), (4.9) the following bounds independently of n

$$\|\boldsymbol{u}_n\|_{L^\infty(0,T;W)} \leq c \tag{4.11}$$

$$\|\varphi_{in}\|_{H^1(0,T;H)\cap L^2(0,T;H^2(\Omega))} \leq c, \quad i = 1,2 \tag{4.12}$$

$$\|\xi_{in}\|_{L^2(0,T;H)} \leq c, \quad i = 1,2. \tag{4.13}$$

By weak and weak star compactness results, we deduce that the following convergence results hold, at least for some suitable subsequences (we still denote by n to simplify notation)

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{weakly star in } L^\infty(0,T;W) \tag{4.14}$$

$$\varphi_{in} \to \varphi_i \text{ weakly in } H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \quad i = 1,2 \tag{4.15}$$

$$\xi_{in} \to \xi_i \text{ weakly in } L^2(0,T;H), \quad i = 1,2. \tag{4.16}$$

In addition, by the strong compactness Aubin-Lions theorem (cf. [15]), for $i = 1,2$, there holds

$$\varphi_{in} \to \varphi_i \text{ strongly in } C^0([0,T];V) \cap L^2(0,T;H). \tag{4.17}$$

First, let us point out that (4.14) and (4.17), through semicontinuity arguments, allow us to identify (see [2])

$$(\xi_1, \xi_2) \in \partial I_K(\varphi_1, \varphi_2). \tag{4.18}$$

Now, we aim to prove also a strong convergence for $\boldsymbol{u}_n$. We proceed directly, showing that it is a Cauchy sequence in $L^\infty(0,T;W)$. We write (3.7), where $(\boldsymbol{\varphi}_{1n}, \boldsymbol{\varphi}_{2n})$ is fixed, for two indices n, m and take the difference. Testing the resulting equation by the difference $\boldsymbol{u}_n - \boldsymbol{u}_m$, we get

$$\begin{aligned}&\langle\langle \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})}(\boldsymbol{u}_n) - \mathscr{A}_{(\boldsymbol{\varphi}_{1m},\boldsymbol{\varphi}_{2m})}(\boldsymbol{u}_m) + \mathscr{B}(\boldsymbol{u}_n - \boldsymbol{u}_m), \boldsymbol{u}_n - \boldsymbol{u}_m\rangle\rangle \\ &= \langle\langle \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})}(\boldsymbol{u}_n) - \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})}(\boldsymbol{u}_m) + \mathscr{B}(\boldsymbol{u}_n - \boldsymbol{u}_m), \boldsymbol{u}_n - \boldsymbol{u}_m\rangle\rangle \\ &+ \langle\langle \mathscr{A}_{((\boldsymbol{\varphi}_{1m},\boldsymbol{\varphi}_{2m})-(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n}))}(\boldsymbol{u}_m), \boldsymbol{u}_n - \boldsymbol{u}_m\rangle\rangle = 0.\end{aligned} \tag{4.19}$$

By monotonicity and (3.5), we can deduce that

$$\langle\langle \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})}(\boldsymbol{u}_n) - \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})}(\boldsymbol{u}_m) + \mathscr{B}(\boldsymbol{u}_n - \boldsymbol{u}_m), \boldsymbol{u}_n - \boldsymbol{u}_m\rangle\rangle \geq c\|\boldsymbol{u}_n - \boldsymbol{u}_m\|_W^2. \tag{4.20}$$

Then, by definition of $\mathscr{A}$, exploiting the strong convergence (4.10), (4.14), and (4.11) (and Sobolev embedding) we have that

$$\lim_{n,m\to+\infty} \langle\langle (\mathscr{A}_{(\boldsymbol{\varphi}_{1m},\boldsymbol{\varphi}_{2m})} - \mathscr{A}_{(\boldsymbol{\varphi}_{1n},\boldsymbol{\varphi}_{2n})})(\boldsymbol{u}_m), \boldsymbol{u}_n - \boldsymbol{u}_m\rangle\rangle = 0. \tag{4.21}$$

Combining (4.19)–(4.21), it follows that $\boldsymbol{u}_n$ is a Cauchy sequence in $L^\infty(0,T;W)$ and thus it strongly converges to $\boldsymbol{u}$. As a consequence, we can extract a subsequence such that div $\boldsymbol{u}_n$ converges a.e. and thus, we can identify the weak (star) limits of $\frac{1}{2}((\mathrm{div}\,\boldsymbol{u}_n)^+)^2$ and $\frac{1}{2}((\mathrm{div}\,\boldsymbol{u}_n)^-)^2$ (in $L^\infty(0,T;H)$) on the right hand sides.

Thus, we are now in the position of passing to the limit in Eqs. (3.7) and (4.3). In particular, it follows that actually (φ_1, φ_2) is a solution to (4.3) whence $\boldsymbol{u}$ is fixed, while $\boldsymbol{u}$ is a solution of (3.7) written for $(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$. As a consequence, (4.17) means that $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is a continuous operator. Now, the Schauder Theorem ensures that $\mathscr{T}$ admits a fix point (φ_1, φ_2) and (letting $\boldsymbol{u} = \mathscr{T}_1(\varphi_1, \varphi_2)$) the corresponding $(\boldsymbol{u}, \varphi_1, \varphi_2)$ is a solution to our original problem, which concludes the proof of the Theorem.

Appendix

The Lamé Parameters

The state quantities φ_i are the Lamé parameters related to the constitutive law

$$\begin{aligned} tr\boldsymbol{\sigma} &= 3\varphi_1 (tr\boldsymbol{\varepsilon})^+ - 3\varphi_2 (tr\boldsymbol{\varepsilon})^- \mathbf{1} + 2\mu tr\boldsymbol{\varepsilon} \\ &= (3\varphi_1 + 2\mu)(tr\boldsymbol{\varepsilon})^+ - (3\varphi_2 + 2\mu)(tr\boldsymbol{\varepsilon})^-, \end{aligned}$$

giving

$$(tr\boldsymbol{\sigma})^{+} = (3\varphi_1 + 2\mu)\,(tr\boldsymbol{\varepsilon})^{+}\,,$$
$$(tr\boldsymbol{\sigma})^{-} = (3\varphi_2 + 2\mu)\,(tr\boldsymbol{\varepsilon})^{-}\,.$$

Consequently, the Poisson coefficients are

$$\nu_1 = \frac{\varphi_1}{2(\varphi_1 + \mu)},\ \nu_2 = \frac{\varphi_2}{2(\varphi_2 + \mu)}.$$

They are negative when the $\varphi_i' s$ are negative. They are different when pulling, $tr\sigma > 0$, and pushing, $tr\sigma < 0$, as required.

The modulus is

$$E = \frac{\mu(3\varphi + 2\mu)}{\varphi + \mu} = 2\mu(1 + \nu).$$

Note that, it is not constant but it is positive.

Hence, one could rewrite the model in terms of ν_i instead of the φ_i $(i = 1, 2)$ as state quantities.

The Evolution of the Parameter φ_1

The evolution of φ_1 is governed by

$$(\theta - \theta_1) - \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^{+}\right)^2$$

which defines two domains separated by a parabola in the plane $(tr\boldsymbol{\varepsilon})\,,\theta)$. Following the line introduced by [8], the free energy may be upgraded by a linear term

$$\hat{k}_1\varphi_1\,(tr\boldsymbol{\varepsilon})^{+} + \hat{k}_2\varphi_2\,(tr\boldsymbol{\varepsilon})^{-}\,,$$

with $\hat{k}_1 \geq 0$ and $\hat{k}_2 \geq 0$. In this case the equation of motion for φ_1

$$\dot{\varphi}_1 - \Delta\varphi_1 + R_1 = (\theta - \theta_1) - \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^{+}\right)^2 - \hat{k}_1\,(tr\boldsymbol{\varepsilon})^{+}\,,$$

is governed by the right hand side

$$(\theta - \theta_1) - \frac{1}{2}\left((tr\boldsymbol{\varepsilon})^{+}\right)^2 - \hat{k}_1\,(tr\boldsymbol{\varepsilon})^{+}$$

which is almost a line (a line in the small perturbation assumption).

The stress becomes

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \varphi_1 \left(tr\boldsymbol{\varepsilon}\right)^+ \mathbf{1} - \varphi_2 \left(tr\boldsymbol{\varepsilon}\right)^- \mathbf{1} + 2\mu\boldsymbol{\varepsilon} + k_1\varphi_1 H\left(tr\boldsymbol{\varepsilon}\right)\mathbf{1} - k_2\varphi_2 H\left(-tr\boldsymbol{\varepsilon}\right)\mathbf{1},$$

where Heaviside graph $H(x)$ is the subdifferential set of the positive part function $(x)^+$ and graph $-H(-x)$ is the subdifferential set of the negative part function $(x)^-$. Note that

$$H\left(tr\boldsymbol{\varepsilon}\right) = H\left(tr\sigma\right),$$
$$H\left(-tr\boldsymbol{\varepsilon}\right) = H\left(-tr\sigma\right),$$

because $(tr\boldsymbol{\sigma})^+ = (3\varphi_1 + 2\mu)\,(tr\boldsymbol{\varepsilon})^+$, $(3\varphi_1 + 2\mu) > 0$ and $(tr\boldsymbol{\sigma})^- = (3\varphi_2 + 2\mu)\,(tr\boldsymbol{\varepsilon})^-$, $(3\varphi_2 + 2\mu) > 0$. Moreover, we have

$$sgn(x) = H(x) - H(-x),$$

where *sgn* is the sign graph, we have

$$sgn(tr\boldsymbol{\sigma}) = sgn(tr\varepsilon(\boldsymbol{u})).$$

Thus, the constitutive law depends on the sign of the trace of the stress which is the sign of the trace of the deformation

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \varepsilon} = \varphi_1 \left(tr\boldsymbol{\varepsilon}\right)^+ \mathbf{1} - \varphi_2 \left(tr\boldsymbol{\varepsilon}\right)^- \mathbf{1} + k_2\varphi_2 sgn(tr\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} + (k_1\varphi_1 - k_2\varphi_2)H\left(tr\boldsymbol{\varepsilon}\right)\mathbf{1},$$

A Related Model of Austenite-Martensite Phase Transition

In this section of Appendix, we propose another possible extension of the model presented in [8], where the austenite-martensite transition was studied by a phase field model and from which this paper has moved. Indeed, in the paper [8], the austenitic phase ends, when the dilatation reaches a default threshold, while the auxetic phase does not cease, when it is subjected to a progressive compression.

It seems appropriate to amend this last feature, considering a lower limit to the auxetic phase: when the compression reaches a threshold value, the body recovers an elastic behavior.

To describe a such new phase, one can combine the motion equation (here we include accelerations)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2}(x,t) = \operatorname{div}\boldsymbol{\sigma}(x,t) + \mathbf{f}(x,t) \tag{5.1}$$

(where the density is taken $\rho = 1$) with the evolution law for phase field $\varphi(x,t)$. The internal variable describes the internal structural state and its evolution is controlled by the Ginzburg-Landau equation (see [11])

$$\gamma(x)\frac{\partial}{\partial t}\,\varphi(x,t) = \lambda(x)\triangle\varphi(x,t) - F'(\varphi(x,t)) + \alpha(tr\boldsymbol{\sigma}(x,t))^2\nu'(\varphi(x,t)) \tag{5.2}$$

where $\gamma(x), \lambda(x)$ and $\alpha(x)$ are positive and

$$F(\varphi) = \frac{\varphi^2}{2}(\varphi^2 - 1). \tag{5.3}$$

Finally, Poisson ratio $\nu(\varphi)$ is defined by

$$\nu(\varphi) = \begin{cases} \frac{1}{2}\,, & |\varphi| > 1 \\ \frac{1}{4}(3\cos\pi(1-\varphi)) - 1)\,, & |\varphi| \leq 1 \end{cases} \tag{5.4}$$

Then, we suggest the following constitutive equation for auxetic materials

$$\frac{d}{dt}\boldsymbol{\varepsilon} = \frac{1}{E}\left[\frac{d}{dt}\boldsymbol{\sigma} - (\frac{d}{dt}\frac{\nu}{2})\,(tr\boldsymbol{\sigma}\,\boldsymbol{I} - \boldsymbol{\sigma}) - \nu\frac{d}{dt}(tr\boldsymbol{\sigma}\,\boldsymbol{I} - \boldsymbol{\sigma})\right] + \alpha\,\frac{\partial}{\partial t}\nu(\varphi)tr\boldsymbol{\sigma}\boldsymbol{I}. \tag{5.5}$$

Note that the phase φ can be supposed a constant function, then Eq. (5.5) is reduced to the classical law for a linear isotropic elastic material

$$\boldsymbol{\varepsilon} = \frac{1}{E}\,[\boldsymbol{\sigma} - \nu(tr\boldsymbol{\sigma}\,\boldsymbol{I} - \boldsymbol{\sigma})]$$

As we consider only isothermal processes, the Second Law of Thermodynamics assumes the Dissipation Principle law by the inequality

$$\dot{\psi} \leq \mathscr{P}^i_\varepsilon + \mathscr{P}^i_\varphi. \tag{5.6}$$

where ψ denotes the free energy and the symbol $\dot{}$ the time derivative. Moreover, $\mathscr{P}^i_\varepsilon$ is the internal mechanical power and $\mathscr{P}^i_\varphi$ the internal structural power, defined by

$$\mathscr{P}^i_\varepsilon = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}\cdot\frac{1}{E}\left[\dot{\boldsymbol{\sigma}} - (\frac{\dot{\nu}(\varphi)}{2})\,(tr\boldsymbol{\sigma}\,\boldsymbol{I} - \boldsymbol{\sigma}) - \nu(\varphi)(tr\dot{\boldsymbol{\sigma}}\,\boldsymbol{I} - \dot{\boldsymbol{\sigma}})\right] + \alpha\dot{\nu}(\varphi)(tr\boldsymbol{\sigma})^2 \tag{5.7}$$

$$\mathscr{P}^i_\varphi = \gamma\dot{\varphi}^2 + \lambda\nabla\varphi\cdot(\nabla\varphi)^{\cdot} + \dot{F}(\varphi) - \alpha(tr\boldsymbol{\sigma}(x,t))^2\dot{\nu}(\varphi) \tag{5.8}$$

So, we have from (5.6)–(5.8)

$$\dot{\psi} \leq \boldsymbol{\sigma} \cdot \frac{1}{E}\left[\dot{\boldsymbol{\sigma}} - (\frac{\dot{\nu}(\varphi)}{2})\,(tr\boldsymbol{\sigma}\,\boldsymbol{I} - \boldsymbol{\sigma}) - \nu(\varphi)(tr\dot{\boldsymbol{\sigma}}\,\boldsymbol{I} - \dot{\boldsymbol{\sigma}})\right] +$$
$$\gamma\dot{\varphi}^2 + \lambda\nabla\varphi \cdot (\nabla\varphi)^{\cdot} + \dot{F}(\varphi)$$

and consequently

$$\left(\frac{\partial\psi}{\partial\varphi} - F'(\varphi) - \frac{\nu'(\varphi)}{2E}\left[\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - (tr\boldsymbol{\sigma})^2\right]\right)\dot{\varphi} + \tag{5.9}$$
$$+\left(\frac{\partial\psi}{\partial\nabla\varphi} - \lambda\nabla\varphi\right)\cdot\nabla\dot{\varphi} + \left(\frac{\partial\psi}{\partial\boldsymbol{\sigma}} - \frac{1}{E}\left[(1+\nu(\varphi))\boldsymbol{\sigma} - \nu(tr\boldsymbol{\sigma})^2\right]\right)\cdot\dot{\boldsymbol{\sigma}} - \gamma\dot{\varphi}^2 \leq 0$$

The state S of this system is defined by the triplet

$$S = (\boldsymbol{\sigma}, \varphi, \nabla\varphi)$$

(5.9) and the arbitrariness of $\dot{\sigma}, \dot{\phi}, \nabla\dot{\phi}$ we obtain the constitutive relations

$$\frac{\partial\psi}{\partial\varphi} = F'(\varphi) + \frac{\nu'(\varphi)}{2E}\left[\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - (tr\boldsymbol{\sigma})^2\right]$$
$$\frac{\partial\psi}{\partial\nabla\varphi} = \lambda\nabla\varphi$$
$$\frac{\partial\psi}{\partial\boldsymbol{\sigma}} = \frac{1}{E}[(1+\nu(\varphi))\boldsymbol{\sigma} - \nu(\varphi)(tr\boldsymbol{\sigma})\mathbf{I}]$$

and $\gamma > 0$.

In our approach we have used the following form for the free energy

$$\psi(\boldsymbol{\sigma}, \varphi, \nabla\varphi) = \frac{1}{2}\lambda(\nabla\varphi)^2 + F(\varphi) + \frac{1}{2E}\left[(1+\nu(\varphi))\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - \nu(\varphi)\,(tr\boldsymbol{\sigma})^2\right].$$

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Global Well-Posedness for a Phase Transition Model with Irreversible Evolution and Acceleration Forces

Giovanna Bonfanti and Fabio Luterotti

Abstract In this paper we investigate a nonlinear PDE system describing irreversible phase transition phenomena where inertial effects are also included. Its derivation comes from the modelling approach proposed by M. Frémond. We obtain a global in time existence and uniqueness result for the related initial and boundary value problem.

Keywords Existence • Irreversibility • Microscopic accelerations • Phase changes • Uniqueness

AMS (MOS) Subject Classification 80A22, 35K55, 35Q79

1 Introduction

This paper is concerned for the analytical investigation of a nonlinear PDE system describing irreversible phase transition phenomena where inertial effects are also included. The underlying model comes from Frémond's theory on phase transitions [17, 18] and it is based on the consideration that microscopic movements and forces give rise to phase changes at the macroscopic level. In particular, in this approach, it is considered a generalized version of the principle of virtual power including the power of the microscopic forces which create and break the microscopic links responsible for the phase transition. Such a model was originally derived in [11] and further refined in [12] where acceleration forces were also encompassed. The main difficulties related to the analytical investigation of Frémond's model on phase transitions are due to the nonlinear character of the resulting PDE system. Actually, the presence of high-order nonlinearities can preclude global in time existence results. The aim of the present contribution is to obtain a *global* in time

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well-posedness result for a phase transition problem where acceleration effects and irreversible evolution are considered.

Let us now briefly recall the derivation of the model and of the corresponding initial and boundary value problem we are dealing with.

On a time interval $(0, T), T > 0$ we investigate the thermal evolution of a two-phase substance located in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ undergoing to an irreversible process of phase transition. As state variables of the model, we introduce the absolute temperature ϑ, an order parameter χ and its gradient $\nabla\chi$. We assume the free energy Ψ in $\Omega \times (0, T)$ as

$$\Psi(\chi, \nabla\chi, \vartheta) := -c_v\vartheta \log\vartheta - \frac{L}{\vartheta_c}(\vartheta - \vartheta_c)\chi + \frac{1}{4}(\chi^2 - 1)^2 + \frac{\nu}{2}|\nabla\chi|^2, \tag{1.1}$$

where $c_v > 0$ stands for the specific heat, $L > 0$ for the latent heat at the critical transition temperature $\vartheta_c > 0$, and $\nu > 0$ is related to the intensity of the local interactions. We note that the double-well potential in (1.1) is one of the standard choices for phase field models (see, e.g., the Caginalp model [14]).

Next, we introduce $\nabla\vartheta$ and $\partial_t\chi$ as dissipative variables of the system. We consider a fairly general expression for the purely thermal contribution in the pseudo-potential of dissipation Φ (cf. with [4–6]) defined in $\Omega \times (0, T)$ as

$$\Phi(\partial_t\chi, \nabla\vartheta) := \frac{\mu}{2}(\partial_t\chi)^2 + I_{[0,+\infty[}(\partial_t\chi) + \frac{\mathsf{K}(\vartheta)}{2\vartheta}|\nabla\vartheta|^2, \tag{1.2}$$

where $\mu > 0$ is a viscosity parameter, $I_{[0,+\infty[}$ denotes the indicator function on the interval $[0, +\infty[$ and the (smooth) and positive function K represents the heat conductivity of the process. We remark that the indicator function $I_{[0,+\infty[}$ in (1.2) compels the time derivative $\partial_t\chi$ to be non-negative and hence renders the irreversible character of the evolution process we are describing. Concerning the function K, we recall that several choices (thermodynamically consistent) are widely adopted in the literature: we quote the case $\mathsf{K}(\vartheta) = k > 0$ corresponding to the standard Fourier law for the heat flux (see our assumption (2.1) below) and the case where $\mathsf{K}(\vartheta) \sim \vartheta^p$, for large values of ϑ and suitable positive values of p (see, e.g., [6, 10, 16]).

Now, the equation for the temperature variable is recovered from the internal energy balance where the power of the microscopic movements is also taken into account. It reads

$$\partial_t e + \operatorname{div} \mathbf{q} = B\partial_t\chi + \mathbf{H}\cdot\nabla\partial_t\chi + f \quad \text{in } \Omega \times (0, T), \qquad \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \times (0, T), \tag{1.3}$$

where $e = \Psi + s\vartheta$ (s stands for the entropy) denotes the internal energy, $\mathbf{q}$ the heat flux, B and $\mathbf{H}$ internal microscopic forces, f an external heat source, and $\mathbf{n}$ the outward normal unit vector to $\partial\Omega$.

The equation governing the evolution of the phase variable comes from the application of a generalized version of the principle of the virtual power where inertial effects are also included. We have

$$\rho_0 \partial_{tt}\chi + B - \operatorname{div} \mathbf{H} = g \quad \text{in } \Omega \times (0,T), \qquad \mathbf{H}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.4}$$

where $\rho_0 > 0$ is proportional to the mass of the microscopic links and g is an external microscopic source.

Next, we have to combine (1.3)–(1.4) with suitable constitutive relations for the involved physical quantities in terms of Ψ and Φ. To this aim, we prescribe for (the dissipative and non-dissipative contributions of) B and $\mathbf{H}$

$$B = B^{\text{nd}} + B^{\text{d}} = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial(\partial_t \chi)}, \tag{1.5}$$

$$\mathbf{H} = \mathbf{H}^{\text{nd}} + \mathbf{H}^{\text{d}} = \frac{\partial \Psi}{\partial \nabla \chi} + \frac{\partial \Phi}{\partial(\nabla(\partial_t \chi))} \tag{1.6}$$

and moreover

$$s = -\frac{\partial \Psi}{\partial \vartheta}, \quad \mathbf{q} = -\vartheta \frac{\partial \Phi}{\partial \nabla \vartheta}. \tag{1.7}$$

With these choices, combining (1.3)–(1.4) with (1.5)–(1.7) and (1.1)–(1.2), we derive the following PDE system

$$c_v \partial_t \vartheta + \frac{L}{\vartheta_c}\vartheta \partial_t \chi - \operatorname{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = f + \mu(\partial_t\chi)^2 + \xi \partial_t \chi \quad \text{in } \Omega \times (0,T), \tag{1.8}$$

$$\rho_0 \partial_{tt}\chi + \mu \partial_t \chi - \nu \Delta \chi + \xi + \chi^3 - \chi = g + \frac{L}{\vartheta_c}(\vartheta - \vartheta_c) \quad \text{in } \Omega \times (0,T), \tag{1.9}$$

$$\xi \in \partial I_{[0,+\infty[}(\partial_t \chi) \quad \text{in } \Omega \times (0,T), \tag{1.10}$$

$$\mathsf{K}(\vartheta)\nabla\vartheta \cdot \mathbf{n} = 0 \qquad \partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.11}$$

which will be supplemented by the following initial conditions

$$\vartheta(\cdot,0) = \vartheta_0 \qquad \chi(\cdot,0) = \chi_0 \qquad \partial_t\chi(\cdot,0) = \chi_1 \quad \text{in } \Omega. \tag{1.12}$$

We recall that the multivalued operator $\partial I_{[0,+\infty[}$ is the subdifferential (in the sense of convex analysis) of the indicator function $I_{[0,+\infty[}$ (i.e. $\partial I_{[0,+\infty[}(u) =]-\infty, 0]$ if $u = 0$, and $\partial I_{[0,+\infty[}(u) = 0$ if $u > 0$) and hence the term $\xi \partial_t \chi$ actually vanishes in (1.8) due to definition $\partial I_{[0,+\infty[}(\partial_t \chi)$.

We remark that the thermodynamical consistency of the model can be proved in the form of the Clausius-Duhem inequality. Combining (1.3) with (1.7), we can equivalently write the internal energy balance as follows

$$\vartheta\left(s_t + \operatorname{div}\frac{\mathbf{q}}{\vartheta} - \frac{f}{\vartheta}\right) = B^{\mathrm{d}}\partial_t\chi - \frac{\mathbf{q}}{\vartheta}\cdot\nabla\vartheta = \partial\Phi(\partial_t\chi,\nabla\vartheta)\cdot(\partial_t\chi,\nabla\vartheta) \geq 0,$$

where the latter inequality is due to the properties of Φ (Φ is convex in all of its variables and $\Phi(0,\mathbf{0}) = 0$). Then the Clausius-Duhem inequality follows, owing to the strict positivity of ϑ.

Finally, we have to remark that a refined version of the (reversible) Caginalp model [14] is derived in the paper [3] where a second-order quadratic term (like $\Delta\chi\partial_t\chi$) in the energy balance equation is the main feature: the local well-posedness for smooth data is obtained and a formal asymptotic analysis towards a sharp interface limit is performed.

Concerning the analytical investigation of system (1.8)–(1.12), we point out that the main difficulty is due to the presence of the quadratic terms in (1.8) which can prevent to deduce global in time existence (and uniqueness) results in the three dimensional setting. As far as we know, no global in time well-posedness results have been established for Frémond's original model derived in [11]. Actually, many papers addressed modified/reduced versions of the original model, in the case with or without acceleration forces. For the sake of completeness, we briefly recall the features of such versions. First, in [11] neglecting the inertial term ($\rho_0 = 0$ in (1.9)) and considering a doubly nonlinear differential inclusion for the phase dynamics, a global existence result was proved under a small perturbations assumption (i.e., ϑ close to the critical temperature ϑ_c, so $\frac{\vartheta}{\vartheta_c}\partial_t\chi$ is replaced by $\partial_t\chi$, and $(\partial_t\chi)^2 \approx 0$). In [22] a further step is done in the direction of dealing with the full equation (1.8), assuming only $(\partial_t\chi)^2 \approx 0$ and allowing the temperature ϑ to be far from the critical value θ_c. The paper [23] proves the existence of a solution in the special case where a finite maximum speed is imposed to the phase transition process. The result of [23] is used in [24] as an approximation tool to prove the local in time existence for the original model. Finally, we mention [16] where an existence result is proved for the original model in the reversible case: here the weak solution satisfies a property of energy conservation together with an entropy inequality (instead of the standard energy balance equation).

Coming back to the models where inertial effects are included, as far as we know, again local in time well-posedness results have been proved. In this concern, we mention [12] where a more general relation of (1.10) was considered in system (1.8)–(1.12). Next, in [7] and [8] a straightforward connection between the models with and without acceleration forces was investigated: starting from the results of [12], an asymptotic analysis was performed as $\rho_0 \to 0$, obtaining a—still—local in time result to Frémond's original model for phase transitions.

Clearly, global existence results to the above problems can be obtained (see [9, 19, 21]) in the one dimensional setting, by exploiting the more regular framework.

Finally, we quote [10] where a global well-posedness result has been obtained for a generalized version of the system (1.8)–(1.12) (with $\rho_0 \geq 0$) assuming that the function K in (1.8) satisfies suitable growth conditions. Now, the aim of the present contribution is to address the system (1.8)–(1.12) in the case where the heat flux is ruled by the classical Fourier law (see our assumption (2.1) below) and the unidirectionality of the process is prescribed by the inclusion (1.10). We note that the presence of the irreversibility entails, on one hand, a further nonlinear feature to deal with, on the other hand it is crucial in the proofs, since some sign and monotonicity properties can be suitably exploited to handle the quadratic terms in (1.8).

The outline of the paper is as follows. The next section is devoted to the assumptions, the notation, and the statements of the results. In Sect. 3, we address to the uniqueness proof. In Sect. 4, we set up a family of approximating problems introducing a suitable regularization of the phase equation, involving two distinct parameters, and we prove a related well-posedness result. Finally, in Sect. 5 we perform the passage to the limit with respect to the approximating parameters and we recover a solution (actually, unique) to the original problem.

2 Statement of the Results

We start by fixing some notation. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We set $Q_t := \Omega \times (0,t)$ for $t \in (0,T)$ and $Q := \Omega \times (0,T)$. Letting $\mathbf{n}$ stand for the outward normal unit vector to $\partial\Omega$, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \left\{u \in H^2(\Omega), \text{ such that } \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\right\},$$

and identify H with its dual space H', so that $W \hookrightarrow V \hookrightarrow H \hookrightarrow V' \hookrightarrow W'$, with dense and compact embeddings. We use the same symbol for the norm of a space of scalar functions and the norm of the space of corresponding vector-valued functions. For instance, $\|\cdot\|_V$ means the norm of both V and V^3. Let the symbol $\|\cdot\|$ indicate the norm of H (or H^3). Henceforth, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V, by $(\cdot,\cdot)$ the scalar product in H and by $((\cdot,\cdot))$ the scalar product in V. Then, the associated Riesz isomorphism $J : V \to V'$ and the scalar product in V', denoted by $((\cdot,\cdot))_*$, can be specified by

$$\langle Jv_1, v_2\rangle := ((v_1, v_2)), \quad ((u_1,u_2))_* := \langle u_1, J^{-1}u_2\rangle \quad \text{for } v_i \in V,\ u_i \in V',\ i = 1,2.$$

In the sequel, we make the following assumptions.

Concerning the thermal coefficient K in (1.8), we choose

$$\mathsf{K}(\vartheta) = k > 0 \tag{2.1}$$

corresponding to the classical Fourier law for the heat flux $\mathbf{q}$ which reads $\mathbf{q} = -k\nabla\vartheta$.

Moreover, as for the Cauchy conditions, we prescribe that

$$\vartheta_0 \in V \;\text{ and } \exists\, c^* > 0 \text{ such that } \;\vartheta_0 \geq c^* \text{ a.e. in } \;\Omega\,, \tag{2.2}$$

$$\chi_0 \in W\,, \quad \chi_1 \in V\,, \tag{2.3}$$

$$\chi_1 \geq 0 \text{ a.e. in } \;\Omega\,. \tag{2.4}$$

Finally, without loss of generality, we set the physical constants occurring in (1.8)–(1.9) and (2.1) equal to 1 and we get rid of the known terms on the right-hand side of (1.8)–(1.9) since they not affect our analysis.

We may now specify the variational problem we are dealing with.

Problem (P) Given a triple of initial data $(\vartheta_0, \chi_0, \chi_1)$ fulfilling (2.2)–(2.4), find a triple (ϑ, χ, ξ) such that

$$\vartheta \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)\,, \tag{2.5}$$

$$\chi \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap L^\infty(0,T;W)\,, \tag{2.6}$$

$$\xi \in L^2(0,T;H)\,, \tag{2.7}$$

fulfilling

$$\partial_t\vartheta + \vartheta\,\partial_t\chi - \Delta\vartheta = (\partial_t\chi)^2 \quad \text{ a.e. in } \;Q\,, \tag{2.8}$$

$$\partial_{tt}\chi + \partial_t\chi + \xi - \Delta\chi + \chi^3 - \chi = \vartheta \quad \text{ a.e. in } \;Q\,, \tag{2.9}$$

$$\xi \in \partial I_{[0,+\infty[}(\partial_t\chi) \quad \text{ a.e. in } \;Q\,, \tag{2.10}$$

$$\vartheta(\cdot,0) = \vartheta_0 \quad \text{ a.e. in } \;\Omega\,, \tag{2.11}$$

$$\chi(\cdot,0) = \chi_0 \quad \text{ a.e. in } \;\Omega\,, \tag{2.12}$$

$$\partial_t\chi(\cdot,0) = \chi_1 \quad \text{ a.e. in } \;\Omega\,. \tag{2.13}$$

Let us observe that the no-flux boundary conditions for ϑ and χ (cf. with (1.11)) are incorporated in (2.5)–(2.6), by the definition of the space W.

With the following result, we state the global well-posedness of Problem **(P)**.

Theorem 1 *Assume (2.2)–(2.4). Then, there exists a unique triple* (ϑ, χ, ξ) *solving Problem* **(P)**. *Moreover, there exists a constant* $c_* > 0$ *such that* $\vartheta \geq c_*$ *a.e. in* Q.

We outline here the strategy of the proof of Theorem 1 which will be shown in the next sections. The uniqueness of the solution is easily proved by suitable contraction estimates. As for the existence, we shall consider a family of auxiliary problems $(\mathbf{P}^h_\varepsilon)$ introducing in (2.9) both a viscosity term (parameter $h > 0$) and the Yosida approximation (parameter $\varepsilon > 0$) of the multivalued operator $\partial I_{[0,+\infty[}$ and we shall prove a related well-posedness result. Next, based on a series of a priori estimates independent of ε, we first perform the passage to the limit as $\varepsilon \searrow 0$ (and h is fixed) obtaining a triple $(\vartheta_h, \chi_h, \xi_h)$ fulfilling an intermediate Problem $(\mathbf{P}^h)$.

Finally, after having derived some other a priori estimates independent of h, we pass to the limit as $h \searrow 0$ and we recover a (unique) solution to the original Problem (**P**).

Remark 1 We warn that, in the proofs, we employ the same symbol c for different constants, even in the same formula, for the sake of simplicity.

Finally, we recall the Young inequality which will be useful in the sequel:

$$ab \le \delta\, a^p + \frac{1}{q}(\delta p)^{-q/p}\, b^q\,, \tag{2.14}$$

for all $a, b \in \mathbb{R}^+$, $\delta > 0$ and $p > 1,\ q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

3 Proof of Uniqueness

We proceed by contradiction. We denote by $\widetilde{\vartheta} = \vartheta_2 - \vartheta_1$, $\widetilde{\chi} = \chi_2 - \chi_1$, $\widetilde{\xi} = \xi_2 - \xi_1$, being $(\vartheta_1, \chi_1, \xi_1)$ and $(\vartheta_2, \chi_2, \xi_2)$ two solutions to Problem (**P**).

We consider the difference between the corresponding equations (2.9), we multiply it by $\partial_t\widetilde{\chi}$ and we integrate over Q_t, with $0 < t < T$. Using the Hölder inequality and recalling the continuous embedding $W \hookrightarrow L^\infty(\Omega)$, we get

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\widetilde{\chi}(t)\|^2 + \|\partial_t\widetilde{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2 + \iint_{Q_t} \widetilde{\xi}\,\partial_t\widetilde{\chi} \le \\
&\le c\|\widetilde{\chi}\|_{L^2(0,t;H)}\|\partial_t\widetilde{\chi}\|_{L^2(0,t;H)} + \|\widetilde{\vartheta}\|_{L^2(0,t;H)}\|\partial_t\widetilde{\chi}\|_{L^2(0,t;H)}\,,
\end{aligned} \tag{3.1}$$

with $c > 0$ depending on $\|\chi_1\|_{L^\infty(Q)}$ and $\|\chi_2\|_{L^\infty(Q)}$. Taking the monotonicity of $\partial I_{[0,+\infty[}$ into account and applying (2.14), we deduce

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\widetilde{\chi}(t)\|^2 + \|\partial_t\widetilde{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2 \le \\
&\le \frac{1}{4}\|\widetilde{\vartheta}\|^2_{L^2(0,t;H)} + c\|\partial_t\widetilde{\chi}\|^2_{L^2(0,t;H)} + c\int_0^t \|\partial_t\widetilde{\chi}\|^2_{L^2(0,s;H)}\,ds\,.
\end{aligned} \tag{3.2}$$

Let us take the difference between the corresponding equations (2.8) and we add (on both sides) $\widetilde{\vartheta}$. Recalling the definition of J, we have

$$\begin{aligned}
&\langle\partial_t\widetilde{\vartheta}, v\rangle + \langle J\widetilde{\vartheta}, v\rangle = \\
&(\widetilde{\vartheta} + \vartheta_1\partial_t\chi_1 - \vartheta_2\partial_t\chi_2 + (\partial_t\chi_2)^2 - (\partial_t\chi_1)^2, v) \quad \forall\, v \in V \ \text{ a.e. in } (0,T)\,.
\end{aligned} \tag{3.3}$$

We choose $v = J^{-1}\widetilde{\vartheta}$ as test function in (3.3) and we integrate from 0 to t, $0 < t < T$. We get

$$\frac{1}{2}\|\widetilde{\vartheta}(t)\|_{V'}^2 + \|\widetilde{\vartheta}\|_{L^2(0,t;H)}^2 \leq \sum_{j=1}^{3} |I_j(t)|, \tag{3.4}$$

where

$$I_1(t) := \int_0^t (\widetilde{\vartheta}, J^{-1}\widetilde{\vartheta}), \tag{3.5}$$

$$I_2(t) := \int_0^t (\vartheta_1 \partial_t \chi_1 - \vartheta_2 \partial_t \chi_2, J^{-1}\widetilde{\vartheta}), \tag{3.6}$$

$$I_3(t) := \int_0^t ((\partial_t \chi_2)^2 - (\partial_t \chi_1)^2, J^{-1}\widetilde{\vartheta}). \tag{3.7}$$

We estimate the latter summands as follows. Using the Hölder inequality, the continuous embedding $V \hookrightarrow L^4(\Omega)$, (2.14), and recalling the definition of J, we get

$$I_1(t) = \|\widetilde{\vartheta}\|_{L^2(0,t;V')}^2, \tag{3.8}$$

$$|I_2(t)| \leq c \int_0^t \left(\|\widetilde{\vartheta}(s)\| \, \|\partial_t \chi_2(s)\|_{L^4(\Omega)} + \|\vartheta_1(s)\|_{L^4(\Omega)} \, \|\partial_t \widetilde{\chi}(s)\| \right) \|\widetilde{\vartheta}(s)\|_{V'} \, ds \leq$$

$$\leq \frac{1}{4}\|\widetilde{\vartheta}\|_{L^2(0,t;H)}^2 + c \left(\|\widetilde{\vartheta}\|_{L^2(0,t;V')}^2 + \|\partial_t \widetilde{\chi}\|_{L^2(0,t;H)}^2 \right), \tag{3.9}$$

$$|I_3(t)| \leq c \left(\|\widetilde{\vartheta}\|_{L^2(0,t;V')}^2 + \|\partial_t \widetilde{\chi}\|_{L^2(0,t;H)}^2 \right), \tag{3.10}$$

for some positive constant c depending on the norms $\|\vartheta_1\|_{L^\infty(0,T;V)}$, $\|\partial_t \chi_1\|_{L^\infty(0,T;V)}$, and $\|\partial_t \chi_2\|_{L^\infty(0,T;V)}$. Now we add (3.2) and (3.4), taking into account (3.8)–(3.10). We apply the Gronwall lemma and we deduce $\widetilde{\vartheta} = \widetilde{\chi} = 0$ a.e. in Q. A comparison in (2.9) gives $\widetilde{\xi} = 0$ a.e. in Q too.

4 Approximating Problems

We shall approximate Problem **(P)** by suitably regularizing the phase equation. More precisely, we shall add in (2.9) a viscosity term, and replace the operator $\partial I_{[0,+\infty[}$ by its Yosida regularization. This approach will enable us to perform enhanced regularity estimates on the (approximate) phase equation and ultimately to deal with the quadratic nonlinearities in the energy equation. For technical reasons (cf. with Remark 2 later on), we shall keep the viscosity parameter (denoted by $h > 0$) distinct from the Yosida regularization parameter (denoted by $\varepsilon > 0$).

Hence, we shall call ($\mathbf{P}_\varepsilon^h$) the initial and boundary value problem for the resulting approximate system and prove that it is well-posed following this outline: first we are going to prove the existence of a local solution by a fixed point argument; next, we are going to extend such a solution to the whole interval $(0, T)$ and finally we shall obtain a related uniqueness result.

In what follows, for the sake of brevity, we denote by α the operator $\partial I_{[0,+\infty[}$ in (2.10) and by $\alpha_\varepsilon : \mathbb{R} \to \mathbb{R}$ its Yosida regularization (cf. [13, Chap. II]). We recall that (cf. [13, Prop. II.2.6]) α_ε is monotone and Lipschitz continuous on $\mathbb{R}$, with Lipschitz constant $1/\varepsilon$. We also denote by $\widehat{\alpha}_\varepsilon$ the Moreau-Yosida approximation of $\widehat{\alpha} := I_{[0,+\infty[}$ and we recall that $\widehat{\alpha}_\varepsilon \in C^1(\mathbb{R})$, with derivative $(\widehat{\alpha}_\varepsilon)' = \alpha_\varepsilon$.

Let us consider

Problem ($\mathbf{P}_\epsilon^h$) Given a triple of initial data $(\vartheta_0, \chi_0, \chi_1)$ fulfilling (2.2)–(2.3), find $(\vartheta_{\varepsilon h}, \chi_{\varepsilon h})$ such that

$$\vartheta_{\varepsilon h} \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W)\,, \tag{4.1}$$

$$\chi_{\varepsilon h} \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W), \tag{4.2}$$

$$\partial_t \vartheta_{\varepsilon h} + \vartheta_{\varepsilon h}\partial_t \chi_{\varepsilon h} - \Delta \vartheta_{\varepsilon h} = (\partial_t \chi_{\varepsilon h})^2 \quad \text{a.e. in } Q\,, \tag{4.3}$$

$$\partial_{tt}\chi_{\varepsilon h} - h\Delta\partial_t\chi_{\varepsilon h} + \partial_t\chi_{\varepsilon h} - \Delta\chi_{\varepsilon h} + \alpha_\varepsilon(\partial_t\chi_{\varepsilon h}) + \chi_{\varepsilon h}^3 - \chi_{\varepsilon h} = \vartheta_{\varepsilon h} \quad \text{a.e. in } Q\,, \tag{4.4}$$

$$\vartheta_{\varepsilon h}(\cdot,0) = \vartheta_0 \quad \text{a.e. in } \Omega\,, \tag{4.5}$$

$$\chi_{\varepsilon h}(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega\,, \tag{4.6}$$

$$\partial_t\chi_{\varepsilon h}(\cdot,0) = \chi_1 \quad \text{a.e. in } \Omega\,. \tag{4.7}$$

The following theorem holds.

Theorem 2 (Global Well-Posedness for Problem ($\mathbf{P}_\varepsilon^h$)) *Let the assumptions* (2.2)–(2.3) *hold. Then, for any* ε, $h > 0$ *there exists a unique solution* $(\vartheta_{\varepsilon h}, \chi_{\varepsilon h})$ *to Problem* ($\mathbf{P}_\varepsilon^h$). *Moreover, there exists a constant* $c_* > 0$ *(independent of* ε *and* h*) such that* $\vartheta_{\varepsilon h} \geq c_*$ *a.e. in* Q.

As already mentioned, we shall first of all prove the existence of a local solution to Problem ($\mathbf{P}_\varepsilon^h$) by applying the Schauder theorem to a suitable operator $\mathcal{T}$ which we construct below. Then we shall show that the local solution extends to a (unique) global one.

4.1 *Local Existence for Problem* ($\mathbf{P}_\varepsilon^h$)

For $R > \|\chi_0\|_{L^4(\Omega)}$, let us define the set $Y(\tau, R)$ by

$$Y(\tau,R) = \left\{ v \in W^{1,4}(0,\tau;L^4(\Omega)) : \|v\|_{W^{1,4}(0,\tau;L^4(\Omega))} \leq R \right\}, \tag{4.8}$$

where $\tau \in (0, T]$ will be determined later in such a way that $\mathcal{T} : Y(\tau, R) \to Y(\tau, R)$ is a compact and continuous operator.

Let $\chi \in Y(\tau, R)$ be fixed and let $\overline{\vartheta} := \mathcal{T}_1(\chi)$ be the unique solution to problem (4.1), (4.3), (4.5), with $\partial_t \chi_{\varepsilon h}$ replaced by $\partial_t \chi$ in (4.3).

Now, given such $\overline{\vartheta}$, let $\overline{\chi} := \mathcal{T}_2(\overline{\vartheta})$ be the unique solution to problem (4.2), (4.4), (4.6), (4.7), with $\vartheta_{\varepsilon h}$ substituted by $\overline{\vartheta}$. We note that the well-posedness of the intermediate problems (4.1), (4.3), (4.5), (with $\partial_t \chi_{\varepsilon h}$ replaced by $\partial_t \chi$) and (4.2), (4.4), (4.6), (4.7) (with $\vartheta_{\varepsilon h}$ substituted by $\overline{\vartheta}$) is ensured by standard results for parabolic equations. Finally, we define the operator $\mathcal{T}$ as the composition $\mathcal{T}_2 \circ \mathcal{T}_1$. In what follows we show that, at least for small times, the map $\mathcal{T}$ complies with the conditions of the Schauder theorem. To do this, we shall derive suitable a priori bounds on $\overline{\vartheta}$ and $\overline{\chi}$.

We multiply (4.3) (with $\partial_t \chi_{\varepsilon h}$ replaced by $\partial_t \chi$) by $\overline{\vartheta}$ and integrate over Q_t, with t arbitrary in $(0, \tau)$. After some integrations by parts, applying the Hölder inequality, we have

$$\frac{1}{2}\|\overline{\vartheta}(t)\|^2 + \|\nabla\overline{\vartheta}\|^2_{L^2(0,t;H)} \leq \frac{1}{2}\|\vartheta_0\|^2 + \\ + \int_0^t \Big(\|\overline{\vartheta}(s)\|_{L^4(\Omega)}\, \|\partial_t\chi(s)\|_{L^4(\Omega)} + \|\partial_t\chi(s)\|^2_{L^4(\Omega)}\Big)\|\overline{\vartheta}(s)\|\, ds\,. \tag{4.9}$$

Next, in order to recover the full V-norm of $\overline{\vartheta}$ on the left-hand side, we add $\|\overline{\vartheta}\|^2_{L^2(0,t;H)}$ to both sides of (4.9). Then, owing to the continuous embedding $V \hookrightarrow L^4(\Omega)$ and using (2.14), we get

$$\|\overline{\vartheta}(t)\|^2 + \|\overline{\vartheta}\|^2_{L^2(0,t;V)} \leq \\ \leq \|\vartheta_0\|^2 + c\int_0^t \Big(\|\partial_t\chi(s)\|^2_{L^4(\Omega)} + 1\Big)\|\overline{\vartheta}(s)\|^2\, ds + c\int_0^t \|\partial_t\chi(s)\|^2_{L^4(\Omega)}\|\overline{\vartheta}(s)\|\, ds\,. \tag{4.10}$$

Recalling the definition of $Y(\tau, R)$ we apply to (4.10) a generalized version of the Gronwall lemma introduced in [1] and we deduce that there exists a positive constant c_1 depending on ϑ_0, T, Ω , and R, such that

$$\|\overline{\vartheta}\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \leq c_1. \tag{4.11}$$

Next, in order to obtain a priori bounds on $\overline{\chi} = \mathcal{T}_2(\overline{\vartheta})$, we multiply (4.4) (with $\vartheta_{\varepsilon h}$ replaced by $\overline{\vartheta}$) by $\partial_t \overline{\chi}$ and we integrate over Q_t, with $0 < t < \tau$. Applying the

Hölder inequality and (2.14), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\overline{\chi}(t)\|^2 + h\|\nabla\partial_t\overline{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_t\overline{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\overline{\chi}(t)\|^2 + \\
&+ \iint_{Q_t} \alpha_\varepsilon(\partial_t\overline{\chi})\partial_t\overline{\chi} + \frac{1}{4}\int_\Omega (\overline{\chi}^2(t)-1)^2\,dx \le \\
&\le \frac{1}{2}\|\chi_1\|^2 + \frac{1}{2}\|\nabla\chi_0\|^2 + c(\|\chi_0\|^4_{L^4(\Omega)} + 1) + \frac{1}{2}\|\overline{\vartheta}\|^2_{L^2(0,t;H)}\,. \qquad (4.12)
\end{aligned}$$

We note that the integral term on the left-hand side of (4.12) is non-negative, thanks to the monotonicity of α_ε and to the fact that $\alpha_\varepsilon(0) = 0$. Then, on account of (4.11), we deduce

$$\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;H)\cap H^1(0,\tau;V)} \le c_2\,, \qquad (4.13)$$

for some positive constant c_2 depending on c_1 and moreover on χ_0, χ_1, and h.

Moreover, we multiply (4.4) (with $\vartheta_{\varepsilon h}$ replaced by $\overline{\vartheta}$) by $-\Delta\partial_t\overline{\chi}$ and we integrate over Q_t, with $0 < t < \tau$. Using the Hölder inequality, some standard continuous embeddings, and (2.14), we have

$$\begin{aligned}
&\frac{1}{2}\|\nabla\partial_t\overline{\chi}(t)\|^2 + h\|\Delta\partial_t\overline{\chi}\|^2_{L^2(0,t;H)} + \|\nabla\partial_t\overline{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\Delta\overline{\chi}(t)\|^2 + \\
&+ \iint_{Q_t} \alpha'_\varepsilon(\partial_t\overline{\chi})|\nabla\partial_t\overline{\chi}|^2 = \frac{1}{2}\|\nabla\chi_1\|^2 + \frac{1}{2}\|\Delta\chi_0\|^2 + \iint_{Q_t}\left(\overline{\chi}^3 - \overline{\chi} - \overline{\vartheta}\right)\Delta\partial_t\overline{\chi} \le \\
&\le \frac{1}{2}\|\nabla\chi_1\|^2 + \frac{1}{2}\|\Delta\chi_0\|^2 + \frac{h}{2}\|\Delta\partial_t\overline{\chi}\|^2_{L^2(0,t;H)} + \\
&+ c(\|\overline{\vartheta}\|^2_{L^2(0,t;H)} + \|\overline{\chi}\|^2_{L^2(0,t;H)} + \|\overline{\chi}\|^6_{L^\infty(0,\tau;V)})\,. \qquad (4.14)
\end{aligned}$$

Thanks to the monotonicity of α_ε, the integral term on the left-hand side of (4.14) is non-negative. On account of (4.11) and (4.13), we infer

$$\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;V)} \le c \quad \text{and} \quad \|\Delta\partial_t\overline{\chi}\|_{L^2(0,\tau;H)} \le c \qquad (4.15)$$

and hence, by elliptic regularity results,

$$\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;V)\cap H^1(0,\tau;W)} \le c_3\,, \qquad (4.16)$$

for some positive constant c_3 having the same dependence of c_2.

Finally, we multiply (4.4) (with $\vartheta_{\varepsilon h}$ replaced by $\overline{\vartheta}$) by $\partial_{tt}\overline{\chi}$ and we integrate over Q_t, with $0 < t < \tau$. We have

$$\|\partial_{tt}\overline{\chi}\|^2_{L^2(0,t;H)} + \frac{h}{2}\|\nabla\partial_t\overline{\chi}(t)\|^2 + \frac{1}{2}\|\partial_t\overline{\chi}(t)\|^2 + \int_\Omega \widehat{\alpha}_\varepsilon(\partial_t\overline{\chi}(t))\,dx \le$$
$$\le \frac{h}{2}\|\nabla\chi_1\|^2 + \frac{1}{2}\|\chi_1\|^2 + \int_\Omega \widehat{\alpha}_\varepsilon(\chi_1)\,dx +$$
$$+ c\Big(\|\Delta\overline{\chi}\|^2_{L^2(0,t;H)} + \|\overline{\chi}^3 - \overline{\chi}\|^2_{L^2(0,t;H)} + \|\overline{\vartheta}\|^2_{L^2(0,t;H)}\Big) + \frac{1}{2}\|\partial_{tt}\overline{\chi}\|^2_{L^2(0,t;H)}\,. \tag{4.17}$$

Then, on account of (4.11) and (4.16), we derive moreover

$$\|\overline{\chi}\|_{H^2(0,\tau;H)} \le c_4\,, \tag{4.18}$$

for some positive constant c_4 depending on c_1, c_3, and ε.

By the Hölder inequality and the continuous embedding $V \hookrightarrow L^4(\Omega)$, we have

$$\|\overline{\chi}\|_{W^{1,4}(0,\tau;L^4(\Omega))} \le c_\Omega\,\tau^{1/4}\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;V)} \le c_\Omega\,c_3\,\tau^{1/4}, \tag{4.19}$$

where c_Ω denotes the embedding constant of V into $L^4(\Omega)$. Thus, choosing $0 < \tau \le (R/c_\Omega\,c_3)^4$, we have that $\overline{\chi} \in Y(\tau, R)$. Eventually, we can choose a smaller $\tau > 0$ such that $\|\chi_0\|_{L^4(\Omega)} \le c_\tau\,R$, where c_τ stands for the embedding constant of $W^{1,4}(0,\tau;L^4(\Omega))$ into $C^0([0,\tau];L^4(\Omega))$ (since we have fixed $R > \|\chi_0\|_{L^4(\Omega)}$, we can take, e.g., $\tau > 0$ such that $c_\tau \ge 1$). All in all, there exists $\tau \in (0, T]$ small enough such that the operator $\mathcal{T} : Y(\tau, R) \to Y(\tau, R)$ turns out to be well defined.

Now, we observe that the same argument leading to (4.16) and (4.18) ensures that $\mathcal{T}$ is a compact operator.

To achieve the proof of the Schauder theorem, it remains to show that $\mathcal{T}$ is continuous with respect to the natural topology induced in $Y(\tau, R)$ by $W^{1,4}(0,\tau;L^4(\Omega))$. To this aim, we consider a sequence χ_n in $Y(\tau, R)$ such that

$$\chi_n \to \chi \text{ strongly in } Y(\tau, R) \quad \text{as } n \to +\infty. \tag{4.20}$$

Now, we set $\overline{\vartheta}_n := \mathcal{T}_1(\chi_n)$, $\overline{\vartheta} := \mathcal{T}_1(\chi)$ and $\widetilde{\vartheta}_n := \overline{\vartheta}_n - \overline{\vartheta}$. Let us take the difference between the corresponding equations (4.3), we add (on both sides) $\widetilde{\vartheta}_n$ and we test by $J^{-1}\widetilde{\vartheta}_n$. Arguing as in the derivation of the estimates (3.4) and (3.8)–(3.10), we end up with

$$\|\widetilde{\vartheta}_n(t)\|^2_{V'} + \|\widetilde{\vartheta}_n\|^2_{L^2(0,t;H)} \le c\int_0^t \|\partial_t\chi_n(s) - \partial_t\chi(s)\|^2_{L^4(\Omega)}\,ds +$$
$$+ c\int_0^t \Big(1 + \|\partial_t\chi_n(s)\|^2_{L^4(\Omega)} + \|\overline{\vartheta}(s)\|^2 + \|\partial_t\chi(s)\|^2_{L^4(\Omega)}\Big)\|\widetilde{\vartheta}_n(s)\|^2_{V'}\,ds\,. \tag{4.21}$$

Then, on account of (4.20), we apply the Gronwall lemma to (4.21), and we conclude

$$\|\widetilde{\vartheta}_n\|_{L^\infty(0,\tau;V')\cap L^2(0,\tau;H)} \to 0\,, \tag{4.22}$$

as $n \to +\infty$.

Now, we set $\overline{\chi}_n := \mathcal{T}_2(\overline{\vartheta}_n) = \mathcal{T}(\chi_n)$, $\overline{\chi} := \mathcal{T}_2(\overline{\vartheta}) = \mathcal{T}(\chi)$, and $\widetilde{\chi}_n := \overline{\chi}_n - \overline{\chi}$.

Next, we consider the difference between the corresponding equations (4.4), we multiply it by $\partial_{tt}\widetilde{\chi}_n$ and we integrate over Q_t. Owing also to the continuous embedding $W \hookrightarrow L^\infty(\Omega)$ (cf. (4.16)), we find

$$\begin{aligned}&\|\partial_{tt}\widetilde{\chi}_n\|^2_{L^2(0,t;H)} + \frac{h}{2}\|\nabla\partial_t\widetilde{\chi}_n(t)\|^2 + \iint_{Q_t}(\alpha_\varepsilon(\partial_t\overline{\chi}_n) - \alpha_\varepsilon(\partial_t\overline{\chi}))\,\partial_{tt}\widetilde{\chi}_n + \\ &+ \frac{1}{2}\|\partial_t\widetilde{\chi}_n(t)\|^2 \le c\iint_{Q_t}|\widetilde{\chi}_n|\,|\partial_{tt}\widetilde{\chi}_n| + \iint_{Q_t}\widetilde{\vartheta}_n\,\partial_t\widetilde{\chi}_n + \iint_{Q_t}\Delta\widetilde{\chi}_n\,\partial_{tt}\widetilde{\chi}_n\,.\end{aligned} \tag{4.23}$$

We estimate the integral terms in (4.23) taking the Lipschitz continuity of α_ε (of Lipschitz constant $1/\varepsilon$) into account, performing some integrations by parts in space and time, and applying (2.14). In particular, we deal with the last term on the right-hand side of (4.23) as follows.

$$\begin{aligned}&\iint_{Q_t}\Delta\widetilde{\chi}_n\,\partial_{tt}\widetilde{\chi}_n = -\iint_{Q_t}\Delta\partial_t\widetilde{\chi}_n\,\partial_t\widetilde{\chi}_n + \int_\Omega\Delta\widetilde{\chi}_n(t)\,\partial_t\widetilde{\chi}_n(t)\,dx = \\ &= \iint_{Q_t}|\nabla\partial_t\widetilde{\chi}_n|^2 - \int_\Omega\nabla\widetilde{\chi}_n(t)\,\nabla\partial_t\widetilde{\chi}_n(t)\,dx \le c\|\nabla\partial_t\widetilde{\chi}_n\|^2_{L^2(0,t;H)} + \frac{h}{4}\|\nabla\partial_t\widetilde{\chi}_n(t)\|^2\,.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{1}{2}\|\partial_{tt}\widetilde{\chi}_n\|^2_{L^2(0,t;H)} + \frac{h}{4}\|\nabla\partial_t\widetilde{\chi}_n(t)\|^2 + \frac{1}{2}\|\partial_t\widetilde{\chi}_n(t)\|^2 \le \\ &\le c\left(\|\partial_t\widetilde{\chi}_n\|^2_{L^2(0,t;H)} + \|\nabla\partial_t\widetilde{\chi}_n\|^2_{L^2(0,t;H)} + \|\widetilde{\vartheta}_n\|^2_{L^2(0,t;H)}\right)\,,\end{aligned} \tag{4.24}$$

where c is a positive constant depending on ε and h (but independent of n) due to (2.14). Finally, we apply the Gronwall lemma to (4.24): the convergence specified by (4.22) allows us to deduce

$$\|\overline{\chi}_n - \overline{\chi}\|_{H^2(0,\tau;H)\cap W^{1,\infty}(0,\tau;V)} \to 0\,, \tag{4.25}$$

as $n \to +\infty$. Hence, in particular

$$\|\overline{\chi}_n - \overline{\chi}\|_{W^{1,4}(0,\tau;L^4(\Omega))} \to 0\,, \tag{4.26}$$

which concludes the proof of the continuity of the operator $\mathcal{T}$. Thus we have shown that $\mathcal{T}$ admits a fixed point in $Y(\tau, R)$ and, hence, that there exists a local-in-time solution $(\vartheta_{\varepsilon h}, \chi_{\varepsilon h})$ to Problem $(\mathbf{P}_\varepsilon^h)$.

Remark 2 A few comments on the approximate Problem $(\mathbf{P}_\varepsilon^h)$ are in order, concerning the technical reasons of the double approximation procedure involving two distinct parameters $\varepsilon > 0$ and $h > 0$.

We recall that in the fixed point scheme, we have performed the estimate (4.23) where the term $\iint_{Q_t} (\alpha_\varepsilon(\partial_t \overline{\chi}_n) - \alpha_\varepsilon(\partial_t \overline{\chi}))\, \partial_{tt} \widetilde{\chi}_n$ cannot be dealt by monotonicity arguments but by exploiting the Lipschitz continuity of α_ε. On the other hand, by introducing the Yosida regularization α_ε, we remove the sign constraint on $\partial_t \chi$ which will turn out to be crucial in dealing with the quadratic term $\vartheta \partial_t \chi$ (cf. with (5.17) below). Thus, we have to perform a first passage to the limit as $\varepsilon \searrow 0$ (and $h > 0$ is fixed): here the enhanced regularity of $\partial_t \chi$ ensured by the presence of the viscosity term (parameter $h > 0$) enables us to control the nonlinearities in the energy equation. Next, taking advantage on the sign constraint on $\partial_t \chi$, we can deduce some other a priori estimates independent of h and ultimately perform the further passage to the limit procedure leading to a solution to the original Problem $(\mathbf{P})$.

4.2 Global Existence (and Uniqueness) for Problem $(\mathbf{P}_\varepsilon^h)$

In order to show that the local solution to Problem $(\mathbf{P}_\varepsilon^h)$ actually extends to the whole time interval $(0, T)$, we shall prove a series of global in time estimates on $(\vartheta_{\varepsilon h}, \chi_{\varepsilon h})$. Then a standard prolongation argument guarantees that, for every $\varepsilon > 0$ and $h > 0$, the local solution extends to the (unique) global solution of Problem $(\mathbf{P}_\varepsilon^h)$. In fact, such global estimates shall be derived independently of $\varepsilon > 0$ so that they will provide the starting point for the passage to the limit as $\varepsilon \searrow 0$ in Sect. 5.1.

To this aim, in what follows, we assume also condition (2.4) and we let ε and h vary, say, in $(0, 1)$. Moreover, in order to simplify notation, we will directly work on the interval $(0, T)$ instead of $(0, \tau)$.

Positivity of the Temperature We show that $\vartheta_{\varepsilon h}$ is bounded from below by a positive constant c_* (independent of ε, h and τ) exploiting the comparison argument developed in [16, Section 4.2.1]. From (4.3) we have

$$\partial_t \vartheta_{\varepsilon h} - \Delta \vartheta_{\varepsilon h} = (\partial_t \chi_{\varepsilon h})^2 - \vartheta_{\varepsilon h} \partial_t \chi_{\varepsilon h} \geq -\frac{1}{2} (\vartheta_{\varepsilon h})^2 .$$

Moreover, we observe that the spatially homogeneous function $r := r(t)$ solving

$$\partial_t r = -\frac{1}{2} r^2 \qquad r(0) = c^*$$

(where $c^* > 0$ is given in (2.2)) is a sub-solution to (4.3) and hence

$$\exists c_* > 0 : \vartheta_{\varepsilon h}(\cdot, t) \geq r(t) \geq c_* \text{ for all } t \in [0, T]\,. \tag{4.27}$$

We note that such a constant $c_* > 0$ is independent of ε, h and τ (e.g. $c_* = r(T)$).

Global A Priori Estimates We first multiply (4.4) by $\partial_t \chi_{\varepsilon h}$; we add the resulting equation to (4.3) and we integrate over Q_t, with $0 < t < T$. Noting that some terms cancel, taking into account the monotonicity of α_ε and the fact that $\alpha_\varepsilon(0) = 0$, we have

$$\int_\Omega \vartheta_{\varepsilon h}(t)\, dx + \frac{1}{2}\|\partial_t \chi_{\varepsilon h}(t)\|^2 + h\|\nabla \partial_t \chi_{\varepsilon h}\|^2_{L^2(0,T;H)} + \frac{1}{2}\|\nabla \chi_{\varepsilon h}(t)\|^2 + \\ + \frac{1}{4}\int_\Omega (\chi^2_{\varepsilon h}(t) - 1)^2\, dx \leq \int_\Omega \vartheta_0\, dx + \frac{1}{2}\|\chi_1\|^2 + \frac{1}{2}\|\nabla \chi_0\|^2 + \frac{1}{4}\int_\Omega (\chi_0^2 - 1)^2\, dx\,. \tag{4.28}$$

We deduce the following upper bounds

$$\|\vartheta_{\varepsilon h}\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,, \tag{4.29}$$

$$\|\chi_{\varepsilon h}\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} \leq c\,, \tag{4.30}$$

$$h^{1/2}\|\partial_t \chi_{\varepsilon h}\|_{L^2(0,T;V)} \leq c\,, \tag{4.31}$$

where the positive constant c depends on the problem data but neither on ε nor h.

Next, we multiply (4.3) by $\vartheta_{\varepsilon h}$ and we integrate over Q_t, with $0 < t < T$. Arguing as in the derivation of (4.11), on account of (4.31), we get

$$\|\vartheta_{\varepsilon h}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c_h\,, \tag{4.32}$$

where the positive constant c_h (here and in the forthcoming estimates (4.33)–(4.35), (4.38), and (4.39)) depends on the problem data and h but it is independent of ε.

Moreover, we multiply (4.4) by $-\Delta \partial_t \chi_{\varepsilon h}$ and we integrate over Q_t, with $0 < t < T$. Developing the very same calculations as for (4.15) and (4.16), thanks to (4.30) and (4.32), we end up with

$$\|\chi_{\varepsilon h}\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \leq c_h\,. \tag{4.33}$$

Now we multiply (4.4) by $\partial_{tt}\chi_{\varepsilon h}$, we integrate over Q_t, with $0 < t < T$, and we proceed as in the derivation of (4.18). Noting that the term $\int_\Omega \widehat{\alpha}_\varepsilon(\chi_1)\, dx = 0$ thanks to (2.4), in view of (4.32) and (4.33), we obtain

$$\|\chi_{\varepsilon h}\|_{H^2(0,T;H)} \leq c_h\,. \tag{4.34}$$

A comparison in (4.4), thanks to (4.32)–(4.34), gives moreover

$$\|\alpha_\varepsilon(\partial_t \chi_{\varepsilon h})\|_{L^2(0,T;H)} \le c_h \,. \tag{4.35}$$

Finally, we multiply (4.3) by $\partial_t \vartheta_{\varepsilon h}$ and we integrate over Q_t, with $0 < t < T$. Applying the Hölder inequality, we have

$$\begin{aligned} &\|\partial_t \vartheta_{\varepsilon h}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla \vartheta_{\varepsilon h}(t)\|^2 \le \frac{1}{2}\|\nabla \vartheta_0\|^2 + \\ &+ \int_0^t \Big(\|\partial_t \chi_{\varepsilon h}(s)\|^2_{L^4(\Omega)} + \|\vartheta_{\varepsilon h}(s)\|_{L^4(\Omega)} \, \|\partial_t \chi_{\varepsilon h}(s)\|_{L^4(\Omega)} \Big) \|\partial_t \vartheta_{\varepsilon h}(s)\| \, ds \,. \end{aligned} \tag{4.36}$$

Next, in order to recover the full V-norm of $\vartheta_{\varepsilon h}$ on the left-hand side, we add $\|\vartheta_{\varepsilon h}(t)\|^2$ to both sides of (4.36). Then, owing to the continuous embedding $V \hookrightarrow L^4(\Omega)$ and using (2.14), we get

$$\begin{aligned} &\frac{1}{2}\|\partial_t \vartheta_{\varepsilon h}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\vartheta_{\varepsilon h}(t)\|^2_V \le \frac{1}{2}\|\nabla \vartheta_0\|^2 + \\ &+ c\int_0^t \|\partial_t \chi_{\varepsilon h}(s)\|^2_{L^4(\Omega)} \|\vartheta_{\varepsilon h}(s)\|^2_V \, ds + c\int_0^t \|\partial_t \chi_{\varepsilon h}(s)\|^4_{L^4(\Omega)} \, ds + c \le \\ &\le \frac{1}{2}\|\nabla \vartheta_0\|^2 + c\|\partial_t \chi_{\varepsilon h}\|^2_{L^\infty(0,T;V)} \, \|\vartheta_{\varepsilon h}\|^2_{L^2(0,T;V)} + c\|\partial_t \chi_{\varepsilon h}\|^4_{L^\infty(0,T;V)} + c \,. \end{aligned} \tag{4.37}$$

Taking (4.32) and (4.33) into account, we infer

$$\|\vartheta_{\varepsilon h}\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \le c_h \,. \tag{4.38}$$

By comparison in (4.3), in view of (4.33) and (4.38), we infer moreover

$$\|\vartheta_{\varepsilon h}\|_{L^2(0,T;W)} \le c_h \,. \tag{4.39}$$

The previous estimates, by a standard prolongation argument, ensure that, for every $\varepsilon > 0$ and $h > 0$, the local solution of Problem $(\mathbf{P}^h_\varepsilon)$ extends to a global one defined on the whole interval $(0, T)$. Actually, such a solution is unique: this can be proved following the very same lines of the uniqueness proof of Problem **(P)** (Sect. 3).

5 Proof of Theorem 1

As already mentioned, we shall obtain a solution of Problem **(P)** by two consecutive limit procedures. In next subsections, we shall pass to the limit in Problem $(\mathbf{P}^h_\varepsilon)$ first as $\varepsilon \searrow 0$ (and $h > 0$ is fixed) and then as $h \searrow 0$.

5.1 Passage to the Limit as $\varepsilon \searrow 0$

First of all, we recall that all the previous estimates are derived independently of ε. Then, letting $h > 0$ be fixed, we collect here the convergences in ε which can be derived from (4.33)–(4.35), (4.38)–(4.39). Well-known weak and weak* compactness results allow us to deduce the following convergences, at least for subsequences

$$\vartheta_{\varepsilon h} \overset{*}{\rightharpoonup} \vartheta_h \ \text{ in } \ H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{5.1}$$

$$\chi_{\varepsilon h} \overset{*}{\rightharpoonup} \chi_h \ \text{ in } \ H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap L^\infty(0,T;W) \tag{5.2}$$

$$\partial_t \chi_{\varepsilon h} \rightharpoonup \partial_t \chi_h \ \text{ in } \ L^2(0,T;W) \tag{5.3}$$

$$\alpha_\varepsilon(\partial_t \chi_{\varepsilon h}) \rightharpoonup \xi_h \ \text{ in } \ L^2(0,T;H)\,, \tag{5.4}$$

as $\varepsilon \searrow 0$.

Moreover, using a classical compactness argument (see [20]) and the generalized Ascoli theorem (see [25, Cor. 4]), we also obtain the following strong convergences

$$\vartheta_{\varepsilon h} \to \vartheta_h \ \text{ in } \ C^0([0,T];H^{1-\delta}(\Omega)) \cap L^2(0,T;H^{2-\delta}(\Omega)) \quad \text{for all } \delta > 0 \tag{5.5}$$

$$\chi_{\varepsilon h} \to \chi_h \ \text{ in } \ C^0([0,T];H^{2-\delta}(\Omega)) \quad \text{for all } \delta > 0 \tag{5.6}$$

$$\partial_t \chi_{\varepsilon h} \to \partial_t \chi_h \ \text{ in } \ C^0([0,T];H^{1-\delta}(\Omega)) \cap L^2(0,T;H^{2-\delta}(\Omega)) \quad \text{for all } \delta > 0\,. \tag{5.7}$$

Thanks to (5.4) and (5.7), it holds

$$\lim_{\varepsilon \to 0} \iint_Q \alpha_\varepsilon(\partial_t \chi_{\varepsilon h})\, \partial_t \chi_{\varepsilon h} = \iint_Q \xi_h\, \partial_t \chi_h \tag{5.8}$$

and then

$$\xi_h \in \alpha(\partial_t \chi_h) \quad \text{ a.e. in } Q \tag{5.9}$$

(i.e. $\xi_h \in \partial I_{[0,+\infty[}(\partial_t \chi_h)$ a.e. in Q), in view of [2, prop.1.1, p.42].

The previous convergences ensure that the triple $(\vartheta_h, \chi_h, \xi_h)$ fulfil

$$\partial_t \vartheta_h + \vartheta_h \partial_t \chi_h - \Delta \vartheta_h = (\partial_t \chi_h)^2 \quad \text{ a.e. in } Q\,, \tag{5.10}$$

$$\partial_{tt} \chi_h - h \Delta \partial_t \chi_h + \partial_t \chi_h - \Delta \chi_h + \xi_h + \chi_h^3 - \chi_h = \vartheta_h \quad \text{ a.e. in } Q\,, \tag{5.11}$$

as well as relation (5.9) and the initial conditions (4.5)–(4.7). Finally, we note that the strong convergence (5.5) guarantees the strict positivity of ϑ_h, since it holds for $\vartheta_{\varepsilon h}$ independently of ε and h (cf. with (4.27)).

Hereafter, we shall call ($\mathbf{P}^h$) the initial boundary value problem associated with (5.10)–(5.11), supplemented by relation (5.9).

5.2 Passage to the Limit as $h \searrow 0$ and Conclusion of the Proof of Theorem 1

We are now going to show that the sequence $\{(\vartheta_h, \chi_h, \xi_h)\}_h$ of solutions to Problem ($\mathbf{P}^h$) obtained in the previous subsection admits a subsequence converging as $h \searrow 0$ to a solution of Problem (**P**). To this aim, we derive here a series of a priori estimates independent of h.

Following the same arguments for (4.29) and (4.30), we infer

$$\|\vartheta_h\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \tag{5.12}$$

$$\|\chi_h\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} \le c\,, \tag{5.13}$$

for a positive constant c independent of h. Moreover, we are going to prove that

$$\|\vartheta_h\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\,, \tag{5.14}$$

$$\|\chi_h\|_{W^{1,\infty}(0,T;V)\cap L^\infty(0,T;W)} \le c\,, \tag{5.15}$$

$$h^{1/2}\|\Delta\partial_t\chi_h\|_{L^2(0,T;H)} \le c\,, \tag{5.16}$$

for some positive constant c independent of h. We combine here two estimates: we multiply (5.10) by ϑ_h, (5.11) by $-\Delta\partial_t\chi_h$, we add the resulting equations, and we integrate over Q_t, with $0 < t < T$. We have

$$\frac{1}{2}\|\vartheta_h(t)\|^2 + \|\nabla\vartheta_h\|^2_{L^2(0,t;H)} + \iint_{Q_t} \vartheta_h^2\partial_t\chi_h + \frac{1}{2}\|\nabla\partial_t\chi_h(t)\|^2 +$$
$$+ h\|\Delta\partial_t\chi_h\|^2_{L^2(0,t;H)} + \|\nabla\partial_t\chi_h\|^2_{L^2(0,t;H)} + \iint_{Q_t} \xi_h(-\Delta\partial_t\chi_h) + \frac{1}{2}\|\Delta\chi_h(t)\|^2 \le$$
$$\le \frac{1}{2}\left(\|\vartheta_0\|^2 + \|\nabla\chi_1\|^2 + \|\Delta\chi_0\|^2\right) + |I_4| + |I_5|\,, \tag{5.17}$$

where the latter summands are estimated as follows by using the Hölder inequality, standard continuous embeddings, (2.14), and (5.13).

$$I_4 = \iint_{Q_t} \vartheta_h(\partial_t\chi_h)^2 \le \int_0^t \|\vartheta_h(s)\|_{L^4(\Omega)}\|\partial_t\chi_h(s)\|_{L^4(\Omega)}\|\partial_t\chi_h(s)\|\,ds \le$$
$$\le \frac{1}{4}\|\vartheta_h\|^2_{L^2(0,t;V)} + c\int_0^t \|\partial_t\chi_h(s)\|^2_V\,\|\partial_t\chi_h(s)\|^2\,ds \le$$

$$
\begin{aligned}
&\leq \frac{1}{4}\|\vartheta_h\|^2_{L^2(0,t;V)} + c\|\partial_t \chi_h\|^2_{L^2(0,t;V)} \\
&I_5 = \iint_{Q_t} \Big(\chi_h^3 - \chi_h - \vartheta_h\Big)(\Delta \partial_t \chi_h) \leq \iint_{Q_t} \Big((3\chi_h^2 + 1)|\nabla \chi_h| + |\nabla \vartheta_h|\Big)|\nabla \partial_t \chi_h| \leq \\
&\leq \frac{1}{4}\|\vartheta_h\|^2_{L^2(0,t;V)} + \\
&+ c\|\nabla \partial_t \chi_h\|^2_{L^2(0,t;H)} + c\int_0^t \Big(\|\chi_h(s)\|^2_{L^6(\Omega)} + 1\Big)\, \|\nabla \chi_h(s)\|_{L^6(\Omega)} \|\nabla \partial_t \chi_h(s)\|\, ds \leq \\
&\leq \frac{1}{4}\|\vartheta_h\|^2_{L^2(0,t;V)} + c\|\chi_h\|^2_{L^2(0,t;W)} + c\|\nabla \partial_t \chi_h\|^2_{L^2(0,t;H)}\,. \qquad (5.18)
\end{aligned}
$$

Now, we remark that the third integral term on the left-hand side of (5.17) is non-negative due to the sign constraint on $\partial_t \chi_h$ ensured by inclusion (5.9). Moreover, thanks to the monotonicity of $\partial I_{[0,+\infty[}$, the seventh integral term is non-negative too [23, Lemma 4.1]. Then, we apply the Gronwall lemma to (5.17) and we deduce (5.14)–(5.16).

Next, arguing as for (4.17), we multiply (5.11) by $\partial_{tt}\chi_h$ and we integrate over Q_t, with $0 < t < T$. We have

$$
\begin{aligned}
&\|\partial_{tt}\chi_h\|^2_{L^2(0,t;H)} + \frac{h}{2}\|\nabla \partial_t \chi_h(t)\|^2 + \frac{1}{2}\|\partial_t \chi_h(t)\|^2 + \int_\Omega \widehat{\alpha}(\partial_t \chi_h(t))\, dx \leq \\
&\leq \frac{h}{2}\|\nabla \chi_1\|^2 + \frac{1}{2}\|\chi_1\|^2 + \int_\Omega \widehat{\alpha}(\chi_1)\, dx + \\
&+ c\Big(\|\Delta \chi_h\|^2_{L^2(0,t;H)} + \|\chi_h^3 - \chi_h\|^2_{L^2(0,t;H)} + \|\vartheta_h\|^2_{L^2(0,t;H)}\Big) + \frac{1}{2}\|\partial_{tt}\chi\|^2_{L^2(0,t;H)}\,, \qquad (5.19)
\end{aligned}
$$

where we have used the chain rule [15, Lemma 4.1] for the functional $\widehat{\alpha}$. We only remark that $\widehat{\alpha}(\chi_1) = 0$ a.e. in Ω because $\chi_1 \geq 0$ a.e. in Ω (cf. with (2.4)). In view of (5.14) and (5.15), we deduce

$$
\|\chi_h\|_{H^2(0,T;H)} \leq c\,, \qquad (5.20)
$$

and, by comparison in (5.11), we also have

$$
\|\xi_h\|_{L^2(0,T;H)} \leq c\,, \qquad (5.21)
$$

for some positive constant c independent of h.

Finally, following the same calculations for (4.38) and (4.39), we deduce

$$\|\vartheta_h\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le c\,, \tag{5.22}$$

for some positive constant c independent of h.

All of the above estimates, the Ascoli-Arzelà theorem, [25, Cor. 4], and standard weak and weak* compactness results yield that there exist a subsequence of $\{(\vartheta_h, \chi_h, \xi_h)\}_h$ (which we do not relabel) and functions (ϑ, χ, ξ) for which (5.1), (5.2), (5.5), (5.6) and

$$\xi_h \rightharpoonup \xi \ \text{ in } \ L^2(0,T;H) \tag{5.23}$$

$$\partial_t \chi_h \to \partial_t \chi \ \text{ in } \ C^0([0,T];H^{1-\delta}(\Omega)) \quad \text{for all } \delta > 0 \tag{5.24}$$

$$h\Delta\partial_t \chi_h \to 0 \ \text{ in } \ L^2(0,T;H)\,, \tag{5.25}$$

hold, as $h \searrow 0$. All the previous convergences ensure the passage to the limit in (5.10)–(5.11). Then, the triple (ϑ, χ, ξ) fulfils (2.5)–(2.9) as well as the initial conditions (2.11)–(2.13). Moreover, the identification (2.10) can be proved by the very same argument leading to (5.9). Thus, we conclude that the triple (ϑ, χ, ξ) is a solution to Problem (**P**). Again, we observe that the strict positivity of ϑ is preserved because the lower bound $c_* > 0$ is independent of h too (cf. with (4.27)). Thus, Theorem 1 is completely proved.

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Perimeter Symmetrization of Some Dynamic and Stationary Equations Involving the Monge-Ampère Operator

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Abstract We apply the perimeter symmetrization to a two-dimensional pseudo-parabolic dynamic problem associated to the Monge-Ampère operator as well as to the second order elliptic problem which arises after an implicit time discretization of the dynamical equation. Curiously, the dynamical problem corresponds to a third order operator but becomes a singular second order parabolic equation (involving the 3-Laplacian operator) in the class of radially symmetric convex functions. Using symmetrization techniques some quantitative comparison estimates and several qualitative properties of solutions are given.

Keywords Perimeter symmetrization • Pseudoparabolic dynamic Monge-Ampère equation • Two-dimensional domain

AMS (MOS) Subject Classification 35K55, 35J65, 35B05

1 Introduction

Starting with the pioneering paper by Giorgio Talenti [37] in 1981, many results were obtained concerning the comparison of solutions to some stationary equations, which can be written in terms of suitable perturbations of the Monge-Ampère operator in a general domain, with the radially symmetric solutions to some auxiliary stationary boundary value problems on an associated ball. In contrast with

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the case of many stationary problems given by operators in divergence form, the main tool is not the Schwarz (neither the Steiner) radially symmetric rearrangement of the solution but now the *perimeter rearrangement* of that function (see, e.g., [9, 12, 13, 19, 41, 42]).

The main difficulty to extend the previous papers concerning several stationary problems to the case of parabolic problems comes from the fact that it seems very complicated to relate the terms

$$\frac{d}{dt}\int_{u<\theta} u(x,t)\,dx \quad \text{and} \quad \frac{d}{dt}\int_{u^\star<\theta} u^\star(x,t)\,dx$$

when $u^\star(\cdot,t)$ is the rearrangement of $u(\cdot,t)$ with respect to the perimeter of its level sets. This contrasts with what happens in the case of the Schwarz radially symmetric rearrangement (since there, by construction, both level sets $\{u<\theta\}$ and $\{u^\star<\theta\}$ keep the same measure): see, e.g. the results relating both time differential terms by Bandle [4, 5], Mossino-Rakotoson [32] and Nagai [33], among many other authors.

Due to that, and following a previous work by Brandolini [10], we shall consider the following dynamic problem associated to the Monge-Ampère operator:

$$\begin{cases} (k_u(x,t)u)_t - \det D^2u = f(x,t) & \text{in } \Omega\times(0,+\infty)\\ u=0 & \text{on } \partial\Omega\times(0,+\infty)\\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here the subscript t means the derivative with respect to the time variable t, Du means the gradient of u with respect to the space variables $x=(x_1,x_2)\in\mathbb{R}^2$, D^2u denotes the Hessian matrix of u with respect to x and $k_u(\cdot,t)=\operatorname{div}\left(\frac{Du(\cdot,t)}{|Du(\cdot,t)|}\right)$ is the curvature of the level line of $u(\cdot,t)$ passing through the point $(\cdot,t)$. As we shall justify later, our main interest will focus on negative convex solutions to problem (1.1).

Notice that problem (1.1) is a pseudoparabolic dynamic problem and that, as we shall see, curiously enough, this third order operator becomes a singular second order parabolic equation (involving the 3-Laplacian operator) in the class of radially symmetric convex functions. For some recent results for other pseudoparabolic problems see e.g. [34]. The main reason for the consideration of the penalization factor $k_u(\cdot,t)$ in the inertia term comes from the fact that, in the class of radially symmetric functions, $\det D^2u$ behaves, formally, in a similar manner to the expression

$$\frac{1}{|x|}\operatorname{div}(|Du|\,Du)$$

and, as we shall show, this is exactly the behavior that brings, in the class of convex radially symmetric functions, the product $k_u\,u$.

The first goal of this work (a preliminary version of an extended paper Brandolini-Díaz [11]) is to obtain some quantitative comparison estimates for the solution u to (1.1) and the solution z to the symmetrized problem, sharpening in this way the results in [10]. Moreover, we shall extend the mentioned comparison result to the case of negative convex solutions to the stationary Dirichlet problem

$$\begin{cases} -\det D^2u + k_u u = f \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We shall give also many indications on the existence and uniqueness of solutions to problem (1.2); nevertheless, for the limited extension of this work, we shall delay to [11] the presentation of the corresponding indications for the dynamic problem (1.1). Finally, we shall apply the rearrangement comparison results in order to get some qualitative properties of solutions to (1.2) and (1.1).

2 Preliminary Results

2.1 Rearrangements and Main Properties

First of all we recall the definition of decreasing rearrangement of a measurable function $\varphi : \Omega \to \mathbb{R}$, where Ω is a bounded open subset of $\mathbb{R}^2$ with measure A. The distribution function of φ is defined by

$$\mu_\varphi(\theta) = |\{x \in \Omega : |\varphi(x)| > \theta\}|, \qquad \theta \geq 0,$$

while the decreasing rearrangement of φ is defined as the generalized left-continuous inverse of μ_φ, i. e.

$$\varphi^*(s) = \inf\{\theta \geq 0 : \mu_\varphi(\theta) < s\}, \qquad s \in [0, +\infty[.$$

Note that $\varphi^*(s) = 0$ if $s \geq A$. By definition, φ and φ^* are equidistributed functions, that is they share the same distribution function. In particular, φ^* is the unique decreasing left-continuous function in $[0, +\infty[$ equidistributed with φ.

By using the previous notions we can also introduce the decreasing spherically symmetric rearrangement of φ, also known as Schwarz symmetrand of φ, as follows

$$\varphi^\sharp(x) = \varphi^*(\pi|x|^2), \qquad x \in \Omega^\sharp,$$

where $\Omega^\sharp$ denotes the disc, centred at the origin, having the same measure A as Ω. By definition, $\varphi^\sharp$ is the unique spherically symmetric function, which is decreasing along the radii and equidistributed with φ.

Being φ, φ^* and $\varphi^\sharp$ equidistributed, if $\varphi \in L^p(\Omega)$ for some $p \in [1, +\infty[$, clearly it holds true that

$$||\varphi||_{L^p(\Omega)} = ||\varphi^*||_{L^p(0,A)} = ||\varphi^\sharp||_{L^p(\Omega^\sharp)}.$$

The theory of rearrangements is well-known and exhaustive treatments can be found, for example, in [31] or [38]. Here we just recall the following celebrated inequality that will be useful in the sequel.

Proposition 1 (Hardy-Littlewood Inequality) *Let φ, ψ be measurable functions in Ω. Then*

$$\int_\Omega |\varphi(x)\psi(x)|\, dx \le \int_0^{+\infty} \varphi^*(s)\psi^*(s)\, ds = \int_{\Omega^\sharp} \varphi^\sharp(x)\psi^\sharp(x)\, dx.$$

The above definitions will be useful in the following sections, but the crucial notion we are dealing with concerns the perimeter $\lambda_\varphi(\theta)$ of the level sets of φ. From now on we consider a bounded, convex, open set Ω in $\mathbb{R}^2$ and we denote by L its perimeter. Let φ be a smooth convex function in Ω, vanishing on the boundary; the sublevel sets of such a function φ are convex subsets of Ω and their perimeter $\lambda_\varphi(\theta)$ coincides with

$$\text{length}\,\{x \in \Omega : \varphi(x) = \theta\}, \qquad \theta \le 0.$$

We define

$$\tilde{\varphi}(s) = \sup\{\theta \le 0 : \lambda_\varphi(\theta) < s\}, \qquad s \in [0, L] \tag{2.3}$$

and the rearrangement of φ with respect to the perimeter of its level sets as

$$\varphi^\star(x) = \tilde{\varphi}(2\pi|x|), \qquad x \in \Omega^\star,$$

where $\Omega^\star$ is the disc, centred at the origin, with the same perimeter L as Ω (we explicitly observe that $\Omega^\sharp \subseteq \Omega^\star$). Differently from φ^*, $\tilde{\varphi}$ is in general not equidistributed with φ. But, the classical isoperimetric inequality states that

$$\mu_\varphi(-\theta) \le \frac{1}{4\pi}\lambda_\varphi(\theta)^2, \qquad \theta \le 0, \tag{2.4}$$

and then $\min_\Omega \varphi = \min_{[0,L]} \tilde{\varphi} = \min_{\Omega^\star} \varphi^\star$, while for every $1 \le p < +\infty$

$$||\varphi||_{L^p(\Omega)} \le \frac{1}{2\pi}||s\tilde{\varphi}(s)||_{L^p(0,L)} = ||\varphi^\star||_{L^p(\Omega^\star)}.$$

The perimeter function $\lambda_\varphi(\theta)$ and the rearrangement $\tilde{\varphi}(s)$ defined by (2.3) satisfy some properties analogous to those ones of the distribution function $\mu_\varphi(\theta)$ and the decreasing rearrangement $\varphi^*(s)$. For the seek of simplicity and completeness we list some of these properties below (see [37, 41, 42]).

Proposition 2 (Regularity Properties) *Let Ω be a bounded, convex, open set in $\mathbb{R}^2$ and let $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ be convex functions, vanishing on the boundary of Ω.*

i) $\lambda_\varphi(\theta) \in C([\min_\Omega \varphi, 0]) \cap C^2([\min_\Omega \varphi, 0))$*; moreover it is an increasing, concave function on the interval* $[\min_\Omega \varphi, 0]$ *and* $\lambda_\varphi(\min_\Omega \varphi) = 0$, $\lambda_\varphi(0) = L$*;*

ii) *for every* $\theta \in (\min_\Omega \varphi, 0)$

$$\lambda'_\varphi(\theta) = \int_{\varphi=\theta} \frac{k_\varphi}{|D\varphi|} \tag{2.5}$$

where, denoted $x = (x_1, x_2) \in \mathbb{R}^2$,

$$k_\varphi = \operatorname{div}\left(\frac{D\varphi}{|D\varphi|}\right) = |D\varphi|^{-3}\left(\begin{pmatrix} \varphi_{x_2x_2} & -\varphi_{x_1x_2} \\ -\varphi_{x_1x_2} & \varphi_{x_1x_1} \end{pmatrix} D\varphi, D\varphi\right) \geq 0$$

is the curvature of the level line $\{\varphi = \theta\}$*;*

iii) $\tilde{\varphi}(\lambda_\varphi(\theta)) = \theta$ *for every* $\theta \in [\min_\Omega \varphi, 0]$*;*

iv) $\tilde{\varphi} \in C([0, L]) \cap C^2([0, L))$*; it is an increasing, convex function on the interval* $[0, L]$ *and* $\tilde{\varphi}(0) = \min_\Omega \varphi$, $\tilde{\varphi}(L) = 0$*; moreover*

$$0 \leq \tilde{\varphi}'(s) \leq \frac{1}{2\pi} \sup_\Omega |D\varphi|, \qquad s \in [\min_\Omega \varphi, 0);$$

v) $\varphi^\star \in C(\overline{\Omega^\star}) \cap C^2(\Omega^\star)$*; moreover it is a convex function on* $\Omega^\star$ *and it vanishes on the boundary of* $\Omega^\star$.

Proposition 3 (General Properties of Rearrangements) *Under the same assumptions of Proposition 2, it holds that:*

vi) *if* $\varphi \leq \psi$ *in* Ω*, then* $\tilde{\varphi} \leq \tilde{\psi}$ *in* $[0, L]$*;*

vii) *for every* $c > 0$, $\widetilde{(c\varphi)} = c\tilde{\varphi}$*;*

viii) *for every* $c \in \mathbb{R}$, $\widetilde{(\varphi + c)} = \tilde{\varphi} + c$*;*

ix) *if* $\Psi : (-\infty, 0] \to (-\infty, 0]$ *is a strictly increasing, continuous, convex function, then*

$$\widetilde{(\Psi \circ \varphi)} = \Psi(\tilde{\varphi});$$

x) *for every* $s \in [0, A]$, $\tilde{\varphi}(2\pi\sqrt{s}) \leq (-\varphi^*(s))$*;*

xi) *the rearrangement operator is continuous from* $L^\infty(\Omega)$ *to* $L^p(0, L)$ *and for every* $s \in [0, L]$

$$|\tilde{\varphi}(s) - \tilde{\psi}(s)| \le ||\varphi - \psi||_{L^\infty(\Omega)}.$$

Proposition 4 *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a convex function, vanishing on the boundary of* Ω*. For every convex subset* E *of* Ω *with perimeter* $P(E)$ *it holds*

$$\int_E \varphi(x)\, dx \ge \frac{1}{2\pi} \int_0^{P(E)} s\, \tilde{\varphi}(s)\, ds.$$

Proposition 5 (Hardy-Littlewood Type Inequality) *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be convex functions, vanishing on the boundary of* Ω*. Then*

$$\int_\Omega \varphi(x)\psi(x)\, dx \le \frac{1}{2\pi} \int_0^L s\, \tilde{\varphi}(s)\tilde{\psi}(s)\, ds = \int_{\Omega^\star} \varphi^\star(x)\psi^\star(x)\, dx. \tag{2.6}$$

Remark 1 Actually, inequality (2.6) can be improved as follows

$$\int_\Omega \varphi(x)\psi(x) dx \le \int_{\Omega^\star} \varphi^\star(x) \left(-\psi^\sharp(x)\right) dx.$$

Proposition 6 (Pólya-Szegő Type Inequality) *Let* Ω *be a bounded, convex, open set in* $\mathbb{R}^2$ *and let* $\varphi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a convex function, vanishing on the boundary of* Ω*. Then*

$$\int_\Omega (-\varphi) \det D^2\varphi\, dx \ge 2\pi \int_0^L (\tilde{\varphi}(s)')^3 ds = \int_{\Omega^\star} (-\varphi^\star) \det D^2\varphi^\star dx,$$

equality holding if Ω *is a disc.*

Proof By a direct computation it is easy to verify that the Hessian determinant of φ can be written in divergence form as follows

$$\det D^2\varphi = \frac{1}{2} \operatorname{div} \left\{ \begin{pmatrix} \varphi_{x_2x_2} & -\varphi_{x_1x_2} \\ -\varphi_{x_1x_2} & \varphi_{x_1x_1} \end{pmatrix} D\varphi \right\}. \tag{2.7}$$

Then, by using divergence theorem and co-area formula, we obtain

$$\int_\Omega (-\varphi) \det D^2\varphi\, dx = \frac{1}{2} \int_\Omega k_\varphi |D\varphi|^3\, dx = \frac{1}{2} \int_{-\infty}^0 d\theta \int_{\varphi=\theta} k_\varphi |D\varphi|^2.$$

By Hölder inequality and (2.5) we get

$$\int_{\varphi=\theta} k_\varphi |D\varphi|^2 \geq \frac{\left(\int_{\varphi=\theta} k\right)^3}{\left(\lambda'_\varphi(\theta)\right)^2}.$$

Gauss-Bonnet theorem ensures that

$$\int_{\varphi=\theta} k = 2\pi.$$

Thus

$$\int_{\varphi=\theta} k_\varphi |D\varphi|^2 \geq \frac{8\pi^3}{\left(\lambda'_\varphi(\theta)\right)^2} = 8\pi^3 (\tilde{\varphi}(s)'|_{s=\lambda_\varphi(\theta)})^2 = \int_{\varphi^\star=\theta} k_{\varphi^\star} |D\varphi^\star|^2,$$

and the thesis immediately follows. □

Proposition 7 *Let Ω be a bounded, convex, open set in $\mathbb{R}^2$ and let $\varphi, \psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ be convex functions, vanishing on the boundary of Ω. Then, the following statements are equivalent:*

1) $\int_0^s r\tilde{\varphi}(r)\,dr \leq \int_0^s r\tilde{\psi}(r)\,dr$, *for* $s \in [0, L]$*;*
2) for every increasing, negative function $\phi \in C^1([0, L])$ *such that* $\phi(L) = 0$,

$$\int_0^L s\tilde{\varphi}(s)\phi(s)\,ds \geq \int_0^L s\tilde{\psi}(s)\phi(s)\,ds.$$

Proof 1) ⇒ 2) is a consequence of the following identity

$$\int_0^L s\tilde{\varphi}(s)\phi(s)\,ds = -\int_0^L \left(\int_0^s r\tilde{\varphi}(r)\,dr\right) d\phi(s) + \phi(L)\int_0^L s\tilde{\varphi}(s)\,ds.$$

2) ⇒ 1) is deduced from Proposition 4 above, after observing that if $\psi = \chi_E$ and $P(E) = s$, then $\tilde{\psi} = -\chi_{[0,s[}$. □

2.1.1 Accretive Operators in Banach Spaces

We start this subsection recalling some definitions contained in [18].

Let $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}(N) \to \mathbb{R}$, where $\mathscr{S}(N)$ is the set of symmetric $N \times N$ matrices. We recall that F is said to be proper if

$$F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever} \quad r \leq s,$$

and F is said degenerate elliptic if

$$F(x,r,p,X) \leq F(x,r,p,Y) \quad \text{whenever} \quad Y \leq X.$$

Lemma 1 *The formal operator*

$$F(u,Du,D^2u) = -\det D^2u + k_u u$$

is degenerate elliptic and proper in the class of C^2, convex and negative functions u.

Proof This property was already shown for the Monge-Ampère part $F_1(u,Du,D^2u) := -\det D^2u$ in [18]. So it remains to prove it for the part

$$F_2(u,Du,D^2u) := k_u\, u = u\,\mathrm{div}\left(\frac{Du}{|Du|}\right),$$

that can be written in the class of negative functions as follows:

$$F_2(u,Du,D^2u) = -\,|u|\ \mathrm{div}\left(\frac{Du}{|Du|}\right).$$

It is well-known that the Laplacian acts as an ordinary differential operator along the lines of steepest descent; more precisely, the value of Δu at a point only involves derivatives of u along the line of steepest descent passing through that point and the mean curvature of the level line through the point:

$$\Delta u = |Du|\mathrm{div}\left(\frac{Du}{|Du|}\right) + \frac{D^2u\, Du \cdot Du}{|Du|^2},$$

that is

$$\mathrm{div}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|}\,\mathrm{trace}\left[\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2u\right].$$

Then, if $r \leq 0$, we get that the operator

$$F_2(r,p,X) = -\frac{|r|}{|p|}\mathrm{trace}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right]$$

is decreasing in X (for r and p prescribed), that is F_2 is degenerate elliptic. Analogously, in the class of convex functions we can assume that $\mathrm{div}\left(\frac{Du}{|Du|}\right) \geq 0$ and then $F_2(r,p,X)$ is increasing in r (for p and X prescribed) so it is proper. □

Now we recall some definitions and properties about accretive and T-accretive operators first abstractly, then in C^0 and L^∞. For all the proofs and applications of the theory of accretive operators to both elliptic and parabolic equations we remind the interested reader for instance to [6, 7, 16, 17, 29].

Let $\mathbb{X}$ be a real Banach space with norm $||\cdot||$ and let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$. A is said to be accretive in $\mathbb{X}$ if

$$||x-\hat{x}|| \le ||x-\hat{x}+\lambda(A(x)-A(\hat{x}))||, \qquad \text{for all } x,\hat{x} \in D(A),\ \lambda > 0.$$

If, in addition, $R(I+\lambda A) = \mathbb{X}$ for some $\lambda > 0$, A is m-accretive, in which case $R(I+\lambda A) = \mathbb{X}$ for all $\lambda > 0$.

For $x, y \in \mathbb{X}$ we define the pairing

$$[y,x]_+ = \inf_{\lambda>0} \frac{||x+\lambda y|| - ||x||}{\lambda}.$$

Clearly, $[\cdot,\cdot]_+ : \mathbb{X}\times\mathbb{X} \to \mathbb{R}$ is upper semicontinuous and A is accretive if and only if

$$[A(x)-A(\hat{x}), x-\hat{x}]_+ \ge 0, \qquad x,\hat{x} \in \mathbb{X}.$$

Moreover, the accretiveness of A in $\mathbb{X}$ can be determined by the normalized duality map. Indeed, if $\mathbb{X}'$ is the dual space of $\mathbb{X}$, then it can be proved that

$$[y,x]_+ = \max_{f\in H(x)} < f, y >_{\mathbb{X}',\mathbb{X}}, \tag{2.8}$$

where $H(x) = \{f \in \mathbb{X}' : ||f||_{\mathbb{X}'} = 1,\ < f, x >_{\mathbb{X}',\mathbb{X}} = ||x||\}$. So, A is accretive if and only if there exists $f \in \mathbb{X}'$, $||f||_{\mathbb{X}'} = 1$, and

$$< f, x-\hat{x} >_{\mathbb{X}',\mathbb{X}} = ||x-\hat{x}||, \quad < f, A(x)-A(\hat{x}) >_{\mathbb{X}',\mathbb{X}} \ge 0, \qquad x,\hat{x} \in \mathbb{X}.$$

Finally, A is said to be T-accretive (T stands for truncation) in $\mathbb{X}$ if

$$||(x-\hat{x})_+|| \le ||(x-\hat{x}+\lambda(A(x)-A(\hat{x})))_+||, \qquad \text{for all } x,\hat{x} \in D(A),\ \lambda > 0.$$

Here $a_+ = \max\{a, 0\}$. Equivalently, A is T-accretive in $\mathbb{X}$ if there exists $f \in \mathbb{X}'$, $f \ge 0$, $||f||_{\mathbb{X}'} = 1$, and

$$< f, x-\hat{x} >_{\mathbb{X}',\mathbb{X}} = ||(x-\hat{x})_+||, \quad < f, A(x)-A(\hat{x}) >_{\mathbb{X}',\mathbb{X}} \ge 0, \qquad x,\hat{x} \in \mathbb{X}.$$

If $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\mathbb{X} = C(\overline{\Omega})$, equipped with the supremum norm, the following representation holds

$$[v,u]_+ = \max\{v(x_0)\mathrm{sign}(u(x_0)) : x_0 \in \overline{\Omega},\ |u(x_0)| = ||u||\},$$

while, when $\mathbb{X} = L^\infty(\Omega)$, we have

$$[v, u]_+ = \lim_{\epsilon \to 0} \text{ess } \sup_{\Omega(u,\epsilon)} v(x)\text{sign}(u(x)), \quad u \not\equiv 0,$$

where $\Omega(u, \epsilon)$ is defined (up to a set of measure zero) by

$$\Omega(u, \epsilon) = \{x \in \Omega : |u(x)| > ||u||_{L^\infty(\Omega)} - \epsilon\}$$

(see [29]). Thus, A is accretive in $L^\infty(\Omega)$ if and only if

$$\lim_{\epsilon \to 0} \text{ess sup}_{x \in \Omega(u-\hat{u},\epsilon)} (A(u(x)) - A(\hat{u}(x)))\text{sign}(u(x) - \hat{u}(x)) \geq 0$$

where

$$\Omega(u - \hat{u}, \epsilon) = \{x \in \Omega : |u(x) - \hat{u}(x)| \geq ||u - \hat{u}||_{L^\infty(\Omega)} - \epsilon\}.$$

Finally, thanks to (2.8), A is T-accretive in $L^\infty(\Omega)$ if and only if there is a finitely additive, absolutely continuous positive set function Φ with total variation 1, such that, for any $u, \hat{u} \in L^\infty(\Omega)$,

$$\int_{u > \hat{u}} (u - \hat{u})(x)\Phi(dx) = ||(u - \hat{u})_+||_{L^\infty(\Omega)}, \quad \int_\Omega (A(u) - A(\hat{u}))(x)\Phi(dx) \geq 0.$$

3 The Stationary Case

In this section we concentrate on the following Dirichlet problem

$$P(\Omega) : \quad \begin{cases} -\det D^2 u + k_u u = f \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad \text{on } \partial\Omega \\ u \text{ convex in } \Omega, \end{cases}$$

where Ω is a planar, bounded, convex, open set. We look for convex, and then negative, solutions; then we need $f \leq 0$ in Ω as compatibility condition. As in Sect. 2, we denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. If $g(x)$ is a smooth, radially symmetric, negative function defined in $\Omega^\star$, which is increasing with respect to the radii, our main goal will be proving a suitable comparison result between u and the solution z to the following symmetrized problem

$$P(\Omega^\star) : \quad \begin{cases} -\det D^2 z + |x|^{-1} z = g \text{ in } \Omega^\star \\ z = 0 \qquad\qquad\qquad \text{on } \partial\Omega^\star \\ z \text{ convex in } \Omega^\star. \end{cases}$$

First of all let us discuss the notion of solutions we shall use in this paper. We immediately note that, as we shall see in the case of the radially symmetric problem, the presence of the term $u\,\mathrm{div}\left(\frac{Du}{|D_u|}\right)$ make quite difficult to get classical solutions (for instance in the radially symmetric case the term $\frac{z(x)}{|x|}$ will never be a bounded function since $z(x)$ will be a bounded function). Then it is natural to start our study by considering the truncated problems

$$P_N(\Omega): \quad \begin{cases} -\det D^2 u_N + T_N(k_{u_N})u_N = f \text{ in } \Omega \\ u_N = 0 \qquad\qquad \text{on } \partial\Omega \\ u_N \text{ convex in } \Omega, \end{cases}$$

and

$$P_N(\Omega^\star): \quad \begin{cases} -\det D^2 z_N + T_N(|x|^{-1})z_N = g \text{ in } \Omega^\star \\ z_N = 0 \qquad\qquad \text{on } \partial\Omega^\star \\ z_N \text{ convex in } \Omega^\star, \end{cases}$$

where

$$T_N(s) = \min\{s, N\} \ \text{ for } \ s \geq 0.$$

Proposition 8 *Given $f \in C(\overline{\Omega})$, $f \leq 0$ in Ω, there exists a unique C-viscosity solution u_N to $P_N(\Omega)$. Moreover*

$$\underline{u} \leq u_N \leq u_{N'} \leq 0 \quad \text{ in } \ \Omega,$$

where $\underline{u}$ is the unique C-viscosity solution to the unperturbed problem

$$P_{M-A}(\Omega): \quad \begin{cases} -\det D^2 \underline{u} = f \ \text{ in } \Omega \\ \underline{u} = 0 \qquad \text{ on } \partial\Omega \\ \underline{u} \text{ convex in } \Omega \end{cases}$$

and $u_{N'}$ is the C-viscosity solution to $P_{N'}(\Omega)$ for $N' > N$.

Proof It is not difficult to verify that the comparison principle holds for problem $P_N(\Omega)$. Thus we can apply the Perron method (see Theorem 4.1 in [18]) starting with the supersolution $\overline{u} = 0$ and the subsolution $\underline{u}$. Moreover since $T_N(s) \leq T_{N'}(s)$ if $N' > N$, we immediately get that $u_N \leq u_{N'}$ in Ω. □

Now we introduce the notion of limit solution.

Definition 1 A function $u \in C(\overline{\Omega}) \cap W^{2,1}_{\mathrm{loc}}(\Omega)$ such that u is convex and $-\det D^2 u + k_u u \in L^\infty(\Omega)$ is called a *limit solution* to $P(\Omega)$ if

$$u(x) = \lim_{N \to +\infty} u_N(x),$$

with u_N solution to $P_N(\Omega)$.

Proposition 9 *Given $f \in C(\overline{\Omega}), f \leq 0$ in Ω, there exists a unique limit solution u_N to $P(\Omega)$.*

Proof It suffices to use the Beppo-Levi monotone convergence theorem and the comparison principle. □

Now we can prove that the following operator (jointly with the Dirichlet boundary condition)

$$\mathscr{A}_N u = -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u|$$

is T-accretive in $C(\overline{\Omega})$ once we define suitably its domain $\mathrm{D}(\mathscr{A}_N)$. Since the *formal* operator

$$F(u, Du, D^2 u) = -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u|$$

is not uniformly elliptic but merely degenerate elliptic we must use the notion of C-viscosity solution for the associated problem (see details and references for instance in [26]).

Definition 2 We say that $u \in \mathrm{D}(\mathscr{A}_N)$ if $u \in C(\overline{\Omega})$ is a convex function, with $u = 0$ on $\partial\Omega$, and there exists a nonpositive continuous function v in Ω such that u is a C-viscosity solution to

$$\begin{cases} -\det \mathrm{D}^2 u - T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right) |u| = v & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

We denote by $\mathscr{A}_N u$ the set of all such $v \in C(\overline{\Omega})$.

Corollary 1 *The operator $\mathscr{A}_N$ is T-accretive in the Banach space $\mathbb{X} = C(\overline{\Omega})$ equipped with the supremum norm.*

Proof It is essentially a consequence of the maximum principle (see Theorem 3.3 and Section 5B in [18]). □

Remark 2 The extension to the accretiveness in $L^\infty(\Omega)$ is standard since the norm is given in a similar way. Notice that without the truncation function $T_N \left(\operatorname{div} \left(\frac{Du}{|Du|} \right) \right)$ the corresponding operator is not well defined as an operator from $\mathbb{X}$ to $\mathbb{X}$ since, as we already pointed out, the expression $\operatorname{div} \left(\frac{Du}{|Du|} \right)$ is in general not an element of $\mathbb{X}$.

We continue this section with some considerations on the existence of solutions to the radially symmetric problem $P(\Omega^\star)$. The convexity condition is not always satisfied. So we shall need some extra conditions on the right hand side. Without any interest in getting the more general result at all, we shall proceed under some additional conditions. We denote by $R^\star$ the radius of $\Omega^\star$ and we assume

$$g(x) = g(|x|), \quad g \in W^{2,1}(\Omega^\star), \quad 0 \geq g(|x|) \geq -\frac{M}{|x|} \text{ for some } M > 0 \tag{3.9}$$

and, with $r = |x|$,

$$g''(r)r + 2g'(r) \geq 0 \quad \text{for a.e. } r \in (0, R^\star). \tag{3.10}$$

We get the following result.

Lemma 2 *Under the assumptions (3.9) and (3.10), there exists a unique convex solution $z \in W_0^{1,3}(\Omega^\star)$ to $P(\Omega^\star)$ with* $\det D^2 z$, $\frac{z}{|x|} \in L^3(\Omega^\star)$.

Proof If we set $w = z + M$, we can equivalently prove the existence of a unique convex solution w to

$$\begin{cases} -\det D^2 w + \dfrac{w}{|x|} = g(|x|) + \dfrac{M}{|x|} & \text{in } \Omega^\star \\ w = M & \text{on } \partial\Omega^\star. \end{cases}$$

From the assumption (3.9) and the maximum principle we know that necessarily $0 \leq w \leq M$ in $\Omega^\star$. Now we define

$$\Gamma(|x|) := g(|x|) + \frac{M}{|x|}$$

and we construct w as the unique solution to the following radially symmetric problem

$$\begin{cases} -\dfrac{1}{2}\Delta_3 w + w = |x|\Gamma(|x|) & \text{in } \Omega^\star \\ w = M & \text{on } \partial\Omega^\star. \end{cases} \tag{3.11}$$

Since $|x|\Gamma(|x|) \in L^\infty(\Omega^\star)$, by well-known results there is a unique (radially symmetric) solution $w \in W^{1,3}(\Omega^\star)$ to problem (3.11). Then $z \in W_0^{1,3}(\Omega^\star)$ and by the Hardy inequality $\frac{z}{|x|} \in L^3(\Omega^\star)$. In order to show that z is convex, since it is radially symmetric, it is enough to show that $z''(r) \geq 0$, being $r = |x|$. But the nonnegative function $\gamma(x) = |x|\,\Gamma(|x|) = |x|\,g(|x|) + M$ is a subsolution to problem (3.11), i.e.

$$\begin{cases} -\dfrac{1}{2}\Delta_3 \gamma + \gamma \leq |x|\,\Gamma(|x|) & \text{in } \Omega^\star \\ \gamma \leq M & \text{on } \partial\Omega^\star. \end{cases}$$

Indeed, by (3.10) we have for a.e. $r \in (0, R^\star)$

$$\begin{aligned}\Delta_3\gamma &= \operatorname{div}(|D\gamma|\,D\gamma) = \frac{d}{dr}\left[\left|g'(r)r + g(r)\right|(g'(r)r + g(r))\right] \\ &= 2|g'(r)r + g(r)|(g''(r)r + 2g'(r)) \geq 0,\end{aligned}$$

while (3.9) implies that $\gamma \leq M$ on $\partial\Omega^\star$. Then, by the maximum principle $w \geq \gamma$ in $\Omega^\star$ and then, since $\Delta_3 w = 2(w-\gamma)$, we get $\Delta_3 w \geq 0$, which shows that $0 \leq w''(r)$ and thus $0 \leq z''(r)$. □

Note that in fact our study of the radially symmetric case did not need to use the truncation argument mentioned at the beginning of this section. Nevertheless, we can easily state a result similar to Proposition 8.

Proposition 10 *Given $g \in C(\overline{\Omega^\star})$, $g \leq 0$ in $\Omega^\star$, there exists a unique C-viscosity solution z_N to $P_N(\Omega^\star)$. Moreover*

$$\underline{z} \leq z_N \leq z_{N'} \leq 0 \quad \text{in } \Omega^\star,$$

where $\underline{z}$ is the unique C-viscosity solution to the unperturbed problem

$$P_{M-A}(\Omega^\star): \quad \begin{cases} -\det D^2\underline{z} = g & \text{in } \Omega^\star \\ \underline{z} = 0 & \text{on } \partial\Omega^\star \\ \underline{z} \text{ convex in } \Omega^\star \end{cases}$$

and $z_{N'}$ is the C-viscosity solution to $P_{N'}(\Omega^\star)$ for $N' > N$.

Corollary 2 *Assume g satisfies (3.9) and (3.10). Then the limit solution to problem $P(\Omega^\star)$ (constructed as in Proposition 9) coincides with the unique solution to $P(\Omega^\star)$ given in Lemma 2.*

By using the notion of rearrangement that we recalled in Sect. 2, we can prove the following result which links the asymmetry of a solution u to problem $P(\Omega)$ to the asymmetry of the datum f. This kind of results is very famous in literature and goes back to authors as celebrated as Pólya, Szegő and Weinberger. It appeared clear that symmetrization techniques are very useful to write explicit and easy to compute estimates of solutions to many variational problems (see for example [39, 40] and the references therein). The first who proved a pointwise comparison result between the Schwarz symmetrands of solutions to Poisson equations was Talenti in 1976 (see [36]). After him, many mathematicians have been interested in symmetrization techniques and have applied them to linear and quasilinear elliptic equations with lower order terms (see for example [1, 2, 40] and the references therein). The case of fully nonlinear equations is different, since preserving the measure of the level sets does not give information on the geometry of a solution. In [37] Talenti faced the Monge-Ampère equation in dimension two and recognized the opportunity of symmetrize preserving the perimeter of the level sets. Then Tso ([42], see also [41]) treated the case of Monge-Ampère equations in dimension n. For related results we refer the reader to [9, 12–14, 19].

Theorem 1 *Let $f \in C(\overline{\Omega})$ be a negative function and let u be the limit solution to problem $P(\Omega)$. Denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. Assume that $g(x)$ is a smooth, radially symmetric, negative function defined in $\Omega^\star$, which is increasing along the radii. Let z be the solution to the symmetrized problem $P(\Omega^\star)$. For $s \in (0, L)$ denote*

$$U(s) = \int_0^s \tilde{u}(\sigma)d\sigma, \qquad Z(s) = \int_0^s \tilde{z}(\sigma)d\sigma,$$

$$F(s) = \int_0^{s^2/4\pi} f^*(\sigma)d\sigma, \; G(s) = \int_0^{s^2/4\pi} g^*(\sigma)d\sigma.$$

Then we have

$$||(Z-U)_+||_{L^\infty(0,L)} \le ||(F-G)_+||_{L^\infty([0,L])}. \tag{3.12}$$

Proof Since $u = \lim_{N\to+\infty} u_N = \sup_N u_N$, it will be enough to get the conclusion by replacing u with u_N in the statement. Notice that, nevertheless, we shall not truncate the radially symmetric problem $P(\Omega^\star)$. Our first argument is that, if θ is a noncritical value for u_N (i.e. $|Du_N| \neq 0$ on $\{x \in \Omega : u_N(x) = \theta\}$), then u_N satisfies

$$-\int_{u_N<\theta} \det D^2 u_N \, dx + \int_{u_N<\theta} T_N\left(k_{u_N}\right) u_N \, dx = \int_{u_N<\theta} f \, dx. \tag{3.13}$$

By using (2.7), divergence theorem, Hölder inequality and (2.5), we obtain

$$\int_{u_N<\theta} \det D^2 u_N \, dx = \frac{1}{2}\int_{u_N=\theta} k_{u_N}|Du_N|^2 \ge \frac{4\pi^3}{\left(\lambda_N'(\theta)\right)^2}, \tag{3.14}$$

where $\lambda_N(\theta) = \lambda_{u_N}(\theta)$. Moreover, by Hardy-Littlewood inequality (2.6) and classical isoperimetric inequality (2.4) we obtain

$$\int_{u_N<\theta} (-f)\, dx \le \int_0^{\lambda_N(\theta)^2/4\pi} f^*(\sigma)\, d\sigma. \tag{3.15}$$

It remains to estimate from above the second integral in the left-hand side of (3.13). To do this, we consider $\varepsilon = \varepsilon(N) > 0$ such that, denoted by x_N^m the minimum point of u_N, it holds that $B_\varepsilon := \{x \in \Omega : |x - x_N^m| > \varepsilon\} \subset \{u_N < \theta, \; k_{u_N} \le N\}$. Then, by co-area formula, we get

$$\int_{u_N<\theta} T_N(k_{u_N})\, u_N \, dx \le \int_{u_N<\theta,\, k_{u_N}\le N} k_{u_N}\, u_N \, dx \le \int_{M_\varepsilon}^{\theta} \tau \lambda_N'(\tau)\, d\tau, \tag{3.16}$$

where $M_\varepsilon = \max_{B_\varepsilon} u_N$. From (3.14) to (3.16) with $s = \lambda_N(\theta)$ we deduce the following inequality involving the rearrangement $\tilde{u}_N(s)$ of the function u_N:

$$4\pi^3 \tilde{u}_N'(s)^2 - \int_{\lambda_N(M_\varepsilon)}^{s} \tilde{u}_N(\sigma)\, d\sigma \le \int_0^{s^2/4\pi} f^*(\sigma)\, d\sigma.$$

Setting

$$U_{N,\varepsilon}(s) = \int_{\lambda_N(M_\varepsilon)}^{s} \tilde{u}_N(\sigma)\, d\sigma, \qquad s \in (\lambda_N(M_\varepsilon), L),$$

we get

$$4\pi^3 U_{N,\varepsilon}''(s)^2 - U_{N,\varepsilon}(s) \le F(s), \qquad s \in (\lambda_N(M_\varepsilon), L). \tag{3.17}$$

Reasoning in an analogous way on the solution z to the symmetrized problem $P(\Omega^\star)$, since all the inequalities become equalities, we get

$$4\pi^3 Z''(s)^2 - Z(s) = G(s), \qquad s \in (0, L). \tag{3.18}$$

Subtracting (3.18) from (3.17) we get

$$4\pi^3 \left(U_{N,\varepsilon}''(s)^2 - Z''(s)^2\right) - (U_{N,\varepsilon}(s) - Z(s)) \le F(s) - G(s), \qquad s \in (0, L),$$

where we extended $U_{N,\varepsilon}$ to zero in $(0, \lambda_N(M_\varepsilon))$. Now we observe that the operator

$$V(s) \to -4\pi^3 V''(s)^2$$

is T-accretive in $L^\infty(0, L)$. Then, by definition, there exists a finitely additive absolutely continuous positive set function Φ with total variation 1, such that

$$\int_{Z > U_{N,\varepsilon}} (Z - U_{N,\varepsilon})(s)\Phi(ds) = ||(Z - U_{N,\varepsilon})_+||_{L^\infty(0,L)}$$

and

$$\int_0^L \left(U_{N,\varepsilon}''(s)^2 - Z''(s)^2\right) \Phi(ds) \ge 0.$$

Then we easily get

$$||(Z - U_{N,\varepsilon})_+||_{L^\infty(0,L)} \le ||(F - G)_+||_{L^\infty(0,L)}.$$

Passing to the limit as ε goes to 0 and N goes to $+\infty$ we obtain (3.12). □

Remark 3 In the particular case when $g(x) = -f^\sharp(x)$, estimate (3.12) immediately gives

$$\int_0^s \tilde{u}(\sigma)\,d\sigma \geq \int_0^s \tilde{z}(\sigma)\,d\sigma, \quad s \in (0, L), \tag{3.19}$$

that can be written as

$$\int_{B(0,r)} u^\star(x)\,dx \geq \int_{B(0,r)} z(x)dx, \qquad r \in [0, R^\star],$$

where $R^\star$ is the radius of $\Omega^\star$. In the case of linear equations (and Schwarz symmetrization) the above inequality is known as "symmetrized mass comparison principle" and it is widely applied to extend estimates on the symmetric function z to the non symmetric function u. The first immediate consequence of (3.19) is the following estimate:

$$||u||_{L^p(\Omega)} \leq ||z||_{L^p(\Omega^\star)}, \quad 1 \leq p \leq +\infty.$$

Remark 4 We explicitly observe that an analogous comparison result between concentrations holds true if u and z are convex, vanishing on the boundary, solutions to the equations

$$-\det D^2u + k_u(-u)^\alpha = f \text{ in } \Omega, \quad -\det D^2z + |x|^{-1}(-z)^\alpha = -f^\sharp \text{ in } \Omega^\star,$$

for some $\alpha > 0$, respectively. More precisely it holds that

$$\int_0^s (-\tilde{u}(\sigma))^\alpha\,d\sigma \leq \int_0^s (-\tilde{z}(\sigma))^\alpha\,d\sigma, \quad s \in (0, L).$$

We end this section with a qualitative property of solutions to $P(\Omega)$ derived trough Theorem 1 and the consideration of this property for the symmetrized problem $P(\Omega^\star)$.

Proposition 11 *Let f be as in Theorem 1. Assume that $f^*(\sigma)$ is strictly monotone. Then no free boundaries (given as the boundary of the subsets where $Du = 0$, with u limit solution to $P(\Omega)$) can be formed.*

Proof Arguing as in [20] it is enough to prove the nonexistence of free boundaries for the radially symmetric solution z to $P(\Omega^\star)$ with $g = -f^\sharp$. In this case, even if the operator Δ_3 is degenerated, it is enough to observe that the right-hand side term in the equation never vanishes (see [20]). □

Remark 5 It is a curious fact that, if $g(|x|) = \frac{z_0}{|x|}$ on a suitable subset of $\Omega^\star$ with positive measure, for some $z_0 < 0$ corresponding to the minimum value of the solution z to $P(\Omega^\star)$, then the set of points where $z(|x|) = z_0$ could also have positive measure and then it could give rise to a free boundary. Anyway, we are talking about unbounded data, something which goes out of the assumptions of this paper.

4 The Evolution Problem

In this section we want to apply symmetrization techniques to the following evolution problem

$$\begin{cases} (k_u u)_t - \det D^2 u = f(x,t) & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(\cdot,t) \text{ convex in } \Omega \end{cases} \tag{4.20}$$

where $f(x,t)$ is a smooth, negative function defined in $\Omega \times (0,+\infty)$ and $u_0(x)$ is a smooth, convex function, vanishing on the boundary of Ω. Concerning the rearrangements theory and the parabolic equations we refer the interested reader to [2, 4, 5, 21, 22, 32, 40, 43] and the references therein.

We remark that, if we proceed as in the stationary case and we integrate the equation in problem (4.20) on the subset of Ω given by $\{x \in \Omega : u(x,t) < \theta\}$ for $\theta < 0$, we obtain

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (k_u\, u)_t\, dx - \int_{\{x\in\Omega:u(x,t)<\theta\}} \det\, D^2 u\, dx = \int_{\{x\in\Omega:u(x,t)<\theta\}} f\, dx. \tag{4.21}$$

By easy calculation we may show that the second integral in the left-hand side of (4.21) can be written in terms of $U(s,t) = \int_0^s \tilde{u}(\sigma,t)\, d\sigma$, where s is the perimeter of $\{x \in \Omega : u(x,t) < \theta\}$. We would like to relate the first one with the derivative of $U(s,t)$ with respect to t. To this aim we recall a derivation formula for a function of the type

$$H(s,t) = \int_{\{x\in\Omega:u(x,t)<\tilde{u}(s,t)\}} h(x,t)\, dx$$

where u and h are smooth functions defined in $\Omega \times [0,+\infty[$ (see [30, 35], see also [3, 10]).

Proposition 12 *Let $u(x,t)$ be a smooth function in $\Omega \times [0,+\infty[$, convex with respect to x in Ω and vanishing on $\partial\Omega \times [0,+\infty[$. If $h \in C^1(\Omega \times [0,+\infty[)$, then for any*

$t \in (0, +\infty)$ *it holds true*

$$\left(\int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} h\, dx\right)_t$$

$$= \int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} h_t\, dx - \int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{h}{|Du|} \times \left\{ \frac{\displaystyle\int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{k_u}{|Du|} u_t}{\displaystyle\int_{\{x\in\Omega: u(x,t)=\tilde{u}(s,t)\}} \frac{k_u}{|Du|}} - u_t \right\}.$$

Remark 6 If $h(x,t) = k_u(x,t)u(x,t)$, then

$$\left(\int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} k_u\, u\, dx\right)_t = \int_{\{x\in\Omega: u(x,t)<\tilde{u}(s,t)\}} (k_u\, u)_t\, dx. \tag{4.22}$$

The following results dealing with comparison between rearrangements will be stated, for simplicity, for classical solutions. Nevertheless, by following the same methods used in the stationary case, the conclusions can be extended to weaker notions of solutions (see [11]).

Theorem 2 *Let u be a classical solution to problem (4.20). Denote by $\Omega^\star$ the disc, centered at the origin, with the same perimeter L as Ω. Assume that $g(x,t)$ is a smooth, negative function defined in $\Omega^\star \times (0, +\infty)$, radially symmetric with respect to the space variables, i. e. $g(x,t) = g(|x|, t)$, and $z_0(x)$ is a smooth, convex function, defined in $\Omega^\star$, vanishing on $\partial\Omega^\star$. Let z be the solution to the following parabolic problem*

$$\begin{cases} |x|^{-1} z_t - \det D^2 z = g(x,t) & \text{in } \Omega^\star \times (0, +\infty) \\ z = 0 & \text{on } \partial\Omega^\star \times (0, +\infty) \\ z(x,0) = z_0(x) & \text{in } \Omega^\star \\ z(\cdot, t) \text{ convex in } \Omega^\star. \end{cases} \tag{4.23}$$

For $s \in (0, L)$ and $t > 0$ denote

$$U(s,t) = \int_0^s \tilde{u}(\sigma,t)d\sigma, \qquad Z(s,t) = \int_0^s \tilde{z}(\sigma,t)d\sigma,$$

$$F(s,t) = \int_0^{s^2/4\pi} f^*(\sigma,t)d\sigma, \quad G(s,t) = \int_0^{s^2/4\pi} g^*(\sigma,t)d\sigma.$$

Then, for every $t > 0$, we have

$$||(Z(\cdot,t) - U(\cdot,t))_+||_{L^\infty(0,L)} \le ||(Z(\cdot,0) - U(\cdot,0))_+||_{L^\infty(0,L)} + \int_0^t ||(F(\cdot,\tau) - G(\cdot,\tau))_+||_{L^\infty([0,L])} d\tau. \tag{4.24}$$

Proof We reason here as in the stationary case. Let $t > 0$ and let us consider a noncritical value $\theta < 0$ (i.e. $|D_x u| \neq 0$ on $\{x \in \Omega : u(x,t) = \theta\}$). We integrate the equation in (4.20) on the sublevel set $\{x \in \Omega : u(x,t) < \theta\}$ obtaining

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (u(x,t)\,k_u(x,t))_t\,dx - \int_{\{x\in\Omega:u(x,t)<\theta\}} \det D^2u\,dx = \int_{\{x\in\Omega:u(x,t)<\theta\}} f(x,t)\,dx. \tag{4.25}$$

As in the stationary case, by using divergence theorem, Hölder inequality and (2.5), we get

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} \det D^2u(x,t)\,dx = \frac{1}{2}\int_{\{x\in\Omega:u(x,t)=\theta\}} k_u(x,t)|Du(x,t)|^2 \geq \frac{4\pi^3}{\left(\frac{\partial\lambda(\theta,t)}{\partial\theta}\right)^2}, \tag{4.26}$$

where $\lambda(\theta,t) = \text{length}\{x \in \Omega : u(x,t) = \theta\}$. Moreover, by using (4.22) we get that the first integral in (4.25) coincides with

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\theta} \tau \frac{\partial\lambda(\tau,t)}{\partial\tau} d\tau\right). \tag{4.27}$$

On the other hand, by Hardy-Littlewood inequality (2.6) and the classical isoperimetric inequality (2.4) we obtain

$$\int_{\{x\in\Omega:u(x,t)<\theta\}} (-f(x,t))\,dx \leq \int_0^{\lambda(\theta,t)^2/4\pi} f^*(\sigma,t)\,d\sigma. \tag{4.28}$$

From (4.26), (4.27) and (4.28) with $s = \lambda(\theta,t)$ we deduce the following inequality involving the rearrangement $\tilde{u}(s,t)$ of the function $u(\cdot,t)$:

$$-\frac{\partial}{\partial t}\left(\int_0^s \tilde{u}(\sigma,t)\,d\sigma\right) + 4\pi^3\left(\frac{\partial\tilde{u}(s,t)}{\partial s}\right)^2 \leq \int_0^{s^2/4\pi} f^*(\sigma,t)\,d\sigma,$$

that is

$$-U_t(s,t) + 4\pi^3 U_{ss}^2(s,t) \leq F(s,t), \qquad s \in (0,L),\ \ t > 0. \tag{4.29}$$

Reasoning in an analogous way on the solution z to the symmetrized problem (4.23), since all the inequalities become equalities, we get

$$-Z_t(s,t) + 4\pi^3 Z_{ss}^2(s,t) = G(s,t), \qquad s \in (0,L),\ t > 0. \tag{4.30}$$

Subtracting (4.30) from (4.29) we get

$$(Z(s,t)-U(s,t))_t+4\pi^3\left(U_{ss}^2(s,t)-Z_{ss}^2(s,t)\right)\le F(s,t)-G(s,t),\qquad s\in(0,L),t>0.$$

Now we observe that the operator

$$U(s,t)\to -4\pi^3U_{ss}^2(s,t)$$

is T-accretive in $L^\infty(0,L)$. Then, by definition, there exists a finitely additive absolutely continuous positive set function Φ with total variation 1, such that

$$\int_{Z>U}(Z-U)(s)\Phi(ds)=||(Z-U)_+||_{L^\infty(0,L)}$$

and

$$\int_0^L\left(U_{ss}^2-Z_{ss}^2\right)\Phi(ds)\ge 0.$$

Then we easily get

$$\int_0^L(Z-U)_t\,\Phi(ds)\le\int_0^L(F-G)\Phi(ds)$$

and finally

$$\frac{d}{dt}||(Z-U)_+||_{L^\infty(0,L)}\le||(F-G)_+||_{L^\infty(0,L)}.$$

Integrating between 0 and t we get the thesis. □

Remark 7 Estimates (4.24) can be read as a continuous dependence on the data symmetry with respect to the spatial variables. Indeed, if $\Omega=\Omega^\star$, the maximal asymmetry of a solution at the time t does not exceed the sum of the asymmetry at the time 0 and the asymmetry of the datum f.

In the particular case when $g(x,t)=-f^\sharp(x,t)$ and $z_0(x)=u_0^\star(x)$, estimate (4.24) immediately implies the comparison result contained in [10, Theorem 3.1], that we state below for completeness.

Theorem 3 *Let u be a classical solution to problem (4.20) and let v be the solution to*

$$\begin{cases}|x|^{-1}v_t-\det D^2v=-f^\sharp(x,t) & \textit{in } \Omega^\star\times(0,+\infty)\\ v=0 & \textit{on } \partial\Omega^\star\times(0,+\infty)\\ v(x,0)=u_0^\star(x) & \textit{in } \Omega^\star\\ v(\cdot,t)\ \textit{convex in } \Omega^\star. & \end{cases}$$

Then, for every $t > 0$ it holds

$$\int_0^s \tilde{u}(\sigma,t)d\sigma \geq \int_0^s \tilde{v}(\sigma,t)d\sigma, \qquad s \in (0,L). \tag{4.31}$$

As an immediate consequence of (4.31) we have the following

Proposition 13 *Under the assumptions of Theorem 3, the following estimates hold true for every $t > 0$:*

$$||u\,(\cdot,t)||_{L^p(\Omega)} \leq ||v\,(\cdot,t)||_{L^p(\Omega^\star)}\,, \qquad 1 \leq p \leq +\infty; \tag{4.32}$$

$$\int_\Omega u(k_u\,u)_t\,dx + \int_\Omega (-u)\det D^2 u\,dx \leq \int_{\Omega^\star} v(k_v\,v)_t\,dx + \int_{\Omega^\star} (-v)\det D^2 v\,dx. \tag{4.33}$$

Proof Estimate (4.32) can be easily deduced from (4.31) and properties of rearrangements.

Multiplying the equation in problem (4.20) by $-u$, integrating over Ω and using (4.31) yield

$$\begin{aligned}
&\int_\Omega u(k_u\,u)_t\,dx + \int_\Omega (-u)\det D^2 u\,dx \\
&\qquad = \int_\Omega f(-u)\,dx \\
&\qquad \leq \frac{1}{2\pi}\int_0^L f^*\left(\frac{s^2}{4\pi},t\right)(-\tilde{u}(s,t))s\,ds \\
&\qquad = \frac{1}{2\pi}\int_0^L \left(\int_0^s (-\tilde{u}(\sigma,t))d\sigma\right)\left(-\frac{d}{ds}\left(f^*\left(\frac{s^2}{4\pi},t\right)s\right)\right)\,ds \\
&\qquad \leq \frac{1}{2\pi}\int_0^L \left(\int_0^s (-\tilde{v}(\sigma,t))d\sigma\right)\left(-\frac{d}{ds}\left(f^*\left(\frac{s^2}{4\pi},t\right)s\right)\right)\,ds \\
&\qquad = \int_{\Omega^\star} f^\sharp(x,t)(-v)\,dx = \int_{\Omega^\star} v(k_v\,v)_t\,dx + \int_{\Omega^\star}(-v)\det D^2 v\,dx,
\end{aligned}$$

that is (4.33). □

A qualitative property typical of some nonlinear models concerns the finite speed of propagation of disturbances: if the initial datum u_0 vanishes on a set of positive measure (i.e. $\operatorname{supt} u_0 \subset \Omega$), then $\operatorname{supt} u(\cdot,t) \subset \Omega$ for any $t > 0$. In our case the following estimate of the perimeter of the zero sublevel set of u can be proved. It will imply that, if $\operatorname{supt} v(\cdot,\bar{t}) = \Omega^\star$ for some $\bar{t} > 0$, then $\operatorname{supt} u(\cdot,\bar{t}) = \Omega$, which means that the equation does not satisfy the finite speed of propagation property. Finally, the following perimeter estimate shows how important is having symmetry conditions on partial differential equations in order to have solutions with small supports.

Proposition 14 *Under the assumptions of Theorem 3, if for every $t > 0$ it holds*

$$\int_{\Omega} (k_u\, u)\, dx = \int_{\Omega^\star} (|x|^{-1} v)\, dx, \tag{4.34}$$

then

$$P(\{x \in \Omega^\star : v(x,t) < 0\}) \le P(\{x \in \Omega : u(x,t) < 0\}), \qquad t > 0.$$

Proof By using co-area formula, assumption (4.34) can be written as

$$\int_{\min_\Omega u}^{0} \lambda_u(\theta,t)\, d\theta = \int_{\min_{\Omega^\star} v}^{0} \lambda_v(\theta,t)\, d\theta \tag{4.35}$$

that is in terms of rearrangements

$$\int_0^L \tilde{u}(\sigma,t)\, d\sigma = \int_0^L \tilde{v}(\sigma,t)\, d\sigma. \tag{4.36}$$

Thus, estimate (4.31) implies

$$\int_s^L \tilde{u}(\sigma,t) d\sigma \le \int_s^L \tilde{v}(\sigma,t) d\sigma.$$

Let $[0, R_u(t)]$ and $[0, R_v(t)]$ denote the support of $\tilde{u}(\cdot,t)$ and $\tilde{v}(\cdot,t)$, respectively, with $0 < R_u(t), R_v(t) \le L$. From (4.36) it immediately follows that $R_v(t) \le R_u(t)$, otherwise

$$0 = \int_{R_u(t)}^{L} \tilde{u}(\sigma,t) d\sigma \le \int_{R_u(t)}^{R_v(t)} \tilde{v}(\sigma,t) d\sigma < 0$$

which is a contradiction. □

Remark 8 When does (4.35) hold? When $t = 0$ it is clearly true since

$$\int_{\min_\Omega u}^{0} \lambda_u(\theta,0) d\theta = \int_{\min_\Omega u_0}^{0} \lambda_{u_0}(\theta) d\theta = \int_{\min_{\Omega^\star} u_0^\star}^{0} \lambda_{u_0^\star}(\theta) d\theta = \int_{\min_{\Omega^\star} v}^{0} \lambda_v(\theta,0) d\theta,$$

that is $\tilde{u}(\cdot,0)$ and $\tilde{v}(\cdot,0)$ have the same L^1 norm. Thus (4.35) is satisfied whenever $\tilde{u}(\cdot,t)$ and $\tilde{v}(\cdot,t)$ preserve the same L^1 norm for every $t > 0$. We stress that this does not mean that $u(\cdot,t)$ and $v(\cdot,t)$ have the same L^1 norm.

Now we want to study the asymptotic behavior of u by proving that the stabilization to a stationary solution requires an infinite time. To this aim we need to introduce the following auxiliary eigenvalue problem

$$\begin{cases} -w'' = \frac{\lambda}{s} w & s \in (0, L) \\ w(0) = 0, w'(L) = 0. \end{cases}$$

By well-known results (see, e.g. [8]) there exists a first eigenvalue $\lambda_1 > 0$ such that the corresponding normalized eigenfunction w_1 satisfies $w_1(s) > 0$ for any $s \in (0, L)$ and $\|w_1\|_\infty = 1$. We also point out that, when $g^*(s,t) \equiv c^2$ $(c > 0)$ for any $s \in (0, L)$ in (4.23), then the problem

$$\begin{cases} -Z_t(s,t) + 4\pi^3 Z_{ss}^2(s,t) = \frac{c^2 s^2}{4\pi} & s \in (0, L),\ t > 0 \\ Z(0,t) = Z_s(L,t) = 0 & t > 0 \\ Z(s,0) = Z_0(s) & s \in (0, L) \end{cases}$$

has the unique stationary solution

$$Z_\infty(s) = \frac{c}{24\pi^2} s \left(s^2 - 3L^2\right)$$

and, in particular, if $Z_0(s) = Z_\infty(s)$ then $Z(s,t) = Z_\infty(s)$ for any $s \in [0, L]$ and $t \geq 0$.

Theorem 4 *Assume $g^*(s) \equiv c^2$ in (4.23) and $F(s,t) = G(s,t)$ for any $s \in (0, L)$, $t > 0$. If there exists $\underline{m} > 0$ such that*

$$U(s,0) \geq Z_\infty(s) + \underline{m} w_1(s) \quad \text{for any } s \in (0, L),$$

then there exists a constant $\varepsilon > 0$ (independent of t) such that for any $t > 0$ and $s \in [0, L]$ we have

$$U(s,t) \geq Z_\infty(s) + \underline{m} w_1(s) e^{-\lambda_1 \varepsilon t}.$$

Proof Arguing as in the proof of Theorem 2, it suffices to show that the function

$$\underline{U}(s,t) := Z_\infty(s) + \underline{m} w_1(s) e^{-\lambda_1 \varepsilon t}$$

is a subsolution to the parabolic problem associated to $U(s,t)$. More precisely, we must check that we can take a constant $\varepsilon > 0$ (independent of t) such that

$$\begin{cases} -\underline{U}_t(s,t) + 4\pi^3 \underline{U}_{ss}^2(s,t) \geq \frac{c^2 s^2}{4\pi} & s \in (0, L),\ t > 0 \\ \underline{U}(0,t) = \underline{U}_s(L,t) = 0 & t > 0 \\ \underline{U}(s,0) \leq U(s,0) & s \in (0, L). \end{cases} \tag{4.37}$$

□

The above result leads to interesting consequences concerning the free boundaries raised by the solution u in the line of papers [25] and [26] (see also [24]).

Another very natural question concerning problem (4.20) is the stabilization of solutions: assumed that

$$f(\cdot,t) \to f_\infty(\cdot) \qquad \text{as} \quad t \to +\infty$$

in suitable functional spaces, is it true that $u(\cdot,t)$ tends to the solution to the associated stationary problem? In the next proposition, reasoning in an analogous way as in the proof of Theorem 4, we obtain the following asymptotic behavior of a solution u to problem (4.20) in a ball.

Proposition 15 *Let B_R be a ball with radius R and let f_∞ be a radially symmetric, negative function defined in B_R. Suppose that $f(x,t) \nearrow f_\infty(x)$ as $t \to +\infty$ for $x \in B_R$. Let u_0 be a convex function in B_R, vanishing on ∂B_R. Let u be a solution to problem (4.20) with Ω replaced by B_R and let ψ be the solution to*

$$\begin{cases} -\det D^2\psi = f_\infty & \text{in } B_R \\ \psi = 0 & \text{on } \partial B_R. \end{cases}$$

Denote

$$U(s,t) = \int_0^s \tilde{u}(\sigma,t)d\sigma, \quad \Psi(s) = \int_0^s \tilde{\psi}(\sigma)d\sigma, \quad U_0(s) = \int_0^s \tilde{u}_0(\sigma)d\sigma.$$

If $U(s,0) \geq \Psi(s)$ for $s \in (0,L)$, then $U(s,t) \geq \Psi(s)$ for every $s \in (0,L)$ and $t > 0$.

Proof It is enough to observe that

$$\begin{aligned} &-U_t + 4\pi^3 U_{ss}^2 \leq F \\ &U(s,0) = U_0(s) \text{ for } s \in (0,L) \\ &U(0,t) = U_s(L,t) = 0 \quad \text{for} \quad t > 0 \end{aligned}$$

and

$$4\pi^3 \Psi_{ss}^2 = F_\infty, \qquad \Psi(0) = \Psi_s(L) = 0,$$

where

$$F(s,t) = \int_0^{s^2/4\pi} f^*(\sigma,t)d\sigma, \quad F_\infty(s) = \int_0^{s^2/4\pi} f_\infty^*(\sigma)\, d\sigma.$$

Since by definition of rearrangement $F(s,t) \leq F_\infty(s)$ for any $t > 0$, then

$$-(U-\Psi)_t + 4\pi^3(U_{ss}^2 - \Psi_{ss}^2) < 0.$$

The thesis follows from the maximum principle. □

We end this paper with some considerations on the existence of solutions for the radially symmetric problem (4.23). As in the stationary case, the convexity condition is not always satisfied. So we shall need some extra conditions on the datum g. Without any interest in getting the more general result at all, we shall proceed under some additional assumptions. Let $\Omega^\star = B_{R^\star}(0)$. Suppose that $z_0 \in W^{2,1}(\Omega^\star) \cap W_0^{1,3}(\Omega^\star)$ is a nonpositive, convex, radially symmetric function such that

$$\Delta_3 z_0 \in L^2(\Omega^\star). \tag{4.38}$$

Suppose also that

$$g(x,t) = g(|x|,t), \quad g \in C([0,T] : L^\infty(\Omega^\star)), \tag{4.39}$$

and, for some $M > 0$,

$$-M(\cosh T)^2 \leq |x|\, g(|x|,t) \leq 0 \quad \text{for any } t \in [0,T], \tag{4.40}$$

$$\Delta_3 z_0(x) + |x|\, g(|x|,t) \geq 0 \quad \text{for a.e. } x \in \Omega^\star \text{ and for any } t \in [0,T]. \tag{4.41}$$

The following result holds.

Lemma 3 *Under the assumptions (4.38), (4.39), (4.40) and (4.41), there exists a unique convex solution* $z \in C([0,T] : L^\infty(\Omega^\star)) \cap L^2([0,T] : W_0^{1,3}(\Omega^\star))$ *to problem (4.23) with* $\frac{z_t(\cdot,t)}{|x|} - \det D^2 z(\cdot,t) \in L^\infty(\Omega^\star)$ *for a.e.* $t \in (0,T)$.

Proof Since the solution must be radially symmetric we know that z is given as the unique solution to the problem

$$\begin{cases} z_t - \frac{1}{2}\Delta_3 z = |x|\, g(|x|,t) & \text{in } \Omega^\star \times (0,T) \\ z = 0 & \text{on } \partial\Omega^\star \times (0,T) \\ z(x,0) = z_0(x) & \text{in } \Omega^\star. \end{cases}$$

From the assumptions (4.39) and (4.40), by the T-accretiveness of the operator $-\Delta_3 z$, we know that

$$\left\| [z(\cdot,t)]_+ \right\|_{L^1(\Omega^\star)} \leq \left\| [z_0]_+ \right\|_{L^1(\Omega^\star)} + \int_0^t |x|\, [g(|x|,s)]_+ \, ds = 0$$

so that $z(x,t) \le 0$ on $\Omega^\star \times (0,T)$. Moreover, in an analogous way, taking

$$\underline{z}(x,t) \equiv -\|z_0\|_{L^\infty(\Omega^\star)} - M \tanh t$$

we get that

$$\begin{cases} \underline{z}_t - \Delta_3 \underline{z} = \rho(t) & \text{in } \Omega^\star \times (0,T) \\ \underline{z} \le 0 & \text{on } \partial\Omega^\star \times (0,T) \\ \underline{z}(x,0) \le z_0(x) & \text{in } \Omega^\star, \end{cases}$$

with

$$\rho(t) = -\frac{M}{(\cosh t)^2}.$$

By (4.40)

$$\rho(t) \le |x|\, g(|x|\,, t) \quad \text{for a.e. } (x,t) \in \Omega^\star \times (0,T),$$

then, by the maximum principle, $\underline{z}(x,t) \le z(x,t) \le 0$ in $\Omega^\star \times (0,T)$. Now, in order to prove the convexity of $z(\cdot,t)$ we argue as in Diaz-Kawohl [27] (see the proof of their Theorem 1). We start by pointing out that $0 \le z''(r,t)$ if and only if $\Delta_3 z(r,t) \ge 0$ so, since $|x|\, g(|x|) \le 0$ we only need to prove that $z_t \ge 0$. But this holds once we have condition (4.41) as in [27].

Since $z_0 \in D(\mathscr{C})$, where $\mathscr{C}$ is the operator on $H = L^2(\Omega^\star)$, given by $\mathscr{C}z = -\Delta_3 z$ and since $\mathscr{C}$ is the subdifferential in $L^2(\Omega^\star)$ of a convex function, we get that $z_t(\cdot,t) \in L^2(\Omega^\star)$, $\Delta_3 z(r,t) \in L^2(\Omega^\star)$ for a.e. $t \in (0,T)$ and the equation takes place for a.e. $x \in \Omega^\star$ and a.e. $t \in (0,T)$. Then by dividing by $|x|$, since we have (4.39), we get that $\frac{z_t(\cdot,t)}{|x|} - \det D^2 z(\cdot,t) \in L^\infty(\Omega^\star)$ for a.e. $t \in (0,T)$. □

Remark 9 We argue as in [15] (Lemma 3.3, p. 73) to get some extra regularity. For instance, by multiplying the equation in (4.23) by z_t we get

$$\int_{\Omega^\star} (z_t)^2 + \frac{1}{6}\frac{d}{dt}\int_{\Omega^\star} |\nabla z|^3 = \int_{\Omega^\star} z_t\, |x|\, g(|x|).$$

This shows that $z \in C([0,T] : W_0^{1,3}(\Omega^\star))$ and, by the Hardy inequality, $\frac{z(\cdot,t)}{|x|} \in L^3(\Omega^\star)$ for any $t \in [0,T]$.

In the special case when $\Omega = \Omega^\star$ and the data $f(x,t)$ and $u_0(x)$ are radially symmetric, but not necessarily decreasing along the radii, it is possible to get some information about how the corresponding solution u is becoming each time more similar to its rearrangement $u^\star$. Some results on the asymptotic stabilization to a stationary solution can be obtained trough similar results for the case of the Δ_3 operator (see, e.g. [23] or [28]).

Proposition 16 *Assume that* $z_0 \in W^{2,1}(\Omega^\star) \cap W_0^{1,3}(\Omega^\star)$ *with* $z_0(x) = z_0(|x|)$, $z_0(x) \le 0$ *in* $\Omega^\star$, *and* $\Delta_3 z_0 \in L^2(\Omega^\star)$. *Suppose also that* $g(x,t) \in L^\infty(\Omega^\star \times (0,+\infty)) \cap W^{1,1}_{loc}((0,+\infty) : L^1(\Omega^\star))$ *satisfies*

$$g(x,t) = g(|x|\,,t)$$

with

$$\int_t^{t+1} |x| \left\| \frac{\partial}{\partial t} g(|x|,s) \right\|_{L^1(\Omega^\star)} ds \le C \quad \text{for any } t > 0$$

for some $C > 0$ *independent of t and*

$$-M \le |x|\, g(|x|\,,t) \le 0 \text{ for a.e. } t \in (0,+\infty).$$

Suppose also that there exists $g_\infty \in L^{3/2}(\Omega^\star)$, *with* $g_\infty(x) = g_\infty(|x|)$, $g_\infty(x) \le 0$ *in* $\Omega^\star$, *such that*

$$\int_t^{t+1} |x|\, \|g(|x|,t) - g_\infty(|x|)\|_{L^{3/2}(\Omega^\star)} \to 0 \text{ as } t \to +\infty.$$

Then, if z is the unique strong solution to the problem

$$\begin{cases} z_t - \frac{1}{2}\Delta_3 z = |x|\, g(|x|\,,t) & \text{in } \Omega^\star \times (0,+\infty) \\ z = 0 & \text{on } \partial\Omega^\star \times (0,+\infty) \\ z(x,0) = z_0(x) & \text{in } \Omega^\star, \end{cases}$$

z is also solution to the problem

$$\begin{cases} |x|^{-1} z_t - \det D_x^2 z = g(x,t) & \text{in } \Omega^\star \times (0,+\infty) \\ z = 0 & \text{on } \partial\Omega^\star \times (0,+\infty) \\ z(x,0) = z_0(x) & \text{in } \Omega^\star. \end{cases} \tag{4.42}$$

Moreover, $z(.,t) \to z_\infty$ *in* $W_0^{1,3}(\Omega^\star)$, *as* $t \to +\infty$, *where* z_∞ *is the unique solution to*

$$\begin{cases} -\det D^2 z_\infty = g_\infty(x) & \text{in } \Omega^\star \\ z_\infty = 0 & \text{on } \partial\Omega^\star. \end{cases}$$

Proof It suffices to apply Lemma 1 and Theorem 1 in [28] to the p-Laplacian operator with $p = 3$. □

Finally, as a simple application of Proposition 16, if we assume for instance that

$$g(x,t) = g_\infty(x) = 0, \tag{4.43}$$

we can give an estimate about the progressive perimeter symmetrization in time of $z(.,t)$ (in a similar manner to the one in Proposition 1 of [21]).

Proposition 17 *Let $z_0(x)$ be as in Proposition 16, with $z_0 \neq z_0^\star$ and suppose that (4.43) holds true. If z is the solution to (4.42) given in Proposition 16 and ζ is the solution to the same problem with $z_0^\star$ as initial datum (always with $g(x,t) = 0$), given $r \geq 1$, for any $q > r$ we have*

$$\|z(\cdot,t) - \zeta(\cdot,t)\|_{L^q(\Omega^\star)} \leq Ct^{-\delta} \|z_0 - z_0^\star\|^{\gamma}_{L^r(\Omega^\star)}$$

with

$$\delta = \frac{2(q-r)}{q(3r+2)} \quad \text{and} \quad \gamma = \frac{r(3q+2)}{q(3r+2)}.$$

Proof Obviously $\zeta(\cdot,t) = \zeta(\cdot,t)^\star$. Then, it suffices to apply the characterization of radially symmetric solutions of the lemma and the regularizing estimate (Théorème III.4) of [44] for the p-Laplacian operator with $p = 3$. □

Remark 10 Note that, according to Proposition 16, $z(\cdot,t)$ and $\zeta(\cdot,t) \to 0$ as $t \to +\infty$ in $W_0^{1,3}(\Omega^\star)$.

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Optimal Boundary Control of a Nonstandard Cahn–Hilliard System with Dynamic Boundary Condition and Double Obstacle Inclusions

Pierluigi Colli and Jürgen Sprekels

Abstract In this paper, we study an optimal boundary control problem for a model for phase separation taking place in a spatial domain that was introduced by P. Podio-Guidugli in Ric. Mat. **55** (2006), pp. 105–118. The model consists of a strongly coupled system of nonlinear parabolic differential inclusions, in which products between the unknown functions and their time derivatives occur that are difficult to handle analytically; the system is complemented by initial and boundary conditions. For the order parameter of the phase separation process, a dynamic boundary condition involving the Laplace–Beltrami operator is assumed, which models an additional nonconserving phase transition occurring on the surface of the domain. We complement in this paper results that were established in the recent contribution appeared in Evol. Equ. Control Theory **6** (2017), pp. 35–58, by the two authors and Gianni Gilardi. In contrast to that paper, in which differentiable potentials of logarithmic type were considered, we investigate here the (more difficult) case of nondifferentiable potentials of double obstacle type. For such nonlinearities, the standard techniques of optimal control theory to establish the existence of Lagrange multipliers for the state constraints are known to fail. To overcome these difficulties, we employ the following line of approach: we use the results contained in the preprint arXiv:1609.07046 [math.AP] (2016), pp. 1–30, for the case of (differentiable) logarithmic potentials and perform a so-called "deep quench limit". Using compactness and monotonicity arguments, it is shown that this strategy leads to the desired first-order necessary optimality conditions for the case of (nondifferentiable) double obstacle potentials.

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote some open, connected and bounded domain with smooth boundary Γ (we should at least have $\Gamma \in C^2$), and let $T > 0$ be a fixed final time and $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$. We denote by $\partial_{\mathbf{n}}$, ∇_Γ, Δ_Γ, the outward normal derivative, the tangential gradient, and the Laplace–Beltrami operator on Γ, in this order. We study in this paper the following optimal boundary control problem:

($\mathscr{P}_0$) Minimize the cost functional

$$\begin{aligned}\mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) := {} & \frac{\beta_1}{2}\,\|\mu-\hat{\mu}_Q\|^2_{L^2(Q)} + \frac{\beta_2}{2}\,\|\rho-\hat{\rho}_Q\|^2_{L^2(Q)} \\ & + \frac{\beta_3}{2}\,\|\rho_\Gamma-\hat{\rho}_\Sigma\|^2_{L^2(\Sigma)} + \frac{\beta_4}{2}\,\|\rho(T)-\hat{\rho}_\Omega\|^2_{L^2(\Omega)} \\ & + \frac{\beta_5}{2}\,\|\rho_\Gamma(T)-\hat{\rho}_\Gamma\|^2_{L^2(\Gamma)} + \frac{\beta_6}{2}\,\|u_\Gamma\|^2_{L^2(\Sigma)} \end{aligned} \tag{1.1}$$

over a suitable set $\mathscr{U}_{\mathrm{ad}} \subset (H^1(0,T;L^2(\Gamma)) \cap L^\infty(\Sigma))$ of admissible controls u_Γ (to be specified later), subject to the state system

$$(1+2g(\rho))\,\mu_t + \mu\,g'(\rho)\,\rho_t - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{1.2}$$

$$\partial_{\mathbf{n}}\mu = 0 \quad \text{a.e. on } \Sigma, \quad \mu(0) = \mu_0 \quad \text{a.e. in } \Omega, \tag{1.3}$$

$$\rho_t - \Delta\rho + \xi + \pi(\rho) = \mu\,g'(\rho) \quad \text{a.e. in } Q, \tag{1.4}$$

$$\xi \in \partial I_{[-1,1]}(\rho) \quad \text{a.e. in } Q, \tag{1.5}$$

$$\partial_{\mathbf{n}}\rho + \partial_t\rho_\Gamma - \Delta_\Gamma\rho_\Gamma + \xi_\Gamma + \pi_\Gamma(\rho_\Gamma) = u_\Gamma, \quad \rho_\Gamma = \rho_{|\Sigma}, \quad \text{a.e. on } \Sigma, \tag{1.6}$$

$$\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma) \quad \text{a.e. on } \Sigma, \tag{1.7}$$

$$\rho(0) = \rho_0 \quad \text{a.e. in } \Omega, \quad \rho_\Gamma(0) = \rho_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{1.8}$$

Here, β_i, $1 \le i \le 6$, are nonnegative weights, and $\hat{\mu}_Q, \hat{\rho}_Q \in L^2(Q)$, $\hat{\rho}_\Sigma \in L^2(\Sigma)$, $\hat{\rho}_\Omega \in L^2(\Omega)$, and $\hat{\rho}_\Gamma \in L^2(\Gamma)$ are prescribed target functions.

The physical background behind the control problem ($\mathscr{P}_0$) is the following: the state system (1.2)–(1.8) constitutes a model for phase separation taking place in the container Ω and originally introduced in [32]. In this connection, the unknowns μ and ρ denote the associated *chemical potential*, which in this particular model has to be nonnegative, and the *order parameter* of the phase separation process, which is usually the volumetric density of one of the involved phases. We assume that ρ is normalized in such a way as to attain its values in the interval $[-1, 1]$. The nonlinearities π, π_Γ, g are assumed to be smooth in $[-1, 1]$, and $\partial I_{[-1,1]}$ denotes the subdifferential of the indicator function of the interval $[-1, 1]$. As is well known, we have that

$$I_{[-1,1]}(\rho) = \begin{cases} 0 & \text{if } \rho \in [-1,1] \\ +\infty & \text{otherwise} \end{cases}, \quad \partial I_{[-1,1]}(\rho) = \begin{cases} (-\infty, 0] & \text{if } \rho = -1 \\ \{0\} & \text{if } -1 < \rho < 1 \\ [0, +\infty) & \text{if } \rho = 1 \end{cases}. \tag{1.9}$$

The state system (1.2)–(1.8) is singular, with highly nonlinear and nonstandard couplings. It has been the subject of intensive study over the past years for the case that (1.6) is replaced by a zero Neumann condition. In this connection, we refer the reader to [6, 8, 9, 11–15]. In [7], an associated control problem with a distributed control in (1.2) was investigated for the special case $g(\rho) = \rho$, and in [10], the corresponding case of a boundary control for μ was studied. A nonlocal version, in which the Laplacian $-\Delta\rho$ in (1.4) was replaced by a nonlocal operator, was discussed in the contributions [22, 23, 26].

In all of the works cited above a zero Neumann condition was assumed for the order parameter ρ. In contrast to this, we study in this paper the case of the dynamic boundary condition (1.6). It models a nonconserving phase transition taking place on the boundary, which could be induced by, e. g., an interaction between bulk and wall. The associated total free energy of the phase separation process is the sum of a bulk and a surface contribution and has the form

$$\begin{aligned} &\mathscr{F}_{\text{tot}}[\mu(t), \rho(t), \rho_\Gamma(t)] \\ &:= \int_\Omega \Big(I_{[-1,1]}(\rho(x,t)) + \hat{\pi}(\rho(x,t)) - \mu(x,t)\, g(\rho(x,t)) + \frac{1}{2}|\nabla\rho(x,t)|^2 \Big)\, \mathrm{d}x \\ &+ \int_\Gamma \Big(I_{[-1,1]}(\rho_\Gamma(x,t)) + \hat{\pi}_\Gamma(\rho_\Gamma(x,t)) - u_\Gamma(x,t)\, \rho_\Gamma(x,t) + \frac{1}{2}|\nabla_\Gamma \rho_\Gamma(x,t)|^2 \Big)\, \mathrm{d}\Gamma, \end{aligned} \tag{1.10}$$

for $t \in [0, T]$, where $\hat{\pi}(r) = \int_0^r \pi(\xi)\mathrm{d}\xi$ and $\hat{\pi}_\Gamma(r) = \int_0^r \pi_\Gamma(\xi)\mathrm{d}\xi$. In the recent contribution [24], the state system (1.2)–(1.8) was studied systematically concerning existence, uniqueness, and regularity. A boundary control problem resembling $(\mathscr{P}_0)$ was solved in [25] for the case of potentials of logarithmic type.

The mathematical literature on control problems for phase field systems involving equations of viscous or nonviscous Cahn–Hilliard type is still scarce and quite recent. We refer in this connection to the works [16, 17, 19, 21, 28, 35]. Control problems for convective Cahn–Hilliard systems were studied in [33, 36, 37], and a few analytical contributions were made to the coupled Cahn–Hilliard/Navier–Stokes system (cf. [27, 29–31]). The contribution [20] dealt with the optimal control of a Cahn–Hilliard type system arising in the modeling of solid tumor growth. For the optimal control of Allen–Cahn equations with dynamic boundary condition, we refer to [5, 18] (see also [4]).

In this paper, we aim to employ the results established in [25] to treat the nondifferentiable double obstacle case when ξ, ξ_Γ satisfy the inclusions (1.5), (1.7). Our approach is guided by a strategy that was introduced in [18] by the present authors and M.H. Farshbaf-Shaker: in fact, we aim to derive first-order necessary optimality conditions for the double obstacle case by performing a so-called "deep quench limit" in a family of optimal control problems with differentiable logarithmic nonlinearities that was treated in [25], and for which the corresponding state systems were analyzed in [24]. The general idea is briefly explained as follows: we replace the inclusions (1.5) and (1.7) by the identities

$$\xi = \varphi(\alpha)\, h'(\rho), \quad \xi_\Gamma = \varphi(\alpha)\, h'(\rho_\Gamma), \tag{1.11}$$

where h is defined by

$$h(\rho) := \begin{cases} (1-\rho)\,\ln(1-\rho) + (1+\rho)\,\ln(1+\rho) & \text{if } \rho \in (-1,1) \\ 2\,\ln(2) & \text{if } \rho \in \{-1,1\} \end{cases}, \tag{1.12}$$

and where φ is continuous and positive on $(0, 1]$ and satisfies

$$\lim_{\alpha \searrow 0} \varphi(\alpha) = 0. \tag{1.13}$$

We remark that we can simply choose $\varphi(\alpha) = \alpha^p$ for some $p > 0$. Now, observe that $h'(y) = \ln\left(\frac{1+y}{1-y}\right)$ and $h''(y) = \frac{2}{1-y^2} > 0$ for $y \in (-1, 1)$. Hence, in particular, we have

$$\lim_{\alpha \searrow 0} \varphi(\alpha)\, h'(y) = 0 \quad \text{for } -1 < y < 1,$$

$$\lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \searrow -1} h'(y)\Big) = -\infty, \quad \lim_{\alpha \searrow 0} \Big(\varphi(\alpha) \lim_{y \nearrow +1} h'(y)\Big) = +\infty. \tag{1.14}$$

We thus may regard the graph $\varphi(\alpha)\, h'$ as an approximation to the graph of the subdifferential $\partial I_{[-1,1]}$.

Now, for any $\alpha > 0$ the optimal control problem (later to be denoted by $(\mathscr{P}_\alpha)$), which results if in $(\mathscr{P}_0)$ the relations (1.5), (1.7) are replaced by (1.11), is of the type for which in [25] the existence of optimal controls $u_\Gamma^\alpha \in \mathscr{U}_{\text{ad}}$ as well as first-order necessary optimality conditions have been derived. Proving a priori estimates (uniform in $\alpha > 0$), and employing compactness and monotonicity arguments, we will be able to show the following existence and approximation result: whenever $\{u_\Gamma^{\alpha_n}\} \subset \mathscr{U}_{\rm ad}$ is a sequence of optimal controls for $(\mathscr{P}_{\alpha_n})$, where $\alpha_n \searrow 0$ as $n \to \infty$, then there exist a subsequence of $\{\alpha_n\}$, which is again indexed by n, and an optimal control $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ of $(\mathscr{P}_0)$ such that

$$u_\Gamma^{\alpha_n} \to \bar{u}_\Gamma \quad \text{weakly-star in } \mathscr{X} \text{ as } n \to \infty, \tag{1.15}$$

where, here and in the following,

$$\mathscr{X} := H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \tag{1.16}$$

will always denote the control space. In other words, optimal controls for $(\mathscr{P}_\alpha)$ are for small $\alpha > 0$ likely to be 'close' to optimal controls for $(\mathscr{P}_0)$. It is natural to ask if the reverse holds, i. e., whether every optimal control for $(\mathscr{P}_0)$ can be approximated by a sequence $\{u_\Gamma^{\alpha_n}\}$ of optimal controls for $(\mathscr{P}_{\alpha_n})$, for some sequence $\alpha_n \searrow 0$.

Unfortunately, we will not be able to prove such a 'global' result that applies to all optimal controls for $(\mathscr{P}_0)$. However, a 'local' result can be established. To this end, let $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ be any optimal control for $(\mathscr{P}_0)$. We introduce the 'adapted' cost functional

$$\widetilde{\mathscr{J}}((\mu,\rho,\rho_\Gamma),u_\Gamma) := \mathscr{J}((\mu,\rho,\rho_\Gamma),u_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} \tag{1.17}$$

and consider for every $\alpha \in (0,1]$ the *adapted control problem* of minimizing $\widetilde{\mathscr{J}}$ subject to $u_\Gamma \in \mathscr{U}_{\rm ad}$ and to the constraint that (μ,ρ,ρ_Γ) solves the approximating system (1.2)–(1.4), (1.6), (1.8), (1.11). It will then turn out that the following is true:

(i) There are some sequence $\alpha_n \searrow 0$ and minimizers $\bar{u}_\Gamma^{\alpha_n} \in \mathscr{U}_{\rm ad}$ of the adapted control problem associated with α_n, $n \in \mathbb{N}$, such that

$$\bar{u}_\Gamma^{\alpha_n} \to \bar{u}_\Gamma \quad \text{strongly in } L^2(\Sigma) \text{ as } n \to \infty. \tag{1.18}$$

(ii) It is possible to pass to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions corresponding to the adapted control problems associated with $\alpha \in (0,1]$ in order to derive first-order necessary optimality conditions for problem $(\mathscr{P}_0)$.

The paper is organized as follows: in Sect. 2, we give a precise statement of the problem under investigation, and we derive some results concerning the state system (1.2)–(1.8) and its α-approximation which is obtained if in $(\mathscr{P}_0)$ the relations (1.5) and (1.7) are replaced by the relations (1.11). In Sect. 3, we then prove the existence of optimal controls and the approximation result formulated above in (i). The final Sect. 4 is devoted to the derivation of the first-order necessary optimality conditions, where the strategy outlined in (ii) is employed.

During the course of this analysis, we will make repeated use of Hölder's inequality, of the elementary Young's inequality

$$ab \le \gamma |a|^2 + \frac{1}{4\gamma}|b|^2 \quad \forall\, a, b \in \mathbb{R} \quad \forall\, \gamma > 0, \tag{1.19}$$

and of the continuity of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $1 \le p \le 6$. We will also use the denotations

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t), \quad \text{for } 0 < t \le T. \tag{1.20}$$

Throughout the paper, for a Banach space X we denote by $\|\cdot\|_X$ its norm and by X^* its dual space. The only exemption from this rule are the norms of the L^p spaces and of their powers, which we often denote by $\|\cdot\|_p$, for $1 \le p \le +\infty$. By $\langle v, w\rangle_X$ we will denote the dual pairing between elements $v \in X^*$ and $w \in X$. About the time derivative of a time-dependent function v, we warn the reader that we may use both the notation $\partial_t v$ and the shorter one v_t.

2 General Assumptions and State Equations

In this section, we formulate the general assumptions of the paper, and we state some preparatory results for the state system (1.2)–(1.8) and its α-approximations. To begin with, we introduce some denotations. We set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{w \in H^2(\Omega) : \partial_{\mathbf{n}} w = 0 \text{ on } \Gamma\},$$

$$H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma), \quad \mathscr{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : v_\Gamma = v_{|\Gamma}\},$$

and endow these spaces with their standard norms. Notice that we have $V \subset H \subset V'$ and $V_\Gamma \subset H_\Gamma \subset V'_\Gamma$, with dense, continuous and compact embeddings.

We make the following general assumptions:

(A1) $\mu_0 \in W$, $\mu_0 \ge 0$ in $\overline{\Omega}$, $\rho_0 \in H^2(\Omega)$, $\rho_{0_\Gamma} := \rho_{0_{|\Gamma}} \in H^2(\Gamma)$, and

$$-1 < \min_{x\in\overline{\Omega}} \rho_0(x), \quad \max_{x\in\overline{\Omega}} \rho_0(x) < +1. \tag{2.1}$$

(A2) $\pi, \pi_\Gamma \in C^2[-1,1]$; $g \in C^3[-1,1]$ is nonnegative and concave on $[-1,1]$.

(A3) $\mathscr{U}_{\text{ad}} = \{u_\Gamma \in \mathscr{X} : u_* \le u_\Gamma \le u^* \text{ a.e. on } \Sigma \text{ and } \|u_\Gamma\|_{\mathscr{X}} \le R_0\}$, where $u_*, u^* \in L^\infty(\Sigma)$ and $R_0 > 0$ are such that $\mathscr{U}_{\text{ad}} \ne \emptyset$.

Now, observe that the set $\mathscr{U}_{\text{ad}}$ is a bounded subset of $\mathscr{X}$. Hence, there exists a bounded open ball in $\mathscr{X}$ that contains $\mathscr{U}_{\text{ad}}$. For later use it is convenient to fix such a ball once and for all, noting that any other such ball could be used instead. In this sense, the following assumption is rather a denotation:

(A4) Let $R > 0$ be such that $\mathscr{U}_{\text{ad}} \subset \mathscr{U}_R := \{u_\Gamma \in \mathscr{X} : \|u_\Gamma\|_{\mathscr{X}} < R\}$.

For the quantities entering the cost functional $\mathscr{J}$ (see (1.1)), we assume:

(A5) The constants β_i, $1 \le i \le 6$, are nonnegative but not all equal to zero, and we have $\hat{\mu}_Q, \hat{\rho}_Q \in L^2(Q)$, $\hat{\rho}_\Sigma \in L^2(\Sigma)$, $\hat{\rho}_\Omega \in L^2(\Omega)$, $\hat{\rho}_\Gamma \in L^2(\Gamma)$.

We observe at this point that if (A1), (A2) and $u_\Gamma \in \mathscr{U}_R$ hold true, then all of the general assumptions made in [24] are satisfied provided we put, in the notation used there, $\hat{\beta} = \hat{\beta}_\Gamma = I_{[-1,1]}$. We thus may conclude from [24, Thm. 2.1 and Rem. 3.1] the following well-posedness result:

Theorem 2.1 *Suppose that the assumptions (A1)–(A4) are fulfilled. Then the state system* (1.2)–(1.8) *has for every* $u_\Gamma \in \mathscr{U}_R$ *a unique solution* (μ, ρ, ρ_Γ) *with* $\mu \ge 0$ *a. e. in Q, which satisfies*

$$\mu \in C^0([0,T];V) \cap L^p(0,T;W) \cap L^2(0,T;W^{2,6}(\Omega)) \cap L^\infty(Q) \quad \forall\, p \in [1,+\infty), \tag{2.2}$$

$$\mu_t \in L^p(0,T;H) \cap L^2(0,T;L^6(\Omega)) \quad \forall\, p \in [1,+\infty), \tag{2.3}$$

$$\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.4}$$

$$\rho_\Gamma \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{2.5}$$

$$\rho \in [-1,1] \quad a.e. \quad in\, Q, \qquad \rho_\Gamma \in [-1,1] \quad a.e.\ on\ \Sigma, \tag{2.6}$$

$$\xi \in L^\infty(0,T;H), \qquad \xi_\Gamma \in L^\infty(0,T;H_\Gamma). \tag{2.7}$$

Moreover, there is a constant $K_1^* > 0$, *which depends only on the data of the state system and on* R, *such that*

$$\|\mu\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)\cap L^\infty(Q)} + \|\rho\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))}$$
$$+ \|\rho_\Gamma\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))} + \|\xi\|_{L^\infty(Q)} + \|\xi_\Gamma\|_{L^\infty(\Sigma)} \le K_1^*, \tag{2.8}$$

whenever (μ, ρ, ρ_Γ) *is a solution to* (1.2)–(1.8) *which corresponds to some* $u_\Gamma \in \mathscr{U}_R$ *and satisfies* (2.2)–(2.7).

Remark 2.2 Thanks to Theorem 2.1, the control-to-state operator $\mathscr{S}_0 : u_\Gamma \mapsto (\mu, \rho, \rho_\Gamma)$ is well defined as a mapping from $\mathscr{U}_R$ into the space specified by the regularity properties (2.2)–(2.5). Moreover, in view of (2.4), it follows from well-known embedding results (see, e. g., [34, Sect. 8, Cor. 4]) that $\rho \in C^0([0,T];H^s(\Omega))$ for $0 < s < 2$. In particular, we have $\rho \in C^0(\overline{Q})$, so that $\rho_\Gamma = \rho_{|\Gamma} \in C^0(\overline{\Sigma})$.

We now turn our interest to the α – approximating system that results if we replace (1.5) and (1.7) by (1.11), with h given by (1.12) and φ satisfying (1.13).

We then obtain the following system of equations:

$$(1+2g(\rho^\alpha))\,\mu_t^\alpha + \mu^\alpha\, g'(\rho^\alpha)\,\rho_t^\alpha - \Delta\mu^\alpha = 0 \quad \text{a.e. in } Q, \tag{2.9}$$

$$\partial_{\mathbf{n}}\mu^\alpha = 0 \quad \text{a.e. on } \Sigma, \quad \mu^\alpha(0) = \mu_0 \quad \text{a.e. in } \Omega, \tag{2.10}$$

$$\rho_t^\alpha - \Delta\rho^\alpha + \varphi(\alpha)\,h'(\rho^\alpha) + \pi(\rho^\alpha) \;=\; \mu^\alpha\, g'(\rho^\alpha) \quad \text{a.e. in } Q, \tag{2.11}$$

$$\partial_{\mathbf{n}}\rho^\alpha + \partial_t\rho_\Gamma^\alpha - \Delta_\Gamma\rho_\Gamma^\alpha + \varphi(\alpha)\,h'(\rho_\Gamma^\alpha) + \pi_\Gamma(\rho_\Gamma^\alpha) = u_\Gamma^\alpha, \quad \rho_\Gamma^\alpha = \rho_{|\Sigma}^\alpha \quad \text{a.e. on } \Sigma, \tag{2.12}$$

$$\rho^\alpha(0) = \rho_0 \quad \text{a.e. in } \Omega, \quad \rho_\Gamma^\alpha(0) = \rho_{0_\Gamma} \quad \text{a.e. on } \Gamma. \tag{2.13}$$

By virtue of [25, Thm. 2.4], the system (2.9)–(2.13) has for every $u_\Gamma^\alpha \in \mathscr{U}_R$ a unique solution $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ satisfying $\mu^\alpha \ge 0$ in Q and (2.2)–(2.5). Moreover, there are constants $r_*(\alpha), r^*(\alpha) \in (-1,1)$, which depend only on R, α, and the data of the system, such that, for all $(x,t) \in \overline{Q}$,

$$-1 < r_*(\alpha) \le \rho^\alpha(x,t) \le r^*(\alpha) < 1, \quad -1 < r_*(\alpha) \le \rho_\Gamma^\alpha(x,t) \le r^*(\alpha) < 1. \tag{2.14}$$

Again it follows (recall Remark 2.2) that $\rho^\alpha \in C^0(\overline{Q})$ and $\rho_\Gamma^\alpha \in C^0(\overline{\Sigma})$. Therefore, we may infer from (A2) that there is a constant $K_2^* > 0$, which depends only on R and the data of the system, such that

$$\max_{0\le i\le 3} \left\|g^{(i)}(\rho^\alpha)\right\|_{C^0(\overline{Q})} + \max_{0\le i\le 2}\left(\left\|\pi^{(i)}(\rho^\alpha)\right\|_{C^0(\overline{Q})} + \left\|\pi_\Gamma^{(i)}(\rho_\Gamma^\alpha)\right\|_{C^0(\overline{\Sigma})}\right) \le K_2^*\,, \tag{2.15}$$

for every solution triple $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ corresponding to some $u_\Gamma \in \mathscr{U}_R$ and any $\alpha \in (0,1]$. Observe that a corresponding estimate cannot be concluded for the derivatives of $\varphi(\alpha)\,h$, since it may well happen that $r_*(\alpha) \searrow -1$ and/or $r^*(\alpha) \nearrow +1$, as $\alpha \searrow 0$.

According to the above considerations, for every $\alpha \in (0,1]$ the solution operator $\mathscr{S}_\alpha : u_\Gamma^\alpha \in \mathscr{U}_R \mapsto (\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ is well defined as a mapping into the space that is specified by the regularity properties (2.2)–(2.5). We now aim to derive some a priori estimates for $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ that are independent of $\alpha \in (0,1]$. We have the following result.

Proposition 2.3 *Suppose that (A1)–(A4) are satisfied. Then there is some constant* $K_3^* > 0$, *which depends only on R and on the data of the system, such that we have: whenever* $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha) = \mathscr{S}_\alpha(u_\Gamma^\alpha)$ *for some* $u_\Gamma^\alpha \in \mathscr{U}_R$ *and some* $\alpha \in (0,1]$, *then it* *holds that*

$$\begin{aligned}
&\|\mu^\alpha\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)\cap L^\infty(Q)} \\
&+ \ \|\rho^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1([0,T];V)\cap L^\infty(0,T;H^2(\Gamma))} \\
&+ \ \|\rho_\Gamma^\alpha\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1([0,T];V_\Gamma)\cap L^\infty(0,T;H^2(\Gamma))} \\
&+ \|\varphi(\alpha)\, h'(\rho^\alpha)\|_{L^\infty(0,T;H)} \ + \ \|\varphi(\alpha)\, h'(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;H_\Gamma)} \ \le \ K_3^* \, .
\end{aligned} \tag{2.16}$$

Proof Let $u_\Gamma^\alpha \in \mathscr{U}_R$ and $\alpha \in (0,1]$ be arbitrary and $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha) = \mathscr{S}_\alpha(u_\Gamma^\alpha)$. The result will be established in a series of a priori estimates. To this end, we will in the following denote by $C > 0$ constants that may depend on the quantities mentioned in the statement, but not on $\alpha \in (0,1]$. For the sake of a better readability, we will omit the superscript α of $(\mu^\alpha, \rho^\alpha, \rho_\Gamma^\alpha)$ during the estimations, writing it only at the end of each estimate. We will also make repeated use of the general bounds (2.15) without further reference.

First estimate:

First, note that $\partial_t((\frac{1}{2} + g(\rho))\,\mu^2) = (1 + 2g(\rho))\,\mu_t\,\mu + g'(\rho)\,\rho_t\,\mu^2$. Thus, multiplying (2.9) by μ and integrating over Q_t, where $t \in (0,T]$, we find the estimate

$$\int_\Omega \big(\tfrac{1}{2} + g(\rho(t))\big)\,|\mu(t)|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \big(\tfrac{1}{2} + g(\rho_0)\big)\,|\mu_0|^2\,\mathrm{d}x\,. \tag{2.17}$$

Hence, as $g(\rho) \ge 0$ by (A2), it follows that

$$\|\mu^\alpha\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.18}$$

Second estimate:

Next, we multiply (2.11) by $\varphi(\alpha)\,h'(\rho^\alpha)$ and integrate over Q_t and by parts, where $t \in (0,T]$. We obtain the identity

$$\begin{aligned}
&\varphi(\alpha)\int_\Omega h(\rho(t))\,\mathrm{d}x + \varphi(\alpha)\int_\Gamma h(\rho_\Gamma(t))\,\mathrm{d}\Gamma + \int_0^t\!\!\int_\Omega |\varphi(\alpha)\,h'(\rho)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t\!\!\int_\Gamma |\varphi(\alpha)\,h'(\rho_\Gamma)|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + \varphi(\alpha)\int_0^t\!\!\int_\Omega h''(\rho)\,|\nabla\rho|^2\,\mathrm{d}x\,\mathrm{d}s \\
&+ \varphi(\alpha)\int_0^t\!\!\int_\Gamma h''(\rho_\Gamma)\,|\nabla_\Gamma\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&= \varphi(\alpha)\int_\Omega h(\rho_0)\,\mathrm{d}x + \varphi(\alpha)\int_\Gamma h(\rho_{0_\Gamma})\,\mathrm{d}\Gamma \\
&\quad + \int_0^t\!\!\int_\Omega (\mu\,g'(\rho) - \pi(\rho))\,\varphi(\alpha)\,h'(\rho)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\!\!\int_\Gamma (u_\Gamma^\alpha - \pi_\Gamma(\rho_\Gamma))\,\varphi(\alpha)\,h'(\rho_\Gamma)\,\mathrm{d}\Gamma\,\mathrm{d}s\,.
\end{aligned} \tag{2.19}$$

Obviously, all of the terms on the left-hand side are nonnegative, while the first two summands on the right-hand side are bounded independently of $\alpha \in (0,1]$. Thus, applying Hölder's and Young's inequalities to the last two integrals in (2.19), and invoking (2.15) and (2.18), we readily find that

$$\left\|\varphi(\alpha)\, h'(\rho^\alpha)\right\|_{L^2(Q)} + \left\|\varphi(\alpha)\, h'(\rho_\Gamma^\alpha)\right\|_{L^2(\Sigma)} \le C \quad \forall\, \alpha \in (0,1]. \tag{2.20}$$

THIRD ESTIMATE:

We now add ρ on both sides of (2.11) and ρ_Γ on both sides of (2.12). Then we multiply the first resulting equation by ρ_t and integrate over Q_t, where $t \in (0,T]$. Employing (2.15), we then obtain an inequality of the form

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega |\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Gamma |\partial_t\rho_\Gamma|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\left(\|\rho(t)\|_V^2 + \|\rho_\Gamma(t)\|_{V_\Gamma}^2\right)\\
&\le \frac{1}{2}\left(\|\rho_0\|_V^2 + \|\rho_{0_\Gamma}\|_{V_\Gamma}^2\right) + \int_0^t\!\!\int_\Omega |\rho_t|\,(|\rho| + |\varphi(\alpha)\,h'(\rho)| + C(1+|\mu|))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^t\!\!\int_\Gamma |\partial_t\rho_\Gamma|\,(|\rho_\Gamma| + |\varphi(\alpha)\,h'(\rho_\Gamma)| + |u_\Gamma^\alpha|)\,\mathrm{d}\Gamma\,\mathrm{d}s\,.
\end{aligned} \tag{2.21}$$

Using (A1), (2.18), and (2.20), and employing Young's inequality and Gronwall's lemma, we thus conclude that

$$\|\rho^\alpha\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\rho_\Gamma^\alpha\|_{H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)} \le C \quad \forall\, \alpha \in (0,1]\,. \tag{2.22}$$

FOURTH ESTIMATE:

We now take advantage of the estimates (2.15), (2.18), (2.20) and (2.22). Indeed, comparison in (2.11) yields that

$$\|\Delta\rho\|_{L^2(Q)} \le C\,. \tag{2.23}$$

Now, observe that, owing to [3, Thm. 3.2, p. 1.79], we have the estimate

$$\int_0^T \|\rho(t)\|_{H^{3/2}(\Omega)}^2\,\mathrm{d}t \le C\int_0^T \left(\|\Delta\rho(t)\|_H^2 + \|\rho_\Gamma(t)\|_{V_\Gamma}^2\right)\mathrm{d}t,$$

so that

$$\|\rho\|_{L^2(0,T;H^{3/2}(\Omega))} \le C. \tag{2.24}$$

Hence, by the trace theorem (cf. [3, Thm. 2.27, p. 1.64]), we infer that

$$\|\partial_{\mathbf{n}}\rho\|_{L^2(0,T;H_\Gamma)} \le C, \tag{2.25}$$

whence, by comparison in (2.12),

$$\|\Delta_\Gamma \rho_\Gamma\|_{L^2(0,T;L^2(\Gamma))} \le C. \tag{2.26}$$

Thus, by the boundary version of elliptic estimates, we deduce that

$$\|\rho_\Gamma\|_{L^2(0,T;H^2(\Gamma))} \le C, \tag{2.27}$$

whence, by virtue of standard elliptic theory, it turns out that

$$\|\rho\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{2.28}$$

Since the embeddings

$$(H^1(0,T;H) \cap L^2(0,T;H^2(\Omega))) \subset C^0([0,T];V)$$

and

$$(H^1(0,T;H_\Gamma) \cap L^2(0,T;H^2(\Gamma))) \subset C^0([0,T];V_\Gamma)$$

are continuous, we have thus shown the estimate

$$\|\rho^\alpha\|_{C^0([0,T];V)\cap L^2(0,T;H^2(\Omega))} + \|\rho^\alpha_\Gamma\|_{C^0([0,T];V_\Gamma)\cap L^2(0,T;H^2(\Gamma))} \le C \quad \forall\, \alpha \in (0,1]. \tag{2.29}$$

FIFTH ESTIMATE:

In this step of the proof, we adopt a formal argument that can be made rigorous by using finite differences in time. Namely, we differentiate (2.11) formally with respect to time, multiply the resulting identity by ρ_t, and integrate over Q_t, where $0 < t \le T$, and (formally) by parts. We then arrive at an inequality of the form

$$\begin{aligned}
&\frac{1}{2}\left(\|\rho_t(t)\|_H^2 + \|\partial_t\rho_\Gamma(t)\|_{H_\Gamma}^2\right) + \int_0^t\!\!\int_\Omega |\nabla\partial_t\rho|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Gamma |\nabla_\Gamma\partial_t\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&+ \varphi(\alpha)\int_0^t\!\!\int_\Omega h''(\rho)|\rho_t|^2\,\mathrm{d}x\,\mathrm{d}s + \varphi(\alpha)\int_0^t\!\!\int_\Gamma h''(\rho_\Gamma)|\partial_t\rho_\Gamma|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\le \frac{1}{2}\left(\|\rho_t(0)\|_H^2 + \|\partial_t\rho_\Gamma(0)\|_{H_\Gamma}^2\right) + \sum_{j=1}^4 I_j,
\end{aligned} \tag{2.30}$$

where the expressions I_j, $1 \le j \le 4$, will be specified and estimated below. Notice that all of the terms on the left-hand side are nonnegative. At first, using (A1), (A2), the trace theorem, and the fact that $u_\Gamma^\alpha \in \mathscr{U}_{\rm ad}$, we find that

$$\|\rho_t(0)\|_H \le \|\Delta\rho_0 - \varphi(\alpha)\, h'(\rho_0) - \pi(\rho_0) + \mu_0\, g'(\rho_0)\|_H \le C,$$
$$\|\partial_t\rho_\Gamma(0)\|_H \le \|\partial_{\mathbf{n}}\rho_0\|_H + \|\Delta_\Gamma\rho_{0_\Gamma} - \varphi(\alpha)\, h'(\rho_{0_\Gamma}) - \pi_\Gamma(\rho_{0_\Gamma}) + u_\Gamma^\alpha(0)\|_H \le C. \tag{2.31}$$

Next, recalling (2.15) and (2.22), we have that

$$I_1 := -\int_0^t\!\!\int_\Omega \pi'(\rho)\, |\rho_t|^2\, {\rm d}x\, {\rm d}s \le C, \tag{2.32}$$

as well as, by also using Young's inequality,

$$I_4 := \int_0^t\!\!\int_\Gamma (\partial_t u_\Gamma^\alpha - \pi_\Gamma'(\rho_\Gamma)\, \partial_t\rho_\Gamma)\, \partial_t\rho_\Gamma\, {\rm d}\Gamma\, {\rm d}s \le C. \tag{2.33}$$

In addition, since $\mu\, g''(\rho) \le 0$, it turns out that

$$I_2 := \int_0^t\!\!\int_\Omega \mu\, g''(\rho)\, |\rho_t|^2\, {\rm d}x\, {\rm d}s \le 0. \tag{2.34}$$

The estimation of the remaining term

$$I_3 := \int_0^t\!\!\int_\Omega \mu_t\, g'(\rho)\, \rho_t\, {\rm d}x\, {\rm d}s$$

is more delicate. To this end, we use the identity (cf. (2.9))

$$\mu_t = (1 + 2g(\rho))^{-1}(\Delta\mu - \mu\, g'(\rho)\, \rho_t),$$

where, obviously, $1/(1 + 2g(\rho)) \le 1$. Substitution of this identity and integration by parts yield that

$$I_3 = \int_0^t\!\!\int_\Omega \frac{1}{1 + 2g(\rho)} \left[\Delta\mu - \mu\, g'(\rho)\, \rho_t\right] g'(\rho)\, \rho_t\, {\rm d}x\, {\rm d}s$$
$$= -\int_0^t\!\!\int_\Omega \nabla\mu(s) \cdot \nabla\Big(\frac{g'(\rho)\, \rho_t}{1 + 2g(\rho)}\Big)\, {\rm d}x\, {\rm d}s - \int_0^t\!\!\int_\Omega \frac{(g'(\rho))^2}{1 + 2g(\rho)}\, \mu\, |\rho_t|^2\, {\rm d}x\, {\rm d}s\,, \tag{2.35}$$

where the second summand on the right is obviously nonpositive. We thus obtain the inequality

$$I_3 \le C\int_0^t\!\!\int_\Omega |\nabla\mu|\, |\nabla\rho_t|\, {\rm d}x\, {\rm d}s + C\int_0^t\!\!\int_\Omega |\nabla\mu|\, |\nabla\rho|\, |\rho_t|\, {\rm d}x\, {\rm d}s := J_1 + J_2\,. \tag{2.36}$$

Obviously, owing to Young's inequality and (2.18), we infer that

$$J_1 \le \frac{1}{4}\int_0^t\int_\Omega |\nabla\rho_t|^2 \,\mathrm{d}x\,\mathrm{d}s + C. \tag{2.37}$$

On the other hand, thanks to Hölder's and Young's inequalities, we also have that

$$\begin{aligned} J_2 &\le C\int_0^t \|\nabla\mu(s)\|_2\, \|\nabla\rho(s)\|_4\, \|\rho_t(s)\|_4 \,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{4}\int_0^t \|\rho_t(s)\|_V^2 \,\mathrm{d}s + C\int_0^t \|\nabla\mu(s)\|_H^2\, \|\nabla\rho(s)\|_V^2 \,\mathrm{d}s \\ &\le C + \frac{1}{4}\int_0^t\int_\Omega |\nabla\rho_t|^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\nabla\mu(s)\|_H^2 \|\nabla\rho(s)\|_V^2 \,\mathrm{d}s\,. \end{aligned} \tag{2.38}$$

The last integral cannot be controlled in this form. We thus try to estimate the expression $\|\nabla\rho(s)\|_V^2$ in terms of the expressions $\|\partial_t\rho(s)\|_H^2$ and $\|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2$ which can be handled using the first summand on the left-hand side of (2.30). To this end, we use the regularity theory for linear elliptic equations and (2.29) to deduce that

$$\|\nabla\rho(s)\|_V^2 \le C\left(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\right) \le C\left(1 + \|\Delta\rho(s)\|_H^2\right). \tag{2.39}$$

We now multiply, just as in the second estimate above, (2.11) by $\varphi(\alpha)\, h'(\rho(s))$, but this time we only integrate over Ω. We then obtain, for almost every $s \in (0, t)$,

$$\begin{aligned} &\|\varphi(\alpha)\, h'(\rho(s))\|_H^2 + \|\varphi(\alpha)\, h'(\rho_\Gamma(s))\|_{H_\Gamma}^2 + \varphi(\alpha)\int_\Omega h''(\rho(s))\, |\nabla\rho(s)|^2 \,\mathrm{d}x \\ &+ \varphi(\alpha)\int_\Gamma h''(\rho_\Gamma(s))\, |\nabla_\Gamma\rho_\Gamma(s)|^2 \,\mathrm{d}\Gamma \\ &= \int_\Omega \varphi(\alpha)\, h'(\rho(s))\left(-\rho_t(s) - \pi(\rho(s)) + \mu(s)\, g'(\rho(s))\right) \mathrm{d}x \\ &\quad + \int_\Gamma \varphi(\alpha)\, h'(\rho_\Gamma(s))\left(-\partial_t\rho_\Gamma(s) - \pi_\Gamma(\rho_\Gamma(s)) + \partial_t u_\Gamma^\alpha(s)\right) \mathrm{d}\Gamma\,, \end{aligned} \tag{2.40}$$

whence, thanks to the already proven estimates and to Young's inequality,

$$\|\varphi(\alpha)\, h'(\rho(s))\|_H^2 + \|\varphi(\alpha)\, h'(\rho_\Gamma(s))\|_{H_\Gamma}^2 \le C\left(1 + \|\partial_t\rho(s)\|_H^2 + \|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2\right)$$
$$\text{for a. e. } s \in (0, t). \tag{2.41}$$

Comparison in (2.11) then yields that

$$\|\Delta\rho(s)\|_H^2 \le C\left(1 + \|\partial_t\rho(s)\|_H^2 + \|\partial_t\rho_\Gamma(s)\|_{H_\Gamma}^2\right) \quad \text{for a. e. } s \in (0, t). \tag{2.42}$$

Combining the estimates (2.36)–(2.42), we have thus shown that

$$I_3 \le C + \frac{1}{2}\int_0^t\int_\Omega |\nabla\rho_t|\,dx\,ds + C\int_0^t \|\nabla\mu(s)\|_H^2 \left(\|\rho_t(s)\|_H^2 + \|\partial_t\rho_\Gamma(s)\|_H^2\right) dx\,ds, \tag{2.43}$$

where the mapping $s \mapsto \|\nabla\mu(s)\|_H^2$ is known to be bounded in $L^1(0,T)$, uniformly with respect to $\alpha \in (0,1]$. We thus may combine (2.30)–(2.34) with (2.43) to infer from Gronwall's lemma that

$$\|\rho^\alpha\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\rho_\Gamma^\alpha\|_{W^{1,\infty}(0,T;H_\Gamma)\cap H^1(0,T;V_\Gamma)} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.44}$$

Therefore, we can conclude from (2.41) and (2.42) that also, for all $\alpha \in (0,1]$,

$$\|\varphi(\alpha)\,h'(\rho^\alpha)\|_{L^\infty(0,T;H)} + \|\varphi(\alpha)\,h'(\rho_\Gamma^\alpha)\|_{L^\infty(0,T;H_\Gamma)} + \|\Delta\rho^\alpha\|_{L^\infty(0,T;H)} \le C. \tag{2.45}$$

Since we already know from (2.29) the bound for $\|\rho_\Gamma^\alpha\|_{C^0([0,T];V_\Gamma)}$, we can follow the same chain of estimates as in the fourth a priori estimate above, eventually obtaining that

$$\|\rho^\alpha\|_{L^\infty(0,T;H^2(\Omega))} + \|\rho_\Gamma^\alpha\|_{L^\infty(0,T;H^2(\Gamma))} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.46}$$

Sixth estimate:

Next, we multiply (2.9) by μ_t and integrate over Q_t, where $t \in (0,T]$. Recalling that $g(\rho)$ is nonnegative, and using Hölder's and Young's inequalities, we obtain from (A1) that

$$\begin{aligned}
&\int_0^t\int_\Omega |\mu_t|^2\,dx\,ds + \frac{1}{2}\|\nabla\mu(t)\|_H^2 \le \frac{1}{2}\|\nabla\mu_0\|_H^2 + C\int_0^t\int_\Omega |\mu_t|\,|\mu|\,|\rho_t|\,dx\,ds \\
&\le C + C\int_0^t \|\mu_t(s)\|_2\,\|\mu(s)\|_4\,\|\rho_t(s)\|_4\,ds \\
&\le C + \frac{1}{2}\int_0^t\int_\Omega |\mu_t|^2\,dx\,ds + C\int_0^t \|\rho_t(s)\|_V^2\,\|\mu(s)\|_V^2\,ds,
\end{aligned} \tag{2.47}$$

where, owing to (2.44), the mapping $s \mapsto \|\rho_t(s)\|_V^2$ is bounded in $L^1(0,T)$, uniformly in $\alpha \in (0,1]$. We thus can infer from Gronwall's lemma that

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le C. \tag{2.48}$$

Comparison in (2.9) then shows that also

$$\|\Delta\mu\|_{L^2(0,T;H)} \le C, \tag{2.49}$$

whence, by virtue of standard elliptic estimates,

$$\|\mu\|_{L^2(0,T;W)} \le C. \tag{2.50}$$

Since the embedding $(H^1(0,T;H) \cap L^2(0,T;H^2(\Omega))) \subset C^0([0,T];V)$ is continuous, we have thus shown the estimate

$$\|\mu^\alpha\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.51}$$

Next, we use the continuity of the embedding

$$(L^\infty(0,T;H) \cap L^2(0,T;V)) \subset L^{7/3}(0,T;L^{14/3}(\Omega)),$$

which, in view of (2.44), implies that

$$\left\|\rho_t^\alpha\right\|_{L^{7/3}(0,T;L^{14/3}(\Omega))} \le C \quad \forall\alpha \in (0,1]. \tag{2.52}$$

With this estimate shown, we may argue as in the proof of [6, Thm. 2.3] to conclude that

$$\|\mu^\alpha\|_{L^\infty(Q)} \le C \quad \forall\,\alpha \in (0,1]. \tag{2.53}$$

Hence, the assertion is completely proved. □

3 Existence and Approximation of Optimal Controls

In this section, we aim to approximate optimal pairs of $(\mathscr{P}_0)$. To this end, we consider for $\alpha \in (0,1]$ the optimal control problem

$(\mathscr{P}_\alpha)$ Minimize the cost functional $\mathscr{J}((\mu^\alpha,\rho^\alpha,\rho_\Gamma^\alpha),u_\Gamma^\alpha)$ for $u_\Gamma^\alpha \in \mathscr{U}_{\rm ad}$, subject to the state system (2.9)–(2.13).

According to [25, Thm. 4.1], this optimal control problem has an optimal pair $((\mu^\alpha,\rho^\alpha,\rho_\Gamma^\alpha),u_\Gamma^\alpha)$, for every $\alpha \in (0,1]$. Our first aim in this section is to prove the following approximation result:

Theorem 3.1 *Suppose that the assumptions (A1)–(A5) are satisfied, and let the sequences* $\{\alpha_n\} \subset (0,1]$ *and* $\{u_\Gamma^{\alpha_n}\} \subset \mathscr{U}_{\rm ad}$ *be given such that* $\alpha_n \searrow 0$ *and* $u_\Gamma^{\alpha_n} \to u_\Gamma$

weakly-star in $\mathscr{X}$ *for some* $u_\Gamma \in \mathscr{U}_{\rm ad}$. *Then it holds, for* $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$, $n \in \mathbb{N}$,

$$\mu^{\alpha_n} \to \mu \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q), \tag{3.1}$$

$$\rho^{\alpha_n} \to \rho \quad \text{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{3.2}$$

$$\rho_\Gamma^{\alpha_n} \to \rho_\Gamma \quad \text{weakly-star in } W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)), \tag{3.3}$$

as well as

$$\varphi(\alpha_n)\, h'(\rho^{\alpha_n}) \to \xi \quad \text{weakly-star in } L^\infty(0,T;H), \tag{3.4}$$

$$\varphi(\alpha_n)\, h'(\rho_\Gamma^{\alpha_n}) \to \xi_\Gamma \quad \text{weakly-star in } L^\infty(0,T;H_\Gamma), \tag{3.5}$$

where $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ *is the unique solution to the state system* (1.2)–(1.8) *associated with* u_Γ. *Moreover, with* $\mathscr{S}_0(u_\Gamma) = (\mu, \rho, \rho_\Gamma)$ *it holds that*

$$\mathscr{J}(\mathscr{S}_0(u_\Gamma), u_\Gamma) \;\le\; \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n}), \tag{3.6}$$

$$\mathscr{J}(\mathscr{S}_0(v_\Gamma), v_\Gamma) \;=\; \lim_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma), v_\Gamma) \quad \forall\, v_\Gamma \in \mathscr{U}_{\rm ad}. \tag{3.7}$$

Proof Let $\{\alpha_n\} \subset (0,1]$ be any sequence such that $\alpha_n \searrow 0$ as $n \to \infty$, and suppose that $u_\Gamma^{\alpha_n} \to u_\Gamma$ weakly-star in $\mathscr{X}$ for some $u_\Gamma \in \mathscr{U}_{\rm ad}$. By virtue of Proposition 2.3, there are a subsequence of $\{\alpha_n\}$, which is again indexed by n, and some quintuple $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ such that the convergence results (3.1)–(3.5) hold true. In particular, we have $\mu(0) = \mu_0$ and $\rho(0) = \rho_0$. Moreover, from standard compact embedding results (cf. [34, Sect. 8, Cor. 4]) we can infer that

$$\mu^{\alpha_n} \to \mu \quad \text{strongly in } C^0(0,T;H) \cap L^2(0,T;V), \tag{3.8}$$

$$\rho^{\alpha_n} \to \rho \quad \text{strongly in } C^0(\overline{Q}), \tag{3.9}$$

also including

$$\rho_\Gamma^{\alpha_n} \to \rho_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{3.10}$$

whence we infer that $\rho_\Gamma = \rho_{|\Sigma}$. Therefore, we obviously have that

$$\Psi(\rho^{\alpha_n}) \to \Psi(\rho) \quad \text{strongly in } C^0(\overline{Q}), \text{for } \Psi \in \{g, g', \pi\}, \tag{3.11}$$

$$\pi_\Gamma(\rho_\Gamma^{\alpha_n}) \to \pi_\Gamma(\rho_\Gamma) \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{3.12}$$

and (3.2) implies that $\partial_{\mathbf{n}}\rho^{\alpha_n} \to \partial_{\mathbf{n}}\rho$ weakly in $L^2(\Sigma)$. Further, we easily verify that, at least weakly in $L^1(Q)$,

$$g(\rho^{\alpha_n})\,\mu_t^{\alpha_n} \to g(\rho)\,\mu_t, \quad \mu^{\alpha_n}\,g'(\rho^{\alpha_n})\,\rho_t^{\alpha_n} \to \mu\,g'(\rho)\,\rho_t, \quad \mu^{\alpha_n}\,g'(\rho^{\alpha_n}) \to \mu\,g'(\rho). \tag{3.13}$$

Combining the above convergence results, we may pass to the limit as $n \to \infty$ in Eqs. (2.9)–(2.13) (written for $\alpha = \alpha_n$) to find that the quintuple $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ satisfies Eqs. (1.2)–(1.4), (1.6), and (1.8). In addition, we have $\mu \ge 0$ in Q, and the properties in (2.6) are fulfilled. We also notice that the regularities in (2.2)–(2.3) follow from $\mu_0 \in W$ (cf. (A1)) and the regularity theory for solutions to linear uniformly parabolic equations with continuous coefficients and right-hand side in $L^\infty(0,T;H) \cap L^2(0,T;L^6(\Omega))$ (comments are given in [24, Section 3, Step 4 and Remark 3.1]). Then, in order to show that the quintuple $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ is in fact the unique solution to problem (1.2)–(1.8) corresponding to u_Γ, it remains to show that $\xi \in \partial I_{[-1,1]}(\rho)$ a. e. in Q and $\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma)$ a. e. in Σ.

Now, recall that h is convex in $[-1,1]$ and both h and φ are nonnegative. We thus have, for every $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\le \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\,h(\rho^{\alpha_n})\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\,h(z)\,\mathrm{d}x\,\mathrm{d}t \,+\, \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\,h'(\rho^{\alpha_n})\,(\rho^{\alpha_n} - z)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\qquad \text{for all } z \in \mathscr{K} := \{v \in L^2(Q) : |v| \le 1 \text{ a.e. in } Q\}\,. \end{aligned} \tag{3.14}$$

Thanks to (1.13), the first integral on the central line of (3.14) tends to zero as $n \to \infty$. Hence, invoking (3.4) and (3.9), the passage to the limit as $n \to \infty$ yields

$$\int_0^T\!\!\int_\Omega \xi\,(\rho - z)\,\mathrm{d}x\,\mathrm{d}t \,\ge 0 \quad \forall z \in \mathscr{K}\,. \tag{3.15}$$

Inequality (3.15) entails that ξ is an element of the subdifferential of the extension $\mathscr{I}$ of $I_{[-1,1]}$ to $L^2(Q)$, which means that $\xi \in \partial\mathscr{I}(\rho)$ or, equivalently (cf. [2, Ex. 2.3.3., p. 25]), $\xi \in \partial I_{[-1,1]}(\rho)$ a. e. in Q. Similarly, we can prove that $\xi_\Gamma \in \partial I_{[-1,1]}(\rho_\Gamma)$ a. e. in Σ.

We have thus shown that, for a suitable subsequence of $\{\alpha_n\}$, we have the convergence properties (3.1)–(3.5), where $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ is a solution to the state system (1.2)–(1.8). But this solution is known to be unique, which entails that the above convergence properties are valid for the entire sequence. This finishes the proof of the first claim of the theorem.

It remains to show the validity of (3.6) and (3.7). In view of (3.1)–(3.3), the inequality (3.6) is an immediate consequence of the weak sequential semicontinuity properties of the cost functional $\mathscr{J}$. To establish the identity (3.7), let $v_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ be arbitrary and put $(\mu^{\alpha^n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(v_\Gamma)$, for $n \in \mathbb{N}$. Taking Proposition 2.3 into account, and arguing as in the first part of this proof, we can conclude that $\mathscr{S}_{\alpha_n}(v_\Gamma)$ converges to $(\mu, \rho, \rho_\Gamma) = \mathscr{S}_0(v_\Gamma)$ in the sense of (3.1)–(3.3) and (3.8)–(3.10). In particular, we have

$$\mathscr{S}_{\alpha_n}(v_\Gamma) \to \mathscr{S}_0(v_\Gamma) \quad \text{strongly in } C^0([0,T];H) \times C^0([0,T];H) \times C^0([0,T];H_\Gamma).$$

As the cost functional $\mathscr{J}$ is obviously continuous in the variables (μ, ρ, ρ_Γ) with respect to the strong topology of $C^0([0,T];H) \times C^0([0,T];H) \times C^0([0,T];H_\Gamma)$, we may thus infer that (3.7) is valid. □

Corollary 3.2 *The optimal control problem* ($\mathscr{P}_0$) *has a least one solution.*

Proof Pick an arbitrary sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$ as $n \to \infty$. Then, by virtue of [25, Thm. 4.1], the optimal control problem ($\mathscr{P}_{\alpha_n}$) has for every $n \in \mathbb{N}$ an optimal pair $((\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n})$, where $u_\Gamma^{\alpha_n} \in \mathscr{U}_{\mathrm{ad}}$ and $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$. Since $\mathscr{U}_{\mathrm{ad}}$ is a bounded subset of $\mathscr{X}$, we may without loss of generality assume that $u_\Gamma^{\alpha_n} \to u_\Gamma$ weakly-star in $\mathscr{X}$ for some $u_\Gamma \in \mathscr{U}_{\mathrm{ad}}$. Then, for the unique solution $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ to (1.2)–(1.8) associated with u_Γ, we conclude from Theorem 3.1 the convergence properties (3.1)–(3.7). Invoking the optimality of $((\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n})$ for ($\mathscr{P}_{\alpha_n}$), we then find, for every $v_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, that

$$\begin{aligned}
\mathscr{J}((\mu, \rho, \rho_\Gamma), u_\Gamma) \;&=\; \mathscr{J}(\mathscr{S}_0(u_\Gamma), u_\Gamma) \;\le\; \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n}) \\
&\le\; \liminf_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma), v_\Gamma) \;=\; \lim_{n\to\infty} \mathscr{J}(\mathscr{S}_{\alpha_n}(v_\Gamma), v_\Gamma) \;=\; \mathscr{J}(\mathscr{S}_0(v_\Gamma), v_\Gamma),
\end{aligned} \tag{3.16}$$

which yields that u_Γ is an optimal control for ($\mathscr{P}_0$) with the associate state $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$. The assertion is thus proved. □

Corollary 3.2 does not yield any information on whether every solution to the optimal control problem ($\mathscr{P}_0$) can be approximated by a sequence of solutions to the problems ($\mathscr{P}_\alpha$). As already announced in the Introduction, we are not able to prove such a general 'global' result. Instead, we can only give a 'local' answer for every individual optimizer of ($\mathscr{P}_0$). For this purpose, we employ a trick due to Barbu [1]. To this end, let $\bar{u}_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ be an arbitrary optimal control for ($\mathscr{P}_0$), and let $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$ be the associated solution quintuple to the state system (1.2)–(1.8) in the sense of Theorem 2.1. In particular, $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma) = \mathscr{S}_0(\bar{u}_\Gamma)$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathscr{J}}((\mu, \rho, \rho_\Gamma), u_\Gamma) := \mathscr{J}((\mu, \rho, \rho_\Gamma), u_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} \tag{3.17}$$

and a corresponding *adapted optimal control problem*,

($\widetilde{\mathscr{P}}_\alpha$) Minimize $\widetilde{\mathscr{J}}((\mu, \rho, \rho_\Gamma), u_\Gamma)$ for $u_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, subject to the condition that (2.9)–(2.13) be satisfied.

With a standard direct argument that needs no repetition here, we can show the following result.

Lemma 3.3 *Suppose that the assumptions (A1)–(A5), (1.12)–(1.13) are satisfied, and let* $\alpha \in (0, 1]$. *Then the optimal control problem* $(\widetilde{\mathscr{P}}_\alpha)$ *admits a solution.*

We are now in the position to give a partial answer to the question raised above. We have the following result.

Theorem 3.4 *Let the assumptions (A1)–(A5), (1.12)–(1.13) be fulfilled, suppose that* $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ *is an arbitrary optimal control of* $(\mathscr{P}_0)$ *with associated state quintuple* $(\bar\mu, \bar\rho, \bar\rho_\Gamma, \bar\xi, \bar\xi_\Gamma)$, *and let* $\{\alpha_n\} \subset (0, 1]$ *be any sequence such that* $\alpha_n \searrow 0$ *as* $n \to \infty$. *Then there exist a subsequence* $\{\alpha_{n_k}\}$ *of* $\{\alpha_n\}$, *and, for every* $k \in \mathbb{N}$, *an optimal control* $u_\Gamma^{\alpha_{n_k}} \in \mathscr{U}_{\rm ad}$ *of the adapted problem* $(\widetilde{\mathscr{P}}_{\alpha_{n_k}})$ *with associated state triple* $(\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}})$ *such that, as* $k \to \infty$,

$$u_\Gamma^{\alpha_{n_k}} \to \bar{u}_\Gamma \quad \text{strongly in } L^2(\Sigma), \tag{3.18}$$

the properties (3.1)–(3.5) *are satisfied, where* $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$

$$\textit{is replaced by } (\bar\mu, \bar\rho, \bar\rho_\Gamma, \bar\xi, \bar\xi_\Gamma) \textit{ and the index } n \textit{ is replaced by } n_k, \tag{3.19}$$

$$\widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \to \mathscr{J}((\mu, \rho, \rho_\Gamma), u_\Gamma)\,. \tag{3.20}$$

Proof Let $\alpha_n \searrow 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we pick an optimal control $u_\Gamma^{\alpha_n} \in \mathscr{U}_{\rm ad}$ for the adapted problem $(\widetilde{\mathscr{P}}_{\alpha_n})$ and denote by $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}) = \mathscr{S}_{\alpha_n}(u_\Gamma^{\alpha_n})$ the associated solution triple of problem (2.9)–(2.13) for $\alpha = \alpha_n$. By the boundedness of $\mathscr{U}_{\rm ad}$, there is some subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that

$$u_\Gamma^{\alpha_{n_k}} \to u_\Gamma \quad \text{weakly-star in } \mathscr{X} \quad \text{as } k \to \infty, \tag{3.21}$$

with some $u_\Gamma \in \mathscr{U}_{\rm ad}$. Thanks to Theorem 3.1, the convergence properties (3.1)–(3.5) hold true, where $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma)$ is the unique solution to the state system (1.2)–(1.8). In particular, the pair $(\mathscr{S}_0(u_\Gamma), u_\Gamma) = ((\mu, \rho, \rho_\Gamma), u_\Gamma)$ is admissible for $(\mathscr{P}_0)$.

We now aim to prove that $u_\Gamma = \bar{u}_\Gamma$. Once this is shown, then the uniqueness result of Theorem 2.1 yields that also $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma) = (\bar\mu, \bar\rho, \bar\rho_\Gamma, \bar\xi, \bar\xi_\Gamma)$, which implies that (3.19) holds true.

Now, observe that, owing to the weak sequential lower semicontinuity of $\widetilde{\mathscr{J}}$, and in view of the optimality property of $((\bar\mu, \bar\rho, \bar\rho_\Gamma), \bar{u}_\Gamma)$ for problem $(\mathscr{P}_0)$,

$$\liminf_{k\to\infty} \widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \geq \mathscr{J}((\mu, \rho, \rho_\Gamma), u_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)}$$

$$\geq \mathscr{J}((\bar\mu, \bar\rho, \bar\rho_\Gamma), \bar{u}_\Gamma) + \frac{1}{2}\,\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)}\,. \tag{3.22}$$

On the other hand, the optimality property of $((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}})$ for problem $(\widetilde{\mathscr{P}}_{\alpha_{n_k}})$ yields that for any $k \in \mathbb{N}$ we have

$$\widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) = \widetilde{\mathscr{J}}(\mathscr{S}_{\alpha_{n_k}}(u_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) \le \widetilde{\mathscr{J}}(\mathscr{S}_{\alpha_{n_k}}(\bar{u}_\Gamma), \bar{u}_\Gamma), \tag{3.23}$$

whence, taking the limit superior as $k \to \infty$ on both sides and invoking (3.7) in Theorem 3.1,

$$\limsup_{k\to\infty} \widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}})$$
$$\le \widetilde{\mathscr{J}}(\mathscr{S}_0(\bar{u}_\Gamma), \bar{u}_\Gamma) = \widetilde{\mathscr{J}}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \mathscr{J}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma). \tag{3.24}$$

Combining (3.22) with (3.24), we have thus shown that $\frac{1}{2}\|u_\Gamma - \bar{u}_\Gamma\|^2_{L^2(\Sigma)} = 0$, so that $u_\Gamma = \bar{u}_\Gamma$ and thus also $(\mu, \rho, \rho_\Gamma, \xi, \xi_\Gamma) = (\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$. Moreover, (3.22) and (3.24) also imply that

$$\mathscr{J}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \widetilde{\mathscr{J}}((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma), \bar{u}_\Gamma) = \liminf_{k\to\infty} \widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}})$$
$$= \limsup_{k\to\infty} \widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}) = \lim_{k\to\infty} \widetilde{\mathscr{J}}((\mu^{\alpha_{n_k}}, \rho^{\alpha_{n_k}}, \rho_\Gamma^{\alpha_{n_k}}), u_\Gamma^{\alpha_{n_k}}), \tag{3.25}$$

which proves (3.20) and, at the same time, also (3.18). This concludes the proof of the assertion. □

4 The Optimality System

In this section, we aim to establish first-order necessary optimality conditions for the optimal control problem $(\mathscr{P}_0)$. This will be achieved by a passage to the limit as $\alpha \searrow 0$ in the first-order necessary optimality conditions for the adapted optimal control problems $(\widetilde{\mathscr{P}}_\alpha)$ that can by derived as in [25] with only minor and obvious changes. This procedure will yield certain generalized first-order necessary optimality conditions in the limit. In this entire section, we assume that h is given by (1.12) and that (1.13) and the general assumptions (A1)–(A5) are satisfied. We also assume that a fixed optimal control $\bar{u}_\Gamma \in \mathscr{U}_{\rm ad}$ for $(\mathscr{P}_0)$ is given, along with the corresponding solution quintuple $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma)$ of the state system (1.2)–(1.8) established in Theorem 2.1. That is, we have $(\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma) = \mathscr{S}_0(\bar{u}_\Gamma)$ as well as $\bar{\xi} \in \partial I_{[-1,1]}(\bar{\rho})$ a. e. in Q and $\bar{\xi}_\Gamma \in \partial I_{[-1,1]}(\bar{\rho}_\Gamma)$ a. e. on Σ.

In order to be able to take advantage of the analysis performed in [25, Sect. 4], we impose the following additional compatibility condition:

(A6) It holds that $(\beta_4(\bar\rho(T)-\hat\rho_\Omega)\,,\,\beta_5(\bar\rho_\Gamma(T)-\hat\rho_\Gamma))\in\mathscr{V}$.

Obviously, (A6) is fulfilled if $\beta_4=\beta_5$ (especially if $\beta_4=\beta_5=0$) and $(\hat\rho_\Omega,\hat\rho_\Gamma)\in\mathscr{V}$. In view of the fact that always $(\bar\rho(T),\bar\rho_\Gamma(T))\in\mathscr{V}$, these conditions for the target functions $\hat\rho_\Omega$ and $\hat\rho_\Gamma$ seem to be quite reasonable.

We begin our analysis by formulating the adjoint state system for the adapted control problem $(\widetilde{\mathscr{P}_\alpha})$. To this end, let us assume that $u_\Gamma^\alpha\in\mathscr{U}_{\rm ad}$ is an arbitrary optimal control for $(\widetilde{\mathscr{P}}_\alpha)$ and that $(\mu^\alpha,\rho^\alpha,\rho_\Gamma^\alpha)$ is the solution triple to the associated state system (2.9)–(2.13). In particular, $(\mu^\alpha,\rho^\alpha,\rho_\Gamma^\alpha)=\mathscr{S}_\alpha(u_\Gamma^\alpha)$, the solution has the regularity properties (2.2)–(2.5), and it satisfies the global bounds (2.15), (2.16), as well as the separation property (2.14). Moreover, it follows from [25, Thm. 4.2] that the associated adjoint system

$$-(1+2g(\rho^\alpha))\,p_t^\alpha-g'(\rho^\alpha)\,\rho_t^\alpha\,p^\alpha-\Delta p^\alpha\;=\;g'(\rho^\alpha)\,q^\alpha+\beta_1(\mu^\alpha-\hat\mu_Q)\quad\text{a.e. in } Q, \tag{4.1}$$

$$\partial_{\mathbf n}p^\alpha=0\quad\text{a.e. on }\Sigma,\qquad p^\alpha(T)=0\quad\text{a.e. in }\Omega, \tag{4.2}$$

$$\begin{aligned}&-q_t^\alpha-\Delta q^\alpha+(\varphi(\alpha)\,h''(\rho^\alpha)+\pi'(\rho^\alpha)-\mu^\alpha\,g''(\rho^\alpha))\,q^\alpha\\&\quad=\;g'(\rho^\alpha)(\mu^\alpha\,p_t^\alpha-\mu_t^\alpha\,p^\alpha)+\beta_2(\rho^\alpha-\hat\rho_Q)\quad\text{a.e. in } Q,\end{aligned} \tag{4.3}$$

$$\partial_{\mathbf n}q^\alpha-\partial_t q^\alpha-\Delta_\Gamma q_\Gamma^\alpha+(\varphi(\alpha)\,h''(\rho_\Gamma^\alpha)+\pi_\Gamma'(\rho_\Gamma^\alpha))\,q_\Gamma^\alpha\;=\;\beta_3(\rho_\Gamma^\alpha-\hat\rho_\Sigma),\quad\text{and }\; q_\Gamma^\alpha=q^\alpha_{|\Sigma},\quad\text{a.e. on }\Sigma, \tag{4.4}$$

$$q^\alpha(T)=\beta_4(\rho^\alpha(T)-\hat\rho_\Omega)\quad\text{a.e. in }\Omega,\qquad q_\Gamma^\alpha(T)=\beta_5(\rho_\Gamma^\alpha(T)-\hat\rho_\Gamma)\quad\text{a.e. on }\Gamma \tag{4.5}$$

has a unique solution $(p^\alpha,q^\alpha,q_\Gamma^\alpha)$ such that

$$p^\alpha\in H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W), \tag{4.6}$$

$$q^\alpha\in H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;H^2(\Omega)), \tag{4.7}$$

$$q_\Gamma^\alpha\in H^1(0,T;H_\Gamma)\cap C^0([0,T];V_\Gamma)\cap L^2(0,T;H^2(\Gamma)). \tag{4.8}$$

In addition, as in the proof of [25, Cor. 4.3], it follows the validity of the variational inequality

$$\int_0^T\!\!\int_\Gamma\big(q_\Gamma^\alpha+\beta_6\,u_\Gamma^\alpha+(u_\Gamma^\alpha-\bar u_\Gamma)\big)(v_\Gamma-u_\Gamma^\alpha)\,\mathrm d\Gamma\,\mathrm dt\;\ge\;0\quad\forall\,v_\Gamma\in\mathscr{U}_{\rm ad}\,. \tag{4.9}$$

We now prove an a priori estimate that will be fundamental for the derivation of the optimality conditions for $(\mathscr{P}_0)$. To this end, we introduce some further function spaces. At first, we put

$$Y := H^1(0,T;V^*) \cap L^2(0,T;V), \quad Y_\Gamma := H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma), \tag{4.10}$$

$$\mathscr{W} := \big(H^1(0,T;V^*) \times H^1(0,T;V_\Gamma^*)\big) \cap L^2(0,T;\mathscr{V}), \tag{4.11}$$

$$\mathscr{W}_0 := \{(\eta,\eta_\Gamma) \in \mathscr{W} : (\eta(0),\eta_\Gamma(0)) = (0,0)\}, \tag{4.12}$$

which are Banach spaces when equipped with the natural norm of $Y \times Y_\Gamma$. Moreover, we have the dense and continuous injections $Y \subset L^2(0,T;V) \subset L^2(Q) \subset L^2(0,T;V^*) \subset Y^*$ and $Y_\Gamma \subset L^2(0,T;V_\Gamma) \subset L^2(\Sigma) \subset L^2(0,T;V_\Gamma^*) \subset Y_\Gamma^*$, where it is understood that

$$\langle z, v\rangle_Y = \int_0^T \langle z(t), v(t)\rangle_V \, \mathrm{d}t \qquad \text{for all } z \in L^2(0,T;V^*) \text{ and } v \in L^2(0,T;V), \tag{4.13}$$

$$\langle z_\Gamma, v_\Gamma\rangle_{Y_\Gamma} = \int_0^T \langle z_\Gamma(t), v_\Gamma(t)\rangle_{V_\Gamma} \, \mathrm{d}t \qquad \text{for all } z_\Gamma \in L^2(0,T;V_\Gamma^*) \text{ and } v_\Gamma \in L^2(0,T;V_\Gamma). \tag{4.14}$$

We also note that the embeddings $Y \subset C^0([0,T];H)$ and $Y_\Gamma \subset C^0([0,T];H_\Gamma)$ are continuous. Likewise, we have the dense and continuous embeddings $\mathscr{W} \subset L^2(0,T;\mathscr{V}) \subset L^2(0,T;H \times H_\Gamma) \subset L^2(0,T;\mathscr{V}^*) \subset \mathscr{W}^*$, as well as the continuous injection $\mathscr{W} \subset C^0([0,T];H \times H_\Gamma)$, which gives the initial condition encoded in (4.12) a proper meaning. Furthermore, since $\mathscr{W}_0$ is a closed subspace of $Y \times Y_\Gamma$, we deduce that the elements $F = (z, z_\Gamma) \in \mathscr{W}_0^*$ are exactly those that are of the form

$$\langle F, (\eta,\eta_\Gamma)\rangle_{\mathscr{W}_0} = \langle z, \eta\rangle_Y + \langle z_\Gamma, \eta_\Gamma\rangle_{Y_\Gamma} \qquad \text{for all } (\eta,\eta_\Gamma) \in \mathscr{W}_0, \tag{4.15}$$

where $z \in Y^*$ and $z_\Gamma \in Y_\Gamma^*$. In particular, for $z \in L^2(0,T;V^*)$ and $z_\Gamma \in L^2(0,T;V_\Gamma^*)$ the formulas (4.13) and (4.14) apply. Observe that these representation formulas allow us to give a proper meaning to statements like

$$(z^\alpha, z_\Gamma^\alpha) \to (z, z_\Gamma) \qquad \text{weakly in } \mathscr{W}_0^*.$$

In addition to the spaces introduced in (4.10)–(4.12), we also define

$$\mathscr{X} := (L^\infty(0,T;H) \times L^\infty(0,T;H_\Gamma)) \cap L^2(0,T;\mathscr{V}), \tag{4.16}$$

which is a Banach spaced when endowed with its natural norm.

We have the following result.

Proposition 4.1 *Let the general assumptions (A1)–(A6), (1.12)–(1.13) be satisfied, and let*

$$(\lambda^\alpha, \lambda_\Gamma^\alpha) := \big(\varphi(\alpha)\, h''(\rho^\alpha)\, q^\alpha\,,\ \varphi(\alpha)\, h''(\rho_\Gamma^\alpha)\, q_\Gamma^\alpha\big) \quad \forall\, \alpha \in (0,1]. \tag{4.17}$$

Then there exists a constant $K_3^* > 0$, which depends only on the data of the system and on R, such that for all $\alpha \in (0,1]$ it holds

$$\|p^\alpha\|_{H^1(0,T;H)\cap C^0([0,T];V)\cap L^2(0,T;W)} + \max_{0\le t\le T} \big(\|q^\alpha(t)\|_H + \|q_\Gamma^\alpha(t)\|_{H_\Gamma}\big)$$

$$+ \|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{V})} + \|(\lambda^\alpha, \lambda_\Gamma^\alpha)\|_{\mathcal{W}_0^*} + \big\|\big(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha\big)\big\|_{\mathcal{W}_0^*} \le K_3^*\,. \tag{4.18}$$

Proof In the following, $C > 0$ denote positive constants that may depend on the data of the system but not on $\alpha \in (0,1]$. We make repeated use of the global estimates (2.15) and (2.16) without further reference.

First, we add p^α on both sides of (4.1), multiply the result by $-p_t^\alpha$, and integrate over $\Omega \times (t,T]$, where $t \in [0,T)$. Using the fact that $p^\alpha(T) = 0$, we obtain the inequality

$$\int_t^T\!\!\int_\Omega |p_t^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\,\|p^\alpha(t)\|_V^2 \le I_1 + I_2 + I_3, \tag{4.19}$$

where the quantities I_j, $1 \le j \le 3$, are specified and estimated below. At first, Young's inequality yields that

$$\begin{aligned} I_1 :&= -\int_t^T\!\!\int_\Omega \big(p^\alpha + \beta_1(\mu^\alpha - \hat\mu_Q)\big)\, p_t^\alpha \,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{1}{5}\int_t^T\!\!\int_\Omega |p_t^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + C + C\int_t^T\!\!\int_\Omega |p^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s\,. \end{aligned} \tag{4.20}$$

Likewise, we have that

$$I_2 := -\int_t^T\!\!\int_\Omega g'(\rho^\alpha)\, q^\alpha\, \rho_t^\alpha \,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{5}\int_t^T\!\!\int_\Omega |\rho_t^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_t^T\!\!\int_\Omega |q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s\,. \tag{4.21}$$

Moreover, by also invoking Hölder's inequality and the continuity of the embedding $V \subset L^4(\Omega)$, we deduce that

$$\begin{aligned} I_3 :&= -\int_t^T\!\!\int_\Omega g'(\rho^\alpha)\, \rho_t^\alpha\, p^\alpha\, p_t^\alpha \,\mathrm{d}x\,\mathrm{d}s \le C\int_t^T \|\rho_t^\alpha(s)\|_4\, \|p^\alpha(s)\|_4\, \|p_t^\alpha(s)\|_2 \,\mathrm{d}s \\ &\le \frac{1}{5}\int_t^T\!\!\int_\Omega |p_t^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + C\int_t^T \|\rho_t^\alpha(s)\|_V^2\, \|p^\alpha(s)\|_V^2 \,\mathrm{d}s\,, \end{aligned} \tag{4.22}$$

where the mapping $s \mapsto \|\rho_t^\alpha(s)\|_V^2$ is bounded in $L^1(0,T)$ uniformly with respect to $\alpha \in (0,1]$.

Next, we multiply (4.3) by q^α and integrate over $\Omega \times (t,T]$, where $t \in [0,T)$. Taking (4.4) into account, we obtain the identity

$$\begin{aligned}
&\frac{1}{2}\left(\|q^\alpha(t)\|_H^2 + \|q_\Gamma^\alpha(t)\|_{H_\Gamma}^2\right) + \int_t^T\!\!\int_\Omega |\nabla q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Gamma |\nabla_\Gamma q_\Gamma^\alpha|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&+ \int_t^T\!\!\int_\Omega \varphi(\alpha)\, h''(\rho^\alpha)\, |q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Gamma \varphi(\alpha)\, h''(\rho_\Gamma^\alpha)\, |q_\Gamma^\alpha|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&= \frac{1}{2}\left(\|q^\alpha(T)\|_H^2 + \|q_\Gamma^\alpha(T)\|_{H_\Gamma}^2\right) \\
&\quad + \int_t^T\!\!\int_\Omega \left(\mu^\alpha g''(\rho^\alpha) - \pi'(\rho^\alpha)\right) |q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Omega \beta_2(\rho^\alpha - \hat{\rho}_Q)\, q^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_t^T\!\!\int_\Gamma \pi_\Gamma'(\rho_\Gamma^\alpha)\, |q_\Gamma^\alpha|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s + \int_t^T\!\!\int_\Gamma \beta_3(\rho_\Gamma^\alpha - \hat{\rho}_\Sigma)\, q_\Gamma^\alpha \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_t^T\!\!\int_\Omega g'(\rho^\alpha)\, \mu^\alpha\, p_t^\alpha\, q^\alpha \,\mathrm{d}x\,\mathrm{d}s - \int_t^T\!\!\int_\Omega g'(\rho^\alpha)\, \mu_t^\alpha\, p^\alpha\, q^\alpha \,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{4.23}$$

Since $\varphi(\alpha)\, h'' \geq 0$, all summands on the left-hand side are nonnegative. Moreover, invoking (4.5) and Young's inequality, it is readily seen that the first five summands on the right-hand side are bounded by an expression of the form

$$C\Big(1 + \int_t^T\!\!\int_\Omega |q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_t^T\!\!\int_\Gamma |q_\Gamma^\alpha|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s\Big)\,. \tag{4.24}$$

It thus remains to estimate the last two summands on the right-hand side, which we denote by J_1 and J_2, respectively. By virtue of Hölder's and Young's inequality, we first have that

$$\begin{aligned}
J_1 &\leq C \int_t^T \|\mu^\alpha(s)\|_\infty\, \|p_t^\alpha(s)\|_2\, \|q^\alpha(s)\|_2 \,\mathrm{d}s \\
&\leq \frac{1}{5}\int_t^T\!\!\int_\Omega |p_t^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s + C \int_t^T\!\!\int_\Omega |q^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned} \tag{4.25}$$

while, also using the continuity of the embedding $V \subset L^4(\Omega)$,

$$\begin{aligned}
J_2 &\leq C \int_t^T \|\mu_t^\alpha(s)\|_2\, \|p^\alpha(s)\|_4\, \|q^\alpha(s)\|_4 \,\mathrm{d}s \\
&\leq \frac{1}{2}\int_t^T \|q^\alpha(s)\|_V^2 \,\mathrm{d}s + C \int_t^T \|\mu_t^\alpha(s)\|_H^2\, \|p^\alpha(s)\|_V^2 \,\mathrm{d}s\,,
\end{aligned} \tag{4.26}$$

where the mapping $s \mapsto \|\mu_t^\alpha(s)\|_H^2$ is known to be bounded in $L^1(0,T)$, uniformly in $\alpha \in (0,1]$. Therefore, combining the estimates (4.19)–(4.26), we obtain from Gronwall's lemma, taken backward in time, the estimate

$$\|p^\alpha\|_{H^1(0,T;H)} + \max_{0\le t\le T} \big(\|p^\alpha(t)\|_V + \|q^\alpha(t)\|_H + \|q_\Gamma^\alpha(t)\|_{H_\Gamma}\big) + \|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{V})} \le C. \tag{4.27}$$

Now, observe that

$$\big\|g'(\rho^\alpha)\,\rho_t^\alpha\,p^\alpha\big\|_{L^2(Q)}^2 \le C\int_0^t\int_\Omega |\rho_t^\alpha|^2\,|p^\alpha|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\int_0^T \|\rho_t^\alpha(s)\|_4^2\,\|p^\alpha(s)\|_4^2\,\mathrm{d}s \le C.$$

Thus, by comparison in (4.1), we find out that $\|\Delta p^\alpha\|_{L^2(Q)} \le C$, whence, by virtue of (4.2) and standard elliptic estimates,

$$\|p^\alpha\|_{L^2(0,T;W)} \le C. \tag{4.28}$$

Next, we derive the bound for the time derivatives. To this end, let $(\eta,\eta_\Gamma) \in \mathscr{W}_0$ be arbitrary. Using the continuity of the embeddings $Y \subset C^0([0,T];H)$ and $Y_\Gamma \subset C^0([0,T];H_\Gamma)$, and invoking the estimate (4.27), we obtain from integration by parts that

$$\langle(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha),(\eta,\eta_\Gamma)\rangle_{\mathscr{W}} = \int_0^T\int_\Omega q_t^\alpha\,\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Gamma \partial_t q_\Gamma^\alpha\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$= \int_\Omega q^\alpha(T)\,\eta(T)\,\mathrm{d}x + \int_\Gamma q_\Gamma^\alpha(T)\,\eta_\Gamma(T)\,\mathrm{d}\Gamma$$
$$- \int_0^T \langle \eta_t(t), q^\alpha(t)\rangle_V\;\mathrm{d}t - \int_0^T \langle \partial_t\eta_\Gamma(t), q_\Gamma^\alpha(t)\rangle_{V_\Gamma}\;\mathrm{d}t$$
$$\le \|q^\alpha(T)\|_H\,\|\eta(T)\|_H + \|q_\Gamma^\alpha(T)\|_{H_\Gamma}\,\|\eta_\Gamma(T)\|_{H_\Gamma}$$
$$+ \int_0^T \|\eta_t(t)\|_{V^*}\,\|q^\alpha(t)\|_V\,\mathrm{d}t + \int_0^T \|\partial_t\eta_\Gamma(t)\|_{V_\Gamma^*}\,\|q_\Gamma^\alpha(t)\|_V\,\mathrm{d}t,$$

whence

$$\langle(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha),(\eta,\eta_\Gamma)\rangle_{\mathscr{W}}$$
$$\le C\max_{0\le t\le T}\,(\|\eta(t)\|_H + \|\eta_\Gamma(t)\|_{H_\Gamma})$$
$$+ C\,\|(q^\alpha, q_\Gamma^\alpha)\|_{L^2(0,T;\mathcal{V})}\big(\|\eta_t\|_{L^2(0,T;V^*)} + \|\partial_t\eta_\Gamma\|_{L^2(0,T;V_\Gamma{}^*)}\big) \le C\,\|(\eta,\eta_\Gamma)\|_{\mathscr{W}_0}\,.$$

We thus have shown that

$$\|(\partial_t q^\alpha, \partial_t q_\Gamma^\alpha)\|_{\mathscr{W}_0^*} \le C. \tag{4.29}$$

Now, let $(\eta, \eta_\Gamma) \in \mathscr{W}_0$ be arbitrary. We define the functions

$$v_1^\alpha := \big(\mu^\alpha g''(\rho^\alpha) - \pi'(\rho^\alpha)\big)\, q^\alpha \;+\; g'(\rho^\alpha)\mu^\alpha p_t^\alpha, \quad v_2^\alpha := -g'(\rho^\alpha)\mu_t^\alpha p^\alpha,$$
$$w^\alpha := -\pi_\Gamma'(\rho_\Gamma^\alpha) q_\Gamma^\alpha\,. \tag{4.30}$$

Multiplying (4.3) by η, and invoking (4.4), we then easily infer the identity

$$\begin{aligned}
&\langle(\lambda^\alpha, \lambda_\Gamma^\alpha), (\eta, \eta_\Gamma)\rangle_{\mathscr{W}_0} \;=\; \int_0^T\!\!\int_\Omega \lambda^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^T\!\!\int_\Gamma \lambda_\Gamma^\alpha\, \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t\\
&= \int_0^T\!\!\int_\Omega \eta\, q_t^\alpha \,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^T\!\!\int_\Gamma \eta_\Gamma\, \partial_t q_\Gamma^\alpha\\
&\quad - \int_0^T\!\!\int_\Omega \nabla q^\alpha \cdot \nabla\eta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma^\alpha \cdot \nabla_\Gamma \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad + \int_0^T\!\!\int_\Omega v_1^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^T\!\!\int_\Omega v_2^\alpha \eta \,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^T\!\!\int_\Gamma w^\alpha\, \eta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad + \int_0^T\!\!\int_\Omega \beta_2(\rho^\alpha - \hat\rho_Q) \,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^T\!\!\int_\Gamma \beta_3(\rho_\Gamma^\alpha - \hat\rho_\Sigma) \,\mathrm{d}\Gamma\,\mathrm{d}t.
\end{aligned} \tag{4.31}$$

Now, observe that v_1^α and w^α are known to be bounded in $L^2(Q)$ and in $L^2(\Sigma)$, respectively, uniformly in $\alpha \in (0, 1]$. Also, using the continuity of the embedding $H^2(\Omega) \subset L^\infty(\Omega)$, we have that

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega v_2^\alpha\, \eta \,\mathrm{d}x\,\mathrm{d}t \;\le\; C \int_0^T \|\mu_t^\alpha(t)\|_2\, \|\eta(t)\|_2\, \|p^\alpha(t)\|_\infty \,\mathrm{d}t\\
&\le\; C \max_{0\le t\le T} \|\eta(t)\|_H\, \|\mu_t^\alpha\|_{L^2(Q)}\, \|p^\alpha\|_{L^2(0,T;H^2(\Omega))} \;\le\; C\,\|\eta\|_Y\,.
\end{aligned} \tag{4.32}$$

Therefore, taking (4.27) and (4.29) into account, we have shown that

$$\|(\lambda^\alpha, \lambda_\Gamma^\alpha)\|_{\mathscr{W}_0^*} \;\le\; C\,. \tag{4.33}$$

This concludes the proof of the assertion. □

After these preliminaries, we are now in a position to establish first-order necessary optimality conditions for $(\mathscr{P}_0)$ by performing a limit as $\alpha \searrow 0$ in the approximating problems. To this end, recall that a fixed optimal control $\bar u_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ for $(\mathscr{P}_0)$, along with a solution quintuple $(\bar\mu, \bar\rho, \bar\rho_\Gamma, \bar\xi, \bar\xi_\Gamma)$ of the associated state system (1.2)–(1.8) is given.

Now, we choose an arbitrary sequence $\{\alpha_n\}$ such that $\alpha_n \searrow 0$ as $n \to \infty$. By virtue of Theorem 3.4, we can find a subsequence, which is again indexed by n, such that, for any $n \in \mathbb{N}$, we can find an optimal control $u_\Gamma^{\alpha_n} \in \mathscr{U}_{\mathrm{ad}}$ for $(\widetilde{\mathscr{P}}_{\alpha_n})$ with associated state triple $(\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n})$ that satisfies the convergence properties (3.18)–(3.20). From [34, Sect. 8, Cor. 4], without loss of generality we may assume that

$$\mu^{\alpha_n} \to \bar{\mu} \quad \text{strongly in } C^0([0,T]; L^p(\Omega)) \quad \text{for } 1 \le p < 6, \tag{4.34}$$

$$\rho^{\alpha_n} \to \bar{\rho} \quad \text{strongly in } C^0(\overline{Q}), \quad \rho_\Gamma^{\alpha_n} \to \bar{\rho}_\Gamma \quad \text{strongly in } C^0(\overline{\Sigma}), \tag{4.35}$$

which entail that

$$\Psi(\rho^{\alpha_n}) \to \Psi(\bar{\rho}) \quad \text{strongly in } C^0(\overline{Q}) \quad \text{for } \Psi \in \{g, g', g'', \pi, \pi'\} \tag{4.36}$$

$$\Psi_\Gamma(\rho_\Gamma^{\alpha_n}) \to \Psi_\Gamma(\bar{\rho}) \quad \text{strongly in } C^0(\overline{\Sigma}) \quad \text{for } \Psi_\Gamma \in \{\pi_\Gamma, \pi'_\Gamma\}. \tag{4.37}$$

Moreover, thanks to Proposition 4.1 and to [34, Sect. 8, Cor. 4], we may assume that the associated adjoint variables $(p^{\alpha_n}, q^{\alpha_n}, q_\Gamma^{\alpha_n})$ satisfy

$$p^{\alpha_n} \to p \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$$
$$\text{and strongly in } C^0([0,T]; L^p(\Omega)) \quad \text{for } 1 \le p < 6, \tag{4.38}$$

$$(q^{\alpha_n}, q_\Gamma^{\alpha_n}) \to (q, q_\Gamma) \quad \text{weakly-star in } \mathscr{X}, \tag{4.39}$$

$$(\partial_t q^{\alpha_n}, \partial_t q_\Gamma^{\alpha_n}) \to (\partial_t q, \partial_t q_\Gamma) \quad \text{weakly in } \mathscr{W}_0^*, \tag{4.40}$$

$$(\lambda^{\alpha_n}, \lambda_\Gamma^{\alpha_n}) \to (\lambda, \lambda_\Gamma) \quad \text{weakly in } \mathscr{W}_0^*, \tag{4.41}$$

for suitable limits (p, q, q_Γ) and $(\lambda, \lambda_\Gamma)$, where $\lambda \in Y^*$ and $\lambda_\Gamma \in Y_\Gamma^*$, as explained around (4.15). Obviously, (4.38) implies that $\partial_{\mathbf{n}} p = 0$ almost everywhere on Σ and $p(T) = 0$ almost everywhere in Ω. Therefore, passing to the limit as $n \to \infty$ in the variational inequality (4.9), written for α_n, $n \in \mathbb{N}$, we obtain that (p, q, q_Γ) satisfies

$$\int_0^T \int_\Gamma (q_\Gamma + \beta_6 \bar{u}_\Gamma)(v_\Gamma - \bar{u}_\Gamma)\, \mathrm{d}\Gamma\, \mathrm{d}t \ge 0 \quad \forall\, v_\Gamma \in \mathscr{U}_{\mathrm{ad}}. \tag{4.42}$$

Next, we aim to show that in the limit as $n \to \infty$ a limiting adjoint system for $(\mathscr{P}_0)$ is satisfied. At first, it easily follows from the convergence properties stated above that

$$g(\rho^{\alpha_n})\, p_t^{\alpha_n} \to g(\bar{\rho})\, p_t, \quad g'(\rho^{\alpha_n})\, \rho_t^{\alpha_n}\, p^{\alpha_n} \to g'(\bar{\rho})\, \bar{\rho}_t\, p, \quad g'(\rho^{\alpha_n})\, q^{\alpha_n} \to g'(\bar{\rho})\, q, \tag{4.43}$$

all weakly in $L^1(Q)$. It thus follows, by taking the limit as $n \to \infty$ in (4.1) and (4.2), that the limits p, q satisfy

$$-(1+2g(\bar\rho))\,p_t - g'(\bar\rho)\,\bar\rho_t\,p - \Delta p \;=\; g'(\bar\rho)\,q + \beta_1(\bar\mu - \hat\mu_Q) \quad \text{a.e. in } Q, \tag{4.44}$$

$$\partial_{\mathbf{n}} p = 0 \quad \text{a.e. on } \Sigma, \quad p(T) = 0 \quad \text{a.e. in } \Omega. \tag{4.45}$$

The limiting equation corresponding to (4.3)–(4.5) has to be formulated in a weak form. To this end, we multiply (4.3), written for $\alpha_n, n \in \mathbb{N}$, by an arbitrary $(\eta, \eta_\Gamma) \in \mathscr{W}_0$ and integrate the resulting equation over Q. Integrating by parts with respect to time and space, and invoking the endpoint conditions for q and q_Γ, as well as the zero initial conditions for (η, η_Γ), we arrive at the identity

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\,\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_0^T \langle \partial_t\eta(t), q^{\alpha_n}(t)\rangle_V\,\mathrm{d}t \\
&+ \int_0^T \langle \partial_t\eta_\Gamma(t), q_\Gamma^{\alpha_n}(t)\rangle_{V_\Gamma}\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \nabla q^{\alpha_n}\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma^{\alpha_n}\cdot\nabla_\Gamma\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&- \int_0^T\!\!\int_\Omega v_1^{\alpha_n}\,\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega v_2^{\alpha_n}\eta\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Gamma w^{\alpha_n}\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&= \beta_2\int_0^T\!\!\int_\Omega (\rho^{\alpha_n} - \hat\rho_Q)\,\eta\,\mathrm{d}x\,\mathrm{d}t + \beta_3\int_0^T\!\!\int_\Gamma (\rho_\Gamma^{\alpha_n} - \hat\rho_\Sigma)\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \beta_4\int_\Omega (\rho^{\alpha_n}(T) - \hat\rho_\Omega)\,\eta(T)\,\mathrm{d}x + \beta_5\int_\Gamma (\rho_\Gamma^{\alpha_n}(T) - \hat\rho_\Gamma)\,\eta_\Gamma(T)\,\mathrm{d}\Gamma\,.
\end{aligned} \tag{4.46}$$

Now, owing to (4.13)–(4.15), the sum of the first two integrals on the left-hand side of (4.46) is equal to $\langle(\lambda^{\alpha_n}, \lambda_\Gamma^{\alpha_n}), (\eta, \eta_\Gamma)\rangle_{\mathscr{W}_0}$, which, by (4.41), converges to $\langle(\lambda, \lambda_\Gamma), (\eta, \eta_\Gamma)\rangle_{\mathscr{W}_0}$. Moreover, it is straightforward to verify (and this may be left to the reader) that also the remaining integrals in (4.46) converge. We therefore obtain, for every $(\eta, \eta_\Gamma) \in \mathscr{W}_0$,

$$\begin{aligned}
&\langle(\lambda, \lambda_\Gamma)(\eta, \eta_\Gamma)\rangle_{\mathscr{W}_0} + \int_0^T \langle \partial_t\eta(t), q(t)\rangle_V\,\mathrm{d}t + \int_0^T \langle \partial_t\eta_\Gamma(t), q_\Gamma(t)\rangle_{V_\Gamma}\,\mathrm{d}t \\
&+ \int_0^T\!\!\int_\Omega \nabla q\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \nabla_\Gamma q_\Gamma\cdot\nabla_\Gamma\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_0^T\!\!\int_\Gamma \pi_\Gamma'(\bar\rho_\Gamma)\,q_\Gamma\,\eta_\Gamma\,\mathrm{d}\Gamma\,\mathrm{d}t \\
&+ \int_0^T\!\!\int_\Omega \big[(\pi'(\bar\rho) - \bar\mu\,g''(\bar\rho))\,q + g'(\bar\rho)\,(\bar\mu_t\,p - \bar\mu\,p_t)\big]\,\eta\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$= \beta_2 \int_0^T\!\!\int_\Omega (\bar\rho - \hat\rho_Q)\, \eta \, dx\, dt \; + \; \beta_3 \int_0^T\!\!\int_\Gamma (\bar\rho_\Gamma - \hat\rho_\Sigma)\, \eta_\Gamma \, d\Gamma\, dt$$

$$+ \; \beta_4 \int_\Omega (\bar\rho(T) - \hat\rho_\Omega)\, \eta(T)\, dx \; + \; \beta_5 \int_\Gamma (\bar\rho_\Gamma(T) - \hat\rho_\Gamma)\, \eta_\Gamma(T)\, d\Gamma\,. \qquad (4.47)$$

Next, we show that the limit pair $((\lambda, \lambda_\Gamma), (q, q_\Gamma))$ satisfies some sort of a complementarity slackness condition. To this end, observe that (cf. (4.17)) for all $n \in \mathbb{N}$ we obviously have

$$\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\, q^{\alpha_n}\, dx\, dt \; = \; \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\, h''(\rho^{\alpha_n})\, |q^{\alpha_n}|^2\, dx\, dt \; \geq \; 0\,.$$

An analogous inequality holds for the corresponding boundary terms. Hence, it is found that

$$\liminf_{n\to\infty} \int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\, q^{\alpha_n}\, dx\, dt \; \geq 0, \quad \liminf_{n\to\infty} \int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}\, q_\Gamma^{\alpha_n}\, d\Gamma\, dt \; \geq \; 0\,. \qquad (4.48)$$

Finally, we derive a relation which gives some indication that the limit $(\lambda, \lambda_\Gamma)$ should somehow be concentrated on the set where $|\bar\rho| = 1$ and $|\bar\rho_\Gamma| = 1$ (which, however, we cannot prove rigorously). To this end, we test the pair $(\lambda^{\alpha_n}, \lambda_\Gamma^{\alpha_n})$ by the function $\big((1 - (\rho^{\alpha_n})^2)\, \phi, (1 - (\rho_\Gamma^{\alpha_n})^2)\, \phi_\Gamma\big)$ that belongs to $\mathscr{V}$, since (ϕ, ϕ_Γ) is any smooth test function satisfying

$$(\phi(0), \phi_\Gamma(0)) = (0, 0), \qquad \int_\Omega (1 - (\rho^{\alpha_n})^2)\, \phi(t)\; dx = 0 \quad \forall\, t \in [0, T]. \qquad (4.49)$$

As $h''(r) = 2/\left(1 - r^2\right)$ for every $r \in (-1, 1)$, we obtain that

$$\lim_{n\to\infty} \left(\int_0^T\!\!\int_\Omega \lambda^{\alpha_n}\, (1 - (\rho^{\alpha_n})^2)\, \phi\, dx\, dt\,, \int_0^T\!\!\int_\Gamma \lambda_\Gamma^{\alpha_n}\, (1 - (\rho_\Gamma^{\alpha_n})^2)\, \phi_\Gamma\, d\Gamma\, dt \right)$$

$$= \lim_{n\to\infty} \left(2 \int_0^T\!\!\int_\Omega \varphi(\alpha_n)\, q^{\alpha_n}\, \phi\, dx\, dt\,, 2 \int_0^T\!\!\int_\Gamma \varphi(\alpha_n)\, q_\Gamma^{\alpha_n}\, \phi_\Gamma\, d\Gamma\, dt \right) \; = \; (0, 0)\,. \qquad (4.50)$$

We now collect the results established above. We have the following statement.

Theorem 4.2 *Let the assumptions (A1)–(A6) and* (1.12)–(1.13) *be satisfied. Moreover, let* $\bar u_\Gamma \in \mathscr{U}_{ad}$ *be an optimal control for* $(\mathscr{P}_0)$ *with the associated quintuple* $(\bar\mu, \bar\rho, \bar\rho_\Gamma, \bar\xi, \bar\xi_\Gamma)$ *solving the corresponding state system (1.2)–(1.8) in the sense of Theorem 2.1. Moreover, let* $\{\alpha_n\} \subset (0, 1]$ *be a sequence with* $\alpha_n \searrow 0$ *as* $n \to \infty$ *such that there are optimal pairs* $((\mu^{\alpha_n}, \rho^{\alpha_n}, \rho_\Gamma^{\alpha_n}), u_\Gamma^{\alpha_n})$ *for the adapted control problem* $(\widetilde{\mathscr{P}}_{\alpha_n})$ *satisfying* (3.18)–(3.20) *(such sequences exist by*

Theorem 3.4) and having the associated adjoint variables $(p^{\alpha_n}, q^{\alpha_n}, q_\Gamma^{\alpha_n})$*. Then, for any subsequence* $\{n_k\}_{k\in\mathbb{N}}$ *of* $\mathbb{N}$*, there are a subsequence* $\{n_{k_\ell}\}_{\ell\in\mathbb{N}}$ *and some quintuple* $(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$ *such that*

$$p \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)),$$

$$(q, q_\Gamma) \in \mathscr{Z}, \quad (\partial_t q, \partial_t q_\Gamma) \in \mathscr{W}_0^*, \quad (\lambda, \lambda_\Gamma) \in \mathscr{W}_0^*, \tag{4.51}$$

and such that the relations (4.38)–(4.41) *are valid (where the sequences are indexed by* n_{k_ℓ} *and the limits are taken as* $\ell \to \infty$*). Moreover, the variational inequality* (4.42) *and the adjoint state equations* (4.44)*,* (4.45)*, and* (4.47) *are satisfied.*

Remark 4.3 Unfortunately, we cannot show that the limit quintuple

$$(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$$

solving the adjoint problem associated with the optimal pair

$$((\bar{\mu}, \bar{\rho}, \bar{\rho}_\Gamma, \bar{\xi}, \bar{\xi}_\Gamma), \bar{u}_\Gamma)$$

is unique. Therefore, it may well happen that the limits differ for different subsequences. However, it turns out that for any such limit $(p, q, q_\Gamma, \lambda, \lambda_\Gamma)$ the component q_Γ should satisfy the variational inequality (4.42).

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Nontrivial Solutions of Quasilinear Elliptic Equations with Natural Growth Term

Marco Degiovanni and Alessandra Pluda

Abstract We prove the existence of multiple solutions for a quasilinear elliptic equation containing a term with natural growth, under assumptions that are invariant by diffeomorphism. To this purpose we develop an adaptation of degree theory.

Keywords Degree theory • Divergence form • Invariance by diffeomorphism • Multiple solutions • Natural growth conditions • Quasilinear elliptic equations

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1 Introduction

Consider the quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}\,[A(x,u)\nabla u] + B(x,u)|\nabla u|^2 = g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded, connected and open subset of $\mathbb{R}^n$ and

$$A, B, g : \Omega \times \mathbb{R} \to \mathbb{R}$$

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are Carathéodory functions such that:

(h_1) *for every* $R > 0$ *there exist* $\beta_R > 0$ *and* $\nu_R > 0$ *satisfying*

$$\nu_R \le A(x,s) \le \beta_R\,, \qquad |B(x,s)| \le \beta_R\,, \qquad |g(x,s)| \le \beta_R\,|s|\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$ *with* $|s| \le R$.

The existence of a weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (1.1), namely a solution of

$$\int_\Omega \left[A(x,u)\nabla u \cdot \nabla v + B(x,u)|\nabla u|^2\, v\right] dx = \int_\Omega g(x,u)v\,dx$$
$$\text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,, \qquad (1.2)$$

follows from the results of [6, 7], provided that a suitable a priori L^∞-estimate holds, possibly related to the existence of a pair of sub-/super-solutions (see e.g. [6, Théorème 2.1] and [7, Theorems 1 and 2]). Moreover, each weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is locally Hölder continuous in Ω (see e.g. [10, Theorem VII.1.1]).

The existence of multiple solutions, in the semilinear case and under suitable regularity assumptions, has also been proved for instance in [2]. Here we are interested in the existence of multiple nontrivial solutions, as (h_1) implies that $g(x,0) = 0$, under assumptions that do not imply an a priori $W^{1,\infty}$-estimate. Let us state our main result.

Theorem 1.1 *Assume* (h_1) *and also that:*

(h_2) *there exist* $\underline{M} < 0 < \overline{M}$ *such that*

$$g(x,\underline{M}) \ge 0 \ge g(x,\overline{M}) \qquad \textit{for a.e. } x \in \Omega\,;$$

(h_3) *the function* $g(x,\cdot)$ *is differentiable at* $s = 0$ *for a.e.* $x \in \Omega$.

Consider the eigenvalue problem

$$\begin{cases} -\mathrm{div}\,[A(x,0)\nabla u] - D_s g(x,0)u = \lambda u & \textit{in } \Omega\,, \\ u = 0 & \textit{on } \partial\Omega\,, \end{cases} \qquad (1.3)$$

and denote by (λ_k), $k \ge 1$, *the sequence of the eigenvalues repeated according to multiplicity.*

If there exists $k \ge 2$ *with* $\lambda_k < 0 < \lambda_{k+1}$ *and* k *even, then problem* (1.2) *admits at least three nontrivial solutions* u_1, u_2, u_3 *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ *with*

$$u_1 < 0 \textit{ in } \Omega\,, \quad u_2 > 0 \textit{ in } \Omega\,, \quad u_3 \textit{ sign-changing.}$$

If A is constant and $B = 0$, the result is essentially contained in [3, 15], which in turn developed previous results of [4]. Actually, in those papers it is enough to assume that $\lambda_2 < 0$, because in that case (1.2) is the Euler-Lagrange equation of a suitable functional and variational methods, e.g. Morse theory, can be applied.

In our case there is no functional and also degree theory arguments cannot be applied in a standard way, because of the presence of the term $B(x,u)|\nabla u|^2$. Let us point out that our assumptions do not imply that the solutions u of (1.2) belong to $W^{1,\infty}(\Omega)$, so that the natural growth term $B(x,u)|\nabla u|^2$ plays a true role.

Let us also mention that our statement has an invariance property.

Remark 1.2 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing C^2-diffeomorphism with $\varphi(0) = 0$. Then the following facts hold:

(a) the functions A, B and g satisfy the assumptions of Theorem 1.1 if and only A^φ, B^φ and g^φ, defined as

$$\begin{aligned} A^\varphi(x,s) &= (\varphi'(s))^2 A(x,\varphi(s))\,, \\ B^\varphi(x,s) &= \varphi'(s)\varphi''(s)A(x,\varphi(s)) + (\varphi'(s))^3 B(x,\varphi(s))\,, \\ g^\varphi(x,s) &= \varphi'(s)g(x,\varphi(s))\,, \end{aligned}$$

do the same; in particular, we have

$$A^\varphi(x,0) = (\varphi'(0))^2 A(x,0)\,, \qquad D_s g^\varphi(x,0) = (\varphi'(0))^2 D_s g(x,0)\,;$$

(b) a function $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.2) if and only if $\varphi^{-1}(u)$ is a solution of (1.2) with A, B and g replaced by A^φ, B^φ and g^φ, respectively.

From the definition of B^φ we see that the term $B(x,u)|\nabla u|^2$ cannot be omitted in the equation, if we want to ensure this kind of invariance.

When (1.2) is the Euler-Lagrange equation associated to a functional, the question of invariance under suitable classes of diffeomorphisms has been already treated in [14], where it is shown that problems with degenerate coercivity can be reduced, in some cases, to coercive problems.

In the next sections we develop an adaptation of degree theory suited for our setting and then we prove Theorem 1.1 by a degree argument. Under assumptions that are not diffeomorphism-invariant, a degree theory for quasilinear elliptic equations with natural growth conditions has been already developed in [1]. Here we find it more convenient to reduce Eq. (1.2) to a variational inequality possessing as obstacles a pair of sub-/super-solutions, according to an approach already considered for instance in [12].

2 Topological Degree in Reflexive Banach Spaces

Let X be a reflexive real Banach space.

Definition 2.1 A map $F : D \to X'$, with $D \subseteq X$, is said to be *of class* $(S)_+$ if, for every sequence (u_k) in D weakly convergent to some u in X with

$$\limsup_k \langle F(u_k), u_k - u\rangle \le 0\,,$$

it holds $\|u_k - u\| \to 0$.

More generally, if T is a metrizable topological space, a map $H : D \to X'$, with $D \subseteq X \times T$, is said to be *of class* $(S)_+$ if, for every sequence (u_k, t_k) in D with (u_k) weakly convergent to u in X, (t_k) convergent to t in T and

$$\limsup_k \langle H_{t_k}(u_k), u_k - u\rangle \le 0\,,$$

it holds $\|u_k - u\| \to 0$ (we write $H_t(u)$ instead of $H(u, t)$).
Assume now that U is a bounded and open subset of X, $F : \overline{U} \to X'$ a bounded and continuous map of class $(S)_+$, K a closed and convex subset of X and $\varphi \in X'$. We aim to consider the variational inequality

$$\begin{cases} u \in K\,, \\ \langle F(u), v - u\rangle \ge \langle \varphi, v - u\rangle \qquad \forall v \in K\,. \end{cases} \tag{2.1}$$

Remark 2.2 It is easily seen that the set

$$\left\{u \in \overline{U} :\ \ u \text{ is a solution of (2.1)}\right\}$$

is compact (possibly empty).
According to [8, 11, 13], if the variational inequality (2.1) has no solution $u \in \partial U$, one can define the topological degree

$$\deg((F, K), U, \varphi) \in \mathbb{Z}\,.$$

Let us recall some basic properties.

Proposition 2.3 *If* (2.1) *has no solution* $u \in \partial U$*, then*

$$\deg((F, K), U, \varphi) = \deg((F - \varphi, K), U, 0).$$

Proposition 2.4 *If* (2.1) *has no solution* $u \in \partial U$, $u_0 \in X$ *and we set*

$$\begin{aligned}\widehat{U} \quad &= \{u - u_0 : \; u \in U\} \,, \\ \widehat{K} \quad &= \{u - u_0 : \; u \in K\} \,, \\ \widehat{F}(u) &= F(u_0 + u) \,,\end{aligned}$$

then

$$\deg((\widehat{F}, \widehat{K}), \widehat{U}, \varphi) = \deg((F, K), U, \varphi) \,.$$

Theorem 2.5 *If* (2.1) *has no solution* $u \in \overline{U}$, *then* $\deg((F, K), U, \varphi) = 0$.

Theorem 2.6 *If* (2.1) *has no solution* $u \in \partial U$ *and there exists* $u_0 \in K \cap U$ *such that*

$$\langle F(v), v - u_0 \rangle \geq \langle \varphi, v - u_0 \rangle \qquad \text{for any } v \in K \cap \partial U \,,$$

then $\deg((F, K), U, \varphi) = 1$.

Theorem 2.7 *If* U_0 *and* U_1 *are two disjoint open subsets of* U *and* (2.1) *has no solution* $u \in \overline{U} \setminus (U_0 \cup U_1)$, *then*

$$\deg((F, K), U, \varphi) = \deg((F, K), U_0, \varphi) + \deg((F, K), U_1, \varphi) \,.$$

Definition 2.8 Let K_k, K be closed and convex subsets of X. The sequence (K_k) is said to be *Mosco-convergent* to K if the following facts hold:

(a) if $k_j \to \infty$, $u_{k_j} \in K_{k_j}$ for any $j \in \mathbb{N}$ and (u_{k_j}) is weakly convergent to u in X, then $u \in K$;
(b) for every $u \in K$ there exist $\overline{k} \in \mathbb{N}$ and a sequence (u_k) strongly convergent to u in X with $u_k \in K_k$ for any $k \geq \overline{k}$.

Theorem 2.9 *Let* W *be a bounded and open subset of* $X \times [0, 1]$, $H : \overline{W} \to X'$ *be a bounded and continuous map of class* $(S)_+$ *and* (K_t), $0 \leq t \leq 1$, *be a family of closed and convex subsets of* X *such that, for every sequence* (t_k) *convergent to* t *in* $[0, 1]$, *the sequence* (K_{t_k}) *is Mosco-convergent to* K_t.

Then the following facts hold:

(a) *the set of pairs* $(u, t) \in \overline{W}$, *satisfying*

$$\begin{cases} u \in K_t \,, \\ \langle H_t(u), v - u \rangle \geq \langle \varphi, v - u \rangle \qquad \forall v \in K_t \,, \end{cases} \tag{2.2}$$

is compact (possibly empty);

(b) *if the problem* (2.2) *has no solution* $(u,t) \in \partial_{X\times[0,1]} W$ *and we set*

$$W_t = \{u \in X : \ (u,t) \in W\} \ ,$$

then $\deg((H_t, K_t), W_t, \varphi)$ *is independent of* $t \in [0,1]$.

Proof If (u_k, t_k) is a sequence in $\overline{W}$ constituted by solutions of (2.2), then up to a subsequence (u_k) is weakly convergent to some u in X and (t_k) is convergent to some t in $[0,1]$. Then $u \in K_t$ and there exists a sequence $(\hat{u}_k)$ strongly convergent to u in X with $\hat{u}_k \in K_{t_k}$. It follows

$$\begin{aligned} \langle H_{t_k}(u_k), u_k - u\rangle &= \langle H_{t_k}(u_k), \hat{u}_k - u\rangle + \langle H_{t_k}(u_k), u_k - \hat{u}_k\rangle \\ &\leq \langle H_{t_k}(u_k), \hat{u}_k - u\rangle + \langle \varphi, u_k - \hat{u}_k\rangle \ , \end{aligned}$$

whence

$$\limsup_k \, \langle H_{t_k}(u_k), u_k - u\rangle \leq 0 \, .$$

Then $\|u_k - u\| \to 0$ and $(u,t) \in \overline{W}$. For every $v \in K_t$ there exists a sequence (v_k) strongly convergent to v in X with $v_k \in K_{t_k}$. From

$$\langle H_{t_k}(u_k), v_k - u_k\rangle \geq \langle \varphi, v_k - u_k\rangle$$

it follows

$$\langle H_t(u), v - u\rangle \geq \langle \varphi, v - u\rangle$$

so that the set introduced in assertion (a) is compact.

Assume now that the problem (2.2) has no solution $(u,t) \in \partial_{X\times[0,1]} W$. It is enough to prove that $\{t \mapsto \deg((H_t, K_t), W_t, \varphi)\}$ is locally constant.

Suppose first that $K_t \neq \emptyset$ for any $t \in [0,1]$. By Michael selection theorem (see e.g. [5, Theorem 1.11.1]) there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(t) \in K_t$ for any $t \in [0,1]$.

If we set

$$\begin{aligned} \widehat{W} \quad &= \{(u - \gamma(t), t) : \ (u,t) \in W\} \ , \\ \widehat{K}_t \quad &= \{u - \gamma(t) : \ u \in K_t\} \ , \\ \widehat{H}_t(u) &= H_t(\gamma(t) + u) \, , \end{aligned}$$

then $\widehat{W}$, $\widehat{K}_t$ and $\widehat{H}$ satisfy the same assumptions and

$$\deg((\widehat{H}_t, \widehat{K}_t), \widehat{W}_t, \varphi) = \deg((H_t, K_t), W_t, \varphi)$$

by Proposition 2.4. Moreover $0 \in \widehat{K}_t$ for any $t \in [0,1]$. Therefore we may assume, without loss of generality, that $0 \in K_t$ for any $t \in [0,1]$.

Given $t \in [0,1]$, there exist a bounded and open subset U of X and $\delta > 0$ such that

$$U \times ([t-\delta, t+\delta] \cap [0,1]) \subseteq W$$

and such that (2.2) has no solution (u, τ) in

$$W \setminus (U \times [t-\delta, t+\delta])$$

with $t - \delta \leq \tau \leq t + \delta$. From Theorem 2.7 we infer that

$$\deg((H_\tau, K_\tau), W_\tau, \varphi) = \deg((H_\tau, K_\tau), U, \varphi) \qquad \text{for any } \tau \in [t-\delta, t+\delta]\,.$$

From [11, Theorem 4.53 and Proposition 4.61] we deduce that $\{\tau \mapsto \deg((H_\tau, K_\tau), U, \varphi)\}$ is constant on $[t-\delta, t+\delta]$.

In general, given $t \in [0,1]$, let us distinguish the cases $K_t \neq \emptyset$ and $K_t = \emptyset$.

If $K_t \neq \emptyset$, by the Mosco-convergence there exists $\delta > 0$ such that $K_\tau \neq \emptyset$ for any $\tau \in [t-\delta, t+\delta]$. By the previous step we infer that $\{\tau \mapsto \deg((H_\tau, K_\tau), W_\tau, \varphi)\}$ is constant on $[t-\delta, t+\delta]$.

If $K_t = \emptyset$, from Theorem 2.5 we infer that $\deg((H_t, K_t), W_t, \varphi) = 0$. Assume, for a contradiction, that there exists a sequence (t_k) convergent to t with $\deg((H_{t_k}, K_{t_k}), W_{t_k}, \varphi) \neq 0$ for any $k \in \mathbb{N}$. Again from Theorem 2.5 we infer that the problem (2.2) has a solution $(u_k, t_k) \in \overline{W}$, in particular $u_k \in K_{t_k}$, for any $k \in \mathbb{N}$. Up to a subsequence, (u_k) is weakly convergent to some u, whence $u \in K_t$ by the Mosco-convergence, and a contradiction follows. □

Now let Ω be a bounded and open subset of $\mathbb{R}^n$, let T be a metrizable topological space and let

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times T) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times T) \to \mathbb{R}$$

be two Carathéodory functions. We will denote by $\| \;\|_p$ the usual norm in L^p and write $a_t(x,s,\xi)$, $b_t(x,s,\xi)$ instead of $a(x,(s,\xi,t))$, $b(x,(s,\xi,t))$.

In this section, we assume that a_t and b_t satisfy the *controllable growth conditions* in the sense of [10], uniformly with respect to t. In a simplified form enough for our purposes, this means that:

(UC) *there exist* $p \in]1,\infty[$, $\alpha^{(0)} \in L^1(\Omega)$, $\alpha^{(1)} \in L^{p'}(\Omega)$, $\beta > 0$ *and* $\nu > 0$ *such that*

$$|a_t(x,s,\xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$|b_t(x,s,\xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$a_t(x,s,\xi) \cdot \xi \geq \nu|\xi|^p - \alpha^{(0)}(x) - \beta\,|s|^p\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $t \in T$; *such a* p *is clearly unique.*

It follows

$$\begin{cases} a_t(x,u,\nabla u) \in L^{p'}(\Omega) \\ b_t(x,u,\nabla u) \in L^{p'}(\Omega) \subseteq W^{-1,p'}(\Omega) \end{cases} \qquad \text{for any } t \in T \text{ and } u \in W_0^{1,p}(\Omega)$$

and the map $H : W_0^{1,p}(\Omega) \times T \to W^{-1,p'}(\Omega)$ defined by

$$H_t(u) = -\mathrm{div}\,[a_t(x,u,\nabla u)] + b_t(x,u,\nabla u)$$

is continuous and bounded on $B \times T$, whenever B is bounded in $W_0^{1,p}(\Omega)$.

Theorem 2.10 *Assume (UC) and also that:*

(UM) *we have*

$$\left[a_t(x,s,\xi) - a_t(x,s,\hat{\xi})\right] \cdot (\xi - \hat{\xi}) > 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \hat{\xi} \in \mathbb{R}^n$, $t \in T$, *with* $\xi \neq \hat{\xi}$.

Then $H : W_0^{1,p}(\Omega) \times T \to W^{-1,p'}(\Omega)$ *is of class* $(S)_+$.

Proof See e.g. [13, Theorem 1.2.1]. □

3 Quasilinear Elliptic Variational Inequalities with Natural Growth Conditions

Again, let Ω be a bounded and open subset of $\mathbb{R}^n$ and let now

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n\,,$$
$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$$

be two Carathéodory functions. In this paper we are interested in the case in which a and b satisfy the *natural growth conditions* in the sense of [10]. More precisely, we assume that:

(N) *there exist* $p \in]1,\infty[$ *and, for every* $R > 0$, $\alpha_R^{(0)} \in L^1(\Omega)$, $\alpha_R^{(1)} \in L^{p'}(\Omega)$, $\beta_R > 0$ *and* $\nu_R > 0$ *such that*

$$|a(x,s,\xi)| \leq \alpha_R^{(1)}(x) + \beta_R\,|\xi|^{p-1}\,,$$
$$|b(x,s,\xi)| \leq \alpha_R^{(0)}(x) + \beta_R\,|\xi|^{p}\,,$$
$$a(x,s,\xi)\cdot\xi \geq \nu_R\,|\xi|^p - \alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ *with* $|s| \leq R$; *such a* p *is clearly unique;*

(M) *we have*

$$\Big[a(x,s,\xi) - a(x,s,\hat{\xi})\Big] \cdot (\xi - \hat{\xi}) > 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \hat{\xi} \in \mathbb{R}^n$ *with* $\xi \neq \hat{\xi}$.

Then we can define a map

$$F : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to W^{-1,p'}(\Omega) + L^1(\Omega)$$

by

$$F(u) = -\mathrm{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u)\,.$$

Remark 3.1 Assume that $\hat{a}$ and $\hat{b}$ also satisfy (*N*) and (*M*). An easy density argument shows that, if

$$\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\big]\,dx = \int_\Omega \big[\hat{a}(x,u,\nabla u)\cdot\nabla v + \hat{b}(x,u,\nabla u)\,v\big]\,dx$$
$$\text{for any } u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and any } v \in C_c^\infty(\Omega)\,,$$

then

$$\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\big]\,dx = \int_\Omega \big[\hat{a}(x,u,\nabla u)\cdot\nabla v + \hat{b}(x,u,\nabla u)\,v\big]\,dx$$
$$\text{for any } u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and any } v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,.$$

Consider also a p-quasi upper semicontinuous function $\underline{u} : \Omega \to \overline{\mathbb{R}}$ and a p-quasi lower semicontinuous function $\overline{u} : \Omega \to \overline{\mathbb{R}}$, and set

$$K = \Big\{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) :\ \underline{u} \le \tilde{u} \le \overline{u}\ p\text{-q.e. in } \Omega\Big\}\,,$$

where $\tilde{u}$ is any p-quasi continuous representative of u (see e.g. [9]).

We aim to consider the solutions of the variational inequality

$$\begin{cases} u \in K\,, \\ \displaystyle\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla(v-u) + b(x,u,\nabla u)\,(v-u)\big]\,dx \ge 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K\,. \end{cases} \tag{VI}$$

We denote by $Z^{tot}(F,K)$ the set of solutions u of (*VI*). We will simply write Z^{tot}, if no confusion can arise.

For every $u \in K$, we also set

$$T_u K = \Big\{ v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \ \tilde{v} \geq 0 \ p\text{-q.e. in } \{\tilde{u} = \underline{u}\} \text{ and } \tilde{v} \leq 0 \ p\text{-q.e. in } \{\tilde{u} = \overline{u}\} \Big\} .$$

Proposition 3.2 *A function $u \in K$ satisfies (VI) if and only if*

$$\int_\Omega \big[a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)\, v\big]\, dx \geq 0 \qquad \textit{for every } v \in T_u K .$$

Proof Assume that u is a solution of (VI) and let $v \in T_u K$ with $v \leq 0$ a.e. in Ω. Since $\max\{k(\underline{u} - \tilde{u}), \tilde{v}\}$ is a nonincreasing sequence of nonpositive p-quasi upper semicontinuous functions converging to $\tilde{v}$ p-q.e. in Ω, by Dal Maso [9, Lemma 1.6] there exists a sequence (v_k) in $W_0^{1,p}(\Omega)$ converging to v in $W_0^{1,p}(\Omega)$ with $\tilde{v}_k \geq \max\{k(\underline{u} - \tilde{u}), \tilde{v}\}$ p-q.e. in Ω. Without loss of generality, we may assume that $\tilde{v}_k \leq 0$ p-q.e. in Ω.

Then it follows that $u + \frac{1}{k} v_k \in K$, whence

$$\int_\Omega \big[a(x, u, \nabla u) \cdot \nabla v_k + b(x, u, \nabla u)\, v_k\big]\, dx \geq 0 .$$

Going to the limit as $k \to \infty$, we get

$$\int_\Omega \big[a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)\, v\big]\, dx \geq 0 .$$

If $v \in T_u K$ with $v \geq 0$ a.e. in Ω, the argument is similar. Since every $v \in T_u K$ can be written as $v = v^+ - v^-$ with $v^+, -v^- \in T_u K$, it follows

$$\int_\Omega \big[a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)\, v\big]\, dx \geq 0 \qquad \text{for every } v \in T_u K .$$

Since $K \subseteq u + T_u K$, the converse is obvious. □

We are also interested in the invariance of the problem with respect to suitable transformations.

Let us denote by Φ the set of increasing C^2-diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and by Θ the set of C^1-functions $\vartheta : \mathbb{R} \to]0, +\infty[$.

For any $\varphi \in \Phi$ and $\vartheta \in \Theta$, we define

$$F^\varphi\,,\ F_\vartheta : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to W^{-1,p'}(\Omega) + L^1(\Omega)$$

by

$$F^\varphi(u) = F(\varphi(u)), \; F_\vartheta(u) = \vartheta(u)\, F(u).$$

If we define the Carathéodory functions

$$a^\varphi, a_\vartheta : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n, \qquad b^\varphi, b_\vartheta : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$$

by

$$\begin{aligned}
a^\varphi(x, s, \xi) &= \varphi'(s)\, a(x, \varphi(s), \varphi'(s)\xi), \\
b^\varphi(x, s, \xi) &= \varphi''(s)\, a(x, \varphi(s), \varphi'(s)\xi) \cdot \xi + \varphi'(s)\, b(x, \varphi(s), \varphi'(s)\xi), \\
a_\vartheta(x, s, \xi) &= \vartheta(s)\, a(x, s, \xi), \\
b_\vartheta(x, s, \xi) &= \vartheta'(s)\, a(x, s, \xi) \cdot \xi + \vartheta(s)\, b(x, s, \xi),
\end{aligned}$$

it easily follows that

$$\begin{aligned}
F^\varphi(u) &= -\mathrm{div}\,[a^\varphi(x, u, \nabla u)] + b^\varphi(x, u, \nabla u), \\
F_\vartheta(u) &= -\mathrm{div}\,[a_\vartheta(x, u, \nabla u)] + b_\vartheta(x, u, \nabla u).
\end{aligned}$$

We also set $u^\varphi = \varphi^{-1}(u)$ and, given a set E of real valued functions, $E^\varphi = \{u^\varphi : \; u \in E\}$.

It is easily seen that

$$(a^\varphi)^\psi = a^{\varphi \circ \psi}, \qquad (a_\vartheta)_\varrho = a_{\vartheta\varrho}, \qquad (u^\varphi)^\psi = u^{\varphi \circ \psi},$$

for every $\varphi, \psi \in \Phi$ and $\vartheta, \varrho \in \Theta$.

We also say that (a, b) is of *Euler-Lagrange type*, if there exists a function

$$L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

such that

(a) $\{x \mapsto L(x, s, \xi)\}$ is measurable for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;
(b) $\{(s, \xi) \mapsto L(x, s, \xi)\}$ is of class C^1 for a.e. $x \in \Omega$;
(c) we have

$$a(x, s, \xi) = \nabla_\xi L(x, s, \xi), \qquad b(x, s, \xi) = D_s L(x, s, \xi),$$

for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Taking into account Proposition 3.2, the next two results are easy to prove.

Proposition 3.3 *For every $\varphi \in \Phi$, the following facts hold:*

(a) the functions a^φ, b^φ satisfy (N) and (M) with the same p;
(b) we have

$$K^\varphi = \left\{ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \; \underline{u}^\varphi \leq \tilde{u} \leq \overline{u}^\varphi \text{ } p\text{-q.e. in } \Omega \right\}$$

and $\underline{u}^\varphi : \Omega \to \overline{\mathbb{R}}$ is p-quasi upper semicontinuous, while $\overline{u}^\varphi : \Omega \to \overline{\mathbb{R}}$ is p-quasi lower semicontinuous (here we agree that $\varphi(-\infty) = -\infty$ and $\varphi(+\infty) = +\infty$);

(c) u is a solution of (VI) if and only if u^φ is a solution of the corresponding variational inequality with a, b and K replaced by a^φ, b^φ and K^φ, respectively;

(d) the pair (a, b) is of Euler-Lagrange type if and only if the pair (a^φ, b^φ) does the same; moreover, if L is associated with (a, b), then

$$L^\varphi(x, s, \xi) = L(x, \varphi(s), \varphi'(s)\xi)$$

is associated with (a^φ, b^φ).

Proposition 3.4 *For every $\vartheta \in \Theta$, the following facts hold:*

(a) the functions a_ϑ, b_ϑ satisfy (N) and (M) with the same p;
(b) u is a solution of (VI) if and only if the same u is a solution of the corresponding variational inequality with a and b replaced by a_ϑ and b_ϑ, respectively.

Remark 3.5 Let $\varphi \in \Phi$ with φ' nonconstant, let $w \in L^2(\Omega)$ and let $a(x, s, \xi) = \varphi'(s)\xi$, $b(x, s, \xi) = -w(x)$.

Then (a, b) is not of Euler-Lagrange type. However, if we take $\vartheta(s) = \varphi'(s)$, then $(a_\vartheta, b_\vartheta)$, which is given by

$$a_\vartheta\,(x, s, \xi) = [\varphi'(s)]^2\xi\,, \qquad b_\vartheta\,(x, s, \xi) = \varphi'(s)\varphi''(s)|\xi|^2 - w(x)\varphi'(s)\,,$$

is of Euler-Lagrange type with

$$L(x, s, \xi) = \frac{1}{2}\,[\varphi'(s)]^2|\xi|^2 - w(x)\varphi(s)\,.$$

Therefore the property of being of Euler-Lagrange type is not invariant under the transformation induced by ϑ, which plays in fact the role of “integrating factor”. By the way, if then we take $\psi = \varphi^{-1}$, we get

$$(a_\vartheta)^\psi(x, s, \xi) = \xi\,, \qquad (b_\vartheta)^\psi(x, s, \xi) = -w(x)\,,$$

which are simply related to

$$L(x,s,\xi) = \frac{1}{2}\,|\xi|^2 - w(x)s\,.$$

Proposition 3.6 *For every $R > 0$ there exist $\vartheta_1, \vartheta_2 \in \Theta$ and $c > 0$, depending only on R, β_R and ν_R, such that*

$$b_{\vartheta_1}(x,s,\xi) \le c\,\alpha_R^{(0)}(x)\,,$$

$$b_{\vartheta_2}(x,s,\xi) \ge -c\,\alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ with $|s| \le R$.

Proof If we set $\vartheta(s) = \exp(\gamma s)$ with $\gamma\,\nu_R \ge \beta_R$, we have

$$\begin{aligned} b_\vartheta(x,s,\xi) &= [\gamma\,a(x,s,\xi)\cdot\xi + b(x,s,\xi)]\exp(\gamma s) \\ &\ge \left[\gamma\,\nu_R\,|\xi|^p - \gamma\,\alpha_R^{(0)}(x) - \alpha_R^{(0)}(x) - \beta_R\,|\xi|^p\right]\exp(\gamma s) \\ &\ge -(\gamma+1)\exp(\gamma R)\,\alpha_R^{(0)}(x)\,, \end{aligned}$$

whence the existence of ϑ_2. The existence of ϑ_1 can be proved in a similar way. □

4 Quasilinear Elliptic Variational Inequalities with Natural Growth Conditions Depending on a Parameter

Again, let Ω be a bounded and open subset of $\mathbb{R}^n$ and let now T be a metrizable topological space and

$$a : \Omega \times (\mathbb{R}\times\mathbb{R}^n\times T) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R}\times\mathbb{R}^n\times T) \to \mathbb{R}$$

be two Carathéodory functions satisfying (N) and (M) uniformly with respect to $t \in T$.

More precisely, we assume that a_t and b_t satisfy (UM) and:

(UN) *there exist $p \in]1,\infty[$ and, for every $R > 0$, $\alpha_R^{(0)} \in L^1(\Omega)$, $\alpha_R^{(1)} \in L^{p'}(\Omega)$, $\beta_R > 0$ and $\nu_R > 0$ such that*

$$|a_t(x,s,\xi)| \le \alpha_R^{(1)}(x) + \beta_R\,|\xi|^{p-1}\,,$$

$$|b_t(x,s,\xi)| \le \alpha_R^{(0)}(x) + \beta_R\,|\xi|^p\,,$$

$$a_t(x,s,\xi)\cdot\xi \ge \nu_R\,|\xi|^p - \alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $t \in T$ with $|s| \leq R$; again, such a p is clearly unique.

Then we can define a map

$$H : \left[W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right] \times T \to \left[W^{-1,p'}(\Omega) + L^1(\Omega)\right]$$

by

$$H_t(u) = -\mathrm{div}\,[a_t(x, u, \nabla u)] + b_t(x, u, \nabla u)\,.$$

Consider also, for each $t \in T$, a p-quasi upper semicontinuous function $\underline{u}_t : \Omega \to \overline{\mathbb{R}}$ and a p-quasi lower semicontinuous function $\overline{u}_t : \Omega \to \overline{\mathbb{R}}$, set

$$K_t = \left\{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) :\ \underline{u}_t \leq \tilde{u} \leq \overline{u}_t \ p\text{-q.e. in } \Omega\right\}$$

and assume the following form of continuity related to the Mosco-convergence:

(MC) *for every sequence (t_k) convergent to t in T, the following facts hold:*

- *if (u_k) is a sequence weakly convergent to u in $W_0^{1,p}(\Omega)$, with $u_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (u_k) bounded in $L^\infty(\Omega)$, then $u \in K_t$;*
- *for every $u \in K_t$ there exist $\overline{k} \in \mathbb{N}$ and a sequence (u_k) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is bounded in $L^\infty(\Omega)$ and strongly convergent to u in $W_0^{1,p}(\Omega)$, with $u_k \in K_{t_k}$ for any $k \geq \overline{k}$.*

Then consider the parametric variational inequality

$$\begin{cases} (u,t) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times T\,, \\ u \in K_t\,, \\ \displaystyle\int_\Omega \left[a_t(x,u,\nabla u) \cdot \nabla(v-u) + b_t(x,u,\nabla u)\,(v-u)\right] dx \geq 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K_t\,. \end{cases} \tag{PVI}$$

Theorem 4.1 *Let (u_k, t_k) be a sequence of solutions of (PVI) with (u_k) bounded in $L^\infty(\Omega)$ and (t_k) convergent to some t in T with $K_t \neq \emptyset$.*

Then (u_k) admits a subsequence strongly convergent in $W_0^{1,p}(\Omega)$ to some u and (u,t) is a solution of (PVI).

Proof Let $w \in K_t$ and let (w_k) be a sequence strongly convergent to w in $W_0^{1,p}(\Omega)$, with $w_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (w_k) bounded in $L^\infty(\Omega)$. If we set

$$\widehat{T} = \mathbb{N} \cup \{\infty\}\,,$$
$$\hat{a}_k(x,s,\xi) = a_{t_k}(x, w_k(x)+s, \nabla w_k(x)+\xi)\,,$$
$$\hat{a}_\infty(x,s,\xi) = a_t(x, w(x)+s, \nabla w(x)+\xi)\,,$$
$$\hat{b}_k(x,s,\xi) = b_{t_k}(x, w_k(x)+s, \nabla w_k(x)+\xi)\,,$$
$$\hat{b}_\infty(x,s,\xi) = b_t(x, w(x)+s, \nabla w(x)+\xi)\,,$$
$$\hat{\underline{u}}_k = \underline{u}_{t_k} - w_k\,, \qquad \hat{\underline{u}}_\infty = \underline{u}_t - w\,,$$
$$\hat{\overline{u}}_k = \overline{u}_{t_k} - w_k\,, \qquad \hat{\overline{u}}_\infty = \overline{u}_t - w\,,$$
$$\hat{u}_k = u_k - w_k\,,$$

and define $\widehat{K}_k$, $\widehat{K}_\infty$ accordingly, it is easily seen that all the assumptions are still satisfied and now $0 \in \widehat{K}_k$. Therefore, we may assume without loss of generality that $0 \in K_{t_k}$ for any $k \in \mathbb{N}$.

Let $R > 0$ be such that $\|u_k\|_\infty \le R$ for any $k \in \mathbb{N}$. We claim that (u_k^+) is bounded in $W_0^{1,p}(\Omega)$. Actually, by Propositions 3.4 and 3.6, we may assume, without loss of generality, that

$$b_t(x,s,\xi) \ge -c\,\alpha_R^{(0)}(x) \qquad \text{whenever } |s| \le R\,.$$

Then the choice $v = -u_k^-$ in (*PVI*) yields

$$0 \ge \int_\Omega \left[a_{t_k}(x, u_k^+, \nabla u_k^+)\cdot \nabla u_k^+ + b_{t_k}(x, u_k^+, \nabla u_k^+)\, u_k^+\right] dx$$
$$\ge \nu_R \int_\Omega |\nabla u_k^+|^p\,dx - \int_\Omega \alpha_R^{(0)}\,dx - c\int_\Omega \alpha_R^{(0)}\, u_k^+\,dx\,,$$

which implies that (u_k^+) is bounded in $W_0^{1,p}(\Omega)$.

In a similar way one finds that (u_k^-) is bounded in $W_0^{1,p}(\Omega)$, so that (u_k) is weakly convergent, up to a subsequence, to some u in $W_0^{1,p}(\Omega)$ and $u \in K_t$. Let (z_k) be a sequence strongly convergent to u in $W_0^{1,p}(\Omega)$, with $z_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (z_k) bounded in $L^\infty(\Omega)$.

Let $\psi : \mathbb{R} \to [0,1]$ be a continuous function such that $\psi(s) = 1$ for $|s| \le R$ and $\psi(s) = 0$ for $|s| \ge R+1$. Then, let

$$\check{a}_t(x,s,\xi) = \psi(s)\, a_t(x,s,\xi) + (1-\psi(s))\,|\xi|^{p-2}\xi\,,$$
$$\check{b}_t(x,s,\xi) = \psi(s)\, b_t(x,s,\xi)\,.$$

It is easily seen that each (u_k, t_k) is also a solution of (*PVI*) with a_t and b_t replaced by $\check{a}_t$ and $\check{b}_t$. Moreover, $\check{a}_t$ and $\check{b}_t$ satisfy both the assumptions (*UN*), (*UM*) and the assumptions of [1, Theorem 4.2]. In particular, there exist $\alpha^{(0)} \in L^1(\Omega)$, $\beta > 0$ and $\nu > 0$ such that

$$\check{a}_t(x, s, \xi) \cdot \xi \geq \nu\, |\xi|^p - \alpha^{(0)}(x)\,, \qquad |\check{b}_t(x, s, \xi)| \leq \alpha^{(0)}(x) + \beta\, |\xi|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $t \in T$.

Now, if $p < n$, the proof of [1, Theorem 4.2] can be repeated in a simplified form, as (u_k) is bounded in $L^\infty(\Omega)$. We have only to observe that, if $\varphi : \mathbb{R} \to \mathbb{R}$ is the solution of

$$\begin{cases} \varphi'(s) = 1 + \dfrac{\beta}{\nu}\, |\varphi(s)|\,, \\ \varphi(0) = 0\,, \end{cases}$$

then there exists $\tau > 0$ such that

$$0 \leq \tau s \varphi(s) \leq s^2 \qquad \text{whenever } |s| \leq R + \sup_k \|z_k\|_\infty\,,$$

so that

$$u_k - \tau\, \varphi(u_k - z_k) \in K_{t_k} \qquad \text{for any } k \in \mathbb{N}\,.$$

It follows

$$\int_\Omega \left[\check{a}_{t_k}(x, u_k, \nabla u_k) \cdot \nabla(\varphi(u_k - z_k)) + \check{b}_{t_k}(x, u_k, \nabla u_k)\, \varphi(u_k - z_k)\right] dx \leq 0$$

and now the proof of [1, Theorem 4.2] can be repeated with minor modifications, showing that (u_k) is strongly convergent to u in $W_0^{1,p}(\Omega)$. If $p \geq n$, the argument is similar and simpler.

It is easily seen that (u, t) is a solution of (*PVI*). □

Remark 4.2 In the previous theorem the assumption $K_t \neq \emptyset$ is crucial to ensure that (u_k) is bounded in $W_0^{1,p}(\Omega)$.

Consider $\Omega =]0, 1[$, $a_t(x, s, \xi) = \xi$, $b_t(x, s, \xi) = 0$, $\underline{u}_t = z - t$ and $\overline{u}_t = z + t$, where $z \in C_c(]0, 1[) \setminus W_0^{1,2}(]0, 1[)$.

If $t_k \to 0$ with $t_k > 0$ and (u_k, t_k) are the solutions of (*PVI*), then the sequence (u_k) is unbounded in $W_0^{1,2}(\Omega)$.

5 Topological Degree for Quasilinear Elliptic Variational Inequalities with Natural Growth Conditions

Consider again the setting of Sect. 3. Throughout this section, we also assume that:

(B) *the functions $\underline{u}$ and $\overline{u}$ are bounded.*

It follows that Z^{tot} is automatically bounded in $L^\infty(\Omega)$.

Theorem 5.1 *The set Z^{tot} is (strongly) compact in $W_0^{1,p}(\Omega)$ (possibly empty).*

Proof If $K = \emptyset$, we have $Z^{tot} = \emptyset$. Otherwise the assertion follows from Theorem 4.1. □

Definition 5.2 We denote by $\mathcal{Z}(F, K)$ the family of the subsets Z of Z^{tot} which are both open and closed in Z^{tot} with respect to the $W_0^{1,p}(\Omega)$-topology. We will simply write $\mathcal{Z}$, if no confusion can arise.
Fix a continuous function $\psi : \mathbb{R} \to [0, 1]$ such that

$$\psi(s) = 1 \quad \text{for } s \leq 1\,, \qquad \psi(s) = 0 \quad \text{for } s \geq 2\,,$$

then set, for any $t \in [0, 1]$ and $s \in \mathbb{R}$,

$$\Psi_t(s) = \psi(t|s|)\, s\,.$$

If we consider $T = [0, 1]$ and define

$$a_t^\psi(x, s, \xi) = \psi(t|s|)\, a(x, s, \xi) + [1 - \psi(t|s|)]\, |\xi|^{p-2}\xi\,,$$
$$b_t^\psi(x, s, \xi) = \Psi_t\,(b(x, s, \xi))\ ,$$

it is easily seen that a_t^ψ, b_t^ψ satisfy (UN) and (UM). Moreover, for every $\underline{t} \in]0, 1[$, they satisfy (UC), if t is restricted to $[\underline{t}, 1]$. In particular, we can define a continuous map $H^\psi : W_0^{1,p}(\Omega)\times]0, 1] \to W^{-1,p'}(\Omega)$ by

$$H_t^\psi(u) = -\text{div}\left[a_t^\psi(x, u, \nabla u)\right] + b_t^\psi(x, u, \nabla u)$$

and, by Theorem 2.10, this map is of class $(S)_+$. We will simply write H, if no confusion can arise.

Proposition 5.3 *For every $Z \in \mathcal{Z}$, the following facts hold:*

(a) there exist a bounded and open subset U of $W_0^{1,p}(\Omega)$ and $\bar{t} \in]0, 1]$ such that

$$Z = Z^{tot} \cap U = Z^{tot} \cap \overline{U}$$

and such that the variational inequality (PVI) *has no solution* $(u, t) \in \partial U \times [0, \bar{t}]$*; in particular, the degree* $\deg((H_t^{\psi}, K), U, 0)$ *is defined whenever* $t \in]0, \bar{t}]$*;*

(b) *if* $\psi_0, \psi_1 : \mathbb{R} \to [0, 1]$ *have the same properties of* ψ *and* U_0*,* $\bar{t}_0$ *and* U_1*,* $\bar{t}_1$ *are as in* (a)*, then*

$$\deg((H_t^{\psi_0}, K), U_0, 0) = \deg((H_\tau^{\psi_1}, K), U_1, 0)$$

for every $t \in]0, \bar{t}_0]$ *and* $\tau \in]0, \bar{t}_1]$*;*

(c) *if* $\hat{a}$ *and* $\hat{b}$ *also satisfy* (N)*,* (M) *and*

$$\int_\Omega \left[a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)\, v\right] dx$$
$$= \int_\Omega \left[\hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u)\, v\right] dx$$
$$\text{for any } u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and any } v \in C_c^\infty(\Omega)\,,$$

then we have

$$\deg((H_t^{\psi}, K), U, 0) = \deg((\widehat{H}_\tau^{\psi}, K), \widehat{U}, 0)$$

for every $t \in]0, \bar{t}]$ *and* $\tau \in]0, \hat{t}]$*, provided that* U*,* $\bar{t}$*,* $\widehat{U}$*,* $\hat{t}$ *are as in* (a) *with respect to* a, b *and* $\hat{a}, \hat{b}$*, respectively.*

Proof By definition of $\mathcal{Z}$, there exists a bounded and open subset U of $W_0^{1,p}(\Omega)$ such that

$$Z = Z^{tot} \cap U = Z^{tot} \cap \overline{U}\,.$$

If (u_k, t_k) is a sequence of solutions of (PVI) with $u_k \in \partial U$ and $t_k \to 0$, then (u_k) is bounded in $L^\infty(\Omega)$ by (B). By Theorem 4.1, up to a subsequence (u_k) is convergent to some u in $W_0^{1,p}(\Omega)$ and u is a solution of (VI). Then $u \in \partial U$ and a contradiction follows. Therefore, there exists $\bar{t} \in]0, 1]$ such that (PVI) has no solution $(u, t) \in \partial U \times [0, \bar{t}]$ and assertion (a) is proved.

To prove (b), define

$$\psi_\mu = (1 - \mu)\psi_0 + \mu\psi_1$$

and consider $H_t^{\psi_\mu}$. Arguing as before, we find $\underline{t} > 0$ with $\underline{t} \le \bar{t}_j$, $j = 0, 1$, such that

$$\begin{cases} u \in K\,, \\ \langle H_t^{\psi_\mu}(u), v - u\rangle \ge 0 \qquad \forall v \in K\,, \end{cases}$$

has no solution with $u \in (\overline{U_0} \setminus U_1) \cup (\overline{U_1} \setminus U_0)$, $t \in [0, \underline{t}]$ and $\mu \in [0, 1]$.

From Theorem 2.7 we infer that

$$\deg((H_{\underline{t}}^{\psi_0}, K), U_0, 0) = \deg((H_{\underline{t}}^{\psi_0}, K), U_0 \cap U_1, 0)\,,$$
$$\deg((H_{\underline{t}}^{\psi_1}, K), U_1, 0) = \deg((H_{\underline{t}}^{\psi_1}, K), U_0 \cap U_1, 0)\,.$$

On the other hand, we have

$$\begin{aligned} \deg((H_{\underline{t}}^{\psi_0}, K), U_0 \cap U_1, 0) &= \deg((H_{\underline{t}}^{\psi_1}, K), U_0 \cap U_1, 0)\,, \\ \deg((H_{\underline{t}}^{\psi_0}, K), U_0, 0) &= \deg((H_{t}^{\psi_0}, K), U_0, 0) && \forall t \in]0, \bar{t}_0]\,, \\ \deg((H_{\underline{t}}^{\psi_1}, K), U_1, 0) &= \deg((H_{\tau}^{\psi_1}, K), U_1, 0) && \forall \tau \in]0, \bar{t}_1]\,, \end{aligned}$$

by Theorem 2.9. Then assertion (b) also follows.

The proof of (c) is quite similar. □

Definition 5.4 For every $Z \in \mathcal{Z}(F, K)$, we set

$$\mathrm{ind}((F, K), Z) = \deg((H_t, K), U, 0)\,,$$

where ψ, U, $\bar{t}$ are as in (a) and $0 < t \le \bar{t}$. We will simply write $\mathrm{ind}(Z)$, if no confusion can arise.

Proposition 5.5 *Assume that a and b satisfy, instead of (N), the more specific controllable growth condition:*

(C) *there exist $p \in]1, \infty[$, $\alpha^{(0)} \in L^1(\Omega)$, $\alpha^{(1)} \in L^{p'}(\Omega)$, $\beta > 0$ and $\nu > 0$ such that*

$$|a(x, s, \xi)| \le \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$
$$|b(x, s, \xi)| \le \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$
$$a(x, s, \xi) \cdot \xi \ge \nu|\xi|^p - \alpha^{(0)}(x) - \beta\,|s|^p\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

Then the map $F : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous, bounded on bounded subsets and of class $(S)_+$. Moreover, if U is a bounded and open subset of $W_0^{1,p}(\Omega)$ such that (VI) has no solution $u \in \partial U$ and

$$Z = \{u \in U : \ u \text{ is a solution of } (VI)\}\,,$$

then $Z \in \mathcal{Z}$ and

$$\mathrm{ind}(Z) = \deg((F, K), U, 0)\,.$$

Proof It is easily seen that this time a_t and b_t satisfy (*UC*) and (*UM*), for t belonging to all $[0, 1]$.

Then the assertions follow from Theorems 2.10 and 2.9. □

Theorem 5.6 *Let* $Z \in \mathcal{Z}$ *with* $\operatorname{ind}(Z) \neq 0$. *Then* $Z \neq \emptyset$.

Proof Let U and $\bar{t}$ be as in (*a*) of Proposition 5.3. If $Z = \emptyset$, from Theorem 4.1 we infer that there exists $t \in]0, \bar{t}]$ such that (*PVI*) has no solution (u, t) with $u \in \overline{U}$. From Theorem 2.5 we deduce that

$$\operatorname{ind}(Z) = \deg((H_t, K), U, 0) = 0$$

and a contradiction follows. □

Along the same line, the additivity property can be proved taking advantage of Theorems 2.7 and 4.1.

Theorem 5.7 *Let* $Z_0, Z_1 \in \mathcal{Z}$ *with* $Z_0 \cap Z_1 = \emptyset$. *Then* $Z_0 \cup Z_1 \in \mathcal{Z}$ *and*

$$\operatorname{ind}(Z_0 \cup Z_1) = \operatorname{ind}(Z_0) + \operatorname{ind}(Z_1)\,.$$

Theorem 5.8 *Let*

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0, 1]) \to \mathbb{R}^n\,,$$
$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0, 1]) \to \mathbb{R}$$

be two Carathéodory functions satisfying (*UN*) *and* (*UM*) *with respect to* $T = [0, 1]$ *and set*

$$H_t(u) = -\operatorname{div}\,[a_t(x, u, \nabla u)] + b_t(x, u, \nabla u)\,.$$

Let also, for each $t \in [0, 1]$, $\underline{u}_t : \Omega \to \overline{\mathbb{R}}$ *be a p-quasi upper semicontinuous function and* $\overline{u}_t : \Omega \to \overline{\mathbb{R}}$ *a p-quasi lower semicontinuous function, define* K_t *as in Sect.* 4 *and assume that:*

- *the functions* $\underline{u}_t, \overline{u}_t$ *are bounded uniformly with respect to* $t \in [0, 1]$*, we have* $K_t \neq \emptyset$ *for any* $t \in [0, 1]$ *and assumption* (*MC*) *is satisfied.*

  Then the following facts hold:

(a) the set

$$\widehat{Z}^{tot} := \left\{(u, t) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times [0, 1] :\ (u, t) \text{ is a solution of } (PVI)\right\}$$

is (strongly) compact in $W_0^{1,p}(\Omega) \times [0, 1]$ *(possibly empty);*

(b) if $\widehat{Z}$ *is open and closed in* $\widehat{Z}^{tot}$ *with respect to the topology of* $W_0^{1,p}(\Omega) \times [0,1]$ *and*

$$\widehat{Z}_t = \left\{ u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \; (u,t) \in \widehat{Z} \right\} ,$$

then $\widehat{Z}_t \in \mathcal{Z}(H_t, K_t)$ *for any* $t \in [0,1]$ *and* $\operatorname{ind}((H_t, K_t), \widehat{Z}_t)$ *is independent of* $t \in [0,1]$.

Proof First of all, the set $\widehat{Z}^{tot}$ is compact by Theorem 4.1. To prove assertion (b), let W be a bounded and open subset of $W_0^{1,p}(\Omega) \times [0,1]$ such that

$$\widehat{Z} = \widehat{Z}^{tot} \cap W = \widehat{Z}^{tot} \cap \overline{W} .$$

In particular, we have $\widehat{Z}_t \in \mathcal{Z}(H_t, K_t)$ for any $t \in [0,1]$.

Let $\widehat{T} = [0,1] \times [0,1]$, let

$$\begin{aligned} a_{t,\tau}(x,s,\xi) &= \psi(\tau|s|)\, a_t(x,s,\xi) + [1 - \psi(\tau|s|)]\, |\xi|^{p-2}\xi , \\ b_{t,\tau}(x,s,\xi) &= \Psi_\tau(b_t(x,s,\xi)) , \end{aligned}$$

for $(t,\tau) \in \widehat{T}$ and define

$$\begin{aligned} H_{t,\tau}(u) &= -\operatorname{div}[a_{t,\tau}(x,u,\nabla u)] + b_{t,\tau}(x,u,\nabla u) , \\ K_{t,\tau} &= K_t . \end{aligned}$$

It is easily seen that $a_{t,\tau}$ and $b_{t,\tau}$ satisfy (UN) and (UM) with respect to $\widehat{T}$, so that we can consider the problem

$$\begin{cases} (u,(t,\tau)) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times \widehat{T} , \\ u \in K_{t,\tau} , \\ \displaystyle\int_\Omega \left[a_{t,\tau}(x,u,\nabla u) \cdot \nabla(v-u) + b_{t,\tau}(x,u,\nabla u)\,(v-u) \right] dx \geq 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K_{t,\tau} . \end{cases} \tag{5.1}$$

Since $[0,1]$ is compact, by Theorem 4.1 there exists $\overline{\tau} \in]0,1]$ such that (5.1) has no solution $(u,(t,\tau))$ with $(u,t) \in \partial W$ and $0 \leq \tau \leq \overline{\tau}$.

By Definition 5.4 we infer that

$$\operatorname{ind}((H_t, K_t), \widehat{Z}_t) = \deg((H_{t,\overline{\tau}}, K_t), W_t, 0) \qquad \text{for any } t \in [0,1]$$

and the assertion follows from Theorem 2.9. □

Remark 5.9 By Theorem 5.8 and Proposition 5.5, $\mathrm{ind}((F, K), Z)$ can be calculated also by other approximation techniques, with respect to the one used in Definition 5.4.

Theorem 5.10 *If $K \neq \emptyset$, then* $\mathrm{ind}(Z^{tot}) = 1$.

Proof Define, for $0 \leq t \leq 1$,

$$a_t(x, s, \xi) = t\, a(x, s, \xi) + (1 - t)\, |\xi|^{p-2}\xi\,,$$
$$b_t(x, s, \xi) = t\, b(x, s, \xi)\,.$$

It is easily seen that a_t and b_t satisfy assumptions (UN) and (UM), so that Theorem 5.8 can be applied. If we take $\widehat{Z} = \widehat{Z}^{tot}$, we get

$$\mathrm{ind}(Z^{tot}) = \mathrm{ind}((H_1, K), \widehat{Z}_1) = \mathrm{ind}((H_0, K), \widehat{Z}_0)\,.$$

Let $u_0 \in K$ and let

$$U = \left\{u \in W_0^{1,p}(\Omega) : \ \|\nabla u\|_2 < r\right\}\,,$$

with r large enough to guarantee that $u_0 \in U, \widehat{Z}_0 \subseteq U$ and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_0)\, dx \geq 0 \qquad \text{for every } v \in \partial U\,.$$

From Proposition 5.5 and Theorem 2.6 we infer that

$$\mathrm{ind}((H_0, K), \widehat{Z}_0) = \deg((H_0, K), U, 0) = 1$$

and the assertion follows. □

Proposition 5.11 *Let $Z \in \mathcal{Z}(F, K)$ and let $\varphi \in \Phi$ and $\vartheta \in \Theta$. Then $Z^\varphi \in \mathcal{Z}(F^\varphi, K^\varphi)$, $Z \in \mathcal{Z}(F_\vartheta, K)$ and*

$$\mathrm{ind}((F, K), Z) = \mathrm{ind}((F^\varphi, K^\varphi), Z^\varphi) = \mathrm{ind}((F_\vartheta, K), Z)\,.$$

Proof If we set

$$\varphi_t(s) = (1 - t)s + t\varphi(s)\,, \qquad \vartheta_t(s) = (1 - t) + t\vartheta(s)\,,$$

the assertion follows from Theorem 5.8. □

6 Proof of Theorem 1.1

We aim to apply the results of the previous sections to

$$a(x,s,\xi) = A(x,s)\xi\,, \qquad b(x,s,\xi) = B(x,s)|\xi|^2 - g(x,s)\,.$$

By hypothesis (h_1), assumptions (N) and (M) are satisfied with $p = 2$. Moreover, if $\underline{M}$ and $\overline{M}$ are as in hypothesis (h_2), then $\underline{u} = \underline{M}$ and $\overline{u} = \overline{M}$ satisfy assumption (B).

Denote by (λ_k), $k \geq 1$, the sequence of the eigenvalues of (1.3), repeated according to multiplicity, and set, for a matter of convenience, $\lambda_0 = -\infty$.

Finally, define F, K, Z^{tot}, $\mathcal{Z}$ and $\mathrm{ind}(Z)$ as before, observe that $K \neq \emptyset$ and set

$$Z_+ = \left\{u \in Z^{tot} \setminus \{0\} : \; u \geq 0 \text{ a.e. in } \Omega\right\}\,,$$
$$Z_- = \left\{u \in Z^{tot} \setminus \{0\} : \; u \leq 0 \text{ a.e. in } \Omega\right\}\,.$$

It is easily seen that

$$a^{\varphi}(x,s,\xi) = A^{\varphi}(x,s)\xi\,, \qquad b^{\varphi}(x,s,\xi) = B^{\varphi}(x,s)|\xi|^2 - g^{\varphi}(x,s)\,.$$

Let us also set

$$A_{\vartheta}(x,s) = \vartheta(s)\,A(x,s)\,, \qquad B_{\vartheta}(x,s) = \vartheta'(s)\,A(x,s) + \vartheta(s)\,B(x,s)\,,$$
$$g_{\vartheta}(x,s) = \vartheta(s)\,g(x,s)\,,$$

so that

$$a_{\vartheta}(x,s,\xi) = A_{\vartheta}(x,s)\xi\,, \qquad b_{\vartheta}(x,s,\xi) = B_{\vartheta}(x,s)|\xi|^2 - g_{\vartheta}(x,s)\,.$$

Proposition 6.1 *For every $R > 0$ there exist $\vartheta_1, \vartheta_2 \in \Theta$, depending only on β_R and ν_R, such that*

$$B_{\vartheta_1}(x,s) \leq 0 \leq B_{\vartheta_2}(x,s) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \leq R\,.$$

Proof It is a simple variant of Proposition 3.6. □

Proposition 6.2 *Let*

$$\hat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$$

be a Carathéodory function such that for every $R > 0$ there exists $\beta_R > 0$ satisfying

$$|\hat{g}(x,s)| \leq \beta_R \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \leq R$$

and such that

$$\hat{g}(x,\underline{M}) \geq 0 \geq \hat{g}(x,\overline{M}) \qquad \textit{for a.e. } x \in \Omega\,.$$

If u is a solution of the variational inequality (VI) with

$$a(x,s,\xi) = A(x,s)\xi\,, \qquad b(x,s,\xi) = B(x,s)|\xi|^2 - \hat{g}(x,s)\,,$$

then u satisfies the equation

$$\int_\Omega \left[A(x,u)\nabla u \cdot \nabla v + B(x,u)\,|\nabla u|^2\,v\right] dx$$
$$= \int_\Omega \hat{g}(x,u)\,v\,dx \qquad \textit{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

Proof Let $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω, let $t > 0$ and let

$$u_t = \min\{u + tv, \overline{M}\}\,.$$

Since $u_t \in K$, it follows

$$\frac{1}{t}\int_\Omega A(x,u)\nabla u \cdot \nabla(u_t - u)\,dx$$
$$\geq -\int_\Omega B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_\Omega \hat{g}(x,u)\,\frac{u_t - u}{t}\,dx$$
$$= -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,\frac{u_t - u}{t}\,dx\,.$$

On the other hand, we have

$$\frac{1}{t}\int_\Omega A(x,u)\nabla u \cdot \nabla(u_t - u)\,dx$$
$$= \int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u \cdot \nabla v\,dx - \frac{1}{t}\int_{\{u+tv\geq\overline{M}\}} A(x,u)|\nabla u|^2\,dx$$
$$\leq \int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u \cdot \nabla v\,dx\,,$$

whence

$$\int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u \cdot \nabla v\,dx$$
$$\geq -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,\frac{u_t - u}{t}\,dx\,.$$

Since $0 \le u_t - u \le tv$, we can go to the limit as $t \to 0^+$, obtaining

$$\begin{aligned}\int_\Omega A(x,u)\nabla u \cdot \nabla v\,dx &= \int_{\{u<\overline{M}\}} A(x,u)\nabla u\cdot\nabla v\,dx\\ &\ge -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,v\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,v\,dx\\ &= -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx\\ &\qquad\qquad - \int_{\{u=\overline{M}\}} \hat{g}(x,\overline{M})\,v\,dx\\ &\ge -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx\,.\end{aligned}$$

Arguing on $u_t = \max\{u - tv, \underline{M}\}$, one can prove in a similar way that

$$\int_\Omega A(x,u)\nabla u\cdot\nabla v\,dx \le -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx\,,$$

whence

$$\int_\Omega \left[A(x,u)\nabla u\cdot\nabla v + B(x,u)\,|\nabla u|^2\,v\right]dx = \int_\Omega \hat{g}(x,u)\,v\,dx$$
$$\text{for any } v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega$$

and the assertion follows. □

Proposition 6.3 *Let Ω be connected and assume that $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfies $u \ge 0$ a.e. in Ω, $u > 0$ on a set of positive measure and*

$$\begin{aligned}&\int_\Omega \left[A(x,u)\nabla u\cdot\nabla v + B(x,u)\,|\nabla u|^2\,v\right]dx\\ &\qquad \ge \int_\Omega g(x,u)\,v\,dx \qquad \text{for any } v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega\,.\end{aligned}$$

Then we have

$$\operatorname*{ess\,inf}_C u > 0 \ \text{ for every compact subset } C \text{ of } \Omega\,.$$

Proof By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s) \le 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \le \|u\|_\infty\,.$$

Then for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω, we have

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v\,dx \geq \int_\Omega g(x,u)\,v\,dx = \int_\Omega \gamma(x,u)\,uv\,dx$$

with $\gamma(x,u) \in L^\infty(\Omega)$.

From [11, Theorem 8.15 and Remark 8.16] the assertion follows. □

Lemma 6.4 *Assume that Ω is connected and that $\lambda_1 < 0$. Moreover, according to (h_1), let $\beta > 0$ be such that*

$$|g(x,s)| \leq \beta|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \leq s \leq \overline{M}\,.$$

Let also $\psi : \mathbb{R} \to [0,1]$ be a continuous function, with $\psi(0) > 0$ and $\psi(s) = 0$ outside $]\underline{M}, \overline{M}[$, and consider the problem

$$\begin{cases} (u,t) \in K \times [0,1]\,, \\ \displaystyle\int_\Omega \left[A(x,u)\nabla u \cdot \nabla(v-u) + B(x,u)\,|\nabla u|^2\,(v-u)\right] dx \geq \int_\Omega g_t(x,u)(v-u) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K\,, \end{cases} \tag{6.1}$$

where

$$g_t(x,s) = g(x,s) + t(\psi(s) + \beta\, s^-)\,.$$

Denote by $\widehat{Z}^{tot}$ the set of solutions (u,t) of (6.1) and let

$$\widehat{Z} = \left\{(u,t) \in \widehat{Z}^{tot} :\ u \geq 0 \text{ a.e. in } \Omega \text{ and } u > 0 \text{ on a set of positive measure}\right\}\,.$$

Then there exist $0 < r_1 < r_2$ such that

$$\begin{aligned} \widehat{Z} &= \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\,dx < r_1^2 < r_2^2 < \int_\Omega (u^+)^2\,dx\right\} \\ &= \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\,dx \leq r_1^2 < r_2^2 \leq \int_\Omega (u^+)^2\,dx\right\}\,. \end{aligned}$$

Proof If we set

$$a_t(x,s,\xi) = A(x,s)\xi\,, \qquad b_t(x,s,\xi) = B(x,s)|\xi|^2 - g_t(x,s)\,,$$

it is easily seen that assumptions (UN) and (UM) are satisfied. From Theorem 5.8 we infer that $\widehat{Z}^{tot}$ is compact in $W_0^{1,p}(\Omega) \times [0,1]$.

First of all, we claim that there exists $r_2 > 0$ such that

$$\widehat{Z} \subseteq \left\{ (u,t) \in \widehat{Z}^{tot} : \; r_2^2 < \int_\Omega (u^+)^2 \, dx \right\} .$$

By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s) \geq 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \leq s \leq \overline{M} .$$

Assume, for a contradiction, that (u_k, t_k) is a sequence in $\widehat{Z}$ with $\|u_k\|_2 \to 0$. Then we may suppose, without loss of generality, that (u_k) is convergent to 0 in $W_0^{1,p}(\Omega)$ and a.e. in Ω and that (t_k) is convergent to some $t \in [0,1]$. Let $u_k = \tau_k z_k$ with $\tau_k = \|\nabla u_k\|_2$ and, up to a subsequence, (z_k) weakly convergent to some z in $W_0^{1,2}(\Omega)$.

If $v \in K \setminus \{0\}$ with $v \geq 0$ a.e. in Ω, we have

$$\int_\Omega \left[A(x,u_k) \nabla z_k \cdot \nabla (v - u_k) + \tau_k \, B(x,u_k) \, |\nabla z_k|^2 \, (v - u_k) \right] dx$$
$$\geq \int_\Omega \frac{g(x, \tau_k z_k)}{\tau_k} \, (v - u_k) \, dx + \frac{t_k}{\tau_k} \int_\Omega \psi(u_k)(v - u_k) \, dx ,$$

which implies that (t_k/τ_k) is bounded hence convergent, up to a subsequence, to some $\sigma \geq 0$.

Then we also get

$$\int_\Omega A(x,0) \nabla z \cdot \nabla v \, dx \geq \int_\Omega D_s g(x,0) z v \, dx + \sigma \, \psi(0) \int_\Omega v \, dx \qquad \text{for any } v \in K ,$$

whence

$$\int_\Omega \left[A(x,0) \nabla z \cdot \nabla v - D_s g(x,0) z v \right] \, dx = \sigma \, \psi(0) \int_\Omega v \, dx$$
$$\text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) .$$

If we choose $v = \varphi_1$, where φ_1 is a positive eigenfunction of (1.3) associated with $\lambda_1 < 0$, we get

$$\lambda_1 \int_\Omega z \varphi_1 \, dx = \sigma \psi(0) \int_\Omega \varphi_1 \, dx ,$$

whence $z = 0$.

Finally, the choice $v = 0$ in (6.1) yields

$$\int_\Omega A(x,u_k)|\nabla z_k|^2\,dx \le \int_\Omega \left[A(x,u_k)|\nabla z_k|^2 + \tau_k\,B(x,u_k)\,|\nabla z_k|^2\,z_k\right]dx$$
$$\le \int_\Omega \frac{g(x,\tau_k z_k)}{\tau_k}\,z_k\,dx + \frac{t_k}{\tau_k}\int_\Omega \psi(u_k)z_k\,dx\,.$$

We infer that $\|\nabla z_k\|_2 \to 0$ and a contradiction follows.

With this choice of r_2, we also have

$$\widehat{Z} \subseteq \left\{(u,t)\in \widehat{Z}^{tot}:\ \int_\Omega (u^-)^2\,dx < r_1^2 < r_2^2 < \int_\Omega (u^+)^2\,dx\right\}$$
$$\subseteq \left\{(u,t)\in \widehat{Z}^{tot}:\ \int_\Omega (u^-)^2\,dx \le r_1^2 < r_2^2 \le \int_\Omega (u^+)^2\,dx\right\}$$

for every $r_1 \in]0, r_2[$. Now we claim that there exists $r_1 \in]0, r_2[$ such that

$$\left\{(u,t)\in \widehat{Z}^{tot}:\ \int_\Omega (u^-)^2\,dx \le r_1^2 < r_2^2 \le \int_\Omega (u^+)^2\,dx\right\} \subseteq \widehat{Z}\,.$$

By Propositions 3.4 and 6.1, now we may assume without loss of generality that

$$B(x,s) \le 0 \qquad \text{for a.e. } x\in\Omega \text{ and every } s\in\mathbb{R} \text{ with } \underline{M}\le s\le \overline{M}\,.$$

Assume, for a contradiction, that (u_k,t_k) is a sequence in the set at the left hand side with $(u_k,t_k)\not\in\widehat{Z}$ and $\|u_k^-\|_2 \to 0$. Then, up to a subsequence, (u_k) is convergent to some $u\in\widehat{Z}$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω, while (t_k) is convergent to some $t\in[0,1]$. By Propositions 6.2 and 6.3 we have $u>0$ a.e. in Ω. If we write $u_k^- = \tau_k z_k$ with $\tau_k = \|\nabla u_k^-\|_2$, we have that (z_k) is weakly convergent, up to a subsequence, to some z in $W_0^{1,2}(\Omega)$ and, on the other hand, (z_k) is convergent to 0 a.e. in Ω, as $u>0$. The choice $v = u_k^+$ in (6.1) yields

$$\int_\Omega \left[A(x,u_k)\nabla u_k\cdot\nabla u_k^- + B(x,u_k)\,|\nabla u_k|^2\,u_k^-\right]dx$$
$$\ge \int_\Omega g(x,u_k)\,u_k^-\,dx + t_k\int_\Omega\left[\psi(u_k)+\beta u_k^-\right]u_k^-\,dx\,,$$

whence

$$-\int_\Omega A(x,u_k)|\nabla z_k|^2\,dx \ge \int_\Omega \frac{g(x,-\tau_k z_k)}{\tau_k}\,z_k\,dx\,.$$

We infer that $\|\nabla z_k\|_2 \to 0$ and a contradiction follows. □

Proposition 6.5 *If Ω is connected and $\lambda_1 < 0$, then we have $Z_+, Z_- \in \mathcal{Z}$ and*

$$\mathrm{ind}\,(Z_+) = \mathrm{ind}\,(Z_-) = 1\,.$$

Proof Let g_t, $\widehat{Z}^{tot}$, $\widehat{Z}$, r_1 and r_2 be as in Lemma 6.4. We aim to apply again Theorem 5.8 with

$$a_t(x,s,\xi) = A(x,s)\xi\,, \qquad b_t(x,s,\xi) = B(x,s)|\xi|^2 - g_t(x,s)\,.$$

By Lemma 6.4 the set $\widehat{Z}$ is open and closed in $\widehat{Z}^{tot}$. From Theorem 5.8 we infer that

$$Z_+ = \widehat{Z}_0 \in \mathcal{Z}(H_0, K) = \mathcal{Z}$$

and that

$$\mathrm{ind}\,(Z_+) = \mathrm{ind}\left((H_0,K),\widehat{Z}_0\right) = \mathrm{ind}\left((H_1,K),\widehat{Z}_1\right)\,.$$

Now we claim that $\widehat{Z}_1 = Z^{tot}(H_1, K)$. By Propositions 3.4 and 6.1 we may assume without loss of generality that

$$B(x,s) \le 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \le s \le \overline{M}\,.$$

If we take $v \in K \setminus \{0\}$ with $v \ge 0$ a.e. in Ω in (6.1), we see that $0 \notin Z^{tot}(H_1, K)$. Moreover, if $u \in Z^{tot}(H_1, K)$, the choice $v = u^+$ yields $(v - u) = u^-$, hence

$$-\int_\Omega A(x,u)|\nabla u^-|^2\,dx \ge \int_\Omega \left[A(x,u)\nabla u\cdot\nabla u^- + B(x,u)\,|\nabla u|^2\,u^-\right]dx$$
$$\ge \int_\Omega g_1(x,u)u^-\,dx \ge 0\,.$$

Therefore $u^- = 0$, whence $u \in \widehat{Z}_1$ and the claim is proved.

From Theorem 5.10 we infer that

$$\mathrm{ind}\left((H_1,K),\widehat{Z}_1\right) = \mathrm{ind}\left((H_1,K), Z^{tot}(H_1,K)\right) = 1$$

and the assertion concerning Z_+ follows.

The assertion concerning Z_- can be proved in a similar way. ◻

Proposition 6.6 *If there exists $k \ge 0$ with $\lambda_k < 0 < \lambda_{k+1}$, then $\{0\} \in \mathcal{Z}$ and*

$$\mathrm{ind}\,(\{0\}) = (-1)^k\,.$$

Proof If we set

$$A_t(x,s) = A(x,ts)\,, \qquad B_t(x,s) = t\,B(x,ts)\,,$$

$$g_t(x,s) = \begin{cases} \dfrac{g(x,ts)}{t} & \text{if } 0 < t \le 1\,,\\ D_s g(x,0)s & \text{if } t = 0\,,\end{cases}$$

$$a_t(x,s,\xi) = A_t(x,s)\xi\,, \qquad b_t(x,s,\xi) = B_t(x,s)|\xi|^2 - g_t(x,s)\,,$$

it is easily seen that assumptions (*UN*) and (*UM*) are satisfied. We aim to apply Theorem 5.8.

We claim that there exists $r > 0$ such that, if $(u,t) \in \widehat{Z}^{tot}$ and $\|\nabla u\|_2 \le r$, then $u = 0$. Assume, for a contradiction, that (u_k, t_k) is a sequence in $\widehat{Z}^{tot}$ with $u_k \neq 0$ and $\|\nabla u_k\|_2 \to 0$. Let $u_k = \tau_k z_k$ with $\tau_k = \|\nabla u_k\|_2$ and, up to a subsequence, (z_k) weakly convergent to some z in $W_0^{1,2}(\Omega)$ and (t_k) convergent to some t in $[0,1]$.

Given $v \in K$, if $t_k > 0$ we have

$$\int_\Omega \left[A(x,t_k u_k)\nabla u_k \cdot \nabla(v-u_k) + t_k\,B(x,t_k u_k)\,|\nabla u_k|^2\,(v-u_k)\right] dx \\ \ge \int_\Omega \frac{g(x,t_k u_k)}{t_k}\,(v-u_k)\,dx\,, \qquad (6.2)$$

whence

$$\int_\Omega \left[A(x,t_k u_k)\nabla z_k \cdot \nabla(v-u_k) + \tau_k t_k\,B(x,t_k u_k)\,|\nabla z_k|^2\,(v-u_k)\right] dx \\ \ge \int_\Omega \frac{g(x,\tau_k t_k z_k)}{\tau_k t_k}\,(v-u_k)\,dx\,.$$

Going to the limit as $k \to \infty$, we get

$$\int_\Omega A(x,0)\nabla z \cdot \nabla v\,dx \ge \int_\Omega D_s g(x,0) z v\,dx \qquad \text{for any } v \in K\,.$$

If $t_k = 0$, we have

$$\int_\Omega A(x,0)\nabla z_k \cdot \nabla(v-u_k)\,dx \ge \int_\Omega D_s g(x,0) z_k (v-u_k)\,dx$$

and the same conclusion easily follows.

Then we infer that

$$\int_\Omega A(x,0)\nabla z\cdot\nabla v\,dx = \int_\Omega D_s g(x,0) z v\,dx \qquad \text{for any } v\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

whence $z=0$, as 0 is not in the sequence (λ_k).

By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s)\geq 0 \qquad \text{for a.e. } x\in\Omega \text{ and every } s\in\mathbb{R} \text{ with } \underline{M}\leq s\leq \overline{M}\,.$$

If $t_k>0$, the choice $v=-u_k^-$ in (6.2) yields

$$\begin{aligned}\int_\Omega A(x,t_k u_k)|\nabla z_k^+|^2\,dx &\leq \int_\Omega \left[A(x,t_k u_k)|\nabla z_k^+|^2 + \tau_k t_k\,B(x,t_k u_k)\,|\nabla z_k|^2\,z_k^+\right]dx\\ &\leq \int_\Omega \frac{g(x,\tau_k t_k z_k)}{\tau_k t_k}\,z_k^+\,dx\,,\end{aligned}$$

which implies that $\|\nabla z_k^+\|_2\to 0$. If $t_k=0$ the argument is analogous and simpler. In a similar way one can show that $\|\nabla z_k^-\|_2\to 0$ and a contradiction follows. Therefore, there exists $r>0$ with the required property.

In particular, we can apply Theorem 5.8 with

$$\widehat{Z}=\{0\}\times[0,1]\,,$$

obtaining

$$\{0\}=\widehat{Z}_1\in\mathcal{Z}(H_1,K)=\mathcal{Z}$$

and

$$\text{ind}\,(\{0\})=\text{ind}\left((H_1,K),\widehat{Z}_1\right)=\text{ind}\left((H_0,K),\widehat{Z}_0\right)=\text{ind}\,((H_0,K),\{0\})\ .$$

On the other hand, if we set

$$U=\left\{u\in W_0^{1,2}(\Omega):\ \|\nabla u\|_2<r\right\}\,,$$

from Proposition 5.5 we infer that

$$\text{ind}((H_0,K),\{0\})=\deg((H_0,K),U,0)\,.$$

Now, if we set

$$K_t=\begin{cases}\frac{1}{t}K & \text{if } 0<t\leq 1\,,\\ W_0^{1,2}(\Omega) & \text{if } t=0\,,\end{cases}$$

from [11, Theorem 4.53 and Proposition 4.61] we deduce that

$$\deg((H_0, K), U, 0) = \deg((H_0, W_0^{1,2}(\Omega)), U, 0) .$$

Finally, from [13, Theorem 2.5.2] it follows that

$$\deg((H_0, W_0^{1,2}(\Omega)), U, 0) = (-1)^k .$$

□

Proof of Theorem 1.1 From Proposition 6.5 we know that

$$\operatorname{ind}(Z_+) = \operatorname{ind}(Z_-) = 1 .$$

By Theorem 5.6 and Propositions 6.2 and 6.3 we infer that there exist at least two solutions $u_1 \in Z_-$ and $u_2 \in Z_+$ of (1.2) with

$$\operatorname*{ess\,sup}_C u_1 < 0 < \operatorname*{ess\,inf}_C u_2 \quad \text{for every compact subset } C \text{ of } \Omega .$$

Assume, for a contradiction, that

$$Z^{tot} = Z_- \cup \{0\} \cup Z_+$$

with $\operatorname{ind}(\{0\}) = 1$ by Proposition 6.6. From Theorems 5.10 and 5.7 we infer that

$$1 = \operatorname{ind}\left(Z^{tot}\right) = 3$$

and a contradiction follows. Therefore there exists

$$u_3 \in Z^{tot} \setminus (Z_- \cup \{0\} \cup Z_+) .$$

By Proposition 6.2 u_3 is a sign-changing solution of (1.2). According to [10, Theorem VII.1.1], each u_j is locally Hölder continuous in Ω. □

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On a Diffuse Interface Model for Tumour Growth with Non-local Interactions and Degenerate Mobilities

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Abstract We study a non-local variant of a diffuse interface model proposed by Hawkins–Daarud et al. (Int. J. Numer. Methods Biomed. Eng. 28:3–24, 2012) for tumour growth in the presence of a chemical species acting as nutrient. The system consists of a Cahn–Hilliard equation coupled to a reaction-diffusion equation. For non-degenerate mobilities and smooth potentials, we derive well-posedness results, which are the non-local analogue of those obtained in Frigeri et al. (European J. Appl. Math. 2015). Furthermore, we establish existence of weak solutions for the case of degenerate mobilities and singular potentials, which serves to confine the order parameter to its physically relevant interval. Due to the non-local nature of the equations, under additional assumptions continuous dependence on initial data can also be shown.

Keywords Degenerate mobility • Non-local Cahn–Hilliard equations • Singular potentials • Tumour growth • Weak solutions • Well-posedness

AMS (MOS) Subject Classification 35D30, 35K55, 35K65, 35K57, 35Q92

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1 Introduction

The tumour model of Hawkins–Daarud et al. [37] is a four-species model consisting of tumour cells, healthy cells, nutrient rich and nutrient poor water. The model is further simplified with the constraint that the total concentration of the cells and of the water remain constant throughout the domain, which then leads to a two-phase model, composed of a Cahn–Hilliard equation coupled to a reaction-diffusion equation. Denoting by φ the difference in volume fractions between the tumour cells and the healthy cells, and by σ the concentration of the nutrient rich water (which we will simply denote as the nutrient), the model equations are (see also [33, §2.5.2])

$$\varphi_t = \operatorname{div}(m(\varphi)\nabla\mu) + P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \tag{1.1a}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi - \chi\sigma, \tag{1.1b}$$

$$\sigma_t = \operatorname{div}(n(\varphi)\nabla(\sigma + \chi(1-\varphi))) - P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \tag{1.1c}$$

where $m(\varphi)$, $n(\varphi)$ are mobilities for φ and σ, respectively, Ψ' is the derivative of a potential Ψ with equal minima at ± 1, A and B are positive constants related to the surface tension and interfacial thickness, $P(\varphi)$ is a non-negative function with the source terms $P(\varphi)(\sigma + \chi(1-\varphi) - \mu)$ motivated from linear phenomenological laws for chemical reactions, and $\chi \geq 0$ is a parameter such that the for $\chi \neq 0$, the terms $\operatorname{div}(n(\varphi)\nabla(\chi\varphi))$ in (1.1c) and $\operatorname{div}(m(\varphi)\nabla(\chi\sigma))$ in (1.1a) (after substituting (1.1b) into (1.1a)) mimic transport mechanisms akin to that of active transport and chemotaxis, respectively, see [33] for more details.

Associated to (1.1) is the free energy

$$\mathscr{E}(\varphi,\sigma) := \int_\Omega A\Psi(\varphi) + \frac{B}{2}|\nabla\varphi|^2 + \frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi)\,dx\,, \tag{1.2}$$

where in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, the first two terms form the well-known Ginzburg–Landau energy, leading to phase separation and surface tension effects. In contrast, it is not expected that the nutrient-rich and nutrient-poor water experience separation akin to that of the cells, and thus the nutrient free energy, modelled by the third and fourth terms, only consists of terms modelling diffusion and interactions with the cells. From now on the model (1.1) with the energy (1.2) will be called local model to distinguish it from the non-local one associated to energy (1.3).

In terms of the analysis for (1.1), the well-posedness of weak and strong solutions with constant mobilities $m(\varphi) = n(\varphi) = 1$ and $\chi = 0$, and the existence of a global attractor have been established in [24] for a large class of nonlinearities Ψ and P. A viscosity regularized version of (1.1) (with constant mobilities and $\chi = 0$) has been the subject of study in [10], where existence and uniqueness of weak solutions and long-time behavior are shown for singular potentials Ψ. Furthermore, for regular quartic potentials, the weak solutions to the viscosity regularized model converge to the model studied in [24] as the viscosity parameter tends to zero. Further analytical results including asymptotic analyses as some small coefficients tend to zero and error estimates have been obtained in the recent works [11, 12] under different assumptions on the bulk potential Ψ including in some cases the singular one (cf., e.g., (1.6)). For the case $\chi \neq 0$, we refer the reader to [29–32, 40] for existence results to similar Cahn–Hilliard systems.

In this work, we study a non-local variant of (1.1), where we replace the Ginzburg–Landau component in $\mathscr{E}$ by a non-local free energy

$$\int_\Omega \int_\Omega \frac{B}{4} J(x-y)(\varphi(x)-\varphi(y))^2 \, dx \, dy + \int_\Omega A\Psi(\varphi)\, dx, \tag{1.3}$$

where J is a symmetric kernel defined on $\Omega \times \Omega$. Then, letting $Q_T := \Omega \times (0,T)$, the non-local variant of (1.1) reads as

$$\varphi_t = \operatorname{div}(m(\varphi)\nabla\mu) + P(\varphi)(\sigma + \chi(1-\varphi) - \mu) \quad \text{in } Q_T, \tag{1.4a}$$

$$\mu = A\Psi'(\varphi) + Ba\varphi - BJ \star \varphi - \chi\sigma \quad \text{in } Q_T, \tag{1.4b}$$

$$\sigma_t = \operatorname{div}(n(\varphi)\nabla(\sigma + \chi(1-\varphi))) - P(\varphi)(\sigma + \chi(1-\varphi) - \mu) \quad \text{in } Q_T, \tag{1.4c}$$

with

$$a(x) := \int_\Omega J(x-y)\, dy, \quad (J \star \varphi)(x,t) := \int_\Omega J(x-y)\varphi(y,t)\, dy.$$

We complement (1.4) with the initial and boundary conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \text{ in } \Omega, \quad \partial_\nu\varphi = \partial_\nu\mu = \partial_\nu\sigma = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.5}$$

where $\partial_\nu f := \nabla f \cdot \nu$ with outer unit normal ν on $\partial\Omega$.

In biological models, non-local interactions have been used to describe competition for space and degradation [47], spatial redistribution [4, 41], and also cell-to-cell adhesion [1, 6, 35]. The model (1.4) which we study belongs to the category of non-local cell-to-cell adhesion, as it is well-known that the Ginzburg–Landau energy leads to separation and surface tension effects, and heuristically this corresponds to the preference of tumour cells to adhere to each other rather than to the healthy cells.

The non-local Cahn–Hilliard equation has been studied intensively by many authors, see for example [2, 3, 26–28]. There has also been some focus towards coupling with fluid equations, such as Brinkman and Hele–Shaw flows [17] or Navier–Stokes flow [9, 20–23, 25]. For the non-local Cahn–Hilliard equation with source terms, analytic results related to well-posedness and long-time behavior have been obtained in [16, 42] for prescribed source terms or Lipschitz source terms depending on the order parameter. Our present contribution aims to extend the study of the non-local Cahn–Hilliard equation to the case where source terms are coupled with other variables.

Our first result concerns the well-posedness of (1.4) with non-degenerate mobilities and regular potentials, which is summarized in Theorems 2.1 and 2.2 below. Due to the non-local nature of the equations, the regularities we obtain here for weak solutions are lower than those satisfied by solutions to the local model studied in [24]. Often in the modelling and in numerical simulations, it is advantageous to consider a singular potential Ψ, which enforces the range of the order parameter φ to lie in the physically relevant interval $[-1,1]$ or $(-1,1)$. One example is the classical logarithmic potential:

$$\Psi_{\log}(\varphi) = \frac{\theta}{2}((1+\varphi)\log(1+\varphi) + (1-\varphi)\log(1-\varphi)) - \frac{\theta_c}{2}\varphi^2, \tag{1.6}$$

for constants $0 < \theta < \theta_c$. Furthermore, depending on the applications one has in mind, a mobility $m(\varphi)$ that is degenerate at $\varphi = \pm 1$ is often considered alongside singular potentials, for example $m(\varphi) = (1-\varphi^2)$ [18, 44, 45]. The degeneracy of the mobility at ± 1 effectively restricts the diffusive mechanisms from the Cahn–Hilliard system to the interfacial region.

In the models of [7, 13, 14, 19, 48] a one-side mobility $m_1(\varphi) = (1+\varphi)_+ = \max(1+\varphi, 0)$ is employed so that the Cahn–Hilliard diffusive mechanisms are switched off in the region of healthy cells $\{\varphi = -1\}$, and the tumour cells are allowed to diffuse. However, in those papers the models are formulated with smooth potentials, and it is not known if the models with a one-sided mobility can be analytically investigated. To the authors' best knowledge, the analytical results concerning local Cahn–Hilliard systems with source terms derived in [8, 10, 15, 24, 29–32, 39, 40, 43] consider positive or constant mobilities. Here, due to the degeneracy of the mobility m, the gradient $\nabla\mu$ is no longer controlled in some Lebesgue space, and thus the Eq. (1.1a) has to be reformulated into a weaker form. In the local setting the main effort lies in deriving high order estimates for φ, which may not be controlled uniformly in a suitable approximation scheme when source terms involving φ and other variables are present.

For our present non-local setting, substituting (1.4b) into (1.4a) and (1.4c) leads to a formulation of (1.4) in which μ does not appear:

$$\begin{aligned}
\varphi_t = &\ \operatorname{div}\left(Am(\varphi)\Psi''(\varphi)\nabla\varphi + m(\varphi)\nabla\left(Ba\varphi - BJ\star\varphi - \chi\sigma\right)\right) \\
&+ P(\varphi)(\sigma + \chi(1-\varphi)) - P(\varphi)\left(A\Psi'(\varphi) + Ba\varphi - BJ\star\varphi - \chi\sigma\right) \text{ in } Q_T, \\
\sigma_t = &\ \operatorname{div}\left(n(\varphi)\nabla(\sigma + \chi(1-\varphi))\right) \\
&- P(\varphi)(\sigma + \chi(1-\varphi)) + P(\varphi)\left(A\Psi'(\varphi) + Ba\varphi - BJ\star\varphi - \chi\sigma\right) \text{ in } Q_T.
\end{aligned}$$

Using the method introduced by Elliott and Garcke in [18] for the Cahn–Hilliard equation with degenerate mobilities, our second main result concerns the existence of weak solutions to (1.4) where the mobility $m(\varphi)$ is degenerate at $\varphi = \pm 1$ and the potential $\Psi : (-1,1) \to \mathbb{R}$ is singular. This is given in Theorem 2.3. Let us point out that we encounter new difficulties in the analysis of the source terms, namely the product $P(\varphi)\Psi'(\varphi)$. Actually, for singular potentials, $\Psi'(s)$ becomes unbounded as $s \to \pm 1$. Hence, to suitably control the product $P\Psi'$, we consider functions $P(s)$ that decay to zero as $s \to \pm 1$ in such a way that the product $P\Psi'$ remains bounded. In the original model of [37], P takes the form $P(s) = (1+s)_+ = \max(1+s, 0)$ (see [33, §2.5.2] for more details) so that the source terms are active only in the tumour region $\{\varphi = 1\}$ and are not active in the healthy cell region $\{\varphi = -1\}$. On the other hand, in the work of [38] the function P is chosen to be a multiple of the potential Ψ (see also [33, §3.3.2]), which is degenerate at ± 1. The effect of the latter choice acts in a similar manner to a two-sided degenerate mobility and restricts the influence of the source terms to the interfacial layer. This effect of localizing the source terms in the interfacial layers is supported by formally matched asymptotic analysis performed in [33, 38]. We also refer the reader to [34] for numerical simulations with a two-sided degenerate P in the multi-component setting.

In contrast to the local version, where uniqueness of solutions to the Cahn–Hilliard equation with degenerate mobilities is still an open question, in the non-local case with degenerate mobilities we can derive a result concerning continuous dependence on initial data when $\chi = 0$. This is given in Theorem 2.4, and can be attributed to the fact that the non-local model is akin to a coupled system of second-order equations. We point out that we have to restrict our analysis to the case $\chi = 0$ as the regularity of the variable σ seems not to be sufficient to control the difference of certain terms.

The remainder of this paper is organized as follows: The assumptions and main results are summarized in Sect. 2. In Sect. 3 we establish existence, regularity and continuous dependence on initial data for weak solutions of (1.4) with non-degenerate mobilities and regular potentials. Then, by an approximation procedure, the existence of weak solutions to the system with degenerate mobilities and singular potentials is treated in Sect. 4, and the continuous dependence on initial data is shown when $\chi = 0$.

For the remainder of this paper, we will use the following notation. The spaces $L^2(\Omega)$ and $H^1(\Omega)$ are denoted by H and V, respectively. For a (real) Banach space X its dual is denoted as X' and $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing between X and X'. The L^2-inner product will be denoted by $(\cdot, \cdot)$. For any $p \in [1, \infty]$ and $k > 0$, we denote $L^p := L^p(\Omega)$ and $W^{k,p} := W^{k,p}(\Omega)$ as the standard Lebesgue spaces and Sobolev spaces equipped with the norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^{k,p}}$. In the case $p = 2$ we use notation $\| \cdot \|_H := \|\cdot\|_{L^2}$ and $\|\cdot\|_V := \|\cdot\|_{H^1}$.

We also recall the following useful inequalities:

- *Young's inequality for convolutions*: For $p, q, r \geq 1$ real numbers with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,
$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$
- The *Gagliardo–Nirenberg interpolation inequality* in dimension d: Let Ω be a Lipschitz bounded domain and $f \in W^{m,r}(\Omega) \cap L^q(\Omega)$, $m \in \mathbb{N}$, $1 \leq q, r \leq \infty$.

For any integer j, $0 \le j < m$, suppose there is $\alpha \in \mathbb{R}$ such that

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \le \alpha \le 1.$$

If $r \in (0, \infty)$ and $m-j-\frac{d}{r}$ is a non-negative integer, then we additionally assume $\alpha \neq 1$. Under these assumptions, there exists a positive constant C depending only on Ω, m, j, q, r, and α such that

$$\|D^j f\|_{L^p(\Omega)} \le C\|f\|^{\alpha}_{W^{m,r}(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)}. \tag{1.7}$$

For the Hilbert triplet (V, H, V') we introduce the Riesz isomorphism $\mathcal{N} : V \to V'$ associated to the standard scalar product of V,

$$\langle \mathcal{N} v, w\rangle_V = \int_\Omega \nabla v \cdot \nabla w + vw\, dx \quad \forall v, w \in V. \tag{1.8}$$

For $u \in D(\mathcal{N}) := \{f \in H^2(\Omega) : \partial_\nu f = 0 \text{ on } \partial\Omega\}$, we have $\mathcal{N} u = -\Delta u + u$, and the restriction of $\mathcal{N}$ to $D(\mathcal{N})$ is an isomorphism from $D(\mathcal{N})$ to H. By the classical spectral theorem, there exists a sequence of eigenvalues λ_j with $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lambda_j \to \infty$, and a family of eigenfunctions $w_j \in D(\mathcal{N})$ such that $\mathcal{N} w_j = \lambda_j w_j$ which forms an orthonormal basis in H and an orthogonal basis in V. Note that the first eigenfunction w_1 is a constant, and hence $\lambda_1 = 1$. Furthermore, the inverse operator $\mathcal{N}^{-1} : V' \to V$ satisfies

$$\begin{aligned} &\langle \mathcal{N} u, \mathcal{N}^{-1} f\rangle_V = \langle f, u\rangle_V, \quad \|\mathcal{N}^{-1} f\|_V \le \|f\|_{V'} \quad \forall u \in V,\ f \in V', \\ &\frac{d}{dt}\|g\|^2_{V'} = 2\langle g_t, \mathcal{N}^{-1} g\rangle_V \quad \forall g \in H^1(0, T; V'). \end{aligned} \tag{1.9}$$

If $f \in H$, by elliptic regularity $\mathcal{N}^{-1} f \in D(\mathcal{N})$ and additionally it holds for some positive constant C depending only on Ω,

$$\|\mathcal{N}^{-1} f\|_{D(\mathcal{N})} \le C\|f\|_H \quad \forall f \in H. \tag{1.10}$$

We will denote the dual space of $D(\mathcal{N})$ by $D(\mathcal{N}^{-1})$.

2 General Assumptions and Main Results

In this section we state the main results on existence, regularity, uniqueness, and continuous dependence of solutions to (1.4)–(1.5) first for the case with non-degenerate mobilities and regular potentials and then for the case of degenerate mobilities and singular potentials. The results are stated for dimension $d = 3$, but similar results also hold for $d = 1, 2$.

2.1 Non-degenerate Mobilities and Regular Potentials

(A1) $m \in C^0(\mathbb{R})$ and there exist constants $m_1, m_2 > 0$ such that

$$m_1 \le m(s) \le m_2 \quad \forall s \in \mathbb{R}.$$

(A2) $n \in C^0(\mathbb{R})$ and there exist constants $n_1, n_2 > 0$ such that

$$n_1 \le n(s) \le n_2 \quad \forall s \in \mathbb{R}.$$

(A3) $J \in W^{1,1}_{loc}(\mathbb{R}^d)$ satisfies

$$J(z) = J(-z), \quad a(x) := \int_\Omega J(x-y)\,dy \ge 0 \text{ a.e. in } \Omega,$$

$$a^* := \sup_{x\in\Omega} \int_\Omega |J(x-y)|\,dy < \infty, \quad b := \sup_{x\in\Omega} \int_\Omega |\nabla J(x-y)|\,dy < \infty.$$

(A4) $\Psi \in C^2(\mathbb{R})$ and there exists $c_0 > \chi^2 \ge 0$ such that

$$A\Psi''(s) + Ba(x) \ge c_0 \quad \forall s \in \mathbb{R} \text{ and for a.e. } x \in \Omega.$$

(A5) There exist $c_1 \in \mathbb{R}$ and

$$c_2 > \frac{1}{2A}\left(B(a^* - a_*) + \chi^2\right), \tag{2.11}$$

such that

$$\Psi(s) \ge c_2\,|s|^2 - c_1 \quad \forall s \in \mathbb{R} \text{ where } a_* := \inf_{x\in\Omega} \int_\Omega J(x-y)\,dy\,.$$

(A6) There exist $z \in (1, 2]$, $c_3 > 0$ and $c_4 \ge 0$ such that

$$\left|\Psi'(s)\right|^z \le c_3\Psi(s) + c_4 \quad \forall s \in \mathbb{R}.$$

(A7) $P \in C^0(\mathbb{R})$ and there exists $c_5 > 0$ such that

$$0 \le P(s) \le c_5\,(1 + |s|^q) \quad \forall s \in \mathbb{R},\ q \in \left[1, \tfrac{10}{3}\right).$$

(A8) $\varphi_0 \in H$ satisfies $\Psi(\varphi_0) \in L^1$ and $\sigma_0 \in H$.

The assumption (A4) imposes the condition that the potential Ψ has to have at least quadratic polynomial growth, and will be essential in the identification of certain limit solutions. We also mention that (A7) is in strong contrast with the growth assumption for $P(\cdot)$ made in [24], where the authors are able to consider polynomial growth up to but not including ninth order. The reason for an upper bound of $\frac{10}{3}$ in the current setting can be seen from the regularity for φ, where in the non-local case one obtains $\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V)$, and in the local case one obtains $\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^3)$. The lower regularity for φ in the non-local case means that we only obtain compactness for the Galerkin approximations of φ in $L^2(0,T;L^r)$ for $r < 6$, which in turn limits the growth assumptions on P.

Definition 2.1 We call a pair (φ, σ) a weak solution to (1.4)–(1.5) on $[0,T]$ if

$$\begin{aligned}
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap W^{1,r}(0,T;D(\mathcal{N}^{-1})),\\
&\sigma \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap W^{1,r}(0,T;D(\mathcal{N}^{-1})),\\
&\mu := Ba\varphi - BJ \star \varphi + A\Psi'(\varphi) - \chi\sigma \in L^2(0,T;V),
\end{aligned}$$

for some $r > 1$, and the following variational formulation is satisfied for a.e. $t \in (0,T)$ and for all $\zeta \in D(\mathcal{N})$,

$$0 = \langle \varphi_t, \zeta \rangle_{D(\mathcal{N})} + (m(\varphi)\nabla\mu, \nabla\zeta) - (P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \zeta), \tag{2.12a}$$

$$0 = \langle \sigma_t, \zeta \rangle_{D(\mathcal{N})} + (n(\varphi)\nabla(\sigma - \chi\varphi), \nabla\zeta) + (P(\varphi)(\sigma + \chi(1-\varphi) - \mu), \zeta), \tag{2.12b}$$

together with

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0.$$

Notice that the regularity properties of a weak solution entail that we have $\varphi, \sigma \in C_w([0,T];H) \cap C^0([0,T];V')$, where $C_w([0,T];H)$ denotes the space of weakly continuous functions on $[0,T]$ with values in the space H. Therefore, the initial conditions make sense.

Theorem 2.1 (Existence and Energy Inequality) *Under Assumptions (A1)–(A8), there exists a weak solution pair (φ, σ) to (1.4) in the sense of Definition 2.1 which satisfies, for all $t > 0$, the following energy inequality*

$$\begin{aligned}
&E(\varphi(t), \sigma(t)) + \|\sqrt{m(\varphi)}\nabla\mu\|^2_{L^2(0,t;H)} + \|\sqrt{n(\varphi)}\nabla(\sigma - \chi\varphi)\|^2_{L^2(0,t;H)}\\
&\quad + \|\sqrt{P(\varphi)}(\sigma + \chi(1-\varphi) - \mu)\|^2_{L^2(0,t;H)} \le E(\varphi_0, \sigma_0),
\end{aligned} \tag{2.13}$$

where

$$E(\varphi, \sigma) = \int_\Omega A\Psi(\varphi) + \frac{B}{2}a(x)\,|\varphi|^2 - \frac{B}{2}\varphi(J \star \varphi) + \frac{1}{2}\,|\sigma|^2 + \chi\sigma(1-\varphi)\,dx\,. \tag{2.14}$$

Furthermore, if (A7) is satisfied with $q \le \frac{4}{3}$ then it holds that

$$\varphi_t, \sigma_t \in L^2(0,T;V'), \quad \varphi, \sigma \in C^0([0,T];H), \quad \varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0 \text{ a.e. in } \Omega,$$

and the energy inequality (2.13) *becomes an equality for all $t > 0$.*
To show continuous dependence on initial data, we make the following assumptions.

(B1) $m = n = 1$.
(B2) $P \in C^{0,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
(B3) In addition to (A4), Ψ also satisfies

$$\left|\Psi'(s_1) - \Psi'(s_2)\right| \le c_6\left(1 + |s_1|^r + |s_2|^r\right)|s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}$$

for some $c_6 > 0$ and $r \in [0, \frac{4}{3}]$.

Under (B2) we see that $\varphi_t, \sigma_t \in L^2(0,T;V')$.

Theorem 2.2 (Continuous Dependence for Constant Mobilities) *Let $(\varphi_i, \sigma_i)_{i=1,2}$ denote two weak solution pairs to* (1.4) *with J satisfying (A3), Ψ satisfying (B3), mobilities m, n equal to 1, and nonlinearity P satisfying (B3), corresponding to initial data $(\varphi_{0,i}, \sigma_{0,i})_{i=1,2}$ satisfying (A8). Then there exists a positive constant C, depending on A, B, a^*, $\|J\|_{W^{1,1}}$, $\|P\|_{L^\infty(\mathbb{R})}$, c_0, χ, c_6, $\|\sigma_i\|_{L^2(0,T;V)}$, $\|\mu_i\|_{L^2(0,T;V)}$, $\|\varphi_i\|_{L^\infty(0,T;H)}$, $\|\varphi_i\|_{L^2(0,T;V)}$ and Ω such that for all $t \in (0,T]$,*

$$\begin{aligned} &\|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 + \|\sigma_1(t) - \sigma_2(t)\|_{V'}^2 \\ &\quad + \|\varphi_1 - \varphi_2\|_{L^2(0,t;H)}^2 + \|\sigma_1 - \sigma_2\|_{L^2(0,t;H)}^2 \\ &\le C\left(\|\varphi_{1,0} - \varphi_{2,0}\|_{V'}^2 + \|\sigma_{1,0} - \sigma_{2,0}\|_{V'}^2\right). \end{aligned} \tag{2.15}$$

Furthermore, if $r \le \frac{2}{3}$ in (B3) then it holds that for all $t \in (0,T]$,

$$\|\mu_1 - \mu_2\|_{L^2(0,t;V')}^2 \le C\left(\|\varphi_{1,0} - \varphi_{2,0}\|_{V'}^2 + \|\sigma_{1,0} - \sigma_{2,0}\|_{V'}^2\right). \tag{2.16}$$

2.2 Degenerate Mobilities and Singular Potentials

We now consider the case where the mobility $m : [-1,1] \to [0,\infty)$ can be degenerate at ± 1, the potential Ψ is singular and defined in $(-1,1)$. The entropy function $M : (-1,1) \to \mathbb{R}$ associated to the mobility m is given by

$$m(s)M''(s) = 1, \quad M(0) = 0, \quad M'(0) = 0.$$

We now make the following assumptions.

(C1) The potential Ψ can be decomposed into $\Psi = \Psi_1 + \Psi_2$ with a regular part $\Psi_2 \in C^2([-1,1])$ and a singular part $\Psi_1 \in C^2(-1,1)$.
(C2) There exists $\varepsilon_0 > 0$ such that Ψ_1'' is non-decreasing in $[1-\varepsilon_0, 1)$ and non-increasing in $(-1, -1+\varepsilon_0]$.
(C3) There exists $c_0 > \chi^2 \geq 0$ such that

$$A\Psi''(s) + Ba(x) \geq c_0 \quad \forall s \in (-1,1) \text{ and for a.e. } x \in \Omega.$$

(C4) $m \in C^0([-1,1])$ with

$$m(s) \geq 0 \quad \forall s \in [-1,1], \quad m(s) = 0 \text{ iff } s = \pm 1, \quad m\Psi'' \in C^0([-1,1]),$$

and there exists $\varepsilon_0 \in (0,1]$ such that m is non-increasing in $[1-\varepsilon_0, 1]$ and non-decreasing in $[-1, -1+\varepsilon_0]$.
(C5) $P \in C^0([-1,1])$, $P \geq 0$, and there exist a positive constant c_7 and $\varepsilon_0 > 0$ such that

$$\sqrt{P(s)} \leq c_7\, m(s) \quad \forall s \in [-1, -1+\varepsilon_0] \cup [1-\varepsilon_0, 1], \quad P\Psi' \in C^0([-1,1]).$$

(C6) $\varphi_0 \in H$ satisfies $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$, $M(\varphi_0) \in L^1$ and $\sigma_0 \in H$.

Remark 1

(1) By (C4), there exists a positive constant C such that $|m(s)\Psi''(s)| \leq C$ for all $s \in [-1,1]$, which in turn implies that $|\Psi''(s)| \leq CM''(s)$ for all $s \in (-1,1)$. Upon integrating from 0 to $s \in (0,1)$, and also from $s \in (-1,0)$ to 0, applying the fundamental theorem of calculus and the conditions $M(0) = M'(0) = 0$ yields

$$|\Psi(s)| \leq |\Psi(0)| + \left|\Psi'(0)\right| |s| + CM(s) \quad \forall s \in (-1,1),$$

and as a consequence of $M(\varphi_0) \in L^1$ we have $\Psi(\varphi_0) \in L^1$.
(2) The assumption (C5) yields the following observations: P is bounded in $[-1,1]$ and thus (A7) is automatically satisfied, and $P(s) = 0$ if and only if $s = \pm 1$.

The degenerate mobility implies that the gradient of the chemical potential μ can no longer be controlled in some L^p space. Thus, we reformulate the definition of the weak solution so that μ does not appear (cf. [18, Theorem 1]).

Definition 2.2 We call a pair (φ, σ) a weak solution to (1.4)–(1.5) on $[0,T]$ if

$$\varphi, \sigma \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'),$$
$$\text{with } \varphi \in L^\infty(Q_T), \quad |\varphi(x,t)| \leq 1 \text{ a.e. in } Q_T,$$

such that for a.e. $t \in (0,T)$ and for all $\zeta \in V$,

$$\begin{aligned} 0 &= \langle \varphi_t, \zeta \rangle_V + (m(\varphi)(A\Psi''(\varphi) + Ba)\nabla\varphi, \nabla\zeta) \\ &\quad + (m(\varphi)(B\varphi\nabla a - B\nabla(J \star \varphi) - \chi\nabla\sigma), \nabla\zeta) \\ &\quad - (P(\varphi)((1+\chi)\sigma + \chi(1-\varphi) - A\Psi'(\varphi) - Ba\varphi + BJ \star \varphi), \zeta), \end{aligned} \tag{2.17a}$$

$$\begin{aligned} 0 &= \langle \sigma_t, \zeta \rangle_V + (n(\varphi)\nabla(\sigma + \chi(1-\varphi)), \nabla\zeta) \\ &\quad + (P(\varphi)((1+\chi)\sigma + \chi(1-\varphi) - A\Psi'(\varphi) - Ba\varphi + BJ \star \varphi), \zeta), \end{aligned} \tag{2.17b}$$

together with $\varphi(0) = \varphi_0$ and $\sigma(0) = \sigma_0$.

Theorem 2.3 (Existence) *Under Assumptions (A2), (A3), (C1)–(C6), there exists a weak solution pair* (φ, σ) *to* (1.4) *in the sense of Definition* 2.2 *such that* $\varphi(0) = \varphi_0$, $\sigma(0) = \sigma_0$ *in* H.

The initial conditions are attained as equalities in H due to the continuous embedding

$$L^2(0,T;V) \cap H^1(0,T;V') \subset C^0([0,T];H).$$

We now state the result regarding the continuous dependence of solutions on initial data. For this we require the following additional assumptions.

(D1) $n = 1, m \in C^{0,1}([-1,1])$, and $\chi = 0$.
(D2) There exist some constants $c_8 > 0$ and $\rho \in [0,1)$ such that

$$\begin{aligned} A\rho\Psi_1''(s) + A\Psi_2''(s) + Ba(x) &\geq 0 \quad \forall s \in (-1,1) \text{ and for a.e. } x \text{ in } \Omega, \\ m(s)\Psi_1''(s) &\geq c_8 \quad \forall s \in [-1,1]. \end{aligned}$$

(D3) The nonlinearity P satisfies $P, P\Psi' \in C^{0,1}([-1,1])$.

We point out that we have to exclude the effects of chemotaxis for the continuous dependence result, as the regularity for σ stated in Theorem 2.3 seems not to be sufficient at handling the differences involving the term $m(\varphi)\chi\nabla\sigma$ in (2.17).

Theorem 2.4 (Continuous Dependence on Initial Data) *Let* $(\varphi_i, \sigma_i)_{i=1,2}$ *denote two solution pairs to* (1.4) *in the sense of Definition* 2.2 *with* J *satisfying (A3), the potential* Ψ*, the mobilities* m*,* n *and nonlinearity* P *satisfying Assumptions (C1)–(C5), (D1)–(D3), corresponding to initial data* $(\varphi_{0,i}, \sigma_{0,i})_{i=1,2}$ *satisfying (C6). Then there exists a positive constant* C*, depending on* A*,* B*,* a^**,* b*,* $\|J\|_{W^{1,1}}$*,* c_8*,* ρ*, the Lipschitz constants of* m *and* $P\Psi'$*,* $\|\sigma_i\|_{L^2(0,T;V)}$*, and* $\|\varphi_i\|_{L^2(0,T;V)}$ *such that for all*

$t \in (0, T]$,

$$\begin{aligned} &\|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 + \|\sigma_1(t) - \sigma_2(t)\|_{V'}^2 \\ &\quad + \|\varphi_1 - \varphi_2\|_{L^2(0,t;H)}^2 + \|\sigma_1 - \sigma_2\|_{L^2(0,t;H)}^2 \\ &\leq C \left(\|\varphi_{1,0} - \varphi_{2,0}\|_{V'}^2 + \|\sigma_{1,0} - \sigma_{2,0}\|_{V'}^2 \right). \end{aligned} \tag{2.18}$$

3 Non-degenerate Mobility and Regular Potential

3.1 Existence

The proof is carried out by means of a Faedo-Galerkin approximation scheme, assuming at first that $\varphi_0 \in D(\mathscr{N})$. The general case $\varphi_0 \in H$ with $\Psi(\varphi_0) \in L^1(\Omega)$ can be handled by means of a density argument and by relying on the fact that Ψ is a quadratic perturbation of a convex function (see [9]). Let $\{w_j\}_{j\in\mathbb{N}}$ denote the set of eigenfunctions of the operator $\mathscr{N}$ introduced in (1.8), which forms an orthonormal basis in H and an orthogonal basis in V. The finite dimensional subspace spanned by the first n eigenfunctions is denoted by W_n, and the projection operator to W_n is denoted by Π_n. For $n \in \mathbb{N}$ fixed, we look for functions of the form

$$\varphi_n(t) = \sum_{k=1}^{n} a_k^n(t) w_k, \quad \mu_n(t) = \sum_{k=1}^{n} b_k^n(t) w_k, \quad \sigma_n(t) = \sum_{k=1}^{n} c_k^n(t) w_k$$

that solve the following approximating problem (with prime denoting derivatives with respect to time)

$$0 = (\varphi_n', \zeta) + (m(\varphi_n)\nabla\mu_n, \nabla\zeta) - (S_n, \zeta), \tag{3.19a}$$

$$0 = (\sigma_n', \zeta) + (n(\varphi_n)\nabla(\sigma_n + \chi(1-\varphi_n)), \nabla\zeta) + (S_n, \zeta), \tag{3.19b}$$

$$\mu_n = \Pi_n \left(A\Psi'(\varphi_n) + Ba\varphi_n - BJ \star \varphi_n - \chi\sigma_n \right), \tag{3.19c}$$

$$S_n = P(\varphi_n)(\sigma_n + \chi(1-\varphi_n) - \mu_n), \tag{3.19d}$$

$$\varphi_n(0) = \Pi_n(\varphi_0), \quad \sigma_n(0) = \Pi_n(\sigma_0), \tag{3.19e}$$

for every $\zeta \in W_n$. Substituting (3.19c) into (3.19a) and (3.19b) leads to a Cauchy problem for a system of ordinary differential equations in the $2n$ unknowns a_k^n and c_k^n. Continuity of Ψ', m, n and P ensures via the Cauchy–Peano theorem that there exists $t_n \in (0, +\infty]$ such that (3.19) has a solution $\mathbf{a}^n = (a_1^n, \ldots, a_n^n)$, $\mathbf{c}^n = (c_1^n, \ldots, c_n^n)$ on $[0, t_n)$ with $\mathbf{a}_n^n, \mathbf{c}_n^n \in C^1([0, t_n); \mathbb{R}^n)$. This in turn yields that $\varphi_n, \sigma_n \in C^1([0, t_n); W_n)$, and defining μ_n via (3.19c) yields that $\mu_n \in C^1([0, t_n); W_n)$. We will now derive a number of a priori estimates, with the symbol C denoting positive constants that may vary line to line, but do not depend on n and T. Positive constants that are independent on n but depend on T will be denoted by C_T.

3.1.1 A Priori Estimates

Substituting $\zeta = \mu_n$ in (3.19a), $\zeta = \sigma_n + \chi(1-\varphi_n)$ in (3.19b), and testing (3.19c) with φ_n', adding the resulting identities together leads to

$$\begin{aligned} &\frac{d}{dt}E_n + \|\sqrt{m(\varphi_n)}\nabla\mu_n\|_H^2 + \|\sqrt{n(\varphi_n)}\nabla(\sigma_n + \chi(1-\varphi_n))\|_H^2 \\ &\quad + \|\sqrt{P(\varphi_n)}(\sigma_n + \chi(1-\varphi_n) - \mu_n)\|_H^2 = 0 \end{aligned} \tag{3.20}$$

where

$$E_n := \int_\Omega A\Psi(\varphi_n) + \frac{B}{2}\left|\sqrt{a}\varphi_n\right|^2 - \frac{B}{2}\varphi_n(J \star \varphi_n) + \frac{1}{2}|\sigma_n|^2 + \chi\sigma_n(1-\varphi_n)\,dx\,.$$

In the above, by the symmetry of J, we have used (suppressing the t-dependence of φ_n)

$$\begin{aligned} &\frac{d}{dt}\frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi_n(x)-\varphi_n(y))^2\,dx\,dy \\ &\quad = \int_\Omega (a(x)\varphi_n(x) - (J\star\varphi_n)(x))\,\varphi_n'(x)\,dx \\ &\quad = \frac{1}{2}\frac{d}{dt}\int_\Omega a(x)\,|\varphi_n(x)|^2 - \varphi_n(x)(J\star\varphi_n)(x)\,dx\,. \end{aligned}$$

Then, by Young's inequality, Young's inequality for convolutions and (A5) we obtain

$$\begin{aligned} E_n &\geq \frac{1}{2}\|\sigma_n\|_H^2 + \left(Ac_2 + a_*\frac{B}{2}\right)\|\varphi_n\|_H^2 - Ac_1\,|\Omega| \\ &\qquad - \chi\|\sigma_n\|_H\left(|\Omega|^{\frac{1}{2}} + \|\varphi_n\|_H\right) - \frac{B}{2}\|\varphi_n\|_H\|J\star\varphi_n\|_H \\ &\geq \eta\|\sigma_n\|_H^2 - \chi\,|\Omega|^{\frac{1}{2}}\,\|\sigma_n\|_H - Ac_1\,|\Omega| \\ &\qquad + \left(Ac_2 + (a_* - a^*)\frac{B}{2} - \frac{\chi^2}{2(1-2\eta)}\right)\|\varphi_n\|_H^2 \\ &\geq \eta_0\|\sigma_n\|_H^2 + \gamma_0\|\varphi_n\|_H^2 - C, \end{aligned} \tag{3.21}$$

where $\eta = \eta_0 \in (0, 1/2)$ is fixed such that the coefficient of $\|\varphi_n\|_H$ is positive (this can be done thanks to (2.11)), and γ_0 is a positive constant depending on η_0.

Furthermore, by (A3) and (A8), the initial energy is bounded:

$$|E_0| \leq A\|\Psi(\varphi_0)\|_{L^1} + (Ba^* + \chi^2)\|\varphi_0\|_H^2 + \|\sigma_0\|_H^2 + \chi^2\,|\Omega|\,.$$

Notice that, since $\varphi_0 \in D(\mathcal{N})$, we have $\varphi_n(0) \to \varphi_0$ in $D(\mathcal{N}) \subset L^\infty(\Omega)$, and hence the sequence of $\|\Psi(\varphi_n(0))\|_{L^1}$ is controlled by $\|\Psi(\varphi_0)\|_{L^1}$. Thus, integrating (3.20) from 0 to t, and using the lower bound (3.21) leads to

$$\begin{aligned} &\|\sigma_n(t)\|_H^2 + \|\varphi_n(t)\|_H^2 + \|\sqrt{m(\varphi_n)}\nabla\mu_n\|_{L^2(0,t;H)}^2 \\ &\quad + \|\sqrt{n(\varphi_n)}\nabla(\sigma_n + \chi(1-\varphi_n))\|_{L^2(0,t;H)}^2 \\ &\quad + \|\sqrt{P(\varphi_n)}(\sigma_n + \chi(1-\varphi_n) - \mu_n)\|_{L^2(0,t;H)}^2 \\ &\leq C\left(1 + \|\varphi_0\|_H^2 + \|\Psi(\varphi_0)\|_{L^1} + \|\sigma_0\|_H^2\right). \end{aligned} \tag{3.22}$$

This estimate yields that $t_n = +\infty$ for every $n \in N$ and thus we can extend the Galerkin functions φ_n, μ_n, σ_n to the interval $[0, +\infty)$. From the definition of E_n it holds that

$$\begin{aligned} \int_\Omega A\Psi(\varphi_n)\,dx &\leq E_n + \frac{1}{2}\|\sigma_n\|_H^2 + Ba^*\|\varphi_n\|_H^2 + \chi\|\sigma_n\|_H\left(|\Omega|^{\frac{1}{2}} + \|\varphi_n\|_H\right) \\ &\leq |E_0| + \frac{1}{2}\|\sigma_n\|_H^2 + Ba^*\|\varphi_n\|_H^2 + \chi\|\sigma_n\|_H\left(|\Omega|^{\frac{1}{2}} + \|\varphi_n\|_H\right). \end{aligned}$$

Thus, using the boundedness of φ_n and σ_n in $L^\infty(0,T;H)$ for $0 < T < \infty$, we obtain that

$$\|\Psi(\varphi_n)\|_{L^\infty(0,T;L^1)} \leq C \quad \forall 0 < T < \infty. \tag{3.23}$$

Furthermore, from (A6) we see that

$$\|\Psi'(\varphi_n)\|_{L^\infty(0,T;L^z)}^z \leq c_3\|\Psi(\varphi_n)\|_{L^\infty(0,T;L^1)} + c_4\,|\Omega| \leq C \quad \forall 0 < T < \infty. \tag{3.24}$$

Using Fubini's theorem and the symmetry of J, we have the relation

$$(J \star \varphi_n, 1) = \int_\Omega\int_\Omega J(y-x)\varphi_n(y)\,dx\,dy = (a\varphi_n, 1),$$

and so, upon integrating (3.19c) over Ω and applying (A6), (3.22) and (3.23), we have

$$\begin{aligned} \left|\int_\Omega \mu_n\,dx\right| &= \left|\int_\Omega A\Psi'(\varphi_n) - \chi\sigma_n\,dx\right| \leq \int_\Omega A\left|\Psi'(\varphi_n)\right| + \chi\,|\sigma_n|\,dx \\ &\leq Ac_3\|\Psi(\varphi_n)\|_{L^1} + Ac_4\,|\Omega| + C\|\sigma_n\|_H \leq C. \end{aligned}$$

The spatial mean of μ_n is bounded uniformly in $L^\infty(0,T)$. Hence, using the uniform boundedness of $\nabla\mu_n$ in $L^2(0,T;H)$ and the Poincaré inequality, we infer that

$$\|\mu_n\|^2_{L^2(0,T;L^2)} \le C\|\nabla\mu_n\|^2_{L^2(0,T;L^2)} + CT$$

and so

$$\|\mu_n\|_{L^2(0,T;V)} \le C_T \quad \forall 0 < T < \infty. \tag{3.25}$$

Multiplying (3.19c) with $-\Delta\varphi_n$, integrating over Ω and applying integration by parts gives

$$\begin{aligned}
&(\nabla\mu_n, \nabla\varphi_n) \\
&\quad = (\nabla\varphi_n, Ba\nabla\varphi_n + B\varphi_n\nabla a + A\Psi''(\varphi_n)\nabla\varphi_n - B\nabla J\star\varphi_n - \chi\nabla\sigma_n) \\
&\quad = (\nabla\varphi_n, (A\Psi''(\varphi_n) + Ba - \chi^2)\nabla\varphi_n + B\varphi_n\nabla a - B\nabla J\star\varphi_n - \chi\nabla(\sigma_n - \chi\varphi_n)) \\
&\quad \ge (c_0 - \chi^2)\|\nabla\varphi_n\|^2_H - \|\nabla\varphi_n\|_H\|B\varphi_n\nabla a - B\nabla J\star\varphi_n - \chi\nabla(\sigma_n - \chi\varphi_n)\|_H,
\end{aligned}$$

where we have used (A4), and in particular, the fact that $c_0 > \chi^2$. By Young's inequality for convolutions, we have that

$$\|\nabla J\star\varphi_n\|_H \le b\|\varphi_n\|_H, \quad \|\varphi_n\nabla a\|_H = \left(\int_\Omega |\varphi_n|^2\,|\nabla(J\star 1)|^2\,dx\right)^{\frac{1}{2}} \le b\|\varphi_n\|_H,$$

and so we obtain for some positive constant C depending on B, χ and b,

$$\begin{aligned}
\|\nabla\mu_n\|_H\|\nabla\varphi_n\|_H &\ge (\nabla\mu_n, \nabla\varphi_n) \\
&\ge (c_0 - \chi^2)\|\nabla\varphi_n\|^2_H - C\|\nabla\varphi_n\|_H\left(\|\varphi_n\|_H + \|\nabla(\sigma_n - \chi\varphi_n)\|_H\right),
\end{aligned}$$

which in turn leads to

$$\|\nabla\varphi_n\|_H \le C\left(\|\nabla\mu_n\|_H + \|\varphi_n\|_H + \|\nabla(\sigma_n + \chi(1-\varphi_n))\|_H\right),$$

and by (3.22) we obtain

$$\|\sigma_n\|_{L^2(0,T;V)} + \|\varphi_n\|_{L^2(0,T;V)} \le C \quad \forall 0 < T < \infty. \tag{3.26}$$

Next, multiplying (3.19c) with $\Pi_n(\Psi'(\varphi_n))$ and integrating over Ω leads to

$$\begin{aligned}
A\|\Pi_n(\Psi'(\varphi_n))\|^2_H &= (\mu_n + \chi\sigma_n - Ba\varphi_n + BJ\star\varphi_n, \Pi_n(\Psi'(\varphi_n))) \\
&\le \left(\|\mu_n + \chi\sigma_n\|_H + 2a^*B\|\varphi_n\|_H\right)\|\Pi_n(\Psi'(\varphi_n))\|_H,
\end{aligned}$$

and by (3.25), (3.26) we see that

$$\|\Pi_n(\Psi'(\varphi_n))\|_{L^2(0,T;H)} \le C_T, \quad \forall 0 < T < \infty. \tag{3.27}$$

Similarly, multiplying (3.19c) with $-\Delta(\Pi_n(\Psi'(\varphi_n))) \in W_n$, integrating over Ω and applying integration by parts leads to

$$\begin{aligned} A\|\nabla \Pi_n(\Psi'(\varphi_n))\|_H^2 &= (\nabla \mu_n + \chi \nabla \sigma_n, \nabla \Pi_n(\Psi'(\varphi_n))) \\ &\quad - B(\varphi_n \nabla a + a \nabla \varphi_n - (\nabla J \star \varphi_n), \nabla \Pi_n(\Psi'(\varphi_n))). \end{aligned}$$

Using the assumption $a \in W^{1,\infty}$ from (A3), applying Young's inequality for convolution and the boundedness of $\{\nabla \mu_n\}_{n\in\mathbb{N}}$, $\{\nabla \sigma_n\}_{n\in\mathbb{N}}$, $\{\nabla \varphi_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(0,T;H)$ leads to

$$\|\nabla \Pi_n(\Psi'(\varphi_n))\|_{L^2(0,T;H)} \le C, \quad \|\Pi_n(\Psi'(\varphi_n))\|_{L^2(0,T;V)} \le C_T \quad \forall 0 < T < \infty. \tag{3.28}$$

We now deduce the estimates for the sequences of time derivatives $\{\varphi_n'\}_{n\in\mathbb{N}}$ and $\{\sigma_n'\}_{n\in\mathbb{N}}$. From the boundedness of $\{\nabla \mu_n\}_{n\in\mathbb{N}}$ and $\{\nabla(\sigma_n + \chi(1-\varphi_n))\}_{n\in\mathbb{N}}$ in $L^2(0,T;H)$, the estimates for the time derivatives come from the estimates for the source term $S_n = P(\varphi_n)(\sigma_n + \chi(1-\varphi_n) - \mu_n)$. Let

$$Q_n := \sqrt{P(\varphi_n)}(\sigma_n + \chi(1-\varphi_n) - \mu_n).$$

Then, from (3.22), we have boundedness of $\{Q_n\}_{n\in\mathbb{N}}$ in $L^2(0,T;H)$ for all $0 < T < \infty$. Now, take a test function $\zeta \in D(\mathcal{N})$ and write it as $\zeta = \zeta_1 + \zeta_2$, where $\zeta_1 \in W_n$ and $\zeta_2 \in W_n^\perp$. We recall that ζ_1, ζ_2 are orthogonal in H, V, and $D(\mathcal{N})$. Then, from (3.19a) we have

$$\langle \varphi_n', \zeta \rangle_{D(\mathcal{N})} = \langle \varphi_n', \zeta_1 \rangle_{D(\mathcal{N})} = -(m(\varphi_n)\nabla \mu_n, \nabla \zeta_1) + (S_n, \zeta_1),$$

and a similar identity follows from (3.19b). Observe now that we have

$$|(S_n, \zeta_1)| \le \|\sqrt{P(\varphi_n)}\|_H \|Q_n\|_H \|\zeta_1\|_{L^\infty} \le C\left(1 + \|\varphi_n\|_{L^q}^{q/2}\right) \|Q_n\|_H \|\zeta\|_{D(\mathcal{N})},$$

where (A7) has been used. From this last estimate, on account also of the bound of Q_n in $L^2(0,T;H)$ and of (3.20), there follows that we need to control the sequence of φ_n in $L^{\gamma q}(0,T;L^q)$, with some $\gamma > 1$, in order to get the control of the sequences of φ_n', σ_n' in $L^r(0,T;D(\mathcal{N}^{-1}))$, with some $r > 1$. On the other hand, we know that φ_n is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$, and thanks to Gagliardo–Nirenberg inequality (1.7), we have

$$L^\infty(0,T;H) \cap L^2(0,T;V) \subset L^{\frac{4q}{3(q-2)}}(0,T;L^q) \quad \text{for } q > 2. \tag{3.29}$$

Therefore, we can see that, thanks to the growth condition $q < \frac{10}{3}$ in assumption (A7), there exists $\gamma > 1$ such that $\frac{4q}{3(q-2)} \geq \gamma q$. This provides the bound for φ_n in $L^{\gamma q}(0,T;L^q)$, with some $\gamma > 1$, and hence the desired bound for the sequences of time derivatives φ_n', σ_n', namely

$$\|\varphi_n'\|_{L^r(0,T;D(\mathscr{N}^{-1}))} + \|\sigma_n'\|_{L^r(0,T;D(\mathscr{N}^{-1}))} \leq C \quad \text{for some } r > 1. \tag{3.30}$$

3.1.2 Passing to the Limit

From the a priori estimates (3.22), (3.25), (3.26), (3.30) and using compactness results (for example [46, §8, Corollary 4]), we obtain for a non-relabelled subsequence and any $s < 6$,

$$\varphi_n \to \varphi \text{ weakly* } \text{ in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap W^{1,r}(0,T;D(\mathscr{N}^{-1})), \tag{3.31a}$$

$$\sigma_n \to \sigma \text{ weakly* } \text{ in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap W^{1,r}(0,T;D(\mathscr{N}^{-1})), \tag{3.31b}$$

$$\mu_n \to \mu \text{ weakly } \quad \text{in } L^2(0,T;V), \tag{3.31c}$$

as well as

$$\varphi_n \to \varphi \text{ strongly } \text{ in } L^2(0,T;L^s) \cap C^0([0,T];V') \text{ and a.e. in } Q_T, \tag{3.32a}$$

$$\sigma_n \to \sigma \text{ strongly } \text{ in } L^2(0,T;L^s) \cap C^0([0,T];V') \text{ and a.e. in } Q_T. \tag{3.32b}$$

To show that the limit functions (φ, μ, σ) satisfy Definition 2.1, we can now proceed by means of a standard argument, which involves multiplying (3.19a) and (3.19b) by $\delta \in C_c^\infty(0,T)$, taking $\zeta \in W_k$, with fixed $k \leq n$, and then passing to the limit as $n \to \infty$, taking the weak/strong convergences above, as well as the density of $\bigcup_{k=1}^\infty W_k$ in $D(\mathscr{N})$ into account. We omit the easy details, and we just sketch the less obvious points.

First, assumption (A1), the a.e. convergence (3.32a), the application of Lebesgue dominated convergence theorem, the weak convergence (3.31c), and estimate (3.20) imply that

$$m(\varphi_n)\nabla\mu_n \to m(\varphi)\nabla\mu \text{ weakly in } L^2(0,T;H).$$

The term involving $n(\cdot)$ can be handled in a similar fashion. Meanwhile, we obtain from (3.28), that

$$\Pi_n(\Psi'(\varphi_n)) \to \xi \text{ weakly in } L^2(0,T;V),$$

for some $\xi \in L^2(0,T;V)$. To identify ξ with $\Psi'(\varphi)$, we first note that by the continuity of Ψ' and the a.e. convergence of φ_n to φ in Q_T, it holds that $\Psi'(\varphi_n)$ converges a.e. to $\Psi'(\varphi)$ in Q_T. Then, thanks to (3.24) we have that

$$\Psi'(\varphi_n) \to \Psi'(\varphi) \text{ weakly* in } L^\infty(0,T;L^z) \text{ for } z \in (1,2],$$

where we used the fact that the weak limit and the pointwise limit must coincide. Using $\zeta \in W_k$ and hence $\zeta = \Pi_n(\zeta)$, for all $n \geq k$, we obtain

$$\begin{aligned}\int_0^T (\Psi'(\varphi), \delta\zeta)\, dt &= \lim_{n\to\infty} \int_0^T (\Psi'(\varphi_n), \delta\zeta)\, dt = \lim_{n\to\infty} \int_0^T (\Psi'(\varphi_n), \delta\Pi_n(\zeta))\, dt \\ &= \lim_{n\to\infty} \int_0^T (\Pi_n(\Psi'(\varphi_n)), \delta\zeta)\, dt = \int_0^T (\xi, \delta\zeta)\, dt\,.\end{aligned}$$

As far as the source terms are concerned, we first see that

$$\varphi_n \to \varphi \text{ strongly in } L^q(Q_T). \tag{3.33}$$

This immediately follows from (3.31a), (3.32a) and the embedding

$$L^\infty(0,T;H) \cap L^2(0,T;V) \subset L^{\frac{10}{3}}(Q_T),$$

which follows from Gagliardo-Nirenberg inequality (recall also that $q < \frac{10}{3}$). Then, (3.33), assumption (A7) and the generalized Lebesgue dominated convergence theorem entail the strong convergence

$$\sqrt{P(\varphi_n)} \to \sqrt{P(\varphi)} \text{ strongly in } L^2(Q_T). \tag{3.34}$$

Next, we see also that

$$\sqrt{P(\varphi_n)}(\sigma_n + \chi(1-\varphi_n) - \mu_n) \to \sqrt{P(\varphi)}(\sigma + \chi(1-\varphi) - \mu) \quad \text{weakly in } L^2(Q_T). \tag{3.35}$$

Indeed, the weak convergence of $\sigma_n + \chi(1-\varphi_n) - \mu_n$ to $\sigma + \chi(1-\varphi) - \mu$ in $L^2(Q_T)$, together with the strong convergence (3.34) imply that the weak convergence (3.35) holds in $L^1(Q_T)$ and, by (3.20), also in $L^2(Q_T)$. Moreover, from the last two convergences we obtain $P(\varphi_n)(\sigma_n + \chi(1-\varphi_n) - \mu_n) \to P(\varphi)(\sigma + \chi(1-\varphi) - \mu)$ weakly in $L^1(Q_T)$, which is enough to pass to the limit in the source terms. Finally, we can also prove that the initial conditions $\varphi(0) = \varphi_0$ and $\sigma(0) = \sigma_0$ are satisfied. Since the argument is standard, we omit the details.

3.1.3 Energy Inequality

In order to prove (2.13) we can argue as follows. We integrate (3.20) between 0 and t, then multiply the resulting identity by an arbitrary $\omega \in \mathscr{D}(0,t)$, with $\omega \geq 0$. By integrating this second identity again in time between 0 and t, we get

$$\begin{aligned}
&\int_0^t E_n(s)\omega(s)\,ds \\
&\quad + \int_0^t \omega(s)\left(\int_0^s \|\sqrt{m(\varphi_n)}\nabla\mu_n\|_H^2 + \|\sqrt{n(\varphi_n)}\nabla(\sigma_n - \chi\varphi_n)\|_H^2\,d\tau\right)ds \\
&\quad + \int_0^t \omega(s)\int_0^s \|\sqrt{P(\varphi_n)}(\sigma_n + \chi(1-\varphi_n) - \mu_n)\|_H^2\,d\tau\,ds \\
&= E_n(0)\int_0^t \omega(s)\,ds.
\end{aligned} \tag{3.36}$$

We now pass to the limit as $n \to \infty$ in this identity. On the left-hand side we use the weak convergences in $L^2(Q_T)$ of $\sqrt{m(\varphi_n)}\nabla\mu_n$ to $\sqrt{m(\varphi)}\nabla\mu$, and of $\sqrt{n(\varphi_n)}\nabla(\sigma_n + \chi(1-\varphi_n))$ to $\sqrt{n(\varphi)}\nabla(\sigma + \chi(1-\varphi))$, (3.35), the weak/strong convergences above for φ_n, σ_n, the lower semicontinuity of the norm and Fatou's lemma. On the right-hand side we use the fact that, since $\varphi_0 \in D(\mathscr{N})$, then $\varphi_n(0) \to \varphi_0$ in L^∞ and hence we have $E_n(0) = E(\varphi_n(0), \sigma_n(0)) \to E(0) = E(\varphi_0, \sigma_0)$. After passing to the limit, from (3.36) we therefore obtain the corresponding inequality for the solution pair (φ, σ), which holds for every $\omega \in \mathscr{D}(0,t)$, with $\omega \geq 0$, and which then yields (2.13).

3.2 Improved Temporal Regularity and Energy Identity

Suppose (A7) is satisfied with $q \leq \frac{4}{3}$, then we have

$$\begin{aligned}
|(S_n, \zeta)| &\leq \|P(\varphi_n)\|_{L^{\frac{3}{2}}}\|\sigma_n + \chi(1-\varphi_n) - \mu_n\|_{L^6}\|\zeta\|_{L^6} \\
&\leq C\|P(\varphi_n)\|_{L^{\frac{3}{2}}}\|\sigma_n + \chi(1-\varphi_n) - \mu_n\|_{L^6}\|\zeta\|_V.
\end{aligned}$$

Furthermore,

$$\|P(\varphi_n)\|_{L^{\frac{3}{2}}} \leq C\left(1 + \|\varphi_n\|_H^q\right), \tag{3.37}$$

which in turn implies that $\{P(\varphi_n)\}_{n\in\mathbb{N}}$ is bounded uniformly in $L^\infty(0,T;L^{\frac{3}{2}})$ by (3.22). This yields that $\{S_n\}_{n\in\mathbb{N}} = \{P(\varphi_n)(\sigma_n + \chi(1-\varphi_n) - \mu_n)\}_{n\in\mathbb{N}}$ is bounded uniformly in $L^2(0,T;V')$ and consequently

$$\|\varphi_n'\|_{L^2(0,T;V')} + \|\sigma_n'\|_{L^2(0,T;V')} \leq C \quad \forall 0 < T < \infty. \tag{3.38}$$

Passing to the limit $n \to \infty$ involves the same argument as in Sect. 3.1.2, but we now have $\varphi_t, \sigma_t \in L^2(0,T;V')$. Furthermore, as $\mu, \sigma, \Psi'(\varphi) \in L^2(0,T;V)$, we obtain, by a similar argument to [18, Proof of Lemma 2(a)],

$$\langle \varphi_t, \mu\rangle_V = \frac{d}{dt}\int_\Omega A\Psi(\varphi) + \frac{B}{2}a(x)\,|\varphi|^2 - \frac{B}{2}\varphi(J\star\varphi)\,dx - \chi\langle\varphi_t,\sigma\rangle_V,$$
$$\frac{d}{dt}\int_\Omega \frac{1}{2}\,|\sigma|^2 + \chi\sigma(1-\varphi)\,dx = \langle\sigma_t, \sigma + \chi(1-\varphi)\rangle_V + \langle\varphi_t, -\chi\sigma\rangle_V.$$

Then, upon adding with the equalities resulting from substituting $\zeta = \mu$ in (2.12a) and $\zeta = \sigma + \chi(1-\varphi)$ in (2.12b), we obtain an identity analogous to (3.20) for (φ, σ). By integrating in time between 0 and t we deduce the energy identity, namely (2.13) holds as an equality for all $t > 0$.

3.3 *Continuous Dependence with Constant Mobilities*

For two weak solutions $(\varphi_i, \sigma_i)_{i=1,2}$ to (1.4) corresponding to initial data $(\varphi_{0,i}, \sigma_{0,i})_{i=1,2}$ satisfying the hypotheses of Theorem 2.2, we define

$$\varphi := \varphi_1 - \varphi_2, \quad \sigma := \sigma_1 - \sigma_2,$$
$$\mu := \mu_1 - \mu_2 = A\Psi'(\varphi_1) - A\Psi'(\varphi_2) + Ba\varphi - BJ\star\varphi - \chi\sigma,$$

which by Theorem 2.1 satisfy

$$\varphi, \sigma \in L^2(0,T;V)\cap H^1(0,T;V')\cap L^\infty(0,T;H), \quad \mu \in L^2(0,T;V),$$

and

$$\begin{aligned}&\langle\varphi_t,\zeta\rangle_V + (\nabla\mu,\nabla\zeta) + (\mu,\zeta) \\ &\quad = ((P(\varphi_1) - P(\varphi_2))(\sigma_2 + \chi(1-\varphi_2) - \mu_2), \zeta) \\ &\qquad + (P(\varphi_1)(\sigma - \chi\varphi - \mu), \zeta) + (\mu,\zeta),\end{aligned} \tag{3.39a}$$

$$\begin{aligned}&\langle\sigma_t,\phi\rangle_V + (\nabla(\sigma-\chi\varphi),\nabla\phi) + (\sigma-\chi\varphi,\phi) \\ &\quad = -((P(\varphi_1) - P(\varphi_2))(\sigma_2 + \chi(1-\varphi_2) - \mu_2), \zeta) \\ &\qquad - (P(\varphi_1)(\sigma - \chi\varphi - \mu), \zeta) + (\sigma-\chi\varphi,\phi),\end{aligned} \tag{3.39b}$$

for all $\zeta, \phi \in V$. Since $\varphi_t, \sigma_t \in L^2(0,T;V')$, we insert $\zeta = \mathcal{N}^{-1}\varphi$ and $\phi = \mathcal{N}^{-1}\sigma$ and employ the relations (1.9), which upon adding leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right) + (\mu,\varphi) + \|\sigma\|_H^2 - (\chi\varphi,\sigma)\\&\quad = (Z, \mathcal{N}^{-1}\varphi - \mathcal{N}^{-1}\sigma) + (\mu, \mathcal{N}^{-1}\varphi) + (\sigma - \chi\varphi, \mathcal{N}^{-1}\sigma)\\&\quad =: I_1 + I_2 + I_3,\end{aligned} \tag{3.40}$$

where

$$Z := (P(\varphi_1) - P(\varphi_2))(\sigma_2 + \chi(1-\varphi_2) - \mu_2) + P(\varphi_1)(\sigma - \chi\varphi - \mu).$$

Using the definition of $\mu = \mu_1 - \mu_2$, the Mean value theorem applied to Ψ', (A4), Young's inequality for convolutions, and Hölder's inequality, we see that

$$\begin{aligned}(\mu - \chi\sigma, \varphi) &= (A(\Psi'(\varphi_1) - \Psi'(\varphi_2)) + Ba\varphi - BJ\star\varphi - 2\chi\sigma, \varphi)\\&\geq c_0\|\varphi\|_H^2 - B\langle\mathcal{N}(J\star\varphi), \mathcal{N}^{-1}\varphi\rangle_V - 2\chi\|\sigma\|_H\|\varphi\|_H\\&\geq c_0\|\varphi\|_H^2 - B\|\mathcal{N}(J\star\varphi)\|_{V'}\|\varphi\|_{V'} - 2\chi\|\sigma\|_H\|\varphi\|_H\\&\geq c_0\|\varphi\|_H^2 - Bb^*\|\varphi\|_H\|\varphi\|_{V'} - 2\chi\|\sigma\|_H\|\varphi\|_H\\&\geq \eta\|\varphi\|_H^2 - \frac{B^2 b^{*2}}{4\eta}\|\varphi\|_{V'}^2 - \frac{\chi^2}{c_0 - 2\eta}\|\sigma\|_H^2,\end{aligned}$$

where $b^* := a^* + b$ and $\eta \in (0, c_0/2)$ is to be fixed. We now insert this last estimate into (3.40) and, owing to the condition $c_0 > \chi^2$, we can fix $\eta = \eta_0$ small enough such that $\delta_0 := 1 - \chi^2/(c_0 - 2\eta_0) > 0$. Therefore, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right) + \eta_0\|\varphi\|_H^2 + \delta_0\|\sigma\|_H^2\\&\quad \leq I_1 + I_2 + I_3 + \frac{B^2 b^{*2}}{4\eta_0}\|\varphi\|_{V'}^2.\end{aligned} \tag{3.41}$$

The right-hand sides I_1, I_2 and I_3 can be estimated as follows: Using (1.9), it holds that

$$|I_3| \leq \left(\|\sigma\|_{V'} + \chi\|\varphi\|_{V'}\right)\|\mathcal{N}^{-1}\sigma\|_V \leq \|\sigma\|_{V'}^2 + \chi\|\varphi\|_{V'}\|\sigma\|_{V'}. \tag{3.42}$$

The estimates for I_1 and I_2 will require an estimate for $\|\mu\|_{V'}$. Recalling $b^* := a^* + b$, we first note that for every $\zeta \in V$ we have

$$|(a\varphi, \zeta)| = |(\varphi, a\zeta)| \leq \|\varphi\|_{V'}\|a\zeta\|_V \leq b^*\|\varphi\|_{V'}\|\zeta\|_V, \tag{3.43}$$

$$|(J\star\varphi, \zeta)| = |(\varphi, J\star\zeta)| \leq \|\varphi\|_{V'}\|J\star\zeta\|_V \leq b^*\|\varphi\|_{V'}\|\zeta\|_V, \tag{3.44}$$

which yield $\|a\varphi\|_{V'} \le b^*\|\varphi\|_{V'}$ and $\|J \star \varphi\|_{V'} \le b^*\|\varphi\|_{V'}$. From (B3), it holds that

$$\|\Psi'(\varphi_1) - \Psi'(\varphi_2)\|_{L^{\frac{6}{5}}} \le C\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\|\varphi\|_H,$$

and so, with the continuous embedding $L^{\frac{6}{5}} \subset V'$ we have that

$$\begin{aligned}\left|(\Psi'(\varphi_1) - \Psi'(\varphi_2), \mathcal{N}^{-1}f)\right| &\le \|\Psi'(\varphi_1) - \Psi'(\varphi_2)\|_{V'}\|\mathcal{N}^{-1}f\|_V \\ &\le C\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\|\varphi\|_H\|f\|_{V'}.\end{aligned} \tag{3.45}$$

Using (A3), we find that

$$\|\mu\|_{V'} \le AC\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\|\varphi\|_H + 2b^*B\|\varphi\|_{V'} + \chi\|\sigma\|_{V'}. \tag{3.46}$$

Immediately, we have

$$\begin{aligned}|I_2| &\le AC\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\|\varphi\|_H\|\varphi\|_{V'} + 2b^*B\|\varphi\|_{V'}^2 + \chi\|\sigma\|_{V'}\|\varphi\|_{V'} \\ &\le C\left(1 + \|\varphi_1\|_{L^{3r}}^{2r} + \|\varphi_2\|_{L^{3r}}^{2r}\right)\|\varphi\|_{V'}^2 + \frac{\eta_0}{4}\|\varphi\|_H^2 + C\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right).\end{aligned} \tag{3.47}$$

For I_1, let us first note that

$$\|P(\varphi_1)f\|_{V'} \le c\|P\|_{L^\infty(\mathbb{R})}\|f\|_{L^{\frac{6}{5}}}.$$

Then, a short computation shows that

$$\|P(\varphi_1)(\sigma - \chi\varphi - Ba\varphi + BJ \star \varphi + \chi\sigma)\|_{V'} \le C\left(\|\sigma\|_H + \|\varphi\|_H\right).$$

Furthermore, by (B3) and the embedding $L^{\frac{6}{5}} \subset V'$,

$$\begin{aligned}A\|P(\varphi_1)(\Psi'(\varphi_1) - \Psi'(\varphi_2))\|_{V'} &\le C\|P(\varphi_1)(\Psi'(\varphi_1) - \Psi'(\varphi_2))\|_{L^{\frac{6}{5}}} \\ &\le C\|P\|_{L^\infty(R)}\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\|\varphi\|_H.\end{aligned}$$

Combining we now obtain

$$\begin{aligned}&\left|(P(\varphi_1)(\sigma - \chi\varphi - \mu), \mathcal{N}^{-1}\varphi - \mathcal{N}^{-1}\sigma)\right| \\ &\quad \le C\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)\left(\|\varphi\|_H + \|\sigma\|_H\right)\left(\|\varphi\|_{V'} + \|\sigma\|_{V'}\right).\end{aligned}$$

Then, thanks to the Lipschitz continuity of P, for I_1 we have

$$
\begin{aligned}
|I_1| &\le \|P(\varphi_1) - P(\varphi_2)\|_H \|\sigma_2 + \chi(1-\varphi_2) - \mu_2\|_{L^3} \|\mathcal{N}^{-1}\varphi - \mathcal{N}^{-1}\sigma\|_{L^6} \\
&\quad + C\left(1 + \|\varphi_1\|_{L^{3r}}^r + \|\varphi_2\|_{L^{3r}}^r\right)(\|\varphi\|_H + \|\sigma\|_H)(\|\varphi\|_{V'} + \|\sigma\|_{V'}) \\
&\le C\left(1 + \|\sigma_2 + \chi(1-\varphi_2) - \mu_2\|_V^2 + \|\varphi_1\|_{L^{3r}}^{2r} + \|\varphi_2\|_{L^{3r}}^{2r}\right)\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right) \\
&\quad + \frac{\eta_0}{4}\|\varphi\|_H^2 + \frac{\delta_0}{2}\|\sigma\|_H^2.
\end{aligned} \tag{3.48}
$$

By Young's inequality, upon substituting (3.42), (3.47), (3.48) into (3.41) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right) + \eta_0\|\varphi\|_H^2 + \delta_0\|\sigma\|_H^2. \\
&\quad \le C\left(1 + \|\varphi_1\|_{L^{3r}}^{2r} + \|\varphi_2\|_{L^{3r}}^{2r} + \|\sigma_2 + \chi(1-\varphi_2) - \mu_2\|_V^2\right)\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right) \\
&\quad =: \mathcal{X}\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right).
\end{aligned}
$$

Now, the prefactor $\mathcal{X}$ for $\left(\|\varphi\|_{V'}^2 + \|\sigma\|_{V'}^2\right)$ on the right-hand side belongs to $L^1(0,T)$, provided $r \le 4/3$. Indeed, employing (3.29) (take $q = 3r$) we have $\varphi_1, \varphi_2 \in L^{\frac{4r}{3r-2}}(0,T;L^{3r})$ and $\frac{4r}{3r-2} \ge 2r$ for $r \le 4/3$. The continuous dependence estimate (2.18) then follows from Gronwall's lemma. If $r \le \frac{2}{3}$, then from (3.46) we have

$$
\begin{aligned}
&\int_0^t \|\mu\|_{V'}^2 \, ds \\
&\quad \le C\left[\left(1 + \sum_{i=1,2}\|\varphi_i\|_{L^\infty(0,T;H)}^{2r}\right)\|\varphi\|_{L^2(0,t;H)}^2 + \|\varphi\|_{L^2(0,t;V')}^2 + \|\sigma\|_{L^2(0,t;V')}^2\right] \\
&\quad \le C\left(\|\varphi(0)\|_{V'}^2 + \|\sigma(0)\|_{V'}^2\right).
\end{aligned}
$$

4 Degenerate Mobility and Singular Potential

4.1 Existence

For $\varepsilon > 0$, we consider the approximate problem (P_ε) given by

$$
\begin{aligned}
\varphi_{\varepsilon,t} &= \operatorname{div}\left(m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon\right) + P_\varepsilon(\varphi_\varepsilon)(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon) && \text{in } Q_T, \\
\mu_\varepsilon &= A\Psi_\varepsilon'(\varphi_\varepsilon) + Ba\varphi_\varepsilon - BJ \star \varphi_\varepsilon - \chi\sigma_\varepsilon && \text{in } Q_T, \\
\sigma_{\varepsilon,t} &= \operatorname{div}\left(n(\varphi_\varepsilon)\nabla(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon))\right) - P_\varepsilon(\varphi_\varepsilon)(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon) && \text{in } Q_T,
\end{aligned} \tag{P_ε}
$$

with Neumann boundary conditions on $\partial\Omega \times (0,T)$ and initial conditions $\varphi_\varepsilon(0) = \varphi_0$, $\sigma_\varepsilon(0) = \sigma_0$, which is obtained by replacing the singular potential Ψ with a regular potential $\Psi_\varepsilon = \Psi_{1,\varepsilon} + \Psi_{2,\varepsilon}$ and the degenerate mobility m by a non-degenerate mobility m_ε given by

$$m_\varepsilon(s) = \begin{cases} m(1-\varepsilon) & \text{for } s \geq 1-\varepsilon, \\ m(s) & \text{for } |s| \leq 1-\varepsilon, \\ m(-1+\varepsilon) & \text{for } s \leq -1+\varepsilon, \end{cases} \tag{4.49a}$$

$$\Psi_{1,\varepsilon}(s) = \begin{cases} \Psi_1(1-\varepsilon) + \Psi_1'(1-\varepsilon)(s-(1-\varepsilon)) & \\ +\frac{1}{2}\Psi_1''(1-\varepsilon)(s-(1-\varepsilon))^2 & \\ +\frac{1}{6}(s-(1-\varepsilon))^3 & \text{for } s \geq 1-\varepsilon, \\ \Psi_1(s) & \text{for } |s| \leq 1-\varepsilon, \\ \Psi_1(-1+\varepsilon) + \Psi_1'(-1+\varepsilon)(s-(\varepsilon-1)) & \\ +\frac{1}{2}\Psi_1''(-1+\varepsilon)(s-(\varepsilon-1))^2 & \\ +\frac{1}{6}\left|s-(\varepsilon-1)\right|^3 & \text{for } s \leq -1+\varepsilon, \end{cases} \tag{4.49b}$$

$$\Psi_{2,\varepsilon}(s) = \begin{cases} \Psi_2(1-\varepsilon) + \Psi_2'(1-\varepsilon)(s-(1-\varepsilon)) & \\ +\frac{1}{2}\Psi_2''(1-\varepsilon)(s-(1-\varepsilon))^2 & \text{for } s \geq 1-\varepsilon, \\ \Psi_2(s) & \text{for } |s| \leq 1-\varepsilon, \\ \Psi_2(-1+\varepsilon) + \Psi_2'(-1+\varepsilon)(s-(\varepsilon-1)) & \\ +\frac{1}{2}\Psi_2''(-1+\varepsilon)(s-(\varepsilon-1))^2 & \text{for } s \leq -1+\varepsilon. \end{cases} \tag{4.49c}$$

Note that $\Psi_{1,\varepsilon}$ is a slightly different variant to the approximation employed in [18]. By (C4) and (4.49a), it holds that m_ε satisfies (A1) for positive ε. We introduce the approximate entropy function $M_\varepsilon \in C^2(\mathbb{R})$ by

$$m_\varepsilon(s)M_\varepsilon''(s) = 1, \quad M_\varepsilon(0) = M_\varepsilon'(0) = 0, \tag{4.50}$$

and the approximate nonlinearity $P_\varepsilon \in C^0(\mathbb{R})$ by

$$P_\varepsilon(s) = \begin{cases} P(1-\varepsilon) & \text{for } s \geq 1-\varepsilon, \\ P(s) & \text{for } |s| \leq 1-\varepsilon, \\ P(-1+\varepsilon) & \text{for } s \leq -1+\varepsilon. \end{cases} \tag{4.51}$$

4.1.1 Properties of the Approximate Functions

In the following, we will derive some properties for the approximating functions Ψ_ε, M_ε, and P_ε, and also some a priori estimates for $\{\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon\}$ that are uniform in ε. For the rest of this section, the symbol C denotes positive constants that may vary line by line but are independent of ε.

(1) **The approximate potential.** We now show that the approximation $\Psi_\varepsilon = \Psi_{1,\varepsilon} + \Psi_{2,\varepsilon}$ satisfies (A4), (A5), (A6). From (C3), (4.49b) and (4.49c) we observe that

$$A\Psi_\varepsilon''(s) + Ba(x) = \begin{cases} A\Psi''(s) + Ba(x) & \text{for } |s| \leq 1-\varepsilon, \\ A\Psi''(1-\varepsilon) + Ba(x) + (s-1+\varepsilon) & \text{for } s > 1-\varepsilon, \\ A\Psi''(-1+\varepsilon) + Ba(x) + |s-\varepsilon+1| & \text{for } s < -1+\varepsilon \end{cases}$$
$$\geq c_0 \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{4.52}$$

which implies that Ψ_ε satisfies (A4) for all $\varepsilon > 0$. Furthermore, (C3) immediately gives a lower bound for Ψ'':

$$\Psi''(s) \geq \frac{1}{A}\left(c_0 - B\|a\|_{L^\infty(\Omega)}\right) =: k \quad \forall s \in (-1,1).$$

Then, we deduce from (4.49b) and (4.49c), and applying Young's inequality, that there exist two constants $k_1 > 0$, $k_2 \in \mathbb{R}$, independent of ε, such that

$$\Psi_\varepsilon(s) \geq k_1 |s|^3 - k_2 \quad \forall s \in \mathbb{R}.$$

By Young's inequality with Hölder exponents, we observe that

$$\Psi_\varepsilon(s) \geq k_1 |s|^3 - k_2 \geq c_2 |s|^2 - C(c_2, k_1, k_2) \quad \forall s \in \mathbb{R},$$

where we can take the constant c_2 such that (2.11) is satisfied. Therefore, (A5) is also satisfied for all $\varepsilon > 0$. Meanwhile, by the definitions (4.49b), (4.49c), Ψ_ε has cubic growth for fixed $\varepsilon > 0$ and thus (A6) is satisfied with $z = \frac{3}{2}$.

(2) **Uniform bounds on the initial energy.** We now establish that $\Psi_\varepsilon(\varphi_0)$ is bounded in $L^1(\Omega)$ independent of ε, see also [23, Proof of Theorem 2] and [20, Proof of Lemma 4]. By Taylor's theorem, for $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is the constant in (C2), we have, for $1-\varepsilon \leq s < 1$

$$\Psi_1(s) = \Psi_1(1-\varepsilon) + \Psi_1'(1-\varepsilon)(s-(1-\varepsilon)) + \frac{1}{2}\Psi_1''(\xi_s)(s-(1-\varepsilon))^2,$$

where $\xi \in (1-\varepsilon, s)$. Then, condition (C2) implies that $\Psi''(\xi_s) \geq \Psi''(1-\varepsilon)$ and so $\Psi_{1,\varepsilon}(s) - (s-(1-\varepsilon))^3/6 \leq \Psi_1(s)$. We argue in a similar fashion for $-1 < s \leq -1+\varepsilon$. Since $\Psi_1(s) = \Psi_{1,\varepsilon}(s)$ for $|s| \leq 1-\varepsilon$, we get the bound

$$\Psi_{1,\varepsilon}(s) \leq \Psi_1(s) + \frac{\varepsilon^3}{6} \quad \forall s \in (-1,1), \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.53}$$

On the other hand, using $\Psi_2 \in C^2([-1,1])$ and a similar argument involving Taylor's theorem, there exist constants $L_1, L_2 > 0$ such that

$$|\Psi_{2,\varepsilon}(s)| \leq L_1 |s|^2 + L_2 \quad \forall s \in \mathbb{R}, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.54}$$

Then, by (C6), (4.53) and (4.54) it holds that

$$\int_\Omega \Psi_\varepsilon(\varphi_0)\, dx \leq \int_\Omega \Psi_1(\varphi_0)\, dx + L_1 \|\varphi_0\|_H^2 + C < \infty \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.55}$$

(3) **The approximate entropy function.** From the definitions (4.49a) and (4.50), we obtain

$$M_\varepsilon(s) = \begin{cases} M(1-\varepsilon) + M'(1-\varepsilon)(s-(1-\varepsilon)) & \\ +\frac{1}{2}M''(1-\varepsilon)(s-(1-\varepsilon))^2 & \text{for } s \geq 1-\varepsilon, \\ M(s) & \text{for } |s| \leq 1-\varepsilon, \\ M(\varepsilon-1) + M'(\varepsilon-1)(s-(\varepsilon-1)) & \\ +\frac{1}{2}M''(\varepsilon-1)(s-(\varepsilon-1))^2 & \text{for } s \leq -1+\varepsilon. \end{cases}$$

Assumption (C4) yields that m is non-increasing in $[1-\varepsilon_0, 1]$ and non-decreasing in $[-1, -1+\varepsilon_0]$. This implies that $M'' = \frac{1}{m}$ is non-decreasing in $[1-\varepsilon_0, 1)$ and non-increasing in $(-1, -1+\varepsilon_0]$. We refer the reader to [5, §3.4], [18, Proof of Lemma 2 c)] and [23, Proof of Theorem 2] for the proof of the following bounds:

$$M_\varepsilon(s) \leq M(s), \quad \left|M'_\varepsilon(s)\right| \leq \left|M'(s)\right| \quad \forall s \in (-1,1) \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{4.56}$$

$$\int_\Omega (|\varphi_\varepsilon| - 1)_+^2\, dx \leq 2 \max(m(-1+\varepsilon), m(1-\varepsilon)) \|M(\varphi_\varepsilon)\|_{L^1}. \tag{4.57}$$

By (4.56), for any initial data φ_0 satisfying (C6), we have

$$\int_\Omega M_\varepsilon(\varphi_0)\, dx \leq \int_\Omega M(\varphi_0)\, dx < \infty. \tag{4.58}$$

(4) **The approximate nonlinearity.** From (4.51) and the expression for M_ε above, we obtain

$$(\sqrt{P_\varepsilon} M'_\varepsilon)(s) = \begin{cases} (\sqrt{P}M')(1-\varepsilon) + \frac{\sqrt{P(1-\varepsilon)}}{m(1-\varepsilon)}(s-(1-\varepsilon)) & \text{for } s \geq 1-\varepsilon, \\ (\sqrt{P}M')(s) & \text{for } |s| \leq 1-\varepsilon, \\ (\sqrt{P}M')(-1+\varepsilon) + \frac{\sqrt{P(-1+\varepsilon)}}{m(-1+\varepsilon)}(s-(\varepsilon-1)) & \text{for } s \leq -1+\varepsilon. \end{cases} \tag{4.59}$$

We now use (C4) and (C5) to estimate the function $\sqrt{P(s)}M'(s)$. For any $s \in [1-\varepsilon_0, 1)$, it holds that

$$\begin{aligned} \left|\sqrt{P(s)}M'(s)\right| &= \left|\sqrt{P(s)}\left(\int_0^{1-\varepsilon_0} \frac{1}{m(r)}\,dr + \int_{1-\varepsilon_0}^{s} \frac{1}{m(r)}\,dr\right)\right| \\ &\leq C + \frac{\sqrt{P(s)}}{m(s)}\,|s-1+\varepsilon_0| \\ &\leq c_7\,|s| + C. \end{aligned}$$

A similar estimate holds for any $s \in (-1, -1+\varepsilon_0]$, and for $|s| \leq 1-\varepsilon_0$ we have

$$\left|\sqrt{P(s)}M'(s)\right| \leq \sqrt{P(s)}\max\left(\int_0^{1-\varepsilon_0} \frac{1}{m(r)}\,dr\,, \int_{-1+\varepsilon_0}^{0} \frac{1}{m(r)}\,dr\right) \leq C,$$

thanks to the fact that $P \in C^0([-1,1])$ and $m(s) > 0$ for all $|s| \leq 1-\varepsilon_0$. Hence, by the explicit form in (4.59) and (C5) there exists a positive constant C such that

$$\left|\sqrt{P_\varepsilon(s)}M'_\varepsilon(s)\right| \leq c_7\,|s| + C \quad \forall s \in \mathbb{R}, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.60}$$

4.1.2 Uniform Estimates

By Theorem 2.1, for fixed $\varepsilon \in (0, \varepsilon_0]$, there exists a pair $(\varphi_\varepsilon, \sigma_\varepsilon)$ such that

$$\varphi_\varepsilon, \sigma_\varepsilon \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'),$$

which satisfies (2.12) with m_ε and Ψ'_ε, and

$$\mu_\varepsilon = A\Psi'_\varepsilon(\varphi_\varepsilon) + Ba\,\varphi_\varepsilon - BJ \star \varphi_\varepsilon - \chi\sigma_\varepsilon. \tag{4.61}$$

Furthermore, the pair $(\varphi_\varepsilon, \sigma_\varepsilon)$ satisfies the energy inequality (2.13) and we then deduce that there exists a positive constant C, independent of ε, such that, for all

$t \in [0, T]$,

$$\begin{aligned} &\|\sigma_\varepsilon(t)\|_H^2 + \|\varphi_\varepsilon(t)\|_H^2 + \|\sqrt{m_\varepsilon(\varphi_\varepsilon)}\nabla\mu_\varepsilon\|_{L^2(0,t;H)}^2 \\ &\quad + \|\sqrt{n(\varphi_\varepsilon)}\nabla(\sigma_\varepsilon - \chi\varphi_\varepsilon)\|_{L^2(0,t;H)}^2 \\ &\quad + \|\sqrt{P_\varepsilon(\varphi_\varepsilon)}(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\|_{L^2(0,t;H)}^2 \\ &\le E_\varepsilon(\varphi_0, \sigma_0) \le C\left(1 + \|\varphi_0\|_H^2 + \|\Psi(\varphi_0)\|_{L^1} + \|\sigma_0\|_H^2\right), \end{aligned} \tag{4.62}$$

where E_ε is given by (2.14) with Ψ replaced with Ψ_ε, and (4.55) has been taken into account. This immediately yields the following uniform estimates with respect to ε:

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;H)} + \|\sigma_\varepsilon\|_{L^\infty(0,T;H)} \le C, \tag{4.63a}$$

$$\|\sqrt{m_\varepsilon(\varphi_\varepsilon)}\nabla\mu_\varepsilon\|_{L^2(0,T;H)} \le C, \tag{4.63b}$$

$$\|\nabla(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon))\|_{L^2(0,T;H)} \le C, \tag{4.63c}$$

$$\|\sqrt{P_\varepsilon(\varphi_\varepsilon)}(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\|_{L^2(0,T;H)} \le C. \tag{4.63d}$$

Recalling the approximate entropy function M_ε from (4.50), since $M_\varepsilon' \in C^1(\mathbb{R})$ and M_ε'' is bounded on $\mathbb{R}$, then we immediately see that $\varphi_\varepsilon \in L^2(0, T; V)$ implies $M_\varepsilon'(\varphi_\varepsilon) \in L^2(0, T; V)$. Since $\varphi_{\varepsilon,t} \in L^2(0, T; V')$ we find from testing the equation for φ_ε with $M_\varepsilon'(\varphi_\varepsilon)$ the following identity:

$$\begin{aligned} &\frac{d}{dt}\int_\Omega M_\varepsilon(\varphi_\varepsilon)\,dx + \int_\Omega m_\varepsilon(\varphi_\varepsilon)M_\varepsilon''(\varphi_\varepsilon)\nabla\mu_\varepsilon\cdot\nabla\varphi_\varepsilon\,dx \\ &\quad = \int_\Omega P_\varepsilon(\varphi_\varepsilon)(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)M_\varepsilon'(\varphi_\varepsilon)\,dx\,. \end{aligned}$$

Using $m_\varepsilon M_\varepsilon'' = 1$ and applying the relation (4.61) to $\nabla\mu_\varepsilon$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega M_\varepsilon(\varphi_\varepsilon)\,dx + \int_\Omega \left(Ba + A\Psi_\varepsilon''(\varphi_\varepsilon)\right)|\nabla\varphi_\varepsilon|^2\,dx \\ &= \int_\Omega M_\varepsilon'(\varphi_\varepsilon)P_\varepsilon(\varphi_\varepsilon)(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\,dx \\ &\quad + \int_\Omega (B\nabla J\star\varphi_\varepsilon + \chi\nabla(\sigma_\varepsilon - \chi\varphi_\varepsilon) - B\varphi_\varepsilon\nabla a)\cdot\nabla\varphi_\varepsilon + \chi^2|\nabla\varphi_\varepsilon|^2\,dx \\ &=: K_1 + K_2. \end{aligned} \tag{4.64}$$

By Young's inequality for convolutions, Hölder's inequality and Young's inequality we see that

$$\begin{aligned}\left|\int_0^t K_2\,dt\right| &\le 2bB\int_0^t \|\varphi_\varepsilon\|_H\|\nabla\varphi_\varepsilon\|_H + \chi^2\|\nabla\varphi_\varepsilon\|_H^2\,dt \\ &\quad + \int_0^t \chi\|\nabla(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon))\|_H\|\nabla\varphi_\varepsilon\|_H\,dt \\ &\le C + \left(\frac{c_0-\chi^2}{2} + \chi^2\right)\|\nabla\varphi_\varepsilon\|_{L^2(0,t;H)}^2\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}\left|\int_0^t K_1\,dt\right| &\le \|\sqrt{P_\varepsilon(\varphi_\varepsilon)}(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\|_{L^2(0,t;H)} \\ &\quad \times \|M_\varepsilon'(\varphi_\varepsilon)\sqrt{P_\varepsilon(\varphi_\varepsilon)}\|_{L^2(0,t;H)} \\ &\le C\|M_\varepsilon'(\varphi_\varepsilon)\sqrt{P_\varepsilon(\varphi_\varepsilon)}\|_{L^2(0,t;H)}.\end{aligned} \tag{4.66}$$

Thus, upon integrating (4.64) over $[0,t]$ for $t\in(0,T]$, and applying (4.52), (4.65) and (4.66), we obtain

$$\begin{aligned}&\int_\Omega M_\varepsilon(\varphi_\varepsilon(t))\,dx + \frac{c_0-\chi^2}{2}\|\nabla\varphi_\varepsilon\|_{L^2(0,t;H)}^2 \\ &\quad\le \int_\Omega M_\varepsilon(\varphi_0)\,dx + C\|M_\varepsilon'(\varphi_\varepsilon)\sqrt{P_\varepsilon(\varphi_\varepsilon)}\|_{L^2(0,t;H)} + C.\end{aligned} \tag{4.67}$$

The first term on the right-hand side is bounded uniformly in ε by (4.58). Moreover, (4.60) together with (4.63a) entail that also the second term on the right-hand side of (4.67) is bounded uniformly in ε. From (C3) we have $c_0 > \chi^2$, and thus, together with (4.63c), we obtain the following uniform estimate

$$\|M_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1)} + \|\nabla\varphi_\varepsilon\|_{L^2(0,T;H)} + \|\nabla\sigma_\varepsilon\|_{L^2(0,T;H)} \le C. \tag{4.68}$$

For estimates on the time derivative $\varphi_{\varepsilon,t}$, we start with the variational formulation for the equation of φ_ε, and applying Hölder's inequality, definition (4.51), (C5) and the fact that m_ε is bounded above uniformly in ε, leads to

$$\begin{aligned}|\langle\varphi_{\varepsilon,t},\zeta\rangle_V| &\le \|\sqrt{m_\varepsilon(\varphi_\varepsilon)}\|_{L^\infty}\|\sqrt{m_\varepsilon(\varphi_\varepsilon)}\nabla\mu_\varepsilon\|_H\|\nabla\zeta\|_H \\ &\quad + \|\sqrt{P_\varepsilon(\varphi_\varepsilon)}(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\|_H\|\sqrt{P_\varepsilon(\varphi_\varepsilon)}\|_{L^3}\|\zeta\|_{L^6} \\ &\le C\left(\|\sqrt{m_\varepsilon(\varphi_\varepsilon)}\nabla\mu_\varepsilon\|_H + \|\sqrt{P_\varepsilon(\varphi_\varepsilon)}(\sigma_\varepsilon + \chi(1-\varphi_\varepsilon) - \mu_\varepsilon)\|_H\right)\|\zeta\|_V,\end{aligned}$$

for all $\zeta \in V$. Squaring, integrating in time and on account of (4.63a), (4.63b), (4.63d), we obtain

$$\|\varphi_{\varepsilon,t}\|_{L^2(0,T;V')} \leq C. \tag{4.69}$$

A similar argument, using (4.63c) and the boundedness of the mobility $n(\cdot)$, yields

$$\|\sigma_{\varepsilon,t}\|_{L^2(0,T;V')} \leq C. \tag{4.70}$$

4.1.3 Passing to the Limit

From the a priori estimates (4.63a), (4.63b), (4.63c), (4.63d), (4.68), (4.69), (4.70) and using compactness results, we obtain for a non-relabelled subsequence and any $s < 6$,

$$\varphi_\varepsilon \to \varphi \text{ weakly* in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'), \tag{4.71a}$$

$$\varphi_\varepsilon \to \varphi \text{ strongly in } L^2(0,T;L^s) \cap C^0([0,T];V') \text{ and a.e. in } Q_T, \tag{4.71b}$$

$$\sigma_\varepsilon \to \sigma \text{ weakly* in } L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'), \tag{4.71c}$$

$$\sigma_\varepsilon \to \sigma \text{ strongly in } L^2(0,T;L^s) \cap C^0([0,T];V') \text{ and a.e. in } Q_T, \tag{4.71d}$$

By (4.57), (4.68), the generalized Lebesgue dominated convergence theorem, and the fact that $m(\pm 1 \mp \varepsilon) \to 0$ as $\varepsilon \to 0$, it holds that

$$\int_\Omega (-\varphi(t) - 1)_+^2 \, dx = 0, \quad \int_\Omega (\varphi(t) - 1)_+^2 \, dx = 0 \quad \text{for a.e. } t \in (0,T),$$

which yields that $|\varphi(x,t)| \leq 1$ for a.e. $(x,t) \in Q_T$. We now multiply the weak formulation of (P_ε) by $\delta \in C_c^\infty(0,T)$ and integrate over $[0,T]$, leading to

$$\begin{aligned} 0 = &\int_0^T \delta \left(\langle \sigma_{\varepsilon,t}, \zeta \rangle_V + \int_\Omega n(\varphi_\varepsilon) \nabla(\sigma_\varepsilon - \chi\varphi_\varepsilon) \cdot \nabla\zeta \, dx \right) dt \\ &+ \int_0^T \int_\Omega \delta P_\varepsilon(\varphi_\varepsilon)((1+\chi)\sigma_\varepsilon + \chi(1-\varphi_\varepsilon))\zeta \, dx \, dt \\ &+ \int_0^T \int_\Omega \delta P_\varepsilon(\varphi_\varepsilon)(-A\Psi_\varepsilon'(\varphi_\varepsilon) - Ba\varphi_\varepsilon + BJ \star \varphi_\varepsilon)\zeta \, dx \, dt \, , \end{aligned} \tag{4.72}$$

and

$$
\begin{aligned}
0 = &\int_0^T \delta \left(\langle \varphi_{\varepsilon,t}, \zeta \rangle_V + \int_\Omega m_\varepsilon(\varphi_\varepsilon) \left(A\Psi_\varepsilon''(\varphi_\varepsilon) + Ba \right) \nabla \varphi_\varepsilon \cdot \nabla \zeta \, dx \right) dt \\
&+ \int_0^T \int_\Omega \delta m_\varepsilon(\varphi_\varepsilon) \left(B\varphi_\varepsilon \nabla a - B(\nabla J \star \varphi_\varepsilon) - \chi \nabla \sigma_\varepsilon \right) \cdot \nabla \zeta \, dx \, dt \\
&- \int_0^T \int_\Omega \delta P_\varepsilon(\varphi_\varepsilon)((1+\chi)\sigma_\varepsilon + \chi(1-\varphi_\varepsilon))\zeta \, dx \, dt \\
&- \int_0^T \int_\Omega \delta P_\varepsilon(\varphi_\varepsilon)(-A\Psi_\varepsilon'(\varphi_\varepsilon) - Ba\varphi_\varepsilon + BJ \star \varphi_\varepsilon)\zeta \, dx \, dt ,
\end{aligned}
\tag{4.73}
$$

for $\zeta \in V$ and we aim to pass to the limit $\varepsilon \to 0$. As the argument for the terms involving the time derivatives, the gradient terms and terms involving J in (4.72) and (4.73) are standard, we will focus on the non-trivial terms involving $m_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon''(\varphi_\varepsilon)$ and $P_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon'(\varphi_\varepsilon)$.

To pass to the limit in $\int_{Q_T} \delta A m_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon''(\varphi_\varepsilon)\nabla\varphi_\varepsilon \cdot \nabla\zeta \, dx \, dt$, it suffices to show that $\delta m_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon''(\varphi_\varepsilon)\nabla\zeta$ converges strongly to $\delta m(\varphi)\Psi''(\varphi)\nabla\zeta$ in $L^2(0,T;H)$. To achieve this we assume that the test function ζ belongs to the space $D(\mathcal{N})$, which is dense in V (see [29, Lemma 3.1]), and then apply a density argument.

Due to the condition $m\Psi'' \in C^0([-1,1])$ and the a.e. convergence $\varphi_\varepsilon \to \varphi$ in Q_T, we observe that (see, e.g., [18])

$$
m_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon''(\varphi_\varepsilon) \to m(\varphi)\Psi''(\varphi) \text{ a.e. in } Q_T. \tag{4.74}
$$

Moreover,

$$
\begin{aligned}
\left| m_\varepsilon(s)\Psi_\varepsilon''(s) \right| \le \; &\| m\Psi'' \|_{L^\infty([-1,1])} + m(1-\varepsilon)(s-(1-\varepsilon))\chi_{[1-\varepsilon,\infty)}(s) \\
&+ m(-1+\varepsilon) \left| s - (-1+\varepsilon) \right| \chi_{(-\infty,-1+\varepsilon]}(s),
\end{aligned}
\tag{4.75}
$$

where χ_E denotes the characteristic function of a set $E \subset \mathbb{R}$. Then, from (4.71a) and the embedding $L^\infty(0,T;H) \cap L^2(0,T;V) \subset L^r(Q_T)$ for $r = 4$ if $d = 2$ and for $r = \frac{10}{3}$ if $d = 3$, we have boundedness of φ_ε in $L^r(Q_T)$. Using the fact that $m(\pm 1 \mp \varepsilon) \to 0$ as $\varepsilon \to 0$, by the generalized Lebesgue dominated convergence theorem from (4.75) we deduce that $(m_\varepsilon\Psi_\varepsilon'')(\varphi_\varepsilon) \to (m\Psi'')(\varphi)$ strongly in $L^r(Q_T)$. Since $\delta\nabla\zeta \in L^6(Q_T)$, we infer the required strong convergence $\delta m_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon''(\varphi_\varepsilon)\nabla\zeta \to \delta m(\varphi)\Psi''(\varphi)\nabla\zeta$ in $L^2(0,T;H)$.

It remains to pass to the limit in $\int_{Q_T} \delta P_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon'(\varphi_\varepsilon)\zeta \, dx \, dt$, and it suffices to show that $P_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon'(\varphi_\varepsilon)$ converges strongly to $P(\varphi)\Psi'(\varphi)$ in $L^s(Q_T)$ for some $s > 1$. By definition of P_ε and Ψ_ε' from (4.49b), (4.49c) and (4.51), and also recalling (C5), we have (for $y = 1 - \varepsilon$)

$$
\begin{aligned}
\left|P_\varepsilon(s)\Psi_\varepsilon'(s)\right| \le\ & 3\|P\Psi'\|_{L^\infty([-1,1])} \\
& + \left|P(y)\Psi''(y)(s-y) + \frac{1}{2}P(y)(s-y)^2\right| \chi_{[y,\infty)}(s) \\
& + \left|P(-y)\Psi''(-y)(s+y) + \frac{1}{2}P(-y)\,|s+y|^2\right| \chi_{(-\infty,-y]}(s) \\
\le\ & 3\|P\Psi'\|_{L^\infty([-1,1])} + C\left|m(y)\Psi''(y)\right| |m(y)(s-y)|\,\chi_{[y,\infty)}(s) \\
& + C\left|P(y)(s-y)^2\right| \chi_{[y,\infty)}(s) \\
& + C\left|m(-y)\Psi''(-y)\right| |m(-y)(s+y)|\,\chi_{(-\infty,-y]}(s) \\
& + C\left|P(-y)\,|s+y|^2\right| \chi_{(-\infty,-y]}(s) \\
\le\ & 3\|P\Psi'\|_{L^\infty([-1,1])} \\
& + C\max\left(m(\pm 1 \mp \varepsilon), P(\pm 1 \mp \varepsilon)\right)\left(1 + |s|^2\right),
\end{aligned}
\tag{4.76}
$$

where we used that $|m(y)\Psi''(y)| \le \|m\Psi''\|_{L^\infty([-1,1])}$ by (C4).

Moreover, due to the condition $P\Psi' \in C^0([-1,1])$ and the a.e. convergence of φ_ε to φ in Q_T, we have analogously to (4.74)

$$P_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon'(\varphi_\varepsilon) \to P(\varphi)\Psi'(\varphi) \text{ a.e. in } Q_T.$$

Then, arguing as in the treatment of the term $m_\varepsilon\Psi_\varepsilon''$, and using again the fact that $m(\pm 1 \mp \varepsilon) \to 0$ as $\varepsilon \to 0$, and the bound for φ_ε in $L^r(Q_T)$ (with r given as above), from (4.76), by the generalized Lebesgue dominated convergence theorem, we get

$$P_\varepsilon(\varphi_\varepsilon)\Psi_\varepsilon'(\varphi_\varepsilon) \to P(\varphi)\Psi'(\varphi) \text{ strongly in } L^{\frac{r}{2}}(Q_T).$$

Thus, passing to the limit $\varepsilon \to 0$ in (4.72) and (4.73) leads to (2.17).

4.2 Continuous Dependence on Initial Data

We follow the ideas in the proof of [36, Theorem 4.1], see also [23, Proof of Proposition 4] and [25, Proof of Theorem 4]. We define

$$\Gamma(s) := \int_0^s m(r)\,dr\,, \text{ and } \Lambda(x,s) := \int_0^s m(r)A\Psi''(r)\,dr\, + Ba(x)\Gamma(s).$$

Then, in the case $\chi = 0$, we can express the first equation of the weak formulation (2.17) as follows:

$$0 = \langle \varphi_t, \zeta \rangle_V + (\nabla \Lambda(\cdot, \varphi), \nabla \zeta) - (\Gamma(\varphi) B \nabla a, \nabla \zeta) + (Bm(\varphi)(\varphi \nabla a - \nabla J \star \varphi), \nabla \zeta) \\ - ((P(\varphi)(\sigma - A\Psi'(\varphi) - Ba\varphi + BJ \star \varphi), \zeta).$$

For any two weak solution pairs $(\varphi_i, \sigma_i)_{i=1,2}$ corresponding to initial data $(\varphi_{0,i}, \sigma_{0,i})_{i=1,2}$ satisfying the hypothesis of Theorem 2.4, let $\varphi := \varphi_1 - \varphi_2$ and $\sigma := \sigma_1 - \sigma_2$ denote their difference. Then, it holds that φ and σ satisfy

$$\begin{aligned} 0 = {} & \langle \varphi_t, \zeta \rangle_V + (\nabla(\Lambda(\cdot, \varphi_1) - \Lambda(\cdot, \varphi_2)), \nabla \zeta) \\ & - ((\Gamma(\varphi_1) - \Gamma(\varphi_2)) B \nabla a, \nabla \zeta) + (m(\varphi_2) B(\varphi \nabla a - \nabla J \star \varphi), \nabla \zeta) \\ & + ((m(\varphi_1) - m(\varphi_2)) B(\varphi_1 \nabla a - \nabla J \star \varphi_1), \nabla \zeta) \\ & - ((P(\varphi_1) - P(\varphi_2))(\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1), \zeta) \\ & - (P(\varphi_2)(\sigma - Ba\varphi + BJ \star \varphi), \zeta) \\ & + A(P(\varphi_1)\Psi'(\varphi_1) - P(\varphi_2)\Psi'(\varphi_2), \zeta), \end{aligned} \tag{4.77}$$

and

$$\begin{aligned} 0 = {} & \langle \sigma_t, \zeta \rangle_V + (\nabla \sigma, \nabla \zeta) - A(P(\varphi_1)\Psi'(\varphi_1) - P(\varphi_2)\Psi'(\varphi_2), \zeta) \\ & + ((P(\varphi_1) - P(\varphi_2))(\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1), \zeta) \\ & + (P(\varphi_2)(\sigma - Ba\varphi + BJ \star \varphi), \zeta). \end{aligned} \tag{4.78}$$

for all $\zeta \in V$. To simplify the subsequent computations, we first analyze the term involving P. By the fact that $|\varphi_2| \leq 1$ a.e. in Q_T and hence $P(\varphi_2)$ is uniformly bounded a.e. in Q_T, and thanks also to the Lipschitz continuity of P and to (3.43), (3.44), we obtain

$$\begin{aligned} |(P(\varphi_2)(\sigma - Ba\varphi + BJ \star \varphi), \zeta)| & \leq \|\sigma - Ba\varphi + BJ \star \varphi\|_{V'} \|P(\varphi_2)\zeta\|_V \\ & \leq C(\|\sigma\|_{V'} + 2b^* B\|\varphi\|_{V'})(1 + \|\nabla \varphi_2\|_H)\|\zeta\|_{D(\mathcal{N})}. \end{aligned} \tag{4.79}$$

Next, using the Lipschitz continuity of P, Young's inequality for convolutions, and assumption (A3), we obtain

$$\begin{aligned} & \|(P(\varphi_1) - P(\varphi_2))(\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1)\|_{V'} \\ & \quad = \sup_{\eta \in V, \|\eta\|_V = 1} \left| \int_\Omega (P(\varphi_1) - P(\varphi_2))(\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1)\eta \, dx \right| \\ & \quad \leq \sup_{\eta \in V, \|\eta\|_V = 1} C\|\varphi\|_H \|\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1\|_{L^3} \|\eta\|_{L^6} \\ & \quad \leq C\|\varphi\|_H (1 + \|\sigma_1\|_{L^3} + \|\varphi_1\|_{L^3}). \end{aligned} \tag{4.80}$$

This in turn implies that

$$\begin{aligned}&|((P(\varphi_1) - P(\varphi_2))(\sigma_1 - Ba\varphi_1 + BJ \star \varphi_1), \zeta)| \\ &\quad \le C\|\varphi\|_H \left(1 + \|\sigma_1\|_{L^3} + \|\varphi_1\|_{L^3}\right) \|\zeta\|_V.\end{aligned} \tag{4.81}$$

Using the property $P\Psi' \in C^{0,1}([-1, 1])$ and by a similar calculation to (4.80) we obtain

$$\|P(\varphi_1)\Psi'(\varphi_1) - P(\varphi_2)\Psi'(\varphi_2)\|_{V'} \le C\|\varphi\|_H,$$

and thus,

$$\left|A(P(\varphi_1)\Psi'(\varphi_1) - P(\varphi_2)\Psi'(\varphi_2), \zeta)\right| \le C\|\varphi\|_H\|\zeta\|_V. \tag{4.82}$$

We now turn our attention to the other terms in (4.77). By the boundedness and Lipschitz continuity of m, Hölder's inequality, Young's inequality for convolutions, Young's inequality and the fact that $|\varphi_i| \le 1$ for $i = 1, 2$, we find that

$$|((\Gamma(\varphi_1) - \Gamma(\varphi_2))\nabla a, \nabla\zeta)| \le C\|\varphi\|_H\|\nabla\zeta\|_H, \tag{4.83}$$

$$|((m(\varphi_1) - m(\varphi_2))(\varphi_2\nabla a - \nabla J \star \varphi_2), \nabla\zeta)| \le C\|\varphi\|_H\|\nabla\zeta\|_H, \tag{4.84}$$

$$|(m(\varphi_2)(\varphi\nabla a - \nabla J \star \varphi), \nabla\zeta)| \le C\|\varphi\|_H\|\nabla\zeta\|_H, \tag{4.85}$$

where the constant C depends on $\|m\|_{L^\infty([-1,1])}$, on the Lipschitz constant of m in $[-1, 1]$ (cf. (D1)), and on b (cf. (A3)). Furthermore, by the property $m\Psi'' \in C^0([-1, 1])$, it holds that

$$\begin{aligned}&|(\Lambda(\cdot, \varphi_1) - \Lambda(\cdot, \varphi_2), \zeta)| \\ &\quad \le B\,|(a(\Gamma(\varphi_1) - \Gamma(\varphi_2)), \zeta)| + A\left|\int_\Omega \int_{\varphi_2}^{\varphi_1} m(r)\Psi''(r)\,dr\,\zeta\,dx\right| \\ &\quad \le Ba^*\|m\|_{L^\infty([-1,1])}\|\varphi\|_H\|\zeta\|_H + A\|m\Psi''\|_{L^\infty([-1,1])}\|\varphi\|_H\|\zeta\|_H \\ &\quad \le C\|\varphi\|_H\|\zeta\|_V.\end{aligned} \tag{4.86}$$

Then, upon adding the identities obtained from substituting $\zeta = \mathcal{N}^{-1}\varphi$ in (4.77) and $\zeta = \mathcal{N}^{-1}\sigma$ in (4.78), using (1.9) and adding the term

$$(\Lambda(\cdot, \varphi) - \Lambda(\cdot, \varphi_2), \mathcal{N}^{-1}\varphi) + (\sigma, \mathcal{N}^{-1}\sigma)$$

to both sides of the equality, we obtain after applying (4.79), (4.81), (4.82), (4.83), (4.84), (4.85), and the estimates $\|\mathcal{N}^{-1}f\|_V \le \|f\|_{V'}$, $\|\mathcal{N}^{-1}f\|_{D(\mathcal{N})} \le C\|f\|_H$

from (1.9) and (1.10),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_{V'}^2+\|\sigma\|_{V'}^2\right)+(\Lambda(\cdot,\varphi_1)-\Lambda(\cdot,\varphi_2),\varphi)+\|\sigma\|_H^2\\ \le(\Lambda(\cdot,\varphi)-\Lambda(\cdot,\varphi_2),\mathcal{N}^{-1}\varphi)+(\sigma,\mathcal{N}^{-1}\sigma)\\ +C\left(\|\sigma\|_{V'}+\|\varphi\|_{V'}\right)\left(1+\|\nabla\varphi_2\|_H\right)\left(\|\varphi\|_H+\|\sigma\|_H\right)\\ +C\left(1+\|\sigma_1\|_{L^3}+\|\varphi_1\|_{L^3}\right)\left(\|\varphi\|_{V'}+\|\sigma\|_{V'}\right)\|\varphi\|_H.\end{aligned} \tag{4.87}$$

From substituting $\zeta=\mathcal{N}^{-1}\varphi$ into (4.86) and also recalling (3.42), we have that

$$\left|(\Lambda(\cdot,\varphi)-\Lambda(\cdot,\varphi_2),\mathcal{N}^{-1}\varphi)+(\sigma,\mathcal{N}^{-1}\sigma)\right|\le C\|\varphi\|_H\|\varphi\|_{V'}+\|\sigma\|_{V'}^2.$$

Moreover, on account of (D2) we find that

$$\begin{aligned}\Lambda(\cdot,\varphi_1)-\Lambda(\cdot,\varphi_2)&=\int_{\varphi_2}^{\varphi_1}m(r)A\Psi''(r)+Ba(x)m(r)\,dr\\ &=\int_0^1 A(m\Psi'')(\theta\varphi_1+(1-\theta)\varphi_2)+Ba(x)m(\theta\varphi_1+(1-\theta)\varphi_2)\,d\theta\\ &\ge\int_0^1 A(1-\rho)(m\Psi_1'')(\theta\varphi_1+(1-\theta)\varphi_2)\,d\theta\\ &=\int_{\varphi_2}^{\varphi_1}A(1-\rho)m(r)\Psi_1''(r)\,dr\ \ge A(1-\rho)c_8\varphi,\end{aligned}$$

and so it holds that

$$(\Lambda(\cdot,\varphi_1)-\Lambda(\cdot,\varphi_2),\varphi)\ge A(1-\rho)c_8\|\varphi\|_H^2. \tag{4.88}$$

Altogether, from (4.87), (4.88) and by using Young's inequality we are led to the following differential inequality

$$\begin{aligned}\frac{d}{dt}\left(\|\varphi\|_{V'}^2+\|\sigma\|_{V'}^2\right)+A(1-\rho)c_8\|\varphi\|_H^2+\|\sigma\|_H^2\\ \le C\left(1+\|\sigma_1\|_{L^3}^2+\|\varphi_1\|_{L^3}^2+\|\nabla\varphi_2\|_H^2\right)\left(\|\varphi\|_{V'}^2+\|\sigma\|_{V'}^2\right).\end{aligned}$$

As the prefactor $(1+\|\sigma_1\|_{L^3}^2+\|\varphi_1\|_{L^3}^2+\|\nabla\varphi_2\|_H^2)$ belongs to $L^1(0,T)$, the application of Gronwall's inequality yields (2.18).

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A Boundary Control Problem for the Equation and Dynamic Boundary Condition of Cahn–Hilliard Type

Takeshi Fukao and Noriaki Yamazaki

Abstract A dynamic boundary condition is a type of partial differential equation that describes the dynamics of a system on the boundary. Combining with the heat equation in a smooth-bounded domain, the characteristic structure of "total mass conservation" appears, namely, the volume in the bulk plus the volume on the boundary is conserved. Based on this interesting structure, an equation and dynamic boundary condition of Cahn–Hilliard type was introduced by Goldstein–Miranville–Schimperna. In this paper, based on the previous work of Colli–Gilardi–Sprekels, a boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type is considered. The optimal boundary control that realizes the minimal cost under a control constraint is determined, and a necessary optimality condition is obtained.

Keywords Boundary control • Cahn–Hilliard system • Dynamic boundary condition

AMS (MOS) Subject Classification 49J20, 35K51, 34G25

1 Introduction

Let us consider the following problem of the equation and dynamic boundary condition of Cahn–Hilliard type: $0 < T < +\infty$, and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\Gamma := \partial\Omega$. Find a quadruplet of functions $(u, u_\Gamma, \mu, \mu_\Gamma)$

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satisfying the following equation and dynamic boundary condition of Cahn–Hilliard type:

$$\frac{\partial u}{\partial t} - \Delta\mu = 0 \quad \text{in } Q := (0,T)\times\Omega, \tag{1}$$

$$\mu = -\Delta u + \mathscr{W}'(u) \quad \text{in } Q, \tag{2}$$

$$u_\Gamma = u_{|\Gamma}, \quad \mu_\Gamma = \mu_{|\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{on } \Sigma := (0,T)\times\Gamma, \tag{3}$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \mathscr{W}_\Gamma'(u_\Gamma) - f_\Gamma \quad \text{on } \Sigma, \tag{4}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{5}$$

$$u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma, \tag{6}$$

where $u_{|\Gamma}$ and $\mu_{|\Gamma}$ denote the traces of u and μ, respectively, u_0 and $u_{0\Gamma}$ are given initial data such that $u_{0\Gamma} = u_{0|\Gamma}$ on Γ, $\partial_{\boldsymbol{\nu}}$ denotes the normal derivative on Γ outward from Ω, and Δ_Γ denotes the Laplace–Beltrami operator on Γ (see, e.g., [15, Chap. 3]). In Eq. (4), the last term $f_\Gamma : \Sigma \to \mathbb{R}$ is a control function on the boundary. In (2) and (4), the functions $\mathscr{W}$ and $\mathscr{W}_\Gamma$, which are called "double-well potentials", play an important role. Because of them, this system becomes the Cahn–Hilliard equation. For example, if we employ $\mathscr{W}(r) := (1/4)(r^2-1)^2$, then $\mathscr{W}'(r) = r^3 - r$ and (1)–(2) gives the prototype Cahn–Hilliard equation (see, e.g., [3, 10]). As another example, the function $\mathscr{W}(r) := (1+r)\ln(1+r) + (1-r)\ln(1-r) - cr^2$, where $c > 0$ is a large constant in order to make a double-well structure, is known as the logarithmic double-well potential. Then, $\mathscr{W}'(r) = \ln((1+r)/(1-r)) - 2cr$. $\mathscr{W}_\Gamma$ is also analogously defined. We can see some symmetry between (1)–(2) in the bulk and (3)–(4) on the boundary. Therefore, we can also interpret the system (1)–(4) as a transmission problem of Cahn–Hilliard equations in the bulk and on the boundary.

The system (1)–(4) was introduced by Goldstein–Miranville–Schimperna in [14] as the prototype Cahn–Hilliard equation with a "non-permeable wall". This gave rise to various studies, leading to the well-posedness with singular potentials being obtained in [6], where the characteristic structure of "total mass conservation" was the key to the proof. Here, total mass conservation means the following conservation law:

$$\int_\Omega u(t)dx + \int_\Gamma u_\Gamma(t)d\Gamma = \int_\Omega u_0 dx + \int_\Gamma u_{0\Gamma} d\Gamma \quad \text{for all } t \in [0,T].$$

Many studies have considered this interesting structure and treated various problems with the dynamic boundary condition [1, 4, 5, 11–13].

Let us consider an optimal control problem. Find an optimal control $\overline{f}_\Gamma : \Sigma \to \mathbb{R}$ in some admissible set $\mathscr{U}_{\mathrm{ad}}$ such that the associated state $(\overline{u}, \overline{u}_\Gamma)$ of $(\overline{u}, \overline{u}_\Gamma, \overline{\mu}, \overline{\mu}_\Gamma)$, which satisfies (1)–(6) with $f_\Gamma = \overline{f}_\Gamma$ in (4), realizes the minimal cost of

$$J\big((u,u_\Gamma),f_\Gamma\big) := \frac{b_Q}{2}\int_Q \big|u(t,x)-u^*(t,x)\big|^2 dxdt + \frac{b_\Sigma}{2}\int_\Sigma \big|u_\Gamma(t,x)-u_\Gamma^*(t,x)\big|^2 d\Gamma dt + \frac{b_0}{2}\int_\Sigma \big|f_\Gamma(t,x)\big|^2 d\Gamma dt,$$

in other words, the state $(\overline{u},\overline{u}_\Gamma)$ is as close as possible to the target state (u^*,u_Γ^*) under the minimal control $\overline{f}_\Gamma$. This kind of boundary control problem for the Cahn–Hilliard equation was treated in a series of papers by Colli–Gilardi–Sprekels [7–9]. Their results provide the essential idea for the proof of our equation and dynamic boundary condition of Cahn–Hilliard type.

The results of the present paper are based on a previous study [8]. The outline of the paper is as follows. In Sect. 2, we present the target problem. We also define the notation that is used in this paper. Moreover, we prove the existence of an optimal control (optimal pair) for system (1)–(6). In Sect. 3, we consider the viscous Cahn–Hilliard system as the approximate problem and discuss the well-posedness. Additionally, we prove the existence of an optimal control (optimal pair) for the viscous Cahn–Hilliard system. Moreover, we state problems (P1) and (P2) concerning the relationship between the original control problem and its approximations. In Sect. 4, we prepare the auxiliary linearized problem and adjoint problem. The differentiability of a control-to-state mapping is then obtained. In Sect. 5, we derive a weak formula for the necessary condition of the original control problem using the limiting observation of approximate situations.

2 Target Problem

In this section, we introduce the target problem.

2.1 Notation

In this paper, we use the spaces $H := L^2(\Omega)$, $V := H^1(\Omega)$, $H_\Gamma := L^2(\Gamma)$, and $V_\Gamma := H^1(\Gamma)$ with the standard norms $|\cdot|_H$, $|\cdot|_V$, $|\cdot|_{H_\Gamma}$, $|\cdot|_{V_\Gamma}$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{H_\Gamma}$, $(\cdot,\cdot)_{V_\Gamma}$, respectively. The symbol V^* denotes the dual space of V, with the duality pairing $\langle\cdot,\cdot\rangle_{V^*,V}$ between V^* and V. Similarly, the symbol V_Γ^* denotes the dual space of V_Γ with the duality pairing $\langle\cdot,\cdot\rangle_{V_\Gamma^*,V_\Gamma}$ between V_Γ^* and V_Γ. Moreover, $\boldsymbol{H} := H \times H_\Gamma$, $\boldsymbol{V} := \{(z,z_\Gamma) \in V \times V_\Gamma : z_\Gamma = z_{|\Gamma}$ a.e. on $\Gamma\}$, and $\boldsymbol{W} := H^2(\Omega) \times H^2(\Gamma)$. Hereafter, we use a bold symbol $\boldsymbol{z}$ as the pair (z,z_Γ) corresponding to the letter. The inner product $(\cdot,\cdot)_{\boldsymbol{H}}$ of $\boldsymbol{H}$ is defined by

$$(\boldsymbol{z},\tilde{\boldsymbol{z}})_{\boldsymbol{H}} := (z,\tilde{z})_H + (z_\Gamma,\tilde{z}_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{z},\tilde{\boldsymbol{z}} \in \boldsymbol{H}.$$

Note that if $\boldsymbol{z} \in \boldsymbol{V}$, then z_Γ is exactly equal to the trace $z_{|\Gamma}$ of z on Γ, whereas if $\boldsymbol{z} \in \boldsymbol{H}$, then $z \in H$ and $z_\Gamma \in H_\Gamma$ are independent. Moreover, $m : \boldsymbol{H} \to \mathbb{R}$ is defined by

$$m(\boldsymbol{z}) := \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma \right\} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $|\Omega| := \int_\Omega 1dx$ and $|\Gamma| := \int_\Gamma 1d\Gamma$. We also define the bilinear form $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u},\boldsymbol{z}) := \int_\Omega \nabla u \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{u},\boldsymbol{z} \in \boldsymbol{V},$$

where ∇_Γ denotes the surface gradient on Γ (see, e.g., [15, Chap. 3]). Finally, we define the subspace $\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$ of $\boldsymbol{H}$ and $\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$ with norms $|\boldsymbol{z}|_{\boldsymbol{H}_0} := |\boldsymbol{z}|_{\boldsymbol{H}}$ for all $\boldsymbol{z} \in \boldsymbol{H}_0$ and $|\boldsymbol{z}|_{\boldsymbol{V}_0} := a(\boldsymbol{z},\boldsymbol{z})^{1/2}$ for all $\boldsymbol{z} \in \boldsymbol{V}_0$, respectively. Under the Poincaré–Wirtinger inequality (see, e.g., [6, Lemma A]), there exists a constant $C_P > 0$ such that $C_P|\boldsymbol{z}|^2_{\boldsymbol{H}_0} \le |\boldsymbol{z}|^2_{\boldsymbol{V}_0}$ for all $\boldsymbol{z} \in \boldsymbol{V}_0$. Then, the function $\boldsymbol{F} : \boldsymbol{V}_0 \to \boldsymbol{V}_0^*$, defined by

$$\langle \boldsymbol{F}\boldsymbol{z}, \tilde{\boldsymbol{z}} \rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} := a(\boldsymbol{z},\tilde{\boldsymbol{z}}) \quad \text{for all } \boldsymbol{z},\tilde{\boldsymbol{z}} \in \boldsymbol{V}_0,$$

is the duality mapping. The inner product in $\boldsymbol{V}_0^*$ is also defined by

$$(\boldsymbol{z}_1^*, \boldsymbol{z}_2^*)_{\boldsymbol{V}_0^*} := \langle \boldsymbol{z}_1^*, \boldsymbol{F}^{-1}\boldsymbol{z}_2^* \rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} \quad \text{for all } \boldsymbol{z}_1^*, \boldsymbol{z}_2^* \in \boldsymbol{V}_0^*.$$

Then, the dense and compact embeddings $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$ hold.

2.2 Target Problem

Recall the known result for the equation and dynamic boundary condition of Cahn–Hilliard type. Find a quadruplet of functions $(u, u_\Gamma, \mu, \mu_\Gamma)$ satisfying the following equation and dynamic boundary condition of Cahn–Hilliard type:

$$\frac{\partial u}{\partial t} - \Delta\mu = 0 \quad \text{a.e. in } Q, \tag{7}$$

$$\mu = -\Delta u + \mathscr{W}'(u) \quad \text{a.e. in } Q, \tag{8}$$

$$u_\Gamma = u_{|\Gamma}, \quad \mu_\Gamma = \mu_{|\Gamma}, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{a.e. on } \Sigma, \tag{9}$$

$$\mu_\Gamma = \partial_{\boldsymbol{\nu}} u - \Delta_\Gamma u_\Gamma + \mathscr{W}_\Gamma'(u_\Gamma) - f_\Gamma \quad \text{a.e. on } \Sigma, \tag{10}$$

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \tag{11}$$

$$u_\Gamma(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma. \tag{12}$$

The well-posedness of the above problem has already been discussed [6]. To obtain a suitable solution, we assume the following:

(A1) $-\infty \le r_- < 0 < r_+ \le +\infty$;
(A2) There exist $\beta, \beta_\Gamma \in C^2(r_-, r_+)$, $\pi, \pi_\Gamma \in C^2(\mathbb{R})$ such that the following conditions hold:

- β and β_Γ are maximal monotone and $\beta(0) = \beta_\Gamma(0) = 0$, β', β'_Γ are bounded from below;
- there exist positive constants $c_1, \tilde{c}_1$ such that $\left|\beta(r)\right| \le c_1 \left|\beta_\Gamma(r)\right| + \tilde{c}_1$ for all $r \in (r_-, r_+)$;
- $\lim_{r \searrow r_-} \beta(r) = \lim_{r \searrow r_-} \beta_\Gamma(r) = -\infty$, $\lim_{r \nearrow r_+} \beta(r) = \lim_{r \nearrow r_+} \beta_\Gamma(r) = +\infty$;
- π, π_Γ are Lipschitz continuous.

Moreover, $\mathscr{W}' = \beta + \pi$, $\mathscr{W}'_\Gamma = \beta_\Gamma + \pi_\Gamma$; $\mathscr{W}, \mathscr{W}_\Gamma \in C^3(r_-, r_+)$ and $\mathscr{W}, \mathscr{W}_\Gamma \ge 0$; $\mathscr{W}(0) = \mathscr{W}_\Gamma(0) = 0$; $\mathscr{W}'', \mathscr{W}''_\Gamma$ are bounded from below; $\lim_{r \searrow r_-} \mathscr{W}'(r) = \lim_{r \searrow r_-} \mathscr{W}'_\Gamma(r) = -\infty$; $\lim_{r \nearrow r_+} \mathscr{W}'(r) = \lim_{r \nearrow r_+} \mathscr{W}'_\Gamma(r) = +\infty$;

(A3) $f_\Gamma \in H^1(0,T;H_\Gamma)$;
(A4) $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{W} \cap \boldsymbol{V}$, $\widehat{\beta}(u_0) \in L^1(\Omega)$, $\widehat{\beta}_\Gamma(u_{0\Gamma}) \in L^1(\Gamma)$ and $m_0 := m(\boldsymbol{u}_0) \in (r_-, r_+)$.

Here, $\widehat{\beta}$ and $\widehat{\beta}_\Gamma$ are the primitives of β and β_Γ, namely $\beta = \partial\widehat{\beta}$ and $\beta_\Gamma = \partial\widehat{\beta}_\Gamma$, respectively.

In this paper, for each $f_\Gamma \in H^1(0,T;H_\Gamma)$, the symbol (P;$f_\Gamma$) denotes the problem (7)–(12) corresponding to the boundary data f_Γ in (10).

Under assumptions (A1)–(A4), as a result of [6, Theorem 2.1] (see also [5, 14, 16]), we can state the following for the solvability of (P;f_Γ).

Proposition 1 (cf. [6, Theorem 2.1]) *Assume* (A1)–(A4). *Then, for each* $f_\Gamma \in H^1(0,T;H_\Gamma)$, *there exist a unique function* $\boldsymbol{u} := (u, u_\Gamma) \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})$ *and a function* $\boldsymbol{\mu} := (\mu, \mu_\Gamma) \in L^2(0,T;\boldsymbol{V})$ *such that*

$$\left\langle \boldsymbol{u}'(t), \boldsymbol{z} \right\rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + a\big(\boldsymbol{\mu}(t), \boldsymbol{z}\big) = 0 \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}, \tag{13}$$

$$\begin{aligned} \big(\boldsymbol{\mu}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} = a\big(\boldsymbol{u}(t), \boldsymbol{z}\big) + \big(\mathscr{W}'\big(u(t)\big), z\big)_H + \big(\mathscr{W}'_\Gamma\big(u_\Gamma(t)\big), z_\Gamma\big)_{H_\Gamma} \\ -\big(f_\Gamma(t), z_\Gamma\big)_{H_\Gamma} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}, \end{aligned} \tag{14}$$

for a.a. $t \in (0,T)$, *and* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *in* $\boldsymbol{H}$. *Moreover, there exists a positive constant* M_1 *such that*

$$|\boldsymbol{u}'|_{L^2(0,T;\boldsymbol{V}^*)} + |\boldsymbol{u}|_{L^\infty(0,T;\boldsymbol{V})} \le M_1\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + |f_\Gamma|_{H^1(0,T;H_\Gamma)}\big), \tag{15}$$

$$\begin{aligned} |\boldsymbol{u}|_{L^2(0,T;\boldsymbol{W})} + |\boldsymbol{\mu}|_{L^2(0,T;\boldsymbol{V})} + \left|\mathscr{W}'(u)\right|_{L^2(0,T;H)} + \left|\mathscr{W}'_\Gamma(u_\Gamma)\right|_{L^2(0,T;H_\Gamma)} \\ \le M_1\big(1 + |\boldsymbol{u}_0|_{\boldsymbol{V}} + |f_\Gamma|_{H^1(0,T;H_\Gamma)}\big). \end{aligned} \tag{16}$$

Remark 1 (cf. [11, Remark 3]) The variational form (13) means that Eqs. (7) and (9) are satisfied in the following variational sense:

$$\big\langle u'(t), z\big\rangle_{V^*,V} + \big\langle u'_\Gamma(t), z_\Gamma\big\rangle_{V^*_\Gamma,V_\Gamma} + \int_\Omega \nabla\mu(t)\cdot\nabla z dx + \int_\Gamma \nabla_\Gamma\mu_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma = 0$$

for all $\boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}$ and for a.a. $t \in (0, T)$.

Definition 1 A mapping $\mathscr{S} : \mathscr{X} \to \mathscr{Y}_1$ is defined by the control-to-state mapping that assigns the unique component $\boldsymbol{u}$, from the solution $(\boldsymbol{u}, \boldsymbol{\mu})$ to (P;f_Γ), to any boundary control f_Γ, that is,

$$\mathscr{S}(f_\Gamma) := \boldsymbol{u} \quad \text{for all } f_\Gamma \in \mathscr{X} := H^1(0, T; H_\Gamma).$$

In this paper, we consider the boundary control problem, that is, for some given target states $u^* \in L^2(0, T; H)$ and $u^*_\Gamma \in L^2(0, T; H_\Gamma)$, define a cost functional

$$J\big((u, u_\Gamma), f_\Gamma\big) := \frac{b_Q}{2}\int_0^T \big|u(t) - u^*(t)\big|_H^2 dt + \frac{b_\Sigma}{2}\int_0^T \big|u_\Gamma(t) - u^*_\Gamma(t)\big|_{H_\Gamma}^2 dt + \frac{b_0}{2}\int_0^T \big|f_\Gamma(t)\big|_{H_\Gamma}^2 dt \tag{17}$$

for all $u \in L^2(0, T; H)$, $u_\Gamma \in L^2(0, T; H_\Gamma)$, and $f_\Gamma \in \mathscr{X}$, where b_Q, b_Σ, and b_0 are given nonnegative constants. Our optimal control problem (OP) is as follows:

(OP) Find a minimizer $((\overline{u}, \overline{u}_\Gamma), \overline{f}_\Gamma)$ of the cost functional (17) subject to the constraint $\overline{f}_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ and the corresponding state $\overline{\boldsymbol{u}} = \mathscr{S}(\overline{f}_\Gamma)$ of (P;$\overline{f}_\Gamma$).

Throughout this paper, $\mathscr{U}_{\mathrm{ad}}$ is an admissible set defined by

$$\mathscr{U}_{\mathrm{ad}} := \big\{\zeta_\Gamma \in \mathscr{X} : |\zeta_\Gamma|_{H^1(0,T;H_\Gamma)} \le M_0\big\}$$

for some positive constant M_0. Note that $\mathscr{U}_{\mathrm{ad}}$ is non-empty, closed, and convex in $\mathscr{X}$.

We now state the first main theorem for the existence of the optimal control:

Theorem 1 *Assume* (A1)–(A4)*,* $u^* \in L^2(0, T; H)$*, and* $u^*_\Gamma \in L^2(0, T; H_\Gamma)$*. Then, there exists at least one* $\overline{f}_\Gamma \in \mathscr{U}_{\mathrm{ad}}$ *such that*

$$J\big(\mathscr{S}(\overline{f}_\Gamma), \overline{f}_\Gamma\big) \le J\big(\mathscr{S}(f_\Gamma), f_\Gamma\big) \quad \text{for all } f_\Gamma \in \mathscr{U}_{\mathrm{ad}}.$$

To prove Theorem 1, we show the following result for the convergence of solutions to (P;$\overline{f}_\Gamma$).

Proposition 2 *Assume* (A1)–(A4)*. Let* $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$*, and let* $\{f_{\Gamma,n}\}_{n\in\mathbb{N}} \subset \mathscr{U}_{\mathrm{ad}}$*. Furthermore, suppose that*

$$f_{\Gamma,n} \to f_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma) \quad \text{as } n \to +\infty. \tag{18}$$

Then, the unique component $\boldsymbol{u}_n = \mathscr{S}(f_{\Gamma,n})$ *of* $(\mathrm{P};f_{\Gamma,n})$ *converges to one* $\boldsymbol{u} = \mathscr{S}(f_\Gamma)$ *to* $(\mathrm{P};f_\Gamma)$ *in the following sense:*

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } C\big([0,T];\boldsymbol{H}\big) \quad \text{as } n \to +\infty. \tag{19}$$

Proof Let $(\boldsymbol{u}_n, \boldsymbol{\mu}_n)$ be the unique solution to $(\mathrm{P};f_{\Gamma,n})$. From the definition of $\mathscr{U}_{\mathrm{ad}}$, we infer from $\{f_{\Gamma,n}\}_{n\in\mathbb{N}} \subset \mathscr{U}_{\mathrm{ad}}$ that $\{f_{\Gamma,n}\}_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H_\Gamma)$. Therefore, we observe from (15)–(16) that $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;\boldsymbol{V}^*)\cap L^\infty(0,T;\boldsymbol{V})\cap L^2(0,T;\boldsymbol{W})$ and $\{\boldsymbol{\mu}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;\boldsymbol{V})$. Namely, there are a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$, a function $\overline{\boldsymbol{u}} \in H^1(0,T;\boldsymbol{V}^*)\cap L^\infty(0,T;\boldsymbol{V})\cap L^2(0,T;\boldsymbol{W})$, and a function $\overline{\boldsymbol{\mu}} \in L^2(0,T;\boldsymbol{V})$ such that $n_k \to \infty$ and

$$\left.\begin{aligned} \boldsymbol{u}_{n_k} \to \overline{\boldsymbol{u}} \quad & \text{weakly in } H^1(0,T;\boldsymbol{V}^*)\cap L^2(0,T;\boldsymbol{W}) \\ & \text{weakly star in } L^\infty(0,T;\boldsymbol{V}), \quad \text{strongly in } C\big([0,T];\boldsymbol{H}\big) \end{aligned}\right\}, \tag{20}$$

$$\boldsymbol{\mu}_{n_k} \to \overline{\boldsymbol{\mu}} \quad \text{weakly in } L^2(0,T;\boldsymbol{V}) \quad \text{as } k \to +\infty, \tag{21}$$

where the standard compactness theorem is used to obtain the strong convergence in (20). Moreover, there exists a function $\boldsymbol{\xi} := (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H})$ such that

$$\mathscr{W}'(u_{n_k}) \to \xi, \quad \beta(u_{n_k}) \to \xi - \pi(\overline{u}) \quad \text{weakly in } L^2(0,T;H), \tag{22}$$

$$\mathscr{W}_\Gamma'(u_{\Gamma,n_k}) \to \xi_\Gamma, \quad \beta_\Gamma(u_{\Gamma,n_k}) \to \xi_\Gamma - \pi_\Gamma(\overline{u}_\Gamma) \quad \text{weakly in } L^2(0,T;H_\Gamma) \tag{23}$$

as $k \to +\infty$. From the demiclosedness of β and β_Γ with (20) and (22)–(23) we infer that $\xi = \mathscr{W}'(\overline{u})$ a.e. in Q, and $\xi_\Gamma = \mathscr{W}_\Gamma'(\overline{u}_\Gamma)$ a.e. on Σ, respectively. Because $(\boldsymbol{u}_{n_k}, \boldsymbol{\mu}_{n_k})$ is the unique solution to $(\mathrm{P};f_{\Gamma,n_k})$, we observe that (cf. (13)–(14)):

$$\int_0^T \big\langle \boldsymbol{u}_{n_k}'(t), \boldsymbol{\zeta}(t)\big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} dt + \int_0^T a\big(\boldsymbol{u}_{n_k}(t), \boldsymbol{\zeta}(t)\big) dt = 0, \tag{24}$$

$$\begin{aligned} \int_0^T \big(\boldsymbol{\mu}_{n_k}(t), \boldsymbol{\zeta}(t)\big)_{\boldsymbol{H}} dt = {} & \int_0^T a\big(\boldsymbol{u}_{n_k}(t), \boldsymbol{\zeta}(t)\big) dt + \int_0^T \big(\mathscr{W}'(u_{n_k}(t)), \zeta(t)\big)_H dt \\ & + \int_0^T \big(\mathscr{W}_\Gamma'(u_{\Gamma,n_k}(t)), \zeta_\Gamma(t)\big)_{H_\Gamma} dt - \int_0^T \big(f_{\Gamma,n_k}(t), \zeta_\Gamma(t)\big)_{H_\Gamma} dt \end{aligned} \tag{25}$$

for all $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V})$, and $\boldsymbol{u}_{n_k}(0) = \boldsymbol{u}_0$ in $\boldsymbol{H}$. Therefore, taking the limits in (24)–(25) as $k \to +\infty$, we observe from (18), (20), and (21) that $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{\mu}})$ is the unique solution to $(\mathrm{P};f_\Gamma)$. By the uniqueness of the solution to $(\mathrm{P};f_\Gamma)$, we conclude that $\overline{\boldsymbol{u}} = \boldsymbol{u} = \mathscr{S}(f_\Gamma)$ and the convergence (19) holds without extracting any subsequence from $\{n\}_{n\in\mathbb{N}}$. □

Now, using Proposition 2, we prove Theorem 1.

Proof (Proof of Theorem 1) Set $\boldsymbol{u} := \mathscr{S}(f_\Gamma)$ and $\overline{J}(f_\Gamma) := J(\mathscr{S}(f_\Gamma), f_\Gamma)$ for all $f_\Gamma \in \mathscr{U}_{\rm ad}$. Moreover, let $\{f_{\Gamma,n}\}_{n\in\mathbb{N}} \subset \mathscr{U}_{\rm ad}$ be the minimizing sequence, namely, $\overline{J}(f_{\Gamma,n}) \to \alpha := \inf_{f_\Gamma \in \mathscr{U}_{\rm ad}} \overline{J}(f_\Gamma)$ as $n \to +\infty$. From the definition of $\mathscr{U}_{\rm ad}$, we see that there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ with $n_k \to +\infty$ as $k \to +\infty$ and $\overline{f}_\Gamma \in \mathscr{X}$ such that

$$f_{\Gamma,n_k} \to \overline{f}_\Gamma \quad \text{weakly in } \mathscr{X} = H^1(0,T;H_\Gamma) \tag{26}$$

as $k \to +\infty$. Therefore, setting $\boldsymbol{u}_{n_k} := \mathscr{S}(f_{\Gamma,n_k})$ and recalling Proposition 2, we observe that there exists $\overline{\boldsymbol{u}} := (\overline{u}, \overline{u}_\Gamma) \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})$ such that $\overline{\boldsymbol{u}} = \mathscr{S}(\overline{f}_\Gamma)$ and $\boldsymbol{u}_{n_k} \to \overline{\boldsymbol{u}}$ in $C([0,T];\boldsymbol{H})$, namely,

$$u_{n_k} \to \overline{u} \quad \text{in } C\big([0,T];H\big), \quad u_{\Gamma,n_k} \to \overline{u}_\Gamma \quad \text{in } C\big([0,T];H_\Gamma\big) \tag{27}$$

as $k \to +\infty$, taking some subsequence if necessary. Thus, the convergence of (26), (27) implies that

$$\begin{aligned}
\alpha &= \inf_{f_\Gamma \in \mathscr{U}_{\rm ad}} J\big(\mathscr{S}(f_\Gamma), f_\Gamma\big) \Big(= \lim_{k\to+\infty} \overline{J}(f_{\Gamma,n_k})\Big) \\
&= \lim_{k\to+\infty} \frac{b_Q}{2} \int_0^T \big|u_{n_k}(t) - u^*(t)\big|_H^2 dt + \lim_{k\to+\infty} \frac{b_\Sigma}{2} \int_0^T \big|u_{\Gamma,n_k}(t) - u_\Gamma^*(t)\big|_{H_\Gamma}^2 dt \\
&\quad + \liminf_{k\to+\infty} \frac{b_0}{2} \int_0^T \big|f_{\Gamma,n_k}(t)\big|_{H_\Gamma}^2 dt \\
&\geq J\big(\mathscr{S}(\overline{f}_\Gamma), \overline{f}_\Gamma\big),
\end{aligned}$$

which implies that $\overline{f}_\Gamma$ is an optimal control for (OP). This completes the proof of Theorem 1. □

Note that, in general, the cost functional J is not convex. Therefore, we do not know whether the optimal control $\overline{f}_\Gamma$ is unique.

3 Viscous Cahn–Hilliard System

In this paper, we obtain some characterization of the optimal control of (OP). To this end, we consider a boundary control problem for the viscous Cahn–Hilliard system.

Let us introduce our viscous Cahn–Hilliard system. For all $\varepsilon \in (0,1]$, find a quadruplet of functions $(u_\varepsilon, u_{\Gamma,\varepsilon}, \mu_\varepsilon, \mu_{\Gamma,\varepsilon})$ satisfying the following equation and dynamic boundary condition of Cahn–Hilliard type:

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta\mu_\varepsilon = 0 \quad \text{a.e. in } Q, \tag{28}$$

$$\mu_\varepsilon = \varepsilon\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \mathscr{W}'(u_\varepsilon) \quad \text{a.e. in } Q, \tag{29}$$

$$u_{\Gamma,\varepsilon} = (u_\varepsilon)_{|\Gamma}, \quad \mu_{\Gamma,\varepsilon} = (\mu_\varepsilon)_{|\Gamma}, \quad \frac{\partial u_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}}\mu_\varepsilon - \Delta_\Gamma\mu_{\Gamma,\varepsilon} = 0 \quad \text{a.e. on } \Sigma, \tag{30}$$

$$\mu_{\Gamma,\varepsilon} = \varepsilon\frac{\partial u_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}}u_\varepsilon - \Delta_\Gamma u_{\Gamma,\varepsilon} + \mathscr{W}_\Gamma'(u_{\Gamma,\varepsilon}) - f_\Gamma \quad \text{a.e. on } \Sigma, \tag{31}$$

$$u_\varepsilon(0) = u_0 \quad \text{a.e. in } \Omega, \tag{32}$$

$$u_{\Gamma,\varepsilon}(0) = u_{0\Gamma} \quad \text{a.e. on } \Gamma. \tag{33}$$

We have already discussed the well-posedness of this problem in terms of the result in [6] as follows:

Proposition 3 (cf. [6]) *For each* $\varepsilon \in (0,1]$ *and* $f_\Gamma \in \mathscr{X}$, *there exist a unique function* $\boldsymbol{u}_\varepsilon := (u_\varepsilon, u_{\Gamma,\varepsilon}) \in W^{1,\infty}(0,T;\boldsymbol{H}) \cap H^1(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{W})$ *and a function* $\boldsymbol{\mu}_\varepsilon := (\mu_\varepsilon, \mu_{\Gamma,\varepsilon}) \in L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})$ *that satisfy* (28)–(33)*. Moreover, there exists a positive constant* M_1 *independent of* $\varepsilon \in (0,1]$ *such that*

$$\varepsilon^{1/2}|\boldsymbol{u}_\varepsilon'|_{L^2(0,T;\boldsymbol{H})} + |\boldsymbol{u}_\varepsilon'|_{L^2(0,T;\boldsymbol{V}^*)} + |\boldsymbol{u}_\varepsilon|_{L^\infty(0,T;\boldsymbol{V})} \le M_1\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + |f_\Gamma|_{\mathscr{X}}\big),$$

$$|\boldsymbol{u}_\varepsilon|_{L^2(0,T;\boldsymbol{W})} + |\boldsymbol{\mu}_\varepsilon|_{L^2(0,T;\boldsymbol{V})} + \big|\mathscr{W}'(u_\varepsilon)\big|_{L^2(0,T;H)} + \big|\mathscr{W}_\Gamma'(u_{\Gamma,\varepsilon})\big|_{L^2(0,T;H_\Gamma)} \le M_1\big(1 + |\boldsymbol{u}_0|_{\boldsymbol{V}} + |f_\Gamma|_{\mathscr{X}}\big).$$

Proof Applying some previous results [6, Theorems 2.2, 4.2] and using (A1)–(A4), we can apply [6, Proposition 4.1] to obtain functions $\boldsymbol{u}_\varepsilon \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})$ and a function $\boldsymbol{\mu}_\varepsilon \in L^2(0,T;\boldsymbol{V})$ satisfying (28)–(33) in the following sense:

$$\big(\boldsymbol{u}_\varepsilon'(t), \boldsymbol{z}\big)_{\boldsymbol{H}} + a\big(\boldsymbol{\mu}_\varepsilon(t), \boldsymbol{z}\big) = 0 \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V},$$

$$\big(\boldsymbol{\mu}_\varepsilon(t), \boldsymbol{z}\big)_{\boldsymbol{H}} = \varepsilon\big(\boldsymbol{u}_\varepsilon'(t), \boldsymbol{z}\big)_{\boldsymbol{H}} + a\big(\boldsymbol{u}_\varepsilon(t), \boldsymbol{z}\big) + \big(\mathscr{W}'(u_\varepsilon(t)), z\big)_H + \big(\mathscr{W}_\Gamma'(u_{\Gamma,\varepsilon}(t)), z_\Gamma\big)_{H_\Gamma} - \big(f_\Gamma(t), z_\Gamma\big)_{H_\Gamma} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V},$$

for a.a. $t \in (0,T)$, and $\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0$ in $\boldsymbol{H}$. The uniform estimates follow from [6, Lemmas 4.1–4.5]. The regularity result follows from [6, Theorem 4.2]. Indeed, because of the viscous term in (31), we do not need to assume a certain boundedness [6, Assumption (A9)] to obtain the key estimate [6, Estimate (4.35), p.429]; we only need the boundedness of $\boldsymbol{u}_0$ in $\boldsymbol{W}$, because $\mathscr{W}'$ and $\mathscr{W}_\Gamma'$ are single-valued and sufficiently smooth. □

Hereafter, for each $f_\Gamma \in \mathscr{X}$ and $\varepsilon \in (0,1]$, the symbol (P; ε,f_Γ) denotes the viscous Cahn–Hilliard system (28)–(33) corresponding to the approximate parameter ε and the control f_Γ in (31). Moreover, for each $\varepsilon \in (0,1]$, we introduce the control-to-state mapping to (P; ε,f_Γ) as follows.

Definition 2 For each $\varepsilon \in (0,1]$, a mapping $\mathscr{S}_\varepsilon : \mathscr{X} \to \mathscr{Y}_2$ is defined by the control-to-state mapping that assigns the unique component $\boldsymbol{u}_\varepsilon$, from the solution $(\boldsymbol{u}_\varepsilon, \boldsymbol{\mu}_\varepsilon)$ to (P; ε,f_Γ), to any boundary control f_Γ, that is,

$$\mathscr{S}_\varepsilon(f_\Gamma) := \boldsymbol{u}_\varepsilon \quad \text{for all } f_\Gamma \in \mathscr{X},$$

where $\mathscr{Y}_2 := H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$.

We now derive the following result for the continuous dependence between (P;f_Γ) and (P; ε,f_Γ) ($\varepsilon \in (0,1]$).

Proposition 4 (cf. [6, Sect. 4]) *Assume* (A1)–(A4), $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, *and* $\{f_{\Gamma,\varepsilon}\}_{\varepsilon \in (0,1]} \subset \mathscr{U}_{\mathrm{ad}}$. *Furthermore, suppose that*

$$f_{\Gamma,\varepsilon} \to f_\Gamma \quad \text{weakly in } L^2(0,T;H_\Gamma) \quad \text{as } \varepsilon \to 0.$$

Then, the unique component $\boldsymbol{u}_\varepsilon = \mathscr{S}_\varepsilon(f_{\Gamma,\varepsilon})$ *of* (P; ε,$f_{\Gamma,\varepsilon}$) *converges to one* $\boldsymbol{u} = \mathscr{S}(f_\Gamma)$ *to* (P;f_Γ) *in the following sense:*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } C\big([0,T];\boldsymbol{H}\big) \quad \text{as } \varepsilon \to 0. \tag{34}$$

Proof Using arguments similar to those in [6, Sect. 4] and Proposition 2, we can prove (34). Therefore, we omit the detailed proof. □

Now, for each $\varepsilon \in (0,1]$, we prepare auxiliary optimal control problems (OP; ε) as follows:

(OP; ε) Find a minimizer $((\overline{u}_\varepsilon, \overline{u}_{\Gamma,\varepsilon}), \overline{f}_{\Gamma,\varepsilon})$ of the cost functional (17) subject to the constraint $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ and the corresponding state $\overline{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})$ of (P; ε,$\overline{f}_{\Gamma,\varepsilon}$).

To prove the necessary optimality condition for the optimal control $\overline{f}_\Gamma$ to (OP) obtained by Theorem 1, we apply the essential idea in [2] (see also [7–9, 18]), namely, we introduce a modified cost functional defined by

$$\tilde{J}_{\overline{f}_\Gamma}\big((u,u_\Gamma),f_\Gamma\big) := J\big((u,u_\Gamma),f_\Gamma\big) + \frac{1}{2}\int_0^T \big|f_\Gamma(t) - \overline{f}_\Gamma(t)\big|_{H_\Gamma}^2 dt \tag{35}$$

for all $u \in L^2(0,T;H)$, $u_\Gamma \in L^2(0,T;H_\Gamma)$, and $f_\Gamma \in \mathscr{X}$. Because of the additional term related to $\overline{f}_\Gamma$, we can characterize the exact optimal control $\overline{f}_\Gamma$ by the sequence of approximated controls. Then, (OP; $\varepsilon,\overline{f}_\Gamma$) becomes the following:

(OP; $\varepsilon,\overline{f}_\Gamma$) Find a minimizer $((\tilde{u}_\varepsilon, \tilde{u}_{\Gamma,\varepsilon}),\tilde{f}_{\Gamma,\varepsilon})$ of the modified cost functional (35) subject to the constraint $\tilde{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ and the corresponding state $\tilde{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\tilde{f}_{\Gamma,\varepsilon})$ of (P; $\varepsilon,\tilde{f}_{\Gamma,\varepsilon}$).

Using the same method as for the proof of Theorem 1, we have the following second main result.

Theorem 2 *For each $\varepsilon \in (0,1]$, there exists $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ such that*

$$J\big(\mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon}),\overline{f}_{\Gamma,\varepsilon}\big) \le J\big(\mathscr{S}_\varepsilon(f_\Gamma),f_\Gamma\big) \quad \textit{for all } f_\Gamma \in \mathscr{U}_{\mathrm{ad}}.$$

Moreover, let $\overline{f}_\Gamma$ be the optimal control of (OP). *For each $\varepsilon \in (0,1]$, there exists $\tilde{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ such that*

$$\tilde{J}_{\overline{f}_\Gamma}\big(\mathscr{S}_\varepsilon(\tilde{f}_{\Gamma,\varepsilon}),\tilde{f}_{\Gamma,\varepsilon}\big) \le \tilde{J}_{\overline{f}_\Gamma}\big(\mathscr{S}_\varepsilon(f_\Gamma),f_\Gamma\big) \quad \textit{for all } f_\Gamma \in \mathscr{U}_{\mathrm{ad}}.$$

Proof The proof is the same as for Theorem 1. Indeed, by an argument similar to that in Proposition 2, we can obtain a result for the convergence of solutions to (P; ε,f_Γ). Therefore, for each $\varepsilon \in (0,1]$, the proof of the existence of an optimal control $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ of (OP; ε) will be a slight modification of that in Theorem 2. Similarly, for each $\varepsilon \in (0,1]$, we can prove the existence of an optimal control $\tilde{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ of (OP; $\varepsilon,\overline{f}_\Gamma$). This completes the proof of Theorem 2. □

In Sect. 5, we derive the following two statements concerning the relationship between (OP) and the approximate control problems:

(P1) For each optimal control $\{\overline{f}_{\Gamma,\varepsilon}\}_{\varepsilon\in(0,1]}$ of (OP; ε), there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ such that $\overline{f}_{\Gamma,\varepsilon_k}$ converges to some optimal control $\overline{f}^*_\Gamma$ of (OP);

(P2) For each optimal control $\overline{f}_\Gamma$ of (OP) and each optimal control $\{\tilde{f}_{\Gamma,\varepsilon}\}_{\varepsilon\in(0,1]}$ of (OP; $\varepsilon,\overline{f}_\Gamma$), there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ such that $\tilde{f}_{\Gamma,\varepsilon_k}$ converges to $\overline{f}_\Gamma$.

4 Auxiliary Problems

In this section, we consider auxiliary problems related to (P; ε, h_Γ) for some $\varepsilon \in (0,1]$ and $h_\Gamma : \Sigma \to \mathbb{R}$. The first auxiliary problem is a linearized problem. The differentiability of J (or $\tilde{J}_{\overline{f}_\Gamma}$) can be proved using this linearized problem. The second auxiliary problem is an adjoint problem. This gives us the necessary optimality condition for the optimal control in a simple form.

4.1 Linearized Problem

Let $f_\Gamma \in \mathscr{U}$ and $h_\Gamma \in \mathscr{U}$ for some open set $\mathscr{U} \subset \mathscr{X}$ with $\mathscr{U}_{\mathrm{ad}} \subset \mathscr{U}$. For all $\varepsilon \in (0,1]$, find a quadruplet of functions $(\xi_\varepsilon, \xi_{\Gamma,\varepsilon}, \eta_\varepsilon, \eta_{\Gamma,\varepsilon})$ satisfying the following linearized problem:

$$\frac{\partial \xi_\varepsilon}{\partial t} - \Delta \eta_\varepsilon = 0 \quad \text{a.e. in } Q, \tag{36}$$

$$\eta_\varepsilon = \varepsilon \frac{\partial \xi_\varepsilon}{\partial t} - \Delta \xi_\varepsilon + \mathscr{W}''(u_\varepsilon)\xi_\varepsilon \quad \text{a.e. in } Q, \tag{37}$$

$$\xi_{\Gamma,\varepsilon} = (\xi_\varepsilon)_{|\Gamma}, \quad \eta_{\Gamma,\varepsilon} = (\eta_\varepsilon)_{|\Gamma}, \quad \frac{\partial \xi_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}} \eta_\varepsilon - \Delta_\Gamma \eta_{\Gamma,\varepsilon} = 0 \quad \text{a.e. on } \Sigma, \tag{38}$$

$$\eta_{\Gamma,\varepsilon} = \varepsilon \frac{\partial \xi_{\Gamma,\varepsilon}}{\partial t} + \partial_{\boldsymbol{\nu}} \xi_\varepsilon - \Delta_\Gamma \xi_{\Gamma,\varepsilon} + \mathscr{W}_\Gamma''(u_{\Gamma,\varepsilon})\xi_{\Gamma,\varepsilon} - h_\Gamma \quad \text{a.e. on } \Sigma, \tag{39}$$

$$\xi_\varepsilon(0) = 0 \quad \text{a.e. in } \Omega, \tag{40}$$

$$\xi_{\Gamma,\varepsilon}(0) = 0 \quad \text{a.e. on } \Gamma. \tag{41}$$

Note that in Eqs. (37) and (39), the function $\boldsymbol{u}_\varepsilon = (u_\varepsilon, u_{\Gamma,\varepsilon})$ appears. This is constructed by $f_\Gamma \in \mathscr{U}$ as $\boldsymbol{u}_\varepsilon = \mathscr{S}_\varepsilon(f_\Gamma)$. From the regularity $\boldsymbol{u}_\varepsilon \in L^\infty(0,T;\boldsymbol{W})$ obtained in Proposition 3, we see that $u_\varepsilon \in L^\infty(Q)$, $u_{\Gamma,\varepsilon} \in L^\infty(\Sigma)$, that is, $\mathscr{W}''(u_\varepsilon) \in L^\infty(Q)$ and $\mathscr{W}_\Gamma''(u_{\Gamma,\varepsilon}) \in L^\infty(\Sigma)$. Therefore, we have also discussed the well-posedness according to [6].

Proposition 5 *For each $\varepsilon \in (0,1]$, $h_\Gamma \in \mathscr{X}$, and $f_\Gamma \in \mathscr{U}$, there exist unique $\boldsymbol{\xi}_\varepsilon := (\xi_\varepsilon, \xi_{\Gamma,\varepsilon}) \in H^1(0,T;\boldsymbol{H}_0) \cap C([0,T];\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W})$ and $\boldsymbol{\eta}_\varepsilon := (\eta_\varepsilon, \eta_{\Gamma,\varepsilon}) \in L^2(0,T;\boldsymbol{V})$ that satisfy (36)–(41). In particular, (36) and (38) are satisfied in the following variational sense:*

$$\int_\Omega \xi_\varepsilon'(t) z dx + \int_\Gamma \xi_{\Gamma,\varepsilon}'(t) z_\Gamma d\Gamma + \int_\Omega \nabla \eta_\varepsilon(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma \eta_{\Gamma,\varepsilon}(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma = 0$$

for all $\boldsymbol{z} \in \boldsymbol{V}$ and for a.a. $t \in (0,T)$. Moreover, there exists a positive constant $M_2(\varepsilon)$ such that

$$|\boldsymbol{\xi}_\varepsilon|_{H^1(0,T;\boldsymbol{H})} + |\boldsymbol{\xi}_\varepsilon|_{L^\infty(0,T;\boldsymbol{V})} \le M_2(\varepsilon)|h_\Gamma|_{\mathscr{X}}. \tag{42}$$

Proof The above linearized problem is equivalent to

$$(\boldsymbol{F}^{-1} + \varepsilon \boldsymbol{I})\boldsymbol{\xi}_\varepsilon'(t) + \partial\varphi\big(\boldsymbol{\xi}_\varepsilon(t)\big) + \boldsymbol{P}\big(\mathscr{W}''\big(u_\varepsilon(t)\big)\xi_\varepsilon(t), \mathscr{W}_\Gamma''\big(u_{\Gamma,\varepsilon}(t)\big)\xi_{\Gamma,\varepsilon}(t)\big)$$
$$= \boldsymbol{P}\big(0, h_\Gamma(t)\big) \quad \text{in } \boldsymbol{H}_0, \quad \text{for a.a. } t \in (0,T),$$

and $\boldsymbol{\xi}_\varepsilon(0) = \mathbf{0}$ in $\boldsymbol{H}_0$, where $\varphi : \boldsymbol{H}_0 \to [0, +\infty]$ is lower semicontinuous and convex, and is defined by

$$\varphi(z) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla z|^2 dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \\ +\infty & \text{otherwise.} \end{cases}$$

The subdifferential $\partial\varphi$ on $\boldsymbol{H}_0$ fulfills $\partial\varphi(\boldsymbol{z}) = (-\Delta z, \partial_\nu z - \Delta_\Gamma z_\Gamma)$ for $\boldsymbol{z} \in D(\partial\varphi) = \boldsymbol{W} \cap \boldsymbol{V}_0$; this is checked in [6, Lemma C]. Moreover, $\boldsymbol{P} : \boldsymbol{H} \to \boldsymbol{H}_0$ is defined by $\boldsymbol{Pz} := \boldsymbol{z} - m(\boldsymbol{z})\mathbf{1}$ for all $\boldsymbol{z} \in \boldsymbol{H}$. The proof of this proposition is essentially the same as that of Proposition 3, because we have $\mathscr{W}''(u_\varepsilon) \in L^\infty(Q)$ and $\mathscr{W}_\Gamma''(u_{\Gamma,\varepsilon}) \in L^\infty(\Sigma)$. To prove the boundedness (42), we use the fact that, from $u_\varepsilon \in L^\infty(Q)$ and $u_{\Gamma,\varepsilon} \in L^\infty(\Sigma)$, there exists a constant $r_\varepsilon \in \mathbb{R}$ such that

$$r_- \le -r_\varepsilon \le u_\varepsilon \le r_\varepsilon \le r_+ \text{ a.e. in } Q, \quad r_- \le -r_\varepsilon \le u_{\Gamma,\varepsilon} \le r_\varepsilon \le r_+ \text{ a.e. on } \Sigma. \tag{43}$$

From (43), we have that the boundedness with the positive constant $M_2(\varepsilon)$ depends on $|\mathscr{W}''|_{C([-r_\varepsilon,r_\varepsilon])}$ and $|\mathscr{W}_\Gamma''|_{C([-r_\varepsilon,r_\varepsilon])}$. □

Hereafter, for each $\varepsilon \in (0, 1]$, $h_\Gamma \in \mathscr{X}$, and $f_\Gamma \in \mathscr{U}$, the symbol (LP; $\varepsilon, h_\Gamma, f_\Gamma$) denotes the linearized problem (36)–(41) corresponding to the approximate parameter ε, the heat source h_Γ in (39), and the coefficients $\mathscr{W}''(u_\varepsilon)$ in (37) and $\mathscr{W}_\Gamma''(u_{\Gamma,\varepsilon})$ in (39) with $\boldsymbol{u}_\varepsilon = \mathscr{S}_\varepsilon(f_\Gamma)$. The solution to (LP; $\varepsilon, h_\Gamma, f_\Gamma$) is useful for characterizing the Fréchet derivative of $\mathscr{S}_\varepsilon$ on some open set $\mathscr{U} \subset \mathscr{X}$.

Proposition 6 *The control-to-state mapping $\mathscr{S}_\varepsilon : \mathscr{U} \subset \mathscr{X} \to \mathscr{Y}_2$ is Fréchet differentiable at each $f_\Gamma \in \mathscr{X}$. Moreover, its Fréchet derivative $D\mathscr{S}_\varepsilon(f_\Gamma) \in \mathscr{L}(\mathscr{X}, \mathscr{Y}_2)$ is characterized by $[D\mathscr{S}_\varepsilon(f_\Gamma)](h_\Gamma) = \boldsymbol{\xi}_\varepsilon$ for all $h_\Gamma \in \mathscr{X}$, where $\boldsymbol{\xi}_\varepsilon$ is the unique component of the solution $(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)$ of* (LP; $\varepsilon, h_\Gamma, f_\Gamma$)*, and $\mathscr{L}(\mathscr{X}, \mathscr{Y}_2)$ is the space of all linear bounded operators from $\mathscr{X}$ to $\mathscr{Y}_2$.*

Proof Consider a mapping that assigns the unique component $\boldsymbol{\xi}_\varepsilon := [\boldsymbol{\xi}_\varepsilon](h_\Gamma)$ of the solution $(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)$ to (LP; $\varepsilon, h_\Gamma, f_\Gamma$) to each $h_\Gamma \in \mathscr{X}$. From (42), we see that $[\boldsymbol{\xi}_\varepsilon](\cdot) \in \mathscr{L}(\mathscr{X}, \mathscr{Y}_2)$, that is, this is a linear and bounded mapping from $\mathscr{X}$ into $\mathscr{Y}_2$. Therefore, it is sufficient to prove that

$$\frac{\mathscr{S}_\varepsilon(f_\Gamma + h_\Gamma) - \mathscr{S}_\varepsilon(f_\Gamma) - \boldsymbol{\xi}_\varepsilon}{|h_\Gamma|_{\mathscr{X}}} \to \mathbf{0} \quad \text{in } \mathscr{Y}_2 = H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$$

as $|h_\Gamma|_{\mathscr{X}} \to 0$. First, let $(\boldsymbol{u}_\varepsilon, \boldsymbol{\mu}_\varepsilon)$ be the solution to (P; ε, f_Γ) and $(\boldsymbol{u}_\varepsilon^h, \boldsymbol{\mu}_\varepsilon^h)$ be the solution to (P; $\varepsilon, f_\Gamma + h_\Gamma$). Moreover, set $\hat{\boldsymbol{u}}_\varepsilon^h := \boldsymbol{u}_\varepsilon^h - \boldsymbol{u}_\varepsilon - \boldsymbol{\xi}_\varepsilon$ and $\hat{\boldsymbol{\mu}}_\varepsilon^h := \boldsymbol{\mu}_\varepsilon^h - \boldsymbol{\mu}_\varepsilon - \boldsymbol{\eta}_\varepsilon$. Then, $\mathscr{S}_\varepsilon(f_\Gamma + h_\Gamma) - \mathscr{S}_\varepsilon(f_\Gamma) - \boldsymbol{\xi}_\varepsilon = \hat{\boldsymbol{u}}_\varepsilon^h$. If we can prove

$$|\hat{\boldsymbol{u}}_\varepsilon^h|_{H^1(0,T;\boldsymbol{H})} + |\hat{\boldsymbol{u}}_\varepsilon^h|_{L^\infty(0,T;\boldsymbol{V})} \le M_3(\varepsilon)|h_\Gamma|_{\mathscr{X}}^2 \tag{44}$$

for some constant $M_3(\varepsilon) > 0$, then we have the desired conclusion. Denote $(\hat{u}_\varepsilon, \hat{u}_{\Gamma,\varepsilon}) := \hat{\boldsymbol{u}}_\varepsilon^h$ and $(\hat{\mu}_\varepsilon, \hat{\mu}_{\Gamma,\varepsilon}) := \hat{\boldsymbol{\mu}}_\varepsilon^h$. From Remark 1, we have

$$\begin{aligned} &\int_\Omega (\hat{u}_\varepsilon^h)'(s) z dx + \int_\Gamma (\hat{u}_{\Gamma,\varepsilon}^h)'(s) z_\Gamma d\Gamma + \int_\Omega \nabla \hat{\mu}_\varepsilon^h(s) \cdot \nabla z dx \\ &+ \int_\Gamma \nabla_\Gamma \hat{\mu}_{\Gamma,\varepsilon}^h(s) \cdot \nabla_\Gamma z_\Gamma d\Gamma = 0 \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V} \quad \text{and for a.a. } s \in (0,T). \end{aligned} \tag{45}$$

Moreover, from (29), (31), (37), and (39), we have

$$\begin{aligned} &\int_\Omega \hat{\mu}_\varepsilon^h(s) z dx + \int_\Gamma \hat{\mu}_{\Gamma,\varepsilon}^h(s) z_\Gamma d\Gamma \\ &= \int_\Omega \Big(\varepsilon (\hat{u}_\varepsilon^h)'(s) - \Delta \hat{u}_\varepsilon^h(s) + \mathscr{W}'\big(u_\varepsilon^h(s)\big) - \mathscr{W}'\big(u_\varepsilon(s)\big) - \mathscr{W}''\big(u_\varepsilon(s)\big)\xi_\varepsilon(s) \Big) z dx \\ &\quad + \int_\Gamma \Big(\varepsilon (\hat{u}_{\Gamma,\varepsilon}^h)'(s) + \partial_{\boldsymbol{\nu}} \hat{u}_\varepsilon^h(s) - \Delta_\Gamma \hat{u}_{\Gamma,\varepsilon}^h(s) \\ &\quad + \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}^h(s)\big) - \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}(s)\big) - \mathscr{W}_\Gamma''\big(u_{\Gamma,\varepsilon}(s)\big)\xi_{\Gamma,\varepsilon}(s) \Big) z_\Gamma d\Gamma \end{aligned} \tag{46}$$

for all $\boldsymbol{z} \in \boldsymbol{H}$ and for a.a. $s \in (0,T)$. We also have $\hat{\boldsymbol{u}}_\varepsilon^h(0) = \boldsymbol{u}_0 - \boldsymbol{u}_0 = \boldsymbol{0}$. Then, the variational formulation (45) is equivalent to

$$\boldsymbol{F}^{-1}(\hat{\boldsymbol{u}}_\varepsilon^h)'(s) + \boldsymbol{P}\hat{\boldsymbol{\mu}}_\varepsilon^h(s) = \boldsymbol{0} \quad \text{in } \boldsymbol{V}_0, \quad \text{for a.a. } s \in (0,T), \tag{47}$$

because $m((\hat{\boldsymbol{u}}_\varepsilon^h)'(s)) = 0$. Therefore, multiplying (47) by $(\hat{\boldsymbol{u}}_\varepsilon^h)'(s) \in \boldsymbol{H}_0$ and integrating the resultant over $[0,t]$ with respect to s, we get

$$\int_0^t \big|(\hat{\boldsymbol{u}}_\varepsilon^h)'(s)\big|_{\boldsymbol{V}_0^*}^2 ds + \int_0^t \big((\hat{\boldsymbol{u}}_\varepsilon^h)'(s), \hat{\boldsymbol{\mu}}_\varepsilon^h(s)\big)_{\boldsymbol{H}} ds = 0. \tag{48}$$

On the other hand, testing (46) using $(\hat{\boldsymbol{u}}_\varepsilon^h)'(s) \in \boldsymbol{H}_0$, integrating the resultants over $[0,t]$ with respect to s, and combining with (48), we obtain

$$\begin{aligned} &\int_0^t \big|(\hat{\boldsymbol{u}}_\varepsilon^h)'(s)\big|_{\boldsymbol{V}_0^*}^2 ds + \varepsilon \int_0^t \big|(\hat{\boldsymbol{u}}_\varepsilon^h)'(s)\big|_{\boldsymbol{H}_0}^2 ds + \frac{1}{2}\big|\hat{\boldsymbol{u}}_\varepsilon^h(t)\big|_{\boldsymbol{V}_0}^2 \\ &= -\int_0^t \Big(\mathscr{W}'\big(u_\varepsilon^h(s)\big) - \mathscr{W}'\big(u_\varepsilon(s)\big) - \mathscr{W}''\big(u_\varepsilon(s)\big)\xi_\varepsilon(s), (\hat{u}_\varepsilon^h)'(s) \Big)_H ds \\ &- \int_0^t \Big(\mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}^h(s)\big) - \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}(s)\big) - \mathscr{W}_\Gamma''\big(u_{\Gamma,\varepsilon}(s)\big)\xi_{\Gamma,\varepsilon}(s), (\hat{u}_{\Gamma,\varepsilon}^h)'(s) \Big)_{H_\Gamma} ds \end{aligned} \tag{49}$$

for all $t \in [0,T]$. To estimate the first term of the right-hand side, recall (43) and the Taylor expansion of $\mathscr{W}' \in C^2([-r_\varepsilon, r_\varepsilon])$. There exists $\sigma_\varepsilon : Q \to \mathbb{R}$ with $-r_\varepsilon \le \sigma_\varepsilon \le r_\varepsilon$ a.e. in Q such that $\mathscr{W}'(u_\varepsilon^h) = \mathscr{W}'(u_\varepsilon) + \mathscr{W}''(u_\varepsilon)(u_\varepsilon^h - u_\varepsilon) + (1/2)\mathscr{W}'''(\sigma_\varepsilon)(u_\varepsilon^h - u_\varepsilon)^2$.

Therefore, from the fact that $u_\varepsilon^h - u_\varepsilon = \hat{u}_\varepsilon^h + \xi_\varepsilon$,

$$
\begin{aligned}
&-\int_0^t \Big(\mathscr{W}'\big(u_\varepsilon^h(s)\big) - \mathscr{W}'\big(u_\varepsilon(s)\big) - \mathscr{W}''\big(u_\varepsilon(s)\big)\xi_\varepsilon(s), (\hat{u}_\varepsilon^h)'(s)\Big)_H ds \\
&\le |\mathscr{W}''|_{C([-r_\varepsilon, r_\varepsilon])} \int_0^t \big|\hat{u}_\varepsilon^h(s)\big|_H \big|(\hat{u}_\varepsilon^h)'(s)\big|_H ds \\
&\quad + \frac{1}{2}|\mathscr{W}'''|_{C([-r_\varepsilon, r_\varepsilon])} \int_0^t \big|u_\varepsilon^h(s) - u_\varepsilon(s)\big|_{L^4(\Omega)}^2 \big|(\hat{u}_\varepsilon^h)'(s)\big|_H ds \\
&\le \frac{\varepsilon}{2}\int_0^t \big|(\hat{u}_\varepsilon^h)'(s)\big|_H^2 ds + \frac{1}{\varepsilon}|\mathscr{W}|_{C^3([-r_\varepsilon, r_\varepsilon])}^2 \int_0^t \big|\hat{u}_\varepsilon^h(s)\big|_H^2 ds \\
&\quad + \frac{1}{4\varepsilon}|\mathscr{W}|_{C^3([-r_\varepsilon, r_\varepsilon])}^2 \int_0^t \big|u_\varepsilon^h(s) - u_\varepsilon(s)\big|_{L^4(\Omega)}^4 ds.
\end{aligned}
$$

Treating the second term similarly, we see that

$$
\begin{aligned}
&-\int_0^t \Big(\mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}^h(s)\big) - \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}(s)\big) - \mathscr{W}_\Gamma''\big(u_{\Gamma,\varepsilon}(s)\big)\xi_{\Gamma,\varepsilon}(s), (\hat{u}_{\Gamma,\varepsilon}^h)'(s)\Big)_{H_\Gamma} ds \\
&\le \frac{\varepsilon}{2}\int_0^t \big|(\hat{u}_{\Gamma,\varepsilon}^h)'(s)\big|_{H_\Gamma}^2 ds + \frac{1}{\varepsilon}|\mathscr{W}_\Gamma|_{C^3([-r_\varepsilon, r_\varepsilon])}^2 \int_0^t \big|\hat{u}_{\Gamma,\varepsilon}^h(s)\big|_{H_\Gamma}^2 ds \\
&\quad + \frac{1}{4\varepsilon}|\mathscr{W}_\Gamma|_{C^3([-r_\varepsilon, r_\varepsilon])}^2 \int_0^t \big|u_{\Gamma,\varepsilon}^h(s) - u_{\Gamma,\varepsilon}(s)\big|_{L^4(\Gamma)}^4 ds. \qquad (50)
\end{aligned}
$$

As $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\Gamma := \partial\Omega$, we infer from the Sobolev embedding theorem that $\boldsymbol{V}$ is continuously embedded in $\boldsymbol{L}^4 := L^4(\Omega) \times L^4(\Gamma)$, and hence $\boldsymbol{V}_0$ is so too. Therefore, combining (49)–(50), we obtain

$$
\begin{aligned}
&\int_0^t \big|(\hat{\boldsymbol{u}}_\varepsilon^h)'(s)\big|_{\boldsymbol{V}_0^*}^2 ds + \frac{\varepsilon}{2}\int_0^t \big|(\hat{\boldsymbol{u}}_\varepsilon^h)'(s)\big|_{\boldsymbol{H}_0}^2 ds + \frac{1}{2}\big|\hat{\boldsymbol{u}}_\varepsilon^h(t)\big|_{\boldsymbol{V}_0}^2 \\
&\le \tilde{M}_3(\varepsilon)\int_0^t \big|\hat{\boldsymbol{u}}_\varepsilon^h(s)\big|_{\boldsymbol{V}_0}^2 ds + \tilde{M}_3(\varepsilon)\int_0^t \big|\boldsymbol{u}_\varepsilon^h(s) - \boldsymbol{u}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^4 ds
\end{aligned} \qquad (51)
$$

for all $t \in [0, T]$, where $\tilde{M}_3(\varepsilon) > 0$ is some constant depending on $|\mathscr{W}|_{C^3([-r_\varepsilon, r_\varepsilon])}$, $|\mathscr{W}_\Gamma|_{C^3([-r_\varepsilon, r_\varepsilon])}$, C_P, ε, and the constant of the embedding of $\boldsymbol{V}_0$.

Next, to estimate the second term of (51), we take the difference in Eqs. (28)–(31) between $(\boldsymbol{u}_\varepsilon^h, \boldsymbol{\mu}_\varepsilon^h)$ and $(\boldsymbol{u}_\varepsilon, \boldsymbol{\mu}_\varepsilon)$. This gives

$$
\boldsymbol{F}^{-1}\big((\boldsymbol{u}_\varepsilon^h)'(s) - \boldsymbol{u}_\varepsilon'(s)\big) + \boldsymbol{P}\big(\boldsymbol{\mu}_\varepsilon^h(s) - \boldsymbol{\mu}_\varepsilon(s)\big) = \boldsymbol{0} \quad \text{in } \boldsymbol{V}_0, \quad \text{and}
$$

$$
\int_\Omega \big(\mu_\varepsilon^h(s) - \mu_\varepsilon(s)\big) z\,dx + \int_\Gamma \big(\mu_{\Gamma,\varepsilon}^h(s) - \mu_\varepsilon(s)\big) z_\Gamma\, d\Gamma
$$

$$
\begin{aligned}
&= \varepsilon \int_\Omega \big((u_\varepsilon^h)'(s) - u_\varepsilon'(s)\big) z dx - \int_\Omega \big(\Delta u_\varepsilon^h(s) - \Delta u_\varepsilon(s)\big) z dx \\
&\quad + \int_\Omega \Big(\mathscr{W}'\big(u_\varepsilon^h(s)\big) - \mathscr{W}'\big(u_\varepsilon(s)\big)\Big) z dx + \varepsilon \int_\Gamma \Big((u_{\Gamma,\varepsilon}^h)'(s) - u_{\Gamma,\varepsilon}'(s)\Big) z_\Gamma d\Gamma \\
&\quad + \int_\Gamma \big(\partial_{\boldsymbol{\nu}} u_\varepsilon^h(s) - \partial_{\boldsymbol{\nu}} u_\varepsilon(s)\big) z_\Gamma d\Gamma - \int_\Gamma \big(\Delta_\Gamma u_{\Gamma,\varepsilon}^h(s) - \Delta_\Gamma u_{\Gamma,\varepsilon}(s)\big) z_\Gamma d\Gamma \\
&\quad + \int_\Gamma \Big(\mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}^h(s)\big) - \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}(s)\big)\Big) z_\Gamma d\Gamma + \int_\Gamma h_\Gamma(s) z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},
\end{aligned}
$$

for a.a. $s \in (0,T)$. Therefore, testing these using $(\boldsymbol{u}_\varepsilon^h)'(s) - \boldsymbol{u}_\varepsilon'(s) \in \boldsymbol{H}_0$ and combining the resultants, we reach

$$
\begin{aligned}
&\big|(\boldsymbol{u}_\varepsilon^h)'(s) - \boldsymbol{u}_\varepsilon'(s)\big|_{\boldsymbol{V}_0^*}^2 + \varepsilon\big|(\boldsymbol{u}_\varepsilon^h)'(s) - \boldsymbol{u}_\varepsilon'(s)\big|_{\boldsymbol{H}_0}^2 + \frac{1}{2}\frac{d}{ds}\big|\boldsymbol{u}_\varepsilon^h(s) - \boldsymbol{u}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 \\
&\quad = -\Big(\mathscr{W}'\big(u_\varepsilon^h(s)\big) - \mathscr{W}'\big(u_\varepsilon(s)\big), (u_\varepsilon^h)'(s) - u_\varepsilon'(s)\Big)_H \\
&\qquad - \Big(\mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}^h(s)\big) - \mathscr{W}_\Gamma'\big(u_{\Gamma,\varepsilon}(s)\big), (u_{\Gamma,\varepsilon}^h)'(s) - u_{\Gamma,\varepsilon}'(s)\Big)_{H_\Gamma} \\
&\qquad - \Big(h_\Gamma(s), (u_{\Gamma,\varepsilon}^h)'(s) - u_{\Gamma,\varepsilon}'(s)\Big)_{H_\Gamma} \\
&\quad \le \frac{\varepsilon}{2}\big|(\boldsymbol{u}_\varepsilon^h)'(s) - \boldsymbol{u}_\varepsilon'(s)\big|_{\boldsymbol{H}_0}^2 + \tilde{M}_3(\varepsilon)\big|\boldsymbol{u}_\varepsilon^h(s) - \boldsymbol{u}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 + \frac{1}{\varepsilon}\big|h_\Gamma(s)\big|_{H_\Gamma}^2
\end{aligned}
$$

for a.a. $s \in (0,T)$; therefore, the Gronwall inequality implies that

$$
\big|\boldsymbol{u}_\varepsilon^h(t) - \boldsymbol{u}_\varepsilon(t)\big|_{\boldsymbol{V}_0}^2 \le \frac{2}{\varepsilon} e^{2\tilde{M}_3(\varepsilon)T} \int_0^t \big|h_\Gamma(s)\big|_{H_\Gamma}^2 ds \quad \text{for all } t \in [0,T]. \tag{52}
$$

Finally, recalling (51)–(52) and using the Gronwall inequality of integral form and the Poincaré–Wirtinger inequality, we conclude that there exists a positive $M_3(\varepsilon)$ depending on $\varepsilon \in (0,1]$ such that (44) holds. □

The following result is related to the necessary optimality condition:

Proposition 7 *For each $\varepsilon \in (0,1]$, let $\overline{f}_{\Gamma,\varepsilon}$ be an optimal control of* (OP; ε). *Moreover, $\overline{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})$. Then,*

$$
\begin{aligned}
b_Q \int_0^T \big(\overline{u}_\varepsilon(t) - u^*(t), \xi_\varepsilon(t)\big)_H dt + b_\Sigma \int_0^T \big(\overline{u}_{\Gamma,\varepsilon}(t) - u_\Gamma^*(t), \xi_{\Gamma,\varepsilon}(t)\big)_{H_\Gamma} dt \\
+ b_0 \int_0^T \big(\overline{f}_{\Gamma,\varepsilon}(t), f_\Gamma(t) - \overline{f}_{\Gamma,\varepsilon}(t)\big)_{H_\Gamma} dt \ge 0
\end{aligned}
\tag{53}
$$

for all $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, where $(\xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ is the solution to (LP; ε, $f_\Gamma - \overline{f}_{\Gamma,\varepsilon}$, $\overline{f}_{\Gamma,\varepsilon}$) *corresponding to $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$.*

Proof Let $\varepsilon \in (0,1]$. Define a new state map $\overline{\mathscr{S}}_\varepsilon : \mathscr{X} \to \mathscr{Y}_2 \times \mathscr{X}$ by $\overline{\mathscr{S}}_\varepsilon(f_\Gamma) := (\mathscr{S}_\varepsilon(f_\Gamma), f_\Gamma)$ for all $f_\Gamma \in \mathscr{X}$. From Proposition 6, we have that $\overline{\mathscr{S}}_\varepsilon$ is Fréchet differentiable at any $f_\Gamma \in \mathscr{U}$ and is characterized by

$$\big[D\overline{\mathscr{S}}_\varepsilon(f_\Gamma)\big](h_\Gamma) = \big(\big[D\mathscr{S}(f_\Gamma)\big](h_\Gamma), h_\Gamma\big) = (\boldsymbol{\xi}_\varepsilon, h_\Gamma) \quad \text{for all } h_\Gamma \in \mathscr{X},$$

where $\boldsymbol{\xi}_\varepsilon$ is the unique component of the solution $(\boldsymbol{\xi}_\varepsilon, \boldsymbol{\eta}_\varepsilon)$ to (LP; $\varepsilon, h_\Gamma, f_\Gamma$). Moreover, recall the cost functional $\overline{J}_\varepsilon(f_\Gamma) = J(\mathscr{S}_\varepsilon(f_\Gamma), f_\Gamma) = (J \circ \overline{\mathscr{S}}_\varepsilon)(f_\Gamma)$. From the Fréchet differentiability for J,

$$\begin{aligned}
\big[D\overline{J}_\varepsilon(f_\Gamma)\big](h_\Gamma) &= \big[DJ\big(\overline{\mathscr{S}}_\varepsilon(f_\Gamma)\big)\big]\big(\big[D\overline{\mathscr{S}}_\varepsilon(f_\Gamma)\big](h_\Gamma)\big) = \big[DJ(\boldsymbol{u}_\varepsilon, f_\Gamma)\big](\boldsymbol{\xi}_\varepsilon, h_\Gamma) \\
&= b_Q \int_0^T \big(u_\varepsilon(t) - u^*(t), \xi_\varepsilon(t)\big)_H dt + b_\Sigma \int_0^T \big(u_{\Gamma,\varepsilon}(t) - u_\Gamma^*(t), \xi_{\Gamma,\varepsilon}(t)\big)_{H_\Gamma} dt \\
&\quad + b_0 \int_0^T \big(f_\Gamma(t), h_\Gamma(t)\big)_{H_\Gamma} dt \quad \text{for all } h_\Gamma \in \mathscr{X}.
\end{aligned}$$

Because $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ is the optimal control of (OP; ε), we observe that

$$\overline{J}_\varepsilon(\overline{f}_\Gamma + f_\Gamma - \overline{f}_\Gamma) = \overline{J}_\varepsilon(f_\Gamma) \geq \overline{J}_\varepsilon(\overline{f}_\Gamma) \quad \text{for all } f_\Gamma \in \mathscr{U}_{\mathrm{ad}}.$$

Therefore, the optimal control $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ satisfies

$$\big[D\overline{J}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})\big](f_\Gamma - \overline{f}_{\Gamma,\varepsilon}) \geq 0 \quad \text{for all } f_\Gamma \in \mathscr{U}_{\mathrm{ad}}.$$

Thus, we have the desired conclusion. □

4.2 Adjoint Evolution Equation

In this subsection, to improve the previous necessary optimality condition, we consider an adjoint evolution equation. For all $\varepsilon \in (0,1]$, let $\overline{f}_{\Gamma,\varepsilon} \in \mathscr{U}_{\mathrm{ad}}$ be the optimal control of (OP;ε) constructed by Proposition 4. Moreover, set $\overline{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})$. Consider the following evolution equation (AD;$\varepsilon, \overline{f}_{\Gamma,\varepsilon}$):

$$\begin{aligned}
&-\big(\boldsymbol{F}^{-1} + \varepsilon \boldsymbol{I}\big)\boldsymbol{q}_\varepsilon'(t) + \partial\varphi\big(\boldsymbol{q}_\varepsilon(t)\big) + \boldsymbol{P}\big(\mathscr{W}''\big(\overline{u}_\varepsilon(t)\big)q_\varepsilon(t), \mathscr{W}_\Gamma''\big(\overline{u}_{\Gamma,\varepsilon}(t)\big)q_{\Gamma,\varepsilon}(t)\big) \\
&= \boldsymbol{P}\big(b_Q\big(\overline{u}_\varepsilon(t) - u^*(t)\big), b_\Sigma\big(\overline{u}_{\Gamma,\varepsilon}(t) - u_\Gamma^*(t)\big)\big) \quad \text{in } \boldsymbol{H}_0, \quad \text{for a.a. } t \in (0,T), \qquad (54)
\end{aligned}$$

$$\boldsymbol{q}_\varepsilon(T) = \mathbf{0} \quad \text{in } \boldsymbol{H}_0. \qquad (55)$$

This is a backward Cauchy problem. Using a change of variable with respect to time, we obtain the following result.

Proposition 8 *For each* $\varepsilon \in (0,1]$*, there exists a unique* $\boldsymbol{q}_\varepsilon := (q_\varepsilon, q_{\Gamma,\varepsilon}) \in H^1(0,T;\boldsymbol{H}_0) \cap C([0,T];\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W})$ *such that* $\boldsymbol{q}_\varepsilon$ *satisfies* (54)–(55).

Proof For each $t \in [0,T]$, set $s := T - t$ and prepare the new function $\check{\boldsymbol{q}}_\varepsilon(s) := \boldsymbol{q}_\varepsilon(T-s)$. Then, the backward Cauchy problem (54)–(55) is equivalent to

$$\begin{aligned}&\bigl(\boldsymbol{F}^{-1}+\varepsilon\boldsymbol{I}\bigr)\check{\boldsymbol{q}}_\varepsilon'(s)+\partial\varphi\bigl(\check{\boldsymbol{q}}_\varepsilon(s)\bigr)+\boldsymbol{P}\bigl(\mathscr{W}''\bigl(\overline{u}_\varepsilon(T-s)\bigr)\check{q}_\varepsilon(s),\mathscr{W}_\Gamma''\bigl(\overline{u}_{\Gamma,\varepsilon}(T-s)\bigr)\check{q}_{\Gamma,\varepsilon}(s)\bigr)\\&\quad=\boldsymbol{P}\bigl(b_Q\bigl(\overline{u}_\varepsilon(T-s)-u^*(T-s)\bigr),b_\Sigma\bigl(\overline{u}_{\Gamma,\varepsilon}(T-s)-u_\Gamma^*(T-s)\bigr)\bigr)\quad\text{in }\boldsymbol{H}_0\end{aligned}\tag{56}$$

for a.a. $s \in (0,T)$ with $\check{\boldsymbol{q}}_\varepsilon(0) = \boldsymbol{0}$ in $\boldsymbol{H}_0$. Therefore, the well-posedness can be discussed by the abstract theory of doubly nonlinear evolution equations like Propositions 3 or 5. Namely, there exists a unique $\check{\boldsymbol{q}}_\varepsilon := (\check{q}_\varepsilon, \check{q}_{\Gamma,\varepsilon}) \in H^1(0,T;\boldsymbol{H}_0) \cap C([0,T];\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W})$ such that $\check{\boldsymbol{q}}_\varepsilon$ satisfies (56). Finally, setting $\boldsymbol{q}_\varepsilon(t) := \check{\boldsymbol{q}}_\varepsilon(T-t)$ for all $t \in [0,T]$ gives the desired conclusion. □

Our third main theorem is related to the necessary optimality condition for the optimal control (OP;ε).

Theorem 3 *For each* $\varepsilon \in (0,1]$*, let* $\overline{f}_{\Gamma,\varepsilon}$ *be an optimal control of* (OP; ε). *Moreover,* $\overline{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})$. *Then,*

$$\int_0^T \bigl(q_{\Gamma,\varepsilon}(t) + b_0\overline{f}_{\Gamma,\varepsilon}(t), f_\Gamma(t) - \overline{f}_{\Gamma,\varepsilon}(t)\bigr)_{H_\Gamma} dt \ge 0$$

for all $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$*, where* $q_{\Gamma,\varepsilon}$ *is the component of the solution* $\boldsymbol{q}_\varepsilon$ *to* (AD; ε, $\overline{f}_{\Gamma,\varepsilon}$). *If* $b_0 > 0$*, then* $\overline{f}_{\Gamma,\varepsilon}$ *is the projection of* $-q_{\Gamma,\varepsilon}/b_0$ *onto* $\mathscr{U}_{\mathrm{ad}}$ *with respect to* $L^2(0,T;H_\Gamma)$*-norm.*

Proof In the proof of Proposition 7, recall the left-hand side of (53). From (54)–(55) and the linearity of $\boldsymbol{F}^{-1}$, we have

$$\begin{aligned}&b_Q\int_0^T(\overline{u}_\varepsilon-u^*,\xi_\varepsilon)_H dt+b_\Sigma\int_0^T(\overline{u}_{\Gamma,\varepsilon}-u_\Gamma^*,\xi_{\Gamma,\varepsilon})_{H_\Gamma}dt+b_0\int_0^T(\overline{f}_{\Gamma,\varepsilon},f_\Gamma-\overline{f}_{\Gamma,\varepsilon})_{H_\Gamma}dt\\&=\int_0^T\bigl(-(\boldsymbol{F}^{-1}+\varepsilon\boldsymbol{I})\boldsymbol{q}_\varepsilon'+\partial\varphi(\boldsymbol{q}_\varepsilon),\boldsymbol{\xi}_\varepsilon\bigr)_{\boldsymbol{H}}dt+\int_0^T\bigl(\mathscr{W}''(\overline{u}_\varepsilon)q_\varepsilon,\xi_\varepsilon\bigr)_H dt\\&\quad+\int_0^T\bigl(\mathscr{W}_\Gamma''(\overline{u}_{\Gamma,\varepsilon})q_{\Gamma,\varepsilon},\xi_{\Gamma,\varepsilon}\bigr)_{H_\Gamma}dt+b_0\int_0^T(\overline{f}_{\Gamma,\varepsilon},f_\Gamma-\overline{f}_{\Gamma,\varepsilon})_{H_\Gamma}dt\\&=\int_0^T\bigl((\boldsymbol{F}^{-1}+\varepsilon\boldsymbol{I})\boldsymbol{q}_\varepsilon,\boldsymbol{\xi}_\varepsilon'\bigr)_{\boldsymbol{H}}dt+\int_0^T(\nabla q_\varepsilon,\nabla\xi_\varepsilon)_{H^d}dt\\&\quad+\int_0^T(\nabla_\Gamma q_{\Gamma,\varepsilon},\nabla_\Gamma\xi_{\Gamma,\varepsilon})_{H_\Gamma^d}dt+\int_0^T\bigl(q_\varepsilon,\mathscr{W}''(\overline{u}_\varepsilon)\xi_\varepsilon\bigr)_H dt\\&\quad+\int_0^T\bigl(q_{\Gamma,\varepsilon},\mathscr{W}_\Gamma''(\overline{u}_{\Gamma,\varepsilon})\xi_{\Gamma,\varepsilon}\bigr)_{H_\Gamma}dt+b_0\int_0^T\bigl(\overline{f}_{\Gamma,\varepsilon},f_\Gamma-\overline{f}_{\Gamma,\varepsilon}\bigr)_{H_\Gamma}dt.\end{aligned}$$

Moreover, using the fact that $\boldsymbol{\xi}_\varepsilon$ is the solution to (LP; ε, $f_\Gamma - \overline{f}_{\Gamma,\varepsilon}$, $\overline{f}_{\Gamma,\varepsilon}$) corresponding to $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, we conclude

$$
\begin{aligned}
0 &\le b_Q \int_0^T (\overline{u}_\varepsilon - u^*, \xi_\varepsilon)_H dt + b_\Sigma \int_0^T (\overline{u}_{\Gamma,\varepsilon} - u^*_\Gamma, \xi_{\Gamma,\varepsilon})_{H_\Gamma} dt \\
&\quad + b_0 \int_0^T (\overline{f}_{\Gamma,\varepsilon}, f_\Gamma - \overline{f}_{\Gamma,\varepsilon})_{H_\Gamma} dt \\
&= \int_0^T \langle \boldsymbol{\xi}'_\varepsilon, \boldsymbol{F}^{-1}(\boldsymbol{q}_\varepsilon)\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} dt \\
&\quad + \int_0^T \left(\boldsymbol{q}_\varepsilon, \varepsilon \boldsymbol{\xi}'_\varepsilon + \partial\varphi(\boldsymbol{\xi}_\varepsilon) + \left(\mathscr{W}''(\overline{u}_\varepsilon)\xi_\varepsilon, \mathscr{W}''_\Gamma(\overline{u}_{\Gamma,\varepsilon})\xi_{\Gamma,\varepsilon}\right)\right)_{\boldsymbol{H}} dt \\
&\quad + b_0 \int_0^T (\overline{f}_{\Gamma,\varepsilon}, f_\Gamma - \overline{f}_{\Gamma,\varepsilon})_{H_\Gamma} dt \\
&= \int_0^T \langle \boldsymbol{\xi}'_\varepsilon, \boldsymbol{F}^{-1}(\boldsymbol{q}_\varepsilon)\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} dt + \int_0^T \langle \boldsymbol{q}_\varepsilon, -\boldsymbol{F}^{-1}\boldsymbol{\xi}'_\varepsilon\rangle_{\boldsymbol{V_0}^*, \boldsymbol{V}_0} dt \\
&\quad + \int_0^T (q_{\Gamma,\varepsilon}, f_\Gamma - \overline{f}_{\Gamma,\varepsilon})_{H_\Gamma} dt + b_0 \int_0^T (\overline{f}_{\Gamma,\varepsilon}, f_\Gamma - \overline{f}_{\Gamma,\varepsilon})_{H_\Gamma} dt \\
&= \int_0^T (q_{\Gamma,\varepsilon} + b_0 \overline{f}_{\Gamma,\varepsilon}, f_\Gamma - \overline{f}_{\Gamma,\varepsilon})_{H_\Gamma} dt
\end{aligned}
$$

for all $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$. If $b_0 > 0$, then the above inequality implies that $-q_{\Gamma,\varepsilon}/b_0 \in \overline{f}_{\Gamma,\varepsilon} + \partial I_{\mathscr{U}_{\mathrm{ad}}}(\overline{f}_{\Gamma,\varepsilon})$ in $L^2(0,T;\boldsymbol{H}_\Gamma)$. Thus, we have the desired conclusion. □

This theorem is the point of emphasis for characterizing the optimal control $\overline{f}_\Gamma$ of (OP) obtained by Theorem 1. If we replace J by $J_{\overline{f}_\Gamma}$ and $\overline{f}_{\Gamma,\varepsilon}$ by $\tilde{f}_{\Gamma,\varepsilon}$ in Propositions 7 and 8, we obtain the following similar results:

Corollary 1 *For each* $\varepsilon \in (0,1]$, *let* $\tilde{f}_{\Gamma,\varepsilon}$ *be an optimal control of* (OP; ε, $\overline{f}_\Gamma$). *Moreover,* $\tilde{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\tilde{f}_{\Gamma,\varepsilon})$. *Then,*

$$
\int_0^T \left(\tilde{q}_{\Gamma,\varepsilon}(t) + b_0 \tilde{f}_{\Gamma,\varepsilon}(t) + \tilde{f}_{\Gamma,\varepsilon}(t) - \overline{f}_\Gamma(t), f_\Gamma(t) - \tilde{f}_{\Gamma,\varepsilon}(t)\right)_{H_\Gamma} dt \ge 0
$$

for all $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$, *where* $\tilde{q}_{\Gamma,\varepsilon}$ *is the component of the solution* $\tilde{\boldsymbol{q}}_\varepsilon$ *to* (AD; ε, $\tilde{f}_{\Gamma,\varepsilon}$).

5 Optimality of (OP)

In this section, we show the necessary optimality condition of (OP).

We begin by proving a result for the relationship between (OP) and (OP;ε) for $\varepsilon \in (0,1]$.

Theorem 4 *For each optimal control* $\{\overline{f}_{\Gamma,\varepsilon}\}_{\varepsilon\in(0,1]}$ *of* (OP; ε)*, there exist a subsequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *satisfying* $\varepsilon_k \searrow 0$ *and a function* $\overline{f}^*_\Gamma \in \mathscr{U}_{\rm ad}$ *such that* $\overline{f}^*_\Gamma$ *is an optimal control of* (OP) *and*

$$\overline{f}_{\Gamma,\varepsilon_k} \to \overline{f}^*_\Gamma \quad \text{weakly in } H^1(0,T;H_\Gamma) \quad \text{as } k \to +\infty.$$

Proof For the optimal controls $\{\overline{f}_{\Gamma,\varepsilon}\}_{\varepsilon\in(0,1]} \subset \mathscr{U}_{\rm ad}$, from the definition of $\mathscr{U}_{\rm ad}$, we see that there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \searrow 0$ and a function $\overline{f}^*_\Gamma \in H^1(0,T;H_\Gamma)$ such that

$$\overline{f}_{\Gamma,\varepsilon_k} \to \overline{f}^*_\Gamma \quad \text{weakly in } H^1(0,T;H_\Gamma) \quad \text{as } k \to +\infty. \tag{57}$$

We show that $\overline{f}^*_\Gamma$ is some optimal control for (OP). From Proposition 4, the corresponding state $\overline{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\overline{f}_{\Gamma,\varepsilon})$ of (P; ε,$\overline{f}_{\Gamma,\varepsilon}$) converges to the unique component $\overline{\boldsymbol{u}} = \mathscr{S}(\overline{f}^*_\Gamma)$ of (P;$\overline{f}^*_\Gamma$) in the following sense:

$$\mathscr{S}_{\varepsilon_k}(\overline{f}_{\Gamma,\varepsilon_k}) \to \mathscr{S}(\overline{f}^*_\Gamma) \quad \text{in } C\big([0,T];\boldsymbol{H}\big) \quad \text{as } k \to +\infty. \tag{58}$$

We now check that $\overline{f}^*_\Gamma$ is an optimal control of (OP). From the fact that $\overline{f}_{\Gamma,\varepsilon}$ is an optimal control of (OP;ε),

$$J\big((\overline{u}_\varepsilon, \overline{u}_{\Gamma,\varepsilon}),\overline{f}_{\Gamma,\varepsilon}\big) \le J\big((u_\varepsilon, u_{\Gamma,\varepsilon}),f_\Gamma\big) \quad \text{for all } f_\Gamma \in \mathscr{U}_{\rm ad},$$

where $\boldsymbol{u}_\varepsilon := \mathscr{S}_\varepsilon(f_\Gamma)$. Therefore, (57)–(58) imply that

$$\begin{aligned} J\big(\mathscr{S}(\overline{f}^*_\Gamma),\overline{f}^*_\Gamma\big) &= J\big((\overline{u},\overline{u}_\Gamma),\overline{f}^*_\Gamma\big) \\ &\le \liminf_{k\to+\infty} J\big((\overline{u}_{\varepsilon_k},\overline{u}_{\Gamma,\varepsilon_k}),\overline{f}_{\Gamma,\varepsilon_k}\big) \\ &\le \liminf_{k\to+\infty} J\big((u_{\varepsilon_k},u_{\Gamma,\varepsilon_k}),f_\Gamma\big) \\ &= \lim_{k\to\infty} J\big((u_{\varepsilon_k},u_{\Gamma,\varepsilon_k}),f_\Gamma\big) \\ &= J\big((u,u_\Gamma),f_\Gamma\big) \quad \text{for all } f_\Gamma \in \mathscr{U}_{\rm ad}, \end{aligned}$$

because $\mathscr{S}_{\varepsilon_k}(f_\Gamma) \to \mathscr{S}(f_\Gamma)$ in $C([0,T];\boldsymbol{H})$ as $k \to +\infty$ (see, e.g., [6, Sect. 4.3]). Thus, $\overline{f}^*_\Gamma$ is an optimal control of (OP). □

The idea of the final main theorem concerning the optimality of (OP) is essentially the same as [8, Theorem 2.7]. Let $\mathscr{Z} := H^1(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0)$ and $\mathscr{Z}_0 := \{\boldsymbol{\zeta} \in \mathscr{Z} : \boldsymbol{\zeta}(0) = \boldsymbol{0}\}$. Moreover, the symbol $\langle\cdot,\cdot\rangle_Q$ denotes the duality pairing between $(H^1(0,T;V^*) \cap L^2(0,T;V))^*$ and $H^1(0,T;V^*) \cap L^2(0,T;V)$. The symbol $\langle\cdot,\cdot\rangle_\Sigma$ denotes the pairing between $(H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma))^*$ and $H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma)$.

Theorem 5 *For each optimal control $\overline{f}_\Gamma$ of* (OP) *and optimal controls $\{\tilde{f}_{\Gamma,\varepsilon}\}_{\varepsilon\in(0,1]}$ of* (OP; $\varepsilon,\overline{f}_\Gamma$)*, there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \searrow 0$ such that*

$$\tilde{f}_{\Gamma,\varepsilon_k} \to \overline{f}_\Gamma \quad \text{in } L^2(0,T;H_\Gamma) \quad \text{as } k \to +\infty. \tag{59}$$

Moreover, there exist two elements

$$\Lambda \in \big(H^1(0,T;V^*) \cap L^2(0,T;V)\big)^*, \quad \Lambda_\Gamma \in \big(H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma)\big)^*$$

and $\tilde{\boldsymbol{q}} \in L^\infty(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0)$ satisfying

$$\begin{aligned} &\int_0^T \big\langle \boldsymbol{\zeta}'(t), \boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}(t)\big\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} dt + \int_0^T a\big(\tilde{\boldsymbol{q}}(t), \boldsymbol{\zeta}(t)\big)dt + \langle \Lambda, \zeta\rangle_Q + \langle \Lambda_\Gamma, \zeta_\Gamma\rangle_\Sigma \\ &= b_Q \int_0^T \big(\overline{u}(t) - u^*(t), \zeta(t)\big)_H dt + b_\Sigma \int_0^T \big(\overline{u}_\Gamma(t) - u_\Gamma^*(t), \zeta_\Gamma(t)\big)_{H_\Gamma} dt \end{aligned} \tag{60}$$

for all $\boldsymbol{\zeta} \in \mathscr{Z}_0$, such that

$$\int_0^T \big(\tilde{q}_\Gamma(t) + b_0\overline{f}_\Gamma(t), f_\Gamma(t) - \overline{f}_\Gamma(t)\big)_{H_\Gamma} dt \ge 0 \tag{61}$$

for all $f_\Gamma \in \mathscr{U}_{\rm ad}$, where $\overline{\boldsymbol{u}} := \mathscr{S}(\overline{f}_\Gamma)$, and $\tilde{q}_\Gamma$ is the component of the solution $\tilde{\boldsymbol{q}}$ satisfying (60)*. If $b_0 > 0$, then $\overline{f}_\Gamma$ is the projection of $-\tilde{q}_\Gamma/b_0$ onto $\mathscr{U}_{\rm ad}$ with respect to the $L^2(0,T;H_\Gamma)$-norm.*

Proof From the fact that $\tilde{f}_{\Gamma,\varepsilon}$ is an optimal control of (OP;$\varepsilon,\overline{f}_\Gamma$),

$$\tilde{J}_{\overline{f}_\Gamma}\big((\tilde{u}_\varepsilon, \tilde{u}_{\Gamma,\varepsilon}), \tilde{f}_{\Gamma,\varepsilon}\big) \le \tilde{J}_{\overline{f}_\Gamma}\big((u_\varepsilon, u_{\Gamma,\varepsilon}), f_\Gamma\big) \quad \text{for all } f_\Gamma \in \mathscr{U}_{\rm ad}, \tag{62}$$

where $\tilde{\boldsymbol{u}}_\varepsilon := \mathscr{S}_\varepsilon(\tilde{f}_{\Gamma,\varepsilon})$ and $\boldsymbol{u}_\varepsilon := \mathscr{S}_\varepsilon(f_\Gamma)$. Moreover, from the definition of $\mathscr{U}_{\rm ad}$, we see that there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \searrow 0$ and a function $\tilde{f}_\Gamma^* \in H^1(0,T;H_\Gamma)$ such that

$$\tilde{f}_{\Gamma,\varepsilon_k} \to \tilde{f}_\Gamma^* \quad \text{weakly in } H^1(0,T;H_\Gamma) \quad \text{as } k \to +\infty. \tag{63}$$

Then, we infer from Proposition 4 that the corresponding state $\tilde{\boldsymbol{u}}_\varepsilon = \mathscr{S}_\varepsilon(\tilde{f}_{\Gamma,\varepsilon})$ of (P; $\varepsilon,\tilde{f}_{\Gamma,\varepsilon}$) converges to the unique component $\tilde{\boldsymbol{u}} \in \mathscr{S}(\tilde{f}_\Gamma^*)$ of (P;$\tilde{f}_\Gamma^*$) in the following sense:

$$\mathscr{S}_{\varepsilon_k}(\tilde{f}_{\Gamma,\varepsilon_k}) \to \mathscr{S}(\tilde{f}_\Gamma^*) \quad \text{in } C\big([0,T];\boldsymbol{H}\big) \quad \text{as } k \to +\infty. \tag{64}$$

Similarly, we observe from Proposition 4 that

$$\mathscr{S}_{\varepsilon_k}(\overline{f}_\Gamma) \to \mathscr{S}(\overline{f}_\Gamma) \quad \text{in } C\big([0,T];\boldsymbol{H}\big) \quad \text{as } k \to +\infty. \tag{65}$$

Next, we check that $\tilde{f}_\Gamma^* = \overline{f}_\Gamma$. From (62)–(65) and the definition of $\tilde{J}_{\overline{f}_\Gamma}$ (cf. (35)), we observe that

$$\begin{aligned}
&\limsup_{k\to+\infty}\left\{\frac{1}{2}\int_0^T \left|\tilde{f}_{\Gamma,\varepsilon_k}(t)-\overline{f}_\Gamma(t)\right|_{H_\Gamma}^2 dt\right\} \\
&= \limsup_{k\to+\infty}\left\{\tilde{J}_{\overline{f}_\Gamma}\big(\mathscr{S}_{\varepsilon_k}(\tilde{f}_{\Gamma,\varepsilon_k}),\tilde{f}_{\Gamma,\varepsilon_k}\big) - J\big(\mathscr{S}_{\varepsilon_k}(\tilde{f}_{\Gamma,\varepsilon_k}),\tilde{f}_{\Gamma,\varepsilon_k}\big)\right\} \\
&\le \limsup_{k\to+\infty}\left\{\tilde{J}_{\overline{f}_\Gamma}\big(\mathscr{S}_{\varepsilon_k}(\overline{f}_\Gamma),\overline{f}_\Gamma\big) - J\big(\mathscr{S}_{\varepsilon_k}(\tilde{f}_{\Gamma,\varepsilon_k}),\tilde{f}_{\Gamma,\varepsilon_k}\big)\right\} \\
&= \lim_{k\to+\infty}\left\{\frac{b_Q}{2}\int_0^T \left|\overline{u}_{\varepsilon_k}(t)-u^*(t)\right|_H^2 dt + \frac{b_\Sigma}{2}\int_0^T \left|\overline{u}_{\Gamma,\varepsilon_k}(t)-u_\Gamma^*(t)\right|_{H_\Gamma}^2 dt\right. \\
&\qquad \left. + \frac{b_0}{2}\int_0^T \left|\overline{f}_\Gamma(t)\right|_{H_\Gamma}^2 dt\right\} - \liminf_{k\to+\infty}\left\{\frac{b_Q}{2}\int_0^T \left|\tilde{u}_{\varepsilon_k}(t)-u^*(t)\right|_H^2 dt\right. \\
&\qquad \left. + \frac{b_\Sigma}{2}\int_0^T \left|\tilde{u}_{\Gamma,\varepsilon_k}(t)-u_\Gamma^*(t)\right|_{H_\Gamma}^2 dt + \frac{b_0}{2}\int_0^T \left|\tilde{f}_{\Gamma,\varepsilon_k}(t)\right|_{H_\Gamma}^2 dt\right\} \\
&\le J\big(\mathscr{S}(\overline{f}_\Gamma),\overline{f}_\Gamma\big) - J\big(\mathscr{S}(\tilde{f}_\Gamma^*),\tilde{f}_\Gamma^*\big).
\end{aligned}$$

This means that

$$\limsup_{k\to+\infty}\left\{\frac{1}{2}\int_0^T \left|\tilde{f}_{\Gamma,\varepsilon_k}(t)-\overline{f}_\Gamma(t)\right|_{H_\Gamma}^2 dt\right\} = 0,$$

because $\overline{f}_\Gamma$ is an optimal control of (OP). Thus,

$$\tilde{f}_{\Gamma,\varepsilon_k} \to \overline{f}_\Gamma = \tilde{f}_\Gamma^* \quad \text{in } L^2(0,T;H_\Gamma) \quad \text{as } k \to +\infty.$$

Namely, $\tilde{\boldsymbol{u}} = \mathscr{S}(\tilde{f}_\Gamma^*) = \mathscr{S}(\overline{f}_\Gamma) = \overline{\boldsymbol{u}}$.

Finally, we show the necessary optimality condition (61). Let us recall the solution $\tilde{\boldsymbol{q}}_\varepsilon$ to (AD;ε,$\tilde{f}_{\Gamma,\varepsilon}$), and obtain some uniform estimates. Recall (54)–(55) at time $s \in (0,T)$. Testing $\tilde{\boldsymbol{q}}_\varepsilon(s)$ and integrating the resultant over $[t,T]$ with respect to time, we have

$$\begin{aligned}
&\frac{1}{2}\left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0^*}^2 + \frac{\varepsilon}{2}\left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{H}_0}^2 + \int_t^T \left|\tilde{\boldsymbol{q}}_\varepsilon(s)\right|_{\boldsymbol{V}_0}^2 ds + \int_t^T\int_\Omega \mathscr{W}''\big(\tilde{u}_\varepsilon(s)\big)\left|\tilde{q}_\varepsilon(s)\right|^2 dxds \\
&\quad + \int_t^T\int_\Gamma \mathscr{W}_\Gamma''\big(\tilde{u}_{\Gamma,\varepsilon}(s)\big)\left|\tilde{q}_{\Gamma,\varepsilon}(s)\right|^2 d\Gamma ds \\
&\le \frac{b_Q^2}{2}\int_t^T \left|\tilde{u}_\varepsilon(s)-u^*(s)\right|_H^2 ds + \frac{b_\Sigma^2}{2}\int_t^T \left|\tilde{u}_{\Gamma,\varepsilon}(s)-u_\Gamma^*(s)\right|_{H_\Gamma}^2 ds \\
&\quad + \frac{1}{2}\int_t^T \left|\tilde{q}_\varepsilon(s)\right|_H^2 ds + \frac{1}{2}\int_t^T \left|\tilde{q}_{\Gamma,\varepsilon}(s)\right|_{H_\Gamma}^2 ds
\end{aligned}$$

for all $t \in [0, T]$. From (A2), there exists a positive constant c_2 such that

$$\int_t^T \int_\Omega \mathscr{W}''\big(\tilde{u}_\varepsilon(s)\big)\big|\tilde{q}_\varepsilon(s)\big|^2 dxds \geq -c_2 \int_t^T \big|\tilde{q}_\varepsilon(s)\big|_H^2 ds,$$

$$\int_t^T \int_\Gamma \mathscr{W}_\Gamma''\big(\tilde{u}_{\Gamma,\varepsilon}(s)\big)\big|\tilde{q}_{\Gamma,\varepsilon}(s)\big|^2 d\Gamma ds \geq -c_2 \int_t^T \big|\tilde{q}_{\Gamma,\varepsilon}(s)\big|_{H_\Gamma}^2 ds.$$

Moreover, $\tilde{\boldsymbol{u}}_\varepsilon = S_\varepsilon(\tilde{f}_{\Gamma,\varepsilon})$ is the unique component of the solution to (P;ε,$\tilde{f}_{\Gamma,\varepsilon}$). Therefore,

$$\begin{aligned}\big|\tilde{u}_\varepsilon(t) - u^*(t)\big|_H^2 &\leq 2\big|\tilde{u}_\varepsilon(t)\big|_H^2 + 2\big|u^*(t)\big|_H^2 \\ &\leq 2M_1^2\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + M_0\big)^2 + 2\big|u^*(t)\big|_H^2,\end{aligned} \tag{66}$$

$$\big|\tilde{u}_{\Gamma,\varepsilon}(t) - u_\Gamma^*(t)\big|_{H_\Gamma}^2 \leq 2M_1^2\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + M_0\big)^2 + 2\big|u_\Gamma^*(t)\big|_{H_\Gamma}^2. \tag{67}$$

In addition, from the interpolation inequality, for each $\delta > 0$, there exists a positive constant C_δ such that $|\boldsymbol{z}|_{\boldsymbol{H}_0}^2 \leq \delta|\boldsymbol{z}|_{\boldsymbol{V}_0}^2 + C_\delta|\boldsymbol{z}|_{\boldsymbol{V}_0^*}^2$ for all $\boldsymbol{z} \in \boldsymbol{V}_0$ (see, e.g., [17, p. 51, Lemme 5.1]). Thus, we have

$$\begin{aligned}&\big|\tilde{\boldsymbol{q}}_\varepsilon(t)\big|_{\boldsymbol{V}_0^*}^2 + \varepsilon\big|\tilde{\boldsymbol{q}}_\varepsilon(t)\big|_{\boldsymbol{H}_0}^2 + 2\int_t^T \big|\tilde{\boldsymbol{q}}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 ds \\ &\leq b_Q^2 \left(2M_1^2\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + M_0\big)^2 T + 2\big|u^*\big|_{L^2(0,T;H)}^2\right) \\ &\quad + b_\Sigma^2 \left(2M_1^2\big(|\boldsymbol{u}_0|_{\boldsymbol{V}} + M_0\big)^2 T + 2\big|u_\Gamma^*(t)\big|_{L^2(0,T;H_\Gamma)}^2\right) \\ &\quad + \delta \int_t^T \big|\tilde{\boldsymbol{q}}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 ds + C_\delta(1 + 2c_2)\int_t^T \big|\tilde{\boldsymbol{q}}_\varepsilon(s)\big|_{\boldsymbol{V}_0^*}^2 ds\end{aligned}$$

for all $t \in [0, T]$. Taking $\delta := 1$ and using the Gronwall inequality, we see that there exists a positive constant M_4 independent of $\varepsilon \in (0, 1]$ such that

$$\big|\tilde{\boldsymbol{q}}_\varepsilon(t)\big|_{\boldsymbol{V}_0^*}^2 + \varepsilon\big|\tilde{\boldsymbol{q}}_\varepsilon(t)\big|_{\boldsymbol{H}_0}^2 \leq M_4, \quad \int_t^T \big|\tilde{\boldsymbol{q}}_\varepsilon(s)\big|_{\boldsymbol{V}_0}^2 ds \leq M_4 \quad \text{for all } t \in [0, T]. \tag{68}$$

Now, set $\boldsymbol{\Lambda}_\varepsilon : \mathscr{Z}_0 \to \mathbb{R}$ as follows:

$$\langle\!\langle \boldsymbol{\Lambda}_\varepsilon, \boldsymbol{\zeta} \rangle\!\rangle := \int_0^T \big(\mathscr{W}''\big(\tilde{u}_\varepsilon(t)\big)\tilde{q}_\varepsilon(t), \zeta(t)\big)_H dt + \int_0^T \big(\mathscr{W}_\Gamma''\big(\tilde{u}_{\Gamma,\varepsilon}(t)\big)\tilde{q}_{\Gamma,\varepsilon}(t), \zeta_\Gamma(t)\big)_{H_\Gamma} dt$$

for all $\boldsymbol{\zeta} \in \mathscr{Z}_0$. Then, from (54)–(55),

$$\begin{aligned}\left|\langle\!\langle \boldsymbol{\Lambda}_\varepsilon, \boldsymbol{\zeta} \rangle\!\rangle\right| \le{}& \int_0^T \left|\boldsymbol{\zeta}'(t)\right|_{\boldsymbol{V}_0^*} \left|\boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}_\varepsilon(t) + \varepsilon\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0} dt \\ &+ \int_0^T \left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0} \left|\boldsymbol{\zeta}(t)\right|_{\boldsymbol{V}_0} dt + \int_0^T \left|b_Q\big(\tilde{u}_\varepsilon(t) - u^*(t)\big)\right|_H \left|\zeta(t)\right|_H dt \\ &+ \int_0^T \left|b_\Sigma\big(\tilde{u}_{\Gamma,\varepsilon}(t) - u_\Gamma^*(t)\big)\right|_{H_\Gamma} \left|\zeta_\Gamma(t)\right|_{H_\Gamma} dt \end{aligned} \tag{69}$$

for all $\boldsymbol{\zeta} \in \mathscr{Z}_0$. Here, from (68), we have

$$\begin{aligned}\int_0^T \left|\boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}_\varepsilon(t) + \varepsilon\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 dt &\le 2\int_0^T \left|\boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 dt + 2\varepsilon^2 \int_0^T \left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 dt \\ &\le 2\int_0^T \left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0^*}^2 dt + 2\int_0^T \left|\tilde{\boldsymbol{q}}_\varepsilon(t)\right|_{\boldsymbol{V}_0}^2 dt \\ &\le 2M_4(T+1). \end{aligned} \tag{70}$$

Thus, using (66)–(70), there exists a positive constant M_5 independent of $\varepsilon \in (0,1]$ such that

$$\left|\langle\!\langle \boldsymbol{\Lambda}_\varepsilon, \boldsymbol{\zeta} \rangle\!\rangle\right| \le M_5 |\boldsymbol{\zeta}|_{\mathscr{Z}} \quad \text{for all } \boldsymbol{\zeta} \in \mathscr{Z}_0, \tag{71}$$

that is, $\boldsymbol{\Lambda}_\varepsilon \in \mathscr{Z}_0^*$. Collecting (68) and (71), we see that there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \searrow 0$, a function $\tilde{\boldsymbol{q}} \in L^\infty(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0)$, and an element $\boldsymbol{\Lambda} \in \mathscr{Z}_0^*$ such that

$$\begin{aligned} \tilde{\boldsymbol{q}}_{\varepsilon_k} \to \tilde{\boldsymbol{q}} &\quad \text{weakly star in } L^\infty(0,T;\boldsymbol{V}_0^*) \cap L^2(0,T;\boldsymbol{V}_0), \\ \varepsilon_k\tilde{\boldsymbol{q}}_{\varepsilon_k} \to \boldsymbol{0} &\quad \text{in } L^\infty(0,T;\boldsymbol{H}_0), \\ \boldsymbol{\Lambda}_{\varepsilon_k} \to \boldsymbol{\Lambda} &\quad \text{weakly star in } \mathscr{Z}_0^* \quad \text{as } k \to +\infty. \end{aligned} \tag{72}$$

Now, recalling a useful property [6, Remark 2], we can extend $\boldsymbol{\zeta}(t) \in \boldsymbol{V}_0^*$ to $\boldsymbol{V}^*$ by

$$\left\langle \boldsymbol{\zeta}(t), \boldsymbol{z} \right\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} := \left\langle \boldsymbol{\zeta}(t), \boldsymbol{P}\boldsymbol{z} \right\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}.$$

Moreover, because $\boldsymbol{V} \subset V \times V_\Gamma$ and considering the Hahn–Banach theorem, we may consider that $\boldsymbol{\Lambda}$ is a linear functional on $L^2(0,T;\boldsymbol{V}_0) \cap (H^1(0,T;V^*) \times H^1(0,T;V_\Gamma^*))$. Thus, there exist Λ and Λ_Γ satisfying

$$\Lambda \in \big(H^1(0,T;V^*) \cap L^2(0,T;V)\big)^*, \quad \Lambda_\Gamma \in \big(H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma)\big)^*$$
$$\langle\!\langle \boldsymbol{\Lambda}, \boldsymbol{\zeta} \rangle\!\rangle = \langle \Lambda, \zeta \rangle_Q + \langle \Lambda_\Gamma, \zeta_\Gamma \rangle_\Sigma \quad \text{for all } \boldsymbol{\zeta} \in \mathscr{Z}_0.$$

Now, we see that $\tilde{\boldsymbol{q}}_{\varepsilon_k}$ satisfies

$$\int_0^T \langle \boldsymbol{\zeta}'(t), \boldsymbol{F}^{-1}\tilde{\boldsymbol{q}}_{\varepsilon_k}(t) + \varepsilon_k \tilde{\boldsymbol{q}}_{\varepsilon_k}(t)\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} dt + \int_0^T a\big(\tilde{\boldsymbol{q}}_{\varepsilon_k}(t), \boldsymbol{\zeta}(t)\big) dt + \langle\langle \boldsymbol{\Lambda}_{\varepsilon_k}, \boldsymbol{\zeta}\rangle\rangle$$
$$= b_Q \int_0^T \big(\tilde{u}_{\varepsilon_k}(t) - u^*(t), \zeta(t)\big)_H dt + b_\Sigma \int_0^T \big(\tilde{u}_{\Gamma,\varepsilon_k}(t) - u_\Gamma^*(t), \zeta_\Gamma(t)\big)_{H_\Gamma} dt$$

for all $\boldsymbol{\zeta} \in \mathscr{Z}_0$. Moreover, from Corollary 1, we see that $\tilde{f}_{\Gamma,\varepsilon_k}$ satisfies

$$\int_0^T \big(\tilde{q}_{\Gamma,\varepsilon_k}(t) + b_0 \tilde{f}_{\Gamma,\varepsilon_k}(t) + \tilde{f}_{\Gamma,\varepsilon_k}(t) - \overline{f}_\Gamma(t), f_\Gamma(t) - \tilde{f}_{\Gamma,\varepsilon_k}(t)\big)_{H_\Gamma} dt \geq 0$$

for all $f_\Gamma \in \mathscr{U}_{\mathrm{ad}}$. Thus, letting $k \to +\infty$, we conclude from (59) and (72) that (60) and (61) hold. □

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New Class of Doubly Nonlinear Evolution Equations Governed by Time-Dependent Subdifferentials

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Abstract We discuss a new class of doubly nonlinear evolution equations governed by time-dependent subdifferentials in uniformly convex Banach spaces, and establish an abstract existence result of solutions. Also, we show non-uniqueness of solution, giving some examples. Moreover, we treat a quasi-variational doubly nonlinear evolution equation by applying this result extensively, and give some applications to nonlinear PDEs with gradient constraint for time-derivatives.

Keywords Doubly nonlinear • Quasi-variational inequalities • Subdifferential • Time-dependent

1 Introduction

This paper is concerned with a new class of doubly nonlinear evolution equations governed by time-dependent subdifferentials. Let H be a real Hilbert space and V be a uniformly convex Banach space such that V is dense in H and the injection from V into H is compact. Also we suppose that the dual space V^* of V is uniformly convex. In this case, identifying H with its dual, we have

$$V \hookrightarrow H \hookrightarrow V^* \text{ with compact embeddings.}$$

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The doubly nonlinear evolution equation, as in the title, is of the following form:

$$(\mathrm{P};f,u_0)\begin{cases}\partial_*\psi^t(u'(t))+\partial_*\varphi^t(u(t))+g(t,u(t))\ni f(t) \text{ in } V^* \text{ for a.e. } t\in(0,T),\\ u(0)=u_0 \ \text{ in } V.\end{cases} \tag{1}$$

Here $0<T<\infty$, $u'=du/dt$ in V, $\psi^t: V\to\mathbb{R}\cup\{\infty\}$ and $\varphi^t: V\to\mathbb{R}\cup\{\infty\}$ are time-dependent proper, l.s.c. (lower semi-continuous) and convex functions on V for each $t\in[0,T]$, $\partial_*\psi^t$ and $\partial_*\varphi^t$ are their subdifferentials from V into V^*, $g(t,\cdot)$ is a single-valued operator from V into V^*, f is a given V^*-valued function and $u_0\in V$ is a given initial datum. Suppose that $\partial_*\varphi^t$ is single-valued, linear and continuous from V into V^*.

The main aim of this paper is to show the existence of a solution to (P;f,u_0) under some additional assumptions. Also, we touch the uniqueness question of solutions to (P;f,u_0), together with an example for non-uniqueness of solutions in the general case. We shall show the uniqueness of solutions under the strong monotonicity of $\partial_*\psi^t$.

Similar types of doubly nonlinear evolution equations have been discussed by many mathematicians, for instance, Akagi [1], Arai [2], Aso et al. [3, 4], Colli [8], Colli–Visintin [9] and Senba [14]. Most of them treated the case

$$\partial\psi^t(u'(t))+\partial\varphi(u(t))\ni f(t) \ \text{ in } H \ \text{ for a.e. } t\in(0,T) \tag{2}$$

and it should be noticed that the second term $\partial\varphi$ in (2) is independent of time and there is no perturbation term g. There has been no theory on nonlinear evolution equations governed by doubly time-dependent subdifferentials because of lack of energy estimate up to date. In this paper we shall establish an abstract approach to (1), specifying the time-dependence of ψ^t and φ^t. As to the application of (1), we can treat nonlinear variational inequalities with gradient constraint for time-derivatives (see Sect. 6), which is a new novelty of this paper.

Another aim of this paper is to treat a doubly nonlinear quasi-variational evolution equation of the form:

$$(\mathrm{QP};f,u_0)\begin{cases}\partial_*\psi^t(u'(t))+\partial_*\varphi^t(u;u(t))+g(t,u(t))\ni f(t) \ \text{ in } V^* \ \text{ for a.e. } t\in(0,T),\\ u(0)=u_0 \ \text{ in } V.\end{cases}$$

The solvability will be discussed in the same framework with (P;f,u_0) by means of a standard fixed-point argument for compact operators. In this formulation, $\varphi^t(v;z)$ is proper, l.s.c. and convex in $z\in V$, and $(t,v)\in[0,T]\times L^2(0,T;V)$ is a parameter which determines the convex function $\varphi^t(v;\cdot)$ on V. The dependence of function v upon $\varphi^t(v;\cdot)$ is allowed to be non-local, in general. Therefore, the expression of (QP;f,u_0) includes an extremely wide class of quasi-linear partial differential equations or variational inequalities.

1.1 Notations

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|_H$. Let V be a uniformly convex (hence reflexive) Banach space with uniformly convex dual space V^*. We denote by $|\cdot|_V$, $|\cdot|_{V^*}$ and $\langle \cdot, \cdot \rangle$ the norms in V, V^* and duality pairing between V^* and V, respectively. Also, suppose that V is dense and embedded compactly in H. Then, identifying H with the dual H^*, we have $V \hookrightarrow H \hookrightarrow V^*$, where $\hookrightarrow$ stands for the compact embedding. Therefore, (V, H, V^*) is the standard triplet and

$$\langle u, v \rangle = (u, v) \ \text{ for } u \in H \text{ and } v \in V.$$

Also, let $F : V \to V^*$ be the duality mapping, which is single-valued and continuous from V onto V^*.

We here prepare some notations and definitions of subdifferential of convex functions. Let $\phi : V \to \mathbb{R} \cup \{\infty\}$ be a proper (i.e., not identically equal to infinity), l.s.c. and convex function. Then, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in V;\ \phi(z) < \infty\}.$$

The subdifferential $\partial_*\phi : V \to V^*$ of ϕ is a possibly multi-valued operator and is defined by:

$$z^* \in \partial_*\phi(z) \iff z^* \in V^*,\ z \in D(\phi),\ \langle z^*, y - z \rangle \leq \phi(y) - \phi(z), \quad \forall y \in V;$$

and the domain of $\partial_*\phi$ is denoted by $D(\partial_*\phi)$, and set as $D(\partial_*\phi) := \{z \in V\ ;\ \partial_*\phi(z) \neq \emptyset\}$. For basic properties and related notions of proper, l.s.c., convex functions and their subdifferentials, we refer to the monographs of Barbu [6, 7].

Next, we recall a notion of convergence for convex functions, developed by Mosco [12]. Let ϕ, ϕ_n $(n \in \mathbb{N})$ be proper, l.s.c. and convex functions on V. Then, we say that ϕ_n converges to ϕ on V in the sense of Mosco [12] as $n \to \infty$, iff. the following two conditions are satisfied:

1. for any subsequence $\{\phi_{n_k}\} \subset \{\phi_n\}$, if $z_k \to z$ weakly in V as $k \to \infty$, then

$$\liminf_{k \to \infty} \phi_{n_k}(z_k) \geq \phi(z);$$

2. for any $z \in D(\phi)$, there is a sequence $\{z_n\}$ in V such that

$$z_n \to z \text{ in } V \text{ as } n \to \infty \quad \text{and} \quad \lim_{n \to \infty} \phi_n(z_n) = \phi(z).$$

2 Main Theorems

We begin with the precise formulation of our problem (P;f, u_0).

We suppose that the duality mapping $F : V \to V^*$ is strongly monotone, more precisely there is a positive constant C_F such that

$$\langle Fz_1 - Fz_2, z_1 - z_2 \rangle \geq C_F |z_1 - z_2|_V^2, \quad \forall z_1,\ z_2 \in V. \tag{3}$$

(*Assumption (A)*)

Let $\psi^t(\cdot)$ be a proper l.s.c. and convex function on V for all $t \in [0, T]$. We assume:

(A1) If $\{t_n\} \subset [0, T]$ and $t \in [0, T]$ with $t_n \to t$ as $n \to \infty$, then $\psi^{t_n}(\cdot) \to \psi^t(\cdot)$ in the sense of Mosco [12] as $n \to \infty$.

(A2) There exist positive constants $C_1 > 0$ and $C_2 > 0$ such that

$$\psi^t(z) \geq C_1 |z|_V^2 - C_2, \quad \forall t \in [0, T],\ \forall z \in D(\psi^t).$$

(A3) $\partial_* \psi^t(0) \ni 0$ for all $t \in [0, T]$ and $\psi^{(\cdot)}(0) \in L^1(0, T)$.

(*Assumption (B)*)

Let $\varphi^t(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ be a non-negative, finite, continuous and convex function with $D(\varphi^t) = V$ for all $t \in [0, T]$. We assume:

(B1) For each $t \in [0, T]$, the subdifferential $\partial_* \varphi^t : D(\partial_* \varphi^t) = V \to V^*$ is linear and uniformly bounded, i.e., there exists a positive constant $C_3 > 0$ such that

$$|\partial_* \varphi^t(z)|_{V^*} \leq C_3 |z|_V, \quad \forall t \in [0, T],\ \forall z \in V.$$

(B2) $\varphi^t(0) = 0$ for all $t \in [0, T]$ and there exists a positive constant $C_4 > 0$ such that

$$\varphi^t(z) \geq C_4 |z|_V^2, \quad \forall t \in [0, T],\ \forall z \in V.$$

(B3) There is a function $\alpha \in W^{1,1}(0, T)$ such that

$$|\varphi^t(z) - \varphi^s(z)| \leq |\alpha(t) - \alpha(s)| \varphi^s(z), \quad \forall s, t \in [0, T],\ \forall z \in V.$$

Remark 1 We derive from (B1) and (B2) that the subdifferential $\partial_* \varphi^t$ satisfies that

$$C_3 |z|_V^2 \geq \langle \partial_* \varphi^t(z), z \rangle \geq \varphi^t(z) \geq C_4 |z|_V^2, \quad \forall z \in V,\ \forall t \in [0, T] \tag{4}$$

and from (B3) that the function $t \to \partial_* \varphi^t(z)$ is weakly continuous from $[0, T]$ into V^*.

Remark 2 The assumption (B3) is the standard time-dependence condition of convex functions (cf. [10, 13, 15]).

(*Assumption (C)*)

Let g be a single-valued operator from $[0, T] \times V$ into V^* such that $g(t, z)$ is strongly measurable in $t \in [0, T]$ for each $z \in V$, and assume:

(C1) For each $t \in [0, T]$, the operator $z \to g(t, z)$ is continuous from V_w into V^*, i.e., if $z_n \to z$ weakly in V as $n \to \infty$, then $g(t, z_n) \to g(t, z)$ in V^* as $n \to \infty$.
(C2) $g(t, \cdot)$ is uniformly Lipschitz from V into V^*, i.e., there is a positive constant $L_g > 0$ such that

$$|g(t, z_1) - g(t, z_2)|_{V^*} \le L_g |z_1 - z_2|_V, \quad \forall t \in [0, T], \ \forall z_i \in V \ (i = 1, 2).$$

Under the above assumptions we define the solution to (P;f, u_0) as follows.

Definition 1 Given $f \in L^2(0, T; V^*)$ and $u_0 \in V$, a function $u : [0, T] \to V$ is called a solution to (P;f, u_0) on $[0, T]$, iff. the following conditions are fulfilled:

(i) $u \in W^{1,2}(0, T; V)$.
(ii) There exists a function $\xi \in L^2(0, T; V^*)$ such that

$$\xi(t) \in \partial_* \psi^t(u'(t)) \ \text{ in } V^* \ \text{ for a.e. } t \in (0, T),$$

$$\xi(t) + \partial_* \varphi^t(u(t)) + g(t, u(t)) = f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0, T).$$

(iii) $u(0) = u_0$ in V.

Now, we mention the first main result of this paper, which is concerned with the existence of a solution to problem (P;f, u_0).

Theorem 1 *Suppose that Assumptions (A), (B) and (C) hold. Then, for each $u_0 \in V$ and $f \in L^2(0, T; V^*)$, there exists at least one solution u to (*P;f, u_0*) on $[0, T]$. Moreover, there exists a positive increasing function $N_0 : \mathbb{R}^3_+ \to \mathbb{R}_+$ with respect to $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ and $|\alpha'|_{L^1(0,T)}$ such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t \in [0,T]} \varphi^t(u(t)) \le N_0 \left(\varphi^0(u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)} \right). \tag{5}$$

In Sect. 3, we shall prove Theorem 1, considering the approximate problems of (P;f, u_0). It is known that the solution to (P;f, u_0) is not unique in general. In Sect. 4, we give an example for non-uniqueness of solutions to (P;f, u_0) in the general case, but we can show the uniqueness under strong monotonicity of $\partial_* \psi^t$, as stated below.

Theorem 2 *Suppose that Assumptions (A), (B) and (C) are fulfilled. In addition, assume that $\partial_* \psi^t$ is strongly monotone in V^*, more precisely,* (A4) *There exists a positive constant* $C_5 > 0$ *such that*

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq C_5 |z_1 - z_2|_V^2, \quad \forall [z_i, z_i^*] \in \partial_* \psi^t \ (i = 1, 2), \ \forall t \in [0, T].$$

Then, the solution to (P;f, u_0) *is unique.*

In Sect. 4, we prove Theorem 2 using the additional assumption (A4) and Gronwall's inequality.

Remark 3 Colli [8, Theorem 5] and Colli–Visintin [9, Remark 2.5] showed several criteria for the uniqueness of solutions to the following type of doubly nonlinear evolution equations:

$$\partial \psi(u'(t)) + \partial \varphi(u(t)) \ni f(t) \ \text{ in } H \ \text{ for a.e. } t \in (0, T). \tag{6}$$

For instance, if $\partial \varphi$ is linear and positive in H and $\partial \psi$ is strictly monotone in H, then the solution to (6) on $[0, T]$ is unique.

3 Existence of Solutions to (P;f, u_0)

In this section, we discuss the solvability of (P;f, u_0) for $f \in L^2(0, T; V^*)$ and $u_0 \in V$.

Throughout this section, we suppose that all the assumptions of Theorem 1 are made. On this basis, we prove Theorem 1 by means of the approximation of (P;f, u_0). Indeed, our approximate problem is of the following form with parameter $\varepsilon \in (0, 1]$:

$$(\mathrm{P};f, u_0)_\varepsilon \begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \ \text{ in } V^* \\ \hfill \text{for a.e. } t \in (0, T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V. \end{cases} \tag{7}$$

We prove the existence-uniqueness of solution to (P;f, u_0)$_\varepsilon$ for each $\varepsilon \in (0, 1]$.

Proposition 1 *Assume (A), (B) and (C) are satisfied. Then, for each* $\varepsilon \in (0, 1]$, $u_0 \in V$ *and* $f \in L^2(0, T; V^*)$, *there exists a unique solution* $u_\varepsilon \in W^{1,2}(0, T; V)$ *to* (P;f, u_0)$_\varepsilon$ *on* $[0, T]$ *satisfying* $u_\varepsilon(0) = u_0$ *in* V *and there exists a function* $\xi_\varepsilon \in L^2(0, T; V^*)$ *such that*

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon'(t)) \ \text{ in } V^* \ \text{ for a.e. } t \in (0, T),$$

$$\varepsilon F u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \ \text{ in } V^* \ \text{ for a.e. } t \in (0, T).$$

Moreover, there exists a positive increasing function N_0 *with respect to* $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ *and* $|\alpha'|_{L^1(0,T)}$, *independent of* $\varepsilon \in (0,1]$, *such that*

$$\int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon(t)) \le N_0\left(\varphi^0(u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)}\right). \tag{8}$$

To show (8), we need the following lemma.

Lemma 1 (cf. [10, Lemma 2.1.1]) *Assume* (*B*). *Let* $v \in W^{1,1}(0,T;V)$. *Then, we have:*

$$\frac{d}{dt}\varphi^t(v(t)) - \langle \partial_*\varphi^t(v(t)), v'(t)\rangle \le |\alpha'(t)|\varphi^t(v(t)), \quad a.e.\ t \in (0,T). \tag{9}$$

Proof We observe from (B3) that $\varphi^t(v(t))$ is absolutely continuous on $[0,T]$ and also observe from the definition of subdifferential that

$$\begin{aligned} &\varphi^t(v(t)) - \varphi^s(v(s)) - \langle \partial_*\varphi^t(v(t)), v(t)-v(s)\rangle \\ \le& \varphi^t(v(s)) - \varphi^s(v(s)) \\ \le& |\alpha(t)-\alpha(s)|\varphi^s(v(s)) \quad \text{for all } s,t \in [0,T]. \end{aligned}$$

Then, we get (9) by dividing the above inequalities by $t-s$ and letting $s \uparrow t$. □

Proof (Proof of Proposition 1) Note that the approximate problem (P;f, u_0)$_\varepsilon$ can be reformulated in the following form:

$$\begin{cases} u_\varepsilon'(t) = (\varepsilon F + \partial_*\psi^t)^{-1}\left(f(t) - \partial_*\varphi^t(u_\varepsilon(t)) - g(t,u_\varepsilon(t))\right) \ \text{ in } V \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V. \end{cases} \tag{10}$$

Here, we put

$$\mathscr{B}(t)z^* := (\varepsilon F + \partial_*\psi^t)^{-1}z^* \ \text{ for all } z^* \in V^*,\ t \in (0,T)$$

and

$$\mathscr{F}(t,z) := f(t) - \partial_*\varphi^t(z) - g(t,z) \ \text{ for all } z \in V,\ t \in (0,T).$$

Now we show that the operator $\mathscr{B}(t)z^* : [0,T]\times V^* \to V$ is Lipschitz in $z^* \in V^*$ and is continuous in $t \in [0,T]$. We first fix any $t \in [0,T]$ to show that $z^* \in V^* \mapsto \mathscr{B}(t)z^* \in V$ is Lipschitz continuous. To this end, put $z_i = \mathscr{B}(t)z_i^*$ $(i = 1,2)$. Then,

$$z_i^* = \varepsilon F z_i + z_{i,*} \ \text{ for some } z_{i,*} \in \partial_*\psi^t(z_i).$$

Hence, we infer from (3) and the monotonicity of $\partial_* \psi^t(\cdot)$ that

$$\begin{aligned}\langle z_1^* - z_2^*, z_1 - z_2 \rangle =& \langle \varepsilon F z_1 + z_{1,*} - \varepsilon F z_2 - z_{2,*}, z_1 - z_2 \rangle \\ \geq& \varepsilon \langle F z_1 - F z_2, z_1 - z_2 \rangle \\ \geq& \varepsilon C_F |z_1 - z_2|_V^2,\end{aligned}$$

which implies that

$$|\mathscr{B}(t) z_1^* - \mathscr{B}(t) z_2^*|_V = |z_1 - z_2|_V \leq \frac{1}{\varepsilon C_F} |z_1^* - z_2^*|_{V^*}.$$

Thus, the operator $\mathscr{B}(t)z^*$ is Lipschitz in $z^* \in V^*$ for all $t \in [0,T]$ with a uniform constant $1/\varepsilon C_F$.

Next, we fix any $z^* \in V^*$ to show that $t \in [0,T] \mapsto \mathscr{B}(t)z^* \in V$ is continuous. Let $z^* \in V^*$ be an arbitrary element and put $z^t := \mathscr{B}(t)z^*$, hence $\varepsilon F z^t + \partial_* \psi^t(z^t) \ni z^*$. Let $\{s_n\} \subset [0,T]$ with $s_n \to t$ (as $n \to \infty$). Note that

$$z^* = \varepsilon F z^{s_n} + z_*^{s_n} \quad \text{for some } z_*^{s_n} \in \partial_* \psi^{s_n}(z^{s_n}). \tag{11}$$

Also, we observe from (A1) that $\partial_* \psi^{s_n}$ converges to $\partial_* \psi^t$ in the sense of graph as $n \to \infty$ (cf. [5, 11]). Therefore, for $[z^t, z^* - \varepsilon F z^t] \in \partial_* \psi^t$, there exists a sequence $\{[z_n, z_n^*]\} \subset V \times V^*$ such that $[z_n, z_n^*] \in \partial_* \psi^{s_n}$ in $V \times V^*$ for all $n \in \mathbb{N}$,

$$z_n \to z^t \text{ in } V \text{ and } z_n^* \to z^* - \varepsilon F z^t \text{ in } V^* \text{ as } n \to \infty. \tag{12}$$

Since the dual space V^* is uniformly convex, the duality mapping F is uniformly continuous on every bounded subset of V. Therefore, we observe from (12) that

$$z_n^* + \varepsilon F z_n \to z^* - \varepsilon F z^t + \varepsilon F z^t = z^* \text{ in } V^* \text{ as } n \to \infty. \tag{13}$$

Hence, we infer from (11), (13) and the monotonicity of $\partial_* \psi^{s_n}$ that

$$\begin{aligned}0 =& \lim_{n\to\infty} \langle z^* - z_n^* - \varepsilon F z_n, z^{s_n} - z_n \rangle \\ =& \lim_{n\to\infty} \langle \varepsilon F z^{s_n} + z_*^{s_n} - z_n^* - \varepsilon F z_n, z^{s_n} - z_n \rangle \\ \geq& \limsup_{n\to\infty} \varepsilon \langle F z^{s_n} - F z_n, z^{s_n} - z_n \rangle \\ \geq& \varepsilon C_F \limsup_{n\to\infty} |z^{s_n} - z_n|_V^2,\end{aligned}$$

which implies from (12) that

$$z^{s_n} = \mathscr{B}(s_n) z^* \to z^t = \mathscr{B}(t) z^* \text{ as } s_n \to t.$$

Thus, the operator $\mathscr{B}(t)z^*$ is continuous in $t \in [0,T]$ for all $z^* \in V^*$.

Furthermore, it follows from (B1), (B3), (C2) and $f \in L^2(0,T;V^*)$ that the operator $\mathscr{F}(t,z) : [0,T] \times V \to V^*$ is (strongly) measurable in $t \in [0,T]$ and Lipschitz in $z \in V$.

Now we show the existence-uniqueness of a solution to (10), i.e., $(\mathrm{P};f,u_0)_\varepsilon$ on $[0,T]$. To this end, for given $u \in C([0,T];V)$, we define the operator $S : C([0,T];V) \to C([0,T];V)$ by:

$$S(u)(t) := u_0 + \int_0^t \mathscr{B}(s)[\mathscr{F}(s,u(s))]ds \quad \text{for all } t \in [0,T].$$

Note that the operator $\mathscr{B}(\cdot)[\mathscr{F}(\cdot,\cdot)] : [0,T] \times V \to V$ satisfies the Carathéodory condition, $\mathscr{B}(\cdot)[\mathscr{F}(\cdot,z)]$ is Lipschitz in $z \in V$ and $\mathscr{B}(\cdot)[\mathscr{F}(\cdot,u)] \in L^1(0,T;V)$ for all $u \in C([0,T];V)$. Therefore, by Cauchy–Lipschitz–Picard's existence theorem, we can prove that S has the fixed point $u \in C([0,T_0];V)$ for some small $T_0 \in (0,T]$, which is a unique solution to $(\mathrm{P};f,u_0)_\varepsilon$ on $[0,T_0]$. By repeating the above argument, we can construct a unique solution u_ε to $(\mathrm{P};f,u_0)_\varepsilon$ on the whole time interval $[0,T]$.

Next we show a priori estimate (8). To this end, multiply (7) by u_ε' to obtain:

$$\begin{aligned} &\langle \varepsilon F u_\varepsilon'(t), u_\varepsilon'(t)\rangle + \langle \xi_\varepsilon(t), u_\varepsilon'(t)\rangle + \langle \partial_*\varphi^t(u_\varepsilon(t)), u_\varepsilon'(t)\rangle \\ &\quad + \langle g(t,u_\varepsilon(t)), u_\varepsilon'(t)\rangle \\ &= \langle f(t), u_\varepsilon'(t)\rangle \quad \text{for a.e. } t \in (0,T), \end{aligned} \tag{14}$$

with $\xi_\varepsilon \in L^2(0,T;V^*)$ satisfying $\xi_\varepsilon(t) \in \partial_*\psi^t(u_\varepsilon'(t))$ in V^* for a.e. $t \in (0,T)$. It follows from the definition of F and $\partial_*\psi^t$, and Lemma 1 that:

$$\langle \varepsilon F u_\varepsilon'(t), u_\varepsilon'(t)\rangle = \varepsilon |u_\varepsilon'(t)|_V^2, \tag{15}$$

$$\langle \xi_\varepsilon(t), u_\varepsilon'(t)\rangle \ge \psi^t(u_\varepsilon'(t)) - \psi^t(0), \tag{16}$$

$$\langle \partial_*\varphi^t(u_\varepsilon(t)), u_\varepsilon'(t)\rangle \ge \frac{d}{dt}\varphi^t(u_\varepsilon(t)) - |\alpha'(t)|\varphi^t(u_\varepsilon(t)) \tag{17}$$

for a.e. $t \in (0,T)$. Also, from (A2), (B2), (C2) and Schwarz's inequality, we observe that

$$\begin{aligned} &|\langle g(t,u_\varepsilon(t)), u_\varepsilon'(t)\rangle| \le |g(t,u_\varepsilon(t))|_{V^*}|u_\varepsilon'(t)|_V \\ \le &\frac{C_1}{4}|u_\varepsilon'(t)|_V^2 + \frac{1}{C_1}|g(t,u_\varepsilon(t))|_{V^*}^2 \\ \le &\frac{1}{4}\psi^t(u_\varepsilon'(t)) + \frac{C_2}{4} + \frac{1}{C_1}\left(|g(t,0)|_{V^*} + L_g|u_\varepsilon(t)|_V\right)^2 \\ \le &\frac{1}{4}\psi^t(u_\varepsilon'(t)) + \frac{C_2}{4} + \frac{2|g(t,0)|_{V^*}^2}{C_1} + \frac{2L_g^2}{C_1C_4}\varphi^t(u_\varepsilon(t)) \end{aligned} \tag{18}$$

and

$$|\langle f(t), u'_\varepsilon(t)\rangle| \le \frac{C_1}{4}|u'_\varepsilon(t)|_V^2 + \frac{1}{C_1}|f(t)|_{V^*}^2 \le \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{1}{C_1}|f(t)|_{V^*}^2 \tag{19}$$

for a.e. $t \in (0, T)$. Thus, using (15)–(19), it follows from (14) that:

$$\begin{aligned} &\varepsilon|u'_\varepsilon(t)|_V^2 + \frac{1}{2}\psi^t(u'_\varepsilon(t)) + \frac{d}{dt}\varphi^t(u_\varepsilon(t)) \\ &\le M_1\left(|\alpha'(t)| + 1)\varphi^t(u_\varepsilon(t)) + M_2(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t,0)|_{V^*}^2 + 1\right) \\ &\qquad \text{for a.e. } t \in (0, T), \end{aligned} \tag{20}$$

where $M_1 > 0$ and $M_2 > 0$ are constants independent of $\varepsilon \in (0, 1]$; for instance, $M_1 = \frac{2L_g^2}{C_1 C_4} + 1$ and $M_2 = \frac{2}{C_1} + \frac{C_2}{2} + 1$. Multiplying (20) by $e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}$, we get

$$\begin{aligned} &\varepsilon e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}|u'_\varepsilon(t)|_V^2 + \frac{1}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(\psi^t(u'_\varepsilon(t)) + C_2) \\ &\qquad + \frac{d}{dt}\left\{e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}\varphi^t(u_\varepsilon(t))\right\} \end{aligned} \tag{21}$$

$$\le \frac{C_2}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau} + M_2 e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t,0)|_{V^*}^2 + 1)$$

$=: M_3(t)$.

Integrating (21) in time, we obtain

$$\begin{aligned} &\int_0^T \psi^t(u'_\varepsilon(t))dt + \sup_{t\in[0,T]}\varphi^t(u_\varepsilon(t)) \\ &\le 3e^{\int_0^T M_1(|\alpha'(\tau)|+1)d\tau}\left\{\varphi^0(u_0) + \int_0^T M_3(\tau)d\tau\right\} =: N_0. \end{aligned}$$

It is easy to see from the above construction of N_0 that N_0 is a positive increasing function with respect to $\varphi^0(u_0)$, $|f|_{L^2(0,T;V^*)}$ and $|\alpha'|_{L^1(0,T)}$, and is independent of $\varepsilon \in (0, 1]$. Thus, the proof of Proposition 1 has been completed. □

Now, let us prove the main Theorem 1.

Proof (Proof of Theorem 1) Let u_ε be a solution to $(\mathrm{P};f, u_0)_\varepsilon$ with initial datum u_0, which is obtained by Proposition 1, and let ξ_ε be a function in $L^2(0, T; V^*)$ such that

$$\xi_\varepsilon(t) \in \partial_*\psi^t(u'_\varepsilon(t)) \ \text{ in } V^* \ \text{ for a.e. } t \in (0, T) \tag{22}$$

and

$$\varepsilon F u'_\varepsilon(t) + \xi_\varepsilon(t) + \partial_*\varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \ \text{ in } V^* \ \text{ for a.e. } t \in (0, T). \tag{23}$$

From (B2), (8) and the Ascoli–Arzelà theorem, we see that there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ (as $n \to \infty$) and a function $u \in W^{1,2}(0,T;V)$ such that

$$\left.\begin{aligned} u_{\varepsilon_n} \to u \ \ &\text{weakly in } W^{1,2}(0,T;V), \ \ \text{in } C([0,T];H) \\ &\text{and weakly-} * \ \text{ in } L^\infty(0,T;V) \text{ as } n \to \infty, \end{aligned}\right\} \tag{24}$$

$$u_{\varepsilon_n}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty, \tag{25}$$

$$\int_0^t \psi^\tau(u'(\tau))d\tau \le \liminf_{n\to\infty} \int_0^t \psi^\tau(u'_{\varepsilon_n}(\tau))d\tau \le N_0 \ \text{ for all } t \in [0,T].$$

Next, we show that $u_{\varepsilon_n} \to u$ in $L^2(0,T;V)$. To this end, we multiply (23) by $u'_{\varepsilon_n} - u'$ to get:

$$\begin{aligned} &\langle \varepsilon_n F u'_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\quad + \langle \partial_* \varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &= \langle f(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \qquad \text{for a.e. } t \in (0,T). \end{aligned} \tag{26}$$

Here, we have by the definition of $\partial_*\psi^t$ (cf. (22)) that

$$\langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \ge \psi^t(u'_{\varepsilon_n}(t)) - \psi^t(u'(t)) \qquad \text{for a.e. } t \in (0,T), \tag{27}$$

and by Lemma 1 that

$$\begin{aligned} &\langle \partial_*\varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &= \langle \partial_*\varphi^t(u_{\varepsilon_n}(t) - u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \partial_*\varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\ge \frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) - |\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\qquad + \langle \partial_*\varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \qquad \text{for a.e. } t \in (0,T). \end{aligned} \tag{28}$$

Therefore, from (26)–(28) we obtain that:

$$\begin{aligned} &\frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\le |\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) + \tilde{L}_{\varepsilon_n}(t) + \psi^t(u'(t)) - \psi^t(u'_{\varepsilon_n}(t)), \end{aligned} \tag{29}$$

for a.e. $t \in (0,T)$, where $\tilde{L}_{\varepsilon_n}(\cdot)$ is a function defined by:

$$\begin{aligned} \tilde{L}_{\varepsilon_n}(t) :=& \langle f(t) - \partial_*\varphi^t(u(t)) - g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &+ \varepsilon_n |F u'_{\varepsilon_n}(t)|_{V^*} |u'_{\varepsilon_n}(t) - u'(t)|_V \qquad \text{for a.e. } t \in (0,T). \end{aligned}$$

Now, just as (20)–(21) in the proof of Proposition 1, by multiplying (29) by $e^{-\int_0^t |\alpha'(\tau)|d\tau}$ and integrating it in time, we get

$$e^{-\int_0^t |\alpha'(\tau)|d\tau}\varphi^t(u_{\varepsilon_n}(t)-u(t))$$
$$\leq \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\tilde{L}_{\varepsilon_n}(s)ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\{\psi^s(u'(s)) - \psi^s(u'_{\varepsilon_n}(s))\}ds.$$

By (24) and (25) the first integral of the right hand side goes to 0 as $n \to \infty$ and by the weak lower semicontinuity of the functional $v \to \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\psi^s(v(s))ds$ on $L^2(0,t;V)$ the limit supremum of the second integral is bounded by 0 as $n \to \infty$. Hence we conclude that

$$\limsup_{n\to\infty} \varphi^t(u_{\varepsilon_n}(t)-u(t)) \leq 0, \quad \text{hence } u_{\varepsilon_n}(t) \to u(t) \text{ in } V, \quad \forall t \in [0,T], \tag{30}$$

so that by the Lebesgue dominated convergence theorem,

$$u_{\varepsilon_n} \to u \ \text{ in } L^2(0,T;V) \ \text{ as } n \to \infty. \tag{31}$$

Now we show that u is a solution of (P;f, u_0) with initial datum u_0. We first note from (B1), (30) and the Lebesgue dominated convergence theorem that

$$\partial_*\varphi^{(\cdot)}(u_{\varepsilon_n}(\cdot)) \to \partial_*\varphi^{(\cdot)}(u(\cdot)) \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty \tag{32}$$

and by (8) that

$$\varepsilon_n F u'_{\varepsilon_n} \to 0 \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{33}$$

By (31)–(33) and (C2),

$$\xi_{\varepsilon_n} = f - \partial_*\varphi^t(u_{\varepsilon_n}) - g(t,u_{\varepsilon_n}) - \varepsilon_n F u'_{\varepsilon_n} \to f - \partial_*\varphi^t(u) - g(t,u) =: \xi \text{ in } L^2(0,T;V^*).$$

Therefore, from the demi-closedness of $\partial_*\psi^t$ in $L^2(0,T;V) \times L^2(0,T;V^*)$ it follows that $\xi(t) \in \partial_*\psi^t(u'(t))$ in V^* for a.e. $t \in (0,T)$ and

$$\xi(t) + \partial_*\varphi^t(u(t)) + g(t,u(t)) = f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0,T).$$

Therefore, we conclude that u is a solution of (P;f, u_0) and from a priori estimate (8) that (5) holds for the same function N_0 as in Proposition 1.

Thus, the proof of Theorem 1 has been completed. □

4 Uniqueness of Solutions to (P;f, u_0)

In this section, we discuss the uniqueness of solutions to (P;f, u_0).

We begin with showing a counterexample for uniqueness of solutions to (P;f, u_0).

Example 4.1 (cf. [8, Section 2]) Let $\Omega = (0, 1)$. Also, let $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Define a closed convex subset K of V by

$$K := \{z \in V\, ;\ |z(x)| \le 1,\ |z_x(x)| \le 1,\ \text{a.e. } x \in \Omega\}.$$

Then, we consider the following variational problem with constraint:

$$\begin{cases} u_t(t) \in K,\ \text{a.e. } t \in (0, T), \\ \displaystyle\int_\Omega u_x(t, x)(u_{xt}(t, x) - v_x(x))dx \le 0, \quad \forall v \in K,\ \text{a.e. } t \in (0, T), \\ u(0, x) = 0, \quad x \in \Omega, \end{cases} \tag{34}$$

where $0 < T < +\infty$.

Here, for each $t \in [0, T]$ we consider the following convex functions:

$$\psi^t(z) = I_K(z), \quad \varphi^t(z) = \frac{1}{2}|z|_V^2, \quad \forall z \in V.$$

Then we have:

1. $z^* \in \partial_* \psi^t(z)$ if and only if $z^* \in V^*,\ z \in K$ and $\langle z^*, v - z\rangle \le 0$ for all $v \in K$,
2. $\langle \partial_* \varphi^t(z), v\rangle = \int_\Omega z(x)v(x)dx + \int_\Omega z_x(x)v_x(x)dx$ for all $v,\ z \in V$,

and problem (34) is reformulated as (P;0, 0) with $g(t, z) = -z$. Therefore, applying Theorem 1, problem (34) has at least one solution u.

Moreover, for each constant $c \in (0, 1)$ the function u_c defined by

$$u_c(t, x) := c(1 - \exp(-t)) \ \text{ for all } (t, x) \in (0, T) \times \Omega$$

is a solution to (34). Indeed, we observe that

$$(u_c)_t(t, x) = c\exp(-t) \in K, \quad (u_c)_x(t, x) = 0, \quad (u_c)_{xt}(t, x) = 0$$

for all $(t, x) \in (0, T) \times \Omega$. Therefore, for each $c \in (0, 1)$, (34) is satisfied. Hence $\{u_c\}_{c\in(0,1)}$ provides with an infinite family of solutions to (34).

Now, we prove Theorem 2 concerning the uniqueness of solutions to (P;f, u_0) under the additional condition (A4) of strict monotonicity of $\partial_* \psi^t$.

Proof (Proof of Theorem 2) Let u_i, $i = 1, 2$, be two solutions to (P;f, u_0) on $[0, T]$. Subtract (P;f, u_0) for $i = 2$ from the one for $i = 1$, and multiply it by $u_1' - u_2'$. Then:

$$\begin{aligned} &\langle \xi_1(t) - \xi_2(t), u_1'(t) - u_2'(t)\rangle + \langle \partial_*\varphi^t(u_1(t)) - \partial_*\varphi^t(u_2(t)), u_1'(t) - u_2'(t)\rangle \\ &\quad + \langle g(t, u_1(t)) - g(t, u_2(t)), u_1'(t) - u_2'(t)\rangle = 0 \quad \text{ for a.e. } t \in (0, T), \end{aligned} \tag{35}$$

where $\xi_i(t) \in \partial_*\psi^t(u_i'(t))$ for a.e. $t \in (0, T)$ $(i = 1, 2)$. From (A4) we observe that

$$\langle \xi_1(t) - \xi_2(t), u_1'(t) - u_2'(t)\rangle \geq C_5|u_1'(t) - u_2'(t)|_V^2 \ \text{ for a.e. } t \in (0, T) \tag{36}$$

and by Lemma 1 that

$$\begin{aligned} &\langle \partial_*\varphi^t(u_1(t)) - \partial_*\varphi^t(u_2(t)), u_1'(t) - u_2'(t)\rangle \\ &= \langle \partial_*\varphi^t(u_1(t) - u_2(t)), u_1'(t) - u_2'(t)\rangle \\ &\geq \frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) - |\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) \quad \text{ for a.e. } t \in (0, T). \end{aligned} \tag{37}$$

Therefore, we observe from (35)–(37) and (C2) with the help of the Schwarz inequality that

$$\begin{aligned} &C_5|u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) \\ &\leq |\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + |g(t, u_1(t)) - g(t, u_2(t))|_{V^*}|u_1'(t) - u_2'(t)|_V \\ &\leq |\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + \frac{1}{2C_5}|g(t, u_1(t)) - g(t, u_2(t))|_{V^*}^2 + \frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 \\ &\leq |\alpha'(t)|\varphi^t(u_1(t) - u_2(t)) + \frac{L_g^2}{2C_5}|u_1(t) - u_2(t)|_V^2 + \frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 \end{aligned}$$

for a.e. $t \in (0, T)$. From the above inequality we infer that

$$\begin{aligned} &\frac{C_5}{2}|u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt}\varphi^t(u_1(t) - u_2(t)) \\ &\leq K_1(|\alpha'(t)| + 1)\varphi^t(u_1(t) - u_2(t)) \quad \text{ for a.e. } t \in (0, T), \end{aligned} \tag{38}$$

for some constant $K_1 > 0$ being independent of u_i $(i = 1, 2)$. Hence, applying the Gronwall inequality to (38), we conclude that

$$u_1(t) - u_2(t) = 0 \ \text{ in } V \text{ for all } t \in [0, T].$$

Thus the proof of Theorem 2 has been completed. □

5 Doubly Nonlinear Quasi-Variational Inequality

In this section we discuss a doubly nonlinear quasi-variational inequality of the form:

$$(\mathrm{QP}; f, u_0)\begin{cases} \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \text{ in } V^* \text{ for a.e. } t \in (0, T), \\ u(0) = u_0 \text{ in } V, \end{cases}$$

where $\psi^t(z)$ and $g(t, z)$ are the same ones as before, and $\varphi^t(v; z)$ is precisely formulated below.

(Assumption (B'))

Putting

$$D_0 := \left\{ v \in W^{1,2}(0, T; V) \;\middle|\; \int_0^T \psi^t(v'(t))dt < \infty \right\},$$

we define a functional $\varphi^t : [0, T] \times D_0 \times V \to \mathbb{R}$ such that $\varphi^t(v; z)$ is non-negative, finite, continuous and convex in $z \in V$ for any $t \in [0, T]$ and any $v \in D_0$, and

$$\varphi^t(v_1; z) = \varphi^t(v_2; z), \ \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0, t],$$

for $v_i \in D_0,\ i = 1, 2$, and assume:

(B1') The subdifferential $\partial_* \varphi^t(v; z)$ of $\varphi^t(v; z)$ with respect to $z \in V$ is linear and bounded from $D(\partial_* \varphi^t(v; \cdot)) = V$ into V^* for each $t \in [0, T]$ and $v \in D_0$, and there is a positive constant C_3' such that

$$|\partial_* \varphi^t(v; z)|_{V^*} \leq C_3' |z|_V, \ \ \forall z \in V, \ \forall v \in D_0, \ \forall t \in [0, T].$$

(B2') If $\{v_n\} \subset D_0$, $\sup_{n \in \mathbb{N}} \int_0^T \psi^t(v_n'(t))dt < \infty$ and $v_n \to v \in C([0, T]; H)$ (as $n \to \infty$), then

$$\partial_* \varphi^t(v_n; z) \to \partial_* \varphi^t(v; z) \ \text{ in } V^*, \ \forall z \in V, \ \forall t \in [0, T] \ \text{ as } n \to \infty.$$

(B3') $\varphi^t(v; 0) = 0$ for all $v \in D_0$ and $t \in [0, T]$. There is a positive constant C_4' such that

$$\varphi^t(v; z) \geq C_4' |z|_V^2, \ \ \forall z \in V, \ \forall v \in D_0, \ \forall t \in [0, T].$$

(B4') There is a function $\alpha \in W^{1,1}(0, T)$ such that

$$|\varphi^t(v; z) - \varphi^s(v; z)| \leq |\alpha(t) - \alpha(s)| \varphi^s(v; z)$$
$$\forall z \in V, \ \ \forall v \in D_0, \ \ \forall s, \ t \in [0, T].$$

We now state the final main theorem of this paper.

Theorem 3 *Suppose that Assumptions (A), (B') and (C) are fulfilled. Let f be any function in $L^2(0, T; V^*)$ and u_0 be any element in V such that*

$$u_0 \in D(\varphi^0(\tilde{v}; \cdot)) \text{ for some } \tilde{v} \in D_0 \textit{ with } \tilde{v}(0) = u_0.$$

Then (QP;f, u_0) *admits at least one solution $u : [0, T] \to V$ in the sense that:*

(i) $u \in D_0$ *with* $u(0) = u_0$ *in* V,
(ii) there is $\xi \in L^2(0, T; V)$ *such that* $\xi(t) \in \partial_* \psi^t(u'(t))$ *in* V^* *for a.e.* $t \in (0, T)$ *and*

$$\xi(t) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) = f(t) \textit{ in } V^* \textit{ for a.e. } t \in (0, T).$$

Proof Let ε be a fixed positive constant in $(0, 1]$ and consider the Cauchy problem for any given $v \in D_0$:

$$\begin{cases} \varepsilon F u'(t) + \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(v; u(t)) + g(t, u(t)) \ni f(t) \text{ in } V^* \\ \hfill \text{for a.e. } t \in (0, T), \\ u(0) = u_0 \text{ in } V. \end{cases} \tag{39}$$

Then, by virtue of Theorems 1 and 2, problem (39) possesses one and only one solution u in the same sense of Definition 1, enjoying the estimate

$$\begin{aligned} &\int_0^T \{\varepsilon |u'(t)|_V^2 + \psi^t(u'(t))\} dt + \sup_{t \in [0,T]} \varphi^t(v; u(t)) \\ &\qquad \le N_0 := N_0(\varphi^0(v; u_0), |f|_{L^2(0,T;V^*)}, |\alpha'|_{L^1(0,T)}). \end{aligned} \tag{40}$$

Now, putting

$$X(u_0) := \left\{ v \in W^{1,2}(0, T; V) \;\middle|\; v(0) = u_0, \; \int_0^T \psi^t(v'(t)) dt \le N_0 \right\},$$

we define a mapping $\mathscr{S} : X(u_0) \to X(u_0)$ which maps each $v \in X(u_0) \subset D_0$ to the unique solution u of (39), namely $\mathscr{S} v = u$; note from (40) that $u \in X(u_0)$. Clearly $X(u_0)$ is non-empty, convex and compact in $C([0, T]; H)$.

Next we show that $\mathscr{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0, T]; H)$. Let $v \in C([0, T]; H)$, and let $\{v_n\}$ be a sequence in $X(u_0)$ such that $v_n \to v$ in $C([0, T]; H)$ (as $n \to \infty$), and put $u_n = \mathscr{S} v_n$. Then we see that $v \in X(u_0)$, $v_n \to v$ weakly in $W^{1,2}(0, T; V)$ and $\sup_{n \in \mathbb{N}} \int_0^T \psi^t(v_n'(t)) dt \le N_0$. From (40) it follows that there is a subsequence of $\{u_n\}$ (not relabeled) and a function $u \in W^{1,2}(0, T; V)$ such that

$$u_n \to u \text{ in } C([0, T]; H), \text{ weakly in } W^{1,2}(0, T; V) \text{ as } n \to \infty$$

and

$$u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0, T] \text{ as } n \to \infty.$$

Also, we have

$$\varepsilon F u_n'(t) + \partial_* \psi^t(u_n'(t)) + \partial_* \varphi^t(v_n; u_n(t)) + g(t, u_n(t)) \ni f(t) \text{ in } V^* \quad \text{for a.e. } t \in (0, T). \tag{41}$$

Just as (30) in the proof of Proposition 1, we obtain by multiplying (41) for $t = s$ by $u_n'(s) - u'(s)$ and using (3) that

$$\begin{aligned} &\varepsilon C_F |u_n'(s) - u'(s)|_V^2 + \frac{d}{ds}\varphi^s(v_n; u_n(s) - u(s)) \\ &\leq |\alpha'(s)|\varphi^s(v_n; u_n(s) - u(s)) + \bar{L}_n(s) \quad \text{for a.e. } s \in (0, T), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \bar{L}_n(s) = &\langle f(s) - g(s, u_n(s)) - \partial_* \varphi^s(v_n; u(s)), u_n'(s) - u'(s) \rangle \\ &- \varepsilon \langle F u'(s), u_n'(s) - u'(s) \rangle + \psi^s(u'(s)) - \psi^s(u_n'(s)) \quad \text{for a.e. } s \in (0, T). \end{aligned}$$

Since $g(\cdot, u_n) \to g(\cdot, u)$ and $\partial_* \varphi^{(\cdot)}(v_n; u) \to \partial_* \varphi^{(\cdot)}(v; u)$ (strongly) in $L^2(0, T; V^*)$ by conditions (C1), (B2') and the functional $w \to \int_0^t \psi^s(w(s))ds$ is lower semicontinuous on $L^2(0, T; V)$, it follows that

$$\limsup_{n \to \infty} \int_0^t \bar{L}_n(s)ds \leq 0, \quad \forall t \in [0, T],$$

so that applying the Gronwall inequality to (42) yields that

$$\limsup_{n \to \infty} \varphi^t(v_n; u_n(t) - u(t)) \leq 0, \text{ i.e. } u_n(t) \to u(t) \text{ in } V, \ \forall t \in [0, T]$$

and $u_n' \to u'$ in $L^2(0, T; V)$ as $n \to \infty$. This implies from (B1') and (B2') that $\partial_* \varphi^t(v_n; u_n(t)) \to \partial_* \varphi^t(v; u(t))$ in V^* for all $t \in [0, T]$, whence

$$\begin{aligned} \varepsilon F u_n'(t) + \partial_* \psi^t(u_n'(t)) \ni \xi_n(t) &:= f(t) - \partial_* \varphi^t(v_n; u_n(t)) - g(t, u_n(t)) \\ &\to f(t) - \partial_* \varphi^t(v; u(t)) - g(t, u(t)) =: \xi(t) \text{ in } V^* \end{aligned}$$

for a.e. $t \in [0, T]$ as $n \to \infty$. Accordingly, by the demi-closedness of maximal monotone mappings, we have $\xi(t) \in \varepsilon F u'(t) + \partial_* \psi^t(u'(t))$ for a.e. $t \in [0, T]$. As a consequence, u satisfies (39), namely $u = \mathscr{S} v$. By the uniqueness of solution to (39) we conclude that $\mathscr{S} v_n = u_n \to u = \mathscr{S} v$ in $C([0, T]; H)$ without extracting any subsequence from $\{u_n\}$. Thus $\mathscr{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0,T];H)$. Therefore, by the Schauder fixed point theorem, $\mathscr{S}$ has at least one fixed point u in $X(u_0)$. This is a solution of (39) with $v = u$.

We showed above that for every small $\varepsilon > 0$ the Cauchy problem

$$\begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \text{ in } V^* \\ \hfill \text{for a.e. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V \end{cases}$$

admits at least one solution $u_\varepsilon \in W^{1,2}(0,T;V)$ enjoying estimate

$$\varepsilon \int_0^T |u_\varepsilon'(t)|_V^2 dt + \int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon; u_\varepsilon(t)) \le N_0, \quad \forall \varepsilon \in (0,1].$$

Therefore, we can choose a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ (as $n \to \infty$) and a function $u \in D_0$ so that

$$\begin{gathered} u_n := u_{\varepsilon_n} \to u \text{ in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;V) \text{ as } n \to \infty, \\ u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty, \\ \varepsilon_n u_n' \to 0 \text{ in } L^2(0,T;V) \text{ as } n \to \infty, \\ \sup_{n\in\mathbb{N}} \int_0^T \psi^t(u_n'(t))dt \le N_0. \end{gathered}$$

Now, in the same way just as in the convergence proof of Theorem 1 again, we can infer from (B2') and (C1) that the limit u satisfies

$$\begin{cases} \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.e. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V. \end{cases}$$

Thus u is a required solution to (QP;f, u_0). □

6 Applications

In this section, we consider two applications of the general results (Theorems 1 and 3).

Let Ω be a bounded domain in $\mathbb{R}^N$ ($1 \le N < \infty$) with a smooth boundary $\Gamma := \partial\Omega$, and let us set

$$V := H_0^1(\Omega), \quad H := L^2(\Omega);$$

note that condition (3) is satisfied with $C_F = 1$.

(Application 1)

Let $T > 0$ be a fixed real number, and let $Q := (0,T) \times \Omega$. Also, let ρ be a prescribed obstacle function in $C(\overline{Q})$ such that

$$(0 <)\rho_* \le \rho(t,x) \le \rho^*, \quad \forall (t,x) \in \overline{Q}, \tag{43}$$

where ρ_* and ρ^* are positive constants.

Now, for each $t \in [0,T]$ define a closed convex set $K(t)$ in V by

$$K(t) := \{z \in V\ ;\ |\nabla z(x)| \le \rho(t,x) \text{ for a.e. } x \in \Omega\}.$$

Then, our variational inequality with constraint is of the form:

$$\left.\begin{array}{c} u_t(t) \in K(t) \ \text{ for a.e. } t \in (0,T), \\ \displaystyle\int_\Omega a(t,x)\nabla u(t,x)\cdot\nabla(u_t(t,x)-v(x))dx + \int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \\ \displaystyle\le \int_\Omega f(t)(u_t(t,x)-v(x))dx \quad \text{for all } v \in K(t) \text{ and a.e. } t \in (0,T), \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{array}\right\} \tag{44}$$

where $g(\cdot,\cdot)$ is a Lipschitz continuous function on $[0,T]\times\mathbb{R}$, f is a function given in $L^2(0,T;H)$, u_0 is an initial datum in V, and $a(\cdot,\cdot)$ is a prescribed function on Q such that

$$(0 <)a_* \le a(t,x) \le a^*, \ \forall (t,x) \in \overline{Q}, \qquad a = a(t) \in W^{1,1}(0,T;C(\overline{\Omega})),$$

where a_* and a^* are positive constants.

Now we show the existence of a solution to (44) on $[0,T]$ by applying the abstract result Theorem 1. To this end, for each $t \in [0,T]$ define proper l.s.c. and convex functions ψ^t, φ^t on V and $\alpha(t)$ by

$$\psi^t(z) := I_{K(t)}(z) = \begin{cases} 0, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise}, \end{cases}, \quad \forall z \in V,\ \forall t \in [0,T], \tag{45}$$

$$\varphi^t(z) := \frac{1}{2}\int_\Omega a(t,x)|\nabla z(x)|^2 dx, \quad \forall z \in V,\ \forall t \in [0,T] \tag{46}$$

and

$$\alpha(t) := \frac{1}{a_*}\int_0^t \left|\frac{\partial}{\partial\tau}a(\tau,x)\right| d\tau, \quad \forall t \in [0,T]. \tag{47}$$

We see easily that

$$z^* \in \partial_* \psi^t(z) \Longleftrightarrow z^* \in V^*,\ z \in K(t) \text{ and } \langle z^*, v - z\rangle \le 0, \quad \forall v \in K(t) \tag{48}$$

and

$$\langle \partial_* \varphi^t(z), v\rangle = \int_\Omega a(t,x)\nabla z(x) \cdot \nabla v(x)dx, \quad \forall z,\ v \in V \tag{49}$$

for all $t \in [0, T]$. In our present case it is easy to check *Assumptions* (A)–(C), except for (A1). We prove (A1) in the following lemma.

Lemma 2 (cf. [11, Lemma 10.1]) *For any sequence* $\{t_n\} \subset [0, T]$ *with* $t_n \to t$ *(as* $n \to \infty$*),* ψ^{t_n} *converges to* ψ^t *on* V *in the sense of Mosco as* $n \to \infty$*.*

Proof Assume that

$$\{z_n\} \subset V, z_n \to z \text{ weakly in } V \text{ and } \liminf_{n\to\infty} \psi^{t_n}(z_n) < \infty. \tag{50}$$

We may assume that $z_n \in K(t_n)$ for all n. By definition

$$|\nabla z_n(x)| \le \rho(t_n, x), \quad \text{a.e. } x \in \Omega. \tag{51}$$

Also, by $\rho \in C(\overline{Q})$, given $\varepsilon > 0$, there exists a positive integer n_ε such that

$$\rho(t_n, x) \le \rho(t, x) + \varepsilon \ \text{ for all } x \in \Omega \text{ and all } n \ge n_\varepsilon. \tag{52}$$

Therefore, it follows from (51) and (52) that

$$|\nabla z_n(x)| \le \rho(t, x) + \varepsilon, \quad \text{a.e. } x \in \Omega \text{ and all } n \ge n_\varepsilon,$$

which implies that

$$z_n \in K_\varepsilon(t) := \{z \in V\ ;\ |\nabla z(x)| \le \rho(t, x) + \varepsilon,\ \text{a.e. } x \in \Omega\} \ \text{ for all } n \ge n_\varepsilon. \tag{53}$$

Note that $K_\varepsilon(t)$ is weakly compact in V, since the set $K_\varepsilon(t)$ is bounded, closed and convex in V. Therefore, it follows from (50) and (53) that

$$z \in K_\varepsilon(t).$$

Since ε is arbitrary, we have $z \in K(t)$. Hence, we observe that

$$\liminf_{n\to\infty} \psi^{t_n}(z_n) = 0 = \psi^t(z).$$

Next, we verify another condition of the Mosco convergence. To this end, assume $z \in K(t)$. Note from $\rho \in C(\overline{Q})$ that for each k, choose a positive integer N_k so that $N_k \geq k$ and

$$\rho(t,x) \leq \rho(t_n,x) + \frac{\rho_*}{k} \quad \text{for all } x \in \Omega \text{ and all } n \geq N_k. \tag{54}$$

Then, we observe from $z \in K(t)$, (43) and (54) that

$$|\nabla z(x)| \leq \rho(t,x) \leq \rho(t_n,x) + \frac{\rho_*}{k} \leq \left(1 + \frac{1}{k}\right)\rho(t_n,x),$$

for a.e. $x \in \Omega$ and all $n \geq N_k$, which implies that

$$\left|\nabla\left(\frac{1}{1+\frac{1}{k}}z(x)\right)\right| \leq \rho(t_n,x), \quad \text{a.e. } x \in \Omega \text{ and all } n \geq N_k. \tag{55}$$

Putting

$$z_n := \begin{cases} \dfrac{1}{1+\frac{1}{k}}z, & \text{if } n \geq N_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{if } 1 \leq n < N_1, \end{cases}$$

we observe from (55) and $z \in K(t)$ that $t_n \to t$ as $n \to \infty$,

$$K(t_n) \ni z_n \to z \ \text{ in } V \ \text{ as } n \to \infty$$

and

$$\lim_{n\to\infty} \psi^{t_n}(z_n) = 0 = \psi^t(z).$$

Thus, ψ^{t_n} converges to ψ^t on V in the sense of Mosco. □

Taking account of (45)–(49), problem (44) can be reformulated in the abstract form (P;f, u_0). Therefore, by Theorem 1, problem (44) admits a solution $u \in W^{1,2}(0,T;V)$.

(Application 2)

Let us consider problem (44) with the diffusion coefficient $a(t,x)$ replaced by $a(t,x,u)$, namely

$$\left.\begin{array}{c} u_t(t) \in K(t) \text{ for a.e. } t \in (0,T), \\ \displaystyle\int_\Omega a(t,x,u(t,x))\nabla u(t,x)\cdot\nabla(u_t(t,x)-v(x))dx \\ \displaystyle + \int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \le \int_\Omega f(t)(u_t(t,x)-v(x))dx \\ \text{for all } v\in K(t) \text{ and a.e. } t\in(0,T), \\ u(0,x)=u_0(x), \quad x\in\Omega, \end{array}\right\} \tag{56}$$

where $K(t)$, f and u_0 are the same as in Application 1; the obstacle function ρ satisfies (43) as well. As to the function $a(t,x,r)$ we suppose that

$$\begin{cases} (0<)a_* \le a(t,x,r) \le a^*, \quad \forall(t,x)\in\overline{Q},\ \forall r\in\mathbb{R}, \\ |a(t_1,x,r_1)-a(t_2,x,r_2)| \le L_a(|t_1-t_2|+|r_1-r_2|), \\ \qquad \forall t_i\in[0,T],\ r_i\in\mathbb{R},\ i=1,2,\ \forall x\in\overline{\Omega}, \end{cases} \tag{57}$$

where $a_*,\ a^*$ and L_a are positive constants. Also, condition (43) is assumed and ψ^t is defined by (45) as well. Furthermore the (t,v)-dependent functional $\varphi^t(v;z)$ is given by

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(t,x,v(t,x))|\nabla z(x)|^2dx, \quad \forall t\in[0,T],\ \forall v\in D_0,\ \forall z\in V, \tag{58}$$

where

$$D_0 = \{v\in W^{1,2}(0,T;V) \mid v'(t)\in K(t) \text{ for a.e. } t\in[0,T]\}.$$

The subdifferential $\partial_*\varphi^t(v;\cdot)$ of $\varphi^t(v;\cdot)$ is given by

$$\langle\partial_*\varphi^t(v;z),w\rangle = \int_\Omega a(t,x,v(t,x))\nabla z(x)\cdot\nabla w(x)dx \tag{59}$$

for all $t\in[0,T]$, $v\in D_0$ and $z,\ w\in V$. Note from (43) that

$$|\nabla v'(t,x)| \le \rho^* \text{ for a.e. } (t,x)\in Q,$$

which implies that

$$\sup_{t\in[0,T]} |v'(t)|_{L^\infty(\Omega)} \le \bar\rho^*, \ \forall v\in D_0, \quad \text{for some constant } \bar\rho^*>0. \tag{60}$$

Therefore, it is easy to check by (57) that *Assumption (B')* holds with

$$C'_3 := a^*,\ C'_4 := \frac{1}{2}a_*,\ \alpha(t) := \frac{1}{a_*}L_a(1+\bar{\rho}^*)t.$$

In fact, (B1') and (B3') are immediately seen from the definition of $\varphi^t(v,z)$. Also, if $v_n \in D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi^t(v'_n(t))dt < \infty$ and $v_n \to v$ in $C([0,T];H)$, then we have

$$\begin{aligned}
&|\langle \partial_*\varphi^t(v_n;z) - \partial_*\varphi^t(v;z), w\rangle| \\
&\le \int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))||\nabla z(x)||\nabla w(x)|dx \\
&\le \left(\int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))|^2|\nabla z(x)|^2dx\right)^{\frac{1}{2}} |w|_V
\end{aligned}$$

and the last integral converges to 0 by the Lebesgue dominated convergence theorem, so that $\partial_*\varphi^t(v_n;z) \to \partial_*\varphi^t(v;z)$ (strongly) in V^*. Thus (B2') holds. Condition (B4') is verified by using (43), (57) and (60) as follows:

$$\begin{aligned}
&|\varphi^t(v;z) - \varphi^s(v;z)| \\
&\le \frac{1}{2}\int_\Omega |a(t,x,v(t,x)) - a(s,x,v(s,x))||\nabla z(x)|^2dx \\
&\le \frac{1}{2}\int_\Omega \int_s^t |a_\tau(\tau,x,v(\tau,x)) + a_v(\tau,x,v(\tau,x))v_\tau(\tau,x)||\nabla z(x)|^2d\tau dx \\
&\le \frac{1}{a_*}(L_a + L_a\bar{\rho}^*)|t-s| \cdot \frac{1}{2}\int_\Omega a(s,x,v(s,x))|\nabla z(x)|^2dx \\
&= \frac{1}{a_*}L_a(1+\bar{\rho}^*)|t-s|\varphi^s(v;z),
\end{aligned}$$

where $a_\tau := \frac{\partial}{\partial\tau}a(\tau,x,v)$ and $a_v := \frac{\partial}{\partial v}a(\tau,x,v)$.

By (58)–(59) problem (56) can be described as

$$\begin{cases}
\partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u;u(t)) + g(t,u(t)) \ni f(t) \text{ in } V^*, \\
u(0) = u_0 \text{ in } V.
\end{cases}$$

By virtue of Theorem 3, this Cauchy problem admits a solution $u \in D_0$, so does problem (56).

Remark 4 (44) is the variational formulation of (P;f, u_0). It seems similar to hyperbolic variational problems and our abstract result might be evolved to the hyperbolic case. However, in this paper, we do not touch it, since the mathematical structure is essentially of parabolic or pseudo-parabolic type.

Remark 5 Problems (P;f, u_0) and (QP;f, u_0) have a wide class of real world applications, for instance, reaction-diffusion systems for multi-species bacteria and solid-liquid phase transition systems with partial irreversibility (cf. [3, 4]). Moreover, when such phenomena are considered in fluid flows, they are coupled with various variational inequalities of the Navier-Stokes type which can be described by our doubly nonlinear evolution equations, too.

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Boundedness of Solutions to a Degenerate Diffusion Equation

Pavel Krejčí

Abstract The diffusion equation with a bounded saturation range under the time derivative and with Robin boundary conditions is shown to admit a regular bounded solution provided that the saturation function and the permeability coefficient have controlled decay at infinity. The result remains valid even if Preisach hysteresis is present in the pressure-saturation relation. The method of proof is based on a Moser-Alikakos iteration scheme which is compatible with a generalized Preisach energy dissipation mechanism.

Keywords Degenerate parabolic equation • Diffusion • Hysteresis

AMS (MOS) Subject Classification 35K65, 47J40

1 Introduction

This note is devoted to the problem

$$\left.\begin{aligned} (f(x,v)+G[v])_t - \operatorname{div}(\mu(x,v)\nabla v) &= h(x,t) && \text{for } (x,t) \in \Omega \times (0,T), \\ \mu(x,v)\nabla v \cdot n &= b(x)(v^* - v) && \text{for } (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) &= v^0(x) && \text{for } x \in \Omega. \end{aligned}\right\} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a bounded Lipschitzian connected domain and $(0, T)$ is a time interval, $f, \mu : \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions, G is a Preisach hysteresis operator which is briefly introduced in Sect. B, n is the unit outward normal vector to $\partial\Omega$, and $h(x, t), b(x), v^*(x, t), v^0(x)$ are given data.

Such problems arise in the study of porous media flow with mechanical interaction, where v represents the capillary pressure, and $s = f(x, v) + G[v]$ is the degree of saturation. For natural reasons, s is bounded from below and from above by, say, 0 and 1, and this explains the term "degenerate" in the title. The function $\mu(x, v) > 0$ corresponds to the permeability of the porous body in the Darcy law, $b(x) \geq 0$ is the permeability of the boundary, $v^*(x, t)$ is the outer pressure, and $h(x, t)$ describes for example volume changes of the void space due to deformations of the solid matrix. For details about the derivation of the model see [2, 12], cf. also [24]. In the engineering literature, there is enough evidence about the presence of hysteresis in porous media flow, see [1, 14, 16], and this is the motivation why hysteresis appears here.

From the physical point of view, it would be more adequate to consider the permeability μ as a function of the concentration s. A mathematical study of the problem involving hysteresis under the divergence operator was carried out in [7, 8] with the result that existence of a solution was proved only if the permeability coefficient was further regularized in time or in space.

From the mathematical point of view, however, hysteresis is not a crucial issue in Problem (1.1) and represents a lower order effect provided sufficient regularity of solutions is available. Readers who are less enthusiastic about hysteresis operators may assume that $G = 0$, which is an admissible choice in Hypothesis 2.1 (iii) below, and skip Sect. B.

Problems of degenerate diffusion constitute a popular topic in the literature over several decades, starting from the most classical publications [4–6, 15] to more modern ones [11, 13], and this list is far from being complete. In particular, the problem treated in [13] is to a large extent much more general than here, but for example boundedness and higher regularity questions (which are essential for the possibility of including hysteresis) are not addressed there.

The text is divided into six sections. In Sect. 2 we specify the hypotheses and state the main results. The first step of the existence proof consists in truncating the nonlinearities and the right hand side in Sect. 3 and proving that the resulting cut off system admits a solution with the desired regularity. Section 4 is devoted to a Moser-Alikakos iteration scheme inspired by [3] which gives uniform estimates independent of the cut off parameters, and the proof of the main theorems on existence, regularity, and uniqueness of solutions is completed in Sect. 5. Sections A and B contain some auxiliary material about Sobolev embeddings and interpolations, and about Preisach hysteresis.

2 Statement of the Problem

We consider Problem (1.1) in variational form for every $\phi \in W^{1,2}(\Omega)$:

$$\left.\begin{aligned} &\int_{\Omega} ((f(x,v) + G[v])_t \phi + \mu(x,v)\nabla v \cdot \nabla\phi)\,\mathrm{d}x + \int_{\partial\Omega} b(x)(v - v^*)\phi\,\mathrm{d}s(x) \\ &\qquad = \int_{\Omega} h\phi\,\mathrm{d}x\,, \\ &v(x,0) = v^0(x) \quad \text{a.e.} \end{aligned}\right\} \tag{2.1}$$

under the following hypotheses.

Hypothesis 2.1 *The data of Problem (2.1) satisfy the following conditions.*

(i) $\mu : \Omega \times \mathbb{R} \to (0,\infty)$ *is Lipschitz continuous on bounded subsets of* $\Omega \times \mathbb{R}$, $\nabla_x \mu(x,\cdot)$ *is continuous for a. e.* $x \in \Omega$, *where* ∇_x *denotes the partial gradient with respect to x, and there exist constants* $\mu^\flat > 0$ *and* $\alpha \in [0,1)$, *and a nondecreasing function* $\mu^\sharp : \mathbb{R} \to (0,\infty)$ *such that*

$$\frac{\mu^\flat}{1+|v|^\alpha} \le \mu(x,v) \le \mu^\sharp(v) \;\; a.e., \tag{2.2}$$

$$|\nabla_x \mu(x,v)| \le \mu^\sharp(v) \;\; a.e.; \tag{2.3}$$

(ii) $f(x,\cdot)$ *is a locally Lipschitz continuous function for a. e.* $x \in \Omega$, $f(\cdot,v)$ *is measurable and bounded for every* $v \in \mathbb{R}$, *and there exists a constant* $c > 0$ *such that*

$$\frac{\partial f}{\partial v}(x,v) \ge \frac{c}{1+|v|^{1+(1-\alpha)(N/2)}} \;\; a.e.$$

with α *from (2.2);*

(iii) G *is a Preisach operator with a given density* $\psi \in L^\infty(\Omega; L^1((0,\infty)\times\mathbb{R}))$ *as in (B.5),* $\psi(x,r,v) \ge 0$ *a. e.;*

(iv) $h \in L^q(\Omega \times (0,T))$ *for some* $q > 1 + (N/2)$*;*

(v) $v^* \in L^\infty(\partial\Omega \times (0,T))$, $|v^*(x,t)| \le \bar{C}$ *a. e. for some* $\bar{C} \ge 1$*;*

(vi) $v^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, $|v^0(x)| \le \bar{C}$ *a. e.;*

(vii) $b \in L^\infty(\partial\Omega)$, $b(x) \ge 0$ *a. e.,* $\int_{\partial\Omega} b(x)\,\mathrm{d}s(x) > 0$.

If $\alpha = 0$, that is, if μ is bounded away from zero, we observe a remarkable coincidence of the exponents $1 + (N/2)$ in Hypotheses 2.1 (ii) and 2.1 (iv). In fact, condition (iv) is the same as in the isotropic special case of [22, Chapter III, §7] for $f(x,v) = v$. It might be interesting to check whether the decay rates in Hypotheses 2.1 (i), (ii) are optimal in general or not.

The main results of the paper read as follows.

Theorem 2.2 (Existence) *Let $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ be a bounded connected Lipschitzian domain, and let Hypothesis 2.1 hold. Then Problem (2.1) has a solution $v \in L^\infty(\Omega \times (0,T))$ with the regularity $v_t \in L^2(\Omega \times (0,T))$, $\nabla v \in L^\infty(0,T; L^2(\Omega))$.*

Theorem 2.3 (Uniqueness) *Let the hypotheses of Theorem 2.2 hold, and let μ have the form of a product $\mu(x,v) = \mu_1(x)\mu_2(v)$. Then the solution v to Problem (2.1) established in Theorem 2.2 is unique.*

Indeed, the L^∞ bound for v is the most technical part of the proof. In order to justify the argument, we first truncate the system, prove the existence and regularity of solutions to the cut-off system by means of an elementary Faedo-Galerkin scheme. Finally, using a kind of Moser-Alikakos iteration technique as in [3], we derive an L^∞ bound for the solution independent of the truncation parameters, so that the solution of the cut-off system satisfies the original system, too.

In [25, Section 4], another case is considered, namely that $f(x,v) = v$, $\mu(x,v)$ may decay polynomially with an arbitrary exponent, $b(x) = 0$, and $h \in L^\infty(0,T; L^q(\Omega))$ with some q sufficiently large. The degeneracy of f changes substantially the situation and it seems that the result of [25], also based on Moser-Alikakos iterations, cannot be directly applied here.

3 Cut Off

Let $K > \bar{C}$ be a constant, and let

$$Q_K(u) = \min\{K, \max\{u, -K\}\} \tag{3.1}$$

for $u \in \mathbb{R}$ be the projection of $\mathbb{R}$ onto the interval $[-K, K]$. We also introduce the dual mapping (dead zone function)

$$P_K(u) = u - Q_K(u) = \min\{u + K, \max\{0, u - K\}\}. \tag{3.2}$$

We choose another constant $R > 0$ and replace (2.1) with the system

$$\left.\begin{aligned} &\int_\Omega ((f_K(x,v) + G[v])_t \phi + \mu_K(x,v)\nabla v \cdot \nabla \phi)\,\mathrm{d}x + \int_{\partial\Omega} b(x)(v - v^*)\phi\,\mathrm{d}s(x) \\ &\qquad = \int_\Omega h_R \phi\,\mathrm{d}x\,, \\ &v(x,0) = v^0(x) \quad \text{a.e.}, \end{aligned}\right\} \tag{3.3}$$

where we set

$$f_K(x, v) = f(x, v) + P_K(v)\,, \tag{3.4}$$

$$\mu_K(x, v) = \mu(x, Q_K(v))\,, \tag{3.5}$$

$$h_R(x, t) = Q_R(h(x, t))\,. \tag{3.6}$$

We have indeed

$$\frac{\partial f_K}{\partial v} \geq c_K^f := \min\left\{1, \frac{c}{1 + K^{1+(1-\alpha)(N/2)}}\right\}. \tag{3.7}$$

For Problem (3.3), the partial Kirchhoff transform

$$w = M_K(x, v) := \int_0^v \mu_K(x, u)\, \mathrm{d}u \tag{3.8}$$

makes sense, since μ_K is bounded by positive constants from above and from below, so that the function $v \mapsto M_K(x, v)$ is Lipschitzian for all $x \in \Omega$ and admits a Lipschitzian partial inverse $v = M_K^{-1}(x, w)$. Put

$$\hat{f}_K(x, w) = f_K(x, M_K^{-1}(x, w))\,, \tag{3.9}$$

$$G_K[w] = G[(M_K^{-1}(x, w)]\,. \tag{3.10}$$

By virtue of Proposition B.2, G_K is still a Preisach operator, and we can rewrite (3.3) in terms of w as

$$\left.\begin{aligned} &\int_\Omega ((\hat{f}_K(x, w) + G_K[w])_t \phi + \nabla w \cdot \nabla \phi)\, \mathrm{d}x + \int_{\partial\Omega} b(x)(M_K^{-1}(x, w) - v^*)\phi\, \mathrm{d}s(x) \\ &\quad = \int_\Omega \nabla_x M_K(x, M_K^{-1}(x, w)) \cdot \nabla\phi\, \mathrm{d}x + \int_\Omega h_R \phi\, \mathrm{d}x\,, \\ &w(x, 0) = w^0(x) := M_K(x, v^0(x)) \quad \text{a. e.} \end{aligned}\right\} \tag{3.11}$$

Note that we have by (3.5), (3.8), (3.9), and (2.3) that

$$\begin{aligned} \frac{\partial \hat{f}_K}{\partial w}(x, w) &= \frac{\frac{\partial f_K}{\partial v}(x, M_K^{-1}(x, w))}{\mu_K(x, M_K^{-1}(x, w))} \\ &= \begin{cases} \dfrac{\frac{\partial f}{\partial v}(x, M_K^{-1}(x, w)) + 1}{\mu(x, K)} & \text{for } |M_K^{-1}(x, w)| \geq K\,, \\ \dfrac{\frac{\partial f}{\partial v}(x, M_K^{-1}(x, w))}{\mu(x, M_K^{-1}(x, w))} & \text{for } |M_K^{-1}(x, w)| < K\,, \end{cases} \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
\left|\nabla_x M_K(x, M_K^{-1}(x,w))\right| &= \left|\int_0^{M_K^{-1}(x,w)} \nabla_x \mu(x, Q_K(v))\,\mathrm{d}v\right| \\
&\le \mu^\sharp(K)\,|M_K^{-1}(x,w)|.
\end{aligned}
\tag{3.13}
$$

We see that there exist positive constants $f_K^\flat < f_K^\sharp$ such that

$$
f_K^\flat \le \frac{\partial \hat{f}_K}{\partial w}(x,w) \le f_K^\sharp \ \text{ a. e.} \tag{3.14}
$$

For this cut off problem, we have the following result.

Proposition 3.1 *Let the hypotheses of Theorem 2.2 be fulfilled. Then for every $R > K > \bar{C}$, Problem (3.11) has a solution $w \in L^\infty(\Omega \times (0,T))$ with the regularity $w_t \in L^2(\Omega \times (0,T))$, $\nabla w \in L^\infty(0,T; L^2(\Omega))$.*

As an immediate consequence, we have

Corollary 3.2 *Let the hypotheses of Theorem 2.2 be fulfilled. Then for every $R > K > \bar{C}$, Problem (3.3) has a solution $v \in L^\infty(\Omega \times (0,T))$ with the regularity $v_t \in L^2(\Omega \times (0,T))$, $\nabla v \in L^\infty(0,T; L^2(\Omega))$.*

Proof of Proposition 3.1 The solution to (3.11) will be constructed by Faedo-Galerkin approximations. We choose the orthonormal basis $\{e_j; j = 0, 1, 2, \dots\}$ of eigenfunctions of the homogeneous Neumann problem

$$
-\Delta e_j = \lambda_j e_j \ \text{ in } \ \Omega\,, \quad \nabla e_j \cdot n = 0 \ \text{ on } \ \partial\Omega\,, \tag{3.15}
$$

and approximate the solution w to (3.11) by a finite series

$$
w^{(m)}(x,t) = \sum_{j=0}^{m} w_j(t) e_j(x)\,. \tag{3.16}
$$

The functions $w_j(t)$ are defined to be solutions of the finite system

$$
\left.
\begin{aligned}
&\int_\Omega (\hat{f}_K(x, w^{(m)}) + G_K[w^{(m)}])_t e_j\,\mathrm{d}x + \lambda_j w_j \\
&\qquad + \int_{\partial\Omega} b(x)(M_K^{-1}(x, w^{(m)}) - v^*) e_j\,\mathrm{d}s(x) \\
&\quad = \int_\Omega \nabla_x M_K(x, M_K^{-1}(x, w^{(m)})) \cdot \nabla e_j\,\mathrm{d}x + \int_\Omega h_R e_j\,\mathrm{d}x\,, \\
&w_j(0) = \int_\Omega w^0(x) e_j(x)\,\mathrm{d}x
\end{aligned}
\right\}
\tag{3.17}
$$

for $j = 0, 1, \dots, m$. Note that for every $x \in \Omega$, the operator $w \mapsto f_K(x,w) + G_K[w]$ is invertible in the space $C[0,T]$ of continuous functions and its inverse is Lipschitz, see [10]. Hence, (3.17) can be viewed as an ODE system with a locally Lipschitzian right hand side, and we conclude that it admits a unique solution for every $m \in \mathbb{N}$.

We test (3.17) with $w_j'(t)$ and sum up over $j = 0, 1, \ldots, m$. By (3.14), Proposition B.2, and (B.8) we have

$$(\hat{f}_K(x, w^{(m)}) + G_K[w^{(m)}])_t w_t^{(m)} \geq f_K^\flat |w_t^{(m)}|^2 \text{ a. e.}$$

Furthermore, we have the identity

$$\nabla_x M_K(x, M_K^{-1}(x, w^{(m)})) \cdot \nabla w_t^{(m)} = \frac{\partial}{\partial t} \left(\nabla_x M_K(x, M_K^{-1}(x, w^{(m)})) \cdot \nabla w^{(m)} \right)$$
$$-\nabla_x \mu_K(x, M_K^{-1}(x, w^{(m)})) \frac{\partial M_K^{-1}}{\partial w}(x, w^{(m)}) w_t^{(m)} \cdot \nabla w^{(m)} \text{ a. e.}$$

Using the inequality $|\nabla_x \mu_K(x, M_K^{-1}(x, w^{(m)}))| \leq \mu^\sharp(K)$ which follows from (2.3), we obtain the estimate

$$\nabla_x M_K(x, M_K^{-1}(x, w^{(m)})) \cdot \nabla w_t^{(m)} \leq \frac{\partial}{\partial t} \left(\nabla_x M_K(x, M_K^{-1}(x, w^{(m)})) \cdot \nabla w^{(m)} \right)$$
$$+ C_K |w_t^{(m)}| \, |\nabla w^{(m)}| \text{ a. e.}$$

Here and in what follows, C_K denotes any constant depending possibly on K and independent of R and m. The norm of h_R in $L^2(\Omega \times (0, T))$ is bounded above independently of R due to Hypothesis 2.1 (iv). Integrating in time from 0 to τ and using (3.13) thus yields for all $\tau \in [0, T]$ that

$$\int_0^\tau \int_\Omega |w_t^{(m)}|^2 \, dx \, dt + \int_\Omega |\nabla w^{(m)}|^2(x, \tau) \, dx + \int_{\partial\Omega} b(x) |w^{(m)}|^2 \, ds(x)$$
$$\leq C_K \left(1 + \int_0^\tau \int_\Omega |w_t^{(m)}| \, |\nabla w^{(m)}| \, dx \, dt + \int_\Omega |w^{(m)}| \, |\nabla w^{(m)}|(x, \tau) \, dx \right).$$

From the Gronwall argument we now obtain the estimate

$$\int_0^T \int_\Omega |w_t^{(m)}|^2 \, dx \, dt \leq C_K \, , \quad \sup_{t \in (0,T)} \int_\Omega |\nabla w^{(m)}|^2 \, dx \leq C_K$$

We find a subsequence of $\{w^{(m)}\}$, still indexed by m, and an element $w \in W^{1,2}(\Omega \times (0, T))$ such that $\nabla w \in L^\infty(0, T; L^2(\Omega))$, and

$$\left. \begin{aligned} w_t^{(m)} &\to w_t \quad \text{weakly in } L^2(\Omega \times (0, T)) \, , \\ \nabla w^{(m)} &\to \nabla w \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) \end{aligned} \right\} \tag{3.18}$$

as $m \to \infty$. By the compact embedding formulas (A.5)–(A.6), selecting again a subsequence, we may assume that

$$\left.\begin{aligned} w^{(m)} &\to w \text{ strongly in } L^2(\Omega; C[0,T])\,, \\ w^{(m)} &\to w \text{ strongly in } L^2(\partial\Omega \times (0,T))\,. \end{aligned}\right\} \tag{3.19}$$

Passing to the limit as $m \to \infty$ in (3.17) we easily check, using also Proposition B.1 in Sect. B below, that w is a solution to (3.11) and, consequently, $v = M_K^{-1}(x, w)$ is a solution to (3.3) and the estimates

$$\int_0^T \int_\Omega |v_t|^2 \,\mathrm{d}x\,\mathrm{d}t \le C_K\,, \qquad \sup_{t\in(0,T)} \int_\Omega |\nabla v|^2 \,\mathrm{d}x \le C_K \tag{3.20}$$

hold with constants C_K independent of R.

We now show that the values of $|v(x,t)|$ are bounded above almost everywhere by a constant. To this end, we put

$$\hat{R} = R \max\left\{1, \frac{1}{c_K^f}\right\} \tag{3.21}$$

with c_K^f as in (3.7), and test equation (3.3) with $\phi = H_\varepsilon(v(x,t) - \hat{R}(1+t))$, where H_ε is the function

$$H_\varepsilon(x) = \begin{cases} 0 & \text{for } x \le 0\,, \\ x/\varepsilon & \text{for } x \in (0,\varepsilon)\,, \\ 1 & \text{for } x \ge \varepsilon \end{cases} \tag{3.22}$$

with a given $\varepsilon > 0$. We have $\nabla v \cdot \nabla H_\varepsilon(v(x,t) - \hat{R}(1+t)) = (\nabla(v(x,t) - \hat{R}(1+t)) \cdot \nabla H_\varepsilon(v(x,t) - \hat{R}(1+t)) \ge 0$ a. e. for every $\varepsilon > 0$, and letting ε tend to 0 we obtain from (3.11) that

$$\begin{aligned} &\int_\Omega (f_K(x,v) + G[v])_t H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x \\ &\qquad + \int_{\partial\Omega} b(x)(v - v^*) H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}s(x) \\ &\le \int_\Omega R H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x, \end{aligned} \tag{3.23}$$

where H is the Heaviside function

$$H(x) = \begin{cases} 0 & \text{for } x \le 0\,, \\ 1 & \text{for } x > 0\,. \end{cases} \tag{3.24}$$

By the choice of $\hat{R}$ in (3.21), by Hypothesis 2.1 (v), and by the fact that $R > K > \bar{C}$, we see that if $v > \hat{R}$, then $v > v^*$, so that $(v - v^*)H(v(x,t) - \hat{R}(1+t)) \ge 0$ a. e. Furthermore, by the Hilpert inequality (B.9), we have

$$(G[v])_t H(v(x,t) - \hat{R}(1+t)) \ge (G[\hat{R}(1+t)])_t H(v(x,t) - \hat{R}(1+t)) + \frac{\partial}{\partial t}\int_0^\infty (\Psi(x,r,\mathfrak{p}_r[v]) - \Psi(x,r,\mathfrak{p}_r[\hat{R}(1+t)]))^+ \,\mathrm{d}r$$

for a. e. $(x,t) \in \Omega \times (0,T)$. From (B.8) it follows that $(G[\hat{R}(1+t)])_t \ge 0$ a. e., and from the choice of the initial state (B.1) of the play operators we infer that

$$(\Psi(x,r,\mathfrak{p}_r[v]) - \Psi(x,r,\mathfrak{p}_r[\hat{R}(1+t)]))^+\big|_{t=0} = 0$$

for all $r > 0$. Indeed, by (3.11) and Hypothesis 2.1 (vi) we have $|v_0(x)| \le \bar{C} < R$ a. e., so that for all $\tau > 0$ and a. e. $x \in \Omega$ we have

$$\int_0^\tau (G_K[v])_t H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}t \ge 0\,. \tag{3.25}$$

We thus obtain from (3.23) for all $\tau > 0$ that

$$\int_0^\tau \int_\Omega (f_K(x,v))_t H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_\Omega R H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x\,\mathrm{d}t, \tag{3.26}$$

hence,

$$\int_0^\tau \int_\Omega (f_K(x,v) - f_K(x,\hat{R}(1+t)))_t H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_\Omega (R - \hat{R}\frac{\partial f_K}{\partial v}(x,\hat{R}(1+t)))H(v(x,t) - \hat{R}(1+t))\,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.27}$$

The right hand side of (3.27) is non-positive by virtue of (3.21), and we conclude for all $\tau > 0$, using also Hypothesis 2.1 (vi), that

$$\int_\Omega (f_K(x,v(x,\tau)) - f_K(x,\hat{R}(1+\tau)))^+\,\mathrm{d}x \le \int_\Omega (f_K(x,v^0(x)) - f_K(x,\hat{R}))^+\,\mathrm{d}x = 0, \tag{3.28}$$

which yields the inequality

$$v(x,t) \le \hat{R}(1+t) \quad \text{a. e.} \tag{3.29}$$

The inequality $v(x,t) \geq -\hat{R}(1+t)$ is fully analogous, and the desired estimate

$$|v(x,t)| \leq \hat{R}(1+t) \quad \text{a. e.} \tag{3.30}$$

is thus proved. This completes the proof of Proposition 3.1. □

The next goal is to pass from the cut off system (3.3) to the original problem (2.1) by letting R tend to infinity and showing that there exists an L^∞ bound for the solution v to (3.3) *independent of R and K*. This will be done in the next section by a kind of the Moser-Alikakos technique, see [3].

4 Moser-Alikakos Iterations

Let $\Omega \subset \mathbb{R}^N$ be as in Sect. 2. We denote here and in the sequel by $|\cdot|_q$ the norm in $L^q(\Omega)$, and by $\|\cdot\|_q$ the norm in $L^q(\Omega \times (0,T))$. We first prove a technical lemma.

Lemma 4.1 *Let $q_0 = 1 + (N/2)$, $q_0' = 1 + (2/N)$, and let s, r be real numbers satisfying the inequalities*

$$\frac{1}{2} \leq s \leq r \leq \frac{N+2s}{N+2} \leq 1\,. \tag{4.1}$$

Assume that a function $v \in L^2(0,T;W^{1,2}(\Omega))$ satisfies for a. e. $t \in (0,T)$ the inequality

$$|v(t)|_{2s}^{2s} + \int_0^T |v(\tau)|_{W^{1,2}(\Omega)}^2 \,\mathrm{d}\tau \leq A \max\left\{B, \|v\|_{2rq'}\right\}^{2r} \tag{4.2}$$

for some $q' < q_0'$ and $A \geq 1$, $B \geq 1$. Then there exists a constant $C \geq 1$ independent of the choice of v, B, and A such that

$$\|v\|_{2rq_0'} \leq C A^{1/(2r)} \max\left\{B, \|v\|_{2rq'}\right\}. \tag{4.3}$$

Proof By virtue of (A.3), there exists a constant $C > 0$ independent of r, s, and of $v \in L^2(0,T;W^{1,2}(\Omega))$ such that

$$|v(t)|_{2rq_0'} \leq C|v(t)|_{2s}^{1-\rho}|v(t)|_{W^{1,2}(\Omega)}^{\rho} \qquad \text{for } t \in (0,T) \tag{4.4}$$

with

$$\rho = \frac{\frac{1}{s} - \frac{1}{rq_0'}}{\frac{1}{s} + \frac{2}{N} - 1}\,, \qquad 1 - \rho = \frac{\frac{1}{rq_0'} + \frac{2}{N} - 1}{\frac{1}{s} + \frac{2}{N} - 1}\,.$$

we easily check that

$$\frac{1}{s} - \frac{1}{rq_0'} \geq \frac{1}{r}\left(1 - \frac{1}{q_0'}\right) = \frac{2}{r(N+2)} > 0\,,$$

$$\frac{1}{rq_0'} + \frac{2}{N} - 1 = \frac{N}{r(N+2)} - \frac{N-2}{N} \geq \frac{N}{N+2} - \frac{N-2}{N} = \frac{4}{N(N+2)} > 0\,,$$

hence $\rho \in (0,1)$. Raising (4.4) to the power $2rq_0'$ and integrating in t yields

$$\|v\|_{2rq_0'}^{2rq_0'} \leq C^{2rq_0'} \sup_{t\in(0,T)} \mathrm{ess}\, |v(t)|_{2s}^{(1-\rho)2rq_0'} \int_0^T |v(t)|_{W^{1,2}(\Omega)}^{2\rho rq_0'}\, \mathrm{d}t. \tag{4.5}$$

We claim that $\rho rq_0' \leq 1$. Indeed,

$$\left(\frac{1}{s} + \frac{2}{N} - 1\right)\left(1 - \rho rq_0'\right) \geq \frac{1}{s} + \frac{2}{N} - 1 - \frac{rq_0'}{s} + 1 = \frac{2}{N} + \frac{1}{s}\left(1 - \frac{r(N+2)}{N}\right)$$
$$= \frac{1}{sN}(N + 2s - r(N+2)) \geq 0$$

by virtue of (4.1). Hence, by Hölder's inequality,

$$\|v\|_{2rq_0'}^{2rq_0'} \leq T^{1-\rho rq_0'} C^{2rq_0'} \sup_{t\in(0,T)} \mathrm{ess}\, |v(t)|_{2s}^{(1-\rho)2rq_0'} \|v\|_{L^2(0,T;W^{1,2}(\Omega))}^{2\rho rq_0'}, \tag{4.6}$$

that is,

$$\|v\|_{2rq_0'}^{2r} \leq T^{(1/q_0')-\rho r} C^{2r} \sup_{t\in(0,T)} \mathrm{ess}\, \left(|v(t)|_{2s}^{2s}\right)^{(1-\rho)r/s} \left(\|v\|_{L^2(0,T;W^{1,2}(\Omega))}^2\right)^{\rho r}. \tag{4.7}$$

We have the straightforward inequality

$$\omega := (1-\rho)\frac{r}{s} + \rho r \leq 1\,, \tag{4.8}$$

which follows from (4.1) and from the relations

$$1 - (1-\rho)\frac{r}{s} - \rho r = 1 - \frac{r}{s} + \frac{\frac{r}{s} - \frac{N}{N+2}}{\frac{1}{s} - \frac{N-2}{N}}\left(\frac{1}{s} - 1\right) = \frac{2(N + 2s - r(N+2))}{(N+2)(N(1-s)+2s)} \geq 0\,.$$

For all positive numbers a, b, c, d we have as a consequence of the Young inequality that

$$c^a d^b \leq \left(\frac{a}{a+b}c + \frac{b}{a+b}d\right)^{a+b},$$

so that (4.7) can be rewritten as

$$\|v\|_{2rq_0'}^{2r} \leq T^{(1/q_0')-\rho r} C^{2r} \left(\sup_{t\in(0,T)} \text{ess}\, |v(t)|_{2s}^{2s} + \|v\|_{L^2(0,T;W^{1,2}(\Omega))}^2 \right)^{\omega}, \tag{4.9}$$

which in turn implies, by Hypothesis (4.2), that

$$\|v\|_{2rq_0'}^{2r} \leq T^{(1/q_0')-\rho r} C^{2r} A^{\omega} \max \left\{B, \|v\|_{2rq'}\right\}^{2r\omega}, \tag{4.10}$$

and (4.3) follows. □

We now derive a series of estimates for the solution of (3.3). By C we denote any constant depending only on the data in Hypothesis 2.1 and independent of R and K. The estimates consist in testing Eq. (3.3) successively by higher and higher powers of v. This is indeed an admissible choice, since by Corollary 3.2 we have $v \in L^{\infty}(\Omega \times 0, T) \cap L^2(0, T; W^{1,2}(\Omega))$.

4.1 Estimate 1

Test (3.3) by $\phi = v \max\{1, |v|^{\alpha}\}$. Using the hysteresis energy inequality (B.10) with $\lambda(v) = v \max\{1, |v|^{\alpha}\}$ as well as the assumption (2.2) we immediately obtain

$$\int_0^T \int_{\Omega} |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\partial\Omega} b(x)|v|^2 \,\mathrm{d}s(x) \,\mathrm{d}t \leq C \left(1 + \|v\|_{q'(1+\alpha)}^{1+\alpha}\right), \tag{4.11}$$

and, by the Poincaré inequality (A.4) together with Hypothesis 2.1 (vi),

$$\|v\|_2^2 \leq C \left(1 + \|v\|_{q'(1+\alpha)}^{1+\alpha}\right), \tag{4.12}$$

where q is as in Hypothesis 2.1 (iv), and $q' = q/(q-1)$ is the conjugate exponent.

4.2 Estimate 2

Test (3.3) by $\phi = v|v|^{N+\alpha}$ and set

$$F_1(x, v) = \int_0^v \frac{\partial f}{\partial v}(x, u) u |u|^{N+\alpha} \,\mathrm{d}u.$$

By Hypothesis 2.1 (ii) we have for a. e. $|v| > 1$ that

$$\frac{\partial f}{\partial v}(x, v) \geq \frac{c}{2|v|^{1+(1-\alpha)(N/2)}},$$

so that

$$F_1(x, v) \geq \frac{c}{2(1+\alpha)(1+(N/2))} \left(|v|^{(1+\alpha)(1+(N/2))} - 1\right). \tag{4.13}$$

This inequality is trivially fulfilled for $|v| \leq 1$, since F_1 is positive, so that (4.13) holds for all $v \in \mathbb{R}$.

The dead zone term containing $P_K(v)$ in (3.4) gives a positive contribution on the left hand side and zero on the right hand side by virtue of Hypothesis 2.1 (vi). The hysteresis term can be again treated using the inequality (B.10), and we obtain

$$\begin{aligned}
&\max_{t\in[0,T]} \int_\Omega F_1(v)(x,t)\,\mathrm{d}x + \frac{\mu^\flat}{2}(N+1+\alpha) \int_0^T \int_\Omega |\nabla v|^2 |v|^N \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \int_{\partial\Omega} b(x)|v|^{N+2}\,\mathrm{d}s(x)\,\mathrm{d}t \\
&\quad \leq C + \int_0^T \int_\Omega |h||v|^{N+1+\alpha}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq C\left(1 + \left(\int_0^T \int_\Omega |v|^{q'(N+1+\alpha)}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/q'}\right).
\end{aligned} \tag{4.14}$$

In the boundary term on the left hand side of (4.14), we have indeed replaced $|v|^{N+2+\alpha} \geq |v|^{N+2} - 1$.

Put $u := v|v|^{N/2}$. We then obtain from (4.14) and (4.13) that

$$\begin{aligned}
&\max_{t\in[0,T]} \int_\Omega |u(x,t)|^{1+\alpha}\,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} b(x)|u|^2\,\mathrm{d}s(x)\,\mathrm{d}t \\
&\quad \leq C\left(1 + \left(\int_0^T \int_\Omega |u|^{q'\frac{2(N+1+\alpha)}{N+2}}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/q'}\right).
\end{aligned} \tag{4.15}$$

By virtue of the Poincaré inequality (A.4), we are in the situation of Lemma 4.1 with $2s = 1+\alpha$ and $r = (N+1+\alpha)/(N+2)$. The assertion of Lemma 4.1 then yields the estimate

$$\|v\|_{(N+1+\alpha)q_0'} \leq C \max\left\{1, \|v\|_{(N+1+\alpha)q'}\right\}. \tag{4.16}$$

Put $p_0 := (N+1+\alpha)q_0' = \frac{(N+1+\alpha)(N+2)}{N}$, $p := (N+1+\alpha)q' < p_0$. We either have $p \leq 2$, or $p > 2$. In the latter case, we use the interpolation inequality (A.1) with $p_1 = 2$ and obtain

$$\|v\|_{p_0} \leq C\left(1 + \|v\|_{p_0}^{\eta}\|v\|_2^{1-\eta}\right) \tag{4.17}$$

with $\eta \in (0,1)$ given by (A.2). In both cases we thus have by (4.12) that

$$\|v\|_{p_0} \le C\left(1 + \|v\|_{p_0}^{\eta}(1 + \|v\|_{q'(1+\alpha)}^{(1-\eta)(1+\alpha)/2}\right) \tag{4.18}$$

with some $\eta \in [0,1)$. We have indeed $q'(1+\alpha) < p_0$, so that (4.18) yields

$$\|v\|_{p_0} \le C\left(1 + \|v\|_{p_0}^{\eta+(1-\eta)(1+\alpha)/2}\right) \tag{4.19}$$

with $\eta + (1-\eta)\frac{1+\alpha}{2} < 1$, and we conclude that

$$\|v\|_{p_0} \le C. \tag{4.20}$$

4.3 Estimate 3

Test (3.3) by $\phi = v|v|^{2k+N+\alpha}$ for some $k > 0$ and set

$$F_k(x,v) = \int_0^v \frac{\partial f}{\partial v}(x,u)u|u|^{2k+N+\alpha}\,\mathrm{d}u.$$

Here, C will denote any constant independent of K, R, and k.

Arguing similarly as in (4.13) we obtain for all $v \in \mathbb{R}$ that

$$F_k(x,v) \ge \frac{c}{4k + (1+\alpha)(N+2)}\left(|v|^{2k+(1+\alpha)(1+(N/2))} - 1\right). \tag{4.21}$$

The hysteresis term can be again handled by the inequality (B.10), and the contribution due to the dead zone function $P_K(v)$ has again the right sign as in Estimate 4.2. The remaining terms yield

$$\begin{aligned}
\max_{t\in[0,T]} \int_\Omega F_k(x,v)(x,t)\,\mathrm{d}x + \frac{\mu^\flat}{2}(2k+N+1)\int_0^T\int_\Omega |\nabla v|^2|v|^{2k+N}\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_0^T\int_{\partial\Omega} b(x)|v|^{2k+N+2}\,\mathrm{d}s(x)\,\mathrm{d}t \\
\le C\left(\bar{C}^{2k+N+2+\alpha} + \int_0^T\int_\Omega |h||v|^{2k+N+1}\,\mathrm{d}x\,\mathrm{d}t\right) \\
\le C\left(\bar{C}^{2k+N+2+\alpha} + \left(\int_0^T\int_\Omega |v|^{q'(2k+N+1+\alpha)}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/q'}\right)
\end{aligned} \tag{4.22}$$

with $\bar{C}$ from Hypothesis 2.1. Put $u = v|v|^{k+(N/2)}$. Then

$$|\nabla u|^2 = (k + 1 + (N/2))^2 |\nabla v|^2 |v|^{2k+N},$$

and (4.22) can be rewritten as

$$\max_{t\in[0,T]} \int_\Omega |u|^{2s_k}(x,t)\,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega} b(x)|u|^2\,\mathrm{d}s(x)\,\mathrm{d}t$$
$$\leq C(1+k)\left(\bar{C}^{2k+N+1+\alpha} + \left(\int_0^T \int_\Omega |u|^{2r_k q'}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/q'}\right) \tag{4.23}$$

(note that $\bar{C}$ and N are constant), with

$$2s_k = \frac{2k + (1+\alpha)(1+(N/2))}{k+1+(N/2)}, \quad 2r_k = \frac{2k+N+1+\alpha}{k+1+(N/2)}. \tag{4.24}$$

We refer again to Lemma 4.1 with $B = \bar{C}^{(2k+N+1+\alpha)/2r_k} = \bar{C}^{k+1+(N/2)}$, $A = 2C(k+1)$, and $s = s_k$, $r = r_k$. Condition (4.1) is fulfilled thanks to the identity

$$\frac{N+2s_k}{N+2} = \frac{k + ((N+1+\alpha)/2)}{k+1+(N/2)} = r_k\,.$$

By Lemma 4.1 we have

$$\max\left\{\bar{C}^{k+1+(N/2)}, \|w\|_{2r_k q_0'}\right\} \leq C(1+k)\max\left\{\bar{C}^{k+1+(N/2)}, \|w\|_{2r_k q'}\right\}. \tag{4.25}$$

Passing back from $u = v|v|^{k+(N/2)}$ to v, we rewrite (4.25) as

$$\max\left\{\bar{C}, \|v\|_{q_k^0}\right\} \leq (C(1+k))^{1/(k+1+(N/2))} \max\left\{\bar{C}, \|v\|_{q_k}\right\}, \tag{4.26}$$

where

$$q_k^0 = q_0'(2k+N+1+\alpha), \quad q_k = q'(2k+N+1+\alpha)\,. \tag{4.27}$$

Put

$$\nu := \frac{q_0'}{q'} - 1 > 0\,, \tag{4.28}$$

and for $j = 0, 1, \ldots$ we define the sequences

$$k_j = \frac{N+1+\alpha}{2}\left((1+\nu)^j - 1\right), \quad p_j := q_{k_j}^0 = q_0'(N+1+\alpha)(1+\nu)^j\,. \tag{4.29}$$

For all $j \in \mathbb{N}$ we have

$$q_{k_j} = \frac{q^0_{k_j}}{1+\nu} = p_{j-1}. \tag{4.30}$$

For $k = k_j$, we can rewrite (4.26) in the form

$$\max\left\{\bar{C}, \|v\|_{p_j}\right\} \le (C(1+k_j))^{1/(k_j+1+(N/2))} \max\left\{\bar{C}, \|v\|_{p_{j-1}}\right\} \tag{4.31}$$

for every $j \in \mathbb{N}$. Put $z_j = \log\max\left\{\bar{C}, \|v\|_{p_j}\right\}$. Then (4.31) has the form

$$z_j - z_{j-1} \le \frac{1}{k_j + 1 + (N/2)} \log(C(1+k_j))\,. \tag{4.32}$$

There exists $\hat{\alpha} > 0$ such that for all $j \in \mathbb{N}$ we have $\hat{\alpha}(1+\nu)^{j-1} \le 1+k_j \le \hat{\alpha}(1+\nu)^j$. This enables us to put

$$C^* := \sum_{j=1}^{\infty} \frac{1}{k_j + 1 + (N/2)} \log(C(1+k_j)) \le \frac{1}{\hat{\alpha}} \sum_{j=1}^{\infty} \frac{j(1+\nu) + \log(\hat{\alpha}C)}{(1+\nu)^{j-1}} < \infty. \tag{4.33}$$

Then (4.32)–(4.33) yield that

$$z_j \le z_0 + C^*, \tag{4.34}$$

where z_0 is bounded independently of K and R by virtue of (4.20). It follows that $\|v\|_{p_j}$ are bounded by a constant independently of K, R, and $j \in \mathbb{N}$. Hence, there exists a constant $B > 0$ independent of K and R such that

$$\|v\|_\infty \le B\,. \tag{4.35}$$

5 Proofs of the Main Results

Proof of Theorem 2.2 We fix the value of $K = B + 1$ with B from (4.35), and let R tend to ∞ in Problem (3.3). Let v_R denote the solutions to (3.3) corresponding to different values of R. By the choice of K and by (4.35), we have that $P_K(v_R) = 0$, hence $Q_K(v_R) = v_R$ and $f_K(x, v_R) = f(x, v_R)$, $\mu_K(x, v_R) = \mu(x, v_R)$. We have the bounds (3.20), (4.35) independent of R and repeating the compactness argument of (3.18)–(3.19) we conclude that the limit v of a subsequence of v_R is a solution of (2.1) with the desired regularity. □

Proof of Theorem 2.3 Let $\mu(x, v) = \mu_1(x)\mu_2(v)$. The uniqueness argument is also based on the Hilpert inequality. Assume that v_1, v_2 are two solutions of (2.1). We test the difference of (2.1) written for v_1 and v_2 by $H_\varepsilon(M(v_1)-M(v_2))$ with $M(v) = \int_0^v \mu_2(u)\,\mathrm{d}u$ similarly as in (3.22)–(3.23). We have

$$\mu_1(x)(\mu_2(v_1)\nabla v_1 - \mu_2(v_2)\nabla v_2)\cdot\nabla H_\varepsilon(M(v_1)-M(v_2)) \geq 0$$

a. e., and letting $\varepsilon \to 0+$ we obtain

$$\int_\Omega (f(v_1) - f(v_2) + G[v_1] - G[v_1])_t H(M(v_1) - M(v_2))\,\mathrm{d}x + \int_{\partial\Omega} b(x)(v_1 - v_2)H(M(v_1) - M(v_2))\,\mathrm{d}s(x) \leq 0\,. \tag{5.1}$$

We have $H(M(v_1) - M(v_2)) = H(v_1 - v_2)$, so that the boundary integral term is nonnegative. Since the initial conditions for v_1 and v_2 coincide, we have by Hilpert's inequality (B.9) that

$$\int_0^\tau \int_\Omega (G[v_1] - G[v_2])_t H(v_1 - v_2)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Hence,

$$\begin{aligned} 0 &\geq \int_0^\tau \int_\Omega (f(x,v_1) - f_K(x,v_2))_t H(v_1 - v_2)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega (f(x,v_1) - f(x,v_2))_t H(f(x,v_1) - f(x,v_2))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega (f(x,v_1) - f(x,v_2))^+(x,\tau)\,\mathrm{d}x\,, \end{aligned} \tag{5.2}$$

which yields that $v_1(x,t) \leq v_2(x,t)$ a. e. Interchanging the indices 1 and 2 we get the opposite inequality, which implies uniqueness. □

A Auxiliary Section A: Embeddings and Interpolations

We summarize here a few standard results from the theory of Lebesgue spaces L^p and Sobolev spaces $W^{k,p}$ than all can be found, e. g., in the monograph [9]. Recall that we denote by $|\cdot|_p$ the norm in $L^p(\Omega)$, and by $\|\cdot\|_p$ the norm in $L^p(\Omega \times (0,T))$. We start with the *interpolation inequality* in Lebesgue spaces which we state in the following form.

Lemma A.1 *Let* $1 \leq p_1 < p < p_0 \leq \infty$ *be given. Then for every* $v \in L^{p_0}(\Omega)$ *we have*

$$|v|_p \leq |v|_{p_0}^\eta |v|_{p_1}^{1-\eta} \tag{A.1}$$

with

$$\eta = \frac{\frac{1}{p_1} - \frac{1}{p}}{\frac{1}{p_1} - \frac{1}{p_0}} \in (0,1). \tag{A.2}$$

Indeed, the same formula holds if Ω is replaced with $\Omega \times (0,T)$ and the norms $|\cdot|$ are replaced with $\|\cdot\|$.

We also refer to the *Gagliardo-Nirenberg inequality*:

Lemma A.2 *There exists a constant $C > 0$ such that for every $w \in W^{1,p}(\Omega)$ the inequality*

$$|w|_q \le C\left(|w|_s + |w|_s^{1-\rho}|\nabla w|_p^{\rho}\right) \tag{A.3}$$

with

$$\rho = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} + \frac{1}{N} - \frac{1}{p}} \in (0,1)$$

is valid for all $1 \le s < q$, $1/q > (1/p) - (1/N)$,
as well as to the *Poincaré inequality*:

Lemma A.3 *Under Hypothesis 2.1 (vii), there exists a constant $C > 0$ such that the inequality*

$$|v|_2^2 \le C\left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_{\partial\Omega} b(x)|v|^2 \,\mathrm{d}s(x)\right) \tag{A.4}$$

holds for every $v \in L^2(\Omega)$.

We define the space

$$X = \{w \in W^{1,2}(\Omega \times (0,T)) : \nabla w \in L^\infty(0,T; L^2(\Omega))\},$$

and denote by the symbol $\hookrightarrow\hookrightarrow$ the compact embedding relation. The following anisotropic compact embeddings are established in [9]:

$$X \hookrightarrow\hookrightarrow L^2(\Omega; C[0,T]), \tag{A.5}$$

and

$$X \hookrightarrow\hookrightarrow L^2(\partial\Omega \times (0,T)). \tag{A.6}$$

B Auxiliary Section B: The Preisach Operator

We use here the variational definition of the Preisach operator which was shown in [19] to equivalent to the original construction in [23]. It is based on the variational inequality for the unknown function ξ_r

$$\left.\begin{array}{ll}|v(t)-\xi_r(t)| \le r & \forall t \in [0,T]\,,\\ \dot\xi_r(t)(v(t)-\xi_r(t)-y) \ge 0 \ \text{ a. e. } & \forall |y| \le r\,,\\ \xi_r(0) = P_r(v(0))\,, & \end{array}\right\} \tag{B.1}$$

where $r > 0$ is a fixed constraint, P_r is the mapping defined by (3.2), $v \in W^{1,1}(0,T)$ is a given input, and the dot denotes time derivative. The mapping $\mathfrak{p}_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$ which with v associates the solution $\xi_r \in W^{1,1}(0,T)$ of (B.1) is called the *play operator* according to [18]. It is proved in [18, §2] that for arbitrary two inputs $v_1, v_2 \in W^{1,1}(0,T)$, the inequality

$$|\mathfrak{p}_r[v_1](t) - \mathfrak{p}_r[v_2](t)| \le \max_{\tau\in[0,t]} |v_1(\tau) - v_2(\tau)| \tag{B.2}$$

holds for all $t \in [0,T]$, hence the play operator can be extended into a Lipschitz continuous mapping $\mathfrak{p}_r : C[0,T] \to C[0,T]$. Furthermore, directly from the definition (B.1) we can infer that the identity

$$\dot\xi_r(t)\dot v(t) = (\dot\xi_r(t))^2 \tag{B.3}$$

holds a. e. for every $v \in W^{1,1}(0,T)$. For inputs v depending on $x \in \Omega$ and $t \in (0,T)$ we define the play operator $\mathfrak{p}_r : L^p(\Omega; C[0,T]) \to L^p(\Omega; C[0,T])$ for $1 \le p \le \infty$ by the formula

$$\mathfrak{p}_r[v](x,t) = \mathfrak{p}_r[v(x,\cdot](t) \tag{B.4}$$

and by virtue of (B.2), the play is Lipschitz continuous in $L^p(\Omega; C[0,T])$ for all $1 \le p \le \infty$.

Given a nonnegative function $\psi \in L^\infty(\Omega; L^1((0,\infty)\times\mathbb{R}))$, the Preisach operator $G : L^p(\Omega; C[0,T]) \to L^p(\Omega; C[0,T])$ is for $(x,t) \in \Omega\times[0,T]$ defined by the formula

$$G[v](x,t) = \int_0^\infty \int_0^{\mathfrak{p}_r[v](x,t)} \psi(x,r,z)\,\mathrm{d}z\,\mathrm{d}r\,, \tag{B.5}$$

and ψ is called the *Preisach density* of G. If we denote for $(x,r,v) \in \Omega\times(0,\infty)\times\mathbb{R}$

$$\Psi(x,r,v) = \int_0^v \psi(x,r,z)\,\mathrm{d}z\,, \tag{B.6}$$

then (B.5) can be written in the form

$$G[v](x,t) = \int_0^\infty \Psi(x, r, \mathfrak{p}_r[v](x,t))\, \mathrm{d}r\,. \tag{B.7}$$

The following statement is an easy consequence of (B.2).

Proposition B.1 *$G : L^p(\Omega; C[0,T]) \to L^p(\Omega; C[0,T])$ is Lipschitz continuous for every $1 \le p \le \infty$.*

From (B.3) it follows that for each $v \in L^2(\Omega; W^{1,1}(0,T))$ we have

$$(G[v])_t v_t \ge 0 \quad \text{a. e.} \tag{B.8}$$

The Preisach operator is monotone in the sense of *Hilpert's inequality*

$$(G[v_1]-G[v_2])_t H(v_1-v_2) \ge \frac{\partial}{\partial t}\int_0^\infty (\Psi(x,r,\mathfrak{p}_r[v_1]) - \Psi(x,r,\mathfrak{p}_r[v_2]))^+\, \mathrm{d}r \tag{B.9}$$

established in [17], which holds a. e. for all $v_1, v_2 \in L^2(\Omega; W^{1,1}(0,T))$ and where H is the Heaviside function (3.24) and $(\cdot)^+$ denotes the positive part. A different proof can be found in [21, Proposition II.2.12].

Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, $\lambda(0) = 0$, and let $v \in L^2(\Omega; W^{1,1}(0,T))$ be given. Put $\xi_r = \mathfrak{p}_r[v]$. By definition (B.1) of the play we have

$$(\xi_r)_t (v - \xi_r) \ge 0 \quad \text{a. e.},$$

hence,

$$(\xi_r)_t (\lambda(v) - \lambda(\xi_r)) \ge 0 \quad \text{a. e.}$$

It follows that

$$(G[v])_t \lambda(v) = \int_0^\infty \psi(x,r,\xi_r)(\xi_r)_t \lambda(v)\, \mathrm{d}r \ge \int_0^\infty \psi(x,r,\xi_r)(\xi_r)_t \lambda(\xi_r)\, \mathrm{d}r \quad \text{a. e.}$$

and we conclude that

$$(G[v])_t \lambda(v) \ge (U_\lambda[v])_t \quad \text{a. e.}, \tag{B.10}$$

where

$$U_\lambda[v] = \int_0^\infty \int_0^{\mathfrak{p}_r[v]} \psi(x,r,z)\lambda(z)\, \mathrm{d}z\, \mathrm{d}r\,. \tag{B.11}$$

This can be interpreted as a generalized hysteresis energy inequality with hysteresis potential U_λ, see [21, Chapter II].

Let us cite also the following result of [20].

Proposition B.2 *Let $J : \Omega \times \mathbb{R} \to \mathbb{R} : (x, v) \mapsto J(x, v)$ be a function such that $\frac{\partial J}{\partial v} \in L^\infty(\Omega \times \mathbb{R})$ is positive almost everywhere, $J(x, 0) = 0$, $J(x, \pm\infty) = \pm\infty$ a. e., and let G be a Preisach operator with Preisach density $\psi \in L^\infty(\Omega; L^1((0, \infty)\times\mathbb{R}))$, $\psi(x, r, v) \geq 0$ a. e. Then the operator G_J defined by the formula*

$$G_J[u](x, t) = G[J(x, u(x, \cdot))](t) \tag{B.12}$$

is a Preisach operator of the form (B.5) *with density*

$$\begin{aligned}\psi_J(x, r, z) = {} & \frac{\partial J}{\partial v}\left(x, \frac{z+r}{2}\right) \frac{\partial J}{\partial v}\left(x, \frac{z-r}{2}\right) \\ & \times\psi\left(x, J\left(x, \frac{z+r}{2}\right) - J\left(x, \frac{z-r}{2}\right), J\left(x, \frac{z+r}{2}\right) + J\left(x, \frac{z-r}{2}\right)\right).\end{aligned} \tag{B.13}$$

Note that we have $\psi_J \geq 0$ and

$$\int_0^\infty \int_{-\infty}^\infty \psi_J(x, r, z)\, \mathrm{d}z\, \mathrm{d}r = \int_0^\infty \int_{-\infty}^\infty \psi(x, r, z)\, \mathrm{d}z\, \mathrm{d}r \quad \text{a. e.} \tag{B.14}$$

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Optimal A Priori Error Estimates of Parabolic Optimal Control Problems with a Moving Point Control

Dmitriy Leykekhman and Boris Vexler

Abstract In this paper we consider a parabolic optimal control problem with a Dirac type control with moving point source in two space dimensions. We discretize the problem with piecewise constant functions in time and continuous piecewise linear finite elements in space. For this discretization we show optimal order of convergence with respect to the time and the space discretization parameters modulo some logarithmic terms. Error analysis for the same problem was carried out in the recent paper (Gong and Yan, SIAM J Numer Anal 54:1229–1262, 2016), however, the analysis there contains a serious flaw. One of the main goals of this paper is to provide the correct proof. The main ingredients of our analysis are the global and local error estimates on a curve, that have an independent interest.

Keywords Discontinuous Galerkin • Error estimates • Finite elements • Moving point control • Optimal control • Parabolic problems • Pointwise error estimates

1 Introduction

In this paper we provide numerical analysis for the following optimal control problem:

$$\min_{q,u} J(q,u) := \frac{1}{2}\int_0^T \|u(t)-\hat{u}(t)\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T |q(t)|^2 dt \tag{1}$$

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subject to the second order parabolic equation

$$u_t(t,x) - \Delta u(t,x) = q(t)\delta_{\gamma(t)}, \qquad (t,x) \in I \times \Omega, \tag{2a}$$

$$u(t,x) = 0, \qquad (t,x) \in I \times \partial\Omega, \tag{2b}$$

$$u(0,x) = 0, \qquad x \in \Omega \tag{2c}$$

and subject to pointwise control constraints

$$q_a \le q(t) \le q_b \quad \text{a. e. in } I. \tag{3}$$

Here $I = (0,T)$, $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain and $\delta_{\gamma(t)}$ is the Dirac delta function at point $x_t = \gamma(t)$ at each t. We will assume:

Assumption 1 $\gamma \in C^1(\bar{I})$ and $\max_{t\in\bar{I}} |\gamma'(t)| \le C_\gamma$.

Assumption 2 $\gamma(t) \subset \overline{\Omega}_0 \subset\subset \Omega_1$, for any $t \in I$, with $\overline{\Omega}_1 \subset\subset \Omega$.

The parameter α is assumed to be positive and the desired state $\hat{u}$ fulfills $\hat{u} \in L^2(I; L^\infty(\Omega))$. The control bounds $q_a, q_b \in \mathbb{R} \cup \{\pm\infty\}$ fulfill $q_a < q_b$. The precise functional-analytic setting is discussed in the next section.

For the discretization, we consider the standard continuous piecewise linear finite elements in space and piecewise constant discontinuous Galerkin method in time. This is a special case ($r = 0$, $s = 1$) of so called dG(r)cG(s) discretization, see e.g. [14] for the analysis of the method for parabolic problems and e.g. [25, 26] for error estimates in the context of optimal control problems. Throughout, we will denote by h the spatial mesh size and by k the size of time steps, see Sect. 3 for details.

The main result of the paper is the following.

Theorem 1 *Let $\bar{q}$ be optimal control for the problem* (1)–(2) *and $\bar{q}_{kh}$ be the optimal dG(0)cG(1) solution. Then there exists a constant C independent of h and k such that*

$$\|\bar{q} - \bar{q}_{kh}\|_{L^2(I)} \le C \left(|\ln h|^3 (k + h^2) + C_\gamma |\ln h| k \right) \left(\|\bar{q}\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^\infty(\Omega))} \right).$$

We would also like to point out that in addition to the optimal order estimate, modulo logarithmic terms, our analysis does not require any relationship between the sizes of the space discretization h and the time steps k.

The problem with fixed location of the point source (i.e. with $\delta_{x_0}(x)$ for some fixed $x_0 \in \Omega$) starting with the work of Lions [23], was investigated in a number of publications, see [2, 3, 10, 12, 28] for the continuous problem and [17, 21, 22] for the finite element approximation and error estimates. There is also a closely related problem of measured valued controls, which received a lot of attention lately [5–8, 20].

The problem with moving Dirac was considered in [9, 27] on a continuous level. The error analysis was carried out in the recent paper [16]. However, the analysis there contains a serious flaw. The last inequality in the estimate (3.33) in [16] is not correct. One of the main goals of this paper is to provide the correct proof. The main ingredients of our analysis are the global and local error estimates on a curve, Theorems 2 and 3, respectively. These results are new and have an independent interest.

Throughout the paper we use the usual notation for Lebesgue and Sobolev spaces. We denote by $(\cdot,\cdot)_\Omega$ the inner product in $L^2(\Omega)$ and by $(\cdot,\cdot)_{\tilde{I}\times\Omega}$ the inner product in $L^2(\tilde{I}\times\Omega)$ for any subinterval $\tilde{I}\subset I$.

The rest of the paper is organized as follows. In Sect. 2 we discuss the functional analytic setting of the problem, state the optimality system and prove regularity results for the state and for the adjoint state. In Sect. 3 we establish important global and local best approximation results along the curve for the heat equation. Finally in Sect. 4 we prove our main result.

2 Optimal Control Problem and Regularity

In order to state the functional analytic setting for the optimal control problem, we first introduce the auxiliary problem

$$\begin{aligned} v_t(t,x)-\Delta v(t,x) &= f(t,x), && (t,x)\in I\times\Omega,\\ v(t,x) &= 0, && (t,x)\in I\times\partial\Omega,\\ v(0,x) &= 0, && x\in\Omega, \end{aligned} \tag{4}$$

with a right-hand side $f\in L^2(I;L^p(\Omega))$ for some $1<p<\infty$. This equation possesses a unique solution

$$v\in L^2(I;H_0^1(\Omega))\cap H^1(I;H^{-1}(\Omega)).$$

Due to the convexity of the polygonal domain Ω the solution v possesses an additional regularity for $p=2$:

$$v\in L^2(I;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(I;L^2(\Omega)),$$

with the corresponding estimate

$$\|v\|_{L^2(I;H^2(\Omega))}+\|v_t\|_{L^2(I;L^2(\Omega))}\le C\|f\|_{L^2(I;L^2(\Omega))}, \tag{5}$$

see, e.g., [15]. From the Sobolev embedding $H^2(\Omega)\hookrightarrow W^{1,s}(\Omega)$ for any $s<\infty$ in two space dimensions and the previous lemma we can establish the following result for $s > 2$,

$$\|v\|_{L^2(I;W^{1,s}(\Omega))} \le Cs\|v\|_{L^2(I;H^2(\Omega))} \le Cs\|f\|_{L^2(I;L^2(\Omega))}. \tag{6}$$

The exact form of the constant can be traced, for example, from the proof of [1, Thm. 10.8]. In addition, there holds the following regularity result (see [21]).

Lemma 1 *If $f \in L^2(I; L^p(\Omega))$ for an arbitrary $p > 1$, then $v \in L^2(I; C(\Omega))$ and*

$$\|v\|_{L^2(I;C(\Omega))} \le C_p\|f\|_{L^2(I;L^p(\Omega))},$$

where $C_p \sim \frac{1}{p-1}$, as $p \to 1$.

We will also need the following local regularity result (see [21]).

Lemma 2 *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $f \in L^2(I; L^2(\Omega)) \cap L^2(I; L^p(\Omega_1))$ for some $2 \le p < \infty$. Then $v \in L^2(I; W^{2,p}(\Omega_0)) \cap H^1(I; L^p(\Omega_0))$ and there exists a constant C independent of p such that*

$$\|v_t\|_{L^2(I;L^p(\Omega_0))} + \|v\|_{L^2(I;W^{2,p}(\Omega_0))} \le Cp(\|f\|_{L^2(I;L^p(\Omega_1))} + \|f\|_{L^2(I;L^2(\Omega))}).$$

To introduce a weak solution of the state equation (2) we use the method of transposition, (cf. [24]). For a given control $q \in Q = L^2(I)$ we denote by $u = u(q) \in L^2(I; L^p(\Omega))$ with $2 \le p < \infty$ a weak solution of (2), if for all $\varphi \in L^2(I; L^{p'}(\Omega))$ with $\frac{1}{p} + \frac{1}{p'} = 1$ there holds

$$\langle u, \varphi\rangle_{L^2(I;L^p(\Omega)),L^2(I;L^{p'}(\Omega))} = \int_I w(t, \gamma(t))q(t)\, dt,$$

where $w \in L^2(I; W^{2,p'}(\Omega) \cap H^1_0(\Omega)) \cap H^1(I; L^{p'}(\Omega))$ is the weak solution of the adjoint equation

$$\begin{aligned} -w_t(t,x) - \Delta w(t,x) &= \varphi(t,x), && (t,x) \in I \times \Omega,\\ w(t,x) &= 0, && (t,x) \in I \times \partial\Omega,\\ w(T,x) &= 0, && x \in \Omega. \end{aligned} \tag{7}$$

The existence of this weak solution $u = u(q)$ follows by duality using the embedding $L^2(I; W^{2,p'}(\Omega)) \hookrightarrow L^2(I; C(\Omega))$ for $p' > 1$. Using Lemma 1 we can prove additional regularity for the state variable $u = u(q)$.

Proposition 2.1 *Without lose of generality we assume $2 \le p < \infty$. Let $q \in Q = L^2(I)$ be given and $u = u(q)$ be the solution of the state equation (2). Then $u \in L^2(I; L^p(\Omega))$ for any $p < \infty$ and the following estimate holds for $p \to \infty$ with a constant C independent of p,*

$$\|u\|_{L^2(I;L^p(\Omega))} \le Cp\|q\|_{L^2(I)}.$$

Proof To establish the result we use a duality argument. There holds

$$\|u\|_{L^2(I;L^p(\Omega))} = \sup_{\|\varphi\|_{L^2(I;L^{p'}(\Omega))}=1} (u,\varphi)_{I\times\Omega}, \quad \text{where} \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Let w be the solution to (7) for $\varphi \in L^2(I;L^{p'}(\Omega))$ with $\|\varphi\|_{L^2(I;L^{p'}(\Omega))} = 1$. From Lemma 1, $w \in L^2(I;C(\Omega))$ and the following estimate holds

$$\|w\|_{L^2(I;C(\Omega))} \le \frac{C}{p'-1}\|\varphi\|_{L^2(I;L^{p'}(\Omega))} = \frac{C}{p'-1} \le Cp, \text{ as } p \to \infty.$$

Thus,

$$\begin{aligned}\|u\|_{L^2(I;L^p(\Omega))} &= \sup_{\|\varphi\|_{L^2(I;L^{p'}(\Omega))}=1} (u,\varphi)_{I\times\Omega} \\ &= \int_I q(t)w(t,\gamma(t))\,dt \le \|q\|_{L^2(I)}\|w\|_{L^2(I;C(\Omega))} \le Cp\|q\|_{L^2(I)}.\end{aligned}$$

Remark 1 We would like to note that the above regularity requires only Assumption 2 on γ. Higher regularity of γ is needed for optimal order error estimates only.

A further regularity result for the state equation follows from [13].

Proposition 2.2 *Let $q \in Q = L^2(I)$ be given and $u = u(q)$ be the solution of the state equation (2). Then for each $1 < s < 2$ there holds*

$$u \in L^2(I;W_0^{1,s}(\Omega)) \quad \textit{and} \quad u_t \in L^2(I;W^{-1,s}(\Omega)).$$

Moreover, the state u fulfills the following weak formulation

$$\langle u_t,\varphi\rangle + (\nabla u,\nabla\varphi) = \int_I q(t)\varphi(t,\gamma(t))\,dt \quad \textit{for all } \varphi \in L^2(I;W_0^{1,s'}(\Omega)),$$

where $\frac{1}{s'}+\frac{1}{s}=1$ and $\langle\cdot,\cdot\rangle$ is the duality product between $L^2(I;W^{-1,s}(\Omega))$ and $L^2(I;W_0^{1,s'}(\Omega))$.

Proof For $s < 2$ we have $s' > 2$ and therefore $W_0^{1,s'}(\Omega)$ is embedded into $C(\bar{\Omega})$. Therefore the right-hand side $q(t)\delta_{\gamma(t)}$ of the state equation can be identified with an element in $L^2(I;W^{-1,s}(\Omega))$. Using the result from [13, Theorem 5.1] on maximal parabolic regularity and exploiting the fact that $-\Delta\colon W_0^{1,s}(\Omega) \to W^{-1,s}(\Omega)$ is an isomorphism, see [19], we obtain

$$u \in L^2(I;W_0^{1,s}(\Omega)) \quad \text{and} \quad u_t \in L^2(I;W^{-1,s}(\Omega)).$$

Given the above regularity the corresponding weak formulation is fulfilled by a standard density argument.

As the next step we introduce the reduced cost functional $j\colon Q \to \mathbb{R}$ on the control space $Q = L^2(I)$ by

$$j(q) = J(q, u(q)),$$

where J is the cost function in (1) and $u(q)$ is the weak solution of the state equation (2) as defined above. The optimal control problem can then be equivalently reformulated as

$$\min j(q), \quad q \in Q_{\text{ad}}, \tag{8}$$

where the set of admissible controls is defined according to (3) by

$$Q_{\text{ad}} = \{q \in Q \mid q_a \le q(t) \le q_b \text{ a. e. in } I\}. \tag{9}$$

By standard arguments this optimization problem possesses a unique solution $\bar{q} \in Q = L^2(I)$ with the corresponding state $\bar{u} = u(\bar{q}) \in L^2(I; L^p(\Omega))$ for all $p < \infty$, see Proposition 2.1 for the regularity of $\bar{u}$. Due to the fact, that this optimal control problem is convex, the solution $\bar{q}$ is equivalently characterized by the optimality condition

$$j'(\bar{q})(\partial q - \bar{q}) \ge 0 \quad \text{for all } \partial q \in Q_{\text{ad}}. \tag{10}$$

The (directional) derivative $j'(q)(\partial q)$ for given $q, \partial q \in Q$ can be expressed as

$$j'(q)(\partial q) = \int_I (\alpha q(t) + z(t, \gamma(t)))\, \partial q(t)\, dt,$$

where $z = z(q)$ is the solution of the adjoint equation

$$-z_t(t,x) - \Delta z(t,x) = u(t,x) - \hat{u}(t,x), \qquad (t,x) \in I \times \Omega, \tag{11a}$$

$$z(t,x) = 0, \qquad (t,x) \in I \times \partial\Omega, \tag{11b}$$

$$z(T,x) = 0, \qquad x \in \Omega, \tag{11c}$$

and $u = u(q)$ on the right-hand side of (11a) is the solution of the state equation (2). The adjoint solution, which corresponds to the optimal control $\bar{q}$ is denoted by $\bar{z} = z(\bar{q})$.

The optimality condition (10) is a variational inequality, which can be equivalently formulated using the projection

$$P_{Q_{\text{ad}}}\colon Q \to Q_{\text{ad}}, \quad P_{Q_{\text{ad}}}(q)(t) = \min\left(q_b, \max(q_a, q(t))\right).$$

The resulting condition reads:

$$\bar{q}(t) = P_{Q_{\text{ad}}}\left(-\frac{1}{\alpha}\bar{z}(t,\gamma(t))\right). \tag{12}$$

In the next proposition we provide regularity results for the solution of the adjoint equation.

Proposition 2.3 *Let $q \in Q$ be given, let $u = u(q)$ be the corresponding state fulfilling (2) and let $z = z(q)$ be the corresponding adjoint state fulfilling (11). Then,*

(a) $z \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega))$ *and the following estimate holds*

$$\|\nabla^2 z\|_{L^2(I;L^2(\Omega))} + \|z_t\|_{L^2(I;L^2(\Omega))} \le C(\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^2(\Omega))}).$$

(b) If $\Omega_0 \subset\subset \Omega$, then $z \in L^2(I; W^{2,p}(\Omega_0)) \cap H^1(I; L^p(\Omega_0))$ for all $2 \le p < \infty$ and the following estimate holds

$$\|\nabla^2 z\|_{L^2(I;L^p(\Omega_0))} + \|z_t\|_{L^2(I;L^p(\Omega_0))} \le Cp^2(\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^\infty(\Omega))}).$$

Proof

(a) The right-hand side of the adjoint equation fulfills $u - \hat{u} \in L^2(I; L^p(\Omega))$ for all $1 < p < \infty$, see Proposition 2.1. Due to the convexity of the domain Ω we directly obtain $z \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega))$ and the estimate

$$\|\nabla^2 z\|_{L^2(I;L^2(\Omega))} + \|z_t\|_{L^2(I;L^2(\Omega))} \le C\|u - \hat{u}\|_{L^2(I;L^2(\Omega))}.$$

The result from Proposition 2.1 leads directly to the first estimate.

(b) From Lemma 2 for $p \ge 2$ we have

$$\|\nabla^2 z\|_{L^2(I;L^p(\Omega_0))} + \|z_t\|_{L^2(I;L^p(\Omega_0))} \le Cp\|u - \hat{u}\|_{L^2(I;L^p(\Omega))}.$$

Hence, by the triangle inequality and Proposition 2.1 we obtain

$$\|u - \hat{u}\|_{L^2(I;L^p(\Omega))} \le C\left(p\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^\infty(\Omega))}\right).$$

That completes the proof.

3 Discretization and the Best Approximation Type Results

3.1 Space-Time Discretization and Notation

For discretization of the problem under the consideration we introduce a partitions of $I = [0,T]$ into subintervals $I_m = (t_{m-1}, t_m]$ of length $k_m = t_m - t_{m-1}$, where $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$. We assume that

$$k_{m+1} \le \kappa k_m, \quad m = 1, \dots, M-1, \quad \text{for some} \quad \kappa > 0. \tag{13}$$

The maximal time step is denoted by $k = \max_m k_m$. The semidiscrete space X_k^0 of piecewise constant functions in time is defined by

$$X_k^0 = \{v_k \in L^2(I; H_0^1(\Omega)) : \ v_k|_{I_m} \in \mathscr{P}_0(I_m; H_0^1(\Omega)),\ m = 1, 2, \dots, M\},$$

where $\mathscr{P}_0(I; V)$ is the space of constant functions in time with values in Banach space V. We will employ the following notation for functions in X_k^0

$$v_m^+ = \lim_{\varepsilon \to 0^+} v(t_m + \varepsilon) := v_{m+1}, \quad v_m^- = \lim_{\varepsilon \to 0^+} v(t_m - \varepsilon) = v(t_m) := v_m, \quad [v]_m = v_m^+ - v_m^-. \tag{14}$$

Let $\mathscr{T}$ denote a quasi-uniform triangulation of Ω with a mesh size h, i.e., $\mathscr{T} = \{\tau\}$ is a partition of Ω into triangles τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\operatorname{diam}(\tau) \le h \le C|\tau|^{\frac{1}{2}}, \quad \forall \tau \in \mathscr{T}$$

hold. Let V_h be the set of all functions in $H_0^1(\Omega)$ that are linear on each τ, i.e. V_h is the usual space of continuous piecewise linear finite elements. We will require the modified Clément interpolant $i_h : L^1(\Omega) \to V_h$ and the L^2-projection $P_h : L^2(\Omega) \to V_h$ defined by

$$(P_h v, \chi)_\Omega = (v, \chi)_\Omega, \quad \forall \chi \in V_h. \tag{15}$$

To obtain the fully discrete approximation we consider the space-time finite element space

$$X_{k,h}^{0,1} = \{v_{kh} \in X_k^0 : \ v_{kh}|_{I_m} \in \mathscr{P}_0(I_m; V_h),\ m = 1, 2, \dots, M\}. \tag{16}$$

We will also need the following semidiscrete projection $\pi_k : C(\bar{I}; H_0^1(\Omega)) \to X_k^0$ defined by

$$\pi_k v|_{I_m} = v(t_m), \quad m = 1, 2, \dots, M, \tag{17}$$

and the fully discrete projection $\pi_{kh} : C(\bar{I}; L^1(\Omega)) \to X_{k,h}^{0,1}$ defined by $\pi_{kh} = i_h \pi_k$.

To introduce the dG(0)cG(1) discretization we define the following bilinear form

$$B(v,\varphi) = \sum_{m=1}^{M} \langle v_t, \varphi \rangle_{I_m \times \Omega} + (\nabla v, \nabla \varphi)_{I\times\Omega} + \sum_{m=2}^{M} ([v]_{m-1}, \varphi^+_{m-1})_\Omega + (v^+_0, \varphi^+_0)_\Omega, \tag{18}$$

where $\langle \cdot,\cdot \rangle_{I_m\times\Omega}$ is the duality product between $L^2(I_m; W^{-1,s}(\Omega))$ and $L^2(I_m; W^{1,s'}_0(\Omega))$. We note, that the first sum vanishes for $v \in X^0_k$. Rearranging the terms, we obtain an equivalent (dual) expression for B:

$$B(v,\varphi) = -\sum_{m=1}^{M} \langle v, \varphi_t \rangle_{I_m \times \Omega} + (\nabla v, \nabla \varphi)_{I\times\Omega} - \sum_{m=1}^{M-1} (v^-_m, [\varphi_k]_m)_\Omega + (v^-_M, \varphi^-_M)_\Omega. \tag{19}$$

In the two following theorems we establish global and local best approximation type results along the curve for the error between the solution v of the auxiliary equation (4) and its dG(0)cG(1) approximation $v_{kh} \in X^{0,1}_{k,h}$ defined as

$$B(v_{kh}, \varphi_{kh}) = (f, \varphi_{kh})_{I\times\Omega} \quad \text{for all } \varphi_{kh} \in X^{0,1}_{k,h}. \tag{20}$$

Since dG(0)cG(1) method is a consistent discretization we have the following Galerkin orthogonality relation:

$$B(v - v_{kh}, \varphi_{kh}) = 0 \quad \text{for all } \varphi_{kh} \in X^{0,1}_{k,h}.$$

3.2 *Discretization of the Curve and the Weight Function*

To define fully discrete optimization problem we will also require a discretization of the curve γ. We define $\gamma_k = \pi_k \gamma$ by

$$\gamma_k|_{I_m} = \gamma(t_m) := \gamma_{k,m} \in \Omega_0, \quad m = 1, 2, \ldots, M, \tag{21}$$

i.e., γ_k is a piecewise constant approximation of γ. Next we introduce a weight function

$$\sigma(t,x) = \sqrt{|x-\gamma(t)|^2 + h^2} \tag{22}$$

and a discrete piecewise constant in time approximation

$$\sigma_k(t,x) = \sqrt{|x-\gamma_k(t)|^2 + h^2}. \tag{23}$$

Define

$$\sigma_{k,m} := \sigma_k|_{I_m} = \sigma_k(t_m, x) = \sigma(t_m, x). \tag{24}$$

One can easily check that σ and σ_k satisfy the following properties for any $(t, x) \in I \times \Omega$,

$$\|\sigma^{-1}(t,\cdot)\|_{L^2(\Omega)}, \|\sigma_k^{-1}(t,\cdot)\|_{L^2(\Omega)} \le C|\ln h|^{\frac{1}{2}}, \quad t \in \bar{I}, \tag{25a}$$

$$|\nabla\sigma(t,x)|,\ |\nabla\sigma_k(t,x)| \le C, \tag{25b}$$

$$|\nabla^2\sigma_k(t,x)| \le C|\sigma_k^{-1}(t,x)|, \tag{25c}$$

$$|\sigma_t(t,x)| \le |\nabla\sigma(t,x)| \cdot |\gamma'(t)| \le CC_\gamma, \tag{25d}$$

$$\max_{x\in\tau} \sigma(x,t) \le C \min_{x\in\tau} \sigma(x,t), \quad \forall \tau \in \mathscr{T}. \tag{25e}$$

3.3 Global Error Estimate Along the Curve

In this section we prove the following global approximation result.

Theorem 2 (Global Best Approximation) *Assume v and v_{kh} satisfy (4) and (20) respectively. Then there exists a constant C independent of k and h such that for any $1 \le p \le \infty$,*

$$\int_I |(v - v_{kh})(t, \gamma_k(t))|^2 dt \le C|\ln h|^2 \times \inf_{\chi \in X_{k,h}^{0,1}} \left(\|v - \chi\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}} \|\pi_k v - \chi\|^2_{L^2(I;L^p(\Omega))} \right).$$

Proof To establish the result we use a duality argument. First, we introduce a smoothed Delta function, which we will denote by $\tilde{\delta}_{\gamma_k}$. This function on each I_m is defined as $\tilde{\delta}_{\gamma_{k,m}}$ and supported in one cell, which we denote by τ_m^0, i.e.

$$(\chi, \tilde{\delta}_{\gamma_{k,m}})_{\tau_m^0} = \chi(\gamma_{k,m}) = \chi(\gamma(t_m)), \quad \forall \chi \in \mathbb{P}^1(\tau_m^0), \quad m = 1, 2, \dots, M.$$

In addition we also have (see [31, Appendix])

$$\|\tilde{\delta}_{\gamma_k}\|_{W_p^s(\Omega)} \le Ch^{-s-2(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \quad s = 0, 1. \tag{26}$$

Thus in particular $\|\tilde{\delta}_{\gamma_k}\|_{L^1(\Omega)} \le C$, $\|\tilde{\delta}_{\gamma_k}\|_{L^2(\Omega)} \le Ch^{-1}$, and $\|\tilde{\delta}_{\gamma_k}\|_{L^\infty(\Omega)} \le Ch^{-2}$.

We define g to be a solution to the following backward parabolic problem

$$\begin{aligned} -g_t(t,x) - \Delta g(t,x) &= v_{kh}(t,\gamma_k(t))\tilde{\delta}_{\gamma_k}(x), && (t,x) \in I \times \Omega, \\ g(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\ g(T,x) &= 0, && x \in \Omega. \end{aligned} \tag{27}$$

There holds

$$\begin{aligned} \int_{I\times\Omega} v_{kh}(t,\gamma_k(t))\tilde{\delta}_{\gamma_k}(x)\varphi_{kh}(t,x)dtdx &= \sum_{m=1}^{M}\int_{I_m} v_{kh}(t,\gamma_k(t))\left(\int_\Omega \tilde{\delta}_{\gamma_k}(x)\varphi_{kh}(t,x)dx\right)dt \\ &= \sum_{m=1}^{M}\int_{I_m} v_{kh}(t,\gamma_k(t))\varphi_{kh}(t,\gamma_k(t))dt \\ &= \int_I v_{kh}(t,\gamma_k(t))\varphi_{kh}(t,\gamma_k(t))dt. \end{aligned}$$

Let $g_{kh} \in X^{0,1}_{k,h}$ be dG(0)cG(1) solution defined by

$$B(\varphi_{kh}, g_{kh}) = (v_{kh}(t,\gamma_k(t))\tilde{\delta}_{\gamma_k}, \varphi_{kh})_{I\times\Omega}, \quad \forall \varphi_{kh} \in X^{0,1}_{k,h}. \tag{28}$$

Then using that dG(0)cG(1) method is consistent, we have

$$\begin{aligned} \int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt &= B(v_{kh}, g_{kh}) = B(v, g_{kh}) \\ &= (\nabla v, \nabla g_{kh})_{I\times\Omega} - \sum_{m=1}^{M}(v_m, [g_{kh}]_m)_\Omega, \end{aligned} \tag{29}$$

where we have used the dual expression (19) for the bilinear form B and the fact that the last term in (19) can be included in the sum by setting $g_{kh,M+1} = 0$ and defining consequently $[g_{kh}]_M = -g_{kh,M}$. The first sum in (19) vanishes due to $g_{kh} \in X^{0,1}_{k,h}$. For each t, integrating by parts elementwise and using that g_{kh} is linear in the spacial variable, by the Hölder's inequality we have

$$(\nabla v, \nabla g_{kh})_\Omega = \frac{1}{2}\sum_\tau (v, [\![\partial_n g_{kh}]\!])_{\partial\tau} \le C\|v\|_{L^\infty(\Omega)} \sum_\tau \|[\![\partial_n g_{kh}]\!]\|_{L^1(\partial\tau)}, \tag{30}$$

where $[\![\partial_n g_{kh}]\!]$ denotes the jumps of the normal derivatives across the element faces.

From Lemma 2.4 in [29] we have

$$\sum_\tau \|[\![\partial_n g_{kh}]\!]\|_{L^1(\partial\tau)} \le C|\ln h|^{\frac12}\left(\|\sigma_k\Delta_h g_{kh}\|_{L^2(\Omega)} + \|\nabla g_{kh}\|_{L^2(\Omega)}\right),$$

where $\Delta_h\colon V_h \to V_h$ is the discrete Laplace operator, defined by

$$-(\Delta_h v_h, \chi)_\Omega = (\nabla v_h, \nabla\chi)_\Omega, \quad \forall \chi \in V_h.$$

To estimate the term involving the jumps in (29), we first use the Hölder's inequality and the inverse estimate to obtain

$$\sum_{m=1}^M (v_m, [g_{kh}]_m)_\Omega \le C\sum_{m=1}^M k_m^{\frac12}\|v_m\|_{L^p(\Omega)} k_m^{-\frac12} h^{-\frac2p}\|[g_{kh}]_m\|_{L^1(\Omega)}. \tag{31}$$

Now we use the fact that Eq. (28) can be rewritten on the each time level as

$$(\nabla\varphi_{kh}, \nabla g_{kh})_{I_m\times\Omega} - (\varphi_{kh,m}, [g_{kh}]_m)_\Omega = (v_{kh}(t,\gamma_k(t))\tilde\delta_{\gamma_k}, \varphi_{kh})_{I_m\times\Omega},$$

or equivalently as

$$-k_m\Delta_h g_{kh,m} - [g_{kh}]_m = k_m v_{kh,m}(\gamma_{k,m})P_h\tilde\delta_{\gamma_{k,m}}, \tag{32}$$

where P_h is the L^2-projection, see (15). From (32) by the triangle inequality, we obtain

$$\|[g_{kh}]_m\|_{L^1(\Omega)} \le k_m\|\Delta_h g_{kh,m}\|_{L^1(\Omega)} + k_m\|P_h\tilde\delta_{\gamma_{k,m}}\|_{L^1(\Omega)}|v_{kh,m}(\gamma_{k,m})|.$$

Using that the L^2-projection is stable in L^1-norm (cf. [11]), we have

$$\|P_h\tilde\delta_{\gamma_{k,m}}\|_{L^1(\Omega)} \le C\|\tilde\delta_{\gamma_{k,m}}\|_{L^1(\Omega)} \le C.$$

Inserting the above estimate into (31) and using (25a), we obtain

$$\sum_{m=1}^M (v_m, [g_{kh}]_m)_\Omega \le Ch^{-\frac2p}\sum_{m=1}^M k_m^{\frac12}\|v_m\|_{L^p(\Omega)}k_m^{\frac12}\left(\|\Delta_h g_{kh,m}\|_{L^1(\Omega)} + |v_{kh,m}(\gamma_{k,m})|\right)$$

$$\le Ch^{-\frac2p}\left(\sum_{m=1}^M k_m\|v_m\|^2_{L^p(\Omega)}\right)^{\frac12}\left(\sum_{m=1}^M k_m\|\Delta_h g_{kh,m}\|^2_{L^1(\Omega)} + k_m|v_{kh,m}(\gamma_{k,m})|^2\right)^{\frac12}$$

$$\le Ch^{-\frac2p}\|\pi_k v\|_{L^2(I;L^p(\Omega))}\left(\int_0^T |\ln h|\|\sigma_k\Delta_h g_{kh}\|^2_{L^2(\Omega)} + |v_{kh}(t,\gamma_k(t))|^2 dt\right)^{\frac12}.$$

Combining (29) and (30) with the above estimates we have

$$\int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt \le C|\ln h|^{\frac{1}{2}} \left(\|v\|_{L^2(I;L^\infty(\Omega))} + h^{-\frac{2}{p}}\|\pi_k v\|_{L^2(I;L^p(\Omega))}\right) \times$$
$$\left(\int_0^T \|\sigma_k \Delta_h g_{kh}\|^2_{L^2(\Omega)} + \|\nabla g_{kh}\|^2_{L^2(\Omega)} + |v_{kh}(t,\gamma_k(t))|^2 dt\right)^{\frac{1}{2}}. \tag{33}$$

To complete the proof of the theorem it is sufficient to show

$$\int_0^T \left(\|\sigma_k \Delta_h g_{kh}\|^2_{L^2(\Omega)} + \|\nabla g_{kh}\|^2_{L^2(\Omega)}\right) dt \le C|\ln h| \int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt. \tag{34}$$

Then from (33) and (34) it would follow that

$$\int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt \le C|\ln h|^2 \left(\|v\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}}\|\pi_k v\|^2_{L^2(I;L^p(\Omega))}\right).$$

Then using that the dG(0)cG(1) method is invariant on $X^{0,1}_{k,h}$, by replacing v an v_{kh} with $v - \chi$ and $v_{kh} - \chi$ for any $\chi \in X_{kh}$, we obtain Theorem 2.

The estimate (34) will follow from the series of lemmas. The first lemma treats the term $\|\sigma_k \Delta_h g_{kh}\|^2_{L^2(I;L^2(\Omega))}$.

Lemma 3 *For any $\varepsilon > 0$ there exists C_ε such that*

$$\int_0^T \|\sigma_k \Delta_h g_{kh}\|^2_{L^2(\Omega)} dt \le C_\varepsilon \int_0^T \left(|v_{kh}(t,\gamma_k(t))|^2 + \|\nabla g_{kh}\|^2_{L^2(\Omega)}\right) dt$$
$$+ \varepsilon \sum_{m=1}^M k_m^{-1} \|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)},$$

where σ_k and $\sigma_{k,m}$ are defined in (23) *and* (24)*, respectively.*

Proof Equation (28) for each time interval I_m can be rewritten as (32). Multiplying (32) with $\varphi = -\sigma_k^2 \Delta_h g_{kh}$ and integrating over $I_m \times \Omega$, we have

$$\begin{aligned}
&\int_{I_m} \|\sigma_{k,m} \Delta_h g_{kh}\|^2_{L^2(\Omega)} dt \\
&\quad = -([g_{kh}]_m, \sigma^2_{k,m} \Delta_h g_{kh,m})_\Omega - (v_{kh}(t,\gamma_{k,m}) P_h \tilde{\delta}_{\gamma_k}, \sigma^2_{k,m} \Delta_h g_{kh})_{I_m \times \Omega} \\
&\quad = -(P_h(\sigma^2_{k,m}[g_{kh}]_m), \Delta_h g_{kh,m})_\Omega - (v_{kh}(t,\gamma_{k,m}) P_h \tilde{\delta}_{\gamma_k}, \sigma^2_{k,m} \Delta_h g_{kh})_{I_m \times \Omega}
\end{aligned}$$

$$= (\nabla(\sigma_{k,m}^2[g_{kh}]_m), \nabla g_{kh,m})_\Omega + (\nabla(P_h - I)(\sigma_{k,m}^2[g_{kh}]_m), \nabla g_{kh,m})_\Omega$$
$$- (v_{kh}(t, \gamma_{k,m}) P_h \tilde{\delta}_{\gamma_k}, \sigma_{k,m}^2 \Delta_h g_{kh})_{I_m \times \Omega}$$
$$= J_1 + J_2 + J_3.$$

We have

$$J_1 = 2(\sigma_{k,m} \nabla \sigma_{k,m} [g_{kh}]_m, \nabla g_{kh,m})_\Omega + (\sigma_{k,m}[\nabla g_{kh}]_m, \sigma_{k,m} \nabla g_{kh,m})_\Omega = J_{11} + J_{12}.$$

By the Cauchy-Schwarz inequality and using (25b) we get

$$J_{11} \le C \|\sigma_{k,m}[g_{kh}]_m\|_{L^2(\Omega)} \|\nabla g_{kh,m}\|_{L^2(\Omega)}.$$

On the other hand we have

$$J_{12} = ([\sigma_k \nabla g_{kh}]_m, \sigma_{k,m} \nabla g_{kh,m})_\Omega + ((\sigma_{k,m} - \sigma_{k,m+1}) \nabla g_{kh,m+1}, \sigma_{k,m} \nabla g_{kh,m})_\Omega$$
$$= J_{121} + J_{122}.$$

Using the identity

$$([w_{kh}]_m, w_{kh,m})_\Omega = \frac{1}{2}\|w_{kh,m+1}\|_{L^2(\Omega)}^2 - \frac{1}{2}\|w_{kh,m}\|_{L^2(\Omega)}^2 - \frac{1}{2}\|[w_{kh}]_m\|_{L^2(\Omega)}^2, \tag{35}$$

we have

$$J_{121} = \frac{1}{2}\|\sigma_{k,m+1} \nabla g_{kh,m+1}\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\sigma_{k,m} \nabla g_{kh,m}\|_{L^2(\Omega)}^2 - \frac{1}{2}\|[\sigma_k \nabla g_{kh}]_m\|_{L^2(\Omega)}^2.$$

By the Cauchy-Schwarz inequality, we obtain

$$J_{122} \le \|(\sigma_{k,m} - \sigma_{k,m+1}) \nabla g_{kh,m+1}\|_{L^2(\Omega)} \|\sigma_{k,m} \nabla g_{kh,m}\|_{L^2(\Omega)}$$
$$\le C C_\gamma k_m \|\nabla g_{kh,m+1}\|_{L^2(\Omega)} \|\sigma_{k,m} \nabla g_{kh,m}\|_{L^2(\Omega)},$$

where in the last step we used that from (25d)

$$|\sigma_{k,m}(x) - \sigma_{k,m+1}(x)| = |\sigma(t_m, x) - \sigma(t_{m+1}, x)| \le C k_m |\sigma_t(\tilde{t}, x)| \le C C_\gamma k_m,$$

for some $\tilde{t} \in I_m$. Using the Young's inequality for J_{11}, neglecting $-\frac{1}{2}\|[\sigma_k \nabla g_{kh}]_m\|_{L^2(\Omega)}^2$, and using the assumption on the time steps $k_m \le \kappa k_{m+1}$

and that $\sigma_k \leq C$, we obtain

$$J_1 \leq \frac{1}{2}\|\sigma_{k,m+1}\nabla g_{kh,m+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sigma_{k,m}\nabla g_{kh,m}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{k_m}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)} + C_\varepsilon k_m\|\nabla g_{kh,m}\|^2_{L^2(\Omega)} + Ck_{m+1}\|\nabla g_{kh,m+1}\|^2_{L^2(\Omega)}. \tag{36}$$

To estimate J_2, first by the Cauchy-Schwarz inequality and the approximation theory we have

$$J_2 = \sum_\tau (\nabla(P_h - I)(\sigma^2_{k,m}[g_{kh}]_m), \nabla g_{kh,m})_\tau \leq Ch\sum_\tau \|\nabla^2(\sigma^2_{k,m}[g_{kh}]_m)\|_{L^2(\tau)}\|\nabla g_{kh,m}\|_{L^2(\tau)}.$$

Using that g_{kh} is piecewise linear we have

$$\nabla^2(\sigma^2[g_{kh}]_m) = \nabla^2(\sigma^2)[g_{kh}]_m + \nabla(\sigma^2)\cdot\nabla[g_{kh}]_m \quad \text{on } \tau.$$

There holds $\partial_{ij}(\sigma^2) = 2(\partial_i\sigma)(\partial_j\sigma) + 2\sigma\partial_{ij}\sigma$ and $\nabla(\sigma^2) = 2\sigma\nabla\sigma$. Thus by the properties of σ (25b) and (25c), we have

$$|\nabla^2(\sigma^2)| \leq C \quad \text{and} \quad |\nabla(\sigma^2)| \leq C\sigma.$$

Same estimates hold for σ_k. Using these estimates, the fact that $h \leq \sigma_k$ and the inverse inequality (in view of (25e) the inverse inequality is valid with σ inside the norm), we obtain

$$\begin{aligned} J_2 &\leq C\sum_\tau \left(h\|[g_{kh}]_m\|_{L^2(\tau)} + h\|\sigma_{k,m}\nabla[g_{kh}]_m\|_{L^2(\tau)}\right)\|\nabla g_{kh,m}\|_{L^2(\tau)} \\ &\leq C\sum_\tau \left(\|\sigma_{k,m}[g_{kh}]_m\|_{L^2(\tau)} + C_{inv}\|\sigma_{k,m}[g_{kh}]_m\|_{L^2(\tau)}\right)\|\nabla g_{kh,m}\|_{L^2(\tau)} \\ &\leq C\sum_\tau \|\sigma_{k,m}[g_{kh}]_m\|_{L^2(\tau)}\|\nabla g_{kh,m}\|_{L^2(\tau)} \\ &\leq C_\varepsilon k_m\|\nabla g_{kh,m}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{k_m}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}. \end{aligned} \tag{37}$$

To estimate J_3 we first notice that

$$\|\sigma_k P_h\tilde{\delta}_{\gamma_k}\|_{L^2(\Omega)} \leq C. \tag{38}$$

The proof is identical to the proof of (3.21) in [21].

By the Cauchy-Schwarz inequality, (38), and the Young's inequality, we obtain

$$J_3 \leq C\int_{I_m}|v_{kh}(t,\gamma_k)|^2dt + \frac{1}{2}\int_{I_m}\|\sigma_{k,m}\Delta_h g_{kh,m}\|^2_{L^2(\Omega)}dt. \tag{39}$$

Using the estimates (36), (37), and (39) we have

$$\int_{I_m} \|\sigma_{k,m}\Delta_h g_{kh}\|^2_{L^2(\Omega)}dt \le C_\varepsilon \int_{I_m} \left(|v_{kh}(t,\gamma_k(t))|^2 + \|\nabla g_{kh}\|^2_{L^2(\Omega)}\right) dt$$
$$+ CC_\gamma \int_{I_{m+1}} \|\nabla g_{kh}\|^2_{L^2(\Omega)}dt + \frac{\varepsilon}{k_m}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}$$
$$+ \frac{1}{2}\|\sigma_{k,m+1}\nabla g_{kh,m+1}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sigma_{k,m}\nabla g_{kh,m}\|^2_{L^2(\Omega)}.$$

Summing over m and using that $g_{kh,M+1} = 0$ we obtain the lemma.

The second lemma treats the term involving jumps.

Lemma 4 *There exists a constant C such that*

$$\sum_{m=1}^{M} k_m^{-1}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)} \le C\int_0^T \left(\|\sigma_k\Delta_h g_{kh}\|^2_{L^2(\Omega)} + |v_{kh}(t,\gamma_k(t)|^2\right) dt.$$

Proof We test (32) with $\varphi|_{I_m} = \sigma^2_{k,m}[g_{kh}]_m$ and obtain

$$\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)} = -(\Delta_h g_{kh}, \sigma^2_{k,m}[g_{kh}]_m)_{I_m\times\Omega}$$
$$- (v_{kh}(t,\gamma_k(t))P_h\tilde{\delta}_{\gamma_k}, \sigma^2_{k,m}[g_{kh}]_m)_{I_m\times\Omega}. \tag{40}$$

The first term on the right hand side of (40) using the Young's inequality can be estimated as

$$(\Delta_h g_{kh}, \sigma^2_{k,m}[g_{kh}]_m)_{I_m\times\Omega} \le Ck_m \int_{I_m} \|\sigma_{k,m}\Delta_h g_{kh}\|^2_{L^2(\Omega)}dt + \frac{1}{4}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}.$$

The last term on the right hand side of (40) can easily be estimated using (38) as

$$(v_{kh}(t,\gamma_{k,m})P_h\tilde{\delta}_{\gamma_k}, \sigma^2_{k,m}[g_{kh}]_m)_{I_m\times\Omega} \le Ck_m \int_{I_m} |v_{kh}(t,\gamma_k(t))|^2dt + \frac{1}{4}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}.$$

Combining the above two estimates we obtain

$$\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)} \le Ck_m \int_{I_m} \left(\|\sigma_{k,m}\Delta_h g_{kh}\|^2_{L^2(\Omega)} + |v_{kh}(t,\gamma_k(t))|^2\right) dt.$$

Summing over m we obtain the lemma.

Lemma 5 *There exists a constant C such that*

$$\|\nabla g_{kh}\|^2_{L^2(I\times\Omega)} \le C|\ln h| \int_0^T |v_{kh}(t,\gamma_k(t))|^2dt.$$

Proof Adding the primal (18) and the dual (19) representation of the bilinear form $B(\cdot,\cdot)$ one immediately arrives at

$$\|\nabla v\|^2_{L^2(I\times\Omega)} \le B(v,v) \quad \text{for all} \ \ v \in X^0_k,$$

see e.g., [25]. Applying this inequality together with the discrete Sobolev inequality, see [4, Lemma 4.9.2], results in

$$\begin{aligned}
\|\nabla g_{kh}\|^2_{L^2(I\times\Omega)} &\le B(g_{kh}, g_{kh}) = (v_{kh}(t,\gamma_k(t))\tilde{\delta}_{\gamma_k}, g_{kh})_{I\times\Omega} \\
&= \int_0^T v_{kh}(t,\gamma_k(t))g_{kh}(t,\gamma_k(t))\,dt \\
&\le \left(\int_0^T |v_{kh}(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \left(\int_0^T |g_{kh}(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \\
&\le \left(\int_0^T |v_{kh}(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \|g_{kh}\|_{L^2(I;L^\infty(\Omega))} \\
&\le c|\ln h|^{\frac{1}{2}} \left(\int_0^T |v_{kh}(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \|\nabla g_{kh}\|_{L^2(I\times\Omega)}.
\end{aligned}$$

This gives the desired estimate.

We proceed with the proof of Theorem 2. From Lemmas 3 to 5. It follows that

$$\begin{aligned}
\int_0^T \left(\|\sigma_k \Delta_h g_{kh}\|^2_{L^2(\Omega)} + \|\nabla g_{kh}\|^2_{L^2(\Omega)}\right) dt \le C_\varepsilon |\ln h| \int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt \\
+ C\varepsilon \int_0^T \|\sigma_k \Delta_h g_{kh}\|^2_{L^2(\Omega)} dt.
\end{aligned}$$

Taking ε sufficiently small we have (34). From (33) we can conclude that

$$\int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt \le C|\ln h|^2 \left(\|v\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}} \|\pi_k v\|^2_{L^2(I;L^p(\Omega))}\right),$$

for some constant C independent of h and k. Using that dG(0)cG(1) method is invariant on $X^{0,1}_{k,h}$, by replacing v and v_{kh} with $v-\chi$ and $v_{kh}-\chi$ for any $\chi \in X^{0,1}_{k,h}$, we obtain

$$\int_0^T |(v_{kh}-\chi)(t,\gamma_k(t))|^2 dt \le C|\ln h|^2 \left(\|v-\chi\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}} \|\pi_k v - \chi\|^2_{L^2(I;L^p(\Omega))}\right).$$

By the triangle inequality and the above estimate we deduce

$$\int_0^T |(v - v_{kh})(t, \gamma_k(t))|^2 dt \le \int_0^T |(v_{kh} - \chi)(t, \gamma_k(t))|^2 dt + \int_0^T |(v - \chi)(t, \gamma_k(t))|^2 dt$$
$$\le C|\ln h|^2 \left(\|v - \chi\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}} \|\pi_k v - \chi\|^2_{L^2(I;L^p(\Omega))} \right).$$

Taking the infimum over χ, we obtain Theorem 2.

3.4 Interior Error Estimate

To obtain optimal error estimates we will also require the following interior result.

Theorem 3 (Interior Approximation) *Let $B_{d,m} := B_d(\gamma(t_m))$ denote a ball of radius d centered at $\gamma(t_m)$. Assume v and v_{kh} satisfy (4) and (20) respectively and let $d > 4h$. Then there exists a constant C independent of h, k and d such that for any $1 \le p \le \infty$*

$$\int_0^T |(v - v_{kh})(t, \gamma_k(t))|^2 dt$$
$$\le C|\ln h|^2 \inf_{\chi \in X^{0,1}_{k,h}} \left\{ \sum_{m=1}^M \left(\|v - \chi\|^2_{L^2(I_m;L^\infty(B_{d,m}))} + h^{-\frac{4}{p}} \|\pi_k v - \chi\|^2_{L^2(I_m;L^p(B_{d,m}))} \right) \right.$$
$$\left. + d^{-2} \left(\|v - \chi\|^2_{L^2(I;L^2(\Omega))} + \|\pi_k v - \chi\|^2_{L^2(I;L^2(\Omega))} + h^2 \|\nabla(v - \chi)\|^2_{L^2(I;L^2(\Omega))} \right) \right\}. \tag{41}$$

Proof To obtain the interior estimate we introduce a smooth cut-off function ω in space and piecewise constant in time, such that $\omega_m := \omega|_{I_m}$,

$$\omega_m(x) \equiv 1, \quad x \in B_{d/2,m} \tag{42a}$$
$$\omega_m(x) \equiv 0, \quad x \in \Omega \setminus B_{d,m} \tag{42b}$$
$$|\nabla \omega_m| \le C d^{-1}, \quad |\nabla^2 \omega_m| \le C d^{-2}, . \tag{42c}$$

As in the proof of Theorem 2 we obtain by (29) that

$$\int_0^T |v_{kh}(t, \gamma_k(t))|^2 dt = B(v_{kh}, g_{kh}) = B(v, g_{kh}) = B(\omega v, g_{kh}) + B((1 - \omega)v, g_{kh}), \tag{43}$$

where g_{kh} is the solution of (28). Note that ωv is discontinuous in time. The first term can be estimated using the global result from Theorem 2. To this end we introduce the solution $\tilde{v}_{kh} \in X^{0,1}_{k,h}$ defined by

$$B(\tilde{v}_{kh} - \omega v, \varphi_{kh}) = 0 \quad \text{for all } \varphi_{kh} \in X^{0,1}_{k,h}.$$

There holds

$$\begin{aligned} B(\omega v, g_{kh}) = B(\tilde{v}_{kh}, g_{kh}) &= \int_0^T v_{kh}(t, \gamma_k(t)) \tilde{v}_{kh}(t, \gamma_k(t))\, dt \\ &\leq \frac{1}{2} \int_0^T |v_{kh}(t, \gamma_k(t))|^2\, dt + \frac{1}{2} \int_0^T |\tilde{v}_{kh}(t, \gamma_k(t))|^2\, dt. \end{aligned}$$

Applying Theorem 2 for the second term, we have

$$\begin{aligned} \int_0^T |\tilde{v}_{kh}(t, \gamma_k(t))|^2 dt &\leq C |\ln h|^2 \left(\|\omega v\|^2_{L^2(I;L^\infty(\Omega))} + h^{-\frac{4}{p}} \|\pi_k(\omega v)\|^2_{L^2(I;L^p(\Omega))} \right) \\ &\leq C |\ln h|^2 \sum_{m=1}^M \left(\|v\|^2_{L^2(I_m;L^\infty(B_{d,m}))} + h^{-\frac{4}{p}} \|\pi_k v\|^2_{L^2(I_m;L^p(B_{d,m}))} \right). \end{aligned}$$

From (43), canceling $\frac{1}{2} \int_0^T |v_{kh}(t, \gamma_k(t))|^2\, dt$ and using the above estimate, we obtain

$$\begin{aligned} \int_0^T |v_{kh}(t, \gamma_k(t))|^2 dt &\leq B((1-\omega)v, g_{kh}) \\ &+ C |\ln h|^2 \sum_{m=1}^M \left(\|v\|^2_{L^2(I_m;L^\infty(B_{d,m}))} + h^{-\frac{4}{p}} \|\pi_k v\|^2_{L^2(I_m;L^p(B_{d,m}))} \right). \end{aligned} \tag{44}$$

It remains to estimate the term $B((1-\omega)v, g_{kh})$. Using the dual expression (19) of the bilinear form B we obtain

$$\begin{aligned} B((1-\omega)v, g_{kh}) &= \sum_{m=1}^M \Big((\nabla((1-\omega_m)v), \nabla g_{kh})_{I_m \times \Omega} - ((1-\omega_m)v_m, [g_{kh}]_m)_\Omega \Big) \\ &= J_1 + J_2. \end{aligned} \tag{45}$$

To estimate J_1 we define $\psi = (1-\omega)v$ and proceed using the Ritz projection $R_h \colon H^1_0(\Omega) \to V_h$ defined by

$$(\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega, \quad \forall \chi \in V_h. \tag{46}$$

There holds

$$\begin{aligned}(\nabla\psi,\nabla g_{kh})_{I_m\times\Omega} &= (\nabla R_h\psi,\nabla g_{kh})_{I_m\times\Omega} = -(R_h\psi,\Delta_h g_{kh})_{I_m\times\Omega}\\ &= -(R_h\psi,\Delta_h g_{kh})_{I_m\times B_{d/4,m}} - (R_h\psi,\Delta_h g_{kh})_{I_m\times\Omega\setminus B_{d/4,m}}\\ &\le \|R_h\psi\|_{L^2(I_m;L^\infty(B_{d/4,m}))}\|\Delta_h g_{kh}\|_{L^2(I_m;L^1(B_{d/4,m}))}\\ &\quad + \|\sigma_{k,m}^{-1}R_h\psi\|_{L^2(I_m\times\Omega\setminus B_{d/4,m})}\|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times\Omega)}.\end{aligned}$$

Using the estimate

$$\begin{aligned}\|\Delta_h g_{kh}\|_{L^2(I_m;L^1(B_{d/4,m}))} &\le \|\sigma_{k,m}^{-1}\|_{L^2(\Omega)}\|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times B_{d/4,m})}\\ &\le C|\ln h|^{\frac{1}{2}}\|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times\Omega)},\end{aligned}$$

where in the last step we used (25a), we obtain

$$\begin{aligned}(\nabla\psi,\nabla g_{kh})_{I_m\times\Omega} \le C|\ln h|^{\frac{1}{2}}&\left(\|R_h\psi\|_{L^2(I_m;L^\infty(B_{d/4,m}))} + \|\sigma_{k,m}^{-1}R_h\psi\|_{L^2(I_m\times\Omega\setminus B_{d/4,m})}\right)\\ &\times\|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times\Omega)}.\end{aligned} \tag{47}$$

By the interior pointwise error estimates from Theorem 5.1 in [30], we have for each $t\in I_m$,

$$\begin{aligned}\|R_h\psi(t)\|_{L^\infty(B_{d/4,m})} &\le c|\ln h|\|\psi(t)\|_{L^\infty(B_{d/2,m})} + Cd^{-1}\|R_h\psi(t)\|_{L^2(B_{d/2,m})}\\ &= Cd^{-1}\|R_h\psi(t)\|_{L^2(B_{d/2,m})},\end{aligned}$$

since the support of $\psi_m = (1-\omega_m)v$ is contained in $\Omega\setminus B_{d/2,m}$. On $\Omega\setminus B_{d/4,m}$ there holds $\sigma_{k,m}\ge d/4$ and therefore for each $t\in I_m$,

$$\|\sigma_{k,m}^{-1}R_h\psi(t)\|_{L^2(\Omega\setminus B_{d/4,m})} \le Cd^{-1}\|R_h\psi(t)\|_{L^2(\Omega\setminus B_{d/4,m})}.$$

Inserting the last two estimates into (47) we get

$$(\nabla\psi,\nabla g_{kh})_{I_m\times\Omega} \le Cd^{-1}|\ln h|^{\frac{1}{2}}\|R_h\psi\|_{L^2(I_m\times\Omega)}\|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times\Omega)}.$$

Using a standard elliptic estimate and recalling $\psi = (1-\omega)v$ we have

$$\begin{aligned}\|R_h\psi(t)\|_{L^2(\Omega)} &\le \|\psi(t)\|_{L^2(\Omega)} + \|\psi(t)-R_h\psi(t)\|_{L^2(\Omega)}\\ &\le \|\psi(t)\|_{L^2(\Omega)} + ch\|\nabla\psi(t)\|_{L^2(\Omega)}\\ &\le \|v(t)\|_{L^2(\Omega)} + ch\|(1-\omega(t))\nabla v(t) - \nabla\omega(t)v(t)\|_{L^2(\Omega)}\\ &\le c\|v(t)\|_{L^2(\Omega)} + ch\|\nabla v(t)\|_{L^2(\Omega)},\end{aligned}$$

where in the last step we used $|\nabla\omega(t)| \le cd^{-1} \le ch^{-1}$. This results in

$$(\nabla\psi, \nabla g_{kh})_{I_m\times\Omega} \le Cd^{-1}|\ln h|^{\frac{1}{2}} \left(\|v\|_{L^2(I_m\times\Omega)} + h\|\nabla v\|_{L^2(I_m\times\Omega)}\right) \|\sigma_{k,m}\Delta_h g_{kh}\|_{L^2(I_m\times\Omega)}.$$

Therefore, we get

$$J_1 \le cd^{-1}|\ln h|^{\frac{1}{2}} \left(\|v\|_{L^2(I;L^2(\Omega))} + h\|\nabla v\|_{L^2(I;L^2(\Omega))}\right) \|\sigma_k\Delta_h g_{kh}\|_{L^2(I;L^2(\Omega))}. \tag{48}$$

For J_2 we obtain

$$\begin{aligned} J_2 &\le \sum_{m=1}^{M} \|\sigma_m^{-1}(1-\omega_m)v_m\|_{L^2(\Omega)} k_m^{\frac{1}{2}} k_m^{-\frac{1}{2}} \|\sigma_{k,m}[g_{kh}]_m\|_{L^2(\Omega)} \\ &\le C\left(\sum_{m=1}^{M} d^{-2}k_m\|(1-\omega_m)v_m\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\sum_{m=1}^{M} k_m^{-1}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\le Cd^{-1}\|\pi_k v\|_{L^2(I;L^2(\Omega))} \left(\sum_{m=1}^{M} k_m^{-1}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \end{aligned} \tag{49}$$

where we used that $supp(1-\omega_m)v_m \subset \Omega \setminus B_{d/2,m}$ and $\sigma_{k,m} \ge d/2$ on this set as well as the definition of π_k (17). Inserting the estimate (48) for J_1 and the estimate (49) for J_2 into (45) we obtain

$$\begin{aligned} B((1-\omega)v, g_{kh}) \le Cd^{-1}|\ln h|^{\frac{1}{2}} &\left(\sum_{m=1}^{M} \|\sigma_{k,m}\Delta_h g_{kh}\|^2_{L^2(I_m\times\Omega)} + k_m^{-1}\|\sigma_{k,m}[g_{kh}]_m\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\times \left(\|v\|_{L^2(I;L^2(\Omega))} + h\|\nabla v\|_{L^2(I;L^2(\Omega))} + \|\pi_k v\|_{L^2(I;L^2(\Omega))}\right). \end{aligned}$$

Using the estimate (34) and Lemma 4

$$\begin{aligned} B((1-\omega)v, g_{kh}) \le Cd^{-1}|\ln h| &\left(\int_0^T |v_{kh}(t,\gamma_k(t))|^2\, dt\right)^{\frac{1}{2}} \\ &\times \left(\|v\|_{L^2(I;L^2(\Omega))} + h\|\nabla v\|_{L^2(I;L^2(\Omega))} + \|\pi_k v\|_{L^2(I;L^2(\Omega))}\right). \end{aligned}$$

Inserting this inequality into (44) we obtain

$$\begin{aligned} \int_0^T |v_{kh}(t,\gamma_k(t))|^2 dt \le C|\ln h|^2 &\left(\sum_{m=1}^{M} \|v\|^2_{L^2(I_m;L^\infty(B_{d,m}))} + h^{-\frac{4}{p}}\|\pi_k v\|^2_{L^2(I_m;L^p(B_{d,m}))}\right) \\ &+ Cd^{-2}|\ln h|^2 \left(\|v\|^2_{L^2(I;L^2(\Omega))} + h^2\|\nabla v\|^2_{L^2(I;L^2(\Omega))} + \|\pi_k v\|^2_{L^2(I;L^2(\Omega))}\right). \end{aligned}$$

Using that the dG(0)cG(1) method is invariant on $X^{0,1}_{k,h}$, by replacing v and v_{kh} with $v - \chi$ and $v_{kh} - \chi$ for any $\chi \in X^{0,1}_{k,h}$, we obtain the estimate in Theorem 3.

4 Discretization of the Optimal Control Problem

In this section we describe the discretization of the optimal control problem (1)–(2) and prove our main result, Theorem 1. We start with discretization of the state equation. For a given control $q \in Q$ we define the corresponding discrete state $u_{kh} = u_{kh}(q) \in X^{0,1}_{k,h}$ by

$$B(u_{kh}, \varphi_{kh}) = \int_0^T q(t)\varphi_{kh}(t, \gamma_k(t))\, dt \quad \text{for all } \; \varphi_{kh} \in X^{0,1}_{k,h}. \tag{50}$$

Using the weak formulation for $u = u(q)$ from Proposition 2.2 we obtain the perturbed Galerkin orthogonality,

$$B(u - u_{kh}, \varphi_{kh}) = \int_0^T q(t)\left(\varphi_{kh}(t, \gamma(t)) - \varphi_{kh}(t, \gamma_k(t))\right)\, dt \quad \text{for all } \; \varphi_{kh} \in X^{0,1}_{k,h}. \tag{51}$$

Note, that the jump terms involving u vanish due to the fact that

$$u \in H^1(I; W^{-1,s}(\Omega)) \hookrightarrow C(I; W^{-1,s}(\Omega))$$

and $\varphi_{kh,m} \in W^{1,\infty}(\Omega)$.

Similarly to the continuous problem, we define the discrete reduced cost functional $j_{kh}\colon Q \to \mathbb{R}$ by

$$j_{kh}(q) = J(q, u_{kh}(q)),$$

where J is the cost function in (1). The discretized optimal control problem is then given as

$$\min j_{kh}(q), \quad q \in Q_{\text{ad}}, \tag{52}$$

where Q_{ad} is the set of admissible controls (9). We note, that the control variable q is not explicitly discretized, cf. [18]. With standard arguments one proves the existence of a unique solution $\bar{q}_{kh} \in Q_{\text{ad}}$ of (52). Due to convexity of the problem, the following condition is necessary and sufficient for the optimality,

$$j'_{kh}(\bar{q}_{kh})(\partial q - \bar{q}_{kh}) \geq 0 \quad \text{for all } \; \partial q \in Q_{\text{ad}}. \tag{53}$$

As on the continuous level, the directional derivative $j'_{kh}(q)(\partial q)$ for given $q, \partial q \in Q$ can be expressed as

$$j'_{kh}(q)(\partial q) = \int_I (\alpha q(t) + z_{kh}(t, \gamma_k(t)))\, \partial q(t)\, dt,$$

where $z_{kh} = z_{kh}(q)$ is the solution of the discrete adjoint equation

$$B(\varphi_{kh}, z_{kh}) = (u_{kh}(q) - \hat{u}, \varphi_{kh})_{I\times\Omega} \quad \text{for all} \;\; \varphi_{kh} \in X^{0,1}_{k,h}. \tag{54}$$

The discrete adjoint state, which corresponds to the discrete optimal control $\bar{q}_{kh}$ is denoted by $\bar{z}_{kh} = z(\bar{q}_{kh})$. The variational inequality (53) is equivalent to the following pointwise projection formula, cf. (12),

$$\bar{q}_{kh}(t) = P_{Q_{\text{ad}}}\left(-\frac{1}{\alpha}\bar{z}_{kh}(t, \gamma_k(t))\right),$$

or

$$\bar{q}_{kh,m} = P_{Q_{\text{ad}}}\left(-\frac{1}{\alpha}\bar{z}_{kh,m}(\gamma_{k,m})\right),$$

on each I_m. Due to the fact that $\bar{z}_{kh} \in X^{0,1}_{k,h}$, we have $\bar{z}_{kh}(t, \gamma_k(t))$ is piecewise constant and therefore by the projection formula also $\bar{q}_{kh}$ is piecewise constant. As a result no explicit discretization of the control variable is required.

To prove Theorem 1 we first need estimates for the error in the state and in the adjoint variables for a given (fixed) control q. Due to the structure of the optimality conditions, we will have to estimate the error $\|z(\cdot, \gamma(\cdot)) - z_{kh}(\cdot, \gamma_k(\cdot))\|_I$, where $z = z(q)$ and $z_{kh} = z_{kh}(q)$. Note, that z_{kh} is not the Galerkin projection of z due to the fact that the right-hand side of the adjoint equation (11) involves $u = u(q)$ and the right-hand side of the discrete adjoint equation (54) involves $u_{kh} = u_{kh}(q)$. To obtain an estimate of optimal order, we will first estimate the error $u - u_{kh}$ with respect to the $L^2(I; L^1(\Omega))$ norm. Note, that an L^2 estimate would not lead to an optimal result.

Theorem 4 *Let $q \in Q$ be given and let $u = u(q)$ be the solution of the state equation (2) and $u_{kh} = u_{kh}(q) \in X^{0,1}_{k,h}$ be the solution of the discrete state equation (50). Then there holds the following estimate*

$$\|u - u_{kh}\|_{L^2(I;L^1(\Omega))} \leq \left(C|\ln h|^2(k + h^2) + C_\gamma |\ln h| k\right) \|q\|_I.$$

Proof We denote by $e = u - u_{kh}$ the error and consider the following auxiliary dual problem

$$\begin{aligned} -w_t(t,x) - \Delta w(t,x) &= b(t,x), && (t,x) \in I \times \Omega, \\ w(t,x) &= 0, && (t,x) \in I \times \partial\Omega, \\ w(T,x) &= 0, && x \in \Omega, \end{aligned}$$

where

$$b(t,x) = \operatorname{sgn}(e(t,x))\|e(t,\cdot)\|_{L^1(\Omega)} \in L^2(I; L^\infty(\Omega))$$

and the corresponding discrete solution $w_{kh} \in X_{k,h}^{0,1}$ defined by

$$B(\varphi_{kh}, w - w_{kh}) = 0, \quad \forall \varphi_{kh} \in X_{k,h}^{0,1}.$$

Using (51) for $e = u - u_{kh}$ and the Galerkin orthogonality for $w - w_{kh}$ we obtain,

$$\begin{aligned}
\int_0^T \|e(t,\cdot)\|_{L^1(\Omega)}^2\,dt &= (e, \operatorname{sgn}(e)\|e(t,\cdot)\|_{L^1(\Omega)})_{I\times\Omega} \\
&= (e,b)_{I\times\Omega} \\
&= B(e,w) \\
&= B(e, w - w_{kh}) + B(e, w_{kh}) \\
&= B(u, w - w_{kh}) + B(e, w_{kh}) \\
&= \int_0^T q(t)(w - w_{kh})(t,\gamma(t))dt + \int_0^T q(t)(w_{kh}(t,\gamma(t)) - w_{kh}(t,\gamma_k(t)))dt \\
&= \int_0^T q(t)(w(t,\gamma(t)) - w_{kh}(t,\gamma_k(t)))dt \\
&\le \|q\|_I \left(\int_0^T |w(t,\gamma(t)) - w_{kh}(t,\gamma_k(t))|^2 dt\right)^{\frac{1}{2}} \\
&\le \|q\|_I \left(\int_0^T \left(|w(t,\gamma(t)) - w(t,\gamma_k(t))|^2 + |(w - w_{kh})(t,\gamma_k(t))|^2\right) dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{55}$$

Using the local estimate from Theorem 3 with $B_{d,m} \subset \Omega_1$ for any $m = 1,\ldots,M$, where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, we obtain

$$\begin{aligned}
&\int_0^T |(w - w_{kh})(t,\gamma_k(t))|^2 dt \\
&\quad\le C|\ln h|^2 \int_0^T \left(\|w - \chi\|_{L^\infty(\Omega_1)}^2 + h^{-\frac{4}{p}}\|\pi_k w - \chi\|_{L^p(\Omega_1)}^2\right) dt \\
&\qquad + C|\ln h|^2 \int_0^T \left(\|w - \chi\|_{L^2(\Omega)}^2 + h^2\|\nabla(w - \chi)\|_{L^2(\Omega)}^2 + \|\pi_k w - \chi\|^2\right) dt \\
&\quad= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

We take $\chi = i_h\pi_k w$, where i_h is the modified Clément interpolant and π_k is the projection defined in (17). Thus, by the triangle inequality, approximation theory, inverse inequality and the stability of the Clément interpolant in L^p norm, we have

$$\begin{aligned} J_1 &\le C|\ln h|^2 \int_0^T \left(\|w - i_h w\|^2_{L^\infty(\Omega_1)} + \|i_h(w-\pi_k w)\|^2_{L^\infty(\Omega_1)}\right) dt \\ &\le C|\ln h|^2 \int_0^T \left(h^{4-\frac{4}{p}}\|w\|^2_{W^{2,p}(\Omega_1)} + h^{-\frac{4}{p}}\|i_h(w-\pi_k w)\|^2_{L^p(\Omega_1)}\right) dt \\ &\le Ch^{-\frac{4}{p}}|\ln h|^2(h^4+k^2)\int_0^T \left(\|w\|^2_{W^{2,p}(\Omega_1)} + \|w_t\|^2_{L^p(\Omega_1)}\right) dt. \end{aligned}$$

J_2 can be estimated similarly since for $\chi = i_h\pi_k w$ by the triangle inequality we have

$$\|\pi_k w - i_h\pi_k w\|_{L^p(\Omega)} \le \|\pi_k w - w\|_{L^p(\Omega)} + \|w - i_h w\|_{L^p(\Omega)} + \|i_h(w-\pi_k w)\|_{L^p(\Omega)}.$$

As a result

$$J_1 + J_2 \le Ch^{-\frac{4}{p}}|\ln h|^2(h^4+k^2)\int_0^T \left(\|w\|^2_{W^{2,p}(\Omega_1)} + \|w_t\|^2_{L^p(\Omega_1)}\right) dt.$$

Using Lemma 2, we obtain

$$\int_0^T \left(\|w\|^2_{W^{2,p}(\Omega_1)} + \|w_t\|^2_{L^p(\Omega_1)}\right) dt \le Cp^2\|b\|^2_{L^2(I;L^p(\Omega))} \le Cp^2\|e\|^2_{L^2(I;L^1(\Omega))}, \tag{56}$$

and hence

$$J_1 + J_2 \le Ch^{-\frac{4}{p}}|\ln h|^2(h^4+k^2)p^2\|e\|^2_{L^2(I;L^1(\Omega))}. \tag{57}$$

For the terms J_3 and J_4 we obtain using an L^2-estimate from [25]

$$\begin{aligned} J_3 + J_4 &\le C|\ln h|^2(h^4+k^2)\left(\|\nabla^2 w\|^2_{L^2(I;L^2(\Omega))} + \|w_t\|^2_{L^2(I;L^2(\Omega))}\right) \\ &\le C|\ln h|^2(h^4+k^2)\|b\|^2_{L^2(I;L^2(\Omega))} \\ &\le C|\ln h|^2(h^4+k^2)\|e\|^2_{L^2(I;L^1(\Omega))}. \end{aligned}$$

J_5 can be estimated similarly since by the triangle inequality

$$\|\pi_k w - i_h\pi_k w\|_{L^2(I\times\Omega)} \le \|\pi_k w - w\|_{L^2(I\times\Omega)} + \|w - i_h\pi_k w\|_{L^2(I\times\Omega)}.$$

On the other hand using that $w \in L^2(I; W^{2,p}(\Omega_0))$ for $p > 2$ and that $W^{2,p}(\Omega_0)) \hookrightarrow C^1(\Omega_0)$ for $p > 2$, and using Assumption 1, we have

$$\begin{aligned}\int_0^T |w(t,\gamma(t)) - w(t,\gamma_k(t))|^2 dt &\le \int_0^T \|w(t,\cdot)\|^2_{C^1(\Omega_0)} |\gamma(t) - \gamma_k(t)|^2 dt \\ &\le C\|\gamma - \gamma_k\|^2_{C^0(I)} \int_0^T \|w(t,\cdot)\|^2_{W^{2,p}(\Omega_0)} dt \\ &\le CC_\gamma^2 k^2 \|w\|^2_{L^2(I;W^{2,p}(\Omega_0))} \\ &\le CC_\gamma^2 k^2 p^2 \|b\|^2_{L^2(I;L^p(\Omega))} \\ &\le CC_\gamma^2 k^2 p^2 \|e\|^2_{L^2(I;L^1(\Omega))},\end{aligned}$$

where in the last two steps we used (56). Combining the estimate for J_1, J_2, J_3, J_4, J_5 and the above estimate and inserting them into (55) we obtain:

$$\|e\|_{L^2(I;L^1(\Omega))} \le \left(C|\ln h|(ph^{-\frac{2}{p}} + 1)(h^2 + k) + C_\gamma pk\right) \|q\|_{L^2(I)}.$$

Setting $p = |\ln h|$ completes the proof.

In the following theorem we provide an estimate of the error in the adjoint state for fixed control q.

Theorem 5 *Let $q \in Q$ be given and let $z = z(q)$ be the solution of the adjoint equation (11) and $z_{kh} = z_{kh}(q) \in X^{0,1}_{k,h}$ be the solution of the discrete adjoint equation (54). Then there holds the following estimate*

$$\left(\int_0^T |z(t,\gamma(t)) - z_{kh}(t,\gamma_k(t))|^2 \, dt\right)^{\frac{1}{2}} \le C\left(|\ln h|^3 (k + h^2) + C_\gamma |\ln h| k\right)\left(\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^\infty(\Omega))}\right).$$

Proof First by the triangle inequality

$$\int_0^T |z(t,\gamma(t)) - z_{kh}(t,\gamma_k(t))|^2 \, dt \le \int_0^T |z(t,\gamma(t)) - z(t,\gamma_k(t))|^2 \, dt + \int_0^T |(z - z_{kh})(t,\gamma_k(t))|^2 \, dt.$$

Using Proposition 2.3 and the assumptions on γ, we have similarly to Theorem 4

$$\begin{aligned}\int_0^T |z(t,\gamma(t)) - z(t,\gamma_k(t))|^2\,dt &\le C\|\gamma-\gamma_k\|^2_{C^0(I)} \int_0^T \|z(t,\cdot)\|^2_{C^1(\Omega_0)}\,dt\\ &\le CC_\gamma^2 k^2 \int_0^T \|z(t,\cdot)\|^2_{W^{2,p}(\Omega_0)}\,dt\\ &\le CC_\gamma^2 pk^2\left(\|q\|^2_{L^2(I)} + \|\hat{u}\|^2_{L^2(I\times\Omega)}\right).\end{aligned}$$

Setting $p = |\ln h|$, we obtain

$$\left(\int_0^T |z(t,\gamma(t)) - z(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \le CC_\gamma|\ln h|k\left(\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I\times\Omega)}\right). \tag{58}$$

Next, we introduce an intermediate adjoint state $\widetilde{z}_{kh} \in X^{0,1}_{k,h}$ defined by

$$B(\varphi_{kh},\widetilde{z}_{kh}) = (u-\hat{u},\varphi_{kh}) \quad \text{for all} \ \ \varphi_{kh}\in X^{0,1}_{k,h},$$

where $u = u(q)$ and therefore $\widetilde{z}_{kh}$ is the Galerkin projection of z. By the local best approximation result of Theorem 3 for any $\chi \in X^{0,1}_{k,h}$ we have

$$\begin{aligned}\int_0^T |(z-\widetilde{z}_{kh})(t,\gamma_k(t))|^2\,dt &\le C|\ln h|^2\int_0^T\left(\|z-\chi\|^2_{L^\infty(\Omega_1)} + h^{-\frac{4}{p}}\|\pi_k z-\chi\|^2_{L^p(\Omega_1)}\right)dt\\ &+ C|\ln h|^2\int_0^T\left(\|z-\chi\|^2_{L^2(\Omega)} + h\|\nabla(z-\chi)\|^2_{L^2(\Omega)} + \|\pi_k z-\chi\|^2_{L^2(\Omega)}\right)dt\\ &= J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

The terms J_1, J_2, J_3, J_4 and J_5 can be estimated the same way as in the proof of Theorem 4 using the regularity result for the adjoint state z from Proposition 2.3. This results in

$$\int_0^T |(z-\widetilde{z}_{kh})(t,\gamma_k(t)|^2\,dt \le C|\ln h|^2(ph^{-\frac{2}{p}}+1)^2(h^2+k)^2\left(\|q\|^2_{L^2(I)} + \|\hat{u}\|^2_{L^2(I;L^\infty(\Omega))}\right).$$

Setting $p = |\ln h|$ and taking square root, we obtain

$$\left(\int_0^T |(z-\widetilde{z}_{kh})(t,\gamma_k(t))|^2\,dt\right)^{\frac{1}{2}} \le C|\ln h|^2(h^2+k)\left(\|q\|_{L^2(I)} + \|\hat{u}\|_{L^2(I;L^\infty(\Omega))}\right). \tag{59}$$

It remains to estimate the corresponding error between $\widetilde{z}_{kh}$ and z_{kh}. We denote $e_{kh} = \widetilde{z}_{kh} - z_{kh} \in X^{0,1}_{k,h}$. Then we have

$$B(\varphi_{kh}, e_{kh}) = (u - u_{kh}, \varphi_{kh})_{I\times\Omega} \quad \text{for all } \varphi \in X^{0,1}_{k,h}.$$

As in the proof of Lemma 5 we use the fact that

$$\|\nabla v\|^2_{L^2(I\times\Omega)} \le B(v, v)$$

holds for all $v \in X^{0,1}_{k,h}$. Applying this inequality together with the discrete Sobolev inequality, see [4], results in

$$\begin{aligned}
\|e_{kh}\|^2_{L^2(I;L^\infty(\Omega))} &\le C|\ln h|\|\nabla e_{kh}\|^2_{L^2(I\times\Omega)}\\
&\le C|\ln h|B(e_{kh}, e_{kh})\\
&= C|\ln h|(u - u_{kh}, e_{kh})_{I\times\Omega}\\
&\le C|\ln h|\|u - u_{kh}\|_{L^2(I;L^1(\Omega))}\|e_{kh}\|_{L^2(I;L^\infty(\Omega))}.
\end{aligned}$$

Therefore

$$\|e_{kh}\|_{L^2(I;L^\infty(\Omega))} \le C|\ln h|\|u - u_{kh}\|_{L^2(I;L^1(\Omega))}.$$

Using Theorem 4 we obtain

$$\|e_{kh}\|_{L^2(I;L^\infty(\Omega))} \le C\left(|\ln h|^3(k + h^2) + C_\gamma|\ln h|k\right)\|q\|_{L^2(I)}.$$

Combining this estimate with (59) we complete the proof.

Using the result of Theorem 5 we proceed with the proof of Theorem 1.

Proof Due to the quadratic structure of discrete reduced functional j_{kh} the second derivative $j'_{kh}(q)(p,p)$ is independent of q and there holds

$$j'_{kh}(q)(p,p) \ge \alpha\|p\|^2_{L^2(I)} \quad \text{for all } p \in Q. \tag{60}$$

Using optimality conditions (10) for $\bar{q}$ and (53) for $\bar{q}_{kh}$ and the fact that $\bar{q}, \bar{q}_{kh} \in Q_{\text{ad}}$ we obtain

$$-j'_{kh}(\bar{q}_{kh})(\bar{q} - \bar{q}_{kh}) \le 0 \le -j'(\bar{q})(\bar{q} - \bar{q}_{kh}).$$

Using the coercivity (60) we get

$$\begin{aligned}
\alpha\|\bar{q} - \bar{q}_{kh}\|^2_{L^2(I)} &\le j'_{kh}(\bar{q})(\bar{q} - \bar{q}_{kh}, \bar{q} - \bar{q}_{kh})_I\\
&= j'_{kh}(\bar{q})(\bar{q} - \bar{q}_{kh}) - j'_{kh}(\bar{q}_{kh})(\bar{q} - \bar{q}_{kh})\\
&\le j'_{kh}(\bar{q})(\bar{q} - \bar{q}_{kh}) - j'(\bar{q})(\bar{q} - \bar{q}_{kh})
\end{aligned}$$

$$= (z(\bar{q})(t, \gamma(t)) - z_{kh}(\bar{q})(t, \gamma_k(t)), \bar{q} - \bar{q}_{kh})_I$$
$$\le \left(\int_0^T |z(\bar{q})(t, \gamma(t)) - z_{kh}(\bar{q})(t, \gamma_k(t))|^2 \, dt \right)^{\frac{1}{2}} \|\bar{q} - \bar{q}_{kh}\|_{L^2(I)}.$$

Applying Theorem 5 completes the proof.

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A Note on the Feedback Stabilization of a Cahn–Hilliard Type System with a Singular Logarithmic Potential

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Abstract This article deals with the internal feedback stabilization of a phase field system of Cahn–Hilliard type involving a logarithmic potential F, and extends the recent results provided in Barbu et al. (J Differ Equ 262:2286–2334, 2017) for the double-well potential. The stabilization is searched around a stationary solution, by a feedback controller with support in a subset ω of the domain. The controller stabilizing the linearized system is constructed as a finite combination of the unstable modes of the operator acting in the linear system and it is further provided in a feedback form by solving a certain minimization problem. Finally, it is proved that this feedback form stabilizes the nonlinear system too, if the stationary solution has not large variations. All these results are provided in the three-dimensional case for a regularization of the singular potential F, and allow the same conclusion for the singular logarithmic potential in the one-dimensional case.

Keywords Cahn–Hilliard system • Closed loop system • Feedback control • Logarithmic potential • Stabilization

AMS (MOS) Subject Classification 93D15, 35K52, 35Q79, 35Q93, 93C20

1 Problem Description

We address the local stabilization of the Cahn–Hilliard system (see [8]) consisting in the equations for the phase field φ and chemical potential μ, (1.2)–(1.3) coupled according to the Caginalp approach (see [6, 7]) with the energy balance

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equation (1.1) for the temperature θ. The system reads

$$(\theta + l_0\varphi)_t - \Delta\theta = 0, \text{ in } (0,\infty)\times\Omega, \tag{1.1}$$

$$\varphi_t - \Delta\mu = 0, \text{ in } (0,\infty)\times\Omega, \tag{1.2}$$

$$\mu = -\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta, \text{ in } (0,\infty)\times\Omega, \tag{1.3}$$

to which we add initial data and homogeneous Neumann boundary conditions

$$\theta(0) = \theta_0, \ \varphi(0) = \varphi_0, \text{ in } \Omega, \tag{1.4}$$

$$\frac{\partial\theta}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = \frac{\partial\mu}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega. \tag{1.5}$$

Here, ν is the outward normal vector to the boundary, l_0, γ_0 are positive constants with some physical meaning, and F' is the derivative of the logarithmic potential

$$F(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r) - ar^2, \text{ for } r\in(-1,1), \tag{1.6}$$

(see e.g., [9, 10]), where a is positive and large enough to prevent the convexity. The space domain Ω is an open bounded connected subset of $\mathbb{R}^d$, $d = 1,2,3$, enough regular, and the time $t \in (0,\infty)$. For some auxiliary results we keep the dimension d up to 3.

We shall investigate the stabilization of this system, around a stationary solution to (1.1)–(1.5), by two controllers (u, v) with the support in an open subset ω of Ω, and acting on the right-hand sides of Eqs. (1.1)–(1.2). After the introduction of the expression of μ given by (1.3) into (1.2) the system under discussion is

$$\varphi_t - \Delta\left(-\nu\Delta\varphi + F'(\varphi) - \gamma_0\theta\right) = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega, \tag{1.7}$$

$$(\theta + l_0\varphi)_t - \Delta\theta = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega, \tag{1.8}$$

$$\varphi(0) = \varphi_0, \ \theta(0) = \theta_0, \text{ in } \Omega, \tag{1.9}$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial(\Delta\varphi)}{\partial\nu} = \frac{\partial\theta}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega, \tag{1.10}$$

where the second boundary condition in (1.10) follows by (1.3) and (1.5).

Here, the function 1^*_ω belongs to $C_0^\infty(\Omega)$, has the support supp(1^*_ω) in ω and it is positive on some subset ω_0 of positive surface measure included in ω.

First of all we mention that we assume that the solutions to the stationary uncontrolled system (1.1)–(1.5) exist. In particular, we observe that it may have constant solutions $(\varphi_\infty, \theta_\infty)$. Anyway, we shall assume that φ_∞ is a regular function, less than 1, but we do not deal here with the proof of the existence of such solutions.

The system will be stabilized exponentially around the stationary solution given by $(\varphi_\infty, \theta_\infty)$, using the feedback control (v, u) computed as a function of φ and θ and this turns out to prove that $\lim_{t\to\infty}(\varphi(t), \theta(t)) = (\varphi_\infty, \theta_\infty)$, with an exponential rate of convergence, whether the initial datum (φ_0, θ_0) is in a certain neighborhood of $(\varphi_\infty, \theta_\infty)$.

The result we are going to prove extends the results provided by Barbu, Colli, Gilardi and Marinoschi in a previous paper (see [5]) for the regular potential $F(\varphi) = \frac{(\varphi^2-1)}{4}$. The technique we shall approach is that introduced in [11] and used then in [1–4] for Navier-Stokes equations and nonlinear parabolic systems and relies on the construction of the feedback controller as a linear combination of the unstable modes of the corresponding linearized system.

We shall follow [5] by making the function transformation $\sigma = \alpha_0(\theta + l_0\varphi)$, with $\alpha_0 := \sqrt{\frac{\gamma_0}{l_0}}$, chosen such that $\frac{\gamma_0}{\alpha_0} = \alpha_0 l_0 =: \gamma > 0$. This is done to enhance the possibility to obtain later a self-adjoint operator acting in the linear part of the system. Denoting $l := \gamma_0 l_0$, we get the equivalent system in the variables φ and σ

$$\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - l\Delta\varphi + \gamma\Delta\sigma = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega, \tag{1.11}$$

$$\sigma_t - \Delta\sigma + \gamma\Delta\varphi = 1^*_\omega \alpha_0 u, \text{ in } (0,\infty)\times\Omega, \tag{1.12}$$

$$\varphi(0) = \varphi_0,\ \sigma(0) = \sigma_0 := \alpha_0(\theta_0 + l_0\varphi_0), \text{ in } \Omega, \tag{1.13}$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0, \text{ in } (0,\infty)\times\partial\Omega. \tag{1.14}$$

For simplicity we shall denote still by u the product $\alpha_0 u$. Next, considering the stationary system in terms of φ_∞ and σ_∞

$$\begin{aligned} \nu\Delta^2\varphi_\infty - \Delta F'(\varphi_\infty) - l\Delta\varphi_\infty + \gamma\Delta\sigma_\infty &= 0, \text{ in } \Omega,\\ -\Delta\sigma_\infty + \gamma\Delta\varphi_\infty &= 0, \text{ in } \Omega,\\ \frac{\partial\varphi_\infty}{\partial\nu} = \frac{\partial\Delta\varphi_\infty}{\partial\nu} = \frac{\partial\sigma_\infty}{\partial\nu} &= 0, \text{ on } \partial\Omega, \end{aligned} \tag{1.15}$$

we compute the difference between system (1.11)–(1.14) and system (1.15) and denoting $y = \varphi - \varphi_\infty$, $z = \sigma - \sigma_\infty$, $y_0 = \varphi_0 - \varphi_\infty$, $z_0 = \sigma_0 - \sigma_\infty$, we get the new nonlinear system

$$y_t + \nu\Delta^2 y - \Delta(F'(y+\varphi_\infty) - F'(\varphi_\infty)) - l\Delta y + \gamma\Delta z = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega, \tag{1.16}$$

$$z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega, \tag{1.17}$$

$$y(0) = y_0,\ z(0) = z_0, \text{ in } \Omega, \tag{1.18}$$

$$\frac{\partial y}{\partial\nu} = \frac{\partial\Delta y}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega. \tag{1.19}$$

As usually in the stabilization studies we shall discuss the zero stabilization of (1.16)–(1.19) around $(y_\infty, z_\infty) = (0, 0)$. Of course, by applying the backward transformations we can obtain without any difficulty the stabilization result for the initial system in (φ, θ).

The proof involves many intermediate results related to the well-posedness and properties of the solution to the linear system, its stabilization by a finite dimensional control, the computation of the feedback control and the stabilization of the nonlinear system by the feedback control. At this point, recalling that F is a singular function, for which we cannot directly apply the results provided in [5], we have to modify system (1.16)–(1.19) and to work first with a regular potential which will be obtained by applying a cut-off function to F.

Thus, for the subsequent part of the paper, let ε_0 be positive and fixed, $\varepsilon_0 \in (0, 1)$. We define $\kappa_{\varepsilon_0} \in C_0^\infty(\mathbb{R})$ such that

$$\kappa_{\varepsilon_0}(r) = \begin{cases} 1, \text{ for } |r| \le 1 - \varepsilon_0 \\ 0, \text{ for } |r| \ge 1 - \frac{\varepsilon_0}{2}, \end{cases}$$

and $0 < \kappa_{\varepsilon_0}(r) \le 1$ for $r \in (-1 + \frac{\varepsilon_0}{2}, -1 + \varepsilon_0] \cup [1 - \varepsilon_0, 1 - \frac{\varepsilon_0}{2})$, and the regularized potential

$$F_{\varepsilon_0}(r) = \begin{cases} F(r), & \text{for } r \in [1 - \varepsilon_0, 1 + \varepsilon_0] \\ F(r)\kappa_{\varepsilon_0}(r), & \text{for } r \in (-1 + \frac{\varepsilon_0}{2}, -1 + \varepsilon_0] \cup [1 - \varepsilon_0, 1 - \frac{\varepsilon_0}{2}) \\ 0, & \text{for } |r| \ge 1 - \frac{\varepsilon_0}{2}, \end{cases}$$

which is of class $C_0^\infty(\mathbb{R})$. The singular function F in system (1.16)–(1.19) will be replaced by the regular function $F_{\varepsilon_0}(r)$ and all results previously mentioned will be shown first for this regularized system.

The assumptions we shall use for the problem with the logarithmic potential are:

$$\varphi_\infty \in C^2(\overline{\Omega}), \quad |\varphi_\infty| < 1 - \varepsilon_0. \tag{1.20}$$

We write the Taylor expansion of $F'_{\varepsilon_0}(y + \varphi_\infty)$ around φ_∞,

$$F'_{\varepsilon_0}(y + \varphi_\infty) = F'_{\varepsilon_0}(\varphi_\infty) + F''_{\varepsilon_0}(\varphi_\infty)y + F_{r,\varepsilon_0}(y),$$

where $F_{r,\varepsilon_0}(y)$ is the nonlinear rest of second order and mention that, since $|\varphi_\infty| < 1 - \varepsilon_0$, all the derivatives of F_{ε_0} computed at φ_∞ coincide with the derivatives of F at φ_∞ and so we can omit for them the subscript ε_0. Therefore, (1.16) becomes

$$y_t + \nu\Delta^2 y - \Delta(F''(\varphi_\infty)y) - l\Delta y + \gamma\Delta z = \Delta F_{r,\varepsilon_0}(y) + 1_\omega^* v. \tag{1.21}$$

Proceeding as in [5], we define

$$\overline{F''_\infty} = \frac{1}{m_\Omega} \int_\Omega F''(\varphi_\infty(\xi))d\xi, \tag{1.22}$$

(the integral exists since $F''(\varphi_\infty)$ is bounded), where m_Ω is the measure of Ω, and we have $F''(\varphi_\infty(x)) = \overline{F''_\infty} + g(x)$, with

$$g(x) := \frac{1}{m_\Omega}\int_\Omega (F''(\varphi_\infty(x)) - F''(\varphi_\infty(\xi)))d\xi. \tag{1.23}$$

Replacing (1.22) in (1.21) we get the following equivalent form of the nonlinear system (1.16)–(1.19):

$$y_t + \nu\Delta^2 y - F_l\Delta y + \gamma\Delta z = \Delta(F_{r,\varepsilon_0}(y) + g(x)y) + 1^*_\omega v, \text{ in } (0,\infty)\times\Omega, \tag{1.24}$$

$$z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega, \tag{1.25}$$

$$y(0) = y_0,\ z(0) = z_0, \text{ in } \Omega, \tag{1.26}$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \text{ in } (0,\infty)\times\partial\Omega, \tag{1.27}$$

where $F_l = \overline{F''_\infty} + l$ depends on l, Ω and $C_{F''} = \|F''\|_{L^\infty(-1+\varepsilon_0,1-\varepsilon_0)}$. We recall that ε_0 is a fixed small value. The transformations of θ into σ and (1.22) allowed to get a linear system with constant coefficients and with a symmetric operator.

The outline of the paper is: in Sect. 2, the well-posedness, the stabilization of the linear system and the construction of the feedback control will be provided by resuming the results given in [5], because the linear system is the same. In Sect. 3, the stabilization of the nonlinear system (1.24)–(1.27) corresponding to the regularized function F_{ε_0} will be studied in the three-dimensional case. This will lead, on the basis of a compactness result working in one-dimension, to the stabilization of the system with the singular function F. We have also to remark that the stabilization theorem we shall obtain for the system (1.16)–(1.19) corresponding to F_{ε_0} can be seen as a stand-alone result which can be applied to other models than Cahn–Hilliard, for instance when treating systems modeling reaction-diffusion processes with nonlinear sources.

2 Stabilization of the Linear System

We observe that the linear system

$$y_t + \nu\Delta^2 y - F_l\Delta y + \gamma\Delta z = 1^*_\omega v, \text{ in } (0,\infty)\times\Omega, \tag{2.1}$$

$$z_t - \Delta z + \gamma\Delta y = 1^*_\omega u, \text{ in } (0,\infty)\times\Omega, \tag{2.2}$$

$$y(0) = y_0,\ z(0) = z_0,\ \text{in } \Omega, \tag{2.3}$$

$$\frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0,\ \text{in } (0,\infty) \times \partial\Omega \tag{2.4}$$

is identical with that obtained in [5], so that all results concerning its stabilization by a feedback controller can be preserved. We shall resume them in this section, without proofs (they are found in [5], Sect. 2). Some slight modifications which may occur will be specified.

Functional Framework Let us denote $H = L^2(\Omega)$, $V = H^1(\Omega)$, with the standard scalar products and set $V' = (H^1(\Omega))'$. Let $A : D(A) \subset H \to H$ be the linear operator

$$A = -\Delta + I,\ \ D(A) = \left\{ w \in H^2(\Omega);\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}. \tag{2.5}$$

It is m-accretive on H and so we can define its fractional powers A^α, for $\alpha \geq 0$, which are linear continuous positive and self-adjoint operators on H, with the domains $D(A^\alpha) = \{w \in H; \|A^\alpha w\|_H < \infty\}$ and the norms $\|w\|_{D(A^\alpha)} = \|A^\alpha w\|_H$. Moreover, $D(A^\alpha) \subset H^{2\alpha}(\Omega)$, with equality if and only if $\alpha > 1/4$.

In the further calculations or formula, we shall denote by C, C_i, c_i, $i = 1, 2, \ldots$ several positive constants possibly depending on the system structure (ν, l, γ), domain, space dimension, and possibly on the norms of some derivatives of F at φ_∞. In the last case this dependence will be specified. The symbol $(\cdot,\cdot)$ is a pair in a product space and $(\cdot,\cdot)_X$ is the scalar product in a Hilbert space X.

The norms in $L^\infty(\Omega)$ and $W^{2,\infty}(\Omega)$ are indicated by $\|\cdot\|_\infty$ and $\|\cdot\|_{2,\infty}$, respectively.

We introduce the self-adjoint operator $\mathscr{A} : D(\mathscr{A}) \subset H \times H \to H \times H$,

$$\mathscr{A} = \begin{bmatrix} \nu\Delta^2 - F_l\Delta & \gamma\Delta \\ \gamma\Delta & -\Delta \end{bmatrix}, \tag{2.6}$$

having the domain

$$D(\mathscr{A}) = \Big\{ w = (y,z) \in H^2(\Omega) \times H^1(\Omega);\ \mathscr{A}w \in H \times H, \\ \frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega \Big\}.$$

Since the domain Ω is regular enough it follows that $D(\mathscr{A}) \subset H^4(\Omega) \times H^2(\Omega)$. We also introduce $\mathscr{H} = H \times H$, $\mathscr{V} = D(A) \times D(A^{1/2})$, $\mathscr{V}' = (D(A) \times D(A^{1/2}))'$, with the standard scalar products and note that $\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}'$ algebraically and topologically, with compact injections.

Then, we can write (2.1)–(2.4) as

$$\frac{d}{dt}(y(t), z(t)) + \mathscr{A}(y(t), z(t)) = 1^*_\omega U(t), \text{ a.e. } t \in (0, \infty), \tag{2.7}$$

$$(y(0), z(0)) = (y_0, z_0), \tag{2.8}$$

where $U(t) = (v(t), u(t))$.

Proposition 2.1 *The operator $\mathscr{A}$ is quasi m-accretive on $\mathscr{H}$ and its resolvent is compact. Let $(y_0, z_0) \in \mathscr{H}$ and $(v, u) \in L^2(0, T; \mathscr{H})$. Then, problem (2.7)–(2.8) has, for all $T > 0$, a unique solution*

$$(y, z) \in C([0, T]; \mathscr{H}) \cap L^2(0, T; \mathscr{V}) \cap W^{1,2}(0, T; \mathscr{V}') \cap C((0, T]; \mathscr{V}).$$

We denote by λ_i and $\{(\varphi_i, \psi_i)\}_{i\geq 1}$ the eigenvalues and eigenvectors of $\mathscr{A}$, respectively. Since $\mathscr{A}$ is self-adjoint and its resolvent $(\lambda I + \mathscr{A})^{-1}$ is compact, the eigenvalues are real and there is a finite number of nonpositive eigenvalues $\lambda_i \leq 0$, each of them possibly having the order of multiplicity l_i, $i = 1, \ldots, p$. We order the sequence $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \leq 0$, where each eigenvalue is counted with its corresponding order of multiplicity and $N = l_1 + l_2 + \ldots + l_p$. The eigenvectors corresponding to distinct eigenvalues are orthogonal and as a matter of fact the system $\{(\varphi_i, \psi_i)\}_i$ can be assumed to be orthonormal.

The controller is searched in the form $1^*_\omega U(t) = \sum_{j=1}^{N} w_j(t) 1^*_\omega(\varphi_j, \psi_j)$ and it is replaced in (2.7), leading to the *open loop linear system*

$$\frac{d}{dt}(y(t), z(t)) + \mathscr{A}(y(t), z(t)) = \sum_{j=1}^{N} w_j(t) 1^*_\omega(\varphi_j, \psi_j), \text{ a.e. } t \in (0, \infty), \tag{2.9}$$

to which we attach an arbitrary initial condition in $\mathscr{H}$,

$$(y(0), z(0)) = (y^0, z^0). \tag{2.10}$$

Proposition 2.2 *There exist $w_j \in L^2(\mathbb{R}^+)$, $j = 1, \ldots, N$, such that the controller*

$$1^*_\omega U(t, x) = \sum_{j=1}^{N} w_j(t) 1^*_\omega(\varphi_j(x), \psi_j(x)), \ t \geq 0, \ x \in \Omega, \tag{2.11}$$

stabilizes exponentially system (2.9)–(2.10), that is, its solution (y, z) satisfies

$$\|y(t)\|_H + \|z(t)\|_H \leq Ce^{-kt}\left(\left\|y^0\right\|_H + \left\|z^0\right\|_H\right), \textit{ for all } t \geq 0. \tag{2.12}$$

Moreover, we have

$$\left(\sum_{j=1}^{N}\int_0^\infty \left|w_j(t)\right|^2 dt\right)^{1/2} \leq C\left(\left\|y^0\right\|_H + \left\|z^0\right\|_H\right). \tag{2.13}$$

In both formulas C and k depend on the problem parameters ν, γ, l, Ω and $C_{F''} = \|F''\|_{L^\infty(-1+\varepsilon_0,1-\varepsilon_0)}$.

In order to compute the expression of the feedback controller we introduce the quadratic minimization problem

$$\Phi(y^0, z^0) \tag{2.14}$$
$$= \underset{W\in L^2(0,\infty;\mathbb{R}^N)}{\text{Min}}\left\{\frac{1}{2}\int_0^\infty\left(\left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2 + \|W(t)\|_{\mathbb{R}^N}^2\right)dt\right\}$$

subject to (2.9)–(2.10). Here W is the function $(w_1, \ldots, w_N) \in L^2(0,\infty;\mathbb{R}^N)$ occurring in (2.9). We note that $D(\Phi) = \{(y^0, z^0) \in H \times H;\ \Phi(y^0, z^0) < \infty\}$.

Proposition 2.3 *For each pair $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$, problem (2.14) has a unique optimal solution*

$$(\{w_j^*\}_{j=1}^N, y^*, z^*) \in L^2(\mathbb{R}^+;\mathbb{R}^N)) \times L^2(\mathbb{R}^+; D(A^{3/2})) \times L^2(\mathbb{R}^+; D(A^{3/4})), \tag{2.15}$$

and

$$c_1\left(\left\|A^{1/2}y^0\right\|_H^2 + \left\|A^{1/4}z^0\right\|_H^2\right) \leq \Phi(y^0, z^0) \leq c_2\left(\left\|A^{1/2}y^0\right\|_H^2 + \left\|A^{1/4}z^0\right\|_H^2\right). \tag{2.16}$$

If $(y^0, z^0) \in D(A) \times D(A^{1/2})$, we have

$$\left(\|Ay^*(t)\|_H^2 + \left\|A^{1/2}z^*(t)\right\|_H^2\right) + \int_0^t\left(\left\|A^2y^*(s)\right\|_H^2 + \|Az^*(s)\|_H^2\right)ds \tag{2.17}$$
$$\leq c_3\left(\left\|Ay^0\right\|_H^2 + \left\|A^{1/2}z^0\right\|_H^2\right), \text{ for all } t \geq 0,$$

where c_1, c_2, c_3 are positive constants (depending on Ω, problem parameters and $C_{F''}$).

This last result implies that there exists a linear functional

$$R : D(A^{1/2}) \times D(A^{1/4}) \to (D(A^{1/2}) \times (D(A^{1/4}))',$$

such that, for all $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$ we have

$$\Phi(y^0, z^0) = \frac{1}{2}\left\langle R(y^0, z^0), (y^0, z^0)\right\rangle_{(D(A^{1/2})\times(D(A^{1/4}))', D(A^{1/2})\times D(A^{1/4})}. \tag{2.18}$$

In fact, $R(y^0, z^0)$ is the Gâteaux derivative of the function Φ at (y^0, z^0),

$$\Phi'(y^0, z^0) = R(y^0, z^0), \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4}). \tag{2.19}$$

Looking back to the open loop system (2.9) we can define the operator $B : \mathbb{R}^N \to H \times H$, and its adjoint $B^* : H \times H \to \mathbb{R}^N$, by

$$BW = \begin{bmatrix} \sum_{i=1}^{N} 1_\omega^* \varphi_i w_i \\ \sum_{i=1}^{N} 1_\omega^* \psi_i w_i \end{bmatrix} \text{ for all } W = \begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix} \in \mathbb{R}^N,$$

and

$$B^* q = \begin{bmatrix} \int_\Omega 1_\omega^* (\varphi_1 q_1 + \psi_1 q_2) dx \\ \dots \\ \int_\Omega 1_\omega^* (\varphi_N q_1 + \psi_N q_2) dx \end{bmatrix} \text{ for all } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in H \times H,$$

in order to write system (2.9)–(2.10) as

$$\frac{d}{dt}(y(t), z(t)) + \mathscr{A}(y(t), z(t)) = BW(t), \text{ a.e. } t > 0, \tag{2.20}$$
$$(y(0), z(0)) = (y^0, z^0).$$

The next proposition provides the form of the feedback controller.

Proposition 2.4 *Let $W^* = \{w_i^*\}_{i=1}^N$ and (y^*, z^*) be optimal for problem (2.14), corresponding to $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$. Then, W^* is expressed as*

$$W^*(t) = -B^* R(y^*(t), z^*(t)), \textit{ for all } t > 0. \tag{2.21}$$

Moreover, R has the following properties

$$2c_1 \left\| (y^0, z^0) \right\|^2_{D(A^{1/2}) \times D(A^{1/4})} \tag{2.22}$$
$$\leq (R(y^0, z^0), (y^0, z^0))_{H \times H} \leq 2c_2 \left\| (y^0, z^0) \right\|^2_{D(A^{1/2}) \times D(A^{1/4})}$$

for all $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/4})$,

$$\left\| R(y^0, z^0) \right\|_{H \times H} \leq C_R \left\| (y^0, z^0) \right\|_{D(A) \times D(A^{1/2})}, \tag{2.23}$$

for all $(y^0, z^0) \in D(A) \times D(A^{1/2})$, *and satisfies the Riccati algebraic equation*

$$2\,(R(\overline{y},\overline{z}),\mathscr{A}(\overline{y},\overline{z}))_{H\times H} + \|B^*R(\overline{y},\overline{z})\|^2_{\mathbb{R}^N} \tag{2.24}$$
$$= \left\|A^{3/2}\overline{y}\right\|^2_H + \left\|A^{3/4}\overline{z}\right\|^2_H,\ \textit{for all}\ (\overline{y},\overline{z}) \in D(A^{3/2}) \times D(A^{3/4}).$$

Here, c_1, c_2, C_R *are constants depending on the problem parameters,* Ω *and* $C_{F''}$.

We mention that in the case with the double-well potential (see [5]) all these constants depended on $\|\varphi_\infty\|_{L^2(\Omega)}$, instead of $C_{F''} = \|F''\|_{L^\infty(-1+\varepsilon_0,1-\varepsilon_0)}$.

In particular, the linear system is stabilized exponentially to $(0,0)$ by the feedback controller just constructed, $1^*_\omega U(t) = -BB^*R(y^*(t), z^*(t))$ (see Remark 2.6 in [5]).

3 Feedback Stabilization of the Nonlinear System

The main result refers to the stabilization of the nonlinear system (1.24)–(1.27) in which the right-hand side $1^*_\omega U(t)$ is replaced by the feedback controller determined in the previous section, that is,

$$1^*_\omega U(t) = -BB^*R(y(t), z(t)). \tag{3.1}$$

The nonlinear system in the abstract form reads

$$\frac{d}{dt}(y(t), z(t)) + \mathscr{A}(y(t), z(t)) = \mathscr{G}_{\varepsilon_0}(y(t)) - BB^*R(y(t), z(t)),\ \text{a.e. } t > 0, \tag{3.2}$$
$$(y(0), z(0)) = (y_0, z_0),$$

where $\mathscr{G}_{\varepsilon_0}(y(t)) = (G_{\varepsilon_0}(y(t)), 0)$ and $G_{\varepsilon_0}(y) = \Delta F_{r,\varepsilon_0}(y) + \Delta(g(x)y)$, by (1.24). We recall that $F_{r,\varepsilon_0}(y)$ is the rest of second order of the Taylor expansion of $F'_{\varepsilon_0}(y+\varphi_\infty)$, considered here in the integral form

$$F_{r,\varepsilon_0}(y) = y^2 \int_0^1 (1-s) F'''_{\varepsilon_0}(\varphi_\infty + sy)ds, \tag{3.3}$$

and g is defined by (1.23). We specify that the derivatives of F up to the fifth order which will be involved in the next computations are continuous on $\{r;\ |r| < 1\}$. The derivatives of F_{ε_0}, i.e., $F'_{\varepsilon_0}, F''_{\varepsilon_0}, F'''_{\varepsilon_0}, F^{(4)}_{\varepsilon_0}, F^{(5)}_{\varepsilon_0}$ are continuous on $\mathbb{R}$, hence, in particular, they are bounded on $|r| \le 1 - \frac{\varepsilon_0}{2}$ and are zero outside this ball. Moreover, these derivatives calculated at $\varphi_\infty \in (1-\varepsilon_0, 1+\varepsilon_0)$ coincide with the derivatives of F at φ_∞.

Also, taking into account the definition of F_{ε_0}, we see that all the derivatives of F_{ε_0} at $r \in (-1+\varepsilon_0/2, 1-\varepsilon_0/2)$ are bounded by constants times the supremum of the same derivatives of F on $(-1+\varepsilon_0/2, 1-\varepsilon_0/2)$.

Let $C_{F'''} = \|F'''\|_{L^\infty(-1+\varepsilon_0/2,1-\varepsilon_0/2)}$, $C_{F^{(i)}} = \left\|F^{(i)}\right\|_{L^\infty(-1+\varepsilon_0/2,1-\varepsilon_0/2)}$, $i = 4, 5$,

$$C_{F'''_{\varepsilon_0}} = \left\|F'''_{\varepsilon_0}\right\|_{L^\infty(-1+\varepsilon_0/2,1-\varepsilon_0/2)} \le CC_{F'''},$$

$$C_{F^{(i)}_{\varepsilon_0}} = \left\|F^{(i)}\right\|_{L^\infty(-1+\varepsilon_0/2,1-\varepsilon_0/2)} \le CC_{F^{(i)}},$$

and, recalling also the definition of $C_{F''}$, let

$$C_F = \max\left\{C_{F''}, C_{F'''}, C_{F^{(4)}}, C_{F^{(5)}}\right\}.$$

This is a constant too, but we mark it as such, in order to show the connection with the L^∞-norms of the derivatives of the function F on intervals strictly included in $(-1, 1)$.

Theorems 3.1 and 3.2 and Corollary 3.3 are the main results of the paper. We set

$$g_\infty := \|\nabla\varphi_\infty\|_\infty + \|\nabla\varphi_\infty\|_\infty^2 + \|\Delta\varphi_\infty\|_\infty. \tag{3.4}$$

Theorem 3.1 *There exists $g_0 > 0$ (depending on the problem parameters, Ω and C_F) such that if $g_\infty < g_0$, one can determine ρ such that for all pairs $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4})$ with $\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \le \rho$, the closed loop system (3.2) has a unique solution*

$$(y,z) \in C([0,\infty); H \times H) \cap L^2(0,\infty; D(A^{3/2}) \times D(A^{3/4})) \\ \cap W^{1,2}(0,\infty; (D(A^{1/2}) \times D(A^{1/4}))'), \tag{3.5}$$

which is exponentially stable, that is

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \le Ce^{-kt}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}). \tag{3.6}$$

The constants C and k depends on the problem parameters, C_F and g_∞.

Proof The proof of this theorem will address the well-posedness of problem (3.2) and the stabilization relation. The proof will follow the steps from [5], but different calculations will be done due to the new form, less explicit, of the nonlinear function $G_{\varepsilon_0}(y)$. Thus, we shall only sketch the proof and insist on the parts involving the computations for the new function F_{ε_0}.

(i) Existence and uniqueness will be proved first on every interval $[0, T]$, by the Schauder fixed point theorem and then will be extended to the whole $[0, \infty)$.

Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/4})$. Let r be positive and bounded by a constant which will be specified later. For $T > 0$ arbitrary, fixed, we introduce the set

$$S_T = \left\{ (y,z) \in L^2(0,T; H \times H);\ \sup_{t\in(0,T)} \left(\|y(t)\|^2_{D(A^{1/2})} + \|z(t)\|^2_{D(A^{1/4})} \right) \right. \tag{3.7}$$
$$\left. + \int_0^T \left(\left\|A^{3/2} y(t)\right\|_H^2 + \left\|A^{3/4} z(t)\right\|_H^2 \right) dt \le r^2 \right\},$$

which is a convex closed subset of $L^2(0,T; D(A^{3/2-\varepsilon'}) \times H)$, for $0 < \varepsilon' < 1/4$. We fix $(\overline{y}, \overline{z}) \in S_T$ and consider the Cauchy problem

$$\frac{d}{dt}(y(t), z(t)) + \mathscr{A}(y(t), z(t)) + BB^* R(y(t), z(t)) = \mathscr{G}_{\varepsilon_0}(\overline{y}(t)), \text{ a.e. } t \in (0,T), \tag{3.8}$$
$$(y(0), z(0)) = (y_0, z_0).$$

We prove that (3.8) is well-posed and then we define

$$\Psi_T : S_T \to L^2(0,T; D(A^{3/2-\varepsilon'}) \times H)$$

by $\Psi_T(\overline{y}, \overline{z}) = (y,z)$, the solution to (3.8). We shall check the conditions required in order to apply the Schauder theorem, that is: $\Psi_T(S_T) \subset S_T$ if r is well chosen; $\Psi_T(S_T)$ is relatively compact in $L^2(0,T; D(A^{3/2-\varepsilon'}) \times H)$; Ψ_T is continuous in the $L^2(0,T; D(A^{3/2-\varepsilon'}) \times H)$ norm.

We begin by showing that $G_{\varepsilon_0}(\overline{y}) \in L^2(0,T;H)$, by computing the norms of all its terms. Thus, writing, for simplicity, $\overline{y}$ instead of $\overline{y}(t)$ we have

$$\nabla F_{r,\varepsilon_0}(\overline{y}) = 2\overline{y}\nabla\overline{y} \int_0^1 (1-s) F'''_{\varepsilon_0}(\varphi_\infty + s\overline{y}) ds + \overline{y}^2 \nabla\overline{y} \int_0^1 (1-s) F^{(4)}_{\varepsilon_0}(\varphi_\infty + s\overline{y}) s ds$$
$$+ \overline{y}^2 \nabla\varphi_\infty \int_0^1 (1-s) F^{(4)}_{\varepsilon_0}(\varphi_\infty + s\overline{y}) ds.$$

Further, also for simplicity, we denote the argument $\zeta_\infty := \varphi_\infty + s\overline{y}$ at which all derivatives of $F_{\varepsilon 0}$ are computed (e.g., we write $F'''_{\varepsilon_0}(\zeta_\infty)$ instead of $F'''_{\varepsilon_0}(\varphi_\infty + s\overline{y})$). We have

$$\Delta F_{r,\varepsilon_0}(\overline{y}) = 2\overline{y}\Delta\overline{y} \int_0^1 (1-s) F'''_{\varepsilon_0}(\zeta_\infty) ds$$
$$+ 2\nabla\overline{y} \cdot \left\{ \nabla\overline{y} \int_0^1 (1-s) F'''_{\varepsilon_0}(\zeta_\infty) ds + \overline{y} \int_0^1 (1-s) F^{(4)}_{\varepsilon_0}(\zeta_\infty) \nabla(\varphi_\infty + s\overline{y}) s ds \right\}$$
$$+ \overline{y}^2 \Delta\overline{y} \int_0^1 (1-s) F^{(4)}_{\varepsilon_0}(\zeta_\infty) s ds + 2\overline{y}\,|\nabla\overline{y}|^2 \int_0^1 (1-s) F^{(4)}_{\varepsilon_0}(\zeta_\infty) s ds$$

$$
+\overline{y}^2\nabla\overline{y}\cdot\int_0^1(1-s)F^{(5)}_{\varepsilon_0}(\zeta_\infty)s\nabla(\varphi_\infty+s\overline{y})ds+\overline{y}^2\Delta\varphi_\infty\int_0^1(1-s)F^{(4)}_{\varepsilon_0}(\zeta_\infty)ds
$$

$$
+\nabla\varphi_\infty\cdot 2\overline{y}\nabla\overline{y}\int_0^1(1-s)F^{(4)}_{\varepsilon_0}(\zeta_\infty)ds
$$

$$
+\overline{y}^2\nabla\varphi_\infty\cdot\int_0^1(1-s)F^{(5)}_{\varepsilon_0}(\zeta_\infty)\nabla(\varphi_\infty+s\overline{y})ds.
$$

We recall that all derivatives above are bounded for $|\zeta_\infty| = |\varphi_\infty+s\overline{y}| \le 1-\frac{\varepsilon_0}{2}$ and are zero for $|\varphi_\infty+s\overline{y}| > 1-\frac{\varepsilon_0}{2}$. So, a nonzero contribution is given by the terms calculated at $|\varphi_\infty+s\overline{y}| \le 1-\frac{\varepsilon_0}{2}$, that is for $|s\overline{y}| \le \min\left\{\left|1-\frac{\varepsilon_0}{2}-\varphi_\infty\right|,\left|-1+\frac{\varepsilon_0}{2}-\varphi_\infty\right|\right\}\le C$. Then,

$$
\begin{aligned}
&\|\Delta F_{r,\varepsilon_0}(\overline{y})\|_H \le C\Big\{C_{F'''}\left(\|\overline{y}\Delta\overline{y}\|_H+\left\||\nabla\overline{y}|^2\right\|_H\right)+C_{F^{(4)}}\left(\|\overline{y}\Delta\overline{y}\|_H+\left\||\nabla\overline{y}|^2\right\|_H\right)\\
&+C_{F^{(5)}}|\nabla\overline{y}|^2_H+\|\nabla\varphi_\infty\|_\infty\left(C_{F^{(4)}}+C_{F^{(5)}}\right)\|\overline{y}\nabla\overline{y}\|_H\\
&+C_{F^{(5)}}\|\nabla\varphi_\infty\|^2_\infty\left\|\overline{y}^2\right\|_H+C_{F^{(4)}}\|\Delta\varphi_\infty\|_\infty\left\|\overline{y}^2\right\|_H\Big\}\\
&\le CC_F\left(\|\overline{y}\Delta\overline{y}\|_H+\left\||\nabla\overline{y}|^2\right\|_H+g_\infty(\|\overline{y}\nabla\overline{y}\|_H+\left\|\overline{y}^2\right\|_H)\right),
\end{aligned}
$$

where g_∞ was defined by (3.4). In the above calculations we used in some terms the estimate $|s\overline{y}|\le C$.

Next, $\Delta(gy)=g\Delta y+2\nabla g\cdot\nabla y+y\Delta g$, and we compute by (1.23)

$$
\begin{aligned}
|g(x)| &= \frac{1}{m_\Omega}\int_\Omega\left|F''(\varphi_\infty(x))-F''(\varphi_\infty(\xi))\right|d\xi \le 2\sup_{|r|\le 1-\varepsilon_0}\left|F'''(r)\right|\|\nabla\varphi_\infty\|_\infty d_\Omega\\
&\le 2C_{F'''}\|\nabla\varphi_\infty\|_\infty d_\Omega,
\end{aligned}
$$

where d_Ω is the supremum of the geodesic distance of Ω,

$$
\nabla g = F'''(\varphi_\infty)\nabla\varphi_\infty,\ \ \Delta g = F'''(\varphi_\infty)\Delta\varphi_\infty+F^{(4)}(\varphi_\infty)|\nabla\varphi_\infty|^2,
$$

so that

$$
\|\Delta(g\overline{y})\|_H \le CC_F g_\infty\|A\overline{y}\|_H,
$$

where $g_\infty := \|\nabla\varphi_\infty\|_\infty+\|\nabla\varphi_\infty\|^2_\infty+\|\Delta\varphi_\infty\|_\infty$. Thus, we finally get

$$
\|G_{\varepsilon_0}(\overline{y})\|_H \le CC_F\left(\|\overline{y}\Delta\overline{y}\|_H+\left\||\nabla\overline{y}|^2\right\|_H+g_\infty(\|\overline{y}\nabla\overline{y}\|_H+\left\|\overline{y}^2\right\|_H+\|A\overline{y}\|_H)\right). \tag{3.9}
$$

Here, C is a constant. Next, we shall estimate the norms on the right hand side of (3.9) and to this end we shall use the following interpolation inequalities and relations involving the powers of A :

$$\|A^{\alpha}w\|_H \le C\,\|A^{\alpha_1}w\|_H^{\lambda}\,\|A^{\alpha_2}w\|_H^{1-\lambda}\,,\ \text{for}\ \alpha=\lambda\alpha_1+(1-\lambda)\alpha_2,\ \lambda\in[0,1], \tag{3.10}$$

$$\|A^{\alpha}w\|_H \le C\left\|A^{\beta}w\right\|_H,\ \text{if}\ \alpha<\beta, \tag{3.11}$$

$$\|A^{\alpha}w\|_{H^{\beta}(\Omega)} \le C\left\|A^{\alpha+\beta/2}w\right\|_H, \tag{3.12}$$

and the Sobolev embedding inequalities

$$\|w\|_{L^{2r}(\Omega)} \le C\,\|w\|_{H^{\alpha}(\Omega)}\,,\ \ \alpha\ge d\left(\frac{1}{2}-\frac{1}{2r}\right), \tag{3.13}$$

$$\|w\|_{\infty} \le C\,\|w\|_{H^2(\Omega)}\,,\ d\le 3, \tag{3.14}$$

with C standing for several constants depending on the domain. We have

$$I_1=\|\overline{y}\Delta\overline{y}\|_H \le C\left\|A^{1/2}\overline{y}\right\|_H\left\|A^{3/2}\overline{y}\right\|_H, \tag{3.15}$$

$$I_2=\left\||\nabla\overline{y}|^2\right\|_H \le C\left\|A^{3/2}\overline{y}\right\|_H\left\|A^{1/2}\overline{y}\right\|_H, \tag{3.16}$$

$$I_3=\|\overline{y}\nabla\overline{y}\|_H \le C\left\|A^{3/2}\overline{y}\right\|_H\left\|A^{1/2}\overline{y}\right\|_H, \tag{3.17}$$

$$I_4=\left\|\overline{y}^2\right\|_H \le C\left\|A^{1/2}\overline{y}\right\|_H^2 \le C\left\|A^{3/2}\overline{y}\right\|_H\left\|A^{1/2}\overline{y}\right\|_H, \tag{3.18}$$

whence $\|G_{\varepsilon_0}(\overline{y})\|_H \le CC_F(I_1+I_2+g_\infty(I_3+I_4)+g_\infty\|A\overline{y}\|_H)$, implying

$$\|G_{\varepsilon_0}(\overline{y})\|_H \le CC_F\left((1+g_\infty)\left\|A^{3/2}\overline{y}\right\|_H\left\|A^{1/2}\overline{y}\right\|_H+g_\infty\|A\overline{y}\|_H\right). \tag{3.19}$$

Finally, taking into account that $\overline{y}\in S_T$ we deduce that

$$\int_0^t \|G_{\varepsilon_0}(\overline{y}(s))\|_H^2\,ds \le CC_F^2\left\{r^4(1+g_\infty)^2+g_\infty^2 r^2\right\} \tag{3.20}$$

and so $G_{\varepsilon_0}(\overline{y})\in L^2(0,T;H)$.

Further, asserting that $\mathscr{A}+BB^*R$ is m-accretive in $H\times H$, as proved in [5], Proposition 2.4, second part, we have for $(y_0,z_0)\in D(A^{1/2})\times D(A^{1/4})\subset H\times H$ that the Cauchy problem (3.8) has a unique solution

$$(y,z)\in C([0,T];\mathscr{H})\cap L^2(0,T;\mathscr{V})\cap W^{1,2}(0,T;\mathscr{V}'). \tag{3.21}$$

Then, $(y(t), z(t)) \in D(A) \times D(A^{1/2})$ and so $R(y(t), z(t)) \in H \times H$ a.e. $t \in (0, T)$. Also, $(y, z) \in L^2(\delta, T; D(\mathscr{A}))$, with $\delta > 0$ arbitrary, which implies that $\mathscr{A}(y(t), z(t)) \in H \times H$ a.e. $t > 0$.

Next, we have to prove that $(y, z) \in S_T$ for a certain r. We multiply (3.8) by $R(y(t), z(t)) \in H \times H$ scalarly in $H \times H$ and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} + (\mathscr{A}(y(t), z(t)), R(y(t), z(t)))_{H\times H} \\
&= -\|B^* R(y(t), z(t))\|^2_{\mathbb{R}^N} + (\mathscr{G}_{\varepsilon_0}(\overline{y}(t)), R(y(t), z(t)))_{H\times H}, \text{ a.e. } t > 0.
\end{aligned}$$

By using the Riccati equation (2.24), we obtain by a few computations involving (2.23), (3.11) and (3.19), that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} \\
&+\frac{1}{2}\left(\left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2 + \|B^* R(y(t), z(t))\|^2_{\mathbb{R}^N}\right) \\
&\leq \|\mathscr{G}_{\varepsilon_0}(\overline{y}(t))\|_{H\times H}\, \|R(y(t), z(t))\|_{\mathscr{H}} \leq C_R \,\|G_{\varepsilon_0}(\overline{y}(t))\|_H \,\|(y(t), z(t))\|_{\mathscr{V}} \\
&\leq CC_R \,\|G_{\varepsilon_0}(\overline{y}(t))\|_H \left(\left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2\right)^{1/2}, \text{ a.e. } t \in (0, T),
\end{aligned}$$

with C_R from (2.23). Integrating over $(0, t)$ and using the Young's inequality, (2.22) and (3.20), we get for all $t \in (0, T)$

$$\begin{aligned}
&\|y(t)\|^2_{D(A^{1/2})} + \|z(t)\|^2_{D(A^{1/4})} + \int_0^t \left(\left\|A^{3/2}y(s)\right\|_H^2 + \left\|A^{3/4}z(s)\right\|_H^2\right) ds \\
&\leq C_1(\|y_0\|^2_{D(A^{1/2})} + \|z_0\|^2_{D(A^{1/4})} + C_F^2((1 + g_\infty)^2 r^4 + g_\infty^2 r^2)),
\end{aligned}$$

with C_1 depending on the problem parameters, Ω and C_F (note that it can be taken greater than 1). Recalling the assumption for the initial condition we have only to impose that

$$\begin{aligned}
&C_1(\|y_0\|^2_{D(A^{1/2})} + \|z_0\|^2_{D(A^{1/4})} + C_F^2((1 + g_\infty)^2 r^4 + g_\infty^2 r^2)) \\
&\leq C_1(\rho^2 + C_F^2((1 + g_\infty)^2 r^4 + g_\infty^2 r^2)) \leq r^2.
\end{aligned} \tag{3.22}$$

This is satisfied if we take for instance $C_1\rho^2 = \frac{r^2}{2}$, that is

$$\rho = \frac{r}{\sqrt{2C_1}} \tag{3.23}$$

and impose

$$(1+g_\infty)^2 r^2 + g_\infty^2 - \overline{C_1} \le 0, \quad \text{with } \overline{C_1} = \frac{1}{2C_1 C_F^2}.$$

The inequality takes place if

$$0 \le g_\infty < \sqrt{\overline{C_1}} \tag{3.24}$$

and

$$r \le r_1 := \frac{\sqrt{\overline{C_1} - g_\infty^2}}{1+g_\infty}. \tag{3.25}$$

We note that r_1 and ρ do not depend on T, but only on the L^∞ norms of the gradient and Laplacian of φ_∞, the problem parameters and C_F. Thus, if g_∞ is small enough, there exists ρ and r such that Ψ_T maps S_T into S_T.

Let $(y,z) = \Psi_T(\overline{y},\overline{z})$, with $(\overline{y},\overline{z}) \in S_T$. We observe that (y,z) and $\frac{d}{dt}(y,z)$ remain bounded in $L^2(0,T;D(A^{3/2}) \times D(A^{3/4}))$ and $W^{1,2}(0,T;(D(A) \times D(A^{1/2}))')$, respectively and since $D(A^{3/2}) \times D(A^{3/4})$ is compactly embedded in $D(A^{3/2-\varepsilon'}) \times H$ it follows by Lions-Aubin lemma that the set $\Psi_T(S_T)$ is relatively compact in $L^2(0,T;D(A^{3/2-\varepsilon'}) \times H)$. The last part is to show that Ψ is continuous and for that we take $(\overline{y_n},\overline{z_n}) \in S_T$, $(\overline{y_n},\overline{z_n}) \to (\overline{y},\overline{z})$ strongly in $L^2(0,T;D(A^{3/2-\varepsilon'}) \times H)$, as $n \to \infty$ and have to show that $\Psi_T(\overline{y_n},\overline{z_n}) \to \Psi_T(\overline{y},\overline{z})$ strongly in $L^2(0,T;D(A^{3/2-\varepsilon'}) \times H)$. This follows exactly as in [5], Theorem 3.1, Step 1, using estimates for the solution (y_n,z_n) and the compactness $D(A^{3/2-\varepsilon'}) \subset H^{3-2\varepsilon'}(\Omega)$. Relying on all these, the Schauder fixed point theorem, applied to the mapping Ψ_T on the space $L^2(0,T;D(A^{3/2-\varepsilon'}) \times H)$, implies that problem (3.8) has at least a solution on the interval $[0,T]$, $(y,z) \in S_T$.

The next step is to prove the solution uniqueness on $[0,T]$ and for that we follow the idea from [5], with slight modifications in the calculations. We work directly with (1.11)–(1.12), rewrite them in terms of the operator A and have in this new case

$$\varphi_t + \nu A^2\varphi + A(\varphi^3 + (l-2\nu)\varphi - \gamma\sigma) - (I-A)(F'_{\varepsilon_0}(\varphi_1) - F'_{\varepsilon_0}(\varphi_2)) \tag{3.26}$$
$$- (l-\nu)\varphi + \gamma\sigma = 1^*_\omega v,$$

$$\sigma_t + A(\sigma - \gamma\varphi) - \sigma + \gamma\varphi = 1^*_\omega u, \tag{3.27}$$

with the boundary and initial conditions (1.13)–(1.14). We assume that there are two solutions (φ^i,σ^i), corresponding to $U_i = (v_i,u_i)$ with $1^*_\omega U_i = -BB^*R(y_i,z_i)$, $i = 1,2$. We take the difference of Eq. (3.26) and test it by $A^{-1}(\varphi^1 - \varphi^2)$. Then, test the difference of Eq. (3.27) by $\lambda(\sigma^1 - \sigma^2)$, where $\lambda > 0$ is a coefficient to be chosen later. We use the notation $\varphi = \varphi^1 - \varphi^2$, $\sigma = \sigma^1 - \sigma^2$, $v = v_1 - v_2$, $u = u_1 - u_2$. Let us compute only one new term, the rest being exactly like in [5]. We have

$$\begin{aligned}
&\left((I - A)(F'_{\varepsilon_0}(\varphi_1) - F'_{\varepsilon_0}(\varphi_2)), A^{-1}\varphi\right)_{H\times H} \\
&= (\left(A^{-1} - I)(F'_{\varepsilon_0}(\varphi_1) - F'_{\varepsilon_0}(\varphi_2)), \varphi\right)_{H\times H} \\
&\leq \left\|(A^{-1} - I)(F'_{\varepsilon_0}(\varphi_1) - F'_{\varepsilon_0}(\varphi_2))\right\|_H \|\varphi\|_H \leq 2\left\|F'_{\varepsilon_0}(\varphi_1) - F'_{\varepsilon_0}(\varphi_2)\right\|_H \|\varphi\|_H \\
&\leq 2C_{F''_{\varepsilon_0}} \|\varphi\|_H^2 \leq \frac{\nu}{k_0}\|\varphi\|_V^2 + C\|\varphi\|_{V'}^2,
\end{aligned}$$

where $C_{F''_{\varepsilon_0}} = \sup_{r\in(-1+\varepsilon_0/2, 1-\varepsilon_0/2)} |F''(r)|$ and k_0 is a suitable integer, such that $1/k_0 << 1$. By a straightforward computation we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varphi(t)\|_{V'}^2 + \lambda\|\sigma(t)\|_H^2\right) + \left(\frac{\nu}{2} - k_1\lambda\right)\|\varphi(t)\|_V^2 + \frac{\lambda}{2}\|\sigma(t)\|_V^2 \\
&\leq C\|\varphi(t)\|_{V'}^2 + C\|\sigma(t)\|_H^2 + C_\lambda\|\varphi(t)\|_{V'}^2 + C_\lambda\|\sigma(t)\|_H^2, \text{ a.e. } t \in (0, T),
\end{aligned}$$

where k_1 can be computed. The symbols C and C_λ denote several positive constants possibly depending on $C_{F''_{\varepsilon_0}}$. Thus, taking $\lambda < \frac{\nu}{2k_1}$, integrating from 0 to t and applying the Gronwall lemma, we get that $\varphi = \sigma = 0$. In the computations one uses the fact that BB^* is linear continuous from $V' \times V' \to V' \times V'$.

Since these results followed for T arbitrary and r and ρ do not depend on T, we can extend the solution on $[0, T]$ to $[0, \infty)$, and set S_∞ as S_T in which $T = \infty$. Hence, the solution (y, z) to (3.2) exists, it is unique and belongs to S_∞.

(ii) For proving the stabilization result we multiply Eq. (3.2) by $R(y(t), z(t))$ scalarly in $H \times H$. Since R is symmetric as an unbounded operator in $H \times H$, we have by the Riccati equation (2.24), and the relations (2.23) and (3.19) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(R(y(t), z(t)), (y(t), z(t)))_{H\times H} \\
&+\frac{1}{2}\left(\left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2 + \|B^*R(y(t), z(t))\|_{\mathbb{R}^N}^2\right) \\
&\leq \|\mathscr{G}_{\varepsilon_0}(y(t))\|_{H\times H}\|R(y(t), z(t))\|_{H\times H} \leq \|G_{\varepsilon_0}(y(t))\|_H \|R(y(t), z(t))\|_{H\times H} \\
&\leq CC_F\left(\left\|A^{3/2}y\right\|_H \left\|A^{1/2}y\right\|_H (1 + g_\infty) + g_\infty\left\|A^{3/2}y\right\|_H\right) \\
&\times C_R\left(\|Ay(t)\|_H + \left\|A^{1/2}z(t)\right\|_H\right),
\end{aligned} \tag{3.28}$$

a.e. $t > 0$. It remains to compute the right-hand side in (3.28). We recall (3.10) and (3.11) and use the Young inequality for the products with the second term in the last parenthesis. For example, we write

$$\left\|A^{3/2}y\right\|_H \left\|A^{1/2}y\right\|_H \left\|A^{1/2}z(t)\right\|_H = \left\|A^{3/2}y\right\|_H \left\|A^{1/2}y\right\|_H^{1/2} \left\|A^{1/2}y\right\|_H^{1/2} \left\|A^{1/2}z(t)\right\|_H$$
$$\leq \left\|A^{3/2}y\right\|_H^2 \left\|A^{1/2}y\right\|_H + \left\|A^{1/2}y\right\|_H \left\|A^{1/2}z(t)\right\|_H^2.$$

We obtain

$$\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} + \left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2 \tag{3.29}$$
$$\leq C_2C_F\left(\left\|A^{3/2}y\right\|_H^2 + \left\|A^{1/2}z\right\|_H^2\right)\left(\left\|A^{1/2}y\right\|_H(1+g_\infty)+g_\infty\right),$$

with $C_2 = CC_R$ which depend on $C_{F''}$. We recall that $\left\|A^{1/2}z\right\|_H \leq \left\|A^{3/4}z\right\|_H$, $y \in S_\infty$, i.e., $\left\|A^{1/2}y(t)\right\|_H \leq r \leq r_1$, and impose

$$C_2C_F\{r(1+g_\infty)+g_\infty\} < 1.$$

This relation takes place if

$$0 \leq g_\infty < \overline{C_2} := \frac{1}{C_2C_F} \tag{3.30}$$

and

$$r < r_2 := \frac{\overline{C_2} - g_\infty}{1+g_\infty}.$$

Recalling (3.24) and (3.25) we set

$$g_0 := \min\left\{\sqrt{\overline{C_1}}, \overline{C_2}\right\}, \tag{3.31}$$

and

$$r < r_0 := \min\{r_1, r_2\} \tag{3.32}$$

and recall that ρ is fixed by (3.23) where $r < r_0$. We note that g_0 depends on C_F, and r and ρ depend on C_F and g_∞. Of course, all these depend on the problem parameters and Ω, too.

We get from (3.29) by using (3.11) and (2.22) that

$$\frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} + C_3c_0(R(y(t),z(t)),(y(t),z(t)))_{H\times H} \tag{3.33}$$
$$\leq \frac{d}{dt}(R(y(t),z(t)),(y(t),z(t)))_{H\times H} + C_3\left(\left\|A^{3/2}y(t)\right\|_H^2 + \left\|A^{3/4}z(t)\right\|_H^2\right) \leq 0,$$

with $C_3 = 1 - C_2 C_F \{r(1 + g_\infty) + g_\infty\}$. It follows that

$$(R(y(t), z(t)), (y(t), z(t)))_{H\times H} \le e^{-2kt}(R(y_0, z_0), (y_0, z_0))_{H\times H} \tag{3.34}$$

where $k := \frac{C_3 c_0}{2}$ and again by (2.22) we deduce the desired result

$$c_1 \|(y(t), z(t))\|^2_{D(A^{1/2})\times D(A^{1/4})} \le c_2 e^{-2kt} \|(y_0, z_0)\|^2_{D(A^{1/2})\times D(A^{1/4})}, \text{ a.e. } t > 0.$$

This leads to (3.6). In conclusion, there is g_0 given by (3.31) such that if $g_\infty < g_0$ (by (3.24) and (3.30)), one can determine r (by (3.32)) and ρ by (3.23), where $r < r_0$, such that (3.6) takes place. This ends the proof. □

This is the stabilization conclusion valid in three-dimensions for the function F_{ε_0}, which together with its derivatives up the the fifth order is continuous and vanish outside a bounded interval of $\mathbb{R}$.

In [5], in the case with the double-well potential, ρ, r, C and k depended on $\|\varphi_\infty\|_\infty$, instead of C_F.

The consequence for the logarithmic function F is further provided.

Theorem 3.2 *Let $\varepsilon_0 \in (0, 1)$ be arbitrary but fixed. There exist $g_0 > 0$ and ρ such that if $g_\infty < g_0$ and if $\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \le \rho$, the closed loop system (3.2) corresponding to the logarithmic potential F has, in the one-dimensional case, a unique solution belonging to the spaces (3.5), which is exponentially stable and satisfies (3.6).*

Proof The assertion in this theorem is based on the compactness $H^1(\Omega)$ in $C(\overline{\Omega})$ true for $d = 1$.

We rely on Theorem 3.1. for the system (1.16)–(1.19) corresponding to the function F_{ε_0}, that is with $F'_{\varepsilon_0}(y + \varphi_\infty)$ in (1.16). As we have already mentioned, in (1.16), $F'_{\varepsilon_0}(\varphi_\infty) = F'(\varphi_\infty)$.

On the one hand, we know by Theorem 3.1, that there exists ρ given by (3.23) such that for any initial data in the ball of radius ρ, we have by (3.6) that

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \le Ce^{-kt}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}) \le Ce^{-kt}\rho.$$

On the other hand, in the one-dimensional case, we have in addition,

$$|y(t)| \le \|y(t)\|_{C(\overline{\Omega})} \le C_\Omega \|y(t)\|_{D(A^{1/2})} \le C_\Omega C e^{-kt}\rho \tag{3.35}$$

the last inequality being due to relation (3.6) and this implies that $|y(t)| \to 0$, as $t \to \infty$. Here, C_Ω is a constant depending on Ω.

Moreover, one can set a new ρ such that the values of the solution remain less that $1 - \varepsilon_0$ for all $t > 0$. Recall that $|\varphi_\infty| < 1 - \varepsilon_0$. To be more precise, we can write $|\varphi_\infty| \le 1 - \varepsilon_0 - \delta_0$, where δ_0 be a positive value, $\delta_0 \in (0, 1 - \varepsilon_0)$. We have by (3.35) that

$$|y(t)| \le C_\Omega C e^{-kt}\rho \text{ for all } t \ge 0,$$

and set a new ρ by the inequality $C_\Omega C e^{-kt}\rho \le \delta_0$, which provides

$$\rho \le \frac{\delta_0}{C_\Omega C} \le \frac{\delta_0}{C_\Omega C} e^{kt}, \text{ for all } t \ge 0.$$

Thus, recalling (3.23) in Theorem 3.1, we can take

$$\rho \le \min\left\{\frac{\delta_0}{C_\Omega C}, \frac{r}{\sqrt{2C_1}}\right\} \tag{3.36}$$

with $r < r_0$ (see (3.31)). Then,

$$|y(t) + \varphi_\infty| < \delta_0 + 1 - \varepsilon_0 - \delta_0 = 1 - \varepsilon_0$$

and we conclude that $F'_{\varepsilon_0}(y + \varphi_\infty) = F'(y + \varphi_\infty)$ in Eq. (1.16). Thus, our solution $y(t)$ actually satisfies the system corresponding to the function F. In conclusion, one can find a small enough ρ, such that for any initial data in the ball with this radius, the system (1.16)–(1.19) corresponding to the logarithmic potential F is exponentially stabilized. □

Finally, we give the stabilization result for the original system in θ and φ, which follows immediately from Theorem 3.2.

Corollary 3.3 *There exists $g_0 > 0$ (depending on the problem parameters, the domain and C_F) such that the following hold true. If $g_\infty < g_0$ there exists ρ such that for all pairs $(\varphi_0, \theta_0) \in D(A^{1/2}) \times D(A^{1/4})$ with*

$$\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|\alpha_0(\theta_0 - \theta_\infty) + \alpha_0 l_0(\varphi_0 - \varphi_\infty)\|_{D(A^{1/4})} \le \rho, \tag{3.37}$$

the closed loop system (1.7)–(1.10) with $(1^*_\omega v, 1^*_\omega u)$ replaced by (3.1) has a unique solution

$$\begin{aligned}(\varphi, \theta) \in C([0,\infty); H \times H) \cap L^2(0,\infty; D(A^{3/2}) \times D(A^{3/4})) \\ \cap W^{1,2}(0,\infty; (D(A^{1/2}) \times D(A^{1/4}))'),\end{aligned} \tag{3.38}$$

which is exponentially stable, that is

$$\begin{aligned}&\|\varphi(t) - \varphi_\infty\|_{D(A^{1/2})} + \|\alpha_0(\theta(t) - \theta_\infty) + \alpha_0 l_0(\varphi(t) - \varphi_\infty)\|_{D(A^{1/4})} \\ &\le Ce^{-kt}(\|\varphi_0 - \varphi_\infty\|_{D(A^{1/2})} + \|\alpha_0(\theta_0 - \theta_\infty) + \alpha_0 l_0(\varphi_0 - \varphi_\infty)\|_{D(A^{1/4})})\end{aligned} \tag{3.39}$$

for all $t \ge 0$ and for some positive constants k and C, depending on the problem parameters, the domain, C_F and g_∞.

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Mathematical Analysis of a Parabolic-Elliptic Model for Brain Lactate Kinetics

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Abstract Our aim in this paper is to study properties of a parabolic-elliptic system related with brain lactate kinetics. These equations are obtained from a reaction-diffusion system, when a small parameter vanishes. In particular, we prove the existence and uniqueness of nonnegative solutions and obtain error estimates on the difference of the solutions to the initial reaction-diffusion system and those to the limit one, on bounded time intervals. We also study the linear stability of the unique spatially homogeneous equilibrium.

Keywords Brain lactate kinetics • Error estimates • Limit system • Linear stability • Nonnegative solutions • Parabolic-elliptic system • Reaction-diffusion system • Well-posedness

AMS (MOS) Subject Classification 35M33, 35K57, 35K67, 35B45

1 Introduction

The following system of ODE's:

$$\frac{du}{dt}+\kappa(\frac{u}{k+u}-\frac{v}{k'+v})=J,\ \kappa,\ k,\ k',\ J>0, \tag{1.1}$$

$$\epsilon\frac{dv}{dt}+Fv+\kappa(\frac{v}{k'+v}-\frac{u}{k+u})=FL,\ \epsilon,\ F,\ L>0, \tag{1.2}$$

where ϵ is a small parameter, was proposed and studied as a model for brain lactate kinetics (see [5, 9, 10] and [11]; see also [4]). In this context, $u=u(t)$ and $v=v(t)$ correspond to the lactate concentrations in an interstitial (i.e., extra-cellular) domain

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and in a capillary domain, respectively. Furthermore, the nonlinear term $\kappa(\frac{u}{k+u} - \frac{v}{k'+v})$ stands for a co-transport through the brain-blood boundary (see [8]). Finally, J and F are forcing and input terms, respectively, assumed frozen (more generally, J depends on t and u and accounts for the interactions with a third intracellular compartment (which includes both neurons and astrocytes), while $F = F(t)$ (an applied electrical stimulus; see [7]) is piecewise linear and periodic). This model has essential applications to the therapeutic management of glioma (also called glial tumors); see [9] for thorough discussions on this issue.

Let us assume that $u(0)$ and $v(0)$ are nonnegative (recall that u and v are concentrations and are thus expected to be nonnegative). Then, noting that, if $u(0) = 0$, then $\frac{du}{dt}(0) > 0$ and, if $v(0) = 0$, then $\frac{dv}{dt}(0) > 0$, it follows from Cauchy–Lipschitz theorem that, for $t > 0$ small, u and v exist and are nonnegative. This also yields that the solutions are defined and remain nonnegative on the whole interval $\mathbb{R}^+$; indeed, it is not difficult to prove that they are bounded on finite time intervals. Furthermore, in [5, 9, 10] and [11], questions related to the stability of the unique equilibrium were addressed. This constitutes an essential point in the modeling, since, as discussed in [9], a therapeutic perspective of such a result is to have the steady state outside the viability domain, where cell necrosis occurs. Finally, in [10], justifications for the dip and buffering which are observed in experiments (see [7]) were given, based on geometrical arguments and averaging theory on a slow manifold.

We can note that the above ODE's model does not account for spatial diffusion. Taking this into account would be relevant and desirable from a biological point of view. The simplest possible corresponding PDE's (reaction-diffusion) system, accounting for spatial diffusion, reads (see also [12])

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \kappa(\frac{u}{k+u} - \frac{v}{k'+v}) = J, \ \alpha > 0, \tag{1.3}$$

$$\epsilon \frac{\partial v}{\partial t} - \beta \Delta v + Fv + \kappa(\frac{v}{k'+v} - \frac{u}{k+u}) = FL, \ \beta > 0, \tag{1.4}$$

where $u = u(x,t)$ and $v = v(x,t)$, which we consider in a bounded and regular domain Ω of $\mathbb{R}^N$, $N = 1, 2$ or 3, together with Neumann boundary conditions,

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma,$$

where $\Gamma = \partial\Omega$ and ν is the unit outer normal vector. Note that the terms $-\alpha\Delta u$ and $-\beta\Delta v$ correspond to random motions. Note however that more precise models should account for the geometry, i.e., the different compartments (interstitial, capillary), so that (1.3)–(1.4) should be viewed as a very first step towards PDE's models for brain lactate kinetics. We will consider more realistic models elsewhere.

We studied in [6] the existence, uniqueness and regularity of nonnegative solutions to (1.3)–(1.4) (note that the mathematical analysis of (1.3)–(1.4) (and, in particular, the well-posedness) appears to be challenging, due to the coupling terms, especially for negative initial data (though biologically irrelevant, this makes sense from a mathematical point of view); this is also the case for the ODE's model (1.1)–(1.2)). We further established the linear (exponential) stability of the unique spatially homogeneous equilibrium. We also mention [13] in which we proved the existence, uniqueness and regularity of the solutions to the following singular reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - \Delta u + Fu + \kappa \frac{u}{k+u} = f(x,t),\ \ F \geq 0, \tag{1.5}$$

corresponding to the case where either u or v is known in (1.3) and (1.4); we can also think of (1.5) as an equation in each compartment, assuming that the lactate concentration is known in the other one.

Our aim in this paper is to study the limit system

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \kappa \left(\frac{u}{k+u} - \frac{v}{k'+v}\right) = J, \tag{1.6}$$

$$-\beta \Delta v + Fv + \kappa \left(\frac{v}{k'+v} - \frac{u}{k+u}\right) = FL, \tag{1.7}$$

corresponding to $\epsilon = 0$ in (1.4). We prove the existence and uniqueness of nonnegative solutions to (1.6)–(1.7). We then prove that the solutions to the initial reaction-diffusion system converge to those to the limit parabolic-elliptic one, on finite time intervals, and provide an error estimate in terms of ϵ. We finally study the linear stability of the unique spatially homogeneous equilibrium. We can note that a similar analysis would also be relevant in the context of the ODE's model (1.1)–(1.2). Though some of the results obtained here could apply (in a simpler way) to this system, this will be considered in more details elsewhere.

Notation

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X and, if X is a Hilbert space, $((\cdot,\cdot))_X$ denotes the associated scalar product.

Throughout the paper, the same letters c, c' and c'' denote positive constants which may vary from line to line. Similarly, the same letter Q denotes continuous and monotone increasing (with respect to each argument) functions which may vary from line to line.

2 The Case $\epsilon > 0$

We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \kappa\left(\frac{u}{k+u} - \frac{v}{k'+v}\right) = J, \tag{2.8}$$

$$\epsilon\frac{\partial v}{\partial t} - \beta \Delta v + Fv + \kappa\left(\frac{v}{k'+v} - \frac{u}{k+u}\right) = FL,\ \epsilon > 0, \tag{2.9}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.10}$$

$$u|_{t=0} = u_0,\ v|_{t=0} = v_0. \tag{2.11}$$

Note that (2.8)–(2.9) are equivalent to

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \kappa\left(\frac{k'}{k'+v} - \frac{k}{k+u}\right) = J, \tag{2.12}$$

$$\epsilon\frac{\partial v}{\partial t} - \beta \Delta v + Fv + \kappa\left(\frac{k}{k+u} - \frac{k'}{k'+v}\right) = FL. \tag{2.13}$$

We assume that

$$(u_0, v_0) \in H^2_{\mathrm{N}}(\Omega)^2,\ u_0 \geq 0,\ v_0 \geq 0 \text{ a.e. } x, \tag{2.14}$$

where

$$H^2_{\mathrm{N}}(\Omega) = \{w \in H^2(\Omega),\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma\}.$$

We proved in [6] the

Theorem 1 *We assume that (2.14) holds. Then, (2.8)–(2.11) possesses a unique strong solution (u, v) such that*

$$u \geq 0,\ v \geq 0 \text{ a.e. } (x,t) \tag{2.15}$$

and, $\forall T > 0$,

$$(u, v) \in L^\infty(0, T; H^2_{\mathrm{N}}(\Omega)^2) \cap L^2(0, T; H^3(\Omega)^2),$$

$$\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) \in L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; H^1(\Omega)^2).$$

Furthermore,

$$\|u(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + (J+\kappa)t,\ t \ge 0,$$

and

$$\|v(t)\|_{L^\infty(\Omega)} \le e^{-\frac{F}{\epsilon}t}\|v_0\|_{L^\infty(\Omega)} + \frac{FL+\kappa}{F},\ t \ge 0.$$

Finally, if $M \ge \frac{FL+\kappa}{F}$ *and* $0 \le v_0 \le M$ *a.e. x, then* $0 \le v \le M$ *a.e.* (x,t).

Remark 1 As far as the above regularity is concerned, the corresponding constants in [6] depend on ϵ, i.e., they are not bounded uniformly with respect to ϵ as this quantity goes to 0. However, it is not difficult, reading the details, to see that most constants can be made independent of ϵ, yielding regularity estimates on u, $\frac{\partial u}{\partial t}$ and v which are uniform with respect to ϵ as $\epsilon \to 0$. Now, we have not been able to derive, at least in a straightforward way, such uniform estimates on $\frac{\partial v}{\partial t}$ which would allow us to pass to the limit in (2.9) (say, in a weak (variational) form) to deduce the existence of a solution to the limit problem corresponding to $\epsilon = 0$ (see however Sect. 4). We will thus give a direct proof of existence for the limit problem which also has an interest on its own.

Remark 2

(i) It follows from the above that the capillary lactate concentration is uniformly (with respect to time) bounded. However, we have not been able to derive a similar upper bound on the interstitial lactate concentration u. We can note that, in the biological model, outside a bounded viability domain, cell necrosis occurs (see [9]), meaning that one expects viable trajectories to be uniformly bounded.

(ii) Multiplying (2.8) by $u+k$, integrating over Ω and by parts, we obtain

$$\frac{dE}{dt} + \alpha\|\nabla u\|^2 + \kappa\|u\|_{L^1(\Omega)} = ((J + \frac{\kappa v}{k'+v}, u+k)),$$

where

$$E = \frac{1}{2}\|u\|^2 + k\|u\|_{L^1(\Omega)}.$$

Noting that v is uniformly bounded (we assume that, say, $0 \le v_0 \le \frac{FL+\kappa}{F}$), we take, for κ, F and L given, J small enough and k' large enough such that

$$J + \frac{\kappa v}{k'+v} < \kappa.$$

We thus deduce that

$$\frac{dE}{dt} + \alpha\|\nabla u\|^2 + c\|u\|_{L^1(\Omega)} \le c',\ c > 0,$$

which yields, noting that

$$\begin{aligned}\alpha\|\nabla u\|^2 + c\|u\|_{L^1(\Omega)} &\ge c'(\|\nabla u\| + \|u\|_{L^1(\Omega)}) - c'' \\ &\ge c'(\|u\| + \|u\|_{L^1(\Omega)}) - c'' \\ &\ge c'(\|u\| + \sqrt{\|u\|_{L^1(\Omega)}}) - c'',\end{aligned}$$

where we have used Young's inequality, the differential inequality

$$\frac{dE}{dt} + c\sqrt{E} \le c',\ c > 0. \tag{2.16}$$

Set $E^* = (\frac{c'}{c})^2$, where c and c' are the same constants as in (2.16), so that

$$\frac{dE^*}{dt} + c\sqrt{E^*} = c'.$$

It then follows from comparison arguments that

$$E(t) \le \max(E(0), E^*),\ t \ge 0, \tag{2.17}$$

and we finally deduce that the L^2-norm of u is uniformly bounded.

3 The Case $\epsilon = 0$

We consider in this section the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \alpha\Delta u + \kappa(\frac{k'}{k' + v} - \frac{k}{k + u}) = J, \tag{3.18}$$

$$-\beta\Delta v + Fv + \kappa(\frac{k}{k + u} - \frac{k'}{k' + v}) = FL, \tag{3.19}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.20}$$

$$u|_{t=0} = u_0. \tag{3.21}$$

We assume that

$$u_0 \in H^2_{\mathrm{N}}(\Omega),\ u_0 \geq 0 \text{ a.e. } x. \tag{3.22}$$

Remark 3 It follows from (3.19) that

$$-\beta\Delta v(0) + Fv(0) - \frac{k'}{k' + v(0)} = FL - \frac{k}{k + u_0}.$$

We will see below that this allows to define in a unique way $v(0)$ such that $v(0) \geq 0$ a.e. x.

3.1 Existence and Uniqueness of Solutions to an Auxiliary Problem

We consider the following modified initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \alpha\Delta u + \kappa\left(\frac{u}{k + |u|} - \frac{v}{k' + |v|}\right) = J, \tag{3.23}$$

$$-\beta\Delta v + Fv + \kappa\left(\frac{v}{k' + |v|} - \frac{u}{k + |u|}\right) = FL, \tag{3.24}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.25}$$

$$u|_{t=0} = u_0. \tag{3.26}$$

We associate with (3.23)–(3.26) the following weak/variational formulation, for $T > 0$ given:

Find $(u, v) : [0, T] \to H^1(\Omega)^2$ such that

$$\frac{d}{dt}((u, \phi)) + \alpha((\nabla u, \nabla\phi)) + ((\varphi_k(u), \phi)) - ((\varphi_{k'}(v), \phi)) \tag{3.27}$$

$$= ((J, \phi)),\ \forall \phi \in H^1(\Omega),$$

$$\beta((\nabla v, \nabla\psi)) + F((v, \psi)) + ((\varphi_{k'}(v), \psi)) - ((\varphi_k(u), \psi)) \tag{3.28}$$

$$= ((FL, \psi)),\ \forall \psi \in H^1(\Omega),$$

in the sense of distributions, and

$$u(0) = u_0 \text{ in } L^2(\Omega), \tag{3.29}$$

where we have set, for $c > 0$ given,

$$\varphi_c(s) = \frac{\kappa s}{c + |s|}, \ s \in \mathbb{R}.$$

We can note that φ_c is bounded (with $|\varphi_c| \leq \kappa$) and of class $\mathscr{C}^1$, with $\varphi_c'(s) = \frac{\kappa c}{(c+|s|)^2}$, so that φ_c is also Lipschitz continuous, with Lipschitz constant $\frac{\kappa}{c}$.

Let then $0 = \lambda_1 < \lambda_2 \leq \cdots$ be the eigenvalues of the minus Laplace operator associated with Neumann boundary conditions and $w_1, w_2, \cdots$ be associated eigenvectors such that the w_j's form an orthonormal in $L^2(\Omega)$ and orthogonal in $H^1(\Omega)$ basis. Setting

$$V_m = \text{Span}(w_1, \cdots, w_m), \ m \in \mathbb{N},$$

we consider the following approximated problem, for $T > 0$ given:

Find $(u_m, v_m) : [0, T] \to V_m \times V_m$ such that

$$\frac{d}{dt}((u_m, \phi)) + \alpha((\nabla u_m, \nabla\phi)) + ((\varphi_k(u_m), \phi)) - ((\varphi_{k'}(v_m), \phi)) \tag{3.30}$$

$$= ((J, \phi)), \ \forall \phi \in V_m,$$

$$\beta((\nabla v_m, \nabla\psi)) + F((v_m, \psi)) + ((\varphi_{k'}(v_m), \psi)) - ((\varphi_k(u_m), \psi)) \tag{3.31}$$

$$= ((FL, \psi)), \ \forall \psi \in V_m,$$

in the sense of distributions, and

$$u_m(0) = u_{0m}, \tag{3.32}$$

where $u_{0m} = P_m u_0$, P_m being the orthogonal projector (for the L^2-norm) from $L^2(\Omega)$ onto V_m.

For $w \in V_m$ given, we consider the following elliptic problem:

Find $z \in V_m$ such that

$$a(z, \phi) + ((\varphi_{k'}(z), \phi)) = ((FL + \varphi_k(w), \phi)), \ \forall \phi \in V_m, \tag{3.33}$$

where

$$a(\cdot, \cdot) = \beta((\nabla\cdot, \nabla\cdot)) + F((\cdot, \cdot))$$

is bilinear, symmetric, continuous and coercive on V_m (and also on $H^1(\Omega)$). Let then $R = R_m$ be the operator defined by

$$R : V_m \to V_m,\ z \mapsto R(z),$$

where

$$((R(z), \phi))_{H^1(\Omega)} = a(z, \phi) + ((\varphi_{k'}(z), \phi)) - ((FL + \varphi_k(w), \phi)),\ \forall \phi \in V_m.$$

It is clear that this operator is well defined and continuous (since $\varphi_{k'}$ is Lipschitz continuous). Furthermore, there holds, for $z \in V_m$,

$$((R(z), z))_{H^1(\Omega)} = a(z, z) + ((\varphi_{k'}(z), z)) - ((FL + \varphi_k(w), z))$$

$$\geq c\|z\|^2_{H^1(\Omega)} - c'\|z\|,\ c > 0$$

(note indeed that w is given and recall that φ_k and $\varphi_{k'}$ are bounded). Therefore,

$$((R(z), z))_{H^1(\Omega)} \geq c\|z\|^2_{H^1(\Omega)} - c',$$

so that

$$((R(z), z))_{H^1(\Omega)} \geq 0 \text{ whenever } \|z\|_{H^1(\Omega)} \geq \sqrt{\frac{c'}{c}}.$$

There thus exists $z \in V_m$, $\|z\|_{H^1(\Omega)} \leq \sqrt{\frac{c'}{c}}$, such that

$$R(z) = 0 \text{ in } V_m,$$

which is equivalent to (3.33). Indeed, otherwise, we can consider, following, e.g., [15], the continuous mapping

$$G : B(0, c'') \to B(0, c''),\ z \mapsto -c'' \frac{R(z)}{\|R(z)\|_{H^1(\Omega)}},$$

where $c'' = \sqrt{\frac{c'}{c}}$ and $B(0, c'')$ is the closed ball in V_m with center 0 and radius c''. It thus follows from the Brouwer fixed point theorem that there exists $z_0 \in V_m$ such that $z_0 = G(z_0)$. This yields that $\|z_0\|_{H^1(\Omega)} = c''$ and $((R(z_0), z_0))_{H^1(\Omega)} = -c''\|R(z_0)\|_{H^1(\Omega)} < 0$, whence a contradiction. Note that all constants here (and also below) are independent of m. This thus defines a mapping $\mathscr{F} = \mathscr{F}_m$,

$$\mathscr{F} : V_m \to V_m,\ w \mapsto z = \mathscr{F}(w).$$

Let then $(w_1, w_2) \in V_m \times V_m$ and set $z_i = \mathscr{F}(w_i)$, $i = 1, 2$. We have, setting $z = z_1 - z_2$ and $w = w_1 - w_2$,

$$a(z, \phi) + ((\varphi_{k'}(z_1) - \varphi_{k'}(z_2), \phi)) = ((\varphi_k(w_1) - \varphi_k(w_2), \phi)), \ \forall \phi \in V_m. \tag{3.34}$$

Taking $\phi = z$ and noting that $\varphi_{k'}$ is monotone increasing and φ_k is Lipschitz continuous, we obtain

$$\|z\|^2_{H^1(\Omega)} \leq c\|w\|\|z\|,$$

whence

$$\|\mathscr{F}(w_1) - \mathscr{F}(w_2)\|_{H^1(\Omega)} \leq c\|w_1 - w_2\|, \tag{3.35}$$

which yields that $\mathscr{F}$ is Lipschitz continuous on V_m (both for the L^2 and H^1-norms); this also yields that $\mathscr{F}$ is indeed a mapping, since $w_1 = w_2$ implies $z_1 = z_2$.

It follows from the above that (3.30)–(3.32) is equivalent to

Find $u_m : [0, T] \to V_m$ such that

$$\frac{d}{dt}((u_m, \phi)) + \alpha((\nabla u_m, \nabla \phi)) + ((\varphi_k(u_m), \phi)) - ((\varphi_{k'} \circ \mathscr{F}(u_m), \phi)) \tag{3.36}$$

$$= ((J, \phi)), \ \forall \phi \in V_m,$$

in the sense of distributions,

$$u_m(0) = u_{0m} \tag{3.37}$$

and then set $v_m = \mathscr{F}(u_m)$.

Since φ_k and $\varphi_{k'}$ are Lipschitz continuous on $\mathbb{R}$ and $\mathscr{F}$ is Lipschitz continuous on V_m with respect to the L^2-norm, it is easy to prove that (3.36)–(3.37) possesses a (unique) solution $u_m \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ (see, e.g., [14]), whence, setting $v_m = \mathscr{F}(u_m)$, the existence of a solution (u_m, v_m) to (3.30)–(3.32) such that $v_m \in L^\infty(0, T; H^1(\Omega))$. We also note that $\frac{\partial u_m}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$, so that $u_m \in \mathscr{C}([0, T]; L^2(\Omega))$.

Writing $u_m(t) = \sum_{i=1}^m d_{i,m}(t) w_i$, taking $\phi = \lambda_i w_i$ in (3.36), multiplying the resulting equality by $d_{i,m}$ and summing over i, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \alpha\|\Delta u_m\|^2 - ((\varphi_k(u_m), \Delta u_m)) + ((\varphi_{k'} \circ \mathscr{F}(u_m), \Delta u_m)) = 0,$$

which yields, recalling that φ_k and $\varphi_{k'}$ are bounded,

$$\frac{d}{dt}\|\nabla u_m\|^2 + \alpha\|\Delta u_m\|^2 \leq c,$$

whence estimates on u_m in $L^\infty(0,T;H^1(\Omega))$ and $L^2(0,T;H^2(\Omega))$. It thus follows that $\frac{\partial u_m}{\partial t} \in L^\infty(0,T;H^{-1}(\Omega)) \cap L^2(0,T;L^2(\Omega))$.

Since the above regularity estimates are uniform with respect to m, we deduce from classical Aubin–Lions compactness theorems that, at least for a subsequence which we do not relabel,

$$u_m \to u \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak star, in } L^2(0,T;H^2(\Omega)) \text{ weak,}$$

$$\text{in } \mathscr{C}([0,T];L^2(\Omega)) \text{ and a.e. } (x,t) \in \Omega \times (0,T),$$

$$v_m \to v \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak star,}$$

for some functions u and v. Actually, since $v_m = \mathscr{F}(u_m)$ and $\mathscr{F}$ is Lipschitz continuous with respect to the L^2-norm, we can see that (v_m) is a Cauchy sequence in $\mathscr{C}([0,T];L^2(\Omega))$ (note indeed that, if $m' \geq m$, then $V_m \subset V_{m'}$ and that the constant c in (3.35) is independent of m), so that

$$v_m \to v \text{ in } \mathscr{C}([0,T];L^2(\Omega)).$$

Recalling finally that φ_k and $\varphi_{k'}$ are Lipschitz continuous, this is sufficient to pass to the limit in (3.30)–(3.32) and deduce the existence of a solution (u,v) to (3.27)–(3.29) (note that the initial condition $u_0 = u(0)$ makes sense; actually, $v(0)$ also makes sense). Indeed, we need to pass to the limit in relations of the form

$$\int_0^T [-((u_m,\phi))\theta'(t) + \alpha((\nabla u_m, \nabla\phi))\theta(t) + ((\varphi_k(u_m),\phi))\theta(t)$$

$$-((\varphi_{k'}(v_m),\phi))\theta(t) - ((J,\phi))\theta(t)]\,dt = 0$$

and

$$\int_0^T [\beta((\nabla v_m, \nabla\psi)) + F((v_m,\psi)) + ((\varphi_{k'}(v_m),\psi)) - ((\varphi_k(u_m),\psi))$$

$$-((FL,\psi))]\theta(t)\,dt = 0,$$

for $(\phi,\psi) \in H^1(\Omega)^2$ and $\theta \in \mathscr{D}(0,T)$. More precisely, we have the

Theorem 2 *We assume that $u_0 \in H^1(\Omega)$. Then,* (3.27)–(3.29) *possesses a unique solution (u,v) such that, $\forall T > 0$,*

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap \mathscr{C}([0,T];L^2(\Omega)),$$

$$v \in L^\infty(0,T;H^1(\Omega)) \cap \mathscr{C}([0,T];L^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^\infty(0, T; H^{-1}(\Omega)) \cap L^2(0, T; L^2(\Omega)).$$

Proof There remains to prove the uniqueness.

Let thus (u_1, v_1) and (u_2, v_2) be two such solutions, with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We have, setting $(u, v) = (u_1 - u_2, v_1 - v_2)$ and $u_0 = u_{0,1} - u_{0,2}$,

$$\frac{d}{dt}((u, \phi)) + \alpha((\nabla u, \nabla \phi)) + ((\varphi_k(u_1) - \varphi_k(u_2), \phi)) - ((\varphi_{k'}(v_1) - \varphi_{k'}(v_2), \phi)) = 0, \ \forall \phi \in H^1(\Omega), \tag{3.38}$$

$$\beta((\nabla v, \nabla \psi)) + F((v, \psi)) + ((\varphi_{k'}(v_1) - \varphi_{k'}(v_2), \psi)) - ((\varphi_k(u_1) - \varphi_k(u_2), \psi)) = 0, \ \forall \psi \in H^1(\Omega), \tag{3.39}$$

$$u(0) = u_0. \tag{3.40}$$

Taking $\phi = u$ and $\psi = v$, we obtain, recalling that φ_k and $\varphi_{k'}$ are monotone increasing and Lipschitz continuous,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|\nabla u\|^2 \le c\|u\|\|v\| \tag{3.41}$$

and

$$\beta\|\nabla v\|^2 + F\|v\|^2 \le c\|u\|\|v\|, \tag{3.42}$$

respectively. In particular, it follows from (3.42) that

$$\|v\| \le c\|u\|, \tag{3.43}$$

which, injected into (3.41), yields

$$\frac{d}{dt}\|u\|^2 \le c\|u\|^2, \tag{3.44}$$

whence, owing to Gronwall's lemma,

$$\|u(t)\| \le e^{ct}\|u_0\|, \ t \ge 0. \tag{3.45}$$

We deduce from (3.43) and (3.45) the uniqueness, as well as the continuous dependence with respect to the initial data in the L^2-norm.

3.2 Existence and Uniqueness of Nonnegative Solutions

We first prove additional regularity results on the solutions to (3.27)–(3.28), assuming that (3.22) holds. This can be fully justified within the Galerkin scheme considered above.

Taking $\psi = -\Delta v$ in (3.28), we have

$$\beta\|\Delta v\|^2 + F\|\nabla v\|^2 = ((\varphi_{k'}(v), \Delta v)) - ((\varphi_k(u), \Delta v)),$$

which yields, recalling that φ_k and $\varphi_{k'}$ are bounded,

$$\frac{\beta}{2}\|\Delta v\|^2 + F\|\nabla v\|^2 \leq c,$$

whence estimates on v in $L^\infty(0, T; H^2(\Omega))$, $\forall T > 0$.

Taking then $\phi = \Delta^2 u$ in (3.27) and $\psi = \Delta^2 v$ in (3.28), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \alpha\|\nabla\Delta u\|^2 = -((\varphi_k(u), \Delta^2 u)) + ((\varphi_{k'}(v), \Delta^2 u))$$

and

$$\beta\|\nabla\Delta v\|^2 + F\|\Delta v\|^2 = -((\varphi_{k'}(v), \Delta^2 v)) + ((\varphi_k(u), \Delta^2 v)),$$

respectively. Noting that

$$|((\varphi_k(u), \Delta^2 u))| = |((\varphi_k'(u)\nabla u, \nabla\Delta u))| \leq c\|\nabla u\|\|\nabla\Delta u\|,$$

we find, proceeding in a similar way for the other terms,

$$\frac{d}{dt}\|\Delta u\|^2 + \alpha\|\nabla\Delta u\|^2 \leq c(\|\nabla u\|^2 + \|\nabla v\|^2)$$

and

$$\frac{\beta}{2}\|\nabla\Delta v\|^2 + F\|\Delta v\|^2 \leq c(\|\nabla u\|^2 + \|\nabla v\|^2).$$

This yields estimates on u and v in $L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega))$ and in $L^\infty(0, T; H^3(\Omega))$, respectively, $\forall T > 0$.

Remark 4 This yields that $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, $\forall T > 0$. We further note that the solution to (3.27)–(3.29) is strong, i.e., (3.23)–(3.26) are satisfied almost everywhere.

We can now prove the

Theorem 3 *We assume that (3.22) holds. Then, (3.18)–(3.21) possesses a unique strong solution* (u, v) *such that* $u \geq 0$, $v \geq 0$ *a.e.* (x, t) *and,* $\forall T > 0$,

$$u \in L^\infty(0, T; H^2_N(\Omega)) \cap L^2(0, T; H^3(\Omega)),$$

$$v \in L^\infty(0, T; H^3(\Omega) \cap H^2_N(\Omega)) \cap \mathscr{C}([0, T]; L^2(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

Proof Let (u, v) be the unique strong solution to (3.23)–(3.26). Multiplying (3.23) by $-u^-$ and (3.24) by $-v^-$, where $x^- = \max(0, -x)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u^-\|^2 + \alpha\|\nabla u^-\|^2 + \kappa\int_\Omega \frac{|u^-|^2}{k + |u|}\,dx + \kappa\int_\Omega \frac{vu^-}{k' + |v|}\,dx \leq 0 \tag{3.46}$$

and

$$\beta\|\nabla u^-\|^2 + F\|u^-\|^2 + \kappa\int_\Omega \frac{|v^-|^2}{k' + |v|}\,dx + \kappa\int_\Omega \frac{uv^-}{k + |u|}\,dx \leq 0, \tag{3.47}$$

respectively. Writing $v = v^+ - v^-$, where $x^+ = \max(0, x)$, we deduce from (3.46) that

$$\frac{1}{2}\frac{d}{dt}\|u^-\|^2 \leq \kappa\int_\Omega \frac{u^- v^-}{k' + |v|}\,dx,$$

whence

$$\frac{d}{dt}\|u^-\|^2 \leq c\|u^-\|\|v^-\|. \tag{3.48}$$

Proceeding in a similar way for (3.47), we find

$$F\|v^-\|^2 \leq c\|u^-\|\|v^-\|,$$

whence

$$\|v^-\| \leq c\|u^-\|. \tag{3.49}$$

Injecting this into (3.48), we deduce that

$$\frac{d}{dt}\|u^-\|^2 \leq c\|u^-\|^2,$$

which yields, owing to Gronwall's lemma,

$$\|u^-(t)\| \le e^{ct}\|u^-(0)\|,\ t \ge 0, \tag{3.50}$$

whence, since $u^-(0) = 0$, $u \ge 0$ a.e. (x,t). This, together with (3.49), yields that $v \ge 0$ a.e. (x,t). Consequently, (u,v) is a strong solution to (3.18)–(3.21), which finishes the proof.

Remark 5 Proceeding exactly as in [6], we can prove that

$$\|u(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + (J+\kappa)t,\ t \ge 0.$$

Furthermore, it follows from (3.19) that

$$-\beta\Delta v + Fv \le FL + \kappa. \tag{3.51}$$

Multiplying (3.51) by v^{m+1}, $m \in \mathbb{N}$, we have

$$\beta(m+1)\int_\Omega v^m|\nabla v|^2\,dx + F\|v\|^{m+2}_{L^{m+2}(\Omega)} \le (FL+\kappa)\int_\Omega v^{m+1}\,dx,$$

which yields

$$F\|v\|^{m+2}_{L^{m+2}(\Omega)} \le (FL+\kappa)\mathrm{Vol}(\Omega)^{\frac{1}{m+2}}\|v\|^{m+1}_{L^{m+2}(\Omega)},$$

whence

$$\|v\|_{L^{m+2}(\Omega)} \le \frac{FL+\kappa}{F}\mathrm{Vol}(\Omega)^{\frac{1}{m+2}}. \tag{3.52}$$

Passing to the limit $m \to +\infty$ in (3.52), we finally obtain (see, e.g., [3])

$$\|v(t)\|_{L^\infty(\Omega)} \le \frac{FL+\kappa}{F},\ t \ge 0,$$

meaning that the capillary lactate concentration is again uniformly (with respect to time) bounded. Also note that (2.17) still holds.

Remark 6

(i) As mentioned in the introduction (for the case $\epsilon > 0$, but the situation is the same here), the existence of solutions for negative initial data is a challenging issue. However, we can prove the following partial result (see also [6] for the case $\epsilon > 0$). Let δ_1 and δ_2 be two positive constants such that $k - \delta_1 > 0$ and $k' - \delta_2 > 0$ and assume that $u_0 \ge -\delta_1$ a.e. x. We then consider the following modified initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \kappa (\frac{u}{k - \delta_1 + |u + \delta_1|} - \frac{v}{k' - \delta_2 + |v + \delta_2|}) = J, \tag{3.53}$$

$$-\beta \Delta v + F v + \kappa (\frac{v}{k' - \delta_2 + |v + \delta_2|} - \frac{u}{k - \delta_1 + |u + \delta_1|}) = FL, \tag{3.54}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.55}$$

$$u|_{t=0} = u_0. \tag{3.56}$$

The existence and uniqueness of the solution to (3.53)–(3.56) can be proved by arguing as above. Next, we set $\tilde{u} = u + \delta_1$ and $\tilde{v} = v + \delta_2$. These functions are solutions to

$$\frac{\partial \tilde{u}}{\partial t} - \alpha \Delta \tilde{u} + \kappa (\frac{\tilde{u}}{k - \delta_1 + |\tilde{u}|} - \frac{\tilde{v}}{k' - \delta_2 + |\tilde{v}|}) = \tilde{J}, \tag{3.57}$$

$$-\beta \Delta \tilde{v} + F \tilde{v} + \kappa (\frac{\tilde{v}}{k' - \delta_2 + |\tilde{v}|} - \frac{\tilde{u}}{k - \delta_1 + |\tilde{u}|}) = \tilde{F}, \tag{3.58}$$

$$\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.59}$$

$$\tilde{u}|_{t=0} = u_0 + \delta_1, \tag{3.60}$$

where

$$\tilde{J} = J + \kappa (\frac{\delta_1}{k - \delta_1 + |\tilde{u}|} - \frac{\delta_2}{k' - \delta_2 + |\tilde{v}|})$$

and

$$\tilde{F} = F(L + \delta_2) - \kappa (\frac{\delta_1}{k - \delta_1 + |\tilde{u}|} - \frac{\delta_2}{k' - \delta_2 + |\tilde{v}|}).$$

Choosing δ_1 and δ_2 such that $\tilde{J} \geq 0$ and $\tilde{F} \geq 0$ (in particular, these hold when δ_1 and δ_2 are small enough) and noting that $\tilde{u}(0) \geq 0$ a.e. x, we can prove, as in the proof of Theorem 3, that $\tilde{u}(x, t) \geq 0$ and $\tilde{v}(x, t) \geq 0$ a.e. (x, t), so that (u, v) is solution to (3.18)–(3.21), with

$$u(x, t) \geq -\delta_1 \text{ and } v(x, t) \geq -\delta_2 \text{ a.e. } (x, t).$$

(ii) Similarly, we can prove that, if δ_1 and δ_2 are positive and small enough, with $u_0 \geq \delta_1$ a.e. x, then

$$u(x,t) \geq \delta_1 \text{ and } v(x,t) \geq \delta_2 \text{ a.e. } (x,t).$$

It follows from the above that we can actually define the Lipschitz continuous (for the L^2 and H^1-norms) mapping

$$\mathscr{F} : H^1(\Omega) \to H^1(\Omega),\ w \mapsto z = \mathscr{F}(w),$$

where z is the unique solution to the following elliptic problem:

$$a(z,\phi) + ((\varphi_{k'}(z),\phi)) = ((FL + \varphi_k(w),\phi)),\ \forall \phi \in H^1(\Omega). \tag{3.61}$$

We then have the

Proposition 1 *The mapping $\mathscr{F}$ is differentiable with respect to the L^2 and H^1-norms.*

Proof Let w_0 and w belong to $H^1(\Omega)$ and set $z_0 = \mathscr{F}(w_0)$ and $z = \mathscr{F}(w)$. We then have

$$a(z-z_0,\phi) + ((\varphi_{k'}(z) - \varphi_{k'}(z_0),\phi)) = ((\varphi_k(w) - \varphi_k(w_0),\phi)),\ \forall \phi \in H^1(\Omega). \tag{3.62}$$

Taking $\phi = z - z_0$ and reacalling that $\varphi_{k'}$ is monotone increasing, this yields

$$\|z - z_0\|_{H^1(\Omega)} \leq c\|\varphi_k(w) - \varphi_k(w_0)\|,$$

whence

$$\|z - z_0\|_{H^1(\Omega)} \leq c\|w - w_0\|. \tag{3.63}$$

Let then $Z \in H^1(\Omega)$ be the solution to the linear elliptic problem (recall that φ_k' is nonnegative)

$$a(Z,\phi) + ((\varphi_{k'}'(z_0)Z,\phi)) = ((\varphi_k'(w_0)(w-w_0),\phi)),\ \forall \phi \in H^1(\Omega). \tag{3.64}$$

Setting $h = w - w_0$, we can see that

$$a(z - z_0 - Z,\phi) + ((\varphi_{k'}(z) - \varphi_{k'}(z_0) - \varphi_{k'}'(z_0)Z,\phi)) \tag{3.65}$$

$$= ((\varphi_k(w) - \varphi_k(w_0) - \varphi_k'(w_0)h,\phi)),\ \forall \phi \in H^1(\Omega).$$

Writing

$$\varphi_{k'}(z) - \varphi_{k'}(z_0) - \varphi'_{k'}(z_0)Z = \varphi'_{k'}(z_0)(z - z_0 - Z) + o(\|z - z_0\|)$$

and

$$\varphi_k(w) - \varphi_k(w_0) - \varphi'_k(w_0)h = o(\|h\|),$$

we obtain, taking $\phi = z - z_0 - Z$ and employing (3.63) (also recall that $\varphi'_{k'} \geq 0$),

$$a(z - z_0 - Z, z - z_0 - Z) \leq |((o(\|h\|), z - z_0 - Z))|,$$

whence

$$\|z - z_0 - Z\|_{H^1(\Omega)} = o(\|h\|).$$

This yields that $\mathscr{F}$ is differentiable at w_0, with $\mathscr{F}'(w_0) \cdot h = Z$, $\mathscr{F}'$ denoting the differential of $\mathscr{F}$.

We deduce from Proposition 1 the

Corollary 1 *Let* (u, v) *be the solution to* (3.18)–(3.21) *given in Theorem 3. Then,* $\forall T > 0$,

$$\frac{\partial v}{\partial t} \in L^\infty(0, T; H^1(\Omega))$$

and

$$\|\frac{\partial v}{\partial t}\|_{H^1(\Omega)} \leq c\|\frac{\partial u}{\partial t}\| \text{ a.e. } t \geq 0. \tag{3.66}$$

Proof It suffices to note that $v = \mathscr{F}(u)$, whence, owing to Proposition 1,

$$\frac{\partial v}{\partial t} = \mathscr{F}'(u) \cdot \frac{\partial u}{\partial t}. \tag{3.67}$$

Indeed, we can note that $u \in H^1(0, T; H^1(\Omega))$, $\forall T > 0$. Furthermore, we have

$$a(\frac{\partial v}{\partial t}, \phi) + ((\varphi'_{k'}(v)\frac{\partial v}{\partial t}, \phi)) = ((\varphi'_k(u)\frac{\partial u}{\partial t}, \phi)),\ \forall \phi \in H^1(\Omega), \tag{3.68}$$

and (3.66) follows, taking $\phi = \frac{\partial v}{\partial t}$ (also recall that $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega))$, $\forall T > 0$).

Remark 7 It follows from standard elliptic regularity results applied to (3.68) (see, e.g., [1] and [2]) that, $\forall T > 0$, $\frac{\partial v}{\partial t} \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega))$, with

$$\|\frac{\partial v}{\partial t}\|_{H^2(\Omega)} \leq c\|\frac{\partial u}{\partial t}\| \text{ a.e. } t \geq 0$$

and

$$\|\frac{\partial v}{\partial t}\|_{L^2(0,T;H^3(\Omega))} \leq Q(T, \|u_0\|_{H^1(\Omega)})\|\frac{\partial u}{\partial t}\|_{L^2(0,T;H^1(\Omega))}.$$

4 Convergence to the Limit Problem

All constants c and c' and functions Q in this section are independent of ϵ.

Let (u^ϵ, v^ϵ) and (u^0, v^0) be the unique strong solutions to the initial and limit problems, respectively, as given in Theorems 1 and 3, where $v_0 = v^0(0)$, i.e.,

$$\frac{\partial u^\epsilon}{\partial t} - \alpha \Delta u^\epsilon + \kappa(\frac{k'}{k' + v^\epsilon} - \frac{k}{k + u^\epsilon}) = J, \tag{4.69}$$

$$\epsilon \frac{\partial v^\epsilon}{\partial t} - \beta \Delta v^\epsilon + F v^\epsilon + \kappa(\frac{k}{k + u^\epsilon} - \frac{k'}{k' + v^\epsilon}) = FL, \tag{4.70}$$

$$\frac{\partial u^\epsilon}{\partial \nu} = \frac{\partial v^\epsilon}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4.71}$$

$$u^\epsilon|_{t=0} = u_0,\ v^\epsilon|_{t=0} = v^0(0), \tag{4.72}$$

and

$$\frac{\partial u^0}{\partial t} - \alpha \Delta u^0 + \kappa(\frac{k'}{k' + v^0} - \frac{k}{k + u^0}) = J, \tag{4.73}$$

$$-\beta \Delta v^0 + F v^0 + \kappa(\frac{k}{k + u^0} - \frac{k'}{k' + v^0}) = FL, \tag{4.74}$$

$$\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4.75}$$

$$u^0|_{t=0} = u_0. \tag{4.76}$$

We have the

Theorem 4 *The following error estimates hold,* $\forall T > 0$*:*

$$\|u^\epsilon(t) - u^0(t)\|_{H^1(\Omega)} \leq Q(T, \|u_0\|_{H^1(\Omega)})\epsilon,$$

$$\|v^\epsilon(t) - v^0(t)\|_{H^1(\Omega)} \leq Q(T, \|u_0\|_{H^1(\Omega)})\sqrt{\epsilon},$$

$t \in [0, T]$, *and*

$$\|u^\epsilon - u^0\|_{L^2(0,T;H^2(\Omega))} \leq Q(T, \|u_0\|_{H^1(\Omega)})\epsilon,$$

$$\|v^\epsilon - v^0\|_{L^2(0,T;H^2(\Omega))} \leq Q(T, \|u_0\|_{H^1(\Omega)})\epsilon.$$

Proof We have, setting $u = u^\epsilon - u^0$ and $v = v^\epsilon - v^0$,

$$\frac{\partial u}{\partial t} - \alpha \Delta u + \varphi_k(u^\epsilon) - \varphi_k(u^0) - \varphi_{k'}(v^\epsilon) + \varphi_{k'}(v^0) = 0, \tag{4.77}$$

$$\epsilon \frac{\partial v}{\partial t} - \beta \Delta v + Fv + \varphi_{k'}(v^\epsilon) - \varphi_{k'}(v^0) - \varphi_k(u^\epsilon) + \varphi_k(u^0) = -\epsilon \frac{\partial v^0}{\partial t},\ \epsilon > 0, \tag{4.78}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4.79}$$

$$u|_{t=0} = 0,\ v|_{t=0} = 0. \tag{4.80}$$

Multiplying (4.77) by u, we obtain, recalling that φ_k is monotone increasing,

$$\frac{d}{dt}\|u\|^2 + \alpha \|\nabla u\|^2 \leq c\|u\|\|v\|. \tag{4.81}$$

Multiplying then (4.78) by v, we find, similarly,

$$\epsilon \frac{d}{dt}\|v\|^2 + c\|v\|^2_{H^1(\Omega)} \leq c'(\|u\|\|v\| + \epsilon^2 \|\frac{\partial v^0}{\partial t}\|^2),\ c > 0. \tag{4.82}$$

Combining (4.81) and (4.82), we have, owing to (3.66),

$$\frac{d}{dt}(\|u\|^2 + \epsilon\|v\|^2) + c(\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}) \tag{4.83}$$

$$\leq c'(\|u\|^2 + \epsilon\|v\|^2 + \epsilon^2 \|\frac{\partial u^\epsilon}{\partial t}\|^2),\ c > 0,$$

from which it follows, owing to Gronwall's lemma,

$$\|u(t)\|^2 + \epsilon\|v(t)\|^2 + c\int_0^T (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Omega)})\, ds \tag{4.84}$$

$$\leq Q(T)\epsilon^2 \|\frac{\partial u^\epsilon}{\partial t}\|^2_{L^2(0,T;L^2(\Omega))},\ c > 0,\ t \in [0, T].$$

Multiplying now (4.69) by $\frac{\partial u^\epsilon}{\partial t}$, we obtain

$$\frac{d}{dt}\|\nabla u^\epsilon\|^2 + c\|\frac{\partial u^\epsilon}{\partial t}\|^2 \leq c',\ c > 0,$$

whence

$$\|\frac{\partial u^\epsilon}{\partial t}\|^2_{L^2(0,T;L^2(\Omega))} \leq Q(T, \|u_0\|_{H^1(\Omega)}). \tag{4.85}$$

We finally deduce from (4.84)–(4.85) that

$$\|u(t)\|^2 + \epsilon\|v(t)\|^2 + c\int_0^T (\|u(s)\|^2_{H^1(\Omega)} + \|v(s)\|^2_{H^1(\Omega)})\, ds$$
$$\leq Q(T, \|u_0\|_{H^1(\Omega)})\epsilon^2,\ c > 0,\ t \in [0, T]. \tag{4.86}$$

Multiplying next (4.77) by $-\Delta u$ and (4.78) by $-\Delta v$, we find, recalling that φ_k and $\varphi_{k'}$ are Lipschitz continuous,

$$\frac{d}{dt}\|\nabla u\|^2 + c\|\Delta u\|^2 \leq c'(\|u\|^2 + \|v\|^2),\ c > 0,$$

and

$$\epsilon\frac{d}{dt}\|\nabla v\|^2 + c\|\Delta v\|^2 \leq c'(\|u\|^2 + \|v\|^2 + \epsilon^2\|\frac{\partial v^0}{\partial t}\|^2),\ c > 0,$$

respectively.

Summing these two inequalities, integrating over $[0, T]$ and proceeding as above, we have, adding the resulting inequality to (4.86),

$$\|u(t)\|^2_{H^1(\Omega)} + \epsilon\|v(t)\|^2_{H^1(\Omega)} + c\int_0^T (\|u(s)\|^2_{H^2(\Omega)} + \|v(s)\|^2_{H^2(\Omega)})\, ds$$

$$\leq c'(\int_0^T (\|u(s)\|^2 + \|v(s)\|^2)\, ds + Q(T, \|u_0\|_{H^1(\Omega)})\epsilon^2),\ c > 0.$$

This yields, employing (4.86) to estimate the right-hand side,

$$\|u(t)\|^2_{H^1(\Omega)} + \epsilon\|v(t)\|^2_{H^1(\Omega)} + c\int_0^T (\|u(s)\|^2_{H^2(\Omega)} + \|v(s)\|^2_{H^2(\Omega)})\, ds$$

$$\leq Q(T, \|u_0\|_{H^1(\Omega)})\epsilon^2,\ c > 0,\ t \in [0, T],$$

which finishes the proof.

Remark 8 We have similar error estimates if we assume that $\|u^\epsilon(0)-u^0(0)\|_{H^1(\Omega)} \leq c\epsilon$ and $\|v^\epsilon(0)-v^0(0)\|_{H^1(\Omega)} \leq c\sqrt{\epsilon}$.

5 A Stability Result

As in [6], (3.18)–(3.19) possesses a unique spatially homogeneous equilibrium $(\overline{u},\overline{v})$ given by

$$\overline{v} = L + \frac{J}{F} > 0$$

and

$$\overline{u} = \frac{k(\frac{J}{\kappa} + \frac{\overline{v}}{k'+\overline{v}})}{1-(\frac{J}{\kappa} + \frac{\overline{v}}{k'+\overline{v}})}.$$

Note that $\overline{u}$ is not necessarily positive. We thus assume in what follows that

$$\overline{u} > 0.$$

The linearized (around $(\overline{u},\overline{v})$) system reads

$$\frac{\partial U}{\partial t} - \alpha\Delta U + \kappa\Big(\frac{k}{(k+\overline{u})^2}U - \frac{k'}{(k'+\overline{v})^2}V\Big) = 0, \tag{5.87}$$

$$-\beta\Delta V + FV + \kappa\Big(\frac{k'}{(k'+\overline{v})^2}V - \frac{k}{(k+\overline{u})^2}U\Big) = 0, \tag{5.88}$$

$$\frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.89}$$

$$U|_{t=0} = U_0. \tag{5.90}$$

It is not difficult here to prove the existence, uniqueness and regularity of the solution to (5.87)–(5.90), assuming that U_0 is regular enough. Furthermore, proceeding as above, we can prove that, if $U_0 \geq 0$ a.e. x, then $U(x,t) \geq 0$ and $V(x,t) \geq 0$ a.e. (x,t).

Multiplying now (5.87) by $\frac{k}{(k+\overline{u})^2}U$ and (5.88) by $\frac{k'}{(k'+\overline{v})^2}V$, we obtain, summing the two resulting equalities,

$$\frac{1}{2}\frac{k}{(k+\overline{u})^2}\frac{d}{dt}\|U\|^2+\frac{\alpha k}{(k+\overline{u})^2}\|\nabla U\|^2+\frac{\beta k'}{(k'+\overline{v})^2}\|\nabla V\|^2+\frac{Fk'}{(k'+\overline{v})^2}\|V\|^2 + \int_\Omega \Big(\frac{k}{(k+\overline{u})^2}U-\frac{k'}{(k'+\overline{v})^2}V\Big)^2\,dx=0. \tag{5.91}$$

It follows from (5.91) that

$$\frac{d}{dt}\|U\|^2\le 0,$$

whence

$$\|U(t)\|\le\|U_0\|,\ t\ge 0. \tag{5.92}$$

Multiplying next (5.88) by V, we easily find

$$\|V\|\le c\|U\|,$$

so that

$$\|V(t)\|\le c\|U_0\|,\ t\ge 0. \tag{5.93}$$

We deduce from (5.92)–(5.93) that $(\overline{u},\overline{v})$ is linearly stable with respect to the L^2-norm. We can also prove the linear stability with respect to the H^1-norm, proceeding in a similar way.

Now, an important question is whether we also have a linear exponential stability as in [6] for the case $\epsilon>0$ (see also [5, 9, 10] and [11] for the ODE's model (1.1)–(1.2)). Indeed, as mentioned in the introduction, a therapeutic perspective of such a result is to have the (spatially homogeneous) steady state outside the viability domain, where cell necrosis occurs (see [9]).

We have, in this direction, the

Theorem 5 *The stationary solution $(\overline{u},\overline{v})$ is linearly exponentially stable, in the sense that all eigenvalues $s\in\mathbb{C}$ associated with the linear system* (5.87)–(5.88) *satisfy $\mathscr{R}e(s)\le-\xi$, for a given $\xi>0$, $\mathscr{R}e$ denoting the real part.*

Proof We first note that it follows from (5.88) that

$$V=k_1(-\beta\Delta+(F+k_2)I)^{-1}U, \tag{5.94}$$

where $k_1 = \frac{\kappa k}{(k+\overline{u})^2}$ and $k_2 = \frac{\kappa k'}{(k'+\overline{v})^2}$. Injecting this into (5.87), we obtain

$$\frac{\partial U}{\partial t} - \alpha \Delta U + k_1 U - k_1 k_2 (-\beta \Delta + (F + k_2) I)^{-1} U = 0. \tag{5.95}$$

We then look for solutions of the form

$$U(x,t) = \hat{U}(x) e^{st},$$

for $s \in \mathbb{C}$, $s = \zeta + i\eta$. Injecting this into (5.95), we find

$$-\alpha \Delta \hat{U} + (s + k_1) \hat{U} - k_1 k_2 (-\beta \Delta + (F + k_2) I)^{-1} \hat{U} = 0, \tag{5.96}$$

where

$$\frac{\partial \hat{U}}{\partial \nu} = 0 \text{ on } \Gamma. \tag{5.97}$$

This yields

$$\alpha\beta \Delta^2 \hat{U} - (\beta s + \alpha F + \beta k_1 + \alpha k_2) \Delta \hat{U} + ((F + k_2) s + k_1 F) \hat{U} = 0, \tag{5.98}$$

where, owing to (5.96) and (5.97),

$$\frac{\partial \hat{U}}{\partial \nu} = \frac{\partial \Delta \hat{U}}{\partial \nu} = 0 \text{ on } \Gamma. \tag{5.99}$$

Multiplying (5.98) by the conjugate of $\hat{U}$, integrating over Ω and by parts and taking the real part, we have

$$\alpha\beta \| |\Delta \hat{U}| \|^2 + (\beta \zeta + \alpha F + \beta k_1 + \alpha k_2) \| |\nabla \hat{U}| \|^2 \tag{5.100}$$

$$+ ((F + k_2) \zeta + k_1 F) \| |\hat{U}| \|^2 = 0.$$

Therefore, when $\zeta \geq 0$, then, necessarily, $\hat{U} \equiv 0$. Furthermore, (5.100) can have nontrivial solutions only when

$$\beta \zeta + \alpha F + \beta k_1 + \alpha k_2 \leq 0$$

or

$$(F + k_2) \zeta + k_1 F \leq 0.$$

Therefore, necessarily,

$$\zeta \leq \max(-\frac{\alpha F + \beta k_1 + \alpha k_2}{\beta}, -\frac{k_1 F}{F + k_2}) < 0, \tag{5.101}$$

which finishes the proof.

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Weak Formulation for Singular Diffusion Equation with Dynamic Boundary Condition

Ryota Nakayashiki and Ken Shirakawa

Abstract In this paper, we propose a weak formulation of the singular diffusion equation subject to the dynamic boundary condition. The weak formulation is based on a reformulation method by an evolution equation including the subdifferential of a governing convex energy. Under suitable assumptions, the principal results of this study are stated in forms of Main Theorems A and B, which are respectively to verify: the adequacy of the weak formulation; the common property between the weak solutions and those in regular problems of standard PDEs.

Keywords Comparison principle • Dynamic boundary condition • Evolution equation • Governing convex energy • Mosco-convergence • Singular diffusion equation

AMS Subject Classification 35K20, 35K67, 49J45

1 Introduction

Let $\varepsilon > 0$, $0 < T < \infty$ and $1 < N \in \mathbb{N}$ be fixed constants. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$, and let n_Γ be the unit outer normal to Γ. Besides, let us denote by $Q := (0, T) \times \Omega$ the product space of the time interval $(0, T)$ and the spatial domain Ω, and let us set $\Sigma := (0, T) \times \Gamma$.

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In this paper, we consider the following initial-boundary value problem of parabolic type:

$$\partial_t u - \mathrm{div}\left(\frac{Du}{|Du|}\right) = \theta(t,x), (t,x) \in Q, \tag{1}$$

$$\partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + (\tfrac{Du}{|Du|})_{|_\Gamma} \cdot n_\Gamma = \theta_\Gamma(t,x_\Gamma), (t,x_\Gamma) \in \Sigma, \tag{2}$$

$$u_{|_\Gamma} = u_\Gamma(t,x_\Gamma), (t,x_\Gamma) \in \Sigma, \tag{3}$$

$$u(0,x) = u_0(x), x \in \Omega, \text{ and } u_\Gamma(0,x_\Gamma) = u_{\Gamma,0}(x_\Gamma), x_\Gamma \in \Gamma, \tag{4}$$

including the singular diffusion $-\mathrm{div}(\frac{Du}{|Du|})$ with the normal derivative $(\frac{Du}{|Du|})_{|_\Gamma} \cdot n_\Gamma$. Here, $\theta \in L^2(0,T;L^2(\Omega))$ and $\theta_\Gamma \in L^2(0,T;L^2(\Gamma))$ are given source terms, and $u_0 \in L^2(\Omega)$ and $u_{\Gamma,0} \in L^2(\Gamma)$ are given initial data. Δ_Γ denotes the Laplace–Beltrami operator on the surface Γ, and "$_{|_\Gamma}$" denotes the trace of a function on Ω. In particular, the boundary conditions (2)–(3) are collectively called *dynamic boundary condition,* and it consists of the part of PDE (2) on the surface Γ, and the part of transmission condition (3) between the PDEs (1)–(2).

The representative characteristics of (1)–(4) is in the point that this problem can be regarded as a type of transmission system, containing the Dirichlet type boundary-value problem of singular diffusion equation (1), (3). So, referring to the previous works [2, 22], one can remark that:

(⋆) the expressions of the singular terms in (1)–(2) and the transmission condition (3) are practically meaningless, and for the treatments in rigorous mathematics, these must be prescribed in a weak variational sense, based on the spatial regularity in the space $BV(\Omega)$ of functions of bounded variations.

To answer the remark **(⋆)**, we here adopt an idea to put:

$$\begin{cases} U_0 := [u_0, u_{\Gamma,0}] \text{ in } \mathscr{H} := L^2(\Omega) \times L^2(\Gamma), \\ U := [u, u_\Gamma] \text{ and } \Theta := [\theta, \theta_\Gamma] \text{ in } L^2(0,T;\mathscr{H}), \end{cases}$$

and to reformulate the transmission system {(1)–(4)} to the Cauchy problem of an evolution equation:

$$\begin{cases} U'(t) + \partial\Phi_*(U(t)) \ni \Theta(t) \text{ in } \mathscr{H}, t \in (0,T), \\ U(0) = U_0 \text{ in } \mathscr{H}; \end{cases} \tag{5}$$

which is governed by the subdifferential $\partial\Phi_*$ of the following convex function Φ_* on $\mathscr{H}$:

$$W = [w, w_\Gamma] \in \mathscr{H} \mapsto \Phi_*(W) = \Phi_*(w, w_\Gamma)$$
$$:= \begin{cases} \displaystyle\int_\Omega |Dw| + \int_\Gamma |w_{|_\Gamma} - w_\Gamma|\, d\Gamma + \frac{\varepsilon^2}{2}\int_\Gamma |\nabla_\Gamma w_\Gamma|^2\, d\Gamma, \\ \qquad \text{if } w \in BV(\Omega) \cap L^2(\Omega), w_\Gamma \in H^1(\Gamma), \\ \infty, \ \text{otherwise}; \end{cases} \tag{6}$$

where $\int_\Omega |Dw|$ denotes the total variation of $w \in BV(\Omega) \cap L^2(\Omega)$, and ∇_Γ and $d\Gamma$ denote the surface gradient and the area element on Γ, respectively. Besides, we simply denote by $\mathscr{W}$ the effective domain of Φ_*, i.e.

$$\mathscr{W} := (BV(\Omega) \cap L^2(\Omega)) \times H^1(\Gamma),$$

and we propose to define a *weak solution*, i.e. the solution to a *weak formulation* to the system (1)–(4), as follows.

Definition 1 A pair of functions $[u, u_\Gamma] \in L^2(0, T; \mathscr{H})$ is called a *weak solution* to (1)–(4), iff. $u \in W^{1,2}(0, T; L^2(\Omega))$, $|Du|(\Omega) \in L^\infty(0, T)$, $u_\Gamma \in W^{1,2}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^1(\Gamma))$ and

$$\begin{aligned} &\int_\Omega \partial_t u(t)(u(t) - z)\, dx + \int_\Omega |Du(t)| + \int_\Gamma |u_{|_\Gamma}(t) - u_\Gamma(t)|\, d\Gamma \\ &+ \int_\Gamma \partial_t u_\Gamma(t)(u_\Gamma(t) - z_\Gamma)\, d\Gamma + \varepsilon^2 \int_\Gamma \nabla_\Gamma u_\Gamma(t) \cdot \nabla_\Gamma(u_\Gamma(t) - z_\Gamma)\, d\Gamma \\ &\le \int_\Omega |Dz| + \int_\Gamma |z_{|_\Gamma} - z_\Gamma|\, d\Gamma, \ \text{for any } [z, z_\Gamma] \in \mathscr{W}. \end{aligned} \tag{7}$$

As a natural consequence, the above Definition 1 will raise some issues concerned with:

(q1) the adequacy of Definition 1 as the variational characterization for the singular transmission system (1)–(4);
(q2) the exemplification of fine properties which sustain common properties between our weak solutions and the solutions to regular transmission systems via the standard dynamic boundary conditions.

In the issue (q1), it will be essential to ensure that:

(★★) the Cauchy problem (5) can be said as an invariant formulation to define the weak solution, i.e. the finding formulation is well-established, if various approximation approaches are applied by using many kinds of relaxation methods, with any convergent orders of the relaxation arguments.

Then, it will be recommended that some of such relaxation methods are involved in the numerical approaches to our singular system.

In view of this, we consider the following regular transmission system via the standard dynamic boundary condition:

$$\partial_t u - \mathrm{div}\left(\nabla f_\delta(\nabla u) + \kappa^2 \nabla u\right) = \theta(t,x), (t,x) \in Q, \tag{8}$$

$$\partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + (\nabla f_\delta(\nabla u) + \kappa^2 \nabla u)_{|_\Gamma} \cdot n_\Gamma = \theta_\Gamma(t, x_\Gamma), (t, x_\Gamma) \in \Sigma, \tag{9}$$

$$u_{|_\Gamma} = u_\Gamma(t, x_\Gamma), (t, x_\Gamma) \in \Sigma, \tag{10}$$

$$u(0,x) = u_0(x), x \in \Omega, \text{ and } u_\Gamma(0, x_\Gamma) = u_{\Gamma,0}(x_\Gamma), x_\Gamma \in \Gamma; \tag{11}$$

as relaxed versions of (1)–(4). Here, $\kappa > 0$ and $\delta > 0$ are given constants, and $\nabla f_\delta \in L^\infty(\mathbb{R}^N)^N$ is the differential (gradient) of a convex function $f_\delta \in W^{1,\infty}(\mathbb{R}^N)$. Besides, the sequence $\{f_\delta\}_{\delta>0}$ is supposed to converge to the Euclidean norm $|\cdot|$, appropriately on $\mathbb{R}^N$, as $\delta \to 0$.

Now, by changing the setting of $\{f_\delta\}_{\delta>0}$ in many ways, we can make various approximating problems that approach to (1)–(4) as $\kappa, \delta \to 0$. Also, we note that the variety of $\{f_\delta\}_{\delta>0}$ can cover typical numerical regularizations for singular diffusions, such as regularization by hyperbola:

$$\omega \in \mathbb{R}^N \mapsto f_\delta(\omega) := \sqrt{|\omega|^2 + \delta^2}, \text{ for } \delta > 0,$$

and the Yosida-regularization of Euclidean norm $|\cdot|$, etc., even if the convergence of $\{f_\delta\}_{\delta>0}$ is restricted to the uniform sense. Incidentally, we can take form any convergent order of the coupling $(\kappa, \delta) \to (0, 0)$, up to the choices of sequences $\{\kappa_n\}_{n=1}^\infty \subset \{\kappa\}$ and $\{\delta_n\}_{n=1}^\infty \subset \{\delta\}$. Such wide flexibility will be reasonable to authorize our weak formulation, and this is the principal reason why we settle the relaxation system as stated in (8)–(11).

In addition, referring to the previous relevant works, e.g. [8–11, 14], we can see that each approximating problem (8)–(11) is equivalent to the Cauchy problem of an evolution equation:

$$\begin{cases} U'(t) + \partial\Phi_\delta^\kappa(U(t)) \ni \Theta(t) \text{ in } \mathscr{H}, t \in (0,T), \\ U(0) = U_0 \text{ in } \mathscr{H}; \end{cases} \tag{12}$$

which is governed by the subdifferential $\partial \Phi_\delta^\kappa$ of a convex function $\Phi_\delta^\kappa : \mathscr{H} \to [0, \infty]$ defined as:

$$V = [v, v_\Gamma] \in \mathscr{H} \mapsto \Phi_\delta^\kappa(V) = \Phi_\delta^\kappa(v, v_\Gamma) \\ := \begin{cases} \displaystyle\int_\Omega \left(f_\delta(\nabla v) + \frac{\kappa^2}{2}|\nabla v|^2 \right) dx + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma v_\Gamma|^2 \, d\Gamma, \\ \qquad \text{if } v \in H^1(\Omega), v_\Gamma \in H^1(\Gamma) \text{ and } v_{|_\Gamma} = v_\Gamma \text{ in } L^2(\Gamma), \\ \infty, \quad \text{otherwise.} \end{cases} \tag{13}$$

Hence, for the verification of (q1), it would be effective to observe the continuous dependence between the Cauchy problems (5) and (12), as $\kappa, \delta \to 0$, for every regularizations $\{f_\delta\}_{\delta>0}$. Furthermore, on account of the general theories of nonlinear evolution equations and their variational convergence [4, 6, 7, 18], the essence of (q1) can be reduced as follows.

(A) An issue to verify that the convex function Φ_* on $\mathscr{H}$, given in (6), is a limit of various sequences of relaxed convex functions $\{\Phi_\delta^\kappa\}_{\kappa,\delta>0}$ on $\mathscr{H}$, in the sense of Mosco [24], as $\kappa, \delta \to 0$.

In the meantime, for the issue (q2), we focus on the comparison principle for the weak solutions to (1)–(4), stated as follows.

(B) If $[u_0^k, u_{\Gamma,0}^k] \in \mathscr{W}$ and $[\theta^k, \theta_\Gamma^k] \in L^2(0, T; \mathscr{H})$, for $k = 1, 2$, and

$$\begin{cases} u_0^1 \le u_0^2 \text{ a.e. in } \Omega, \theta^1 \le \theta^2 \text{ a.e. in } Q, \\ u_{\Gamma,0}^1 \le u_{\Gamma,0}^2 \text{ a.e. on } \Gamma, \theta_\Gamma^1 \le \theta_\Gamma^2 \text{ a.e. on } \Sigma, \end{cases}$$

then, it holds that:

$$u^1 \le u^2 \text{ a.e. in } Q, \text{ and } u_\Gamma^1 \le u_\Gamma^2 \text{ a.e. on } \Sigma,$$

where for every $k = 1, 2$, $[u^k, u_\Gamma^k] \in L^2(0, T; \mathscr{H})$ is a solution to (1)–(4) in the case when $[u_0, u_{\Gamma,0}] = [u_0^k, u_{\Gamma,0}^k]$ and $[\theta, \theta_\Gamma] = [\theta^k, \theta_\Gamma^k]$.

Indeed, in regular systems like (8)–(11), the property kindred to (B) can be verified, immediately, by applying usual methods as in [2, 6, 7, 18, 22]. But in our study, the issue of comparison principle (B) will be delicate, because the boundary integral $\int_\Gamma |w_{|_\Gamma} - w_\Gamma| \, d\Gamma$ as in (6) will bring non-trivial interaction between the unknowns u and u_Γ in the transmission system (1)–(4).

In view of these, the discussions for the above (A) and (B) are developed in accordance with the following contents. In Sect. 2, we prepare preliminaries of this study, and in Sect. 3, we state the results of this paper. The principal part of our results are stated as Main Theorems A and B, and these correspond to the issues (A) and (B), respectively. Then, the continuous dependence between Cauchy problems (5) and (12) will be mentioned as a Corollary of Main Theorem A. The results are proved through the following Sects. 4 and 5, which are assigned to the preparation of Key-Lemmas, and to the body of the proofs of Main Theorems and the corollary, respectively. Furthermore, in the final Sect. 6, we mention about an advanced issue as the future prospective of this study.

2 Preliminaries

In this section, we outline some basic matters, as preliminaries of our study.

Notation 1 (Notations in Real Analysis) For arbitrary $a, b \in [-\infty, \infty]$, we define:

$$a \vee b := \max\{a, b\} \text{ and } a \wedge b := \min\{a, b\};$$

and in particular, we write $[a]^+ := a \vee 0$ and $[b]^- := -(0 \wedge b)$.

Let $d \in \mathbb{N}$ be any fixed dimension. Then, we simply denote by $|x|$ and $x \cdot y$ the Euclidean norm of $x \in \mathbb{R}^d$ and the standard scalar product of $x, y \in \mathbb{R}^d$, respectively. Also, we denote by $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$ the d-dimensional unit open ball centered at the origin, and its boundary, respectively, i.e.:

$$\mathbb{B}^d := \left\{ x \in \mathbb{R}^d \;\middle|\; |x| < 1 \right\} \text{ and } \mathbb{S}^{d-1} := \left\{ x \in \mathbb{R}^d \;\middle|\; |x| = 1 \right\}.$$

In particular, when $d > 1$, we let:

$$\begin{cases} x \vee y := \left[x_1 \vee y_1, \dots, x_d \vee y_d\right], x \wedge y := \left[x_1 \wedge y_1, \dots, x_d \wedge y_d\right], \\ [x]^+ := \left[[x_1]^+, \dots, [x_d]^+\right] \text{ and } [y]^- := \left[[y_1]^-, \dots, [y_d]^-\right], \end{cases} \text{for all } x, y \in \mathbb{R}^d.$$

Besides, we often describe a d-dimensional vector $x = [x_1, \dots, x_d] \in \mathbb{R}^d$ as $x = [\tilde{x}, x_d]$ by putting $\tilde{x} = [x_1, \dots, x_{d-1}] \in \mathbb{R}^{d-1}$. As well as, we describe the gradient $\nabla = [\partial_1, \dots, \partial_d]$ as $\nabla = [\tilde{\nabla}, \partial_d]$ by putting $\tilde{\nabla} = [\partial_1, \dots, \partial_{d-1}]$, and additionally, we describe ∇_x, ∂_t, ∂_{x_i}, $i = 1, \dots, d$, and so on, when we need to specify the variables of differentials.

Notation 2 (Notations of Functional Analysis) For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by ${}_{X^*}\langle \cdot, \cdot \rangle_X$ the duality pairing between X and the dual space X^* of X. In particular, when X is a Hilbert space, we denote by $(\,\cdot\,,\,\cdot\,)_X$ the inner product in X.

Notation 3 (Notations in Convex Analysis) Let X be an abstract real Hilbert space.

For any closed and convex set $\mathscr{C} \subset X$, we denote by $\pi_{\mathscr{C}} : X \to \mathscr{C}$ the orthogonal projection onto $\mathscr{C}$.

For any proper lower semi-continuous (l.s.c. from now on) and convex function Ψ defined on X, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product space $X \times X$. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0), \text{ for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{z \in X \,|\, \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$, and the notation "$[z_0, z_0^*] \in \partial\Psi$ in $X \times X$" is often rephrased as "$z_0^* \in \partial\Psi(z_0)$ in X with $z_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

On this basis, we here recall the notion of Mosco-convergence for sequences of convex functions.

Definition 2 (Mosco-Convergence: cf. [24]) Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^{\infty}$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco [24], as $n \to \infty$, iff. the following two conditions are fulfilled.

(M1) Lower-bound condition: $\varliminf_{n\to\infty} \Psi_n(\check{z}_n) \geq \Psi(\check{z})$, if $\check{z} \in X$, $\{\check{z}_n\}_{n=1}^{\infty} \subset X$, and $\check{z}_n \to \check{z}$ weakly in X as $n \to \infty$.

(M2) Optimality condition: for any $\hat{z} \in D(\Psi)$, there exists a sequence $\{\hat{z}_n\}_{n=1}^{\infty} \subset X$ such that $\hat{z}_n \to \hat{z}$ in X and $\Psi_n(\hat{z}_n) \to \Psi(\hat{z})$, as $n \to \infty$.

Notation 4 (Notations in Basic Measure Theory: cf. [1, 5]) For any $d \in \mathbb{N}$, the d-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$, and unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observations on a C^1-surface S, the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on S is denoted by dS.

Let $d \in \mathbb{N}$ be any dimension, and let $A \subset \mathbb{R}^d$ be any open set. We denote by $\mathscr{M}(A)$ (resp. $\mathscr{M}_{\mathrm{loc}}(A)$) the space of all finite Radon measures (resp. the space of all Radon measures) on A. In general, the space $\mathscr{M}(A)$ (resp. $\mathscr{M}_{\mathrm{loc}}(A)$) is known as the dual of the Banach space $C_0(A)$ (resp. dual of the locally convex space $C_{\mathrm{c}}(A)$).

Notation 5 (Notations in BV-Theory: cf. [1, 5, 12, 15]) Let $d \in \mathbb{N}$ be a dimension of the Euclidean space $\mathbb{R}^d$, and let $A \subset \mathbb{R}^d$ be an open set. A function $u \in L^1(A)$ (resp. $u \in L^1_{\mathrm{loc}}(A)$) is called a function of bounded variation, or a BV-function (resp. a function of locally bounded variation, or a $\mathrm{BV}_{\mathrm{loc}}$-function) on A, iff. its distributional differential Du is a finite Radon measure on A (resp. a Radon measure on A), namely $Du \in \mathscr{M}(A)$ (resp. $Du \in \mathscr{M}_{\mathrm{loc}}(A)$). We denote by $BV(A)$ (resp. $BV_{\mathrm{loc}}(A)$) the space of all BV-functions (resp. all $\mathrm{BV}_{\mathrm{loc}}$-functions) on A. For any $u \in BV(A)$, the Radon measure Du is called the variation measure of u, and its total variation $|Du|$ is called the total variation measure of u. Additionally, the value $|Du|(A)$, for any $u \in BV(A)$, can be calculated as follows:

$$|Du|(A) = \sup \left\{ \int_A u \operatorname{div} \varphi \, dy \middle| \varphi \in C_c^1(A)^d \text{ and } |\varphi| \leq 1 \text{ on } A \right\}.$$

The space $BV(A)$ is a Banach space, endowed with the following norm:

$$|u|_{BV(A)} := |u|_{L^1(A)} + |Du|(A), \text{ for any } u \in BV(A).$$

Also, $BV(A)$ is a metric space, endowed with the following distance:

$$[u, v] \in BV(A)^2 \mapsto |u - v|_{L^1(A)} + \left| \int_A |Du| - \int_A |Dv| \right|.$$

The topology provided by this distance is called the *strict topology* of $BV(A)$ and the convergence of sequence in the strict topology is often phrased as "strictly in $BV(A)$".

In particular, if $d > 1$, if the open set A is bounded, and if the boundary ∂A is Lipschitz, then the space $BV(A)$ is continuously embedded into $L^{d/(d-1)}(A)$ and compactly embedded into $L^q(A)$ for any $1 \leq q < d/(d-1)$ (cf. [1, Corollary 3.49] or [5, Theorem 10.1.3–10.1.4]). Besides, there exists a (unique) bounded linear operator $\mathscr{T}_{\partial A} : BV(A) \mapsto L^1(\partial A)$, called *trace*, such that $\mathscr{T}_{\partial A}\varphi = \varphi|_{\partial A}$ on ∂A for any $\varphi \in C^1(\overline{A})$. Hence, in this paper, we shortly denote the value of trace $\mathscr{T}_{\partial A}u \in L^1(\partial A)$ by $u_{|_{\partial A}}$. Additionally, if $1 \leq r < \infty$, then the space $C^\infty(\overline{A})$ is dense in $BV(A) \cap L^r(A)$ for the *intermediate convergence* (cf. [5, Definition 10.1.3. and Theorem 10.1.2]), i.e. for any $u \in BV(A) \cap L^r(A)$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty(\overline{A})$ such that $u_n \to u$ in $L^r(A)$ and $\int_A |\nabla u_n| dx \to |Du|(A)$ as $n \to \infty$.

Remark 1 (cf. [1, Theorem 3.88]) Let $1 < d \in \mathbb{N}$, and let $A \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary ∂A. Then, it holds that:

$$\int_{\partial A} u_{|_{\partial A}} (\psi \cdot n_{\partial A}) \, d\mathscr{H}^{d-1} = \int_A u \operatorname{div} \psi \, dx + \int_A \psi \cdot Du, \text{ for any } \psi \in C_c^1(\mathbb{R}^d)^d,$$

where $n_{\partial A}$ denotes the unit outer normal on ∂A. Moreover, the trace $\mathscr{T}_{\partial A} : BV(A) \to L^1(\partial A)$ is continuous with respect to the strict topology of $BV(A)$. Namely, the convergence of continuous dependence holds:

$$\mathscr{T}_{\partial A}u_n \to \mathscr{T}_{\partial A}u \text{ as } n \to \infty, \text{ for } u \in BV(A) \text{ and } \{u_n\}_{n=1}^\infty \subset BV(A), \tag{14}$$

in the topology of $L^1(\partial A)$, if $u_n \to u$ strictly in $BV(A)$. However, in contrast with the traces on Sobolev spaces, it must be noted that the convergence (14) is not guaranteed, if $u_n \to u$ weakly-$*$ in $BV(A)$, and even if we adopt any weak topology for (14) (including the distributional one).

Notation 6 (Extensions of Functions: cf. [1, 5]) Let $d \in \mathbb{N}$, and let $B \subset \mathbb{R}^d$ be a Borel set. For any $\mu \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^d)$ and any μ-measurable function $u : B \to \mathbb{R}$, we denote by $[u]^{\mathrm{ex}}$ an extension of u over $\mathbb{R}^N$. More precisely, $[u]^{\mathrm{ex}} : \mathbb{R}^N \to \mathbb{R}$ is a Lebesgue measurable function such that $[u]^{\mathrm{ex}}$ has an expression as a μ-measurable function on B, and $[u]^{\mathrm{ex}} = u$ μ-a.e. in B. In general, the extension of $[u]^{\mathrm{ex}} : \mathbb{R}^N \to \mathbb{R}$ is not unique, for each $u : B \to \mathbb{R}$.

Remark 2 Let $1 < d \in \mathbb{N}$, and let $A \subset \mathbb{R}^d$ be a bounded open set with a C^1-boundary ∂A. Then, for the extensions of functions in $BV(A)$ and $H^{\frac{1}{2}}(\partial A)$, we can check the following facts.

(Fact 1) (cf. [1, Proposition 3.21]) There exists a bounded linear operator $\mathscr{E}_A : BV(A) \to BV(\mathbb{R}^N)$, such that:

- $\mathscr{E}_A$ maps any function $u \in BV(A)$ to an extension $[u]^{\mathrm{ex}} \in BV(\mathbb{R}^N)$;
- for any $1 \leq q < \infty$, $\mathscr{E}_A(W^{1,q}(A)) \subset W^{1,q}(\mathbb{R}^N)$, and the restriction $\mathscr{E}_A|_{W^{1,q}(A)} : W^{1,q}(A) \to W^{1,q}(\mathbb{R}^N)$ forms a bounded and linear operator with respect to the (strong-)topologies of the restricted Sobolev spaces.

(Fact 2) (cf. [5, Theorem 5.4.1 and Proposition 5.6.3]) There exists a bounded linear operator $\mathscr{E}_{\partial A} : H^{\frac{1}{2}}(\partial A) \to H^1(\mathbb{R}^N)$, which maps any function $\varrho \in H^{\frac{1}{2}}(\partial A)$ to an extension $[\varrho]^{\mathrm{ex}} \in H^1(\mathbb{R}^N)$.

Next, we prepare the notations for the spatial domain Ω and functions and measures on this domain.

Notation 7 (Notations for the Spatial Domain) Throughout this paper, let $1 < N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^∞-boundary $\Gamma := \partial\Omega$ and the unit outer normal $n_\Gamma \in C^\infty(\Gamma)^N$. Besides, we suppose that Ω and Γ fulfill the following two conditions.

($\boldsymbol{\Omega}$0) There exists a small constant $r_\Gamma > 0$, and the mapping

$$d_\Gamma : x \in \overline{\Omega} \mapsto \inf_{y \in \Gamma} |x - y| \in [0, \infty),$$

forms a smooth function on the neighborhoods of Γ:

$$\Gamma(r) := \left\{ x \in \Omega \middle| d_\Gamma(x) < r \right\}, \text{ for every } r \in (0, r_\Gamma].$$

($\boldsymbol{\Omega}$1) There exists a small constant $r_* \in (0, r_\Gamma]$, and for any $x_\Gamma \in \Gamma$ and arbitrary $\rho, r \in (0, r_*]$, the neighborhood:

$$G_{x_\Gamma}(\rho, r) := \left\{ y + x_\Gamma + \tau n_\Gamma \,\middle|\, \begin{array}{l} \tau \in (-r, r), y \in \Gamma - x_\Gamma, \text{ and} \\ \left| y - \big(y \cdot n_\Gamma(x_\Gamma)\big) n_\Gamma(x_\Gamma) \right| < \rho \end{array} \right\},$$

is transformed to a cylinder:

$$\Pi_0(\rho, r) := \left\{ \xi = [\tilde{\xi}, \xi_N] \in \mathbb{R}^N \middle| \tilde{\xi} \in \rho\mathbb{B}^{N-1} \text{ and } \xi_N \in (-r, r) \right\},$$

by using a uniform C^∞-diffeomorphism $\varXi_{x_\Gamma} : G_{x_\Gamma}(r_*, r_*) \to \varPi_0(r_*, r_*)$. Additionally, for any $x_\Gamma \in \Gamma$, there exists a function $\gamma_{x_\Gamma} \in C^\infty(r_*\overline{\mathbb{B}^{N-1}})$, a congruence transform $\varLambda_{x_\Gamma} : \mathbb{R}^N \to \mathbb{R}^N$ and a C^∞-diffeomorphism $H_{x_\Gamma} : \varLambda_{x_\Gamma} G_{x_\Gamma}(r_*, r_*) \to \varPi_0(r_*, r_*)$ such that:

($\boldsymbol{\omega}0$) $\varXi_{x_\Gamma} = H_{x_\Gamma} \circ \varLambda_{x_\Gamma}$ as a mapping from $G_{x_\Gamma}(r_*, r_*)$ onto $\varPi_0(r_*, r_*)$;
($\boldsymbol{\omega}1$) $\gamma_{x_\Gamma}(0) = 0$, and $\nabla \gamma_{x_\Gamma}(0) = 0$ in $\mathbb{R}^{N-1}$;
($\boldsymbol{\omega}2$) for every $\rho, r \in (0, r_*]$,

$$\varLambda_{x_\Gamma} G_{x_\Gamma}(\rho, r) = Y_{x_\Gamma}(\rho, r) := \left\{ y = [\tilde{y}, y_N] \in \mathbb{R}^N \middle| \; [\tilde{y}, y_N - \gamma_{x_\Gamma}(\tilde{y})] \in \varPi_0(\rho, r) \right\},$$

and in particular,

$$\varLambda_{x_\Gamma}\big(\Gamma \cap G_{x_\Gamma}(\rho, r)\big) = \left\{ y = [\tilde{y}, \gamma_{x_\Gamma}(\tilde{y})] \in \mathbb{R}^N \middle| \; \tilde{y} \in \rho\mathbb{B}^{N-1} \right\};$$

($\boldsymbol{\omega}3$) for every $\rho, r \in (0, r_*]$,

$$H_{x_\Gamma} : y = [\tilde{y}, y_N] \in Y_{x_\Gamma}(\rho, r) \mapsto \xi = H_{x_\Gamma} y := [\tilde{y}, y_N - \gamma_{x_\Gamma}(\tilde{y})] \in \varPi_0(\rho, r).$$

Remark 3 From ($\varOmega$0), we may further suppose the following condition.

($\boldsymbol{\varOmega}2$) For any $\sigma > 0$, there exists a constant $\rho_*^\sigma \in (0, r_*]$ such that:

$$\begin{gathered} \rho_*^\sigma \le \sigma, \; |\gamma_{x_\Gamma}|_{C^1(\rho\overline{\mathbb{B}^{N-1}})} \le \sigma \text{ and} \\ \left\{ \varXi_{x_\Gamma}^{-1}[\tilde{\xi}, \gamma_{x_\Gamma}(\tilde{\xi}) + r_*] \middle| \; \tilde{\xi} \in \rho\overline{\mathbb{B}^{N-1}} \right\} \cap \overline{\Gamma(r_*/2)} = \emptyset, \\ \text{for any } x_\Gamma \in \Gamma \text{ and any } \rho \in (0, \rho_*^\sigma]. \end{gathered}$$

Notation 8 (Notations of Surface-Differentials) Under the assumption ($\varOmega$0) in Notation 7, we can define the Laplacian Δ_Γ on the surface Γ, i.e. the so-called Laplace–Beltrami operator, as the composition $\Delta_\Gamma := \mathrm{div}_\Gamma \circ \nabla_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ of the *surface gradient:*

$$\nabla_\Gamma \varphi := \nabla[\varphi]^{\mathrm{ex}} - (\nabla d_\Gamma \otimes \nabla d_\Gamma)\nabla[\varphi]^{\mathrm{ex}}, \text{ for any } \varphi \in C^\infty(\Gamma),$$

and the *surface-divergence:*

$$\mathrm{div}_\Gamma \omega := \mathrm{div}[\omega]^{\mathrm{ex}} - \nabla([\omega]^{\mathrm{ex}} \cdot \nabla d_\Gamma) \cdot \nabla d_\Gamma, \text{ for any} \omega = [\omega_1, \dots, \omega_N] \in C^\infty(\Gamma)^N.$$

As is well-known (cf. [25]), the values $\nabla_\Gamma \varphi$ and $\mathrm{div}_\Gamma \omega$ are determined independently with respect to the choices of the extensions $[\varphi]^{\mathrm{ex}} \in C^\infty(\mathbb{R}^N)$ and $[\omega]^{\mathrm{ex}} = \big[[\omega_1]^{\mathrm{ex}}, \dots, [\omega_N]^{\mathrm{ex}}\big] \in C^\infty(\mathbb{R})^N$, and moreover, the operator $-\Delta_\Gamma$ can be extended to a duality map between $H^1(\Gamma)$ and $H^{-1}(\Gamma)$, via the following variational identity:

$$ {}_{H^{-1}(\Gamma)}\langle -\Delta_\Gamma \varphi, \psi \rangle_{H^1(\Gamma)} = (\nabla_\Gamma \varphi, \nabla_\Gamma \psi)_{L^2(\Gamma)^N}, \text{ for all } [\varphi, \psi] \in H^1(\Gamma)^2. $$

Finally, we prescribe some specific notations.

Notation 9 Let $R_\Omega > 0$ be a sufficiently large constant, such that $\mathbb{B}_\Omega := R_\Omega \mathbb{B}^N \supset \overline{\Omega}$. Besides, for any $u \in BV(\Omega)$ and any $g \in H^{\frac{1}{2}}(\Gamma)$, we denote by $[u]_g^{\mathrm{ex}} \in BV(\mathbb{B}_\Omega) \cap H^1(\mathbb{B}_\Omega \setminus \overline{\Omega})$ an extension of u, provided as:

$$ x \in \mathbb{R}^N \mapsto [u]_g^{\mathrm{ex}}(x) := \begin{cases} u(x), \text{ if } x \in \Omega, \\ [g]^{\mathrm{ex}}(x), \text{ if } x \in \mathbb{B}_\Omega \setminus \overline{\Omega}, \end{cases} \tag{15} $$

with the use of an extension $[g]^{\mathrm{ex}} \in H^1(\mathbb{R}^N)$ of g.

Remark 4 As consequences of BV-theory (cf. [1, Corollary 3.89], [5, Example 10.2.1] and [12, Theorem 5.8]) and Remark 2, we can verify the following facts.

(Fact 3) For any $u \in BV(\Omega)$ and any $g \in H^{\frac{1}{2}}(\Gamma)$, it holds that:

$$ |D[u]_g|(B) = \int_{B \cap \Omega} |Du| + \int_{B \cap \Gamma} |u_{|_\Gamma} - g| \, d\Gamma + \int_{B \setminus \overline{\Omega}} |\nabla [g]^{\mathrm{ex}}| \, dx, $$
for any Borel set $B \subset \mathbb{B}_\Omega$, and any extension $[g]^{\mathrm{ex}} \in H^1(\mathbb{R}^N)$ of g.

(Fact 4) For any $g \in H^{\frac{1}{2}}(\Gamma)$, the functional:

$$ u \in L^1(\Omega) \mapsto \left| D[u]_g^{\mathrm{ex}} \right| (\overline{\Omega}) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_\Gamma |u_{|_\Gamma} - g| \, d\Gamma = |D[u]_g^{\mathrm{ex}}|(\mathbb{B}_\Omega) - |D[g]^{\mathrm{ex}}|(\mathbb{B}_\Omega \setminus \overline{\Omega}), \\ \qquad \text{if } u \in BV(\Omega), \\ \infty, \quad \text{otherwise,} \end{cases} $$

forms a single-valued proper l.s.c. and convex function on $L^1(\Omega)$.

(Fact 5) (cf. [2, 3, 28]) $|D[u_n]_g^{\mathrm{ex}}|(\overline{\Omega}) \to |D[u]_g^{\mathrm{ex}}|(\overline{\Omega})$ as $n \to \infty$, whenever $\{u_n\}_{n=1}^\infty \subset BV(\Omega) \cap L^2(\Omega)$, $u \in BV(\Omega) \cap L^2(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ and strictly in $BV(\Omega)$ as $n \to \infty$.

Remark 5 From the definition (6), we easily see that Φ_* is proper and convex. Also, the above Remark 4 (Fact 4)–(Fact 5) lead to the lower semi-continuity of this Φ_*. In fact, taking arbitrary $W = [w, w_\Gamma] \in \mathscr{H}$ and $\{W_n = [w_n, w_{\Gamma,n}]\}_{n=1}^\infty \subset \mathscr{W}$, such that:

$$ W_n = [w_n, w_{\Gamma,n}] \to W = [w, w_\Gamma] \text{ in } \mathscr{H}, \text{ as } n \to \infty, $$

we immediately see from Remark 4 (Fact 5) that:

$$\begin{aligned}\varliminf_{n\to\infty} \Phi_*(W_n) &\geq \varliminf_{n\to\infty} |D[w_n]^{\mathrm{ex}}_{w_\Gamma}|(\overline{\Omega}) - \lim_{n\to\infty} \int_\Gamma |w_{\Gamma|_\Gamma} - w_\Gamma| \, d\Gamma \\ &\quad + \frac{\varepsilon^2}{2} \varliminf_{n\to\infty} \int_\Gamma |\nabla_\Gamma w_{\Gamma,n}|^2 \, d\Gamma \\ &\geq |D[u]^{\mathrm{ex}}_{w_\Gamma}|(\overline{\Omega}) + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma w_\Gamma|^2 \, d\Gamma = \Phi_*(W).\end{aligned}$$

3 The Results of This Paper

First, we prescribe, anew, the product Hilbert space $\mathscr{H} := L^2(\Omega) \times L^2(\Gamma)$, with the inner product:

$$\big([z^1, z^1_\Gamma], [z^2, z^2_\Gamma]\big)_{\mathscr{H}} := (z^1, z^2)_{L^2(\Omega)} + (z^1_\Gamma, z^2_\Gamma)_{L^2(\Gamma)}, \\ \text{for all } [z^k, z^k_\Gamma], k = 1, 2.$$

As is mentioned in Introduction, the Hilbert space $\mathscr{H}$ is to be the base-space of the convex functions as in (6) and (13), and the Cauchy problems (5) and (12). Also, let $\mathscr{W} := (BV(\Omega) \cap L^2(\Omega)) \times H^1(\Gamma)$ be the effective domain of the convex function Φ_*, given in (6), and let $\mathscr{V}$ be a closed linear subspace in the product Hilbert space $H^1(\Omega) \times H^1(\Gamma)$, defined as:

$$\mathscr{V} := \left\{ [v, v_\Gamma] \in \mathscr{H} \;\middle|\; \begin{array}{l} v \in H^1(\Omega), v_\Gamma \in H^1(\Gamma) \\ \text{and } v_{|_\Gamma} = v_\Gamma \text{ a.e. on } \Gamma \end{array} \right\}.$$

Next, we prescribe the assumptions in our study.

(A0) $\varepsilon > 0$ is a fixed constant, and $\delta > 0$ and $\kappa > 0$ are given constants. Besides, $1 < N \in \mathbb{N}$ is a fixed constant, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ and the unit outer normal n_Γ, that fulfills the conditions (Ω0)–(Ω1) in Notation 7.

(A1) $\{f_\delta\}_{0<\delta\leq 1} \subset W^{1,\infty}(\mathbb{R}^N)$ is a sequence of convex functions such that

$$f_\delta(0) = 0 \text{ and } f_\delta(\omega) \geq 0, \text{ for any } 0 < \delta \leq 1 \text{ and any } \omega \in \mathbb{R}^N, \\ \text{and } f_\delta \to |\cdot| (= |\cdot|_{\mathbb{R}^N}), \text{ uniformly on } \mathbb{R}^N, \text{ as } \delta \to 0.$$

Remark 6 The assumptions (A0)–(A1) cover the setting of $\{f_\delta\}_{\delta>0} = \{|\cdot|\}$, and this setting is just the case that was mainly dealt with in the previous work [11].

Now, the results of this paper are stated as follows.

Main Theorem A (Mosco-Convergence) *Under (A1)–(A0), let* $\Phi_* : \mathscr{H} \to [0,\infty]$ *be the functional given in* (6)*, and for every* $\delta > 0$ *and* $\kappa > 0$*, let* $\Phi_\delta^\kappa : \mathscr{H} \to [0,\infty]$ *be the proper l.s.c. and convex function given in* (13)*. Then, for every sequences* $\{\delta_n\}_{n=1}^\infty \subset (0,1]$ *and* $\{\kappa_n\}_{n=1}^\infty \subset (0,1]$*, such that:*

$$\delta_n \to 0 \text{ and } \kappa_n \to 0, \text{ as } n \to \infty, \tag{16}$$

the sequence $\{\Phi_n\}_{n=1}^\infty$ *of convex functions* $\Phi_n := \Phi_{\delta_n}^{\kappa_n} : \mathscr{H} \to [0,\infty]$, $n \in \mathbb{N}$, *converges to the convex function* Φ_* *on* $\mathscr{H}$*, in the sense of Mosco, as* $n \to \infty$.

Corollary 1 (Continuous Dependence of Cauchy Problems) *Let* $0 < T < \infty$*, and for every* $U_0 = [u_0, u_{\Gamma,0}] \in \mathscr{W}$ *and* $\Theta = [\theta, \theta_\Gamma] \in L^2(0,T;\mathscr{H})$*, let* $U = [u, u_\Gamma] \in L^2(0,T;\mathscr{H})$ *be the solution to* (5)*. Also, for every* $n \in \mathbb{N}$, $U_0^n := [u_0^n, u_{\Gamma,0}^n] \in \mathscr{V}$*, and* $\Theta^n := [\theta^n, \theta_\Gamma^n] \in L^2(0,T;\mathscr{H})$*, let* $U^n := [u^n, u_\Gamma^n] \in W^{1,2}(0,T;\mathscr{H}) \cap L^\infty(0,T;\mathscr{V})$ *be the solution to* (12) *in the case when* $\delta = \delta_n$ *and* $\kappa = \kappa_n$*, i.e.:*

$$\begin{cases} (U^n)'(t) + \partial\Phi_n(U^n(t)) \ni \Theta^n(t) \text{ in } \mathscr{H}, \text{ a.e. } t \in (0,T), \\ U^n(0) = U_0^n \text{ in } \mathscr{H}. \end{cases}$$

On this basis, let us assume that:

$$U_0^n \to U_0 \text{ in } \mathscr{H} \text{ and } \Theta^n \to \Theta \text{ in } L^2(0,T;\mathscr{H}), \text{ with } (16).$$

Then, the sequence $\{U^n = [u^n, u_\Gamma^n]\}_{n=1}^\infty$ *converges to* $U = [u, u_\Gamma]$ *in the sense that:*

$$\begin{aligned} U^n \to U \ &\text{ in } C([0,T];\mathscr{H}), \\ &\text{weakly in } W^{1,2}(0,T;\mathscr{H}), \text{ as } n \to \infty, \end{aligned}$$

and

$$\int_0^T \Phi_n(U^n(t))\,dt \to \int_0^T \Phi_*(U(t))\,dt, \text{ as } n \to \infty.$$

Main Theorem B (Comparison Principle) *For every* $k = 1, 2$*, let* $[u_0^k, u_{\Gamma,0}^k] \in \mathscr{W}$ *be given initial data, let* $[\theta^k, \theta_\Gamma^k] \in L^2(0,T;\mathscr{H})$ *be a given source term, and let* $[u^k, u_\Gamma^k] \in L^2(0,T;\mathscr{H})$ *be a weak solution to* $\{$(8)–(11)$\}$ *in the case when* $[u_0, u_{\Gamma,0}] = [u_0^k, u_{\Gamma,0}^k]$ *and* $[\theta, \theta_\Gamma] = [\theta^k, \theta_\Gamma^k]$*. Then, it holds that:*

$$\begin{aligned} &\left|[u^1 - u^2]^+(t)\right|_{L^2(\Omega)}^2 + \left|[u_\Gamma^1 - u_\Gamma^2]^+(t)\right|_{L^2(\Gamma)}^2 \\ \le &e^t \left(\left|[u_0^1 - u_0^2]^+\right|_{L^2(\Omega)}^2 + \left|[u_{\Gamma,0}^1 - u_{\Gamma,0}^2]^+\right|_{L^2(\Gamma)}^2\right) \end{aligned}$$

$$+\int_0^t e^{t-\tau}\Big(\big|[\theta^1-\theta^2]^+(\tau)\big|_{L^2(\Omega)}^2+\big|[\theta_\Gamma^1-\theta_\Gamma^2]^+(\tau)\big|_{L^2(\Gamma)}^2\Big)\,d\tau, \qquad (17)$$
$$\textit{for all } t\in[0,T].$$

Remark 7 In Main Theorem B, we can suppose the well-posedness for the weak formulation (7), because the Definition 1 lets the well-posedness be just a straightforward consequence of the general theory of nonlinear evolution equations [6, 7, 18]. Also, we note that the comparison principle (B), mentioned in Introduction, is immediately deduced from the inequality (17).

4 Key-Lemmas

In Main Theorem A, the keypoint is in the construction method of approximating sequences for BV-functions, which is stated in the following Key-Lemma A.

Key-Lemma A *For any* $[\hat{u},\hat{u}_\Gamma]\in\mathscr{W}$, *there exists a sequence* $\{\hat{u}_\ell\}_{\ell=1}^\infty\subset H^1(\Omega)$, *such that:*

$$\hat{u}_{\ell|_\Gamma}=\hat{u}_\Gamma \textit{ in } H^{\frac{1}{2}}(\Gamma), \textit{ for any } \ell\in\mathbb{N}, \qquad (18)$$

$$\hat{u}_\ell\to\hat{u} \textit{ in } L^2(\Omega) \textit{ and } \int_\Omega|\nabla\hat{u}_\ell|\,dx\to\int_\Omega|D\hat{u}|+\int_\Gamma|\hat{u}_{|_\Gamma}-\hat{u}_\Gamma|\,d\Gamma, \quad \textit{as } \ell\to\infty. \qquad (19)$$

Meanwhile, the keypoint of Main Theorem B is in the so-called *T-monotonicity* of the subdifferential $\partial\Phi_*$, which is stated in the following Key-Lemma B.

Key-Lemma B *Let* Φ_* *be the convex function given in* (6). *Then, the subdifferential* $\partial\Phi_*$ *fulfills the following inequality of T-monotonicity:*

$$\begin{aligned}\big(U^{*,1}-U^{*,2},[U^1-U^2]^+\big)_{\mathscr{H}}&=(u^{*,1}-u^{*,2},[u^1-u^2]^+)_{L^2(\Omega)}\\&\quad+(u_\Gamma^{*,1}-u_\Gamma^{*,2},[u_\Gamma^1-u_\Gamma^2]^+)_{L^2(\Gamma)}\geq 0,\\ \textit{for all } [U^k,U^{*,k}]=\big[[u^k,u_\Gamma^k],[u^{*,k},u_\Gamma^{*,k}]\big]&\in\partial\Phi_* \textit{ in } \mathscr{H}\times\mathscr{H}, k=1,2.\end{aligned} \qquad (20)$$

Now, before the proofs of these Key-Lemmas, we prepare some auxiliary lemmas and remarks.

Lemma 1 *Let $\mathbb{R}^N_+$ be the upper half-space of $\mathbb{R}^N$, i.e.:*

$$\mathbb{R}^N_+ := \left\{ [\tilde{\xi}, \xi_N] \in \mathbb{R}^N \middle| \tilde{\xi} \in \mathbb{R}^{N-1} \text{ and } \xi_N > 0 \right\}.$$

Then, for any $\varpi \in H^1(\mathbb{R}^{N-1}) \cap BV(\mathbb{R}^{N-1})$, there exists a sequence $\{[\![\varpi]\!]^{\rm ex}_r\}_{r>0} \subset H^1(\mathbb{R}^N_+) \cap BV(\mathbb{R}^N_+)$, and for any $\tau > 0$, there exists a small constant $r^\tau_\varpi \in (0, r_*]$, such that:

$$\begin{gathered} r^\tau_\varpi \leq \tau \text{ and } [\![\varpi]\!]^{\rm ex}_r(\tilde{\xi}, \xi_N) = 0, \text{ for any } r \in (0, r^\tau_\varpi] \\ \text{and a.e. } [\tilde{\xi}, \xi_N] \in \mathbb{R}^N_+, \text{ satisfying } \xi_N > r, \end{gathered} \tag{21}$$

$$[\![\varpi]\!]^{\rm ex}_r|_{\mathbb{R}^{N-1}} = \varpi \text{ in } H^{\frac{1}{2}}(\mathbb{R}^{N-1}) \text{ for any } r \in (0, r^\tau_\varpi], \tag{22}$$

and

$$\begin{gathered} |[\![\varpi]\!]^{\rm ex}_r|_{L^2(\mathbb{R}^N_+)} \leq \tau \text{ and } |D[\![\varpi]\!]^{\rm ex}_r|(\mathbb{R}^N_+) \leq |\varpi|_{L^1(\mathbb{R}^{N-1})} + \tau, \\ \text{for any } r \in (0, r^\tau_\varpi]. \end{gathered} \tag{23}$$

Proof Let us define:

$$\begin{gathered} [\![\varpi]\!]^{\rm ex}_r(\xi) = [\![\varpi]\!]^{\rm ex}_r(\tilde{\xi}, \xi_N) := [1 - r^{-1}\xi_N]^+ \varpi(\tilde{\xi}), \\ \text{for a.e. } \tilde{\xi} \in \mathbb{R}^{N-1}, \text{ a.e. } \xi_N \geq 0 \text{ and any } r > 0. \end{gathered} \tag{24}$$

Then, from the assumption $\varpi \in H^1(\mathbb{R}^{N-1}) \cap BV(\mathbb{R}^{N-1})$, we immediately check that $\{[\![\varpi]\!]^{\rm ex}_r\}_{r>0} \subset H^1(\mathbb{R}^N_+) \cap BV(\mathbb{R}^N_+)$.

On this basis, for any $\tau > 0$, let us take a small constant $r^\tau_\varpi \in (0, r_*]$, such that:

$$r^\tau_\varpi \in (0, \tau], \ \sqrt{\frac{r^\tau_\varpi}{3}} |\varpi|_{L^2(\mathbb{R}^{N-1})} < \tau \ \text{ and } \ \frac{r^\tau_\varpi}{2} \int_{\mathbb{R}^{N-1}} |\nabla \varpi| \, d\tilde{\xi} < \tau. \tag{25}$$

Then, we can see the conditions (21)–(22) by means of (24)–(25) and a standard argument of the trace. Additionally, with (24)–(25) in mind, we can verify the remaining (23) as follows.

$$\begin{aligned} |[\![\varpi]\!]^{\rm ex}_r|^2_{L^2(\mathbb{R}^N_+)} &= \int_{\mathbb{R}^N_+} \left| [1 - r^{-1}\xi_N]^+ \varpi(\tilde{\xi}) \right|^2 d\xi \\ &= \left(\int_0^r (1 - r^{-1}\xi_N)^2 \, d\xi_N \right) \left(\int_{\mathbb{R}^{N-1}} |\varpi(\tilde{\xi})|^2 \, d\tilde{\xi} \right) \\ &= \frac{r}{3} |\varpi|^2_{L^2(\mathbb{R}^{N-1})} \leq \tau^2, \text{ for any } r \in (0, r^\tau_\varpi], \end{aligned}$$

and

$$
\begin{aligned}
|D[\![\varpi]\!]_r^{\rm ex}|(\mathbb{R}_N^+) &= \int_{\mathbb{R}_+^N} |(\nabla[\![\varpi]\!]_r^{\rm ex})(\xi)|\, d\xi \\
&\leq \int_{\mathbb{R}_+^N} |(\tilde{\nabla}[\![\varpi]\!]_r^{\rm ex})(\xi)|\, d\xi + \int_{\mathbb{R}_+^N} |(\partial_N[\![\varpi]\!]_r^{\rm ex})(\xi)|\, d\xi \\
&= \int_{\mathbb{R}_+^N} \left|[1 - r^{-1}\xi_N]^+ \tilde{\nabla}\varpi(\tilde{\xi})\right| d\xi + \int_{\mathbb{R}_+^N} \left|-r^{-1}\chi_{(0,r)}(\xi_N)\varpi(\tilde{\xi})\right| d\xi \\
&= \frac{r}{2}\int_{\mathbb{R}^{N-1}} |\tilde{\nabla}\varpi|\, d\tilde{\xi} + |\varpi|_{L^1(\mathbb{R}^{N-1})} \\
&= |\varpi|_{L^1(\mathbb{R}^{N-1})} + \tau, \text{ for any } r \in (0, r_\varpi^\tau].
\end{aligned}
$$

□

Lemma 2 *For any $\hat{v}_\Gamma \in H^1(\Gamma)$ and any $\ell \in \mathbb{N}$, there exists a function $\hat{v}_\ell \in H^1(\Omega)$ such that*

$$\hat{v}_{\ell|_\Gamma} = \hat{v}_\Gamma \ \textit{ in } H^{\frac{1}{2}}(\Gamma), \textit{ for } \ell = 1, 2, 3, \dots, \tag{26}$$

$$\hat{v}_\ell(x) = 0, \ \textit{for a.e. } x \in \Omega \setminus \Gamma(2^{-\ell}) \textit{ and } \ell = 1, 2, 3, \dots, \tag{27}$$

and

$$|\hat{v}_\ell|_{L^2(\Omega)} \leq 2^{-\ell} \textit{ and } |D\hat{v}_\ell|(\Omega) \leq |\hat{v}_\Gamma|_{L^1(\Gamma)} + 2^{-\ell}, \textit{ for } \ell = 1, 2, 3, \dots. \tag{28}$$

Proof Let $\sigma > 0$ be arbitrary, and let ρ_*^σ be the constant as in (Ω2). Since $\Gamma \subset \mathbb{R}^{N-1}$ is compact, we can take a large number $m_\Omega^\sigma \in \mathbb{N}$ and a finite sequence $\{x_{\Gamma,1}^\sigma, \dots, x_{\Gamma,m_\Omega^\sigma}^\sigma\} \subset \Gamma$, such that:

$$
\begin{aligned}
&\overline{\Gamma(r_*/2)} \subset G_*^\sigma := \bigcup_{i=1}^{m_\Omega^\sigma} G_i^\sigma, \text{ with the neighborhoods} \\
&G_i^\sigma := G_{x_{\Gamma,i}^\sigma}(\rho_*^\sigma, r_*),\ i = 1, \dots, m_\Omega^\sigma, \text{ as in}(\Omega 1);
\end{aligned} \tag{29}
$$

and then, we can take the partition of unity $\{\eta_i^\sigma\}_{i=1}^{m_\Omega^\sigma} \subset C_{\rm c}^\infty(\mathbb{R}^N)$ for the covering G_*^σ, such that:

$$0 \leq \eta_i^\sigma \in C_{\rm c}^\infty(G_i^\sigma) \text{ for } i = 1, \dots, m_\Omega^\sigma, \text{ and } \sum_{i=1}^{m_\Omega^\sigma} \eta_i^\sigma = 1 \text{ on } \overline{\Gamma(r_*/2)}. \tag{30}$$

Next, let us take any $\tau > 0$, and with $(\Omega 1)$ and Lemma 1 in mind, let us set:

$$\Xi_i^\sigma := \Xi_{x_{\Gamma,i}^\sigma}, \text{ with } \Lambda_i^\sigma := \Lambda_{x_{\Gamma,i}^\sigma} \text{ and } H_i^\sigma := H_{x_{\Gamma,i}^\sigma}, \ i = 1, \dots, m_\Omega^\sigma, \tag{31}$$

$$\varpi_i^\sigma(\tilde{\xi}) := \begin{cases} (\eta_i^\sigma \hat{v}_\Gamma)\big((\Xi_i^\sigma)^{-1}\tilde{\xi}\big), \\ \quad \text{if } \tilde{\xi} \in \rho_*^\sigma \mathbb{B}^{N-1} \text{ and } i = 1, \dots, m_\Omega^\sigma, \text{ for a.e. } \tilde{\xi} \in \mathbb{R}^{N-1}, \\ 0, \ \text{otherwise}, \end{cases} \tag{32}$$

and

$$\hat{r}_\sigma^\tau := \min \left\{ r_{\varpi_i^\sigma}^\tau \,\middle|\, i = 1, \dots, m_\Omega^\sigma \right\}. \tag{33}$$

Based on these, we define a class of functions $\{\hat{v}_\sigma^\tau \mid \sigma, \tau > 0\}$, as follows:

$$\hat{v}_\sigma^\tau(x) := \begin{cases} \displaystyle\sum_{i=1}^{m_\Omega^\sigma} [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}\big(\Xi_i^\sigma x\big), \\ \quad \text{if } x \in G_i^\sigma, \text{ for some } i \in \{1, \dots, m_\Omega^\sigma\}, \\ 0, \ \text{otherwise}, \end{cases} \\ \text{for a.e. } x \in \Omega \text{ and all } \sigma, \tau > 0. \tag{34}$$

Then, as direct consequences of (29)–(34) and Lemma 1, it is inferred that:

$$\hat{v}_\sigma^\tau \in H^1(\Omega), \ \hat{v}_{\sigma|\Gamma}^\tau = \hat{v}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma), \\ \text{and } \hat{v}_\sigma^\tau = 0 \text{ a.e. on } \Omega \setminus \Gamma(\tau), \ \text{for all } \sigma, \tau > 0. \tag{35}$$

Also, in the light of (23), $(\Omega 2)$ and Lemma 1, we compute that:

$$|\hat{v}_\sigma^\tau|_{L^2(\Omega)} = \left[\int_\Omega \left| \sum_{i=1}^{m_\Omega^\sigma} [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_i^\sigma x) \right|^2 dx \right]^{\frac{1}{2}} \leq \sum_{i=1}^{m_\Omega^\sigma} \left[\int_{\mathbb{R}_+^N} \left| [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi) \right|^2 d\xi \right]^{\frac{1}{2}} \\ \leq m_\Omega^\sigma \tau, \ \text{for all } \sigma, \tau > 0, \tag{36}$$

and

$$\int_\Omega |\nabla_x \hat{v}_\sigma^\tau(x)| \, dx \leq \sum_{i=1}^{m_\Omega^\sigma} \int_{G_i^\sigma \cap \Omega} \left| \nabla_x [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_i^\sigma x) \right| dx \\ = \sum_{i=1}^{m_\Omega^\sigma} \int_{Y_i^\sigma \cap (\Lambda_i^\sigma \Omega)} \left| \nabla_y [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(H_i^\sigma y) \right| dy \\ \leq \sum_{i=1}^{m_\Omega^\sigma} (1 + |\nabla \gamma_{x_\Gamma}|_{C(\rho_*^\sigma \overline{\mathbb{B}^{N-1}})}) \int_{\mathbb{R}_+^N} \left| \nabla_\xi [\![\varpi_i^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi) \right| d\xi$$

$$\leq (1+\sigma)\sum_{i=1}^{m_\Omega^\sigma}\left(\int_{\mathbb{R}^{N-1}} |\varpi_i^\sigma(\tilde{\xi})|\,d\tilde{\xi} + \tau\right)$$

$$\leq (1+\sigma)\sum_{i=1}^{m_\Omega^\sigma}\left(\int_{G_i^\sigma \cap \Gamma} \eta_i^\sigma |\hat{v}_\Gamma|\,d\Gamma + \tau\right)$$

$$\leq (1+\sigma)|\hat{v}_\Gamma|_{L^1(\Gamma)} + m_\Omega^\sigma \tau(1+\sigma), \text{ for all } \sigma, \tau > 0. \tag{37}$$

Now, for any $\ell \in \mathbb{N}$, let us take two constants $\sigma_\ell, \tau_\ell \in (0, 1]$, such that:

$$\begin{cases} (1+\sigma_\ell)|\hat{v}_\Gamma|_{L^1(\Gamma)} \leq |\hat{v}_\Gamma|_{L^1(\Gamma)} + 2^{-\ell-1}, \\ \tau_\ell + m_\Omega^{\sigma_\ell}\tau_\ell(1+\sigma_\ell) \leq 2^{-\ell-1}, \end{cases} \text{for } \ell = 1, 2, 3, \dots. \tag{38}$$

Then, on account of (35)–(38), we will conclude that the function $\hat{v}_\ell := \hat{v}_{\sigma_\ell}^{\tau_\ell} \in H^1(\Omega)$, for each $\ell \in \mathbb{N}$, will fulfill the required condition (26)–(28). □

Proof of Key-Lemma A The proof is a modified version of [22, Theorem 6]. Let $u \in BV(\Omega) \cap L^2(\Omega)$ be arbitrary. Then, by the smoothness of Γ as in (Ω1)–(Ω2), we can apply the standard regularization method of BV-functions (cf. [5, Theorem 10.1.2]), and can find a sequence $\{\hat{\varphi}_\ell\}_{\ell=1}^\infty \subset C^\infty(\overline{\Omega})$, such that:

$$\hat{\varphi}_\ell \to \hat{u} \text{ in } L^2(\Omega) \text{ and strictly in } BV(\Omega), \text{ as } \ell \to \infty. \tag{39}$$

Besides, from Remark 1, it follows that:

$$\hat{\varphi}_{\ell|_\Gamma} \to \hat{u}_{|_\Gamma} \text{ in } L^1(\Gamma), \text{ as } \ell \to \infty. \tag{40}$$

Next, for any $\ell \in \mathbb{N}$, we apply Lemma 2 as the case when $\hat{v}_\Gamma := \hat{u}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}$ in $H^{\frac{1}{2}}(\Gamma)$, and then, we can take a function $\hat{\psi}_\ell \in H^1(\Omega)$, such that:

$$\begin{cases} \hat{\psi}_{\ell|_\Gamma} = \hat{u}_\Gamma - \hat{\varphi}_{\ell|_\Gamma} \text{ in } H^{\frac{1}{2}}(\Gamma), \\ |\hat{\psi}_\ell|_{L^2(\Omega)} \leq 2^{-\ell} \text{ and } \displaystyle\int_\Omega |\nabla \hat{\psi}_\ell|\,dx \leq \int_\Gamma \left|\hat{u}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}\right|\,d\Gamma + 2^{-\ell}. \end{cases} \tag{41}$$

Based on these, let us define:

$$\hat{u}_\ell := \hat{\varphi}_\ell + \hat{\psi}_\ell \text{ in } L^2(\Omega), \text{ for } \ell = 1, 2, 3, \dots. \tag{42}$$

Then, in the light of (39)–(41), it is computed that:

$$\hat{u}_{\ell|_\Gamma} = \hat{\varphi}_{\ell|_\Gamma} + \hat{\psi}_{|_\Gamma} = \hat{\varphi}_{\ell|_\Gamma} + (\hat{u}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}) = \hat{u}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma), \text{ for } \ell = 1, 2, 3, \dots, \tag{43}$$

$$\begin{aligned}|\hat{u}_\ell - \hat{u}|_{L^2(\Omega)} &= |(\hat{\varphi}_\ell - \hat{u}) + \hat{\psi}_\ell|_{L^2(\Omega)} \\ &\le |\hat{\varphi}_\ell - \hat{u}|_{L^2(\Omega)} + 2^{-\ell} \to 0 \text{ as } \ell \to \infty,\end{aligned} \tag{44}$$

and

$$\begin{aligned}\overline{\lim_{\ell\to\infty}} \int_\Omega |\nabla \hat{u}_\ell|\,dx &\le \lim_{\ell\to\infty} \int_\Omega |\nabla \hat{\varphi}_\ell|\,dx + \overline{\lim_{\ell\to\infty}} \int_\Omega |\nabla \hat{\psi}_\ell|\,dx \\ &\le \int_\Omega |D\hat{u}| + \lim_{\ell\to\infty} \left(\int_\Gamma |\hat{u}_\Gamma - \hat{\phi}_{\ell|_\Gamma}|\,d\Gamma + 2^{-\ell} \right) \\ &= \int_\Omega |D\hat{u}| + \int_\Gamma |\hat{u}_{|_\Gamma} - \hat{u}_\Gamma|\,d\Gamma.\end{aligned} \tag{45}$$

Additionally, having in mind Remark 4 (Fact 4) and (43)–(44), one can also see that:

$$\begin{aligned}\underline{\lim_{\ell\to\infty}} \int_\Omega |\nabla \hat{u}_\ell|\,dx &= \underline{\lim_{\ell\to\infty}} \left(\int_\Omega |\nabla \hat{u}_\ell|\,dx + \int_\Gamma |\hat{u}_{\ell|_\Gamma} - \hat{u}_\Gamma|\,d\Gamma \right) \\ &\ge \int_\Omega |D\hat{u}| + \int_\Gamma |\hat{u}_{|_\Gamma} - \hat{u}_\Gamma|\,d\Gamma.\end{aligned} \tag{46}$$

On account of (43)–(46), we conclude that the sequence $\{\hat{u}_\ell\}_{\ell=1}^\infty \subset H^1(\Omega)$, given by (42), is the required sequence, fulfilling (18)–(19). □

Proof of Key-Lemma B Let us set:

$$\mathscr{K}_0 := \left\{ W = [w, w_\Gamma] \in \mathscr{H} \,\middle|\, w \le 0, \text{ a.e. in } \Omega \text{ and } w_\Gamma \le 0, \text{ a.e. on } \Gamma \right\}.$$

Then, by using the orthogonal projection $\pi_{\mathscr{K}_0} : \mathscr{H} \to \mathscr{K}_0$, we can reformulate the conclusion (20) to the following equivalent form:

$$\begin{aligned}&\left(U^{*,1} - U^{*,2}, (U^1 - U^2) - \pi_{\mathscr{K}_0}(U^1 - U^2)\right)_{\mathscr{H}} \ge 0, \\ &\text{for all } [U^k, U^{*,k}] \in \partial\Phi_* \text{ in } \mathscr{H} \times \mathscr{H}, k = 1, 2.\end{aligned} \tag{47}$$

Here, according to the general theory of T-monotonicity [19], the above (47) is equivalent to:

$$\begin{aligned}&\Phi_*(W^1 - \pi_{\mathscr{K}_0}(W^1 - W^2)) + \Phi_*(W^2 + \pi_{\mathscr{K}_0}(W^1 - W^2)) \\ &\le \Phi_*(W^1) + \Phi_*(W^2), \text{ for all } W^k \in \mathscr{W}, k = 1, 2.\end{aligned}$$

Additionally, from the definition of $\mathscr{K}_0$, one can easily check that:

$$\begin{cases} W^1 - \pi_{\mathscr{K}_0}(W^1 - W^2) = W^1 \vee W^2, \\ W^2 + \pi_{\mathscr{K}_0}(W^1 - W^2) = W^1 \wedge W^2, \end{cases} \text{for all } W^k \in \mathscr{W}, k = 1, 2.$$

Based on these, our goal can be reduced to the verification of:

$$\Phi_*(W^1 \vee W^2) + \Phi_*(W^1 \wedge W^2) \leq \Phi_*(W^1) + \Phi_*(W^2),\\ \text{for all } W^k \in D(\Phi_*), k = 1, 2. \tag{48}$$

Now, to verify (48), we apply Key-Lemma A, and we can prepare two sequences $\{V_\ell^k = [v_\ell^k, v_{\Gamma,\ell}^k]\}_{\ell=1}^\infty \subset \mathscr{V}$, $k = 1, 2$, such that:

$$v_{\ell}^k{}_{|\Gamma} = v_{\Gamma,\ell}^k = w_\Gamma^k \text{ in } H^{\frac{1}{2}}(\Gamma), \text{ for every } \ell \in \mathbb{N} \text{ and } k = 1, 2, \tag{49}$$

$$v_\ell^k \to w^k \text{ in } L^2(\Omega) \text{ and } \int_\Omega |\nabla v_\ell^k|\, dx \to \int_\Omega |Dw^k| + \int_\Gamma |w^k{}_{|\Gamma} - w_\Gamma^k|\, d\Gamma,\\ \text{as } \ell \to \infty, \text{ for every } k = 1, 2. \tag{50}$$

Subsequently, we compute that:

$$\begin{aligned}
&\Phi_*(V_\ell^1 \vee V_\ell^2) + \Phi_*(V_\ell^1 \wedge V_\ell^2)\\
&= \int_\Omega |\nabla v_\ell^1|\, dx + \int_\Omega |\nabla v_\ell^2|\, dx + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma w_\Gamma^1|^2\, d\Gamma + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma w_\Gamma^2|^2\, d\Gamma,\\
&\qquad \text{for any } \ell \in \mathbb{N}.
\end{aligned}$$

Now, taking into account (49)–(50) and the convergences:

$$V_\ell^1 \vee V_\ell^2 \to W^1 \vee W^2 \text{ and } V_\ell^1 \wedge V_\ell^2 \to W^1 \wedge W^2 \text{ in } \mathscr{H} \text{ as } \ell \to \infty,$$

the inequality (48) is deduced as follows:

$$\begin{aligned}
&\Phi_*(W^1 \vee W^2) + \Phi_*(W^1 \wedge W^2)\\
&\leq \varliminf_{\ell \to \infty} \left(\int_\Omega |\nabla v_\ell^1|\, dx + \int_\Omega |\nabla v_\ell^2|\, dx \right) + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma w_\Gamma^1|^2\, d\Gamma + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma w_\Gamma^2|^2\, d\Gamma\\
&= \Phi_*(W^1) + \Phi_*(W^2).
\end{aligned}$$

□

5 Proofs of the Results

In this section, we prove the results by means of the lemmas and remarks prepared in previous sections.

Proof of Main Theorem A We begin with the verification of the part of lower-bound condition of Mosco-convergence.

Let us take any $\check{W} = [\check{w}, \check{w}_\Gamma] \in \mathscr{H}$ and any sequence $\{\check{W}_n = [\check{w}_n, \check{w}_{\Gamma,n}]\}_{n=1}^\infty \subset \mathscr{V}$ such that

$$\check{W}_n = [\check{w}_n, \check{w}_{\Gamma,n}] \to \check{W} = [\check{w}, \check{w}_\Gamma] \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty.$$

Then, for the verification of the inequality of lower-bound condition:

$$\varliminf_{n\to\infty} \Phi_n(\check{W}_n) \geq \Phi_*(\check{W}), \tag{51}$$

the situation can be restricted to the case that:

$$\begin{aligned} &\lim_{\ell\to\infty} \Phi_{n_\ell}(\check{V}_\ell) = \varliminf_{n\to\infty} \Phi_n(\check{W}_n) < \infty, \text{ for some subsequences } \{n_\ell\}_{\ell=1}^\infty \subset \{n\}, \\ &\qquad \text{and } \{\check{V}_\ell = [\check{v}_\ell, \check{v}_{\Gamma,\ell}]\}_{\ell=1}^\infty := \{\check{W}_{n_\ell} = [\check{w}_{n_\ell}, \check{w}_{\Gamma,n_\ell}]\}_{\ell=1}^\infty \subset \{\check{W}_n\}, \end{aligned} \tag{52}$$

because the other ones can be said as trivial. Also, from (A1), we can see that:

$$\begin{aligned} \Phi_*(\check{V}_\ell) &\leq \int_\Omega |\nabla \check{v}_\ell|\, dx + \frac{\kappa_{n_\ell}^2}{2} \int_\Omega |\nabla \check{v}_\ell|^2\, dx + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma \check{v}_{\Gamma,\ell}|^2\, d\Gamma \\ &\leq \Phi_{n_\ell}(\check{V}_\ell) + \mathscr{L}^N(\Omega) \sup_{\omega\in\mathbb{R}^N} \left| f_{\delta_{n_\ell}}(\omega) - |\omega| \right|, \text{ for } \ell = 1, 2, 3, \ldots. \end{aligned} \tag{53}$$

The conditions (52)–(53) imply the boundedness of the sequence $\{\check{V}_\ell\}_{\ell=1}^\infty$ $(\subset \mathscr{V})$ in $\mathscr{W}$, and in addition, the assumption (A1) and the lower semi-continuity of Φ_* leads to the inequality (51) of lower-bound condition, via the following calculation:

$$\begin{aligned} \varliminf_{n\to\infty} \Phi_n(\check{W}_n) &= \lim_{\ell\to\infty} \Phi_{n_\ell}(\check{V}_\ell) \\ &\geq \varliminf_{\ell\to\infty} \Phi_*(\check{V}_\ell) - \mathscr{L}^N(\Omega) \lim_{\ell\to\infty} \sup_{\omega\in\mathbb{R}^N} \left| f_{\delta_{n_\ell}}(\omega) - |\omega| \right| \geq \Phi_*(\check{W}). \end{aligned}$$

Next, we show the part of optimality condition. This part can be obtained by applying (A1), Key-Lemma A and the diagonal argument.

Let us fix any function $\hat{W} = [\hat{w}, \hat{w}_\Gamma] \in \mathscr{W}$. Then, Key-Lemma A enables us to take a sequence $\{\hat{V}_\ell = [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}]\}_{\ell=1}^\infty \subset \mathscr{V}$, such that:

$$\hat{v}_{\ell|_\Gamma} = \hat{v}_{\Gamma,\ell} = \hat{w}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma), \text{ for } \ell = 1, 2, 3, \ldots, \tag{54}$$

$$\begin{cases} \hat{V}_\ell = [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}] \to \hat{W} = [\hat{w}, \hat{w}_\Gamma] \text{ in } \mathscr{H}, \\ \displaystyle\int_\Omega |\nabla \hat{v}_\ell|\, dx \to \int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|_\Gamma} - \hat{w}_\Gamma|\, d\Gamma, \end{cases} \text{ as } \ell \to \infty. \tag{55}$$

Here, for any $\ell \in \mathbb{N}$, let us take a large number $\hat{n}_\ell \in \mathbb{N}$ such that:

$$\frac{\kappa_n^2}{2} \int_\Omega |\nabla \hat{v}_\ell|^2 \, dx \leq 2^{-\ell}, \text{ for any } n \geq \hat{n}_\ell. \tag{56}$$

Besides, we define a sequence $\{\hat{W}_n = [\hat{w}_n, \hat{w}_{\Gamma,n}]\}_{n=1}^\infty \subset \mathscr{V}$, by letting

$$\hat{W}_n = [\hat{w}_n, \hat{w}_{\Gamma,n}] := \begin{cases} \hat{V}_\ell = [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}] \text{ in } \mathscr{V}, \\ \quad \text{if } \hat{n}_\ell \leq n < \hat{n}_{\ell+1}, \text{ for some } \ell \in \mathbb{N}, \\ \hat{V}_1 = [\hat{v}_1, \hat{v}_{\Gamma,1}] \text{ in } \mathscr{V}, \\ \quad \text{if } 1 \leq n < \hat{n}_1, \end{cases} \text{for } n = 1, 2, 3, \ldots. \tag{57}$$

Then, on account of the (54)–(57), it is inferred that

$$\begin{aligned} &\left| \Phi_n(\hat{W}_n) - \Phi_*(\hat{W}) \right| \\ &\leq \left| \int_\Omega \left(f_{\delta_n}(\nabla \hat{w}_n) + \frac{\kappa_n^2}{2} |\nabla \hat{w}_n|^2 \right) dx - \left(\int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|_\Gamma} - \hat{w}_\Gamma| \, d\Gamma \right) \right| \\ &\qquad + \frac{\varepsilon^2}{2} \left| \int_\Gamma \left(|\nabla_\Gamma \hat{w}_{\Gamma,n}|^2 - |\nabla_\Gamma \hat{w}_\Gamma|^2 \right) d\Gamma \right| \\ &\leq \left| \int_\Omega |\nabla \hat{w}_n| \, dx - \left(\int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|_\Gamma} - \hat{w}_\Gamma| \, d\Gamma \right) \right| \\ &\qquad + \mathscr{L}^N(\Omega) \sup_{\omega \in \mathbb{R}^N} \left| f_{\delta_n}(\omega) - |\omega| \right| + 2^{-\ell}, \\ &\qquad\qquad \text{for any } \hat{n}_\ell \leq n < \hat{n}_{\ell+1}, \ell = 1, 2, 3, \ldots, \end{aligned}$$

and it implies the convergence $\lim_{n \to \infty} \Phi_n(\hat{W}_n) = \Phi_*(\hat{W})$, required in optimality condition.

Thus, we conclude Main Theorem A. □

Remark 8 Let us simply denote by $\Phi_0 := \Phi_*|_{\mathscr{V}}$ the restriction of Φ_* onto $\mathscr{V}$, more precisely:

$$V = [v, v_\Gamma] \in \mathscr{V} \mapsto \Phi_0(V) = \Phi_0(v, v_\Gamma) := \int_\Omega |\nabla v| \, dx + \frac{\varepsilon^2}{2} \int_\Gamma |\nabla_\Gamma v_\Gamma|^2 \, d\Gamma.$$

Then, as a consequence of Main Theorem A, one can observe that Φ_* coincides with the lower semi-continuous envelope $\overline{\Phi_0}$ of the restriction Φ_0, i.e.:

$$\Phi_*(W) = \overline{\Phi_0}(W) := \inf \left\{ \varliminf_{n \to \infty} \Phi_0(V_n) \,\middle|\, \begin{array}{l} \{V_n\}_{n=1}^\infty \subset \mathscr{V} \text{ and} \\ V_n \to W \text{ in } \mathscr{H} \text{ as } n \to \infty \end{array} \right\}, \quad \text{for any } W \in \mathscr{H}. \tag{58}$$

In fact, from (58) of $\overline{\Phi_0}$, we see that the lower semi-continuous envelope $\overline{\Phi_0}$ is a maximal l.s.c. function supporting Φ_0 on $\mathscr{V}$. So, we immediately have:

$$\Phi_* \leq \overline{\Phi_0} \text{ on } \mathscr{H}, \text{ and } D(\overline{\Phi_0}) \subset D(\Phi_*) = \mathscr{W}. \tag{59}$$

Meanwhile, for any $\hat{W} = [\hat{w}, \hat{w}_\Gamma] \in D(\overline{\Phi_0})$, taking the sequence $\{\hat{V}_\ell = [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}]\}_{\ell=1}^\infty \subset \mathscr{V}$, as in (54)–(55), enables us to deduce that:

$$\overline{\Phi_0}(\hat{W}) \leq \varliminf_{\ell\to\infty} \Phi_0(\hat{V}_\ell) = \Phi_*(\hat{W}). \tag{60}$$

(59) and (60) imply the coincidence $\Phi_* = \overline{\Phi_0}$ on $\mathscr{H}$.

Proof of Corollary 1 This corollary will be obtained as straightforward consequences of Main Theorem A and the general theories of abstract evolution equations and their variational convergences, e.g. [4, 6, 7, 18], and so on. □

Proof of Main Theorem B By the assumption, we find two functions $U^{*,k} \in L^2(0, T; \mathscr{H})$, $k = 1, 2$, such that:

$$\begin{gathered} U^{*,k}(t) \in \partial\Phi_*(U^k(t)) \text{ and } (U^k)'(t) + U^{*,k}(t) = \Theta^k(t) \text{ in } \mathscr{H}, \\ \text{for a.e. } t \in (0, T), k = 1, 2. \end{gathered} \tag{61}$$

Here, taking the difference between the equations in (61) and multiplying the both sides by $[U^1 - U^2]^+(t)$, one can see that:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big|[U^1 - U^2]^+(t)\big|_{\mathscr{H}}^2 &+ \big((U^{*,1} - U^{*,2})(t), [U^1 - U^2]^+(t)\big)_{\mathscr{H}} \\ &= \big((\Theta^1 - \Theta^2)(t), [U^1 - U^2]^+(t)\big)_{\mathscr{H}}, \text{ a.e. } t \in (0, T). \end{aligned} \tag{62}$$

Also, from Key-Lemma B, it immediately follows that:

$$\big((U^{*,1} - U^{*,2})(t), [U^1 - U^2]^+(t)\big)_{\mathscr{H}} \geq 0. \tag{63}$$

Thus, Main Theorem B will be concluded by using the standard method, i.e. by applying (63), Young's inequality and Gronwall's lemma to (62). □

Remark 9 In the proofs of Main Theorems A and B, the essentials will be in the fixed-situations of boundary data for approximating functions, as in (18), (49) and (54). Then, the auxiliary Lemmas 1–2 are to support the presence of such approximations, and proofs of these can be said as some simplified version of the regularization method developed by Gagliardo [13]. But, the original method by Gagliardo [13] would be available just for the regularizations of BV-functions by $W^{1,1}$-functions, and it would not support the regularizations by other kinds of functions, so immediately. Hence, for the H^1-regularizations required in this study, the simplified construction (21)–(23) would be essential, and then, the H^1-regularity of the boundary data would be needed to be the assumptions, as in Key-Lemma A and Lemmas 1–2.

6 Future Prospective

One of the possible prospectives is to apply our theory to the phase-field system of grain boundary motion, known as "Kobayashi–Warren–Carter model", cf. [20, 21]. Indeed, the Kobayashi–Warren–Carter model is derived as a gradient system of a governing energy, including a generalized (unknown-dependent) total variation. In this light, the objective of this issue will be in the enhancement of the mathematical method for grain boundary phenomena, if we can combine our results and the line of relevant works to the Kobayashi–Warren–Carter model, e.g. [16, 17, 20, 21, 23, 26, 27].

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Smooth and Broken Minimizers of Some Free Discontinuity Problems

Danilo Percivale and Franco Tomarelli

Abstract We show that minimizers of free discontinuity problems with energy dependent on jump integrals and Dirichlet boundary conditions are smooth provided a smallness condition is imposed on data. We examine in detail two examples: the elastic-plastic beam and the elastic-plastic plate with free yield lines. In both examples there is a gap between the condition for solvability (safe load condition) and this smallness condition (load regularity condition) which imply regularity and uniqueness of minimizers. Such gap allows the existence of damaged/creased minimizers. Eventually we produce explicit examples of irregular solutions when the load is in the gap.

Keywords Bounded Hessian functions • Free discontinuity problem • Safe load condition • Regular minimizers • Broken minimizers • Plastic hinges in a beam

AMS (MOS) Subject Classification (2010) 49J10, 74K20, 74K30, 74R99, 74C99

1 Introduction

Free discontinuity problems related to image segmentation and inpainting achieve minimum regardless to the size of the data, due to the structural growth of the forcing term [13, 17, 20, 26, 27]. Free discontinuity problems in continuum mechanics have minimizers only if the loads are small, that is a suitable safe load condition is satisfied [11, 34, 39–41]. Strong solutions of free discontinuity problems without jump integrals over the singular set were proven to exist by showing partial

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regularity of weak solutions under higher integrability assumptions on data [4, Chaps. 7–8], [10, 12, 14, 15, 21, 28].

In this paper we show that some functionals, which allow free discontinuity and pay jump integrals over the singular set, do have minimizers provided the forcing term is sufficiently small say it fulfils an explicit safe load condition; this condition is also a necessary condition when load does not change sign. Moreover we prove that the minimizers have empty discontinuity set when the load is smaller than required by safe load: e.g. admissible small load deform an elastic-plastic plate or beam in the elastic range without occurrence of plastic yield (see Sects. 4 and 5). We call load regularity condition this more stringent inequality.

In many cases there is a gap between the safe load condition and the load regularity condition: in this situation strong inequality in the safe load (sufficient condition for existence) allows existence of non regular solutions for suitable load in between: we show explicit examples of broken/creased solutions, when the load stays in this gap. The main results (Theorems 2, 5, 9, 10, Remark 6) show a detailed analysis of the solutions structure.

We focus our analysis on the problems listed below: they are all related to deformations of an elastic body which undergoes free damage and is subject to Dirichlet boundary conditions. Boundary conditions are imposed by allowing variations defined in the whole Euclidean space which can be different from Dirichlet datum only in the reference bounded set.

The main focus of paper is a variational approach for detecting elastic-plastic yielding of beams and plates: Problems II, III below. The tools for the analysis are suggested by simpler first order Problem I, which is a toy problem, inspired by Barenblatt approach [6] and aiming to describe some effects of mesoscopic damage by using only macroscopic variables, without ambition to grasp whole complexity of real phenomenon (see [5, 6, 9, 22–25, 29, 30, 32, 33, 37, 48]).

I. First order model problem (elastic rod with free damage under traction):

Minimize the following functional over scalar functions $w \in SBV(\mathbb{R})$ s.t. $\mathrm{spt}(w - w_0) \subset [0, L]$:

$$\mathscr{F}_1(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\,|\dot{w}|^2 - fw \right) dx \;+\; \alpha\,\sharp(S_w) \;+\; \gamma \sum_{S_w} |[w]|\,. \tag{1.1}$$

Here $f \in L^1(\mathbb{R})$ with $\mathrm{spt} f \subset [0, L]$ is the traction load, $w_0 \in SBV(\mathbb{R})$ represents the boundary traction and fulfils $w_0 \in \mathrm{dom}\,\mathscr{F}_1$, $\sharp$ denotes the counting measure, $\alpha > 0,\ \gamma > 0$, w represents the axial displacement of the rod and S_w is the singular set of w. From now on $\dot{w}$ denotes the absolutely continuous part of the distributional derivative w' , $[w]$ denotes the jump $w_+ - w_-$ of the function w, where $w_\pm = w(x_\pm)$, while its positive part is denoted by $w^+ = w \vee 0$.

II. Second order problem (elastic plastic beam under transverse load):

Minimize the following functional over scalar functions $w \in SBH(\mathbb{R})$ s.t. $\mathrm{spt}(w - w_0) \subset [0, L]$:

$$\mathscr{F}_2(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\, |\ddot{w}|^2 - fw \right) dx \;+\; \beta\, \sharp(S_{\dot{w}}) \;+\; \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \,. \tag{1.2}$$

Here $f \in \mathscr{M}(\mathbb{R})$ with $\mathrm{spt} f^s \subset (0, L)$ is the transverse load, $w_0 \in SBH(\mathbb{R}) \cap \mathrm{dom}\, \mathscr{F}_2$ provides the boundary condition, $\beta > 0$, $\gamma > 0$, w represents the transverse displacement of the plate and $S_{\dot{w}}$ is the singular set of $\dot{w}$ (see [39, 41]).

III. Clamped elastic plastic plate (Kirchhoff-Love plate with plastic yield along free lines):

Minimize the functional

$$\mathscr{P}_{KL}(w) \;=\; \frac{2}{3}\mu \int_{\mathbb{R}^2} \left(\left|(D^2 w)^a\right|^2 + \frac{\lambda}{\lambda + 2\mu}\, |\Delta^a w|^2 - fw \right) d\mathbf{x} \,+ \\ +\, \beta\, \mathscr{H}^1(S_{Dw}) \;+\; \gamma \int_{S_{Dw}} |[Dw]|\, d\mathscr{H}^1 \tag{1.3}$$

over scalar functions $w \in SBH(\mathbb{R}^2)$ such that $\mathrm{spt}\, w \subset \overline{\Sigma}$.

In (1.3): $\Sigma \subset\subset \mathbb{R}^2$ is either a connected C^4 open set or an open convex polygon, $f \in L^p(\mathbb{R}^2)$ with $\mathrm{spt} f \subset \overline{\Sigma}$ is the transverse load, ∇ denotes the absolutely continuous part of the distributional gradient D, $\Delta^a w$ is the trace of ∇Dw, S_{Dw} is the singular set of Dw , $\alpha > 0$, $\gamma > 0$, $\mu > 0$, $\lambda + \mu > 0$, $p > 1$ and $\mathscr{H}^1$ is the length (1 dimensional Hausdorff measure).

Here Σ is the reference configuration of an elastic thin plate, w the transverse displacement of the plate. The functional $\mathscr{P}_{KL}$ represents the mechanical energy of the deformed plate, subject to transverse dead load f, with free plastic yield lines whose pattern (the set S_{Dw}) is “a priori” unknown [11, 40, 46]. Constants λ, μ denote Lamé coefficients; β and γ are respectively, per unit length, the energy released by formations of plastic yield lines and the energy released by folding the corresponding plastic hinge.

Plastic yield (or damage) may be located also at the boundary (say $\{0, L\}$ in Problems I, II and in $\partial\Sigma$ in Problem III). Actually in some particular 1-dimensional case we show in [43] that damage may take place only at the boundary.

For each one of the above problems we give an explicit safe load condition and prove that it entails the existence of a finite minimum (Lemmas 1, 5, 10); then we prove an excess estimate (Lemmas 7, 14) say a comparison with the energy of the solutions of the related elastic problems (e.g. minimizers of exactly the same functionals over the smaller domain of competing functions which are regular); hence we deduce regularity conditions which have an implicit form since they refer to the solution of the associated purely elastic problem (condition on stress for Problem I: Remark 2; condition on bending moment for beam and plate in Problems II, III: Lemmas 8, 15); eventually we prove load regularity conditions explicitly dependent only on data for each Problem: respectively (3.17), (4.4), (5.19).

The method of calibrations, successfully applied to first order problems with free discontinuity [2], looks difficult to implement on Problems II and III, which involve second derivatives: this difficulty is circumvented by introducing a *calibration by comparison* for testing minimizers based on *comparison with purely elastic solutions*, through a suitable compliance identity and an *energy excess estimate*: the method works whenever a DuBois-Raymond equation holds true at least for variations which are as regular as the purely elastic solution is (see Lemma 11).

Euler equations are derived explicitly in 1-dimensional Problems I and II together with several qualitative properties of free discontinuity set of minimizers: some of them reminds of Weierstrass-Erdman corner condition for these functionals with free discontinuity, say $\dot{w}$ is continuous in $(0, L)$ for minimizers of $\mathscr{F}_1$ and $\ddot{w}$, $\dddot{w}$ are continuous in $(0, L)$ for minimizers of $\mathscr{F}_2$ (see Lemmas 2, 6).

An interesting issue about consistency of these models with experiments in the safe load range [48] is achieved in the 1d frame by the analysis of minimizers structure: we prove that minimizers of $\mathscr{F}_1$ may break at one single point at most and that minimizers of $\mathscr{F}_2$ may exhibit no more than two crease points.

Explicit examples of load producing damaged minimizer of $\mathscr{F}_1$ and creased minimizers of $\mathscr{F}_2$ are shown when the load belongs to the narrow gap between safe load condition and regularity load condition: Examples 3, 4, Theorem 6. In order to achieve these examples, a careful estimate of this gap is obtained by showing: first, sharp Poincaré inequalities (see (3.4), Lemma 4), then stress estimate (3.16) for rod and bending estimates (4.24), (5.28) for beam and plate by mean of Green function (Lemma 9) and elliptic regularity (Lemma 13). Conditions for development of plastic yield lines in a plate (functional $\mathscr{P}$) follow by Lemma 14.

Some of the results which are proven here about plate and beam were announced in [42, 44]. We refer to [43] for a deeper analysis of the elastic-plastic beam and asymptotic analysis of Problem II as the parameter $\beta \to 0_+$, in the framework of L^∞ load.

We emphasize that the analysis of Problem I in the framework of L^∞ load would provide the same qualitative picture of rod deformation proven here for L^1 or measure load, since the constants in related safe load regularity conditions are the same (except for the different homogeneity in L) so that they coincide on constant load.

On the contrary the behavior of the beam (Problem II) does change a lot in the framework of L^∞ load since optimal constant (besides different homogeneity in L) in the $L^1 - BH$ Poincaré inequality (Lemma 2.1 in [43]) is quite different ($L/8$ here vs $L^2/16$ there) with respect to the one appearing in $L^\infty - BH$ Poincaré inequality (Lemma 4.1 in present paper): hence (see [43]) we can show that there are choices of constant load fulfilling the appropriate L^∞ safe load condition but not the L^∞ regularity load condition (respectively (2.5), (3.13) in [43]) which produce plastic hinges at both endpoints of the beam. While, in the present context, we show that increasing the intensity of a concentrated load with support contained in $(0, L)$ does not produce symmetric plastic hinges at endpoints before collapse (Theorem 7). Moreover Theorem 8 entails that symmetric loads of constant sign and fulfilling the total mass safe load condition (4.3) do not produce plastic hinges at all. About skew-symmetric load analysis for the elastic plastic beam we refer to [42] and to a forthcoming paper. Asymptotic analysis as $\beta \to 0_+$ for Problem II was studied in [43]. The issue of uniqueness, which is quite delicate in free discontinuity problems [3, 8], here finds a positive answer for Problems I, II, III provided the load regularity condition is fulfilled: respectively (3.17), (4.4), (5.19).

2 Notation

We denote the total variation of μ in E by $\|\mu\|_{T(E)}$, for any measure $\mu \in \mathscr{M}(\Omega)$ and Borel set $E \subset \Omega$; we write shortly $\|\mu\|_T = \|\mu\|_{T(\mathbb{R}^n)}$ when $E = \Omega = \mathbb{R}^n$. We write $\int_0^L f(x)v(x)dx$ in place of $\int_{\mathbb{R}} v(x)df(x)$ for any $f \in \mathscr{M}(\mathbb{R})$ with $spt\, f^s \subset\subset (0, L)$ and any $v \in L^1_f(\mathbb{R})$.

Any $\mu \in \mathscr{M}(\mathbb{R})$ can be split into three parts, say $\mu = \mu^a + \mu^s = \mu^a + \mu^j + \mu^c$ where μ^a is the absolutely continuous part, μ^s is the singular part, μ^j is the purely atomic part and μ^c is the diffuse singular one (the Cantor part of μ): the decomposition is unique.

Analogously, if I is an interval, then any $w \in BV(I)$ can be represented by $w = w_a + w_j + w_c$ where w_a has an absolutely continuous distributional derivative $(w_a)' = (w')^a \in L^1(I)$, w_j is a piece-wise constant function and $(w_j)' = (w')^j$ is purely atomic, w_c is a Cantor-type function (i.e. $(w_c)' = (w')^c$): for any $w \in BV(I)$ these three functions are uniquely defined up to additive constants ([4], Corollary 3.33), the constants are 0 when the support of w is a compact subset of I. We label $\dot{w} = (w_a)'$ the absolutely continuous part of distributional derivative w', hence we write as follows the unique decomposition of the derivative for a BV function with compact support: $w' = \dot{w} + (w_j)' + (w_c)'$. Approximate discontinuity sets of w and $\dot{w}$ (see [4]) are labeled by S_w, $S_{\dot{w}}$ and are shortly referred to as singular set of w, $\dot{w}$. Symbols $\sharp$ and $\sharp\llcorner E$ respectively denote the counting measure and its restriction to $E \subset \mathbb{R}$. Symbols $[\ ]$, $\otimes$ and $\odot$ denote respectively jumps, the tensor product and its symmetric part.

About the case of several variables we denote respectively by Dv and ∇v the distributional gradient and the approximate gradient of v. For any open set $\Omega \subset \mathbb{R}^n$ we denote:

$$
\begin{aligned}
&\mathscr{M}(\Omega) = \{\mu :\ \text{real valued Radon measures in } \Omega\}\,,\\
&BV(\Omega) = \{v \in L^1(\Omega) : Dv \in \mathscr{M}\},\\
&SBV(\Omega) = \{v \in BV(\Omega) :\ Dv \text{ has no Cantor part }\},\\
&BH(\Omega) = \{v \in W^{1,1}(\Omega) :\ D^2 v \in \mathscr{M}\},\\
&SBH(\Omega) = \{v \in BH(\Omega) :\ D^2 v \text{ has no Cantor part }\}\,.
\end{aligned}
$$

S_v denotes the singular set of v (set of points in Ω where v is not approximately continuous). $AC(I)$ denotes the space of real-valued absolutely continuous functions in the interval $I \subset \mathbb{R}$. For definition and properties of the above function spaces we refer to [4, 17, 23, 47].

3 (Pb I) Elastic Rod with Free Damage Under Traction

In this Section we study the minimization of the functional

$$\mathscr{F}_1(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\,|\dot{w}|^2 - fw \right) dx \;+\; \alpha\, \sharp(S_w) \;+\; \gamma \sum_{S_w} |[w]| \tag{3.1}$$

over scalar functions w in $SBV(\mathbb{R})$ such that $\mathrm{spt}(w-w_0) \subset [0,L]$.

Here α, γ are given constants, w_0 is a given function and $\sharp$ is the counting measure.
All along this Section we assume

$$\begin{cases} \alpha > 0,\ \gamma > 0, & \Sigma = (0,L), \quad f \in L^1(\mathbb{R}), \quad \mathrm{spt} f \subset \overline{\Sigma}, \\ w_0 \in SBV(\mathbb{R}), & \sharp(S_{w_0}) \,<\, +\infty, \quad S_{w_0} \subset \overline{\Sigma}, \quad \mathscr{F}_1(w_0) \in \mathbb{R}\,. \end{cases} \tag{3.2}$$

Functional (3.1) describes the total energy of an elastic rod which may undergo damage at free locations and is subject to given traction body force f and given boundary traction expressed by $w_0(0_-)$ and $w_0(L_+)$. The damage is a priori unknown and its location is given by the singular set of optimal axial displacement w.

Functional (3.1) is a crude simplification of more realistic models involving a concave interface energy contribution in place of $\gamma \sum_{S_{\dot{w}}} |[\dot{w}]|$ (see [7, 25, 30, 35, 36]). Nevertheless (3.1) provides a simple framework in which we are able to describe completely the structure of minimizers, moreover (3.1) proved very helpful in suggesting the techniques to tackle the harder and more significant models of elastic plastic beams and plates faced in Sects. 4 and 5.

We introduce a localization of the functional: for any Borel set $A \subset \mathbb{R}$ we set

$$\mathscr{F}_1(w,A) \;=\; \int_A \left(\frac{1}{2}\,|\dot{w}|^2 - fw \right) dx \;+\; \alpha\, \sharp(S_w \cap A) \;+\; \gamma \sum_{S_w \cap A} |[w]|.$$

At first we prove that a smallness condition (safe load condition) on f entails the existence of minimizers, while a violation of the safe load may lead to collapse.

Lemma 1 *Assume* (3.2) *and*

$$\|f\|_{L^1(\Sigma)} \;<\; 2\,\gamma \qquad\qquad (\mathscr{F}_1 \textbf{ safe load condition})\,. \tag{3.3}$$

Then $\mathscr{F}_1$ achieves a finite minimum among $w \in SBV(\mathbb{R})$ s.t. $\mathrm{spt}(w-w_0) \subset [0,L]$.

Proof If $\|f\|_{L^1(\Sigma)} < 2\gamma$, then we can apply the direct method since $\mathscr{F}_1$ is coercive in BV: in fact by the fundamental Theorem of calculus

$$w(x_-) = \frac{1}{2}\Big(w_0(0_-) + w'\big([0,x)\big)\Big) + \frac{1}{2}\Big(w_0(L_+) - w'\big([x,L]\big)\Big)$$

$$w(x_+) = \frac{1}{2}\Big(w_0(0_-) + w'\big([0,x]\big)\Big) + \frac{1}{2}\Big(w_0(L_+) - w'\big((x,L]\big)\Big)$$

$$\|w\|_{L^\infty(0,L)} \le \tfrac{1}{2}\Big(\|w'\|_{T(\overline{\Sigma})} + |w_0(0_-)| + |w_0(L_+)|\Big) \qquad \forall w \in BV(\mathbb{R}) : \ \mathrm{spt}(w - w_0) \subset [0,L]\,. \tag{3.4}$$

Hence for any admissible w

$$-\int_\Sigma fw \ \ge\ -\|f\|_{L^1(\Sigma)}\|w\|_{L^\infty(\Sigma)} \ \ge\ -\frac{1}{2}\|f\|_{L^1(\Sigma)}\Big(\|w'\|_{T(\overline{\Sigma})} + w_0(0_-) + w_0(L_+)\Big).$$

Moreover

$$\mathscr{F}_1(w) = \mathscr{F}_1(w,\mathbb{R}) = \mathscr{F}_1(w_0, \mathbb{R}\setminus\overline{\Sigma}) + \mathscr{F}_1(w,\overline{\Sigma})$$

Then by integrating over Σ the Young inequality $|\dot w|^2/2 \ge \gamma|\dot w| - \gamma^2/2$, we have

$$\mathscr{F}_1(w_0) \ \ge\ \mathscr{F}_1(w) \ \ge\ \alpha\,\sharp\,(S_w) + \left(\gamma - \frac{1}{2}\|f\|_{L^1(\Sigma)}\right)\|w'\|_{T(\overline{\Sigma})}$$

$$- \frac{L}{2}\gamma^2 - \frac{\|f\|_{L^1(\Sigma)}}{2}\Big(|w_0(0-)| + |w_0(L+)|\Big) + \mathscr{F}_1(w_0,\mathbb{R}\setminus\overline{\Sigma}).$$

Due to the inequality $2\gamma - \|f\|_{L^1} > 0$, the functional is bounded from below and, the elements of any minimizing sequence eventually fulfil the estimate (3.4). By w^*BV compactness and l.s.c properties [4] the existence of minimizers follows. □

The following Remark shows that the constant 2γ in the safe load (3.3) cannot be improved for generic L^1 load.

Remark 1 If $\|f\|_{L^1(\Sigma)} > 2\gamma$, f does not change sign and $w_0 \equiv 0$, then $\inf \mathscr{F}_1 = -\infty$.

In fact: if $f \ge 0$, $\|f\|_{L^1(\Sigma)} \ge 2\gamma + \varepsilon$ and $\varepsilon > 0$, set $v_t(x) = t > 0$ if $x \in \Sigma$, $v_t(x) = 0$ if $x \notin \Sigma$; then $J_{v_t} = \{0, L\}$, $\dot v \equiv 0$, $\displaystyle\int_{\mathbb{R}} f v_t\,dx \ge (2\gamma+\varepsilon)t$ and $\mathscr{F}_1(v_t) = 2\alpha - \varepsilon t \to -\infty$ as $t \to \infty$.

Lemma 2 (Euler Equations for $\mathscr{F}_1$) *Assume* (3.2) *and w is a minimizer of $\mathscr{F}_1$ among v s.t. $v = w_0$ on $\mathbb{R}\setminus\overline{\Sigma}$.*

Then $w' = \dot w \in AC(I)$ *for any interval* I *contained in* $\Sigma \setminus S_w$ *and, by setting* $w_\pm(x) = w(x_\pm)$, $\dot w_\pm(x) = \dot w(x_\pm)$, *the following equalities hold true*

$$\begin{array}{llll} (i) & -w'' &= f & \text{in } (0,L)\setminus S_w\,, \\ (ii) & \dot w_- &= \gamma\,\mathrm{sign}([w]) & \text{in } S_w \cap (0,L]\,, \\ (iii) & \dot w_+ &= \gamma\,\mathrm{sign}([w]) & \text{in } S_w \cap [0,L)\,, \end{array}$$

$$(iv)\ \int_0^L (\dot w(\dot z - \dot w) - f(z-w))\,dx - \gamma \sum_{S_w} |[z-w]| = 0,\ \forall z \in SBV(\mathbb{R}) \colon \mathrm{spt}(z-w) \subset \overline{\Sigma},\ S_z \subset S_w\,.$$

Hence $-(\dot w)' = f$ *in* $\mathscr{D}'(0,L)$ *and* $\dot w \in AC(0,L)$ *even if* $S_w \cap (0,L)$ *is not empty; nevertheless the continuity of* $\dot w$ *may fail at* 0 *and* L.

Proof By choosing $\varphi \in C^\infty(\mathbb{R}\setminus S_w) \cap SBV(\mathbb{R})$ with $\mathrm{spt}\,\varphi \subset \overline{\Sigma}$, and with C^∞ limit from both sides at any point in S_w, we get $\mathscr{F}_1(w) \le \mathscr{F}_1(w+\varepsilon\varphi)$. By convexity and taking into account that $w \in SBV$ entails $\dot w = w'$ in $(0,L)\setminus S_w$ and $\dot\varphi = \varphi' - \sum_{S_\varphi} [\varphi]\, \sharp \llcorner S_\varphi$, we get,

for $0 < \varepsilon < \min_{S_w} |[w]| / \|\varphi\|_{L^\infty}$:

$$\begin{aligned} 0 &\le \varepsilon \int_\Sigma (\dot w \dot\varphi - f\varphi) dx + \alpha\Big(\sharp(S_{w+\varepsilon\varphi}) - \sharp(S_w)\Big) + \gamma \sum_{S_w} (|[w+\varepsilon\varphi]| - |[w]|) + o(\varepsilon) = \\ &= \varepsilon \Bigg(\int_\Sigma (-w'' - f)\varphi dx + (\varphi(L_-)\dot w(L_-) - (\varphi(0_+)\dot w(0_+)) + \\ &\quad + \sum_{S_w \cap (0,L)} ((\varphi_- \dot w_-) - (\varphi_+ \dot w_+)) + \gamma \sum_{S_w} [\varphi]\ \mathrm{sign}([w]) \Bigg) + o(\varepsilon) = \\ &= \varepsilon \Bigg(\int_\Sigma (-w'' - f)\varphi dx + (\varphi(L_-)\dot w(L_-) - (\varphi(0_+)\dot w(0_+)) + \\ &\quad + \sum_{S_w \cap (0,L)} ((\varphi_- \dot w_-) - (\varphi_+ \dot w_+)) + \gamma \sum_{S_w} (\varphi_+ - \varphi_-)\ \mathrm{sign}([w]) \Bigg) + o(\varepsilon) \\ &= \varepsilon \Bigg(\int_\Sigma (-w'' - f)\varphi dx \\ &\quad + \varphi(0_+)\Big(\gamma\,\mathrm{sign}([w](0)) - \dot w(0_+)\Big) - \varphi(L_-)\Big(\gamma\,\mathrm{sign}([w](L)) - \dot w(L_-)\Big) + \\ &\quad + \sum_{S_w \cap (0,L)} \Big(\varphi_+(\gamma\,\mathrm{sign}([w]) - \dot w_+) - \varphi_-(\gamma\,\mathrm{sign}[w] - \dot w_-)\Big) \Bigg) + o(\varepsilon)\,. \end{aligned}$$

By choosing all φ with compact support in an interval contained in $(0,L)\setminus S_w$ we get the differential identity $-w'' = f$ in $(0,L)\setminus S_w$. Then for any fixed $x_k \in S_w$ we can choose at first (if $x_k < L$) all φ with compact support in $[x_k, x_{k+1})$ where x_{k+1} is the closest singular point bigger than x_k if any or L else, and then (if $0 < x_k$) all φ

with compact support in $[x_k - 1, x_k)$ where x_{k-1} is the closest singular point smaller than x_k if any or 0 else: this provides the values of $\dot{w}_\pm$ in S_w. The derivation of (ii), (iii) at $0, L$ is analogous.
The statement about continuity of $\dot{w}$ is straightforward, since $\dot{w} = w'$ in open interval where w is continuous, then in such intervals w is AC $w' \in L^1$ and w minimizes the Dirichlet integral (hence w is C^1), $\dot{w}_\pm$ exist in these intervals and $\dot{w}_\pm = w'$; while $\dot{w}_+ - \dot{w}_- = (\gamma - \gamma)\,\mathrm{sign}[w] = 0$ in $S_w \cap (0, L)$. Du Bois-Raymond equation (iv) follows in the same way by minimality of w with respect to variations $w + \varepsilon(z - w)$. $\square$

Lemma 3 (Compliance Identity for Functional $\mathscr{F}_1$) *Assume that w fulfils Euler equations* (i), (ii) *and* (iii) *in Lemma 2 and* $\mathrm{spt}(w - w_0) \subset [0, L]$. *Then*

$$\mathscr{F}_1(w) \;=\; -1/2 \int_0^L |\dot{w}|^2 \,+\, \alpha\, \sharp\,(S_w) \,-\, \dot{w}(0_+)\, w_0(0_-) \,+ \dot{w}(L_-)\, w_0(L_+)\,. \qquad (3.5)$$

Proof By Euler equations $(\dot{w})' = -f$ in $\mathscr{D}'(0, L)$, then by taking into account the identity $w' = \dot{w} + \sum_{S_w} [w]\, d\, \sharp \llcorner S_w$ in $\mathscr{D}'(0, L)$, we get

$$\int_{\mathbb{R}} fw = \int_0^L fw = -\int_0^L (\dot{w})'w = \int_0^L (\dot{w})w' + \; \dot{w}(0_+)w(0_+) - \dot{w}(L_-)w(L_-) \;=$$

$$= \int_0^L |\dot{w}|^2 dx + \; \dot{w}(0_+)w(0_+) - \dot{w}(L_-)w(L_-) + \sum_{S_w \cap (0,L)} \dot{w}[w]\,.$$

By recalling $\dot{w} = \gamma\,\mathrm{sign}[w]$ in $S_w \cap (0, L)$, $\dot{w}(0_+) = \gamma\,\mathrm{sign}([w](0))$, $\dot{w}(L_-) = \gamma\,\mathrm{sign}([w](L))$, $w(0_+) = [w](0) + w_0(0_-)$, $w(L_-) = -[w](L) + w_0(L_+)$ and (by $S_w \subset S_{w_0} \subset [0, L]$) we get

$$\int_0^L fw = \int_0^L |\dot{w}|^2 + \gamma \sum_{S_w} |[w]| + \; \dot{w}(0_+)w_0(0_-) - \dot{w}(L_-)w_0(L_+)$$

and thesis follows by the definition of $\mathscr{F}_1$. $\square$

Theorem 1 *Assume* (3.2) *and w is a minimizer of* $\mathscr{F}_1$ *with Dirichlet datum* w_0.
Then there is at most one crack, say $\sharp\,(S_w) \leq 1$.

Proof By contradiction assume $\sharp(S_w) \geq 2$. The we can choose $x_1, x_2 \in S_w$ with $x_1 \neq x_2$ and set

$$\tilde{w}(x) \;=\; \begin{cases} w(x) & \text{if } x \notin [x_1, x_2]\,, \\ w(x) + [w](x_2) & \text{if } x \in [x_1, x_2]\,. \end{cases}$$

Then $S_{\tilde{w}} = S_w \setminus \{x_2\}$ so that, by compliance identity (3.5) (and Euler equation (ii) when $x_2 = L$), we deduce the contradiction $\mathscr{F}_1(\widetilde{w}) = \mathscr{F}_1(w) - \alpha < \mathscr{F}_1(w)$. $\square$

Theorem 2 (Structure of $\mathscr{F}_1$ Minimizers) *Assume (3.2), (3.3) and u is the solution of*

$$u \in H^1(\Sigma), \qquad -u'' = f \text{ in } \Sigma, \qquad u(0_+) = w_0(0_-),\ u(L_-) = w_0(L_+), \tag{3.6}$$

hence $u' = \dot{u} \in C^0(\overline{\Sigma})$ and u has an extension, still denoted by u, s.t. $u \in SBV(\mathbb{R}) \cap C^0(\mathbb{R})$ and $u \equiv w_0$ in $\mathbb{R} \setminus \overline{\Sigma}$ (boundary values of u are always understood as interior traces). Then

$$\min_{spt(v-w_0)\subset[0,L]} \mathscr{F}_1(v) \;=\; \mathscr{F}_1(u) + \min\left(0\,,\, \alpha - \frac{L}{2}\big((\|u'\|_{L^\infty} - \gamma)^+\big)^2\right), \tag{3.7}$$

$$\operatorname*{argmin}_{spt(\cdot - w_0)\subset[0,L]} \mathscr{F}_1 = \begin{cases} \{\,u\,\} & \text{if } \|u'\|_{L^\infty} < \gamma + \sqrt{2\alpha/L} \\ \{\,u + z_t :\ |u'(t)| = \|u'\|_{L^\infty}\} & \text{if } \|u'\|_{L^\infty} > \gamma + \sqrt{2\alpha/L} \\ \{\,u\,\} \cup \{\,u + z_t :\ |u'(t)| = \|u'\|_{L^\infty}\} & \text{if } \|u'\|_{L^\infty} = \gamma + \sqrt{2\alpha/L} \end{cases} \tag{3.8}$$

where the function $z_t \in SBV(\mathbb{R})$ is defined as follows

$$z_t(x) \;=\; \begin{cases} -\text{sign}(u'(t))\,(|u'(t)| - \gamma)^+\, x & x \in [0,t)\,, \\ -\text{sign}(u'(t))\,(|u'(t)| - \gamma)^+\,(x - L) & x \in (t, L]\,, \\ 0 & x \notin [0, L]\,. \end{cases} \tag{3.9}$$

Proof Let S be the set of all $v \in$ SBV $(\mathbb{R})$ having no more than one jump point and fulfilling the Euler conditions (i), (ii), (iii) of Lemma 2.

Then all minimizers belong to S, due to Lemma 2, Theorem 1. Moreover we claim that

$$S = \{u + z_t :\ t \in [0, L]\} \tag{3.10}$$

Indeed it is obvious that $u + z_t \in S,\ \forall t \in [0, L]$. Conversely let $v \in S$, then either $v \equiv u$ or $S_v = \{t\}$ for some $t \in [0, L]$. In the second case by Euler equations we get

$$\begin{cases} \dot{v} - u' \in AC(0, L), \quad (\dot{v} - u')' \;=\; 0 \text{ in (0,L)}, \qquad \text{hence:} \\ \dot{v} - u' \;\equiv\; \text{const} \;\equiv\; \dot{v}(t_\pm) - u'(t) \;=\; \gamma\,\text{sign}\big([v](t)\big) - u'(t) \text{ in } (0, L) \\ S_{v-u} \;=\; S_v \;=\; \{t\} \end{cases} \tag{3.11}$$

and taking into account that $v = u = w_0$ in $\mathbb{R} \setminus [0, L]$ we get

$$z_t(x) \;=\; v(x) - u(x) \;=\; \begin{cases} (\gamma\text{sign}([v](t)) - u'(t))\; x & x \in [0,t)\,, \\ (\gamma\text{sign}([v](t)) - u'(t))\;(x - L) & x \in (t, L]\,, \\ 0 & x \notin [0, L]\,. \end{cases} \tag{3.12}$$

Since $[v](t) = [v-u](t) = -L\,(\gamma \operatorname{sign}([v])(t) - u'(t))$ we have

$$\begin{array}{l} [v](t) > 0 \;\Leftrightarrow\; u'(t) > \gamma > 0 \\ [v](t) < 0 \;\Leftrightarrow\; u'(t) < -\gamma < 0 \end{array} \tag{3.13}$$

hence, in both cases, $|u'(t)| > \gamma$ and $\operatorname{sign}(u'(t)) = \operatorname{sign}([v](t))$.

By summarizing, if v is a minimizer, then

$$v(x) - u(x) \;=\; \begin{cases} -\big((|u'(t)| - \gamma)^{+}\operatorname{sign}(u'(t))\big)\; x & x \in [0,t) \\ -\big((|u'(t)| - \gamma)^{+}\operatorname{sign}(u'(t)\big)\; (x-L) & x \in (t,L] \\ 0 & x \notin [0,L] \end{cases} \tag{3.14}$$

say $v(x) = u(x) + z_t(x)$ and

$$\min_{\operatorname{spt}(v-w_0)\subset[0,L]} \mathscr{F}_1 = \min_S \mathscr{F}_1 = \min_{t\in[0,L]} \mathscr{F}_1(u+z_t). \tag{3.15}$$

By taking into account that u and $u + z_t$ fulfil the Euler equations in $[0,L]$, we deduce $\dot z_t(x) = -(|u'(t)| - \gamma)^{+}\operatorname{sign}(u'(t)$ in $(0,L)$; hence, by compliance identity (Lemma 3),

$$\begin{aligned} &\mathscr{F}_1(u+z_t) \;=\; \alpha\, H(|u'(t)| - \gamma) - \frac{1}{2}\int_0^L |u' + \dot z_t|^2\, dx + \\ &+(u' + \dot z_t)(L_-)w_0(L_+) - (u' + \dot z_t)(0_+)w_0(0_-) = \alpha\, H(|u'(t)| - \gamma) + \\ &-\frac{1}{2}\int_0^L |u'|^2\, dx + (u')(L_-)w_0(L_+) - (u')(0_+)w_0(0_-) - \frac{1}{2}\int_0^L |\dot z_t|^2\, dx \\ &= \mathscr{F}_1(u) + \alpha\, H(|u'(t)| - \gamma) - \frac{L}{2}\big((|u'(t)| - \gamma)^{+}\big)^2 \end{aligned}$$

where the Heaviside function is point-wise defined in this way: $H(0) = 0$. Then

$$\begin{aligned} &\min_S \mathscr{F}_1(v) = \min\{\mathscr{F}_1(u+z_t) : t \in [0,L]\} = \\ &= \mathscr{F}_1(u) + \min_t\{\alpha\, H(|u'(t)| - \gamma) - \frac{L}{2}\big((|u'(t)| - \gamma)^{+}\big)^2\} = \\ &= \mathscr{F}_1(u) + \min\{\alpha\, H(\xi - \gamma) - \frac{L}{2}((\xi - \gamma)^{+})^2 : 0 \le \xi \le \|u'\|_\infty\} = \\ &= \mathscr{F}_1(u) + \min\{0, \alpha - \frac{L}{2}[(\|u'\|_{L^\infty} - \gamma)^{+}]^2\} \end{aligned}$$

and (3.7), (3.8) follow. □

A straightforward consequence of (3.7) and (3.8) is the subsequent remark.

Remark 2 (Stress Regularity Condition for Functional $\mathscr{F}_1$) Assume (3.2), (3.3).

If the solution u of (3.6) fulfils

$$\|u'\|_{L^\infty(0,L)} \;\le\; \gamma + \sqrt{2\alpha/L} \tag{3.16}$$

then $u \in \operatorname{argmin} \mathscr{F}_1$ and u is the unique minimizer when the inequality is strict.

Corollary 1 (Load Regularity Conditions for Functional $\mathscr{F}_1$) *Assume (3.2), (3.3).*

If

$$\|f\|_{L^1(0,L)} \;+\; \frac{1}{L}\,|w_0(L_+) - w_0(0_-)| \;\le\; \gamma + \sqrt{2\alpha/L} \tag{3.17}$$

then the regular solution u of (3.6) is the unique minimizer of $\mathscr{F}_1$.

If

$$\frac{L}{2}\,\|f\|_{L^\infty(0,L)} \;+\; \frac{1}{L}|w_0(L_+) - w_0(0_-)| \;\le\; \gamma + \sqrt{2\alpha/L} \tag{3.18}$$

then the solution u of (3.6) is a regular minimizer of $\mathscr{F}_1$, moreover if the inequality in condition (3.18) *is strict then the minimizer is unique.*

Proof By using the Green representation formula for the solution of (3.6) we get

$$u'(x) = \int_0^L J(x,y)f(y)\,dy \;+\; \frac{1}{L}\big(w_0(L_+) - w_0(0_-)\big) \tag{3.19}$$

where

$$J(x,y) = \begin{cases} 1 - y/L & \text{if } y \in [x, L] \\ -y/L & \text{if } y \in [0, x)\,. \end{cases} \tag{3.20}$$

If $f \in L^1$, taking into account that $|J(x,y)| \le 1$ and $x \in [0,1]$ entails $|J(x,y)| < 1$ for all $y \in [0,1]$ but at most one value, from (3.17) we get

$$\|u'\|_{L^\infty(0,L)} \;<\; \|f\|_{L^1(0,L)} + \frac{1}{L}|w_0(L_+) - w_0(0_-)| \;\le\; \gamma + \sqrt{2\alpha/L}\,,$$

hence the regularity stress condition is fulfilled by u and we can apply Remark 2 to achieve the claim about f in L^1.

When $f \in L^\infty(0,L)$, we have

$$\begin{aligned} |u'(x)| &\le \|f\|_{L^\infty(0,L)} \int_0^L |J(x,y)|\,dy + \frac{|w_0(L_+) - w_0(0_-)|}{L} \\ &= \|f\|_{L^\infty(0,L)} \frac{2x^2 + L^2 - 2Lx}{2L} + \frac{|w_0(L_+) - w_0(0_-)|}{L} \le \\ &\le \frac{L}{2}\|f\|_{L^\infty(0,L)} + \frac{|w_0(L_+) - w_0(0_-)|}{L}\,, \end{aligned} \tag{3.21}$$

so, by taking into account (3.18), we obtain

$$\|u'\|_{L^\infty(0,L)} \;\leq\; \frac{1}{2}\,\|f\|_{L^\infty(0,L)} + \frac{1}{L}|w_0(L_+) - w_0(0_-)| \;\leq\; \gamma + \sqrt{2\alpha/L}\,,$$

hence the regularity stress condition is fulfilled by u and we can apply Remark 2 and the claim about $f \in L^\infty$ follows. □

Remark 3 By summarizing: the safe load condition (3.3) entails existence of a minimizer, while the load regularity condition (3.17) entails existence, regularity and uniqueness.

Assumptions (3.2), (3.3), (3.6), $\|u'\|_{L^\infty} > \gamma + \sqrt{2\alpha/L}$ imply that all the minimizers have exactly one crack: both uniqueness and non uniqueness of minimizers are possible, depending on the cardinality of the set $\{t \in [0,L] :\ |u'(t)| = \|u'\|_{L^\infty} > \gamma\}$.

Assumptions (3.2), (3.3), (3.6), $\|u'\|_{L^\infty} = \gamma + \sqrt{2\alpha/L}$ entail that both u and every $v_t = u + z_t$ with $|u'(t)| = \|u'\|_{L^\infty} = \gamma$ are minimizers and $\min \mathscr{F}_1 = \mathscr{F}_1(v_t) = \mathscr{F}_1(u)$.
The point break of a broken minimizer may be placed anywhere in $[0,L]$ and uniqueness of minimizer is not expected in general (see Example 4). Though no crack is allowed if $\|f\|_{L^1} < \gamma + \sqrt{2\alpha/L}$ and $w_0 \equiv 0$, we emphasize that a load in the gap, e.g. the condition

$$\gamma + \sqrt{2\alpha/L} \;<\; \|f\|_{L^1(0,L)} < 2\,\gamma \quad \text{with} \quad w_0 \equiv 0\,, \tag{3.22}$$

does not force the minimizers to be discontinuous in all cases.
Statements analogous to Theorem 2 are difficult to achieve in Problems II and III which are studied in the next sections; nevertheless a suitable "calibration by comparison" can be proven also in those cases and this is enough to achieve explicit conditions for regularity of solutions, similar to (3.17). We end this section by showing explicit examples of minimizers with exactly one crack when the load regularity condition (3.17) fails.

Example 3 **A broken minimizer of $\mathscr{F}_1$ with non trivial f verifying safe load (3.3) and homogeneous Dirichlet datum: uniqueness and crack at the boundary.**
Assume $f(x) = 2cx,\ c > 0,$ s.t. $3\gamma/2 < cL^2 < 2\gamma$, and $w_0 \equiv 0$.

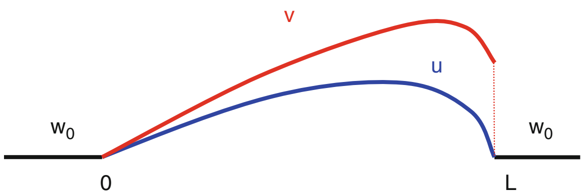
Then safe load condition (3.3), $cL^2 \leq 2\gamma$, holds true while regularity load condition (3.17), $cL^2 \leq \gamma$, fails to be true. The purely elastic solution is $u(x) = cx(L^2 - x^2)/3$ if $x \in (0,L)$, $u(x) = 0$ else, hence $\|u'\|_\infty = |u'(L)| = 2cL^2/3 > \gamma$ and L is the only point t where $|u'(t)| = \|u'\|_{L^\infty}$. By Theorem 2 the function v below is the unique candidate broken minimizer:

$$v(x) = (cL^2 - \gamma)x - cx^3/3 = u(x) + \big(2cL^2/3 - \gamma\big)x \ \text{ if } x \in (0,L),\ v(x) = 0 \text{ elsewhere;}$$

then $S_v = \{L\}$, $[v](L) = -v(L_-) = (\gamma - 2cL^2/3)L < 0$, $\dot v(L) = -\gamma$ and, by summarizing:

- (i) if $0 < \alpha < \frac{2}{9}c^2L^5 + \frac{\gamma^2 L}{2} - \frac{2}{3}c\gamma L^3 = \frac{L}{2}\left(\gamma - \frac{2}{3}cL^2\right)^2$
  then $\mathscr{F}_1(v) < \mathscr{F}_1(u)$ and, by Theorem 2, v is the unique minimizer .
- (ii) if $\alpha = \frac{L}{2}\left(\gamma - \frac{2}{3}cL^2\right)^2$
  then both v and u are minimizers and, by Theorem 2, there are no more.
- (iii) if $\alpha > \frac{L}{2}\left(\gamma - \frac{2}{3}cL^2\right)^2$
  then u is the unique minimizer by Lemma 2, Theorem 2.

Figure below shows the broken minimizer v of $\mathscr{F}_1$ in cases *(i), (ii)* above: $\|u'\|_{L^\infty} > \gamma$.



Example 4 **Existence of infinitely many broken minimizers of $\mathscr{F}_1$ under null load and non-homogeneous Dirichlet datum.**
Assume $f \equiv 0$, $w_0(x) = 0$ *if* $x < 0$ $w_0(x) = h > 0$ *if* $x > L$.

Then the safe load condition (3.3) is always (for any h) fulfilled, while the load condition (3.17) reads $h \le \gamma L + \sqrt{2\alpha L}$. The purely elastic solution is $u(x) = h\,x/L$ and $\|u'\|_{L^\infty} = h/L$. Theorem 2 provides a full description of minimizers as traction h at point L increases:

if $h \le \gamma L$, then the only solution is the continuous one, $u(x) = (L/h)x$;

if $\gamma L < h < 2\gamma L$ and $0 < (h - \gamma L)^2/2L < \alpha$, again there is the unique solution u;

if $\gamma L < h < 2\gamma L$ and $0 < \alpha < (h - \gamma L)^2/2L$, then there are infinitely many solutions v_t, all of them with a single break-point: $v_t(x) = 0$ if $x < 0$, $v_t(x) = \gamma x + (h - \gamma L)\,\chi_{[t,L]}(x)$ if $0 < x < L$, $v_t(x) = h$ if $L < x$;

if $\gamma L < h < 2\gamma L$ and $0 < \alpha = (h - \gamma L)^2/2L$, then the continuous solution u and all functions v_t above with a crack at t are minimizers.

Eventually, when $h \ge 2\gamma L$: if $h < \gamma L + \sqrt{2\alpha L}$ then u is the unique solution; if $h > \gamma L + \sqrt{2\alpha L}$ then there are infinitely many broken solutions and no regular solution; if $h = \gamma L + \sqrt{2\alpha L}$ then both u and the broken solutions are present.

4 (Pb II) Elastic Plastic Beam Under Transverse Load

In this Section we study the minimization of the functional

$$\mathscr{F}_2(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\,|\ddot{w}|^2 - fw \right) dx \;+\; \beta\, \sharp(S_{\dot{w}}) \;+\; \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \tag{4.1}$$

over scalar functions w such that $w \in SBH(\mathbb{R})$ and $\operatorname{spt} w \subset \overline{\Sigma}$.

All along this Section α, γ are given constants, $\sharp$ is the counting measure and we assume

$$\beta > 0,\; \gamma > 0, \quad \Sigma = (0, L), \quad f \in \mathscr{M}(\mathbb{R}), \quad \operatorname{spt} f \subset \overline{\Sigma},\; \operatorname{spt} f^s \subset\subset \Sigma. \tag{4.2}$$

Functional (4.1) describes the total energy associated to deformations of an elastic-plastic beam which is clamped at both endpoints; w is the vertical displacement of the beam under the action of the transverse load f. The crease points set $S_{\dot{w}}$ of a minimizer w may be interpreted as location of plastic hinges in the beam at equilibrium: functional (4.1) takes into account that the energy released in the deformation of an elastic plastic beam is the sum of elastic bending energy and of energy concentrated at plastic hinges. Jump points are not allowed (say $S_w = \emptyset$) for admissible displacements w which must be continuous since $SBH(\mathbb{R}) \subset C^0(\mathbb{R})$.

The main result of this Section is Theorem 5 below, for the homogeneous Dirichlet problem, which provides an explicit condition on the load for existence of minimizers and a stronger condition which entails their regularity: creased minimizers do appear in the gap, as shown by subsequent Theorem 6.

Theorem 5 *Assume* (4.1)*,* (4.2) *and*

$$\|f\|_{T(\overline{\Sigma})} \;<\; \frac{8\,\gamma}{L} \qquad (\mathscr{F}_2 \textbf{ safe load condition})\,. \tag{4.3}$$

Then $\mathscr{F}_2$ achieves a finite minimum among $w \in SBH(\mathbb{R})$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$.

If the inequality (4.3) *is substituted by the following stronger one*

$$\|f\|_{T(\overline{\Sigma})} \;\le\; \frac{27}{4}\,\frac{\gamma}{L} \qquad (\textbf{load regularity condition for functional } \mathscr{F}_2)\,, \tag{4.4}$$

then $\mathscr{F}_2$ has unique minimizer among $w \in SBH(\mathbb{R})$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$, this minimizer belongs to $H^2(\mathbb{R})$ and coincides in $[0, L]$ with the unique solution u of

$$\left\{ u \in H^2_0(0, L) \;:\; u'''' = f \quad in\ (0, L) \right\}\,. \tag{4.5}$$

The proof of Theorem 5 is postponed until several preliminary Lemmas are achieved.

Lemma 4 (**$L^\infty - BH$ Poincarè Inequality**) *Let* $v \in BH(\mathbb{R})$ *with* $\operatorname{spt} v \subset [0, L]$. *Then*

$$\|v\|_{L^\infty(0,L)} \le \frac{L}{8}\|v''\|_{T([0,L])}\,. \tag{4.6}$$

The equality in (4.6) *holds true iff* $v = r_s$ *(roof-function), for some* $s \in \mathbb{R}$*:*

$$r_s(x) = s\left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^{+}. \tag{4.7}$$

Proof Fix $v \in \mathscr{K}^* = \{v \in BH(\mathbb{R})\ s.t.\ \operatorname{spt} v \subset [0, L]\}$. Without loss of generality we assume $v \not\equiv 0$. Then define

$$\tilde v(x) = \begin{cases} \text{convex envelope of } -|v| \text{ evaluated at } x & \text{if } x \in [0, L] \\ 0 & \text{if } x \notin [0, L]\,. \end{cases}$$

We claim that $\tilde v$ fulfils

$$\begin{cases} \tilde v \in BH(\mathbb{R})\,, \quad \operatorname{spt} \tilde v \subset [0, L]\,, \quad v \le 0\,, \quad \tilde v \text{ convex in } [0, L]\,, \\ \|\tilde v\|_{L^\infty} = \|v\|_{L^\infty}\,, \qquad \|\tilde v''\|_T \le \|v''\|_T\,. \end{cases} \tag{4.8}$$

The only non trivial point in (4.8) is the estimate of total variation ($\|\tilde v''\|_T \le \|v''\|_T$) which can be proven by exploiting chain-rule for superposition of *BH* functions (Theorems 1, 4 in [45]) as in Lemma 2.1 of [42]. By (4.8) we get

$$\inf\left\{\frac{\|v''\|_T}{\|v\|_{L^\infty}} : v \in \mathscr{K}^*\right\} = \inf\left\{\frac{\|v''\|_T}{\|v\|_{L^\infty}} : v \in \mathscr{K}^*,\ v \text{ convex in } [0, L]\right\}. \tag{4.9}$$

If we take $v \in \mathscr{K}^*$, v convex in $[0, L]$ and $v \not\equiv 0$, then

$$-\infty < v'_+(0) \le 0\,, \qquad 0 \le v'_-(L) < +\infty$$

and we can define

$$\check v(x) = (v'_+(0)x) \vee \big((v'_-(L)(x - L)\big) \ \text{ if } x \in [0, L] \qquad \text{and } \check v(x) \equiv 0 \ \text{ otherwise}\,.$$

Then $\check v \le v$, $\|\check v\|_{L^\infty} \ge \|v\|_{L^\infty}$ and $\|\check v''\|_{T(\mathbb{R})} = 2(v_-(L) - v'_+(0)) = \|v''\|_{T([0,L])}$ by convexity.

$$\begin{aligned} &\inf\{\,\|v''\|_T / \|v\|_{L^\infty} : v \in SBH,\ \operatorname{spt} v \subset [0, L],\ v \text{ convex in } [0, L]\,\} \ge \\ &\ge \inf\{\,\|v''\|_T / \|v\|_{L^\infty} : v(x) = (-ax) \vee (b(x - L)),\ a > 0,\ b > 0\,\} = \\ &= \min_{a>0,\, b>0} \frac{2(a + b)^2}{abL} = 8/L\,. \end{aligned} \tag{4.10}$$

Actually the infimum in (4.10) is a minimum since it is achieved at $a = b$ say when v is a roof function. By summarizing (4.9), (4.10) prove (4.6). About the fact that only roof functions (4.7) achieve the equality in (4.6) we emphasize that: the map $v \to \tilde{v}$ strictly reduces the relevant quotient $\|v''\|_T \,/\, \|v\|_{L^\infty}$ whenever $|v| \not\equiv |\tilde{v}|$, since in such case $\|(\cdot)''\|_T$ strictly decreases, while also the map $v \to \check{v}$ strictly reduces the relevant quotient for v convex in $[0, L]$ and $|v| \not\equiv |\check{v}|$, since in such case $\|\cdot\|_{L^\infty}$ strictly increases. □

For a different proof of (4.6) see [38].

Now we can prove that a smallness condition (safe load condition) on f entails existence of minimizers (for any boundary datum), while a violation of the safe load may lead to collapse.

Lemma 5 *Assume* (4.2), (4.3) *and* $w_0 \in SBH(\mathbb{R})$ *with* $\mathscr{F}_2(w_0) < +\infty$.

Then $\mathscr{F}_2$ *achieves a finite minimum among* $w \in SBH(\mathbb{R}^2)$ *with* $\operatorname{spt}(w - w_0) \subset \overline{\Sigma}$.

Proof We use the direct method. First we show that $\mathscr{F}_2$ is coercive: by Lemma 4,

$$\|z\|_{L^\infty} \le \frac{L}{8}|z''|_{T(\overline{\Sigma})} \qquad \forall z \in SBH(\mathbb{R}) \ s.t. \ \operatorname{spt} \subset \overline{\Sigma} \tag{4.11}$$

$$\begin{aligned}
\left|\int_0^L fw\,dx\right| \;&\le\; \left|\int_0^L f(w - w_0)\,dx\right| + \left|\int_0^L fw_0\,dx\right| \;\le \\
&\le\; \|f\|_{T(\overline{\Sigma})}\,\|w - w_0\|_{L^\infty} + \left|\int_0^L fw_0\,dx\right| \le \\
&\le\; \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\Big(\|w''\|_{T(\overline{\Sigma})} + \|w_0''\|_{T(\overline{\Sigma})}\Big) + \left|\int_\Sigma fw_0\,dx\right| \;= \\
&=\; \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\left(\sum_{S_{\dot{w}}}[\dot{w}] + \int_0^L |\ddot{w}|\,dx + \|w_0''\|_{T(\overline{\Sigma})}\right) + \left|\int_0^L fw_0\,dx\right| \\
&\qquad\qquad \forall\, w \text{ s.t. } \operatorname{spt}(w - w_0) \subset [0, L],
\end{aligned} \tag{4.12}$$

and by Young inequality

$$\frac{1}{2}\int_0^L |\ddot{w}|^2\,dx \;\ge\; \gamma\int_0^L |\ddot{w}|\,dx - \frac{L}{2}\gamma^2$$

hence, by denoting $\mathscr{F}_2(\cdot, A)$ the localized functional in a Borel set $A \subset \mathbb{R}$

$$\mathscr{F}_2(w, A) \;=\; \int_A\left(\frac{1}{2}\,|\ddot{w}|^2 - fw\right)\,dx \;+\; \beta\,\sharp(S_{\dot{w}} \cap A) \;+\; \gamma\sum_{S_w \cap A} |[\dot{w}]|,$$

for any w in a minimizing sequence of $\mathscr{F}_2$, we have ultimately

$$\begin{aligned}
&\mathscr{F}_2(w_0,\overline{\Sigma}) + \mathscr{F}_2(w_0,\mathbb{R}\backslash\overline{\Sigma}) \;=\; \mathscr{F}_2(w_0) \;\geq \\
&\quad\geq\; \mathscr{F}_2(w) \;\;=\; \mathscr{F}_2(w,\overline{\Sigma}) + \mathscr{F}_2(w_0,\mathbb{R}\backslash\overline{\Sigma}) \;\geq \\
&\quad\geq\; \beta\sharp(S_{\dot w}) + \left(\gamma - \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\right)\|w''\|_{T(\overline{\Sigma})} + \\
&\quad -\frac{L}{2}\gamma^2 - \left|\int_{\mathbb{R}} fw_0\,dx\right| - \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\|w_0''\|_{T(\overline{\Sigma})} + \mathscr{F}_2(w_0,\mathbb{R}\setminus\overline{\Sigma}).
\end{aligned}$$

Hence by (4.3) the functional is bounded from below. The existence of minimizers for $\mathscr{F}_2$ follows by sequential compactness of minimizing sequences and BH^* sequential lower semi-continuity of $\mathscr{F}_2$ [11, 12, 39]. □

For sake of simplicity we study only the homogeneous case ($w_0 \equiv 0$) in the sequel.

The constant $8\gamma/L$ in the safe load (4.3) cannot be improved for generic $\mathscr{M}$ or L^1 load as shown by the following Remark. Nevertheless for L^∞ load we refer to [43] where we prove a safe load condition which turns out to be less stringent on uniform loads.

Remark 4 There are examples with $\|f\|_{T(\overline{\Sigma})} > 8\,\gamma/L$, such that $\inf \mathscr{F}_2 \;= -\infty$:

choose $f \;=\; (8\gamma/L + \varepsilon)\;\delta(x - L/2)$, $\varepsilon > 0$, set $w_t(x) \;=\; t(L/2 - |x - L/2|)^+$. Then $J_{\dot w_t} = \{0, L/2, L\}$, $\ddot w_t \equiv 0$, $\langle f\, w_t\rangle = t\,(4\gamma + \varepsilon L/2)$, $\mathscr{F}_2(w_t) = 3\beta - \varepsilon Lt/2 \to -\infty$ as $t \to +\infty$.

Lemma 6 (**$\mathscr{F}_2$ Euler Equations**) *Assume* (4.1), (4.2) *and w minimizes $\mathscr{F}_2$ among v belongs to $SBH(\mathbb{R})$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$ *. Then* $\ddddot w = (\ddot w)'' = f \in \mathscr{M}$ *in* $(0, L)$, $\dddot w = (\ddot w)'$ *belongs to $AC(I)$ for any interval $I \subset \Sigma \setminus S_{\dot w}$ and*

$$\begin{array}{llll}
(i) & w'''' & = f & in\ (0,L)\setminus S_{\dot w}\\
(ii) & \dddot w_- & = \gamma\,\mathrm{sign}([\dot w]) & in\ S_{\dot w}\cap(0,L]\\
(iii) & \dddot w_+ & = \gamma\,\mathrm{sign}([\dot w]) & in\ S_{\dot w}\cap[0,L)\\
(iv) & \dddot w_- & = \dddot w_+ & in\ (0,L).
\end{array}$$

$$(v)\ \int_0^L \big(\ddot w\,(\ddot z - \ddot w) - f(z - w)\big)dx - \gamma\sum_{S_{\dot w}}|[\dot z - \dot w]| = 0,\ \ \forall z \in SBH(\mathbb{R}):\ \operatorname{spt}(z - w) \subset \overline{\Sigma},\ S_{\dot z} \subset S_{\dot w}\,.$$

In particular $\ddot w \in BH(0, L)$, hence $\ddot w$ and $\dddot w = (\ddot w)'$ are continuous in $(0, L)$ but may be discontinuous at 0 and L, even if these points do not belong to $S_{\dot w}$.

Proof The proof can be achieved by running the same steps in the proof of Lemma 2.

Lemma 7 *Assume* (4.1), (4.2) *and u is the extension by zeroes of the unique solution of* (4.5)*. Then we have the following properties.*

Excess estimate for $\mathscr{F}_2$ **:**

$$\mathscr{F}_2(v) - \mathscr{F}_2(u) \;\geq\; \beta \,\sharp\, (S_{\dot v}) + \left(\sum_{S_{\dot v}} \gamma \,|[\dot v]| - u''\,[\dot v] \right) \qquad \forall v \in SBH(\mathbb{R}) : \; \operatorname{spt} v \subset \overline{\Sigma}\,. \tag{4.13}$$

Excess identity for minimizers of $\mathscr{F}_2$ **:**
If v *minimize* $\mathscr{F}_2$ *among* $v \in SBH(\mathbb{R})$ *s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$, *then*

$$\mathscr{F}_2(v) - \mathscr{F}_2(u) \;=\; \beta \,\sharp\, (S_{\dot v}) + \frac{1}{2} \left(\sum_{S_{\dot v}} \gamma \,|[\dot v]| - u''\,[\dot v] \right). \tag{4.14}$$

Necessary conditions for existence of creased minimizers of $\mathscr{F}_2$ **:**
If v *minimize* $\mathscr{F}_2$ *among* $w \in SBH(\mathbb{R})$ *s.t.* $\operatorname{spt} w \subset \overline{\Sigma}$, *and* $S_{\dot v} \neq \emptyset$, *then*

$$\|u''\|_{L^\infty(\Sigma)} > \gamma\,, \tag{4.15}$$

$$\sum_{S_{\dot v}} [\dot v] \,\big(\gamma \;\operatorname{sign}[\dot v] - u''\big) = \sum_{S_{\dot v}} \big(\gamma\,|[\dot v]| - u''\,[\dot v]\big) \leq -2\beta \,\sharp\, (S_{\dot v}) < 0\,, \tag{4.16}$$

$$\beta \;\leq\; \frac{1}{2} L\,\gamma^2\,. \tag{4.17}$$

By (4.16)*, if the set* $S_{\dot v}$ *contains exactly in one point* $\overline{x}$ *then* $|u''(\overline{x})| \;>\; \gamma$.

Proof By exploiting $u'' \in C(\overline{\Sigma})$, $\ddot v \;=\; v'' \;-\; [\dot v]\, d\sharp \llcorner S_{\dot v} \cap (0, L)$ in $\mathscr{D}'(0, L)$, $u \in H^2_0(\Sigma)$, $u'''' = f$ in Σ and $u = 0$ on $\mathbb{R} \setminus \Sigma$, the convexity of $s \to s^2/2$ and

$$\int_0^L u''(v-u)''\,dx \;=\; \int_0^L u''''(v-u)\,dx \;-u''(L)\,[\dot v](L) \;-\; u''(0)\,[\dot v](0)$$

we have, for every $v \in SBH(\mathbb{R})$ s.t. $\operatorname{spt} v \subset \overline{\Sigma}$,

$$\begin{aligned}
\mathscr{F}_2(v) \geq\; & \mathscr{F}_2(u) + \int_0^L u''(\ddot v - u'')dx - \int_0^L f(v-u)dx + \beta\sharp(S_{\dot v}) + \gamma \sum_{S_{\dot v}} |[\dot v]| = \\
=\; & \mathscr{F}_2(u) + \int_0^L u''(v'' - u'')dx - \int_0^L f(v-u)dx + \beta\sharp(S_{\dot v}) + \gamma \sum_{S_{\dot v}} |[\dot v]| - \sum_{S_{\dot v} \cap (0,L)} u''[\dot v] = \\
=\; & \mathscr{F}_2(u) + \beta\sharp(S_{\dot v}) + \sum_{S_{\dot v}} (\gamma|[\dot v]| - u''[\dot v])
\end{aligned}$$

Then (4.13) is proved.

If $v \in \operatorname{argmin} \mathscr{F}_2$ and u solves (4.5), then $\ddot{v}$ is continuous in $(0, L)$ by Theorem 6 and $u = v$, and $u'' = v'' = \ddot{v}$ hold true in $\mathbb{R} \setminus \overline{\Sigma}$ while Du-Bois Raymond equation (v) relative to variations $v_\varepsilon = v + \varepsilon(u - v)$ yields

$$\int_\Sigma \big(\ddot{v}\,(u'' - \ddot{v}) - f(u - v)\big)\; dx - \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| \;=\; 0\,. \tag{4.18}$$

Hence

$$\mathscr{F}_2(v) - \mathscr{F}_2(u) = \frac{1}{2}\int_\Sigma |\ddot{v}|^2\, dx + \beta\, \sharp(S_{\dot{v}}) + \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| - \int_\Sigma f v\, dx - \frac{1}{2}\int_\Sigma |u''|^2 + \int_\Sigma fu =$$

$$= \frac{1}{2}\int_\Sigma \big(\ddot{v} + u''\big)\,(\ddot{v} - u'') + \beta\, \sharp(S_{\dot{v}}) + \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| - \int_\Sigma f v\, dx + \int_\Sigma fu =$$

$$= \frac{1}{2}\int_\Sigma f(v - u) - \int_\Sigma f(v - u) - \frac{\gamma}{2}\sum_{S_{\dot{v}}} |[\dot{v}]| + \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| + \frac{1}{2}\int_\Sigma u''(\ddot{v} - u'') + \beta\, \sharp(S_{\dot{v}}) =$$

$$= \frac{\gamma}{2}\sum_{S_{\dot{v}}} |[\dot{v}]| - \frac{1}{2}\int_\Sigma f(v - u) + \frac{1}{2}\int_\Sigma u''(\ddot{v} - u'') + \beta\, \sharp\,(S_{\dot{v}})\,.$$

Since $u'' \in C(\overline{\Sigma})$, $\ddot{v} \;=\; v'' - [\dot{v}]\, d\, \sharp \llcorner \big(S_{\dot{v}} \cap (0, L)\big)$ in $\mathscr{D}'(0, L)$, $u \in H^2_0(\Sigma)$, $u'''' = f$ in Σ and $u = v$, on $\partial\Sigma$, $\dot{v}_+(0) - u'(0) = [\dot{v}(0)]$, $\dot{v}_-(L) - u'(L) = -[\dot{v}(L)]$, we get

$$\mathscr{F}_2(v) - \mathscr{F}_2(u) =$$

$$= \frac{1}{2}\int_\Sigma u''(v'' - u'') - \frac{1}{2}\int_{S_{\dot{v}}} f(v - u) + \frac{\gamma}{2}\sum_{S_{\dot{v}}} |[\dot{v}]| - \frac{1}{2}\sum_{S_{\dot{v}} \cap (0,L)} u''[\dot{v}] + \beta \sharp(S_{\dot{v}}) =$$

$$= \beta\, \sharp(S_{\dot{v}}) \;+\; \frac{1}{2}\left(\sum_{S_{\dot{v}}} \gamma |[\dot{v}]| - \sum_{S_{\dot{v}} \cap (0,L)} u''[\dot{v}]\right) \;-\; \frac{1}{2}\, u''(0)\, [\dot{v}](0) \;-\; \frac{1}{2}\, u''(L)\, [\dot{v}](L)$$

$$= \beta\, \sharp(S_{\dot{v}}) \;+\; \frac{1}{2}\sum_{S_{\dot{v}}} \big(\gamma |[\dot{v}]| - u''[\dot{v}]\big)\,.$$

Necessary conditions (4.16), (4.15) for creased minimizers follow by inserting $\sharp(S_{\dot{v}}) \geq 1$ in (4.14).

By $\mathscr{F}_2(v) \leq \mathscr{F}_2(0) = 0$, $L^\infty - BV$ Poincaré inequality (4.6), safe load (4.3) and Young inequality we get

$$\begin{aligned} &\frac{1}{2}\int_0^L |\ddot{v}|^2 dx + \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| + \beta \sharp(S_{\dot{v}}) \leq \int_0^L f v dx \leq \|f\|_{L^1} \|v\|_{L^\infty} \leq \\ &\leq \frac{L}{8}\|f\|_{L^1}\left(\int_0^L |\ddot{v}| + \sum_{S_{\dot{v}}} |[\dot{v}]|\right) \leq \gamma \left(\int_0^L |\ddot{v}| + \sum_{S_{\dot{v}}} |[\dot{v}]|\right) \leq \\ &\leq \frac{1}{2}\int_0^L |\ddot{v}|^2 dx + \frac{1}{2} L\gamma^2 dx + \gamma \sum_{S_{\dot{v}}} |[\dot{v}]| \end{aligned} \tag{4.19}$$

and, if $\sharp(S_{\dot{v}}) \geq 1$ then $\beta \leq \beta\, \sharp(S_{\dot{v}}) \leq \frac{1}{2} L \gamma^2$ say (4.17). $\square$

Lemma 8 (L^∞ Bending Moment Regularity Condition for Clamped Beam) *Assume* (4.1), (4.2) *and the unique solution u of* (4.5) *fulfils*

$$\|u''\|_{L^\infty(0,L)} \leq \gamma \,. \tag{4.20}$$

Then u is also a minimizer of $\mathscr{F}_2$ *. Moreover u is the unique minimizer of* $\mathscr{F}_2$*.*

Proof Straightforward consequence of necessary condition (4.15)

Lemma 9 (Green Representation Formulae) *Assume u solves* (4.5)*. Then*

$$u''(x) = \int_0^L K(x,y) f(y)\, dy \tag{4.21}$$

where

$$K(x,y) = \frac{1}{2L^3}(2x - L)\, y^2\, (3L - 2y) - \frac{1}{2L} y^2 + (y-x)^+ \,. \tag{4.22}$$

Moreover

$$\max_{x,y\in[0,L]} |K(x,y)| = \frac{4}{27} L \,, \tag{4.23}$$

$$\|u''\|_{L^\infty} \leq \frac{4}{27} L \, \|f\|_{T(\overline{\Sigma})} \,. \tag{4.24}$$

The equality in (4.24) *can be achieved: the constant* $4L/27$ *is optimal for* (4.24)*.* $\square$

Proof We perform the computations by assuming $f \in L^1$; the general case can be handled exactly in the same way, since $\mathrm{spt} f^s \subset\subset \Sigma$.

The classical Green formula provides the standard representation

$$u(x) \;=\; \int_0^L \mathscr{G}(x,y)\, f(y)\, dy \tag{4.25}$$

where we denote by $\mathscr{G}(x,y)$ the Green function associated to the operator $(d/dx)^4$ in$(0,L)$ with homogeneous boundary conditions:

$$\mathscr{G}_{xxxx}(x,y) = \delta(x-y) \text{ in } (0,L)^2\,, \quad \mathscr{G}(0,y) = \mathscr{G}_x(0,y) = \mathscr{G}(L,y) = \mathscr{G}_x(L,y) = 0 \text{ in } (0,L)\,.$$

Moreover, by setting $P_3(y) = \; (3L-2y)\, y^2/L^3, \quad P_1(y) = y/L$, we get

$$P_3(y) + P_3(L-y) = 1 \tag{4.26}$$

and, by setting

$$J_3(x,y) := \begin{cases} P_3(y) & \text{if } 0 \le y \le x \le L, \\ -P_3(L-y) & \text{if } 0 \le x < y \le L, \end{cases}$$

$$J_1(x,y) := \begin{cases} P_1(y) & \text{if } 0 \le y \le x \le L, \\ -P_1(L-y) & \text{if } 0 \le x < y \le L, \end{cases}$$

$$u'''(x) \;=\; \int_0^L J_3(x,y)\, df(y) \;=\; \int_0^x P_3(\tau)\, df(\tau) - \int_x^L P_3(L-\tau) f(\tau)\, d\tau \tag{4.27}$$

$$\begin{aligned} u''(x) \;&=\; \int_0^L J_1(x,y)\, u'''(y)\, dy \;= \\ =\; \int_0^L J_1(x,y) &\left(\int_0^y P_3(\tau) f(\tau)\, d\tau - \int_y^L P_3(L-\tau) f(\tau)\, d\tau \right) dy = \\ &=\; \int_0^L K(x,y)\, f(y)\, dy \end{aligned} \tag{4.28}$$

Hence (4.21). By standard computations, which exploit also the Green function G for the operator $(d/dx)^2$, we get

$$\max_{x,y\in[0,L]^2} |K| \;=\; \max\{\max_y |K(y,y)|\,,\, \max_y |K(0,y)|\,,\, \max_y |K(L,y)|\,\}\,.$$

Then $K(y,y) = -\frac{2y^2}{L^3}\,(L-y)^2\,,\; K(0,y) = y\left(1-\frac{y}{L}\right)^2\,,\; K(L,y) = \frac{y^2}{L} - \frac{y^3}{L^2}$ together entail

$$\max_y |K(y,y)| \;=\; \frac{L}{8}\,, \qquad \max_y |K(0,y)| \;=\; \max_y |K(L,y)| \;=\; \frac{4}{27}\, L$$

and (4.23) follows. Estimate (4.24) follows by (4.28), (4.23). The equality is achieved in (4.24) since $f(y) = \delta(y - 2L/3)$ entails $\|u''\| = 4L/27$. □

Proof of Theorem 5 Assume (4.3); then the first statement follows by Lemma 5, with $w_0 \equiv 0$. Now assume (4.4) and u is the solution of (4.5). Hence by (4.24)

$$\|u''\|_{L^\infty} \;\le\; \frac{4\,L}{27}\,\|f\|_{T(\overline{\Sigma})} \;\le\; \gamma$$

and Lemma 8 give the conclusion, since the bending moment condition is fulfilled by u. □

Actually non regular minimizers of $\mathscr{F}_2$ do appear in some cases: we recall additional properties of minimizers followed by some examples.

Remark 5 (Compliance Identity for Functional $\mathscr{F}_2$) Assume (4.1), (4.2), w satisfies Euler conditions (i), (ii), (iii) of Theorem 6 and spt $w \subset \overline{\Sigma}$. Then (by Lemma 3.5 in [42])

$$\mathscr{F}_2(w) \;=\; -\frac{1}{2}\int_0^L |\ddot{w}|^2\,dx \;+\; \beta\,\sharp\,(S_{\dot{w}}). \tag{4.29}$$

Remark 6 If (4.1), (4.2) hold and v minimizes $\mathscr{F}_2$ among functions with spt $\subset \overline{\Sigma}$, then $\sharp(S_{\dot{v}}) \le 2$ (this fact can be shown as in the proof of Theorem 4.1 in [42]).

Next we show an example of creased minimizer of $\mathscr{F}_2$ with homogeneous boundary condition and load f fulfilling the safe load condition (4.3): all properties claimed below can be proven by the same computations done in the analysis of Example 4.1 in [42], which dealt the case of concentrated load.

*Example 6 (**A load which leads to creased minimizers**)* If we choose parameters δ, k and load $f \in L^1$ such that

$$f \;=\; \frac{k\,\gamma}{2\,\delta\,L}\,\chi_{[2L/3-\delta,2L/3+\delta]}\,, \qquad 0<\delta<L/3\,, \qquad 27/4<k<8\,, \tag{4.30}$$

then load f fulfills the safe load condition (4.3), violates the stress regularity condition (e.g., the solution of (4.5) fulfils $\|u''\|_{L^\infty} > \gamma$) for the related non creased solution and the minimizers of $\mathscr{F}_2$ among functions with spt $\subset \overline{\Sigma}$, have a crease whenever

$$0 \;<\; \beta \;<\; \frac{L}{4}(\ddot{u}(L)-\gamma)^2\,.$$

We notice that the necessary condition (4.17) for the existence of a crease (say $2\,\beta \le L\,\gamma^2$) is fulfilled if $0 < \beta < (\ddot{u}(L)-\gamma)^2$.
We end this Section by showing that, if total mass safe load condition (4.3) holds true and the load does not change sign, then the minimizer cannot exhibit symmetric plastic hinges at endpoints (see Theorem 7); moreover any symmetric load of

constant sign and fulfilling the total mass safe load condition (4.3) do not produce plastic hinges at all (see Theorem 8).

Theorem 7 *Assume* (4.1)–(4.3)*,* v *minimizes* $\mathscr{F}_2$ *among functions with* spt $\subset \overline{\Sigma}$ *and either* $f \geq 0$ *or* $f \leq 0$*. Then* $S_{\dot v} \neq \{0, L\}$

Proof Assume by contradiction $S_{\dot v} = \{0, L\}$. Then by Euler equations

$$\begin{cases} (\ddot v)'' = f & (0,L) \\ \ddot v(0_+) = \gamma \ \mathrm{sign}[\dot v(0)]\,, \quad \ddot v(L_-) = \gamma \ \mathrm{sign}[\dot v(L)]\,, \quad v(0) = v(L) = 0 \end{cases}$$

and (up to interchanging boundary values at 0 and L, or changing sign in both boundary values) only two cases may occur: either

$$\begin{cases} (\ddot v)'' = f \quad (0,L) \\ \ddot v(0_+) = +\gamma\,,\ \dot v(0_+) > 0\,,\ \ddot v(L_-) = -\gamma\,, \quad \dot v(L_-) > 0\,,\ v(0) = v(L) = 0 \end{cases} \tag{4.31}$$

or

$$\begin{cases} (\ddot v)'' = f \quad (0,L) \\ \ddot v(0_+) = \ddot v(L_-) = +\gamma\,, \quad \dot v(0_+) > 0\,, \quad \dot v(L_-) < 0\,, \quad v(0) = v(L) = 0\,. \end{cases} \tag{4.32}$$

We show that both cases lead to a contradiction.

In case (4.31), we claim

$$[\dot v](0)\,[\dot v](L) \ < \ 0 \ \Rightarrow \ \begin{cases} [\dot v](0)\,\big(\gamma\, \mathrm{sign}[\dot v](0) - u''(0)\big) \ < \ 0 \\ [\dot v](L)\,\big(\gamma\, \mathrm{sign}[\dot v](L) - u''(L)\big) \ < \ 0 \end{cases} \tag{4.33}$$

We set $s_0 = [\dot v](0),\ s_L = [\dot v](L)$. Since $\big(\ddot v - u''\big)'' \equiv 0$ $(0,L)$ and $v'' = \ddot v + s_0\delta_0 + s_L\delta_L$, there exist c, d s.t.

$$\ddot v(x) - u''(x) = (cx + d)\,\chi_{[0,L]}(x). \tag{4.34}$$

By integration over $\mathbb{R}$ we get

$$\frac{c}{2}L^2 + d\,L + s_0 + s_L \ = \ 0 \tag{4.35}$$

by integrating (4.34) twice we get

$$c\,L^3/6 + d\,L^2/2 + L\,s_0 \ = \ 0\,. \tag{4.36}$$

Euler equations

$$\ddot v(0) \ = \ \gamma \ \mathrm{sign}\, s_0 \qquad v''(L)\ddot v(L) \ = \ \gamma \ \mathrm{sign}\, s_L \tag{4.37}$$

entail

$$\ddot{v}(0) - u''(0) \;=\; \gamma \,\mathrm{sign}\, s_0 - u''(0) \;=\; d \tag{4.38}$$

$$\ddot{v}(L) - u''(L) \;=\; \gamma \,\mathrm{sign}\, s_L - u''(L) \;=\; c\,L + d\,. \tag{4.39}$$

By solving (4.35), (4.36) we get

$$c \;=\; 6\,(s_0 - s_L)/L^2\,, \quad d \;=\; 2(s_L - 2\,s_0)/L\,. \tag{4.40}$$

By setting $a = [\dot{v}](0)\,\big(\gamma\,\mathrm{sign}[\dot{v}](0) - u''(0)\big) = s_0(\gamma\,\mathrm{sign}\,s_0 - u''(0))$,
$b = [\dot{v}](L)\,\big(\gamma\,\mathrm{sign}[\dot{v}](L) - u''(L)\big) = s_L(\gamma\,\mathrm{sign}\,s_L - u''(L))$,
the thesis of claim (4.33) reads

$$a = s_0 d \;<\; 0\,, \qquad b = s_L(cL + d) \;<\; 0\,, \tag{4.41}$$

and since (4.31) entails $s_L < 0 < s_0$ we get

$$a \;=\; 2s_0(s_L - 2s_0)\,/\,L \;<\; 0\,, \qquad b \;=\; 2s_L(s_0 - 2s_L)\,/\,L < 0 \tag{4.42}$$

(notice that also in the other case with $s_0 s_L < 0$, e.g. $s_0 < 0 < s_L$, we get (4.42)) hence (4.41) and the claim (4.33) is proven. By using claim (4.33) we get

$$u''(L) - u''(0) \;<\; \gamma\,(s_L - s_0) \;=\; -2\,\gamma, \tag{4.43}$$

notice that also in the other case with $s_0 s_L < 0$, e.g. $s_0 < 0 < s_L$, we get

$$u''(L) - u''(0) \;>\; \gamma\,(s_L - s_0) \;=\; +2\,\gamma. \tag{4.44}$$

In any case by (4.21), (4.22)

$$2\gamma \;<\; \left|u''(L) - u''(0)\right| \;<\; \left|\int_0^L (K(L,y) - K(0,y))\, f(y)\, dy\right| \tag{4.45}$$

$$K(L,y) - K(0,y) \;=\; \frac{1}{L^2}\left(3Ly^2 - 2y^3 - L^2 y\right) \tag{4.46}$$

$$|K(L,y) - K(0,y)| \le |K(L, L(1 - 1/\sqrt{3}) - K(0, L(1 - 1/\sqrt{3})| = L(3 - \sqrt{3} - 2/\sqrt{3})/3 \tag{4.47}$$

By (4.45), (4.47), (4.3) we get the contradiction

$$2\gamma < \left|u''(L) - u''(0)\right| < \frac{L}{3}\left(3 - \sqrt{3} - 2/\sqrt{3}\right) \|f\|_{L^1} \;<\; \frac{8}{3}\left(3 - \sqrt{3} - 2/\sqrt{3}\right)\gamma < 2\gamma\,.$$

In case (4.32), by using Green function G for d^2/dx^2 with homogeneous boundary conditions, we get $\ddot{v}(x) = \gamma + \int_0^L G(x,y)f(y)\,dy$. Hence $\dot{v}(L) - \dot{v}(0) = \gamma L + \int_0^L \int_0^L G(x,y)dy\,dx$. Since

$$\max_{y\in[0,L]} \left| \int_0^L G(x,y)\,dx \right| = \frac{L^2}{8} \tag{4.48}$$

by Fubini-Tonelli and (4.3) we get

$$\left| \int_0^L \int_0^L G(x,y)f(y)\,dy\,dx \right| \le \frac{L^2}{8}\|f\|_{L^1} < \frac{L^2}{8}\frac{8\gamma}{L} = \gamma L \tag{4.49}$$

hence the contradiction $\dot{v}(L_-) - \dot{v}(0_+) > 0$. □

Theorem 8 *Assume* (4.1)–(4.3)*, f does not change sign and is symmetric:*

$$f(x) = f(L-x)\ x \in (0,L) \ \text{ and either } f \ge 0 \text{ or } f \le 0\,. \tag{4.50}$$

Then there is a unique minimizer of $\mathscr{F}_1$ among v s.t. $\operatorname{spt} v \subset [0,L]$*, moreover such minimizer is the regular solution of* (4.5)*.*

Proof By taking into account the symmetry of f and K

$$f(y) = f(L-y) \qquad K(x,y) = K(L-x,L-y) \tag{4.51}$$

in the Green representation (4.27) we get, for $x \in (0,L)$,

$$\begin{aligned} u''(x) &= \int_0^L K(x,y)f(y)\,dy = \int_0^L K(L-x,L-y)f(y)\,dy = \\ &= \int_0^L K(L-x,y)f(y)\,dy = u''(L-x)\,. \end{aligned} \tag{4.52}$$

Then u'' is even with respect to $L/2$.
Therefore $\|u''\|_{L^\infty} = \max\{|u''(L)|, |u''(L/2)|\}$ and by (4.21), (4.22), (4.3) we get

$$\begin{aligned} |\,u''(L)\,| &= \left| \int_0^L K(L,y)f(y)\,dy \right| = \\ &= \left| \frac{1}{L^2}\int_0^{L/2} y^2(L-y)f(y)\,dy + \frac{1}{L^2}\int_{L/2}^L y(L-y)^2 f(y)\,dy \right| = \\ &= \frac{1}{L}\left| \int_0^{L/2} y(L-y)f(y)\,dy \right| \le \frac{1}{L}\frac{L^2}{4}\frac{1}{2}\|f\|_{L^1(0,L)} = \frac{L}{8}\|f\|_{L^1(0,L)} < \gamma\,, \end{aligned} \tag{4.53}$$

$$
\begin{aligned}
&\left|u''\left(\frac{L}{2}\right)\right| = \left|\int_0^L \left((y-L/2)^+ \; - \; \frac{y^2}{2L}\right)\right| \; = \\
&= \frac{1}{2L}\left|\int_0^{L/2} y^2 f(y)\, dy \; + \; \int_{L/2}^{L} (L-y)^2 f(y)\, dy\right| \\
&= \frac{1}{L}\left|\int_0^{L/2} y^2 f(y)\, dy\right| \; \le \; \frac{1}{L}\,\frac{L^2}{4}\,\frac{1}{2}\,\|f\|_{L^1(0,L)} \; = \; \frac{L}{8}\,\|f\|_{L^1(0,L)} \; < \; \gamma\,,
\end{aligned}
\tag{4.54}
$$

$$
\|u''\|_{L^\infty} \; < \; \gamma \tag{4.55}
$$

and the thesis follows by Lemma 8. □

5 (Pb III) Clamped Kirchhoff-Love Plate with Plastic Yield Along Free Lines

In this Section we study the minimization of the functional

$$
\mathscr{P}_{KL}(w) \; = \; \frac{2}{3}\mu\int_{\Sigma}(|\nabla^2 w|^2+\frac{\lambda}{\lambda+2\mu}|\Delta^a w|^2)d\mathbf{x}+\beta\mathscr{H}^1(S_{\nabla w})+\gamma\int_{S_{Dw}}|[Dw]|\,d\mathscr{H}^1-\int_{\Sigma} fw d\mathbf{x}
$$

over scalar functions $w \in SBH(\mathbb{R}^2)$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$.
Here ∇ denotes the absolutely continuous part of the distributional gradient D, while Δ^a denotes the absolutely continuous part of the distributional Laplace operator, say $\Delta^a w \; = \; \mathrm{Tr}\left(\nabla D w\right) \; = \; (\Delta w)^a$, β, γ are given constants, $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure and f is a given transverse load. All along this section we assume

$$
\Sigma \subset \mathbb{R}^2 \text{ connected Lipschitz open set}\,, \tag{5.1}
$$

$$
\beta > 0,\; \gamma > 0, \tag{5.2}
$$

$$
\mu > 0,\; 2\mu + 3\lambda > 0\,, \tag{5.3}
$$

$$
f \in \mathscr{M}(\mathbb{R}),\; \operatorname{spt} f \subset \overline{\Sigma}\,. \tag{5.4}
$$

Notice that for any $w \in SBH$ we have (see [4, 11, 31]):

$$
\nabla w = Dw\,, \qquad S_{\nabla w} = S_{Dw}\,, \tag{5.5}
$$

$$
S_{Dw} \;\text{ is a countably } \mathscr{H}^1 \text{ rectifiable set} \tag{5.6}
$$

$$\begin{cases} S_{Dw} \text{ has an approximate normal vector } \nu_{S_{Dw}} \;\; \mathscr{H}^1 \text{a.e. in } S_{Dw} \text{ and} \\ \nu_{S_{Dw}} \;\text{ is uniquely defined up to the orientation at every point where it is defined,} \end{cases} \tag{5.7}$$

$$[Dw] \;=\; [\nabla w] \;=\; \left[\frac{\partial w}{\partial \nu_{S_{Dw}}}\right] \nu_{S_{Dw}}\,, \qquad |[Dw]| \;=\; \left|\left[\frac{\partial w}{\partial \nu_{S_{Dw}}}\right]\right|\,, \tag{5.8}$$

$$\text{both } D^2 w \text{ and } \nabla^2 w \text{ are symmetric}\,. \tag{5.9}$$

Instead of confining the analysis only to the quadratic form

$$\mathrm{Q}_{KL}(\mathbb{M}) = \frac{2}{3}\left(\mu|\mathbb{M}|^2 + \frac{\mu\lambda}{\lambda+2\mu}|\;\mathrm{Tr}\,\mathbb{M}|^2\right) \tag{5.10}$$

associated to Kirchhoff-Love plate energy $\mathscr{P}_{KL}$, we consider a generic positive definite quadratic Q form evaluated on $\nabla^2\mathbf{v}$ and study the whole class of functionals $\mathscr{P}$ (including $\mathscr{P}_{KL}$)

$$\mathscr{P}(w) = \int_{\Sigma} \big(\,\mathrm{Q}\big(\nabla^2 w\big) \,-\, fw\big)\, d\mathbf{x} + \beta\mathscr{H}^1(S_{\nabla w}) + \gamma\int_{S_{Dw}} |[Dw]|\, d\mathscr{H}^1\,, \tag{5.11}$$

to be minimized among $w \in SBH(\mathbb{R}^2)$ s.t. $\mathrm{spt}\, w \subset \overline{\Sigma}$. We assume that Q fulfils

$$\begin{cases} \exists\, q_{ijhk} \in \mathbb{R},\; q_{ijhk} = q_{hkij} \;:\;\; \mathrm{Q}(\mathbb{M}) = \displaystyle\sum_{i,j,h,k=1}^{2} q_{ijhk}\,\mathbb{M}_{ij}\,\mathbb{M}_{hk} \quad \forall\, \mathbb{M}\,, \\ \exists\, a,\, A\,,\; 0 < a \le A < +\infty \;:\quad a\,|\mathbb{M}|_2^2 \;\le\; \mathrm{Q}(\mathbb{M}) \;\le\; A\,|\mathbb{M}|_2^2 \quad \forall\, \mathbb{M}\,, \end{cases} \tag{5.12}$$

here and in the following $\mathbb{A}:\mathbb{B} = \sum_{i,j} \mathbb{A}_{ij}\mathbb{B}_{ij}$ and $|\mathbb{M}|_p$ denotes the l^p norm, for 2×2 real symmetric matrices $\mathbb{A}, \mathbb{B}, \mathbb{M}$. We denote $\mathrm{Q}' = \partial\, \mathrm{Q}/\partial\mathbb{M}$ so that, by (5.12) we get

$$(\,\mathrm{Q}'(\mathbb{M}))_{hk} = 2\sum_{ij=1}^{2} q_{ijhk}\,\mathbb{M}_{ij}\,, \qquad \text{hence} \qquad \frac{|\,\mathrm{Q}'(\mathbb{M})|}{|\mathbb{M}|} \le 2A \;\;\forall\, \mathbb{M}\,. \tag{5.13}$$

In the particular case of $\mathrm{Q} = \mathrm{Q}_{KL}$ we have

$$\mathrm{Q}'_{KL}(\mathbb{M}) = \frac{4}{3}\left(\mu\mathbb{M} + \frac{\mu\lambda}{\lambda+2\mu}(\mathrm{Tr}\,\mathbb{M})\mathbb{I}\right)\,. \tag{5.14}$$

The main results of this Section are the two statements below: Theorem 9 which shows partial regularity of any solution (e.g. smoothness of creased plates outside closed yield lines of finite length) if $\mathrm{Q}(\mathbb{M}) = \mathbb{M}:\mathbb{M}$ together with a mild integrability condition on the load; Theorem 10 which provides a smallness condition on the load entailing global regularity and uniqueness of minimizers for $\mathscr{P}$.

Theorem 9 (Partial Regularity for Elastic-Plastic Clamped Plate) *Assume* (5.1)*,* (5.2)*,* $\|f\|_{L^1(\Sigma)} < 4\gamma$ *and there is* $p > 2$ *such that* $f \in L^p(\Sigma)$*. Then*

$$\int_{\Sigma} \left(|\nabla^2 v|^2 - f v\right) d\mathbf{x} + \beta \mathscr{H}^1(S_{Dv}) + \gamma \int_{S_{Dv}} |[Dv]| \, d\mathscr{H}^1$$

achieves a finite minimum among $v \in SBH(\mathbb{R}^2)$ *s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$*, moreover every minimizer w is also a strong solution, say:*

$$w \in C^0(\overline{\Sigma}) \cap C^2(\Sigma \setminus \overline{S_{Dw}}), \tag{5.15}$$

$$\mathscr{H}^1\left(\overline{S_{Dw}} \setminus S_{Dw}\right) = 0 \tag{5.16}$$

and the pair $(\overline{S_{Dw}}, w)$ *minimizes the functional*

$$P(K, v) = \int_{\Sigma \setminus K} \left(|D^2 v|^2 - f v\right) d\mathbf{x} + \beta \mathscr{H}^1(K \cap \overline{\Sigma}) + \gamma \int_{K \cap \overline{\Sigma}} |[Dv]| \, d\mathscr{H}^1 \tag{5.17}$$

among pairs (K, v) *such that* $K \subset \mathbb{R}^2$ *is a closed set and* $v \in C^0(\overline{\Sigma}) \cap C^2(\Sigma \setminus K)$*.*

Theorem 10 (Load Regularity Condition for Clamped Plate $\mathscr{P}$) *Assume:*

$$\partial\Sigma \text{ is either a convex polygonal or a } C^4 \text{ simple closed curve,} \tag{5.18}$$

(5.2)*,* (5.11)*,* (5.12)*,* $\|f\|_{L^1(\Sigma)} < 4\gamma$ *and there is* $p \in (1, +\infty)$ *such that*

$$\|f\|_{L^p(\Sigma)} \leq \frac{\gamma}{2 A C_2} \tag{5.19}$$

where $C_2 = C_2(\Sigma, p, a, A)$ *is the constant appearing in the estimate* (5.28)*.*

Then the solution u of

$$u \in H^2_0(\Sigma), \qquad \operatorname{div}\operatorname{div} Q'(D^2 u) = f \text{ in } \Sigma, \tag{5.20}$$

minimizes energy $\mathscr{P}$ *among scalar functions in* $SBH(\mathbb{R}^2)$ *with support in* Σ*.*

Moreover u is the unique minimizer of $\mathscr{P}$ *in this class.*

The proofs are postponed after some preliminary Lemmas. We are not able to prove (and even to write) the complete system of Euler equations (analogous to Lemma 6 for the beam) for functional (5.11), since we cannot hope to have enough regularity of minimizers v to give meaning to the product $(\nabla^2 v : \mu)$ when μ is a matrix-valued measure; moreover for a general minimizer v the set S_{Dv} is not smooth enough to perform integration by parts. The difference with respect to beam problem faced in Sect. 3 is that weak and strong formulation of free gradient discontinuity problems coincide only in dimension $n = 1$. Nevertheless we can prove something similar to Du Bois-Raymond equation, by considering only particular variations $\varepsilon(w-v)$, where w belongs to $C^2(\overline{\Sigma}) \cap SBH(\mathbb{R}^2)$, $\operatorname{spt} w \subset \overline{\Sigma}$, $v \in \operatorname{argmin} \mathscr{P}$ and $\varepsilon \in \mathbb{R}$, as stated by the Lemma 11 below. So we get Euler equation (5.48) only in the set $\Sigma \setminus \overline{S_{Dv}}$ and the compliance identity as stated in Lemma 12. Moreover additional assumptions on f and $\partial\Sigma$ allow proof of basic relationship: a sufficiently small load f in $L^p(\Sigma)$ with $p > 1$ entails excess identity (5.34) and the regularity Theorem 10. In a different perspective any $f \in L^p(\Sigma)$ with $p > 2$ leads to partial regularity result stated in Theorem 9.

Lemma 10 *Assume* (5.1), (5.2), (5.4), (5.11), (5.12) *and*

$$\|f\|_T(\overline{\Sigma}) \ < \ 4\gamma \qquad \textbf{(safe load condition for clamped plate)} \tag{5.21}$$

Then $\mathscr{P}$ *achieves a finite minimum over* $w \in SBH(\mathbb{R}^2)$ *with* $\operatorname{spt} w \subset \overline{\Sigma}$.

Proof It is a particular case of Theorem 8.3 in [16]. □

Lemma 11 *Assume* (5.1), (5.2), (5.4), (5.11), (5.12). *Then, for any* $w \in C^2(\overline{\Sigma}) \cap SBH(\mathbb{R}^2)$ *with* $\operatorname{spt} w \subset \overline{\Sigma}$, *and* $v \in \operatorname{argmin} \mathscr{P}$ *(minimizing over* $SBH(\mathbb{R}^2)$ *with* $\operatorname{spt} \subset \overline{\Sigma}$*) we have*

$$\int_\Sigma \big(Q'(\nabla^2 v) : \big(D^2 w - \nabla^2 v\big) - f(w-v)\big)\, d\mathbf{x} \ - \ \gamma \int_{S_{Dv}} |[Dv]|\, d\mathscr{H}^1 \ = \ 0\,. \tag{5.22}$$

Proof Identity (5.22) is a kind of Du Bois-Raymond equation and can be proved by exactly the same procedure of item (*iv*) of Lemma 2, by performing variations of v of the kind $v + \varepsilon(z - v)$, with $|\varepsilon| < 1$ to avoid cancelation of the singular set, then exploiting minimality of v, convexity of Q and $\nabla^2 v = D^2 v - [Dv]\, d\mathscr{H}^1 \llcorner S_{Dv}$.

Lemma 12 (Compliance Identity for Elastic-Plastic Plate) *Assume* (5.1), (5.2), (5.4), (5.11), (5.12). *Then*

$$2\int_\Sigma Q(\nabla^2 v)\, d\mathbf{x} \ = \ \int_\Sigma f\, v\, d\mathbf{x} - \gamma \int_{S_{Dv}} |[Dv]|\, d\mathscr{H}^1 \qquad \forall v \in \operatorname{argmin} \mathscr{P}\,. \tag{5.23}$$

Hence the following compliance identity holds true

$$\mathscr{P}(v) \ = \ - \int_\Sigma Q(\nabla^2 v)\, d\mathbf{x} \ + \ \beta\, \mathscr{H}^1(S_{Dv}) \qquad \forall v \in \operatorname{argmin} \mathscr{P}\,. \tag{5.24}$$

Proof Choose $w = 0$ in (5.22). □

Lemma 13 (Elliptic Regularity) *Assume* (5.12), (5.18) *and*

$$f \ \in \ L^p\,, \quad 1 < p < +\infty\,. \tag{5.25}$$

Then the elliptic problem of fourth order (5.20) *has unique solution u which is also the unique minimizer of* $\int_\Sigma \big(Q(D^2 v) - fv \big)\, d\mathbf{x}$ *over* $v \in H^2_0(\Sigma)$, *moreover u fulfils the associate compliance identity*

$$\int_\Sigma Q(D^2 u)\, d\mathbf{x} \;=\; \frac{1}{2}\int_\Sigma fu\, d\mathbf{x}\,, \qquad e.g. \qquad \mathscr{P}(u) \;=\; -\frac{1}{2}\int_\Sigma fu\, d\mathbf{x}\,, \tag{5.26}$$

u belongs to $W^{4,p}(\Sigma)$ *and there are two constants* $C_1,\ C_2$, *with* $C_1 = C_1(\Sigma, p, a, A)$ *and* $C_2 = C_2(\Sigma, p, a, A)$ *such that*

$$\|u\|_{W^{4,p}(\Sigma)} \;\le\; C_1\, \|f\|_{L^p(\Sigma)}\,, \tag{5.27}$$

$$\|D^2 u\|_{C^0(\overline{\Sigma})} \;\le\; C_2\, \|f\|_{L^p(\Sigma)}\,. \tag{5.28}$$

If $Q = Q_{KL}$ *(Kirchhoff-Love elastic plate) then problem* (5.20) *reads as follows:*

$$u \in H^2_0(\Sigma)\,, \qquad \Delta^2 u \;=\; \frac{3\,(\lambda + 2\mu)}{8\,\mu\,(\lambda+\mu)}\, f \text{ in } \Sigma\,. \tag{5.29}$$

Proof Since $L^p(\Sigma) \subset H^{-2}(\Sigma)$, by denoting C_3 the related embedding constant and applying standard Hilbert technique for elliptic equations, we get existence and uniqueness of solution for (5.20), minimizing the purely elastic energy $\int_\Sigma \big(\mathrm{Q}(D^2 v) - fv \big)\, d\mathbf{x}$ and fulfilling (5.26) together with the following estimates (due to (5.12)), $\Sigma \subset \mathbb{R}^2$, $\operatorname{spt} u \subset \overline{\Sigma}$):

$$\|u\|_{L^\infty(\Sigma)} \le \frac{1}{4}\|Du^2\|_{T(\overline{\Sigma})} \;\le\; \frac{1}{4}|\Sigma|^{1/2}\|D^2 u\|_{L^2(\overline{\Sigma})} \;\le$$

$$\le\; \frac{|\Sigma|^{1/2}}{4\sqrt{a}}\left(\int_\Sigma \mathrm{Q}(D^2 u)\, d\mathbf{x}\right)^{1/2} \;=\; \frac{|\Sigma|^{1/2}}{4\sqrt{a}}\left(\frac{1}{2}\int_\Sigma fu\, d\mathbf{x}\right)^{1/2} \;\le$$

$$\le\; \frac{|\Sigma|^{1/2}}{4\sqrt{2a}}\, \|f\|^{1/2}_{L^p(\Sigma)}\, \|u\|^{1/2}_{L^{p'}(\Sigma)} \;\le\; \frac{|\Sigma|^{\frac{1}{2}+\frac{1}{2p'}}}{4\sqrt{2a}}\, \|f\|^{1/2}_{L^p(\Sigma)}\, \|u\|^{1/2}_{L^\infty(\Sigma)}\,,$$

hence, by $H^2_0(\Sigma) \subset L^\infty(\Sigma)$, we get

$$\|u\|_{L^\infty(\Sigma)} \;\le\; \frac{|\Sigma|^{2-1/p}}{32\, a}\|f\|_{L^p(\Sigma)}\,,$$

$$\|u\|_{L^p(\Sigma)} \;\le\; |\Sigma|^{1/p}\, \|u\|_{L^\infty(\Sigma)} \;\le\; \frac{|\Sigma|^2}{32\, a}\|f\|_{L^p(\Sigma)}\,. \tag{5.30}$$

Regularity $W^{4,p}$ of solution u in (5.20) follows by standard interior regularity and use of Lemma 4.2 p. 414 of [1] (with $m = j = 2$) on a finite atlas of the boundary $\partial\Sigma$ in the convex polygon case, and by Theorem 8.1 p. 443 of [1] in the C^4 boundary case: hence in both cases:

$$\|u\|_{W^{4,p}(\Sigma)} \leq C_0 \left(\|f\|_{L^p(\Sigma)} + \|u\|_{L^p(\Sigma)}\right) . \tag{5.31}$$

Then (5.30), (5.31) entail (5.27) with $C_1 = \left(1 + |\Sigma|^2/(32a)\right)C_0$.

Estimate (5.27) together with Sobolev inequality entail estimate (5.28) with $C_2 = C_1 C_3 = \left(1 + |\Sigma|^2/(32a)\right)C_0 C_3$ where C_3 is the embedding constant: $\|D^2 u\|_{C^0(\Sigma)} \leq C_3 \|u\|_{W^{4,p}(\Sigma)}$. □

Lemma 14 *Assume* (5.2)*,* (5.11)*,* (5.12)*,* (5.18)*,* (5.25) *and u is the unique solution of*

$$u \in H^2_0(\Sigma)\,, \quad \operatorname{spt} u \subset \overline{\Sigma}\,, \quad \operatorname{div}\operatorname{div}\; Q'(D^2 u) \;=\; f \text{ in } \Sigma\,. \tag{5.32}$$

Then extension by zeroes of u is in $C^2(\overline{\Sigma}) \cap C^1(\mathbb{R}^n)$ *and the following statements hold true.*

Excess estimate for $\mathscr{P}$ *: If u solves* (5.32) *then for all* $v \in SBH(\mathbb{R}^2)$ *s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$

$$\mathscr{P}(v) - \mathscr{P}(u) \;\geq\; \beta\,\mathscr{H}^1(S_{Dv}) + \int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv] \otimes \nu_{S_{Dv}})\Big)\, d\mathscr{H}^1 . \tag{5.33}$$

Excess identity for minimizers of $\mathscr{P}$ *: If v minimize* $\mathscr{P}$ *among* $v \in SBH(\mathbb{R}^2)$ *such that* $\operatorname{spt} v \subset \overline{\Sigma}$ *and u solves* (5.32) *then*

$$\mathscr{P}(v) - \mathscr{P}(u) \;=\; \beta\,\mathscr{H}^1(S_{Dv}) + \frac{1}{2}\int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv] \otimes \nu_{S_{Dv}})\Big)\, d\mathscr{H}^1. \tag{5.34}$$

Necessary conditions for existence of creased minimizers of $\mathscr{P}$ *: If v minimize* $\mathscr{P}$ *among* $v \in SBH(\mathbb{R}^2)$ *s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$, $S_{Dv} \neq \emptyset$*, and u solves* (5.32)*, then*

$$\|\, Q'(D^2u)\|_{L^\infty\left(\Sigma, \mathscr{L}(\mathbb{R}^2,\mathbb{R}^2)\right)} > \gamma\,, \tag{5.35}$$

$$\int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv] \otimes \nu_{S_{Dv}})\Big)\, d\mathscr{H}^1 \leq -2\beta\,\mathscr{H}^1(S_{Dv}) < 0\,. \tag{5.36}$$

We emphasize that the excess estimate (5.33) *holds true also under weaker assumptions: Q convex and* C^2 *; while the excess identity* (5.34) *and its consequence, say the fact that* (5.35)*,* (5.36) *are necessary conditions for creased minimizers, require the quadratic structure* (5.12) *of Q.*

Proof By (5.25), (5.18), (5.32) and Lemma 13 we know: $D^2u \in C(\overline{\Sigma})$, $u \in H^2_0(\Sigma)$; hence $u \in C^2(\overline{\Sigma}) \cap C^1(\mathbb{R}^n)$. For simplicity, we will write shortly ν instead of $\nu_{S_{Dv}}$ in the proof.

By convexity of Q we get

$$\mathscr{P}(v) - \mathscr{P}(u) = \beta \mathscr{H}^1(S_{\nabla v}) + \int_\Sigma \mathrm{Q}(\nabla^2 v)\, dx + \gamma \int_{S_{Dv}} \left| \left[\frac{\partial v}{\partial \nu} \right] \right| d\mathscr{H}^1 + \\ - \int_\Sigma f v\, dx - \int_\Sigma \mathrm{Q}(D^2 u)\, dx + \int_\Sigma f u\, dx \geq \\ \geq \beta \mathscr{H}^1(S_{Dv}) + \gamma \int_{S_{\nabla v}} \left| \left[\frac{\partial v}{\partial \nu} \right] \right| d\mathscr{H}^1 - \int_\Sigma f(v-u)\, dx + \int_\Sigma \mathrm{Q}'(D^2 u) : (\nabla^2 v - D^2 u)\, dx. \tag{5.37}$$

Thanks to Lemma 13

$$D^2 u \in C^0(\overline{\Sigma}) \tag{5.38}$$

so that we can apply Lemma 11 with $w = u$ and $v \in \operatorname{argmin} \mathscr{P}$.
By (5.8), (5.20), (5.32), (5.38) and Theorems 2.15, 6.3, 6.4 of [11] we have:

$$\begin{aligned} \nabla^2 v &= D^2 v - [Dv] \otimes \nu\, d\mathscr{H}^1 \llcorner S_{Dv} \cap \Sigma = \\ &= D^2 v - \left[\frac{\partial v}{\partial \nu} \right] \otimes \nu\, d\mathscr{H}^1 \llcorner S_{Dv} \cap \Sigma \qquad \text{in } \mathscr{D}'(\Sigma), \end{aligned} \tag{5.39}$$

$$[Dv] \otimes \nu\, d\mathscr{H}^1 \llcorner S_{Dv} \cap \Sigma = \left[\frac{\partial v}{\partial \nu} \right] \otimes \nu\, d\mathscr{H}^1 \llcorner S_{Dv} \cap \Sigma \tag{5.40}$$

$$f = div\, div\ \mathrm{Q}'(D^2 u) \qquad \text{say} \qquad f = \frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 u \quad \text{if } \mathrm{Q} = \mathrm{Q}_{KL}.$$

Hence, integrating by parts twice and taking into account $u = v = 0$ on $\partial\Sigma$, we get

$$\begin{aligned} &\int_\Sigma \mathrm{Q}'(D^2 u) : (D^2 v - D^2 u)\, d\mathbf{x} = \\ &= - \int_\Sigma \operatorname{div}\ \mathrm{Q}'(D^2 u) \cdot D(v-u)\, d\mathbf{x} + \int_{\partial\Sigma} \mathrm{Q}'(D^2 u) : \left(\frac{\partial v}{\partial \nu_\Sigma} \nu_\Sigma \otimes \nu_\Sigma \right) = \\ &= \int_\Sigma \operatorname{div}\operatorname{div}\ \mathrm{Q}'(D^2 u) \cdot (v-u)\, d\mathbf{x} + \int_{\partial\Sigma} \mathrm{Q}'(D^2 u) : \left(\frac{\partial v}{\partial \nu_\Sigma} \nu_\Sigma \otimes \nu_\Sigma \right) = \\ &= \int_\Sigma f(v-u)\, d\mathbf{x} + \int_{\partial\Sigma} \mathrm{Q}'(D^2 u) : \left(\frac{\partial v}{\partial \nu_\Sigma} \nu_\Sigma \otimes \nu_\Sigma \right) \end{aligned} \tag{5.41}$$

where ν_Σ is the outward normal to $\partial\Sigma$. We choose $\nu = \nu_{S_{Dv}} = \nu_\Sigma$ on $\partial\Sigma \cap S_{Dv}$ and, abusing notation we define $\left(\frac{\partial v}{\partial \nu} \nu \otimes \nu\right) = \mathbb{O}$ on $\partial\Sigma \setminus S_{Dv}$; with this convention, by denoting $|_{out}$ and $|_{in}$ respectively the outer and inner traces at $\partial\Omega$ and taking into account that $\partial/\partial\nu_\Sigma$ stands for the inner trace of the derivative in the direction of outer normal, we get

$$\left[\frac{\partial v}{\partial \nu} \right] = \left. \frac{\partial v}{\partial \nu} \right|_{out} - \left. \frac{\partial v}{\partial \nu} \right|_{in} = - \left. \frac{\partial v}{\partial \nu} \right|_{in} = - \frac{\partial v}{\partial \nu_\Sigma} \quad \text{and} \quad \frac{\partial v}{\partial \nu_\Sigma} \nu_\Sigma \otimes \nu_\Sigma = - \left[\frac{\partial v}{\partial \nu} \right] \nu \otimes \nu$$

so that (5.41) reads as follows

$$\int_{\Sigma} \mathrm{Q}'(D^2u) : (D^2v - D^2u)\, d\mathbf{x} \; = \int_{\Sigma} f(v-u)\, d\mathbf{x} - \int_{\partial\Sigma} \mathrm{Q}'(D^2u) : \left(\left[\frac{\partial v}{\partial \nu}\right] \nu \otimes \nu\right). \tag{5.42}$$

By substituting (5.39) in (5.37) and taking into account (5.40), (5.42) and $Dv = \nabla v$ we get

$$\begin{aligned}
&\mathscr{P}(v) - \mathscr{P}(u) \;\geq\; \beta \mathscr{H}^1(S_{Dv}) + \int_{\Sigma} \mathrm{Q}'(D^2u) : (D^2v - D^2u) + \\
&\quad - \int_{\Sigma} f(v-u) + \gamma \int_{S_{Dv}\cap\Sigma} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 - \int_{S_{Dv}\cap\Sigma} \mathrm{Q}'(D^2u) : [\nabla v] \otimes \nu\, d\mathscr{H}^1 = \\
&= \beta \mathscr{H}^1(S_{Dv}) + \int_{\Sigma} f(v-u)\, d\mathbf{x} - \int_{\partial\Sigma} \mathrm{Q}'(D^2u) : \left(\left[\frac{\partial v}{\partial \nu}\right] \nu \otimes \nu\right) + \\
&\quad - \int_{\Sigma} f(v-u) + \gamma \int_{S_{Dv}\cap\Sigma} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 - \int_{S_{Dv}\cap\Sigma} \mathrm{Q}'(D^2u) : [\nabla v] \otimes \nu\, d\mathscr{H}^1 = \\
&= \beta \mathscr{H}^1(S_{Dv}) + \left(\int_{S_{Dv}} \gamma \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 - \int_{S_{Dv}} \mathrm{Q}'(D^2u) : [\nabla v] \otimes \nu\, d\mathscr{H}^1 \right) \\
&= \beta \mathscr{H}^1(S_{Dv}) + \left(\int_{S_{Dv}} \gamma \left|\left[\frac{\partial v}{\partial \nu}\right]\right| - \mathrm{Q}'(D^2u) : \left[\frac{\partial v}{\partial \nu}\right] \nu \otimes \nu\, d\mathscr{H}^1 \right)
\end{aligned} \tag{5.43}$$

so that (5.33) follows by(5.40). Since Q is a symmetric quadratic form we get

$$\mathrm{Q}(\mathbb{A}) - \mathrm{Q}(\mathbb{B}) = \frac{1}{2}(\mathrm{Q}'(\mathbb{A}) + \mathrm{Q}'(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \tag{5.44}$$

hence by using (5.39), (5.40), (5.42), (5.44) and eventually (5.22) we get (5.34) as follows:

$$\begin{aligned}
&\mathscr{P}(v) - \mathscr{P}(u) \;= \\
&= \int_{\Sigma} (\mathrm{Q}(\nabla^2 v) - \mathrm{Q}(D^2u))\, d\mathbf{x} - \int_{\Sigma} f(v-u)\, d\mathbf{x} + \gamma \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 + \beta \mathscr{H}^1(S_{Dv}) = \\
&\quad \frac{1}{2} \int_{\Sigma} (\mathrm{Q}'(\nabla^2 v) + \mathrm{Q}'(D^2u)) : (\nabla^2 v - D^2u) \\
&\quad - \int_{\Sigma} f(v-u)\, d\mathbf{x} + \gamma \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 + \beta \mathscr{H}^1(S_{Dv}) = \\
&= \frac{\gamma}{2} \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathscr{H}^1 - \frac{1}{2} \int_{\Sigma} f(v-u)\, d\mathbf{x} + \beta \mathscr{H}^1(S_{Dv}) \\
&\quad + \frac{1}{2} \int_{\Sigma} \mathrm{Q}'(D^2u) : (D^2v - D^2u) - \frac{1}{2} \int_{S_{Dv}\cap\Sigma} \mathrm{Q}'(D^2u) : ([Dv] \otimes \nu)\, d\mathscr{H}^1 \; = \\
&= \beta \mathscr{H}^1(S_{Dv}) + \frac{1}{2} \int_{S_{Dv}} \left\{ \gamma \left|\left[\frac{\partial v}{\partial \nu}\right]\right| - \mathrm{Q}'(D^2u) : ([Dv] \otimes \nu) \right\} d\mathscr{H}^1 .
\end{aligned} \tag{5.45}$$

Then (5.35) follows by (5.2), (5.34) and minimality of v, (5.36) follow from (5.34), (5.35). □

As a consequence of Lemma 14 we can prove the following result (announced in Th. 2.2 of [42], for the particular case of Kirchhoff-Love plate $\mathscr{P}_{KL}$) which states that the minimizers of (5.11) do not exhibit any plastic yield whenever the purely elastic solution has small second derivatives.

Lemma 15 (Bending Moment Regularity Condition for Clamped Plate) *Assume* (5.1), (5.2), (5.11), (5.12), (5.18), (5.25) *and the solution u of purely elastic problem* (5.20) *fulfils*

$$\| \, Q'(D^2u)\|_{L^\infty\left(\Sigma,\mathscr{L}(\mathbb{R}^2,\mathbb{R}^2)\right)} \leq \gamma \, . \tag{5.46}$$

Then $u \in \operatorname{argmin} \mathscr{P}(w)$ and u is the unique minimizer of $\mathscr{P}$.

Explicitly, in the case of Kirchhoff-Love plate (functional $\mathscr{P}_{KL}$, corresponding to $Q = Q_{KL}$) condition (5.46) *reads*

$$\left\| \mu\, D^2u \, + \, \frac{\mu\,\lambda}{\lambda + 2\mu}\,(Tr\, D^2u)\, I \right\|_{L^\infty(\Sigma)} \leq \frac{3}{4}\,\gamma \, . \tag{5.47}$$

Proof By (5.33) and (5.46) we get

$$\mathscr{P}(v) - \mathscr{P}(u) \;\geq\; \beta\mathscr{H}^1(S_{Dv}) + \left(\int_{S_{Dv}} \big(\gamma\,|[Dv]| - \,\mathrm{Q}'(D^2u) : ([Dv] \otimes \nu_{S_{Dv}})\big)\, d\mathscr{H}^1 \right) \geq$$

$$= \beta\mathscr{H}^1(S_{Dv}) + \left(\int_{S_{Dv}} (\gamma - \| \, \mathrm{Q}'(D^2u)\|_\infty)\,|[Dv]| \; d\mathscr{H}^1 \right) \geq \beta\mathscr{H}^1(S_{Dv}) \geq 0 \, .$$

and the last inequality is strict if $\mathscr{H}^1(S_{Dv}) > 0$. □

Proof of Theorem 10 Safe load condition $\|f\|_{L^1(\Sigma)} < 4\,\gamma$ entails existence of minimizers of $\mathscr{P}$ over $w \in SBH$ with $\operatorname{spt} w \subset \overline{\Sigma}$, thanks to Lemma 10. Inequalities (5.13), (5.19), (5.28) entail (5.46), hence we can apply Lemma 15 to achieve the claim. □

Lemma 16 (Euler Equation for $\mathscr{P}$) *Assume* (5.1), (5.2), (5.11), (5.12), (5.25), (5.18), (5.32) *and w minimizes $\mathscr{P}$ among v in $SBH(\mathbb{R}^2)$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$. *Then*

$$\operatorname{div}\operatorname{div}\, Q'(D^2w) \;=\; f \qquad\qquad \Sigma \setminus \overline{S_{Dw}} \, . \tag{5.48}$$

Proof Perform smooth variations with support in $\Sigma \setminus \overline{S_{Dw}}$. □

Proof of Theorem 9 Safe load condition $\|f\|_{L^1(\Sigma)} < 4\,\gamma$ together with Lemma 10 entail the existence of minimizers for $\mathscr{P}$. So we can apply Corollary 4.14 and Theorem 4.15 in [12] to any minimizer of $\mathscr{P}$ and get interior partial regularity in Σ, then repeat the technique of [18] in this simpler case (homogeneous Dirichlet datum, free discontinuity allowed only for derivatives) to prove partial regularity up to the boundary $\partial \Sigma$.□

Remark 7 About plastic yield lines analysis we emphasize the similarity of their properties with free discontinuity set in Blake & Zisserman functional in image segmentation: we refer to [17] and [19] for geometric properties of crease set, squared-hessian jump, stress concentration and asymptotic expansion around crease-tip of a minimizer.

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Stability Results for Abstract Evolution Equations with Intermittent Time-Delay Feedback

Cristina Pignotti

Abstract We consider abstract evolution equations with on–off time delay feedback. Without the time delay term, the model is described by an exponentially stable semigroup. We show that, under appropriate conditions involving the delay term, the system remains asymptotically stable. Under additional assumptions exponential stability results are also obtained. Concrete examples illustrating the abstract results are finally given.

Keywords Delay feedbacks • Evolution equations • Stabilization

1 Introduction

In this paper we study the stability properties of abstract evolution equations in presence of a time delay term.

In particular, we include into the model an on–off time delay feedback, i.e. the time delay is intermittently present.

Let $\mathscr{H}$ be a Hilbert space, with norm $\|\cdot\|$, and let $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ be a dissipative operator generating a C_0-semigroup $(S(t))_{t\geq 0}$ exponentially stable, namely there are two positive constants M and μ such that

$$\|S(t)\|_{\mathscr{L}(\mathscr{H})} \leq Me^{-\mu t}, \quad \forall\, t \geq 0, \tag{1}$$

where $\mathscr{L}(\mathscr{H})$ denotes the space of bounded linear operators from $\mathscr{H}$ into itself.

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We consider the following problem

$$\begin{cases} U_t(t) = \mathscr{A}U(t) + \mathscr{B}(t)U(t-\tau) & t > 0, \\ U(0) = U_0, \end{cases} \tag{2}$$

where τ, the time delay, is a fixed positive constant, the initial datum U_0 belongs to $\mathscr{H}$ and, for $t > 0$, $\mathscr{B}(t)$ is a bounded operator from $\mathscr{H}$ to $\mathscr{H}$.

In particular, we assume that there exists an increasing sequence of positive real numbers $\{t_n\}_n$, with $t_0 = 0$, such that

$$\begin{aligned} &1) \quad \mathscr{B}(t) = 0 \quad \forall\, t \in I_{2n} = [t_{2n}, t_{2n+1}), \\ &2) \quad \mathscr{B}(t) = \mathscr{B}_{2n+1} \quad \forall\, t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}). \end{aligned}$$

We denote $B_{2n+1} = \|\mathscr{B}_{2n+1}\|_{\mathscr{L}(\mathscr{H})}$, $n \in \mathbb{N}$, where the symbol $\mathbb{N}$ denotes the set of the natural numbers starting from 0. Moreover, denoted by T_n the length of the interval I_n, that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}, \tag{3}$$

we assume

$$T_{2n} \geq \tau, \quad \forall\, n \in \mathbb{N}. \tag{4}$$

Time delay effects are frequently present in applications and concrete models and it is now well-understood that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly stable in absence of delay (see e.g. [7, 8, 22, 30]).

We want to show that, under appropriate assumptions involving the delay feedback coefficients, the size of the time intervals where the delay appears and the parameters M and μ in (1), the considered model is asymptotically stable or exponentially stable, in spite of the presence of the time delay term.

Stability results for second-order evolution equations with intermittent damping were first studied by Haraux, Martinez and Vancostenoble [14], without any time delay term. They considered a model with intermittent on–off or with positive–negative damping and gave sufficient conditions ensuring that the behavior of the system in the time intervals with the standard dissipative damping, i.e. with positive coefficient, prevails over the *bad* behavior in remaining intervals where the damping is no present or it is present with the negative sign, namely as anti-damping. Therefore, asymptotic/exponential stability results were obtained.

More recently Nicaise and Pignotti [23, 24] considered second-order evolution equations with intermittent delay feedback. These results have been improved and extended to some semilinear equations in [9]. In the studied models, when the delay term (which possess a destabilizing effect) is not present, a not-delayed damping acts. Under appropriate sufficient conditions, stability results are then obtained. Related results for wave equations with intermittent delay feedback have been obtained, in 1-dimension, in [12, 13] and [3] by using a different approach based on the D'Alembert formula. However, this last approach furnishes stability results only for particular choices of the time delay.

In the recent paper [28], the intermittent delay feedback is compensated by a viscoelastic damping with exponentially decaying kernel.

The asymptotic behavior of wave-type equations with infinite memory and time delay feedback has been studied by Guesmia in [11] (cfr. [15]) via a Lyapunov approach and by Alabau-Boussouira et al. [2] by combining multiplier identities (cfr. [1]) and perturbative arguments.

We refer also to Day and Yang [6] for the same kind of problem in the case of finite memory. In these papers the authors prove exponential stability results if the coefficient of the delay damping is sufficiently small. These stability results could be easily extended to a variable coefficient $b(\cdot) \in L^\infty(0, +\infty)$ under a suitable *smallness* assumption on the L^∞-norm of $b(\cdot)$.

In [28], instead, asymptotic stability results are obtained without smallness conditions related to the L^∞-norm of the delay coefficient. On the other hand, the analysis is restricted to intermittent delay feedback. Asymptotic stability is proved when the coefficient of the delay feedback belongs to $L^1(0, +\infty)$ and the length of the time intervals where the delay is not present is sufficiently large. The same paper considers also problems with on–off anti-damping instead of a time delay feedback. Stability results are obtained even in this case under analogous assumptions.

The idea is here to generalize the results of [28] by considering abstract evolution equations for which, without considering the intermittent delay term, the associated operator generates an exponentially stable C_0-semigroup.

For such a class of evolution equations we already know that, under a suitable smallness condition on the delay feedback coefficient, an exponential stability result holds true (see [25]). We want to show that stability results are available also under a condition on the L^1-norm of the delay coefficient, without restriction on the pointwise L^∞-norm.

The paper is organized as follows. In Sect. 2 we give a well-posedness result. In Sects. 3 and 4 we prove asymptotic and exponential stability results, respectively, for the abstract model under appropriate conditions. Stability results are established also for a problem with intermittent anti-damping instead of delay feedback in Sect. 5. Finally, in Sect. 6, we give some concrete applications of the abstract results.

2 Well-posedness

In this section we illustrate a well-posedness results for problem (2).

Theorem 1 *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique (mild) solution $U \in C([0,\infty);\mathcal{H})$ of problem* (2). *Moreover,*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\mathcal{B}(s)U(s-\tau)\,ds. \tag{5}$$

Proof We prove the existence and uniqueness result on the interval $[0,t_2]$; then the global result follows by translation (cfr. [23]). In the time interval $[0,t_1]$, since $\mathcal{B}(t) = 0\ \forall\ t \in [0,t_1)$, then there exists a unique solution $U \in C([0,\tau],\mathcal{H})$ satisfying (5). The situation is different in the time interval $[t_1,t_2]$ where the delay feedback is present. In this case, we decompose the interval $[t_1,t_2]$ into the successive intervals $[t_1+j\tau, t_1+(j+1)\tau)$, for $j = 0,\dots,N-1$, where N is such that $t_1+(N+1)\tau \geq t_2$, and $[t_1+N\tau, t_2]$. Now, first we look at the problem on the interval $[t_1, t_1+\tau]$. Here $U(t-\tau)$ can be considered as a known function. Indeed, for $t \in [t_1,t_1+\tau]$, then $t-\tau \in [0,t_1]$, and we know the solution U on $[0,t_1]$ by the first step. Thus, problem (2) may be reformulated on $[t_1,t_1+\tau]$ as

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + g_0(t) & t \in (\tau, 2\tau), \\ U(\tau) = U(\tau_-), \end{cases} \tag{6}$$

where $g_0(t) = \mathcal{B}(t)U(t-\tau)$. This problem has a unique solution $U \in C([\tau,2\tau],\mathcal{H})$ (see e.g. Th. 1.2, Ch. 6 of [27]) given by

$$U(t) = S(t-\tau)U(\tau_-) + \int_\tau^t S(t-s)g_0(s)\,ds, \quad \forall\ t \in [\tau,2\tau].$$

Proceedings analogously in the successive time intervals $[t_1+j\tau, t_1+(j+1)\tau)$, we obtain a solution on $[0,t_2]$. ■

3 Asymptotic Stability Results

Let T^* be defined as

$$T^* := \frac{1}{\mu}\ln M, \tag{7}$$

where M and μ are the constants in (1), that is T^* is the time for which $Me^{-\mu T^*} = 1$.

We can state a first estimate on the intervals I_{2n} where the delay feedback is not present.

Proposition 1 *Assume $T_{2n} > T^*$. Then, there exists a constant $c_n \in (0, 1)$ such that*

$$\|U(t_{2n+1})\|^2 \le c_n \|U(t_{2n})\|^2, \tag{8}$$

for every solution of problem (2).

Proof Observe that in the time interval $I_{2n} = [t_{2n}, t_{2n+1}]$ the delay feedback is not present since $\mathscr{B}(t) \equiv 0$. Thus, (8) easily follows from (1) with $\sqrt{c_n} = Me^{-\mu T_{2n}} < Me^{-\mu T^*} = 1$. ■

Let us now introduce the Lyapunov functional

$$F(t) = F(U, t) := \frac{1}{2}\|U(t)\|^2 + \frac{1}{2}\int_{t-\tau}^{t} \|\mathscr{B}(s+\tau)\|_{\mathscr{L}(\mathscr{H})}\|U(s)\|^2\, ds. \tag{9}$$

Proposition 2 *Assume* 1), 2). *Moreover, assume $T_{2n} \ge \tau, \ \forall\, n \in \mathbb{N}$. Then,*

$$F'(t) \le B_{2n+1}\|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \ \forall\, n \in \mathbb{N}. \tag{10}$$

for any solution of problem (2).

Proof By differentiating the energy $F(\cdot)$, we have

$$F'(t) = \langle U(t), \mathscr{A}U(t)\rangle + \langle U(t), \mathscr{B}(t)U(t-\tau)\rangle + \frac{1}{2}\|\mathscr{B}(t+\tau)\|_{\mathscr{L}(\mathscr{H})}\|U(t)\|^2 \\ -\frac{1}{2}\|\mathscr{B}(t)\|_{\mathscr{L}(\mathscr{H})}\|U(t-\tau)\|^2.$$

Then, since the operator $\mathscr{A}$ is dissipative, one can estimate

$$F'(t) \le \|\mathscr{B}(t)\|_{\mathscr{L}(\mathscr{H})}\|U(t)\|\|U(t-\tau)\| + \frac{1}{2}\|\mathscr{B}(t+\tau)\|_{\mathscr{L}(\mathscr{H})}\|U(t)\|^2 \\ -\frac{1}{2}\|\mathscr{B}(t)\|_{\mathscr{L}(\mathscr{H})}\|U(t-\tau)\|^2. \tag{11}$$

Therefore, from Cauchy–Schwarz inequality,

$$F'(t) \le \frac{1}{2}\|\mathscr{B}(t)\|_{\mathscr{L}(\mathscr{H})}\|U(t)\|^2 + \frac{1}{2}\|\mathscr{B}(t+\tau)\|_{\mathscr{L}(\mathscr{H})}\|U(t)\|^2.$$

Now, observe that, since $T_{2n} \ge \tau$, for every $n \in \mathbb{N}$, if t belongs to I_{2n+1} then $t+\tau$ belongs to I_{2n+1} or to I_{2n+2}. In the first case $\|\mathscr{B}(t+\tau)\|_{\mathscr{L}(\mathscr{H})} = B_{2n+1}$ while, in the second case $\|\mathscr{B}(t+\tau)\|_{\mathscr{L}(\mathscr{H})} = 0$. Thus (10) is proved. ■

Theorem 2 *Assume* 1), 2) *and* $T_{2n} \geq \tau$ *for all* $n \in \mathbb{N}$. *Moreover assume* $T_{2n} > T^*$, *for all* $n \in \mathbb{N}$, *where* T^* *is the time defined in* (7). *Then, if*

$$\sum_{n=0}^{\infty} \ln \left[e^{2B_{2n+1}T_{2n+1}} (c_n + T_{2n+1}B_{2n+1}) \right] = -\infty, \tag{12}$$

Eq. (2) *is asymptotically stable, namely any solution* U *of* (2) *satisfies* $\|U(t)\| \to 0$ *for* $t \to +\infty$.

Proof Note that from (10) we obtain

$$F'(t) \leq 2B_{2n+1}F(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \; n \in \mathbb{N}.$$

Then, by integrating on the time interval I_{2n+1},

$$F(t) \leq e^{2B_{2n+1}T_{2n+1}} F(t_{2n+1}), \quad \forall\, t \in [t_{2n+1}, t_{2n+2}], \; \forall\, n \in \mathbb{N}. \tag{13}$$

From the definition of the Lyapunov functional F,

$$F(t_{2n+1}) = \frac{1}{2}\|U(t_{2n+1})\|^2 + \frac{1}{2}\int_{t_{2n+1}-\tau}^{t_{2n+1}} \|\mathscr{B}(s+\tau)\|_{\mathscr{L}(H)} \|U(s)\|^2 ds. \tag{14}$$

Note that, for $t \in [t_{2n+1} - \tau, t_{2n+1})$, then $t + \tau \in [t_{2n+1}, t_{2n+1} + \tau)$ and therefore, since $|I_{2n+2}| \geq \tau$ it results $t + \tau \in I_{2n+1} \cup I_{2n+2}$. Now, if $t + \tau \in I_{2n+2}$, then $\mathscr{B}(t + \tau) = 0$. Otherwise, if $t + \tau \in I_{2n+1}$, then $\|\mathscr{B}(t + \tau)\| = B_{2n+1}$. Then, from (14) we deduce

$$F(t_{2n+1}) = \frac{1}{2}\|U(t_{2n+1})\|^2 + \frac{1}{2}B_{2n+1}\int_{t_{2n+1}-\tau}^{\min(t_{2n+2}-\tau, t_{2n+1})} \|U(s)\|^2 ds, \tag{15}$$

since if $t_{2n+1} > t_{2n+2} - \tau$, then $\mathscr{B}(t) = 0$ for all $t \in [t_{2n+2}, t_{2n+1} + \tau) \subset [t_{2n+2}, t_{2n+3})$.

Then, since $\|U(\cdot)\|$ is decreasing in the intervals I_{2n} (the operator $\mathscr{A}$ is dissipative and $\mathscr{B}(t) \equiv 0$), we deduce

$$\begin{aligned} F(t_{2n+1}) &\leq \frac{1}{2}\|U(t_{2n+1})\|^2 + \frac{1}{2}T_{2n+1}B_{2n+1}\|U(t_{2n+1} - \tau)\|^2 \\ &\leq \frac{1}{2}\|U(t_{2n+1})\|^2 + \frac{1}{2}T_{2n+1}B_{2n+1}\|U(t_{2n})\|^2. \end{aligned} \tag{16}$$

Using this last estimate in (13), we obtain

$$\|U(t_{2n+2})\|^2 \leq 2F(t_{2n+2}) \leq e^{2B_{2n+1}T_{2n+1}}(c_n + T_{2n+1}B_{2n+1})\|U(t_{2n})\|^2, \quad \forall\, n \in \mathbb{N}, \tag{17}$$

where we have used also the estimate (8). By iterating this argument we arrive at

$$\|U(t_{2n+2})\|^2 \le \Pi_{k=0}^{n} e^{2B_{2k+1}T_{2k+1}}(c_k + T_{2k+1}B_{2k+1})\|U_0\|^2, \quad \forall\, n \in \mathbb{N}. \tag{18}$$

Now observe that $\|U(t)\|$ is not decreasing in the whole $(0, +\infty)$. However, it is decreasing for $t \in [t_{2n}, t_{2n+1})$, $n \in \mathbb{N}$, where the destabilizing delay feedback does not act and so

$$\|U(t)\| \le \|U(t_{2n})\|, \quad \forall\, t \in [t_{2n}, t_{2n+1}). \tag{19}$$

Moreover, from (16), for $t \in [t_{2n+1}, t_{2n+2})$ we have

$$\|U(t)\|^2 \le 2F(t) \le e^{2B_{2n+1}T_{2n+1}}(c_n + B_{2n+1}T_{2n+1})\|U(t_{2n})\|^2, \tag{20}$$

where in the second inequality we have used (8).

Then, we have asymptotic stability if

$$\Pi_{k=0}^{n} e^{2B_{2k+1}T_{2k+1}}(c_k + T_{2k+1}B_{2k+1}) \longrightarrow 0, \quad \text{for } n \to \infty,$$

or equivalently

$$\ln\left[\Pi_{k=0}^{n} e^{2B_{2k+1}T_{2k+1}}(c_k + T_{2k+1}B_{2k+1})\right] \longrightarrow -\infty, \quad \text{for } n \to \infty,$$

namely under the assumption (12). This concludes the proof. ■

Remark 1 In particular, (12) is verified if the following conditions are satisfied:

$$\sum_{n=0}^{\infty} B_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln c_n = -\infty. \tag{21}$$

Indeed, it is easy to see that (21) is equivalent to

$$\sum_{n=0}^{\infty} B_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln(c_n + B_{2n+1}T_{2n+1}) = -\infty \tag{22}$$

and that (22) implies (12).

Therefore, from (21), we have stability if $\|\mathscr{B}(t)\| \in L^1(0, +\infty)$ and, for instance, the length of the *good* intervals I_{2n} is greater than a fixed time $\bar{T}$, $\bar{T} > T^*$ and $\bar{T} \ge \tau$, namely

$$T_{2n} \ge \bar{T}, \quad \forall\, n \in \mathbb{N}.$$

Indeed, in this case there exists $\bar{c} \in (0, 1)$ such that $0 < c_n < \bar{c}$.

If we assume that the length of the delay intervals, namely the time intervals where the delay feedback is present, is lower than the time delay τ, that is

$$T_{2n+1} \leq \tau, \quad \forall n \in \mathbb{N}. \tag{23}$$

we can prove another asymptotic stability result which is, in some sense, complementary to the previous one.

In this case we can directly work with $\|U(t)\|$ instead of passing through the function $F(\cdot)$. We can give the following preliminary estimates on the time intervals I_{2n+1}, $n \in \mathbb{N}$.

Proposition 3 *Assume* 1), 2). *Moreover assume* $T_{2n+1} \leq \tau$ *and* $T_{2n} \geq \tau$, $\forall\, n \in \mathbb{N}$. *Then, for* $t \in I_{2n+1}$,

$$\frac{d}{dt}\|U(t)\|^2 \leq B_{2n+1}\|U(t)\|^2 + B_{2n+1}\|U(t_{2n})\|^2. \tag{24}$$

Proof By differentiating $\|U(t)\|^2$ we get

$$\frac{d}{dt}\|U(t)\|^2 = 2\langle U(t), \mathscr{A}U(t)\rangle + 2\langle U(t), \mathscr{B}(t)U(t-\tau)\rangle.$$

Then, by using the dissipativity of the operator $\mathscr{A}$,

$$\frac{d}{dt}\|U(t)\|^2 \leq 2\langle U(t), \mathscr{B}(t)U(t-\tau)\rangle.$$

Hence, from 2),

$$\frac{d}{dt}\|U(t)\|^2 \leq B_{2n+1}\|U(t)\|^2 + B_{2n+1}\|U(t-\tau)\|^2.$$

We can now conclude observing that since $T_{2n+1} \leq \tau$ and $T_{2n} \geq \tau$, then for $t \in I_{2n+1}$ it is $t-\tau \in I_{2n}$. Then, since $\|U(t)\|$ is decreasing in I_{2n}, the estimate in the statement is proved. ∎

The stability result follows.

Theorem 3 *Assume* 1), 2), $T_{2n+1} \leq \tau$ *and* $T_{2n} \geq \tau$, $\forall\, n \in \mathbb{N}$. *Moreover assume* $T_{2n} > T^*$, *for all* $n \in \mathbb{N}$, *where* T^* *is the time defined in* (7). *If*

$$\sum_{n=0}^{\infty} \ln\left[e^{B_{2n+1}T_{2n+1}}(c_n + 1 - e^{-B_{2n+1}T_{2n+1}})\right] = -\infty, \tag{25}$$

then every solution U *of* (2) *satisfies* $\|U(t)\| \to 0$ *for* $t \to +\infty$.

Proof For $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, from estimate (24) we have

$$\|U(t)\|^2 \le e^{B_{2n+1}(t-t_{2n+1})}\Big\{\|U(t_{2n+1})\|^2 + B_{2n+1}\int_{t_{2n+1}}^{t} \|U(t_{2n})\|^2 e^{-B_{2n+1}(s-t_{2n+1})}ds\Big\}.$$

Then we deduce

$$\|U(t)\|^2 \le e^{B_{2n+1}T_{2n+1}}\|U(t_{2n+1})\|^2 + e^{B_{2n+1}(t-t_{2n+1})}\|U(t_{2n})\|^2\Big[1 - e^{-B_{2n+1}(t-t_{2n+1})}\Big],$$

and therefore

$$\|U(t)\|^2 \le e^{B_{2n+1}T_{2n+1}}\|U(t_{2n+1})\|^2 + e^{B_{2n+1}T_{2n+1}}\|U(t_{2n})\|^2 - \|U(t_{2n})\|^2,$$

for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}),\ n \in \mathbb{N}$.

Now we use the estimate (8) obtaining

$$\|U(t_{2n+2})\|^2 \le e^{B_{2n+1}T_{2n+1}}\left(c_n + 1 - e^{-B_{2n+1}T_{2n+1}}\right)\|U(t_{2n})\|^2, \quad n \in \mathbb{N}.$$

Thus,

$$\|U(t_{2n+2})\| \le \Big[\Pi_{k=0}^{n} e^{B_{2k+1}T_{2k+1}}(c_k + 1 - e^{-B_{2k+1}T_{2k+1}})\Big]^{\frac{1}{2}}\|U_0\|. \tag{26}$$

Then the asymptotic stability result follows if

$$\Pi_{k=0}^{n} e^{B_{2k+1}T_{2k+1}}\left(c_k + 1 - e^{-B_{2k+1}T_{2k+1}}\right) \to 0, \quad \text{for } n \to \infty,$$

namely if

$$\sum_{n=0}^{\infty} \ln\Big[e^{B_{2n+1}T_{2n+1}}(c_n + 1 - e^{-B_{2n+1}T_{2n+1}})\Big] \to -\infty, \quad \text{for } n \to \infty. \qquad \blacksquare$$

Remark 2 Observe that, when the odd intervals I_{2n+1} have length lower or equal than the time delay τ, the assumption (25) is a bit less restrictive than (12). Indeed,

$$e^{B_{2n+1}T_{2n+1}}(c_n + 1 - e^{-B_{2n+1}T_{2n+1}}) < e^{2B_{2n+1}T_{2n+1}}(c_n + B_{2n+1}T_{2n+1}), \quad \forall n \in \mathbb{N}.$$

Remark 3 Arguing as in Remark 1 we can show that condition (25) is verified, in particular, if (21) holds true.

4 Exponential Stability

Under additional assumptions on the coefficients T_n, B_{2n+1}, c_n, exponential stability results hold true.

Theorem 4 *Assume* 1), 2). *Moreover, assume*

$$T_{2n} = T^0 \quad \forall\, n \in \mathbb{N}, \tag{27}$$

with $T^0 \ge \tau$ and $T^0 > T^*$, where T^* is the constant defined in (7)*,*

$$T_{2n+1} = \tilde{T} \quad \forall\, n \in \mathbb{N} \tag{28}$$

and

$$\sup_{n\in\mathbb{N}} e^{2B_{2n+1}\tilde{T}}(c + B_{2n+1}\tilde{T}) = d < 1, \tag{29}$$

where $c = Me^{-\mu T^0}$. Then, there exist two positive constants C, α such that

$$\|U(t)\| \le Ce^{-\alpha t}\|U_0\|, \quad t > 0, \tag{30}$$

for any solution of problem (2).

Proof Note that, from the definition of the constant c, estimate (8) holds with $c_n = c,\ \forall\, n \in \mathbb{N}$. Thus, from (29) and (17) we obtain

$$\|U(T^0 + \tilde{T})\| \le d^{\frac{1}{2}}\|U_0\|,$$

and then,

$$\|U(n(T^0 + \tilde{T}))\| \le d^{\frac{n}{2}}\|U_0\|, \quad \forall n \in \mathbb{N}.$$

Therefore, $\|U(t)\|$ satisfies an exponential estimate like (30) (see Lemma 1 of [12]). ■

Concerning the case of *small* delay intervals, namely $|I_{2n+1}| \le \tau,\ \forall n \in \mathbb{N}$, one can state the following asymptotic stability result.

Theorem 5 *Assume* 1), 2). *Moreover assume*

$$T_{2n} = T^0 \quad \forall\, n \in \mathbb{N},$$

with $T^0 \ge \tau$ and $T^0 > T^*$, where the time T^* is defined in (7)*,*

$$T_{2n+1} = \tilde{T}, \quad \text{with } \tilde{T} \le \tau \quad \forall\, n \in \mathbb{N} \tag{31}$$

and

$$\sup_{n\in\mathbb{N}} e^{B_{2n+1}\tilde{T}}(c+1-e^{-B_{2n+1}\tilde{T}}) = d < 1, \tag{32}$$

where $c = Me^{-\mu T^0}$. *Then, there exist two positive constants* C, α *such that*

$$\|U(t)\| \le Ce^{-\alpha t}\|U_0\|, \quad t > 0, \tag{33}$$

for any solution of (2).

Proof The proof is analogous to the one of Theorem 4. ∎

5 Intermittent Anti-damping

With analogous technics we can also deal with an intermittent anti-damping term. More precisely, let us consider the model

$$\begin{cases} U_t(t) = \mathscr{A}U(t) + \mathscr{B}(t)U(t) \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{34}$$

where τ is the time delay, the initial datum U_0 belongs to $\mathscr{H}$ and, for $t > 0$, $\mathscr{B}(t)$ is a bounded operator from $\mathscr{H}$ such that

$$\langle \mathscr{B}(t)U, U\rangle \ge 0, \quad \forall\, U \in \mathscr{H}.$$

Thus $\mathscr{B}(t)U(t)$ is an anti-damping term (cfr. [14]). In particular we consider an intermittent feedback, that is we assume that there exists an increasing sequence of positive real numbers $\{t_n\}_n$, with $t_0 = 0$, such that

3) $\mathscr{B}(t) = 0 \quad \forall\, t \in I_{2n} = [t_{2n}, t_{2n+1})$,
4) $\mathscr{B}(t) = \mathscr{D}_{2n+1} \quad \forall\, t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$.

We denote $D_{2n+1} = \|\mathscr{D}_{2n+1}\|_{\mathscr{L}(\mathscr{H})}$, $n \in \mathbb{N}$.

As before, denote by T_n the length of the interval I_n, that is

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}.$$

Note that Proposition 1, which gives an observability estimate on the intervals I_{2n} where the anti-damping is not present, still holds true. Concerning the time intervals I_{2n+1} where the anti-damping acts one can obtain the following estimate.

Proposition 4 *Assume* 3) *and* 4). *For every solution of problem* (34),

$$\frac{d}{dt}\|U(t)\|^2 \le 2D_{2n+1}\|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \ \ \forall\, n \in \mathbb{N}.$$

Proof Being $\mathscr{A}$ dissipative, the estimate follows immediately from 3). ■

From Proposition 4 we deduce an asymptotic stability result.

Theorem 6 *Assume* 3), 4). *Moreover assume* $T_{2n} > T^*$, *for all* $n \in \mathbb{N}$, *where* T^* *is the time defined in* (7). *If*

$$\sum_{n=0}^{\infty} \ln\left(e^{2D_{2n+1}T_{2n+1}} c_n\right) = -\infty, \tag{35}$$

then the problem (34) *is asymptotically stable, that is any solution* U *of* (34) *satisfies* $\|U(t)\| \to 0$ *for* $t \to +\infty$.

Proof From Proposition 4 we have the estimate

$$\frac{d}{dt}\|U(t)\|^2 \le 2D_{2n+1}\|U(t)\|^2, \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}], \ \ \forall\, n \in \mathbb{N}.$$

This implies

$$\|U(t_{2n+2})\|^2 \le e^{2D_{2n+1}T_{2n+1}}\|U(t_{2n+1})\|^2, \quad \forall\, n \in \mathbb{N}. \tag{36}$$

Then, from estimate (8) which is always valid of course in the time intervals without damping,

$$\|U(t_{2n+2})\|^2 \le e^{2D_{2n+1}T_{2n+1}} c_n \|U(t_{2n})\|^2, \quad \forall\, n \in \mathbb{N}. \tag{37}$$

By repeating this argument we obtain

$$\|U(t_{2n+2})\|^2 \le \Pi_{k=0}^{n} e^{2D_{2k+1}T_{2k+1}} c_k \|U_0\|^2, \quad \forall\, n \in \mathbb{N}. \tag{38}$$

Therefore, asymptotic stability is ensured if

$$\Pi_{k=0}^{n} e^{2D_{2k+1}T_{2k+1}} c_k \longrightarrow 0, \quad \text{for } n \to \infty,$$

or equivalently

$$\ln\left(\Pi_{k=0}^{n} e^{2D_{2k+1}T_{2k+1}} c_k\right) \longrightarrow -\infty, \quad \text{for } n \to \infty.$$

This concludes. ■

Remark 4 In particular (35) is verified under the following assumptions:

$$\sum_{n=0}^{\infty} D_{2n+1}T_{2n+1} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \ln c_n = -\infty. \tag{39}$$

Under additional assumptions on the problem coefficients T_n, D_{2n+1}, c_n, an exponential stability result holds.

Theorem 7 *Assume* 3), 4) *and*

$$T_{2n} = T^0 \quad \forall\, n \in \mathbb{N}, \tag{40}$$

with $T^0 > T^*$, where the time T^* is defined in (7). *Assume also that*

$$T_{2n+1} = \tilde{T} \quad \forall\, n \in \mathbb{N} \tag{41}$$

and

$$\sup_{n\in\mathbb{N}} e^{2D_{2n+1}\tilde{T}} c = d < 1, \tag{42}$$

where, $c = Me^{-\mu T^0}$. Then, there exist two positive constants C, α such that

$$\|U(t)\| \le Ce^{-\alpha t}\|U_0\|, \quad t > 0, \tag{43}$$

for any solution of problem (34).

6 Concrete Examples

In this section we illustrate some examples falling into the previous abstract setting.

6.1 Viscoelastic Wave Type Equation

Let H be a real Hilbert space and let $A : \mathcal{D}(A) \to H$ be a positive self-adjoint operator with a compact inverse in H. Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$.

Let us consider the problem

$$u_{tt}(x,t) + Au(x,t) - \int_0^\infty \mu(s)Au(x,t-s)ds + b(t)u_t(x,t-\tau) = 0 \quad t > 0, \tag{44}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{45}$$

$$u(x,t) = u_0(x,t) \quad \text{in} \quad \Omega \times (-\infty, 0]; \tag{46}$$

where the initial datum u_0 belongs to a suitable space, the constant $\tau > 0$ is the time delay, and the memory kernel $\mu : [0, +\infty) \to [0, +\infty)$ satisfies

i) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$;
ii) $\mu(0) = \mu_0 > 0$;
iii) $\int_0^{+\infty} \mu(t)dt = \tilde{\mu} < 1$;
iv) $\mu'(t) \leq -\delta\mu(t), \quad$ for some $\delta > 0$.

Moreover, the function $b(\cdot) \in L^\infty_{loc}(0, +\infty)$ is a function which is zero intermittently. That is, we assume that for all $n \in \mathbb{N}$ there exists $t_n > 0$, with $t_0 = 0$ and $t_n < t_{n+1}$, such that

$$
\begin{aligned}
&1_w) \quad b(t) = 0 \quad \forall\, t \in I_{2n} = [t_{2n}, t_{2n+1}),\\
&2_w) \quad |b(t)| \leq b_{2n+1} \neq 0 \quad \forall\, t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}).
\end{aligned}
$$

Stability result for the above problem were firstly obtained in [28]. We want to show that they can also be obtained as particular case of previous abstract setting.

To this aim, following Dafermos [5], we can introduce the new variable

$$
\eta^t(x, s) := u(x, t) - u(x, t - s). \tag{47}
$$

Then, problem (44)–(46) may be rewritten as

$$
\begin{aligned}
u_{tt}(x, t) = -(1 - \tilde{\mu})Au(x, t) - \int_0^\infty \mu(s)A\eta^t(x, s)ds&\\
-b(t)u_t(x, t - \tau) \quad \text{in } \Omega \times (0, +\infty),& \quad (48)
\end{aligned}
$$

$$
\eta^t_t(x, s) = -\eta^t_s(x, s) + u_t(x, t) \quad \text{in } \Omega \times (0, +\infty) \times (0, +\infty), \tag{49}
$$

$$
u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{50}
$$

$$
\eta^t(x, s) = 0 \quad \text{in } \partial\Omega \times (0, +\infty),\ t \geq 0, \tag{51}
$$

$$
u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \tag{52}
$$

$$
\eta^0(x, s) = \eta_0(x, s) \quad \text{in } \Omega \times (0, +\infty), \tag{53}
$$

where

$$
\begin{aligned}
&u_0(x) = u_0(x, 0), \quad x \in \Omega,\\
&u_1(x) = \tfrac{\partial u_0}{\partial t}(x, t)|_{t=0}, \quad x \in \Omega,\\
&\eta_0(x, s) = u_0(x, 0) - u_0(x, -s), \quad x \in \Omega,\ s \in (0, +\infty).
\end{aligned} \tag{54}
$$

Set $L^2_\mu((0, \infty); V)$ the Hilbert space of V-valued functions on $(0, +\infty)$, endowed with the inner product

$$
\langle \varphi, \psi \rangle_{L^2_\mu((0,\infty);V)} = \int_0^\infty \mu(s)\langle A^{1/2}\varphi(s), A^{1/2}\psi(s)\rangle_H ds.
$$

Let $\mathscr{H}$ be the Hilbert space

$$\mathscr{H} = V \times H \times L^2_\mu((0,\infty);V),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathscr{H}} := (1-\tilde{\mu})\langle A^{1/2}u, A^{1/2}\tilde{u}\rangle_H + \langle v, \tilde{v}\rangle_H + \int_0^\infty \mu(s)\langle A^{1/2}w, A^{1/2}\tilde{w}\rangle_H ds. \tag{55}$$

Denoting by U the vector $U = (u, u_t, \eta)$, the above problem can be rewritten in the form (2), where $\mathscr{B}U = B(u, v, \eta) = (0, -bv, 0)$ and $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} := \begin{pmatrix} v \\ -(1-\tilde{\mu})Au - \int_0^\infty \mu(s)Aw(s)ds \\ -w_s + v \end{pmatrix}, \tag{56}$$

with domain (cfr. [26])

$$\mathscr{D}(\mathscr{A}) := \Big\{ (u, v, \eta)^T \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2_\mu((0,+\infty); H^1_0(\Omega)) \ : \ (1-\tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H^1_0(\Omega), \ \eta_s \in L^2_\mu((0,+\infty); H^1_0(\Omega)) \Big\}. \tag{57}$$

It has been proved in [10] that the above system is exponentially stable, namely that the operator $\mathscr{A}$ generates a strongly continuous semigroup satisfying the estimate (1), for suitable constants. Moreover, it is well-known that, the operator $\mathscr{A}$ is dissipative. Therefore, our previous results apply to this model.

As a concrete example we can consider the wave equation with memory. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$. Let us consider the initial boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty \mu(s)\Delta u(x,t-s)ds + b(t)u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{58}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty), \tag{59}$$

$$u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty, 0]. \tag{60}$$

This problem enters in previous form (44)–(46), if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathscr{D}(A) \to H \; : \; u \to -\Delta u,$$

where $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathscr{D}(A^{1/2}) = H_0^1(\Omega)$.

Under the same conditions that before on the memory kernel $\mu(\cdot)$ and on the function $b(\cdot)$, previous asymptotic/exponential stability results are valid. The case b constant has been studied in [2]. In particular, we have proved that the exponential stability is preserved, in presence of the delay feedback, if the coefficient b of this one is sufficiently small. The choice b constant was made only for the sake of clearness. The result in [2] remains true if instead of b constant we consider $b = b(t)$, under a suitable smallness condition on the L^∞-norm of $b(\cdot)$. On the contrary here we give stability results without restrictions on the L^∞-norm of $b(\cdot)$, even if only for on–off $b(\cdot)$.

Our results also apply to Petrovsky system with viscoelastic damping with Dirichlet and Neumann boundary conditions:

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \int_0^\infty \mu(s)\Delta^2 u(x,t-s)ds + b(t)u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{61}$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,+\infty), \tag{62}$$

$$u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty, 0]. \tag{63}$$

This problem enters into the previous abstract framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathscr{D}(A) \to H \; : \; u \to \Delta^2 u,$$

where $\mathscr{D}(A) = H_0^2(\Omega) \cap H^4(\Omega)$, with

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) \; : \; v = \frac{\partial v}{\partial \nu} = 0 \; \text{ on } \; \partial\Omega \right\}.$$

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathscr{D}(A^{1/2}) = H_0^2(\Omega)$.

Therefore, under the same conditions that before on the memory kernel $\mu(\cdot)$ and on the function $b(\cdot)$, previous asymptotic/exponential stability results are valid.

6.2 Locally Damped Wave Equation

Here we consider the wave equation with local internal damping and intermittent delay feedback. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary $\partial\Omega$ of class C^2. Denoting by m the standard multiplier $m(x) = x - x_0,\ x_0 \in \mathbb{R}^n$, let ω_1 be the intersection of Ω with an open neighborhood of the subset of $\partial\Omega$

$$\Gamma_0 = \{\, x \in \partial\Omega \ :\ m(x) \cdot \nu(x) > 0 \,\}, \tag{64}$$

where $\nu(x)$ denotes the outer unit normal vector to $x \in \partial\Omega$. Fixed any subset $\omega_2 \subseteq \Omega$, let us consider the initial boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_{\omega_1} u_t(x,t) + b(t)\chi_{\omega_2} u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{65}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{66}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{67}$$

where χ_{ω_i} denotes the characteristic function of ω_i, $i = 1, 2$, a is a positive number and b in $L^\infty(0,+\infty)$ is an on–off function satisfying (1_w) and (2_w) of Sect. 6.1. The initial datum (u_0, u_1) belongs to $H_0^1(\Omega) \times L^2(\Omega)$.

This problem enters into our previous framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathscr{D}(A) \to H \ :\ u \to -\Delta u,$$

where $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

Now, denoting $U = (u, u_t)$, the problem can be restated in the abstract form (2) where $\mathscr{B}U = B(u,v) = (0, -b(t)\chi_{\omega_2} v)$ and $\mathscr{A}$ is defined by

$$\mathscr{A} \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ -Au - a\chi_{\omega_1} v \end{pmatrix}, \tag{68}$$

with domain $\mathscr{D}(A) \times L^2(\Omega)$ in the Hilbert space $\mathscr{H} = H \times H$.

Concerning the part without delay feedback, namely the locally damped wave equation

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_{\omega_1} u_t(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty), \tag{69}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,+\infty), \tag{70}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{71}$$

it is well-known that, under the previous Lions geometric condition on the set ω_1 (or under the more general assumption of control geometric property [4]) where the frictional damping is localized, an exponential stability result holds (see e.g. [4, 16–21, 29, 31]). This is equivalent to say that the strongly continuous semigroup generated by the operator $\mathscr{A}$ associated to (69)–(71), namely the one defined in (68), satisfies (1). As well-known, the operator $\mathscr{A}$ is dissipative. Thus previous abstract stability results are valid also for this model. We emphasize the fact that the set ω_2 may be any subset of Ω, not necessarily a subset of ω_1. On the contrary, in previous stability results for damped wave equation and intermittent delay feedback (see e.g. [9, 24]) the set ω_2 has to be a subset of ω_1. On the other hand, now the standard (not delayed) frictional damping is always present in time while in the quoted papers it is on–off like the delay feedback and it acts only on the complementary time intervals with respect to this one.

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From Visco-Energetic to Energetic and Balanced Viscosity Solutions of Rate-Independent Systems

Riccarda Rossi and Giuseppe Savaré

Abstract This paper focuses on weak solvability concepts for rate-independent systems in a metric setting. *Visco-Energetic* solutions have been recently obtained by passing to the time-continuous limit in a time-incremental scheme, akin to that for Energetic solutions, but perturbed by a 'viscous' correction term, as in the case of Balanced Viscosity solutions. However, for Visco-Energetic solutions this viscous correction is tuned by a *fixed* parameter μ. The resulting solution notion is characterized by a stability condition and an energy balance analogous to those for Energetic solutions, but, in addition, it provides a fine description of the system behavior at jumps as Balanced Viscosity solutions do. Visco-Energetic evolution can be thus thought as 'in-between' Energetic and Balanced Viscosity evolution. Here we aim to formalize this intermediate character of Visco-Energetic solutions by studying their singular limits as $\mu \downarrow 0$ and $\mu \uparrow \infty$. We shall prove convergence to Energetic solutions in the former case, and to Balanced Viscosity solutions in the latter situation.

Keywords Balanced viscosity solutions • Energetic solutions • Rate-independent systems • Singular limits • Time discretization • Vanishing viscosity • Visco-Energetic solutions

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1 Introduction

A large class of *rate-independent systems* are driven by

- a time-dependent energy functional $\mathscr{E} : [0,T] \times X \to (-\infty,\infty]$, with $[0,T]$ the time span during which the system is observed, and X the space of the states of the system,
- a (positive) dissipation functional $\mathscr{D} : X \times X \to [0,\infty)$, keeping track of the energy dissipated by the curve $u : [0,T] \to X$ describing the evolution of the system, that satisfies suitable structural properties peculiar of rate-independence.

When X is a (separable) Banach space, a natural class of dissipations is provided by translation invariant functionals of the form $\mathscr{D}(u_1,u_2) := \Psi(u_2-u_1)$, where $\Psi : X \to [0,\infty)$ is a (convex, lower semicontinuous) dissipation potential, positively homogeneous of degree 1, namely $\Psi(\lambda v) = \lambda\Psi(v)$ for all $\lambda \geq 0$ and $v \in X$. The evolution of the rate-independent system is governed by the doubly nonlinear differential inclusion

$$\partial\Psi(u'(t)) + \partial_u\mathscr{E}(t,u(t)) \ni 0 \quad \text{in } X^* \quad \text{for a.a. } t \in (0,T), \tag{1.1}$$

where $\partial\Psi : X \rightrightarrows X^*$ is the subdifferential in the sense of convex analysis, while $\partial_u\mathscr{E} : [0,T] \times X \rightrightarrows X^*$ is a suitable notion of subdifferential of $\mathscr{E}$ w.r.t. the variable u. As it will be apparent from the forthcoming discussion, in general (1.1) is only formally written.

More generally, throughout this paper we shall assume that the dissipation $\mathscr{D}$ is induced by a distance d on the space X, such that

$$(X,\mathsf{d}) \text{ is a } \textit{complete} \text{ metric space.} \tag{X}$$

We will henceforth denote a (metric) rate-independent system by $(X,\mathscr{E},\mathsf{d})$.

Rate-independent evolution occurs in manifold problems in physics and engineering, cf. [9, 10] for a survey. In addition to its wide range of applicability, over the last two decades the analysis of rate-independent systems has attracted considerable interest due to its intrinsic mathematical challenges: first and foremost, the quest of a proper solvability concept for the system $(X,\mathscr{E},\mathsf{d})$. In fact, since the dissipation potential has linear growth at infinity, one can in general expect only BV-time regularity for the curve u (unless the energy functional is uniformly convex). Thus u may have jumps as a function of time. Therefore, the pointwise derivative u' in the subdifferential inclusion (1.1) in the Banach setting, and the metric derivative $|u'|$ in the general metric setup (X), need not be defined. This calls for a suitable weak formulation of rate-independent evolution, also able to satisfactorily capture the behavior of the system in the jump regime.

In what follows we illustrate the three solution concepts this paper is concerned with, referring to Sects. 2 and 3 for more details and precise statements.

1.1 Energetic, Balanced Viscosity, and Visco-Energetic Solutions

The pioneering papers [11, 12] advanced the by now classical concept of (Global) *Energetic* solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ (cf. also the notion of 'quasistatic evolution' in the realm of crack propagation, dating back to [4]), which can be in fact given in a more general topological setting [8]. It is a curve $u : [0, T] \to X$ with bounded variation, complying for every $t \in [0, T]$ with

- the global stability condition

$$\mathscr{E}(t, u(t)) \le \mathscr{E}(t, v) + \mathsf{d}(u(t), v) \quad \text{for every } v \in X, \tag{S_d}$$

- the energy balance

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_\mathsf{d}(u, [0, t]) = \mathscr{E}(0, u(0)) + \int_0^t \partial_t \mathscr{E}(s, u(s))\,\mathrm{d}s\,. \tag{E_d}$$

Here, $\mathrm{Var}_\mathsf{d}(u, [0, t])$ denotes the (pointwise) total variation of the curve u induced by the metric d, which is related to 'energy dissipation': in fact, (E_d) balances the stored energy at the process time t and the energy dissipated up to t with the initial energy and the work of the external loadings, encoded in the second integral on the right-hand side. Existence results for Energetic solutions may be proved by resorting to a well understood time discretization procedure. Indeed, for every fixed partition $\mathscr{T}_\tau := \{t_\tau^0 = 0 < t_\tau^1 < \ldots < t_\tau^{N-1} < t_\tau^N = T\}$ of the interval $[0, T]$, with fineness $\tau := \max_{i=1,\ldots,N}(t_\tau^i - t_\tau^{i-1})$, discrete solutions $(U_n^\tau)_{n=1}^N$ are constructed as solutions of the time-incremental minimization scheme

$$\min_{U \in X} \left(\mathscr{E}(t_\tau^n, U) + \mathsf{d}(U_\tau^{n-1}, U) \right). \tag{IM_τ}$$

Under suitable conditions it can be shown that, for every null sequence $(\tau_k)_k$, up to a subsequence the piecewise constant interpolants $(\overline{U}_{\tau_k})_k$ of the discrete solutions converge to an Energetic solution. While widely applied, the Energetic concept has also been criticized on the grounds that the global stability condition (S_d) is too strong a requirement, when dealing with nonconvex energies. To avoid violating it, the system may in fact have to change instantaneously in a very drastic way, jumping into very far-apart energetic configurations, possibly 'too early'. In this connection, we refer to the discussions from [6, Ex. 6.3], [13, Ex. 6.1], as well as to [20], providing a characterization of Energetic solutions to one-dimensional rate-independent systems (i.e., with $X = \mathbb{R}$), driven by a fairly broad class of nonconvex energies. In [20], the input-output relation associated with the Energetic concept is shown to be related to the so-called *Maxwell rule* for hysteresis processes [22]. These features are also reflected in the jump conditions satisfied by an Energetic solution u at every jump point $t \in \mathrm{J}_u$ ($u(t-)$, $u(t+)$ denoting the left/right limits of u at t and J_u its jump set), namely

$$\begin{aligned} \mathsf{d}(u(t-), u(t)) &= \mathscr{E}(t, u(t-)) - \mathscr{E}(t, u(t)), \\ \mathsf{d}(u(t), u(t+)) &= \mathscr{E}(t, u(t)) - \mathscr{E}(t, u(t+)), \end{aligned} \tag{1.2}$$

which show the influence of the global energy landscape of $\mathscr{E}$.

The global stability condition (S_{d}) in fact stems from the global minimization problem (IM_τ), whereas a scheme based on *local* minimization would be preferable, cf. [3] for a first discussion of this in the realm of crack propagation, and [5] in the frame of abstract (finite-dimensional) rate-independent systems. This localization can be achieved by perturbing the variational scheme (IM_τ) with a term, modulated by a viscosity parameter ε, which penalizes the squared distance from the previous step U_τ^{n-1}. One is thus led to consider the time-incremental minimization

$$\min_{U \in X} \left(\mathscr{E}(t_\tau^n, U) + \mathsf{d}(U_\tau^{n-1}, U) + \frac{\varepsilon}{2\tau} \mathsf{d}^2(U_\tau^{n-1}, U) \right), \tag{$\mathrm{IM}_{\varepsilon,\tau}$}$$

which may be considered as a *viscous* approximation of (IM_τ). For fixed $\varepsilon > 0$, the limit passage as $\tau \downarrow 0$ in ($\mathrm{IM}_{\varepsilon,\tau}$) leads to solutions (of the metric formulation) of the Generalized Gradient System $(X, \mathscr{E}, \mathsf{d}, \psi_\varepsilon)$, where the dissipation function $\psi_\varepsilon : [0, \infty) \to [0, \infty)$ is given by

$$\psi_\varepsilon(r) = r + \frac{\varepsilon}{2} r^2 = \frac{1}{\varepsilon} \psi(\varepsilon r) \qquad \text{with} \quad \psi(r) = r + \frac{1}{2} r^2 . \tag{1.3}$$

We refer to [21] for existence results for gradient systems in metric spaces, driven by dissipation potentials with superlinear growth at infinity like ψ_ε. In turn, it has been shown in [13] (cf. also [15]) that, under suitable conditions on the energy functional, time-continuous solutions (to the metric formulation) of $(X, \mathscr{E}, \mathsf{d}, \psi_\varepsilon)$ converge as $\varepsilon \downarrow 0$, up to reparameterization, to a <u>*Balanced Viscosity*</u> (BV) solution of the rate-independent system $(X, \mathscr{E}, \mathsf{d})$. The latter is a curve $u \in \mathrm{BV}([0, T]; X)$ satisfying

– the *local* stability condition

$$|\mathrm{D}\mathscr{E}|(t, u(t)) \le 1 \quad \text{for every } t \in [0, T] \setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathsf{d},\mathrm{loc}}$}$$

– the energy balance

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u, [0, t]) = \mathscr{E}(0, u(0)) + \int_0^t \partial_t \mathscr{E}(s, u(s))\, \mathrm{d}s \quad \text{for all } t \in [0, T] . \tag{$\mathrm{E}_{\mathsf{d},\mathsf{v}}$}$$

Here, $|D\mathscr{E}| : [0, T] \times X \to [0, \infty]$ is the *metric slope* of the energy functional $\mathscr{E}$, namely

$$|D\mathscr{E}|(t, u) := \limsup_{v \to u} \frac{(\mathscr{E}(t, u) - \mathscr{E}(t, v))_+}{\mathsf{d}(u, v)}, \tag{1.4}$$

and $\mathrm{Var}_{\mathsf{d,v}}$ is a suitably *augmented* notion of total variation, fulfilling $\mathrm{Var}_{\mathsf{d,v}}(u, [a, b]) \geq \mathrm{Var}_{\mathsf{d}}(u, [a, b])$ for all $[a, b] \subset [0, T]$, which measures the energy dissipated along the jump, at a point $t \in \mathrm{J}_u$, by means of the cost

$$\begin{aligned} \mathsf{v}(t, u(t-), u(t+)) := \inf\Big\{ \int_{r_0}^{r_1} |\vartheta'|(r) \, (|D\mathscr{E}|(t, \vartheta(r)) \vee 1) \, \mathrm{d}r \ : \\ \vartheta \in \mathrm{AC}([r_0, r_1]; X), \ \vartheta(r_0) = u(t-), \ \vartheta(r_1) = u(t+) \Big\} \end{aligned} \tag{1.5}$$

that is reminiscent of the viscous approximation ($\mathrm{IM}_{\varepsilon,\tau}$). Indeed, it is possible to show (cf. (1.6) ahead) that every BV solution to $(X, \mathscr{E}, \mathsf{d})$ complies with the jump conditions

$$\begin{aligned} \mathscr{E}(t, u(t-)) - \mathscr{E}(t, u(t+)) &= \mathsf{v}(t, u(t-), u(t+)) \\ &= \int_{r_0}^{r_1} |\vartheta'|(r) \, (|D\mathscr{E}|(t, \vartheta(r)) \vee 1) \, \mathrm{d}r \end{aligned} \tag{1.6}$$

at every jump point $t \in \mathrm{J}_u$, with ϑ an optimal jump transition between $u(t-)$ and $u(t+)$. Any optimal transition can be decomposed into an (at most) countable collection of *sliding* transitions, evolving in the rate-independent mode, and *viscous* transitions, i.e. (metric) solutions of the Generalized Gradient System $(X, \mathscr{E}, \mathsf{d}, \psi)$ with the superlinear ψ from (1.3), and where the time variable in the energy functional is frozen at the jump time t. Therefore, BV solutions account for the onset of viscous behavior at jumps of the system, which can be in fact interpreted as fast transitions (possibly) governed by viscosity. The characterization in the one-dimensional case, with a nonconvex driving energy, from [20] reveals that the input-output relation underlying BV solutions follows the *delay rule* [22], as they tend to jump 'as late as possible'.

A notable feature of BV solutions is that they can be directly obtained as limits of the discrete solutions arising from the perturbed scheme ($\mathrm{IM}_{\varepsilon,\tau}$), when the parameters ε and τ *jointly* tend to zero with convergence rates such that

$$\lim_{\varepsilon,\tau \downarrow 0} \frac{\varepsilon}{\tau} = +\infty; \tag{1.7}$$

the argument developed in [14, 17] in the Banach setting can be in fact easily extended to the metric framework, cf. the discussion in Sect. 3.1. This remarkable property has somehow inspired the approach in [19]. There, a new notion of rate-independent evolution has been obtained in the time-continuous limit, as $\tau \downarrow 0$, of the perturbed time-incremental minimization scheme

$$\min_{U\in X}\Big(\mathscr{E}(t^n_\tau, U) + \mathsf{d}(U^{n-1}_\tau, U) + \frac{\mu}{2}\mathsf{d}^2(U^{n-1}_\tau, U)\Big) \quad \text{with } \mu > 0 \textit{ fixed} \tag{IM$_\mu$}$$

as a parameter. The analysis carried out in [19] in fact covers a more general, topological setting, akin to that of [8], with a general viscous correction $\delta : X\times X \to [0,\infty)$ compatible, in a suitable sense, with the metric d: a particular case is in fact $\delta(u,v) = \frac{\mu}{2}\mathsf{d}^2(u,v)$ as in (IM$_\mu$). In the simplified metric setting of (X), under the same conditions ensuring the existence of Energetic solutions it is possible to show that the (piecewise constant interpolants of the) discrete solutions arising from (IM$_\mu$) converge, as $\tau \downarrow 0$ and $\mu > 0$ is fixed, to a (μ-)*Visco-Energetic* solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$. In what follows, we will simply speak of *Visco-Energetic* (VE) solutions, and often highlight their dependence on the parameter μ in the acronym VE$_\mu$. A VE$_\mu$ solution is a curve $u \in \mathrm{BV}([0,T];X)$ complying with the

– 'perturbed', still global, stability condition

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) + \frac{\mu}{2}\mathsf{d}^2(u(t),v) \qquad \text{for every } v\in X \text{ and for every } t \in [0,T]\setminus \mathrm{J}_u, \tag{S$_\mathsf{D}$}$$

– the energy balance

$$\begin{aligned}&\mathscr{E}(t,u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u,[0,t]) \\ &= \mathscr{E}(0,u(0)) + \int_0^t \partial_t\mathscr{E}(s,u(s))\,\mathrm{d}s \quad \text{for all } t\in[0,T]\,.\end{aligned} \tag{E$_{\mathsf{d},\mathsf{c}}$}$$

Here, $\mathrm{Var}_{\mathsf{d},\mathsf{c}}$ is an alternative augmented total variation functional, again estimating the total variation induced by d, but featuring a different notion of jump dissipation cost. In analogy with (1.5), the visco-energetic cost c (we shall often write c_μ to highlight its dependence on the parameter μ, and accordingly write (E$_{\mathsf{d},\mathsf{c}_\mu}$)), is still obtained by minimizing a suitable transition cost $\mathrm{Trc}_{\mathrm{VE}}$ over a class of continuous, but not necessarily absolutely continuous, curves $\vartheta : E \to X$, with E an arbitrary compact subset of $\mathbb{R}$ having a possibly more complicated structure than that of an interval. The transition cost $\mathrm{Trc}_{\mathrm{VE}}$ evaluates (1) the d-total variation $\mathrm{Var}_\mathsf{d}(\vartheta, E)$ of ϑ over E; (2) a quantity related to the "gaps" of the set E; (3) a quantity measuring the violation of the (global) stability condition (S$_\mathsf{D}$) along the jump transition ϑ, cf. [19] and Sect. 2.2 ahead for all details and precise formulae. In this context as well, it can be proved (cf. [19, Prop. 3.8]) that any VE solution u satisfies at its jump points $t \in \mathrm{J}_u$ the jump conditions

$$\mathscr{E}(t, u(t-)) - \mathscr{E}(t, u(t+)) = \mathsf{c}(t, u(t-), u(t+)) = \mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E) \qquad (1.8)$$

with $\vartheta : E \to X$ an optimal transition curve between $u(t-)$ and $u(t+)$. Furthermore, any optimal transition can be decomposed into an (at most countable) collection of *sliding transitions*, parameterized by a continuous variable and fulfilling the stability condition (S_{D}), and *pure jump transitions*, defined on discrete subsets of E, along which the stability (S_{D}) may be violated. A notable property of VE solutions is that, if an optimal jump transition $\vartheta : E \to X$ at a jump point t does not comply with the stability condition (S_{D}) at some $s \in E$, then s is isolated and, denoting by $s_- := \max(E \cap (-\infty, s))$, there holds

$$\vartheta(s) \in \mathrm{Argmin}_{y \in X} \left\{ \mathscr{E}(t, y) + \mathsf{d}(\vartheta(s_-), y) + \frac{\mu}{2} \mathsf{d}^2(\vartheta(s_-), y) \right\} .$$

A complete characterization of VE solutions to one-dimensional rate-independent systems has been recently provided in [18], showing that their behavior strongly depends on the parameter μ. When $\mu = 0$, VE solutions coincide with Energetic solutions and therefore they satisfy the *Maxwell rule*. For a sufficiently 'strong' viscous correction, i.e. with μ above a certain threshold depending on the (nonconvex) driving energy, VE solutions exhibit a behavior akin to that of BV solutions, and follow the *delay rule*. With a 'weak' correction, VE solutions have an intermediate character between Energetic and BV solutions.

1.2 Main Results

In this paper, we aim to gain further insight into this in-between quality of VE solutions and into the role of the tuning parameter μ, revealed by the analysis in [18], in a more general context. To this end, we shall study the singular limits of VE_μ solutions to the (metric) rate-independent system $(X, \mathscr{E}, \mathsf{d})$ as $\mu \downarrow 0$ and $\mu \uparrow \infty$.

With **Theorem 1** we will show that, any sequence $(u_n)_n$ of μ_n solutions corresponding to a null sequence $\mu_n \downarrow 0$ converges, up to a subsequence, to an Energetic solution of $(X, \mathscr{E}, \mathsf{d})$. **Theorem 2** will address the behavior of a sequence $(u_n)_n$ of VE_{μ_n} solutions with parameters $\mu_n \uparrow \infty$. In this case, in accordance with condition (1.7), we expect to obtain BV solutions. We will prove indeed that, up to a subsequence, as $\mu_n \uparrow \infty$ VE_{μ_n} solutions converge to a BV solution of $(X, \mathscr{E}, \mathsf{d})$.

While referring to Sects. 4 and 5 for further comments and all details, let us mention here that the proof of Theorem 2 is quite challenging. In fact, it involves passing from the transitions that describe the jump behavior of a sequence of VE_{μ_n} solutions, and that are given by a collection of 'sliding pieces' and discrete trajectories, to the jump transitions for BV solutions, that are instead *absolutely continuous* curves. This can be achieved by means of a careful reparameterization technique, combined with a delicate compactness argument for transition curves in *varying* domains.

Plan of the Paper

In Sect. 2 we collect some preliminary results, set up the basic assumptions on the energy functional $\mathscr{E}$, and give the precise definitions of Energetic, Balanced Viscosity, and Visco-Energetic solutions to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$. In Sect. 3 we recapitulate the existence results for the three solution concepts, and state our own Theorems 1 and 2, whose proof is developed throughout Sects. 4 and 5, also resorting to some auxiliary results stated and proved in the Appendix.

2 Preliminary Results and Overview of the Solution Concepts for Rate-Independent Systems

We start by fixing some notation: for a given arbitrary $E \subset \mathbb{R}$, we shall denote by

$$\begin{aligned} &\mathfrak{P}_f(E) \text{ the collection of all finite subsets of } E, \\ &E^- := \inf E, \quad E^+ := \sup E\,. \end{aligned} \tag{2.1}$$

Kuratowski Convergence of Sets

In view of the compactness argument developed in Sect. 5 ahead, we provide a minimal aside on the notion of *Kuratowski* convergence of sets, confining the discussion to *closed* sets, and referring to [1] for all details. We say that a sequence $(C_n)_n$ of closed subsets of X converge in the sense of Kuratowski to a closed set C, if

$$\mathrm{Li}_{n\to\infty} C_n = \mathrm{Ls}_{n\to\infty} C_n = C, \tag{2.2}$$

where

$$\mathrm{Li}_{n\to\infty} C_n := \{x \in X \,:\, \exists\, x_n \in C_n \text{ such that } x_n \to x\}, \tag{2.3a}$$

$$\begin{aligned} \mathrm{Ls}_{n\to\infty} C_n := \{x \in X \,:\, &\exists\, j \mapsto n_j \text{ increasing and} \\ & x_{n_j} \in C_{n_j} \text{ such that } x_{n_j} \to x\}. \end{aligned} \tag{2.3b}$$

If all the closed sets C_n are contained in a compact set K, then Kuratowski convergence coincides with the convergence induced by the Hausdorff distance [1, Prop. 4.4.14]. That is why, the *Blaschke* Theorem (cf., e.g., [1, Thm. 4.4.15]) is applicable, ensuring that, if $K \subset X$ is a fixed compact set, then every sequence of closed sets $(C_n)_n \subset K$ admits a subsequence converging in the Kuratowski sense to a closed set $C \subset K$. If the sets C_n are connected, then C is also connected.

2.1 Preliminaries on Functions of Bounded Variation and Absolutely Continuous Functions

Let us first recall some preliminary definitions and properties related to functions of bounded variation with values in the metric space (X, d). The *pointwise* total variation $\mathrm{Var}_{\mathsf{d}}(u, E)$ of a function $u : E \to X$ is defined by

$$\mathrm{Var}_{\mathsf{d}}(u, E) := \sup \left\{ \sum_{j=1}^{M} \mathsf{d}(u(t_{j-1}), u(t_j)) \ : \ t_0 < t_1 < \ldots < t_M,\ \{t_j\}_{j=0}^{M} \in \mathfrak{P}_f(E) \right\}, \tag{2.4}$$

with $\mathrm{Var}_{\mathsf{d}}(u, \emptyset) := 0$. We define the space of functions with bounded variation via

$$\mathrm{BV}_{\mathsf{d}}(E; X) := \{u : E \to X \ : \ \mathrm{Var}_{\mathsf{d}}(u, E) < \infty\}.$$

For every $u \in \mathrm{BV}_{\mathsf{d}}(E; X)$ we may introduce the function

$$V_u : [E^-, E^+] \to [0, \infty) \quad \text{given by} \quad V_u(t) := \mathrm{Var}_{\mathsf{d}}(u, E \cap [E^-, t]). \tag{2.5}$$

Observe that V_u is monotone nondecreasing and satisfies

$$\mathsf{d}(u(t_0), u(t_1)) \le \mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1]) = V_u(t_1) - V_u(t_0) \quad \text{for all } t_0, t_1 \in E \text{ with } t_0 \le t_1.$$

Since the metric space (X, d) is complete, every function $u \in \mathrm{BV}_{\mathsf{d}}(E; X)$ is *regulated*, i.e. at every $t \in E$ the left and right limits $u(t-)$ and $u(t+)$ exist (with obvious adjustments at E^- and E^+). We recall that u only has jump discontinuities, and that its (at most) *countable* jump set J_u coincides with the jump set of V_u.

We will also consider the distributional derivative ν_u of the function V_u and recall that the Borel measure ν_u can be decomposed into the sum

$$\nu_u = \nu_u^{\mathrm{d}} + \nu_u^{\mathrm{J}} \tag{2.6}$$

with ν_u^{d} the diffuse part of ν_u (i.e. the sum of its absolutely continuous and Cantor parts), fulfilling $\nu_u^{\mathrm{d}}(\{t\}) = 0$ for every $t \in [E^-, E^+]$, and ν_u^{J} its jump part, concentrated on the set J_u, so that

$$\nu_u^{\mathrm{J}}(\{t\}) = \mathsf{d}(u(t-), u(t)) + \mathsf{d}(u(t)), u(t+)) \quad \text{for every } t \in \mathrm{J}_u.$$

Therefore we have

$$\mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1]) = \nu_u^{\mathrm{d}}([t_0, t_1]) + \mathrm{Jmp}_{\mathsf{d}}(u; [t_0, t_1]) \tag{2.7}$$

for every interval $[t_0, t_1] \subset E$, with the jump contribution

$$\begin{aligned} \mathrm{Jmp}_{\mathsf{d}}(u; [t_0, t_1]) :=& \mathsf{d}(u(t_0)), u(t_0+)) + \mathsf{d}(u(t_1-)), u(t_1)) \\ &+ \sum_{t \in \mathrm{J}_u \cap (t_0, t_1)} (\mathsf{d}(u(t-), u(t)) + \mathsf{d}(u(t), u(t+))) \ . \end{aligned} \tag{2.8}$$

In the definition of Balanced Viscosity and Visco-Energetic solutions, there will come into play an alternative notion of total variation for a curve $u \in \mathrm{BV}([0, T]; X)$, which will reflect the energetic behavior of the (Balanced Viscosity/Visco-Energetic) solution at jump points. It will be obtained by suitably modifying the jump contribution to the total variation induced by d, cf. (2.7), in terms of a (general) *cost function* $\mathsf{e} : [0, T] \times X \times X \to [0, \infty]$, with $\mathsf{e} \geq \mathsf{d}$, that shall measure the energy dissipated along a jump. Thus, hereafter we will refer to e as *jump dissipation cost*. As particular cases of e, we will consider

- the *viscous (jump dissipation) cost* v, cf. (2.21) ahead, in the case of Balanced Viscosity solutions;
- the *visco-energetic (jump dissipation) cost* c, cf. (2.28) ahead, in the case of Visco-Energetic solutions.

With the jump dissipation cost e we associate the *incremental cost*

$$\Delta_{\mathsf{e}} : [0, T] \times X \times X \to [0, \infty], \quad \Delta_{\mathsf{e}}(t, u_-, u_+) := \mathsf{e}(t, u_-, u_+) - \mathsf{d}(u_-, u_+) \tag{2.9}$$

for all $t \in [0, T]$ and $u_-,\ u_+ \in X$, where the notation $u_-,\ u_+$ is suggestive of the fact that, in the definition of the total variation functional induced by e, the incremental cost will be evaluated at the left and right limits $u(t-)$ and $u(t+)$ at a jump point of a curve u. We will also use the notation

$$\Delta_{\mathsf{e}}(t, u_-, u, u_+) := \Delta_{\mathsf{e}}(t, u_-, u) + \Delta_{\mathsf{e}}(t, u, u_+) \, .$$

We are now in a position to introduce the *augmented total variation* functional induced by e.

Definition 1 Given a (jump dissipation) cost function e and the associated incremental cost Δ_{e}, and given a curve $u \in \mathrm{BV}([0, T]; X)$, we define the *incremental* *jump variation* of u on a sub-interval $[t_0, t_1] \subset [0, T]$ by

$$\begin{aligned}\mathrm{Jmp}_{\Delta_{\mathsf{e}}}(u; [t_0, t_1]) &:= \Delta_{\mathsf{e}}(t_0, u(t_0)), u(t_0+)) \\ &+ \Delta_{\mathsf{e}}(t_1, u(t_1-)), u(t_1)) + \sum_{t \in \mathrm{J}_u \cap (t_0, t_1)} \Delta_{\mathsf{e}}(t, u(t-), u(t), u(t+)) \, .\end{aligned} \tag{2.10}$$

This induces the *augmented total variation* functional

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [t_0, t_1]) := \mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1]) + \mathrm{Jmp}_{\Delta_{\mathsf{e}}}(u; [t_0, t_1]) \tag{2.11}$$

along any sub-interval $[t_0, t_1] \subset [0, T]$.
Since we have subtracted from the e-jump contribution the d-distance of the jump end-points, cf. (2.9), the d-jump contribution (2.8) to $\mathrm{Var}_{\mathsf{d}}$ cancels out, and in fact only the *diffuse* contribution $\nu_u^{\mathrm{d}}([t_0, t_1])$ remains. In fact, one could rewrite $\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [t_0, t_1])$ as

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [t_0, t_1]) = \nu_u^{\mathrm{d}}([t_0, t_1]) + \mathrm{Jmp}_{\mathsf{e}}(u; [t_0, t_1]), \tag{2.12}$$

with $\mathrm{Jmp}_{\mathsf{e}}(u; [t_0, t_1])$ defined by (2.10) with the "whole" cost e in place of its incremental version Δ_{e}, i.e.

$$\begin{aligned}\mathrm{Jmp}_{\mathsf{e}}(u; [t_0, t_1]) :=& \mathsf{e}(t_0, u(t_0), u(t_0+)) + \mathsf{e}(t_1, u(t_1-), u(t_1)) \\ &+ \sum_{t \in \mathrm{J}_u \cap (t_0, t_1)} \big(\mathsf{e}(t, u(t-), u(t)) + \mathsf{e}(t, u(t), u(t+))\big) \, .\end{aligned} \tag{2.13}$$

Clearly, $\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [t_0, t_1]) \geq \mathrm{Var}_{\mathsf{d}}(u, [t_0, t_1])$, and they coincide if $\mathsf{e} = \mathsf{d}$, or when $\mathrm{J}_u = \emptyset$. Moreover, as observed in [19], although it need not be induced by a distance on X, $\mathrm{Var}_{\mathsf{d},\mathsf{e}}$ still enjoys the additivity property

$$\mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [a, c]) = \mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [a, b]) + \mathrm{Var}_{\mathsf{d},\mathsf{e}}(u, [b, c]) \qquad \text{for all } 0 \leq a \leq b \leq c \leq T.$$

Finally, we recall that a curve $u : [0, T] \to X$ is absolutely continuous (and write $u \in \mathrm{AC}([0, T]; X)$) if there exists $m \in L^1(0, T)$ such that

$$\mathsf{d}(u(s), u(t)) \leq \int_s^t m(r) \, \mathrm{d}r \qquad \text{for all } 0 \leq s \leq t \leq T. \tag{2.14}$$

For every $u \in \mathrm{AC}([0, T]; X)$, the limit

$$|u'|(t) = \lim_{s \to t} \frac{\mathsf{d}(u(s), u(t))}{|t - s|} \quad \text{exists for a.a. } t \in (0, T), \tag{2.15}$$

cf. [2, Sec. 1.1]. We will refer to it as the *metric derivative* of u at t. The map $t \mapsto |u'|(t)$ belongs to $L^1(0, T)$ and it is minimal within the class of functions $m \in L^1(0, T)$ fulfilling (2.14).

2.2 Energetic, Balanced Viscosity, and Visco-Energetic Solutions at a Glance

We now give a quick overview of the notions of rate-independent evolution this paper is concerned with. We aim to somehow motivate the various solution concepts and in addition highlight both the common points, and the differences, in their structure.

Underlying the upcoming definitions, there will be the following basic conditions on the energy functional $\mathscr{E}$. Let us mention in advance that we in fact allow for a possibly nonsmooth time-dependence $t \mapsto \mathscr{E}(t,u)$. However, in what follows for simplicity we will confine our analysis to the case in which the domain of $\mathscr{E}(t,\cdot)$ in fact coincides with X for every $t \in [0,T]$, referring to [19, Rmk. 2.7] for a discussion of the more general case in which $\mathrm{dom}(\mathscr{E}(t,\cdot))$ is a proper subset of X (still independent of the time variable).

Basic Assumptions on the Energy

Throughout the paper, we will require that $\mathscr{E}$ complies with two basic properties, involving the perturbed energy functional $\mathscr{F} : [0,T] \times X \to \mathbb{R}$

$$\mathscr{F}(t,u) := \mathscr{E}(t,u) + \mathsf{d}(x_o,u) \quad \text{with } x_o \text{ a given reference point in } X \tag{2.16}$$

and its sublevel sets $S_C := \{(t,u) \in [0,T] \times X \,:\, \mathscr{F}(t,u) \le C\}$. Namely,
Lower semicontinuity and compactness: for all $C \in \mathbb{R}$

$$\mathscr{E} \text{ is lower semicontinuous on } S_C \text{ and the sets } S_C \text{ are compact in } [0,T] \times X; \tag{E$_1$}$$

Power control: there exists a map $\mathscr{P} : [0,T] \times X \to \mathbb{R}$ fulfilling

$$\begin{gathered}\liminf_{s\uparrow t} \frac{\mathscr{E}(t,u)-\mathscr{E}(s,u)}{t-s} \ge \mathscr{P}(t,u) \ge \limsup_{s\downarrow t} \frac{\mathscr{E}(s,u)-\mathscr{E}(t,u)}{s-t} \\ \text{for all } (t,u) \in [0,T] \times X, \\ \exists\, C_P > 0 \;\; \forall\, (t,u) \in [0,T] \times X : \quad |\mathscr{P}(t,u)| \le C_P \mathscr{F}(t,u)\,. \end{gathered} \tag{E$_2$}$$

We may understand the power functional $\mathscr{P}$ as a sort of "time superdifferential" of the energy functional, surrogating its partial time derivative in the case where the functional $t \mapsto \mathscr{E}(t,u)$ is not differentiable at every point of $[0,T] \times X$. This for instance occurs for reduced energies having the form $\mathscr{E}(t,u) = \min_{\varphi\in\Phi} \mathscr{I}(t,\varphi,u)$ and such that the set of minimizers does not reduce to a singleton, as considered, e.g., in [7, 15, 16, 19]. By repeating the very same arguments as in [19], we may deduce from (E_1) & (E_2) that

$$\begin{aligned} &\text{the function } t \mapsto \mathscr{E}(t,u) \text{ is Lipschitz continuous for every } u \in X, \text{ with} \\ &\mathscr{P}(t,u) = \partial_t \mathscr{E}(t,u) \qquad \text{for almost all } t \in (0,T) \text{ and for all } u \in X. \end{aligned} \tag{2.17}$$

Therefore,

$$\mathscr{E}(t,u) = \mathscr{E}(s,u) + \int_s^t \mathscr{P}(r,u)\,\mathrm{d}r \qquad \text{for every } [s,t] \subset [0,T]. \tag{2.18}$$

Combining this with the power control estimate in (E_2) and exploiting the Gronwall Lemma, we conclude that

$$\mathscr{F}(t,u) \le \mathscr{F}(s,u) \exp\left(C_P|t-s|\right) \quad \text{for all } s,\ t \in [0,T]. \tag{2.19}$$

That is why, it is significant (and notationally convenient) to work with the functional $\mathscr{F}_0(u) := \mathscr{F}(0,u)$, which controls $\mathscr{F}(t,u)$, and thus the power functional $\mathscr{P}(t,u)$, at all $t \in [0,T]$.

We are now in a position to give the concept of **Energetic** solution, dating back to [11, 12], cf. also [9].

Definition 2 (Energetic Solution) A curve $u \in \mathrm{BV}([0,T];X)$ is an Energetic solution of the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if it satisfies for every $t \in [0,T]$

– the global stability condition

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,v) + \mathsf{d}(u(t),v) \quad \text{for every } v \in X, \tag{S_d}$$

– the energy balance

$$\mathscr{E}(t,u(t)) + \mathrm{Var}_\mathsf{d}(u,[0,t]) = \mathscr{E}(0,u(0)) + \int_0^t \mathscr{P}(s,u(s))\,\mathrm{d}s\,. \tag{E_d}$$

For later use, we introduce the d-stable set

$$\mathscr{S}_\mathsf{d} := \{(t,u) \in [0,T]\times X \ :\ \mathscr{E}(t,u) \le \mathscr{E}(t,v) + \mathsf{d}(u,v) \text{ for all } v \in X\},$$

with its time-dependent sections $\mathscr{S}_\mathsf{d}(t) := \{u \in X \ :\ (t,u) \in \mathscr{S}_\mathsf{d}\}$. We postpone to Sect. 3.1 a discussion on the existence of Energetic solutions.

As already mentioned in the Introduction, **Balanced Viscosity** solutions arise in the time-continuous limit of the time-incremental scheme ($IM_{\varepsilon,\tau}$), when the parameters ε and τ both tend to zero with $\frac{\varepsilon}{\tau} \uparrow \infty$ cf. (1.7). They fulfill the *local version* of the stability condition (S_d), involving the *metric slope* of the energy functional $\mathscr{E}$, cf. (1.4). The "viscous" character of the approximation that underlies condition (1.7), is also reflected in the *viscous jump dissipation cost*. Indeed, at fixed process time $t \in [0, T]$, $\mathsf{v}(t, u_-, u_+)$ is obtained by minimizing the *transition cost*

$$\mathrm{Trc}_{\mathrm{BV}}(t, \vartheta, [r_0, r_1]) := \int_{r_0}^{r_1} |\vartheta'|(r)\,(|\mathrm{D}\mathscr{E}|(t, \vartheta(r)) \vee 1)\,\mathrm{d}r \tag{2.20}$$

over all *absolutely continuous* curves ϑ on an interval $[r_0, r_1]$, connecting the two points u_- and u_+, where we recall that $|\vartheta'|$ is the (almost everywhere defined) *metric derivative* of the curve ϑ. Namely,

$$\begin{aligned} \mathsf{v}(t, u_-, u_+) := \inf \Big\{ & \mathrm{Trc}_{\mathrm{BV}}(t, \vartheta, [r_0, r_1]) : \\ & \vartheta \in \mathrm{AC}([r_0, r_1]; X),\ \vartheta(r_0) = u_-,\ \vartheta(r_1) = u_+ \Big\}. \end{aligned} \tag{2.21}$$

We can then introduce the incremental cost Δ_{v} (2.9) and the jump variation $\mathrm{Jmp}_{\Delta_{\mathsf{v}}}$ (2.10) associated with v, and thus arrive at the induced augmented total variation $\mathrm{Var}_{\mathsf{d},\mathsf{v}}$ (2.11), which enters into the energy balance involved in the Balanced Viscosity concept.

Definition 3 (Balanced Viscosity Solution) A curve $u \in \mathrm{BV}([0, T]; X)$ is a Balanced Viscosity (BV) solution of the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if it satisfies

– the local stability condition

$$|\mathrm{D}\mathscr{E}|(t, u(t)) \le 1 \quad \text{for every } t \in [0, T] \setminus \mathrm{J}_u, \tag{$\mathrm{S}_{\mathsf{d},\mathrm{loc}}$}$$

– the energy balance

$$\begin{aligned} &\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u, [0, t]) \\ &= \mathscr{E}(0, u(0)) + \int_0^t \mathscr{P}(s, u(s))\,\mathrm{d}s \quad \text{for all } t \in [0, T]. \end{aligned} \tag{$\mathrm{E}_{\mathsf{d},\mathsf{v}}$}$$

The notion of **Visco-Energetic** solution features a modified concept of stability which also involves the viscous correction $\delta(u, v) = \frac{\mu}{2}\mathsf{d}^2(u, v)$. We then define the functional

$$\mathsf{D}(u, v) := \mathsf{d}(u, v) + \delta(u, v) = \mathsf{d}(u, v) + \frac{\mu}{2}\mathsf{d}^2(u, v) \tag{2.22}$$

and we say that a point $(t, x) \in [0, T] \times X$ is D-stable if

$$\mathscr{E}(t, x) \le \mathscr{E}(t, y) + \mathsf{D}(x, y) = \mathscr{E}(t, y) + \mathsf{d}(x, y) + \frac{\mu}{2}\mathsf{d}^2(x, y) \quad \text{for all } y \in X. \tag{2.23}$$

We denote by $\mathscr{S}_{\mathsf{D}}$ the collection of all D-stable points, and by $\mathscr{S}_{\mathsf{D}}(t)$ its section at time $t \in [0,T]$. We also introduce the *residual stability function* $\mathscr{R} : [0,T] \times X \to \mathbb{R}$ given by

$$\begin{aligned} \mathscr{R}(t,x) : &= \sup_{y\in X} \{\mathscr{E}(t,x) - \mathscr{E}(t,y) - \mathsf{D}(x,y)\} \\ &= \mathscr{E}(t,x) - \inf_{y\in X} \{\mathscr{E}(t,y) + \mathsf{D}(x,y)\} \end{aligned} \tag{2.24}$$

(for simplicity, we choose to neglect the μ-dependence of the functionals D and $\mathscr{R}$ in their notation). Observe that

$$\begin{aligned} &\mathscr{R}(t,x) \geq 0 \quad \text{for all } (t,x) \in [0,T] \times X \quad \text{with} \\ &\mathscr{R}(t,x) = 0 \text{ if and only if } (t,x) \in \mathscr{S}_{\mathsf{D}}\,, \end{aligned} \tag{2.25}$$

so that $\mathscr{R}$ may be interpreted as "measuring the failure" of the stability condition at a given point $(t,x) \in [0,T] \times X$. It can be straightforwardly checked that, under the basic lower semicontinuity assumption (E_1) on $\mathscr{E}$, the functional $\mathscr{R}$ is lower semicontinuous on $[0,T] \times X$.

We now have all the ingredients to define the jump-dissipation cost for Visco-Energetic solutions. In the same way as for Balanced Viscosity solutions, such a cost is obtained by minimizing a suitable transition cost over a class of curves connecting the two end-points of the jump. However, such curves, while still continuous, need not be absolutely continuous. Further, they are in general defined on a *compact* subset $E \subset \mathbb{R}$ that may have a more complicated structure than that of an interval. To describe it, we introduce

$$\text{the collection } \mathfrak{h}(E) \text{ of the connected components of the set } [E^-,E^+] \setminus E, \tag{2.26}$$

where we recall that $E^- = \inf E$ and $E^+ = \sup E$. Since $[E^-,E^+] \setminus E$ is an open set, $\mathfrak{h}(E)$ consists of at most countably many open intervals, which we will often refer to as the "holes" of E. Hence, the transition cost at the basis of the concept of Visco-Energetic solution evaluates (1) the d-total variation of a continuous curve defined on a set E, (2) the sum, over all the holes of E, of a quantity related to the gaps (3) the measure of "how much" the curve ϑ fails to comply with the D-stability condition (2.23) at the points in $E \setminus \{E^+\}$.

Definition 4 Let E be a compact subset of $\mathbb{R}$ and $\vartheta \in \mathrm{C}(E;X)$. For every $t \in [0,T]$ we define the *transition cost function*

$$\mathrm{Trc}_{\mathsf{VE}}(t,\vartheta,E) := \mathrm{Var}_{\mathsf{d}}(\vartheta,E) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta,E) + \sum_{s\in E\setminus E^+} \mathscr{R}(t,\vartheta(s))\,, \tag{2.27}$$

with

1. $\mathrm{Var}_{\mathsf{d}}(\vartheta, E)$ from (2.4);
2. $\mathrm{GapVar}_{\mathsf{d}}(\vartheta, E) := \sum_{I \in \mathfrak{h}(E)} \frac{\mu}{2}\mathsf{d}^2(\vartheta(I^-), \vartheta(I^+))$;
3. the (possibly infinite) sum

$$\sum_{s \in E \setminus E^+} \mathscr{R}(t, \vartheta(s)) := \begin{cases} \sup\{\sum_{s \in P} \mathscr{R}(t, \vartheta(s)) \ : \ P \in \mathfrak{P}_f(E)\} & \text{if } E \setminus E^+ \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

(recall that $\mathfrak{P}_f(E)$ denotes the collection of all finite subsets of E).

Along with [19], we observe that, for every fixed $t \in [0, T]$ and $\vartheta \in \mathrm{C}(E; X)$, the transition cost fulfills the additivity property

$$\mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E \cap [a, c]) = \mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E \cap [a, b]) + \mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E \cap [b, c])$$

for all $a < b < c$. We are now in a position to define the associated *visco-energetic jump dissipation cost* $\mathsf{c} : [0, T] \times X \times X \to [0, \infty]$ via

$$\begin{aligned} \mathsf{c}(t, u_-, u_+) := \inf\{&\mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E) \ : \\ & E \Subset \mathbb{R},\ \vartheta \in \mathrm{C}(E; X),\ \vartheta(E^-) = u_-,\ \vartheta(E^+) = u_+\}, \end{aligned} \tag{2.28}$$

whence the incremental dissipation cost Δ_{c} according to (2.9), the jump variation $\mathrm{Jmp}_{\Delta_{\mathsf{c}}}$ as in (2.10), and the augmented total variation $\mathrm{Var}_{\mathsf{d},\mathsf{c}}$ as in (2.11).

We can now give the following

Definition 5 (Visco-Energetic Solution) A curve $u \in \mathrm{BV}([0, T]; X)$ is a Visco-Energetic (VE) solution of the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if it satisfies

– the D-stability condition

$$\begin{aligned} \mathscr{E}(t, u(t)) \le \mathscr{E}(t, v) + \mathsf{d}(u(t), v) + \frac{\mu}{2}\mathsf{d}^2(u(t), v) \\ \text{for every } v \in X \text{ and for every } t \in [0, T] \setminus \mathrm{J}_u, \end{aligned} \tag{S_{D}}$$

– the energy balance

$$\begin{aligned} &\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u, [0, t]) \\ &= \mathscr{E}(0, u(0)) + \int_0^t \mathscr{P}(s, u(s))\,\mathrm{d}s \quad \text{for all } t \in [0, T]\,. \end{aligned} \tag{$\mathrm{E}_{\mathsf{d},\mathsf{c}}$}$$

3 Main Results

Prior to stating our own results on the singular limits of VE solutions in Sect. 3.2, in Sect. 3.1 below we recall the known existence results for Energetic, BV, and VE solutions. Under the same conditions ensuring the existence for the two former solution concepts, we will prove our convergence statements for VE_μ solutions in the limits $\mu \downarrow 0$ and $\mu \uparrow \infty$, respectively.

3.1 A Survey on Existence Results

In what follows, in addition to the basic conditions (E_1) and (E_2), we will introduce further assumptions on the energy functional $\mathscr{E}$ that will be at the core of the upcoming existence results for Energetic (Theorem 1), BV (Theorem 2), and VE (Theorem 3) solutions. We will also illustrate the main ideas underlying their proofs.

Energetic Solutions

For the existence of Energetic solutions in the metric setting of (X) we refer to [8, Thm. 4.5], cf. also [9] and [10, Sec. 2.1]. In accordance with these results, in addition to the coercivity (E_1) and the power control (E_2), we require that
Upper semicontinuity of the power: $\mathscr{P} : [0,T] \times X \to \mathbb{R}$ satisfies the *conditional upper semicontinuity* condition

$$
\begin{aligned}
&((t_n, u_n) \to (t,u) \text{ in } [0,T] \times X,\ \mathscr{E}(t_n,u_n) \to \mathscr{E}(t,u)) \\
&\Longrightarrow \limsup_{n\to\infty} \mathscr{P}(t_n,u_n) \le \mathscr{P}(t,u)\,.
\end{aligned}
\tag{E$_3$}
$$

We thus have

Theorem 1 *Let* $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ *comply with* (E_1), (E_2) *and* (E_3). *Then, for every initial datum* u_0 *stable at* $t = 0$, *i.e.* $u_0 \in \mathscr{S}_{\mathsf{d}}(0)$, *there exists at least one Energetic solution to the rate-independent system* $(X, \mathscr{E}, \mathsf{d})$ *with* $u(0) = u_0$.
The *proof* is based on a (by now standard in the frame of rate-independent systems) time-discretization procedure, with the discrete solutions constructed by recursively solving the time-incremental minimization scheme (IM_τ). Their (piecewise constant) interpolants are shown to comply with the discrete versions of the stability condition (S_{d}) and of the upper energy estimate in (E_{d}), whence all a priori estimates stem, also based on the power control (E_2). With a Helly-type compactness result, crucially relying on (E_1), we thus infer that the approximate solutions pointwise converge to a curve $u \in \mathrm{BV}([0,T];X)$. The continuity (cf. (2.17)) and lower

semicontinuity properties

$$\begin{aligned} &t_n \to t \;\Rightarrow\; \mathscr{E}(t_n, y) \to \mathscr{E}(t, y) \quad \text{for all } y \in X, \\ &(t_n \to t,\; u_n \to u) \;\Rightarrow\; \liminf_{n\to\infty} \mathscr{E}(t_n, u_n) \geq \mathscr{E}(t, u) \end{aligned} \tag{3.1}$$

ensure the closedness of the stable set $\mathscr{S}_{\mathsf{d}}$, which allows us to pass to the limit in the discrete stability condition and conclude that u complies with (S_{d}). Lower semicontinuity arguments, joint with (E_3), lead to the limit passage in the discrete upper energy estimate, so that u complies with the upper energy estimate $\leq$ of (E_{d}). The lower energy estimate $\geq$ can be then deduced from the stability condition either via a Riemann-sum argument, formalized in, e.g., [10, Prop, 2.1.23], or by applying [19, Lemma 6.2].

Balanced Viscosity Solutions

Along the footsteps of [15, Thm. 4.2], for the existence of Balanced Viscosity solutions, in addition to (E_1) and (E_2), we again need to impose the (conditional) upper semicontinuity of the power functional and, *in addition*, the lower semicontinuity of the slope along sequences *with bounded energy and slope*. These requirements are subsumed by the following condition:
Upper semicontinuity of the power, lower semicontinuity of the slope: $\mathscr{E} : [0, T] \times X \to \mathbb{R}$ and $\mathscr{P} : [0, T] \times X \to \mathbb{R}$ satisfy

$$\begin{aligned} &\Big((t_n, u_n) \to (t, u) \text{ in } [0, T] \times X,\; \sup_{n\in\mathbb{N}} \mathscr{F}_0(u_n) < \infty,\; \sup_{n\in\mathbb{N}} |\mathrm{D}\mathscr{E}|(t_n, u_n) < \infty\Big) \\ &\Longrightarrow \begin{cases} \liminf_{n\to\infty} |\mathrm{D}\mathscr{E}|(t_n, u_n) \geq |\mathrm{D}\mathscr{E}|(t, u), \\ \limsup_{n\to\infty} \mathscr{P}(t_n, u_n) \leq \mathscr{P}(t, u)\,. \end{cases} \end{aligned} \tag{E_3'}$$

The last, key condition underlying the existence of Balanced Viscosity solutions is that $\mathscr{E}$ complies with the
Chain-rule inequality: for every curve $u \in \mathrm{AC}([0, T]; X)$ the function $t \mapsto \mathscr{E}(t, u(t))$ is absolutely continuous on $[0, T]$, and there holds

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t, u(t)) + \mathscr{P}(t, u(t)) \leq |u'|(t)|\mathrm{D}\mathscr{E}|(t, u(t)) \qquad \text{for a.a. } t \in (0, T)\,. \tag{E_4}$$

Under these conditions, the following existence result was proved in [15].

Theorem 2 *Let $\mathscr{E} : [0, T] \times X \to \mathbb{R}$ comply with (E_1), (E_2), (E_3'), and (E_4). Then, for every $u_0 \in X$ there exists at least one Balanced Viscosity solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ with $u(0) = u_0$.*

As mentioned in the Introduction, in the *proof* of [15, Thm. 4.2] (cf. also [13]), BV solutions arise by taking the vanishing-viscosity limit, as $\varepsilon \downarrow 0$, of the time-continuous solutions of the Gradient Systems $(X, \mathscr{E}, \mathsf{d}, \psi_\varepsilon)$ with ψ_ε from (1.3). Nonetheless, exploiting the arguments from [14, 17] in the Banach setting, the vanishing-viscosity analysis developed in [15] could be easily adapted to the direct limit passage in the time-discretization scheme ($\mathrm{IM}_{\varepsilon,\tau}$). In fact, the lower semicontinuity of the slope from (E_3') serves to the purpose of passing to the limit in the dissipation term in the discrete energy-dissipation inequality arising from the scheme ($\mathrm{IM}_{\varepsilon,\tau}$). This leads to the total variation term $\mathrm{Var}_{\mathsf{d},\mathsf{v}}(u, [0, t])$ in the energy balance ($\mathrm{E}_{\mathsf{d},\mathsf{v}}$). Instead, the upper semicontinuity of the power allows us to take the limit in the power term of the discrete energy inequality. In this way, it is possible to conclude that any limit curve $u \in \mathrm{BV}([0, T]; X)$ of the discrete solutions complies with the local stability condition ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$) and with the upper energy estimate

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u, [0, t]) \le \mathscr{E}(0, u(0)) + \int_0^t \mathscr{P}(s, u(s))\,\mathrm{d}s\,. \qquad (\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}})$$

Unlike the case of Energetic solutions, where the validity of global stability condition (S_{d}) was sufficient to conclude the lower energy estimate for (E_{d}), ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$) is not strong enough to lead to the converse inequality of ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$). This is instead ensured by a chain-rule argument based on (E_4), cf. [14, Prop. 4.2, Thm. 4.3].

Finally, let us mention that, under the very assumptions for the existence Theorem 2, trivially adapting the argument for [17, Thm. 3.15] it can be shown that a curve $u \in \mathrm{BV}([0, T]; X)$ is a BV solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if and only if it satisfies ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$), the localized energy inequality

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d}}(u, [s, t]) \le \mathscr{E}(s, u(s)) + \int_s^t \mathscr{P}(r, u(r))\,\mathrm{d}r \qquad (3.2)$$

for all $0 \le s \le t \le T$, and the jump conditions

$$\begin{aligned} \mathscr{E}(t, u(t-)) - \mathscr{E}(t, u(t)) &= \mathsf{v}(t, u(t-), u(t)), \\ \mathscr{E}(t, u(t)) - \mathscr{E}(t, u(t+)) &= \mathsf{v}(t, u(t), u(t+)), \\ \mathscr{E}(t, u(t-)) - \mathscr{E}(t, u(t+)) &= \mathsf{v}(t, u(t-), u(t+))\,. \end{aligned} \qquad (3.3)$$

Visco-Energetic Solutions

As already hinted, Visco-Energetic solutions were introduced in [19] within a more complex topological setting, featuring an *asymmetric* distance and a topology σ, involved in the coercivity condition on the energy functional. It turns out that, in the present metric setting where σ is the topology induced by d, (E_1), (E_2) and (E_3') coincide with the conditions required on the energy functional $\mathscr{E}$ within [19, Assumption $< A >$, Sect. 2.2]. Furthermore, the particular choice $\delta(u, v) =$

$\frac{\mu}{2}\mathsf{d}^2(u, v)$ for the viscous correction ensures the validity of [19, Assumption $< B >$, Sect. 3.1]. In particular, condition [19, $< B.3 >$, Sect. 3.1] is fulfilled, namely D-stability implies local d-stability, as it can be straightforwardly checked. Finally, thanks to the lower semicontinuity of the residual functional $\mathscr{R}$ from (2.24), also [19, Assumption $< C >$, Sect. 3.3] is fulfilled. Therefore, [19, Thm. 3.9] applies, ensuring the convergence of the time-incremental scheme (IM_μ), with $\mu > 0$ fixed, to a Visco-Energetic solution. In particular, we have the following existence result, under the *same* conditions on the energy functional as in the existence Theorem 1 for Energetic solutions.

Theorem 3 *Let* $\mathscr{E} : [0, T] \times X \to \mathbb{R}$ *comply with* (E_1)*,* (E_2) *and* (E_3)*. Then, for every* $\mu > 0$ *and every initial datum* $u_0 \in X$ *there exists at least one* VE_μ *solution to the rate-independent system* $(X, \mathscr{E}, \mathsf{d})$ *with* $u(0) = u_0$*.*
The outline of the existence argument is the same as for Energetic solutions, though the technical difficulties attached to the single steps are peculiar of the Visco-Energetic case. The D-stability condition (S_D) and the upper energy estimate in ($\mathrm{E}_{\mathsf{d,c}}$) are derived by passing to the limit in their discrete versions, valid for the discrete solutions to the time-incremental scheme (IM_μ). As shown in [19, Thm. 6.5], the lower energy estimate can then be derived from (S_D) by applying [19, Lemma 6.2].

Under the same conditions as for the existence Theorem 3, we have the following 'stability' result for VE solutions with respect to convergence of the parameters μ_n to some *strictly positive* μ.

Proposition 1 *Let* $\mathscr{E} : [0, T]\times X \to \mathbb{R}$ *comply with* (E_1)*,* (E_2) *and* (E_3)*. Let* $(\mu_n)_n \subset$ *fulfill*

$$\mu_n \to \mu > 0 \qquad \text{as } n \to \infty.$$

Let $(u_n^0)_n,\ u_0 \subset X$ *fulfill*

$$u_n^0 \to u_0 \quad \text{and} \quad \mathscr{E}(0, u_n^0) \to \mathscr{E}(0, u_0) \text{ as } n \to \infty. \tag{3.4}$$

Then, there exist a subsequence $(u_{n_k})_k$ *and a curve* $u \in \mathrm{BV}([0, T]; X)$ *such that* $u(0) = u_0$*,*

$$u_{n_k}(t) \to u(t) \ \text{ and } \ \mathscr{E}(t, u_{n_k}(t)) \to \mathscr{E}(t, u(t)) \quad \text{for every } t \in [0, T], \tag{3.5}$$

and u *is a* VE_μ *solution to the rate-independent system* $(X, \mathscr{E}, \mathsf{d})$*.*
We will outline the *proof* of Proposition 1 at the end of Sect. 5.1.

We conclude this section by recalling that, VE solutions as well can be characterized in terms of suitable jump conditions. Namely, it was proved in [19, Prop. 3.8] that a curve $u \in \mathrm{BV}([0, T]; X)$ is a VE solution to the rate-independent system $(X, \mathscr{E}, \mathsf{d})$ if and only if it satisfies (S_D), the energy-dissipation inequality (3.2), and the jump conditions

$$
\begin{aligned}
\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t)) &= \mathsf{c}(t,u(t-),u(t)),\\
\mathscr{E}(t,u(t)) - \mathscr{E}(t,u(t+)) &= \mathsf{c}(t,u(t),u(t+)),\\
\mathscr{E}(t,u(t-)) - \mathscr{E}(t,u(t+)) &= \mathsf{c}(t,u(t-),u(t+))\,.
\end{aligned}
\tag{3.6}
$$

3.2 Main Results: Singular Limits of Visco-Energetic Solutions

We now consider a sequence $(\mu_n)_n \subset (0,\infty)$, either converging to 0, or diverging to ∞. Accordingly, let $(u_n^0)_n \subset X$ be a sequence of initial data for the rate-independent system $(X,\mathscr{E},\mathsf{d})$. Under conditions ($E_1$), ($E_2$) and ($E_3$), there exists a corresponding sequence of Visco-Energetic solutions $(u_n)_n \subset \mathrm{BV}([0,T];X)$ to the rate-independent system $(X,\mathscr{E},\mathsf{d})$, arising from the viscous corrections $\delta_n(u,v) = \frac{\mu_n}{2}\mathsf{d}^2(u,v)$ and satisfying the initial condition $u_n(0) = u_n^0$.

Our first result addresses the behavior of the sequence $(u_n)_n$ in the case $\mu_n \downarrow 0$, under the *sole* conditions (E_1), (E_2) and (E_3) guaranteeing the existence of Visco-Energetic and Energetic solutions, cf. Theorems 1 and 3.

Theorem 1 (Convergence to Energetic Solutions as $\mu \downarrow 0$) *Let $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ comply with (E_1), (E_2) and (E_3). Let $(u_n^0)_n$, $u_0 \subset X$ fulfill (3.4) and suppose that $u_0 \in \mathscr{S}_{\mathsf{d}}(0)$. Let $(\mu_n)_n \subset (0,\infty)$ be a null sequence, and, correspondingly, let $(u_n)_n \subset \mathrm{BV}([0,T];X)$ be a sequence of VE_{μ_n} solutions to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ fulfilling $u_n(0) = u_n^0$.*

Then, there exist a subsequence $(u_{n_k})_k$ and a curve $u \in \mathrm{BV}([0,T];X)$ such that $u(0) = u_0$, convergences (3.5) hold, and and u is an Energetic solution to $(X,\mathscr{E},\mathsf{d})$.

We will prove the convergence (along a subsequence) of a sequence of VE_{μ_n} solutions, as $\mu_n \uparrow \infty$, to a Balanced Viscosity solution, under the same conditions as in the existence Theorem 2 for Balanced Viscosity solutions. Hence we need to strengthen (E_3) with (E_3'), and require the chain-rule inequality (E_4) as well.

Theorem 2 (Convergence to Balanced Viscosity Solutions as $\mu \uparrow \infty$) *Let $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ comply with (E_1), (E_2), (E_3'), and (E_4). Let $(u_n^0)_n$, $u_0 \subset X$ fulfill (3.4). Let $(\mu_n)_n \subset (0,\infty)$ be a diverging sequence, and, correspondingly, let $(u_n)_n \subset \mathrm{BV}([0,T];X)$ be a sequence of VE_{μ_n} solutions to the rate-independent system $(X,\mathscr{E},\mathsf{d})$ fulfilling $u_n(0) = u_n^0$.*

Then, there exist a subsequence $(u_{n_k})_k$ and a curve $u \in \mathrm{BV}([0,T];X)$ such that $u(0) = u_0$, convergences (3.5) hold, and u is an Balanced Viscosity solution to $(X,\mathscr{E},\mathsf{d})$.

Both proofs will be carried out throughout Sects. 4 and 5.

4 Proofs of Theorems 1 and 2

A preliminary Compactness Result

We start with a Helly-type compactness result for a sequence of VE_{μ_n} solutions, associated with parameters $(\mu_n)_n$, which applies both to the limit $\mu_n \downarrow 0$, and to the limit $\mu_n \uparrow \infty$, under the basic conditions (E_1) and (E_2) on $\mathscr{E}$. The key starting observation is that, since

$$\mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u,[0,t]) \geq \mathrm{Var}_{\mathsf{d}}(u,[0,t]) \qquad \text{for every } u \in \mathrm{BV}([0,T];X) \text{ and every } \mu > 0, \tag{4.1}$$

every VE solution complies with the upper energy estimate of the energy balance (E_{d}), cf. (4.2) below, where the (either vanishing or blowing up) parameters μ_n no longer feature. From this energy estimate there stem all the a priori estimates and compactness properties common to the two singular limits $\mu_n \downarrow 0$ and $\mu_n \uparrow \infty$.

Proposition 2 (A Priori Estimates and Compactness) *Let $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ comply with (E_1) and (E_2). Consider a sequence $(u_n)_n \subset \mathrm{BV}([0,T];X)$ of curves starting from initial data $(u_0^n)_n \subset X$ converging to some $u_0 \in X$ as in (3.4). Suppose that the curves u_n fulfill for every $n \in \mathbb{N}$ the upper energy estimate*

$$\mathscr{E}(t,u_n(t)) + \mathrm{Var}_{\mathsf{d}}(u_n,[0,t]) \leq \mathscr{E}(0,u_0^n) + \int_0^t \mathscr{P}(s,u_n(s))\,\mathrm{d}s \tag{4.2}$$

for all $t \in [0,T]$. Set $V_n := V_{u_n}$ (cf. (2.5)).

Then,

$$\exists\, C > 0\ \forall\, n \in \mathbb{N} : \qquad \sup_{t\in[0,T]} \mathscr{F}_0(u_n(t)) + V_n(T) \leq C\,. \tag{4.3}$$

Furthermore, there exist a subsequence $k \mapsto n_k$ and functions $u \in \mathrm{BV}([0,T];X)$, E, $\mathsf{V} \in \mathrm{BV}([0,T])$, and $\mathsf{P} \in L^\infty(0,T)$, such that

$$u_{n_k}(t) \to u(t) \qquad \text{for all } t \in [0,T], \tag{4.4a}$$

$$\mathscr{E}(t,u_{n_k}(t)) \to \mathsf{E}(t) \qquad \text{for all } t \in (0,T], \tag{4.4b}$$

$$V_{n_k}(t) \to \mathsf{V}(t) \qquad \text{for all } t \in (0,T], \tag{4.4c}$$

$$\mathscr{P}(t,u_{n_k}(t)) \rightharpoonup^* \mathsf{P} \qquad \text{in } L^\infty(0,T), \tag{4.4d}$$

so that $u(0) = u_0$ and there hold

$$\mathsf{d}(u(s),u(t)) \leq \mathsf{V}(t) - \mathsf{V}(s) \qquad \text{for all } 0 \leq s \leq t \leq T, \tag{4.5a}$$

$$\mathsf{E}(t) \geq \mathscr{E}(t,u(t)) \qquad \text{for all } t \in (0,T],\ \text{with } \mathsf{E}(0) = \mathscr{E}(0,u_0). \tag{4.5b}$$

Furthermore, for every $t \in \mathrm{J}_u$ *there exist two sequences* $\alpha_k \uparrow t$ *and* $\beta_k \downarrow t$ *such that*

$$u_{n_k}(\alpha_k) \to u(t-) \quad \textit{and} \quad u_{n_k}(\beta_k) \to u(t+)\,. \tag{4.6}$$

Finally, the functions $(u, \mathsf{E}, \mathsf{V}, \mathsf{P})$ *comply with*

$$\mathsf{E}(t) + \mathsf{V}(t) = \mathsf{E}(s) + \mathsf{V}(s) + \int_s^t \mathsf{P}(r)\,\mathrm{d}r \quad \textit{for all } 0 \le s \le t \le T\,. \tag{4.7}$$

The *proof* follows by trivially adapting the argument for [19, Thm. 7.2]. Let us only mention that estimate (4.3) derives from (4.2), where the integral term on the right-hand side involving the power functional is estimated by resorting to the power control (E_2). As for (4.6), it can be shown by suitably adapting the Helly-type compactness argument yielding (4.4a).

In the next Sects. 4.1 and 4.2, we will carry out the proof of Theorem 1 and, respectively, outline the argument for Theorem 2. In fact, in Sect. 5 we will develop the proof of the main technical lower semicontinuity result underlying the limit passage as $\mu_n \uparrow \infty$ in the Visco-Energetic energy balance ($\mathrm{E}_{\mathsf{d},\mathsf{c}_{\mu_n}}$) and leading to the upper energy estimate ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$).

4.1 Proof Theorem 1

We apply Proposition 2 and deduce that there exist a subsequence $(u_{n_k})_k$ of $\mathrm{VE}_{\mu_{n_k}}$ solutions, and a curve $u \in \mathrm{BV}([0,T];X)$, such that (4.4), (4.5), and (4.7) hold. In what follows, for simplicity we shall denote the sequence of curves $(u_{n_k})_k$ by $(u_k)_k$ and accordingly write μ_k in place of μ_{n_k}. We split the argument for proving that the limiting curve u is an Energetic solution in some steps.

Claim 1*: there holds*

$$\begin{aligned} \mathsf{E}(t) = \mathscr{E}(t,u(t)), \quad & \limsup_{k\to\infty} \mathscr{P}(t,u_k(t)) \le \mathscr{P}(t,u(t)) \\ & \text{for all } t \in [0,T] \setminus \widetilde{\mathrm{J}} \text{ with } \widetilde{\mathrm{J}} := \cap_{m\in\mathbb{N}} \cup_{k\ge m} \mathrm{J}_{u_k}\,, \end{aligned} \tag{4.8}$$

i.e., the countable set $\widetilde{\mathrm{J}}$ *is the* lim sup *of the sets* $(\mathrm{J}_{u_k})_k$. As a result,

$$\mathsf{P}(t) \le \mathscr{P}(t,u(t)) \quad \text{for a.a. } t \in (0,T). \tag{4.9}$$

To prove (4.8) at a fixed $t \in [0,T] \setminus \widetilde{\mathrm{J}}$, we observe that, since $t \in [0,T] \setminus \mathrm{J}_{u_k}$ for every $k \ge m$ and $m \in \mathbb{N}$ a given index (only) depending on t, the stability condition for all $y \in X$ and for all $k \ge m$

$$\mathscr{E}(t,u_k(t)) \le \mathscr{E}(t,y) + \mathsf{d}(u_k(t),y) + \frac{\mu_k}{2}\mathsf{d}^2(u_k(t),y) \tag{4.10}$$

holds. We choose $y = u(t)$ in (4.10) and thus deduce that $\limsup_{k\to\infty} \mathscr{E}(t, u_k(t)) \leq \mathscr{E}(t, u(t))$. Hence, we conclude the energy convergence

$$\mathscr{E}(t, u_k(t)) \to \mathscr{E}(t, u(t)) \qquad \text{for all } t \in [0, T] \setminus \widetilde{\mathrm{J}}, \tag{4.11}$$

whence the first of (4.8). The lim sup inequality for the power term in (4.8) follows from (E_3). Then, since the set $\widetilde{\mathrm{J}}$ is negligible, we have for every $t \in (0, T)$ and $r \in (0, (T-t) \wedge t)$

$$\int_{t-r}^{t+r} \mathsf{P}(s)\,\mathrm{d}s \leq \limsup_{k\to\infty} \int_{t-r}^{t+r} \mathscr{P}(s, u_k(s))\,\mathrm{d}s \leq \int_{t-r}^{t+r} \mathscr{P}(s, u(s))\,\mathrm{d}s, \tag{4.12}$$

where the second inequality follows from the second of (4.8) and the Fatou Lemma, taking into account that $\sup_{t\in[0,T]} \mathscr{P}(t, u_k(t)) \leq C_P \sup_{t\in[0,T]} \mathscr{F}(t, u_k(t)) \leq C$ by virtue of (E_2), (2.19), and estimate (4.3). Therefore, (4.9) ensues upon dividing (4.12) by r and taking the limit as $r \downarrow 0$.

Claim* 2*: the curve u complies with

$$\begin{aligned} &\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d}}(u, [s, t]) \leq \mathscr{E}(s, u(s)) + \int_s^t \mathscr{P}(r, u(r))\,\mathrm{d}r \\ &\qquad\qquad\qquad \text{for all } t \in (0, T],\ s \in (0, t) \setminus \widetilde{\mathrm{J}}, \text{ and } s = 0. \end{aligned} \tag{4.13}$$

The upper energy estimate (4.13) ensues from (4.7), taking into account (4.5), (4.8), and (4.9).

Claim* 3*:

$$u(t) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in [0, T] \setminus \widetilde{\mathrm{J}}. \tag{4.14}$$

It follows from passing to the limit as $k \to \infty$ in the stability condition (4.10).

Claim* 4*:

$$\begin{aligned} &u(t-),\ u(t+) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in (0, T), \\ &u(0+) \in \mathscr{S}_{\mathsf{d}}(0), \qquad u(T-) \in \mathscr{S}_{\mathsf{d}}(T). \end{aligned} \tag{4.15}$$

Let us only prove the assertion at $t \in (0, T)$ and for $u(t+)$: since the latter right limit exists, we have that $u(t+) = \lim_{s\downarrow t,\, s\in(t,T)\setminus\widetilde{\mathrm{J}}} u(s)$. Therefore, $u(t+) \in \mathscr{S}_{\mathsf{d}}(t)$ follows from the previously obtained (4.14), combined with the closedness of the stable set $\mathscr{S}_{\mathsf{d}}$, cf. (3.1).

Claim* 5*:

$$u(t) \in \mathscr{S}_{\mathsf{d}}(t) \quad \text{for every } t \in (0, T] \cap \widetilde{\mathrm{J}}. \tag{4.16}$$

Therefore, u complies with the stability condition (S_{d}).

We consider the upper energy estimate (4.13) written on the interval $[s,t]$, for every $s \in (0,t) \setminus \widetilde{\mathrm{J}}$, and then take the limit of the right-hand side as $s \uparrow t$. We use that $u(t-) = \lim_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} u(s)$, and that

$$\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(s)) \leq \mathscr{E}(t,u(t-)). \tag{4.17}$$

This follows from applying the stability condition $u(s) \in \mathscr{S}_{\mathsf{d}}(s)$, which holds at all $s \in (0,t) \setminus \widetilde{\mathrm{J}}$, with competitor $y = u(t-)$. Therefore $\mathscr{E}(s,u(s)) \leq \mathscr{E}(s,u(t-)) + \mathsf{d}(u(s),u(t-))$, which yields

$$\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(s)) \leq \limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \mathscr{E}(s,u(t-)). \tag{4.18}$$

In turn,

$$\begin{aligned} &\limsup_{s\uparrow t,\, s\in(0,t)\setminus\widetilde{\mathrm{J}}} \left(\mathscr{E}(s,u(t-)) - \mathscr{E}(t,u(t-))\right) \\ &\overset{(1)}{\leq} \limsup_{s\uparrow t} \int_s^t |\mathscr{P}(r,u(t-))|\,\mathrm{d}r \\ &\overset{(2)}{\leq} C \limsup_{s\uparrow t}(t-s) = 0 \end{aligned} \tag{4.19}$$

with (1) due to (2.18) and (2) to the power-control estimate

$$|\mathscr{P}(r,u(t-))| \leq C\mathscr{F}_0(u(t-)) \leq C\,. \tag{4.20}$$

In (4.20) the first inequality ensues from (E_2) and (2.19), while the second one from the lower semicontinuity of $u \mapsto \mathscr{F}_0(u)$, which gives $\mathscr{F}_0(u(t-)) \leq \liminf_{s\uparrow t} \mathscr{F}_0(u(s)) \leq C$ thanks to the energy bound $\sup_{t\in[0,T]} \mathscr{F}_0(u(t)) \leq C$, deriving from estimate (4.3) by the lower semicontinuity of $\mathscr{F}_0$. Combining (4.18) with (4.19) we thus conclude (4.17). We also observe that

$$\liminf_{s\uparrow t} \mathrm{Var}_{\mathsf{d}}(u,[s,t]) \geq \mathsf{d}(u(t-),u(t))\,. \tag{4.21}$$

On account of (4.17) and (4.21), from (4.13) we deduce the jump estimate

$$\mathscr{E}(t,u(t)) + \mathsf{d}(u(t-),u(t)) \leq \mathscr{E}(t,u(t-)) \qquad \text{for every } t \in (0,T] \cap \widetilde{\mathrm{J}}. \tag{4.22}$$

We combine this with the previously obtained stability condition (4.15) to conclude (4.16).

Claim 6*:* the curve u complies with the lower energy estimate

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d}}(u, [0, t]) \geq \mathscr{E}(0, u(0)) + \int_0^t \mathscr{P}(r, u(r))\,\mathrm{d}r \tag{4.23}$$

for all $t \in [0, T]$, *and thus with the energy balance* (E_{d}).

We either apply [10, Prop. 2.1.23] or [19, Lemma 6.2, Thm. 6.5], to conclude (4.23) from the previously obtained (S_{d}).

Claim 7*:* the convergence of the energies $\mathscr{E}(t, u_k(t)) \to \mathscr{E}(t, u(t))$ holds at every $t \in [0, T]$.

It follows from (4.4b) and (4.5b) that $\liminf_{k\to\infty} \mathscr{E}(t, u_k(t)) \geq \mathscr{E}(t, u(t))$ for every $t \in [0, T]$. To prove the converse inequality for the lim sup, we resort to a by now classical argument based on the comparison of the energy balances (E_{d}) and ($\mathrm{E}_{\mathsf{d},\mathsf{c}}$). Indeed, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \mathscr{E}(t, u_k(t)) \\
&\overset{(1)}{\leq} \limsup_{k\to\infty} \mathscr{E}(0, u_k^0) + \limsup_{k\to\infty} \int_0^t \mathscr{P}(r, u_k(r))\,\mathrm{d}r - \liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k, [0, t]) \\
&\overset{(2)}{\leq} \mathscr{E}(0, u_0) + \int_0^t \mathscr{P}(r, u(r))\,\mathrm{d}r - \mathrm{Var}_{\mathsf{d}}(u, [0, t]) \overset{(3)}{=} \mathscr{E}(t, u(t))\,,
\end{aligned}$$

with (1) due to ($\mathrm{E}_{\mathsf{d},\mathsf{c}}$), (2) following from the assumed convergence of the initial data (3.4), from (4.4d) combined with (4.9), and from (4.1) and, finally, (3) due to the just obtained energy balance (E_{d}).

This concludes the proof of Theorem 1. ∎

4.2 *Proof Theorem 2*

Proposition 2 ensures that any sequence $(u_n)_n$ of VE solutions, corresponding to parameters $\mu_n \to \infty$, admits a subsequence $(u_{n_k})_k$ converging to a curve $u \in \mathrm{BV}([0, T]; X)$ in the sense of (4.4) and (4.5); as in the proof of Theorem 1, hereafter we will write u_k, μ_k, and c_k in place of u_{n_k}, μ_{n_k}, and c_{μ_k}, respectively. Thanks to the chain rule from condition (E_4), in order to prove that u is a BV solution it is sufficient to verify the local stability ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$) and the upper energy estimate ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$), cf. [14, Prop. 4.2, Thm. 4.3]. The convergence of the energies $\mathscr{E}(t, u_k(t)) \to \mathscr{E}(t, u(t))$ holds at every $t \in [0, T]$ will then follow from comparing the energy balances ($\mathrm{E}_{\mathsf{d},\mathsf{c}}$) and ($\mathrm{E}_{\mathsf{d},\mathsf{v}}$), similarly as in Claim 7 of the proof of Theorem 1.

The Local Stability Condition ($\mathrm{S}_{\mathsf{d},\mathrm{loc}}$)

As in the proof of Theorem 1, we introduce the set $\widetilde{\mathrm{J}} := \cap_{m\in\mathbb{N}} \cup_{k\geq m} \mathrm{J}_{u_k}$. Since D-stability implies local stability, we have that for every $t \in [0,T] \setminus \widetilde{\mathrm{J}}$ there holds

$$|\mathrm{D}\mathscr{E}|(t, u_k(t)) \leq 1 \quad \text{for all } k \geq m, \tag{4.24}$$

with $m \in \mathbb{N}$ depending on t. Taking into account the energy bound (4.3) as well, we are in a position to exploit the lower semicontinuity property ensured by (E_3'). Taking the $\liminf_{k\to\infty}$ of (4.24), we thus deduce that

$$|\mathrm{D}\mathscr{E}|(t, u(t)) \leq 1 \quad \text{for all } t \in [0,T] \setminus \widetilde{\mathrm{J}}. \tag{4.25}$$

We also conclude that

$$|\mathrm{D}\mathscr{E}|(t, u(t-)),\ |\mathrm{D}\mathscr{E}|(t, u(t+)) \leq 1 \quad \text{for all } t \in (0,T), \tag{4.26}$$

and analogously for $|\mathrm{D}\mathscr{E}|(0, u(0+))$ and $|\mathrm{D}\mathscr{E}|(T, u(T-))$, by arguing in the very same way as for **Claim** 4 in the proof of Theorem 2. Clearly, we then have the local stability condition at all points in $[0,T] \setminus \mathrm{J}_u$.

The Upper Energy Estimate ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$)

Combining the energy bound (4.3) and the slope estimate (4.24) with convergence (4.4a) and resorting to (E_3'), we conclude that $\limsup_{k\to\infty} \mathscr{P}(t, u_k(t)) \leq \mathscr{P}(t, u(t))$ for all $t \in [0,T] \setminus \widetilde{\mathrm{J}}$. Therefore, the very same argument as for **Claim** 1 in the proof of Theorem 2 yields that $\mathsf{P}(t) \leq \mathscr{P}(t, u(t))$ for almost all $t \in (0,T)$. All in all, taking the $\liminf_{k\to\infty}$ in ($\mathrm{E}_{\mathsf{d},\mathsf{c}_{\mu_k}}$) and exploiting the initial data convergence (3.4), the previously obtained (4.5b), and the above estimate for P, we infer that

$$\mathscr{E}(T, u(T)) + \liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k, [0,T]) \leq \mathscr{E}(0, u(0)) + \int_0^T \mathscr{P}(r, u(r))\,\mathrm{d}r\,.$$

In order to conclude ($\mathrm{E}_{\mathsf{d},\mathsf{v}}^{\mathrm{ineq}}$), it thus remains to show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_{\mu_k}}(u_k, [0,T]) \geq \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u, [0,T])\,.$$

This will be guaranteed by the upcoming result, whose proof will be developed throughout Sect. 5. ∎

Theorem 4 *Let* $\mathscr{E} : [0,T] \times X \to \mathbb{R}$ *comply with* (E$_1$), (E$_2$), *and* (E$'_3$). *Let* $\mu_k \uparrow \infty$ *and* $(u_k)_k$, $u \in \mathrm{BV}([0,T];X)$ *fulfill*

$$\exists\, C_F > 0\ \forall\, k \in \mathbb{N}\ :\ \sup_{t\in[0,T]} \mathscr{F}_0(u_k(t)) \leq C_F\,, \tag{4.27a}$$

$$u_k(t) \to u(t) \quad \textit{for every } t \in [0,T], \tag{4.27b}$$

$$\begin{aligned} &\forall\, t \in \mathrm{J}_u\ \exists\, (\alpha_k)_k,\ (\beta_k)_k \subset [0,T] \textit{ with} \\ &\alpha_k \uparrow t,\ \beta_k \downarrow t \textit{ and } u_k(\alpha_k) \to u(t-),\ u_k(\beta_k) \to u(t+). \end{aligned} \tag{4.27c}$$

Then,

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \geq \mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[a,b]) \qquad \textit{for all } [a,b] \subset [0,T]. \tag{4.28}$$

5 Proof of Theorem 4

Let us mention in advance the argument for proving the lower semicontinuity inequality (4.28) follows the same steps, outlined below, as those for the lower semicontinuity result [17, Prop. 7.3] in the context of the limit passage from ‘viscous’ gradient systems to BV solutions. Nevertheless, we have to cope with the (nontrivial) technical issues peculiar of the fact that the kind of transitions describing the system behavior at jumps changes upon passing from VE to BV solutions. This problem will be addressed in the proof of Proposition 3 ahead.

Outline of the Proof of Theorem 4

Up to the extraction of a (not relabeled) subsequence and modifying the constant C_F from (4.27a), we may suppose that

$$\sup_k \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \leq C_F, \tag{5.1}$$

too. We introduce a sequence of non-negative and bounded Borel measures η_k by defining them on intervals via

$$\eta_k([a,b]) := \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[a,b]) \quad \text{for all } [a,b] \subset [0,T].$$

In view of (5.1), we have that, up to a further extraction, there exists a Borel measure η such that $\eta_k \rightharpoonup^* \eta$ in duality with $\mathrm{C}([0,T])$. Observe that, by (4.1), we have

$$\eta([a,b]) \geq \limsup_{k\to\infty} \eta_k([a,b]) \geq \limsup_{k\to\infty} \mathrm{Var}_{\mathsf{d}}(u_k,[a,b]) \geq \mathrm{Var}_{\mathsf{d}}(u,[a,b]) \geq \nu_u^{\mathrm{d}}([a,b]),$$

with ν_u^{d} the diffuse measure associated with u via (2.6). Therefore we obtain

$$\eta \geq \nu_u^{\mathrm{d}}\,. \tag{5.2}$$

We now exploit Proposition 3 ahead to conclude that, for every $t \in \mathrm{J}_u$ and any two sequences $\alpha_k \uparrow t$ and $\beta_k \downarrow t$ fulfilling (4.27c), there holds

$$\eta(\{t\}) \geq \limsup_{k\to\infty} \eta_k([\alpha_k,\beta_k]) \geq \liminf_{k\to\infty} \eta_k([\alpha_k,\beta_k]) \geq \mathsf{v}(t,u(t-),u(t+))\,. \tag{5.3}$$

Analogously, we can prove that

$$\begin{aligned} &\limsup_{k\to\infty} \eta_k([\alpha_k,t]) \geq \mathsf{v}(t,u(t-),u(t)),\\ &\limsup_{k\to\infty} \eta_k([t,\beta_k]) \geq \mathsf{v}(t,u(t),u(t+))\,. \end{aligned} \tag{5.4}$$

Arguing in the very same way as in the proof of [17, Prop. 7.3], we combine (5.2)–(5.4) with the representation

$$\begin{aligned} &\mathrm{Var}_{\mathsf{d},\mathsf{v}}(u,[a,b])\\ &= \nu_u^{\mathrm{d}}([a,b]) + \mathrm{Jmp}_{\mathsf{v}}(u;[a,b])\\ &= \nu_u^{\mathrm{d}}([a,b]) + \mathsf{v}(a,u(a),u(a+)) + \mathsf{v}(b,u(b-),u(b))\\ &\qquad + \sum_{t\in \mathrm{J}_u\cap(a,b)} (\mathsf{v}(t,u(t-),u(t)) + \mathsf{v}(t,u(t),u(t+)))\,, \end{aligned}$$

cf. (2.12), to conclude the desired lower semicontinuity inequality (4.28). ■

The proof of the upcoming result is developed throughout Sect. 5.1.

Proposition 3 *Let* $\mathscr{E} : [0,T]\times X \to \mathbb{R}$ *comply with* (E_1), (E_2)*, and* (E_3')*. Let* $\mu_k \uparrow \infty$ *and* $(u_k)_k$, $u \in \mathrm{BV}([0,T];X)$ *fulfill* (4.27) *and* (5.1)*. For every* $t \in \mathrm{J}_u$*, pick two sequences* $(\alpha_k)_k$, $(\beta_k)_k$ *converging to t and fulfilling* (4.27c)*. Then,*

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k,[\alpha_k,\beta_k]) \geq \mathsf{v}(t,u(t-),u(t+))\,. \tag{5.5}$$

5.1 Proof of Proposition 3

We split the argument in some steps, some of which in turn rely on some technical results proved in the Appendix.

Step 1: Reparameterization

The curve u_k has at most countably many jump points $(t^k_m)_{m\in M_k}$ between the points α_k and β_k. We now suitably reparameterize both the continuous pieces of the trajectory u_k, as well as the optimal transitions ϑ^k_j connecting the left and right limits $u_k(t^k_j-)$ and $u_k(t^k_j+)$ at a jump point t^k_j. We will then glue all of them together to obtain a sequence of curves $(\mathfrak{u}_k)_k$, defined on compact sets $(C_k)_k$, which shall enjoy suitable estimates (cf. Step 2), allowing for a refined compactness argument both for the curves $\mathfrak{u}_k$ and for the sets C_k.

We set

$$\mathfrak{m}_k := \beta_k - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) + \sum_{m\in M_k} 2^{-m}$$

and define the rescaling function $\mathfrak{s}_k : [\alpha_k, \beta_k] \to [0, \mathfrak{m}_k]$ by

$$\mathfrak{s}_k(t) := t - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, t]) + \sum_{\{m\in M_k :\, t^k_m \le t\}} 2^{-m} .$$

Observe that $\mathfrak{s}_k$ is strictly increasing, with jump set $\mathrm{J}_{\mathfrak{s}_k} = (t^k_m)_{m\in M_k}$. We introduce the notation

$$I^k_m := (\mathfrak{s}_k(t^k_m-), \mathfrak{s}_k(t^k_m+)), \quad I_k := \cup_{m\in M_k} I^k_m, \quad \Lambda_k := [\mathfrak{s}_k(\alpha_k), \mathfrak{s}_k(\beta_k)].$$

On $\Lambda_k \setminus I_k$ the inverse $\mathfrak{t}_k : \Lambda_k \setminus I_k \to [\alpha_k, \beta_k]$ of $\mathfrak{s}_k$ is well defined and Lipschitz continuous. We set

$$\mathfrak{u}_k(s) := (u_k \circ \mathfrak{t}_k)(s) \qquad \text{for all } s \in \Lambda_k \setminus I_k . \tag{5.6}$$

The curve $\mathfrak{u}_k$ is also Lipschitz, and satisfies

$$\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(\mathfrak{u}_k, [s_0, s_1]) \le (s_1 - s_0) \quad \text{for all } [s_0, s_1] \subset \Lambda_k \setminus I_k . \tag{5.7}$$

We check (5.7) in the case in which $s_0 = \mathfrak{s}_k(t_0)$ and $s_1 = \mathfrak{s}_k(t_1)$, with $t_0 < t_1$ belonging to the same connected component of $[\alpha_k, \beta_k] \setminus (t^k_m)_{m\in M_k}$ (the other case is completely analogous). Then, we observe that

$$s_1 - s_0 = \mathfrak{s}_k(t_1) - \mathfrak{s}_k(t_0) = t_1 - t_0 + \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [t_0, t_1]) \ge \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(\mathfrak{u}_k, [s_0, s_1]) .$$

We now recall [19, Thm. 3.14], ensuring that at every jump point t_m^k there exists an optimal transition ϑ_m^k that is continuous on a compact set E_m^k, *tight* (i.e. it fulfills $\vartheta_m^k(J^-) \neq \vartheta_m^k(J^+)$ for every "hole" $J \in \mathfrak{h}(E_m^k)$), and such that

$$\begin{aligned}
& u(t_m^k-) = \vartheta_m^k((E_m^k)^-), \qquad u(t_m^k+) = \vartheta_m^k((E_m^k)^+), \qquad u(t_m^k) \in \vartheta_m^k(E_m^k), \\
& \mathscr{E}(t_m^k, u(t_m^k-)) - \mathscr{E}(t_m^k, u(t_m^k+)) = \mathsf{c}(t_m^k, u(t_m^k-), u(t_m^k+)) \\
& = \mathrm{Trc}_{\mathrm{VE}}(t_m^k, \vartheta_m^k, E_m^k) \\
& = \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \sum_{r \in E_m^k \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)) \,.
\end{aligned} \tag{5.8}$$

We adapt the calculations from [19, Lemma 5.1] and define the rescaling function $\boldsymbol{\sigma}_m^k$ on E_m^k by

$$\begin{aligned}
\boldsymbol{\sigma}_m^k(t) := & \frac{1}{2^m} \frac{t - (E_m^k)^-}{(E_m^k)^+ - (E_m^k)^-} + \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k \cap [(E_m^k)^-, t]) \\
& + \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k \cap [(E_m^k)^-, t]) + \sum_{r \in [(E_m^k)^-, t] \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)) + \mathfrak{s}_k(t_m^k-)
\end{aligned}$$

for all $t \in E_m^k$. It can be checked that $\boldsymbol{\sigma}_m^k$ is continuous and strictly increasing, with image a compact set $S_m^k \subset I_m^k$ such that

$$\begin{aligned}
& (S_m^k)^- = \boldsymbol{\sigma}_m^k((E_m^k)^-) = \mathfrak{s}_k(t_m^k-) \quad \text{and} \\
& (S_m^k)^+ = \boldsymbol{\sigma}_m^k((E_m^k)^+) \\
& = \frac{1}{2^m} + \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k) + \sum_{r \in E_m^k \setminus (E_m^k)^+} \mathscr{R}(t_m^k, \vartheta_m^k(r)) + \mathfrak{s}_k(t_m^k-) \\
& = \mathfrak{s}_k(t_m^k+) \,.
\end{aligned}$$

The inverse function $\boldsymbol{\tau}_m^k : S_m^k \to E_m^k$ is Lipschitz continuous.

We then introduce the set

$$C_k := (\Lambda_k \setminus I_k) \cup (\cup_{m \in M_k} S_m^k) \,.$$

It is not difficult to check that C_k is a closed subset of Λ_k. We extend the functions t_k and u_k, so far defined on $\Lambda_k \setminus I_k$, only, to the set C_k by setting

$$\mathsf{t}_k(s) \equiv t_m^k \quad \text{and} \quad \mathsf{u}_k(s) := \vartheta_m^k(\boldsymbol{\tau}_m^k(s)) \qquad \text{whenever } s \in S_m^k \text{ for some } m \in M_k.$$

Since $u(t_m^k-) = \vartheta_m^k((E_m^k)^-)$ and $u(t_m^k+) = \vartheta_m^k((E_m^k)^+)$, we have that the extended curve $\mathfrak{u}_k \in \mathrm{C}(C_k; X)$. Furthermore, $\mathfrak{u}_k \in \mathrm{BV}(C_k; X)$: indeed,

$$\begin{aligned} \mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k) = \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k), \qquad \mathrm{GapVar}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k) = \mathrm{GapVar}_{\mathsf{d}}(\vartheta_m^k, E_m^k), \\ \sum_{s \in S_m^k \setminus \{(S_m^k)^+\}} \mathscr{R}(t_m^k, \mathfrak{u}_k(s)) = \sum_{r \in E_m^k \setminus \{(E_m^k)^+\}} \mathscr{R}(t_m^k, \vartheta_m^k(r)), \end{aligned} \tag{5.9a}$$

as well as

$$\mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, S_m^k \cap [s_0, s_1]) \le (s_1 - s_0) \qquad \text{for all } s_0, s_1 \in S_m^k \text{ with } s_0 < s_1. \tag{5.9b}$$

Step 2: A Priori Estimates

It follows from (5.1) and from the fact that $(\beta_k - \alpha_k) \downarrow 0$, that

$$C_k^+ = \mathfrak{m}_k \le \beta_k - \alpha_k + \mathrm{Var}_{\mathsf{d},\mathsf{c}}(u_k, [\alpha_k, \beta_k]) + 2 \le 2C_F \tag{5.10}$$

(up to modifying the constant C_F). Moreover, in view of (5.1), (5.7), and (5.9b) we have

$$\sup_{k \in \mathbb{N}} \mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, C_k) \le C, \tag{5.11a}$$

$$\mathrm{Var}_{\mathsf{d}}(\mathfrak{u}_k, C_k \cap [s_0, s_1]) \le (s_1 - s_0) \quad \text{for all } s_0, s_1 \in C_k,\ s_0 < s_1,\ \text{and } k \in \mathbb{N}. \tag{5.11b}$$

Finally, we remark that

$$\sup_{k \in \mathbb{N}} \sup_{s \in C_k} \mathscr{F}_0(\mathfrak{u}_k(s)) \le C_F. \tag{5.11c}$$

Indeed, we have that

$$\sup_{s \in \Lambda_k \setminus I_k} \mathscr{F}_0(\mathfrak{u}_k(s)) = \sup_{t \in [\alpha_k, \beta_k] \setminus (t_m^k)_{m \in M_k}} \mathscr{F}_0(u_k(t)) \le C_F$$

in view of (4.27). Furthermore, it follows from [19, Thm. 3.16] that for all $r \in E_m^k$ there holds

$$\begin{aligned} \mathscr{E}(t_m^k, \vartheta_m^k(r)) + \mathsf{d}(\vartheta_m^k(r), \vartheta_m^k((E_m^k)^-)) &\le \mathscr{E}(t_m^k, \vartheta_m^k(r)) + \mathrm{Var}_{\mathsf{d}}(\vartheta_m^k, E_m^k \cap [(E_m^k)^-, r]) \\ &\le \mathscr{E}(t_m^k, \vartheta_m^k((E_m^k)^-)) = \mathscr{E}(t_m^k, u_k(t_m^k-)) \,. \end{aligned}$$

Therefore,

$$\sup_{s\in S_m^k} \mathscr{F}_0(\mathfrak{u}_k(s)) = \sup_{r\in E_m^k} \mathscr{F}_0(\vartheta_m^k(r)) \le \mathscr{F}_0(u_k(t_m^k-)) \le C_F\,.$$

All in all, we conclude (5.11c).

Step 3: Compactness

By virtue of estimates (5.11), we are in a position to apply the compactness result [19, Thm. 5.4] and conclude that there exist a (not relabeled) subsequence, a compact set $C \subset [0, 2C_F]$, and a function $\mathfrak{u} \in \mathrm{BV}(C; X)$ such that, as $k \to \infty$, there hold

1. $C_k \to C$ à la Kuratowski;
2. $\mathrm{graph}(\mathfrak{u}) \subset \mathrm{Li}_{k\to\infty}\mathrm{graph}(\mathfrak{u}_k)$;
3. whenever $(s_k)_k \in C_k$ converge to $s \in C$, then $\mathfrak{u}_k(s_k) \to \mathfrak{u}(s)$;
4. $\mathfrak{u}_k((C_k)^\pm) \to \mathfrak{u}(C^\pm)$.

Therefore, $\mathfrak{u}(C^-) = u(t-)$, and $\mathfrak{u}(C^+) = u(t+)$. Furthermore, it follows from (5.11b) that the curve $\mathfrak{u}$ is Lipschitz on C. Finally, for later use let us point out that, since the functions $\mathfrak{t}_k$ take values in the intervals $[\alpha_k, \beta_k]$ shrinking to the singleton $\{t\}$, there holds

$$\lim_{k\to\infty} \sup_{s\in C_k} |\mathfrak{t}_k(s) - t| = 0. \tag{5.12}$$

Step 4: Connectedness of *C*

Observe that, since the sets C_k are not, in general, connected, we cannot immediately deduce that C is connected. We will however show that,

$$\forall\, I \in \mathfrak{h}(C) \text{ there holds } \mathfrak{u}(I^-) = \mathfrak{u}(I^+) =: \mathfrak{u}_I. \tag{5.13}$$

In view of this, we may extend $\mathfrak{u}$ to the whole interval $[0, C^+]$ by defining

$$\mathfrak{u}(s) := \mathfrak{u}_I \qquad \text{for all } s \in I \quad \text{for all } I \in \mathfrak{h}(C).$$

Hereafter, we will replace C by $[0, C^+]$. We will split the proof of (5.13) in two claims.

Claim 1**:** for every $I \in \mathfrak{h}(C)$ there exist J_k such that

$$J_k \in \mathfrak{h}(C_k) \text{ and } \lim_{k\to\infty} J_k^- = I^-, \quad \lim_{k\to\infty} J_k^+ = I^+. \tag{5.14}$$

This follows by repeating the very same arguments as in the proof of [19, Thm. 5.3].

Claim* 2*: there holds $\mathfrak{u}(I^-) = \mathfrak{u}(I^+)$. In view of the compactness property (3) from Step 3, there holds $\mathfrak{u}_k(J_k^\pm) \to \mathfrak{u}(I^\pm)$. Therefore,

$$\begin{aligned}
\mathsf{d}(\mathfrak{u}(I^-), \mathfrak{u}(I^+)) &= \lim_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(J_k^-), \mathfrak{u}_k(J_k^+)) \\
&\leq \limsup_{k\to\infty} \frac{1}{2\mu_k^{1/2}} \left(\mu_k \mathsf{d}^2(\mathfrak{u}_k(J_k^-), \mathfrak{u}_k(J_k^+)) + 1\right) \\
&\leq \limsup_{k\to\infty} \frac{1}{\mu_k^{1/2}} \left(\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) + 1\right) = 0,
\end{aligned}$$

where we have used Young's equality and estimate (5.1).

Step 5: Estimate of the Transition Cost and Conclusion of the Proof

With Steps 3 and 4 we have shown that the Lipschitz continuous curve $\mathfrak{u}$ is defined on the interval $[0, C^+]$ and connects the left and right limits $u(t-)$ and $u(t+)$. We now aim to prove that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) \geq \mathrm{Trc}_{\mathrm{BV}}(t, \mathfrak{u}, [0, C^+]) \geq \mathsf{v}(t, u(t-), u(t+)), \tag{5.15}$$

which will lead to (5.5).

Indeed, it follows from Lemma 1 that

$$\begin{aligned}
\mathrm{Trc}_{\mathrm{BV}}(t, \mathfrak{u}, [0, C^+]) &= \int_0^{C^+} |\mathfrak{u}'|(s) \left(|\mathrm{D}\mathscr{E}|(t, \mathfrak{u}(s)) \vee 1\right) \mathrm{d}s \\
&= \sup \Bigg\{ \sum_{i=1}^N \mathsf{d}(\mathfrak{u}(\sigma_{i-1}), \mathfrak{u}(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} \left(|\mathrm{D}\mathscr{E}|(t, \mathfrak{u}(\sigma)) \vee 1\right) \ : \\
&\qquad\qquad (\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0, C^+]) \Bigg\}.
\end{aligned} \tag{5.16}$$

Therefore, in what follows we will prove that

$$\begin{aligned}
&\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) \\
&\geq \sum_{i=1}^N \mathsf{d}(\mathfrak{u}(\sigma_{i-1}), \mathfrak{u}(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} \left(|\mathrm{D}\mathscr{E}|(t, \mathfrak{u}(\sigma)) \vee 1\right)
\end{aligned} \tag{5.17}$$

for every $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0, C^+])$.

Let us consider a given partition $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0, C^+])$ and fix an index $j \in \{1, \ldots, N\}$. Preliminarily, we observe that, by the compactness property (1) in

Step 3, there exist sequences $(\sigma_{j-1}^k)_k$, $(\sigma_j^k)_k \subset C_k$ such that, as $k \to \infty$, there holds

$$\sigma_{j-1}^k \to \sigma_{j-1},\ \sigma_j^k \to \sigma_j \quad \text{and} \quad \mathfrak{u}_k(\sigma_{j-1}^k) \to \mathfrak{u}(\sigma_{j-1}),\ \mathfrak{u}_k(\sigma_j^k) \to \mathfrak{u}(\sigma_j) \tag{5.18}$$

where the second convergence follows from the compactness property (3). We now distinguish two cases

1. $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) \vee 1) = 1$;
2. $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} |\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > 1$.

Clearly, the second case is equivalent to $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) \vee 1) > 1$.
Case (1): In view of (5.18), we have

$$\begin{aligned} &\mathsf{d}(\mathfrak{u}(\sigma_{j-1}),\mathfrak{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) \vee 1) \\ &= \lim_{k\to\infty} \mathsf{d}(\mathfrak{u}_k(\sigma_{j-1}^k),\mathfrak{u}_k(\sigma_j^k))\,. \end{aligned} \tag{5.19}$$

Case (2): We have that $|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > \delta > 1$ for all $\sigma \in [\sigma_{j-1},\sigma_j]$. First of all, we observe that

$$\exists\,\bar\delta \in (1,\delta) \quad \exists\,\bar k \in \mathbb{N} \quad \inf_{k\geq\bar k}\ \inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \geq \bar\delta\,. \tag{5.20}$$

To show this, we argue by contradiction and suppose that there exists a (not relabeled) subsequence along which $\inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \leq 1$. Since for every $k \in \mathbb{N}$ the inf on the compact set $[\sigma_{j-1}^k,\sigma_j^k] \cap C_k$ is attained by lower semicontinuity of the map $\sigma \mapsto |\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma))$, we deduce that there exists a sequence $(\tilde\sigma_k)_k$ with $|\mathrm{D}\mathscr{E}|(\mathfrak{t}_k(\tilde\sigma_k),\mathfrak{u}_k(\tilde\sigma_k)) \leq 1$, converging up to a subsequence to some $\tilde\sigma \in [\sigma_{j-1},\sigma_j]$. Now, $\mathfrak{t}_k(\tilde\sigma_k) \to t$ by (5.12) and $\mathfrak{u}_k(\tilde\sigma_k) \to \mathfrak{u}(\tilde\sigma)$ by the compactness property (3) from Step 3. Hence, using the lower semicontinuity of $|\mathrm{D}\mathscr{E}|$ granted by (E_3') we conclude that $|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\tilde\sigma)) \leq 1$, in contradiction with the assumption that $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} |\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > 1$.

Observe that (5.20) implies that $\mathscr{R}(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) > 0$ for all $\sigma \in [\sigma_{j-1}^k,\sigma_j^k] \cap C_k$ and all $k \geq \bar k$. We now deduce the *uniform* positivity property

$$\exists\, r > 0 \qquad \inf_{k\geq\bar k}\ \inf_{\sigma\in[\sigma_{j-1}^k,\sigma_j^k]\cap C_k} \mathscr{R}(\mathfrak{t}_k(\sigma),\mathfrak{u}_k(\sigma)) \geq r\,. \tag{5.21}$$

Indeed, as for (5.20) we proceed by contradiction: if (5.21) did not hold, there would exist a sequence $(\tilde\sigma_k)_k$ with $\mathscr{R}(\mathfrak{t}_k(\tilde\sigma_k),\mathfrak{u}_k(\tilde\sigma_k)) \to 0$, converging to some $\tilde\sigma \in [\sigma_{j-1},\sigma_j]$ that would fulfill $\mathscr{R}(t,\mathfrak{u}(\tilde\sigma)) = 0$ by the lower semicontinuity of $\mathscr{R}$. Now, by property (2.25), $\mathscr{R}(t,\mathfrak{u}(\tilde\sigma)) = 0$ would imply that $(t,\mathfrak{u}(\tilde\sigma))$ belongs to the stable set $\mathscr{S}_{\mathsf{D}}$. In turn, the D-stability condition (2.23) would imply that $|\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\tilde\sigma)) \leq 1$, against the standing assumption that $\inf_{\sigma\in[\sigma_{j-1},\sigma_j]} |\mathrm{D}\mathscr{E}|(t,\mathfrak{u}(\sigma)) > 1$.

Now, (5.21) entails that $\mathsf{t}_k(\sigma) \in (t^k_m)_{m\in M_k}$ for all $\sigma \in [\sigma^k_{j-1}, \sigma^k_j] \cap C_k =: \mathscr{L}_k$. But then, it is not difficult to realize that the function t_k must be constant on $\mathscr{L}_k$. Namely, there exists $m_k \in M_k$ such that $\mathsf{t}_k(\sigma) \equiv t^k_{m_k}$ for all $\sigma \in \mathscr{L}_k$. It was observed in [19, Rmk. 3.15] that the set $C^{\mathscr{R}}_k := \{s \in S^k_{m_k} \setminus \{(S^k_{m_k})^+\} : \mathscr{R}(t^k_{m_k}, \mathsf{u}_k(s)) > 0\}$ is discrete. Trivially adapting the argument from [19, Rmk. 3.15], from (5.21) we in fact conclude that for all $k \geq \bar{k}$ the set $\mathscr{L}_k \subset C^{\mathscr{R}}_k$ consists of finitely many points $(r^k_\ell)^{L_k}_{\ell=1}$, and that the cardinality L_k of the sets $\mathscr{L}_k$ is uniformly bounded with respect to k, i.e.

$$\sup_{k\geq\bar{k}} L_k \leq C < \infty. \tag{5.22}$$

Furthermore, notice that r^k_ℓ is the extremum of a hole of C_k for every $\ell = 1, \ldots, L_k$. The compactness statement from Step 3 (cf. again [19, Thm. 5.4]) applies, yielding that, up to a subsequence,

1. the sets $(\mathscr{L}_k)_k$ converge in the sense of Kuratowski to a finite, thanks to (5.22), set $\mathscr{L} = (r_l)^L_{l=1} \subset [\sigma_{j-1}, \sigma_j]$, such that $\sigma_{j-1}, \sigma_j \in \mathscr{L}$.
2. for every $r_l \in \mathscr{L}$ there exists a sequence $(r^k_{\ell_k}(l))_k$, with $r^k_{\ell_k}(l) \in \mathscr{L}_k$ for every $k \in \mathbb{N}$, such that $\mathsf{u}_k(r^k_{\ell_k}(l)) \to \mathsf{u}(r_l)$. From now on, we will use the simplified notation $r_k(l)$ in place of $r^k_{\ell_k}(l)$;
3. whenever $r^{k_n}_{\ell_n} \in \overline{\mathscr{L}_{k_n}}$ converge to some $r_l \in \mathscr{L}$ as $n \to \infty$, then $\mathsf{u}_{k_n}(r^{k_n}_{\ell_n}) \to \mathsf{u}(r_l)$.

We now estimate $\mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\sigma)) \vee 1)$ by interpolating between the points σ_{j-1} and σ_j the points $\mathscr{L} = (r_l)^L_{l=1}$. Thus we have

$$\begin{aligned}
&\mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\sigma)) \vee 1) \\
&\leq \mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) + \mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) \inf_{\sigma\in[\sigma_{j-1},\sigma_j]} (|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(\sigma)) - 1) \\
&\leq \mathsf{d}(\mathsf{u}(\sigma_{j-1}), \mathsf{u}(\sigma_j)) + \sum_{l=1}^{L} \mathsf{d}(\mathsf{u}(r_{l-1}), \mathsf{u}(r_l)) \, (|\mathrm{D}\mathscr{E}|(t, \mathsf{u}(r_l)) - 1) \\
&\overset{(1)}{\leq} \liminf_{k\to\infty} \mathsf{d}(\mathsf{u}_k(\sigma^k_{j-1}), \mathsf{u}_k(\sigma^k_j)) \\
&\qquad + \sum_{l=1}^{L} \liminf_{k\to\infty} \mathsf{d}(\mathsf{u}_k(r_k(l-1)), \mathsf{u}_k(r_k(l))) \sqrt{2\mu_k \mathscr{R}(t^k_{m_k}, \mathsf{u}_k(r_k(l)))} \\
&\overset{(2)}{\leq} \liminf_{k\to\infty} \mathsf{d}(\mathsf{u}_k(\sigma^k_{j-1}), \mathsf{u}_k(\sigma^k_j)) + \liminf_{k\to\infty} \sum_{l=1}^{L} \frac{\mu_k}{2} \mathsf{d}^2(\mathsf{u}_k(r_k(l-1)), \mathsf{u}_k(r_k(l))) \\
&\qquad\qquad + \liminf_{k\to\infty} \sum_{l=1}^{L} \mathscr{R}(t^k_{m_k}, \mathsf{u}_k(r_k(l))) \, .
\end{aligned} \tag{5.23}$$

For (1), we have used that for every $l = 1, \ldots, L$ there exists a sequence $(r_k(l))_k$ fulfilling the aforementioned convergence property (2), and applied the forthcoming Lemma 3 with the choice $\psi(r) = r + \frac{1}{2}r^2$ (cf. (1.3)), so that $\psi^*(S) = \frac{1}{2}((S-1)_+)^2$, with $\tau_k := \mu_k^{-1}$, with $t_k := t^k_{m_k} \to t$ as $k \to \infty$, and with $u_k := \mathrm{u}_k(r_k(l)) \to \mathrm{u}(r_l)$. We then conclude that (cf. (A.5) ahead for the definition of the generalized Moreau-Yosida approximation $\mathscr{Y}^{\psi}_{\mu_k^{-1}}(\mathscr{E})$)

$$
\begin{aligned}
(|\mathrm{D}\mathscr{E}|(t, \mathrm{u}(r_l)) - 1) &= (|\mathrm{D}\mathscr{E}|(t, \mathrm{u}(r_l)) - 1)_+ \\
&\leq \liminf_{k \to \infty} \sqrt{2\mu_k \left(\mathscr{E}(t^k_{m_k}, \mathrm{u}_k(r_k(l))) - \mathscr{Y}^{\psi}_{\mu_k^{-1}}(\mathscr{E})(t^k_{m_k}, \mathrm{u}_k(r_k(l))) \right)} \\
&= \liminf_{k \to \infty} \sqrt{2\mu_k \mathscr{R}(t, \mathrm{u}_k(r_k(l)))} \qquad \text{for all } l = 1, \ldots, L.
\end{aligned}
\tag{5.24}
$$

Finally, for (2) in (5.23) we have applied Young's inequality.

Observe that the term multiplied by μ_k featuring on the right-hand side of (5.23) involves points that are extrema of holes in C_k. Therefore, it is estimated by $\mathrm{GapVar}_{\mathsf{d}}(\mathrm{u}_k, C_k)$, whereas the third term is bounded by $\sum_{s \in S^k_{m_k} \setminus \{(S^k_{m_k})^+\}} \mathscr{R}(t^k_{m_k}, \mathrm{u}_k(s))$. Combining (5.19), and (5.23), and summing over all the points of $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([0, C^+])$, we conclude the desired (5.17). This finishes the proof of Theorem 4. ■

We conclude this section by giving the

Outline of the Proof of Proposition 1

The argument borrows some ideas both from the proof of Theorem 1, and of Theorem 2. Let us briefly sketch its steps.

- **Compactness:** We again apply Proposition 2 and deduce the existence of a subsequence $(u_{n_k})_k$ converging to some $u \in \mathrm{BV}([0, T]; X)$ in the sense of (4.4) and (4.5); hereafter we will again use the short-hands u_k, μ_k, and c_k in place of u_{n_k}, μ_{n_k}, and c_{μ_k}, respectively. We will use the notation

$$
\mathsf{D}_{\mu_k}(u, v) := \mathsf{d}(u, v) + \tfrac{\mu_k}{2}\mathsf{d}^2(u, v), \qquad \mathsf{D}_{\mu}(u, v) := \mathsf{d}(u, v) + \tfrac{\mu}{2}\mathsf{d}^2(u, v),
$$

  and write $\mathrm{GapVar}^{\mu_k}_{\mathsf{d}}$, $\mathrm{GapVar}^{\mu}_{\mathsf{d}}$, $\mathscr{R}^{\mu_k}$, $\mathscr{R}^{\mu}$.
- **The D_μ-stability condition:** As in *Claim* 1 within the proof of Theorem 1, we introduce the set $\widetilde{J} = \cap_{m \in \mathbb{N}} \cup_{k \geq m} \mathrm{J}_{u_k}$. First, we prove that the limit curve u fulfills the stability condition ($\mathrm{S}_{\mathsf{D}_\mu}$) at every $t \in [0, T] \setminus \widetilde{J}$ by passing to the limit as $k \to \infty$ in the D_{μ_k}-stability condition for the curves u_k, holding on $[0, T] \setminus \mathrm{J}_{u_k}$. Secondly, we deduce the validity of the D_μ-stability condition at every $t \in [0, T] \setminus \mathrm{J}_u$ by density argument, similarly as in the proof of Theorem 1,

Claim 4. Here we exploit the closedness of the D_μ-stable set $\mathscr{S}_{\mathsf{D}_\mu}$, which is in turn ensured by the lower semicontinuity of $\mathscr{R}^\mu$.

- **The upper energy estimate $\leq$ in** ($\mathrm{E}_{\mathsf{d},\mathsf{c}_\mu}$): We show that for all $t \in [0, T]$

$$\mathscr{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u, [0, t]) \leq \mathscr{E}(0, u(0)) + \int_0^t \mathscr{P}(s, u(s))\,\mathrm{d}s \qquad (5.25)$$

by taking the $\liminf_{k\to\infty}$ in the analogous upper energy estimate for the curves $(u_k)_k$. Let us only comment on the proof of the key lower semicontinuity inequality

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [a, b]) \geq \mathrm{Var}_{\mathsf{d},\mathsf{c}_\mu}(u, [a, b]) \qquad \text{for all } [a, b] \subset [0, T], \qquad (5.26)$$

since for dealing with the other terms in (5.25) we repeat the very same arguments as in the proofs of Theorems 1 and 2.

First of all, we may suppose that the sequence $(u_k)_k$ complies with the conditions (4.27) of Theorem 4. Along the footsteps of the proof of Theorem 4, we introduce the Borel measures $\eta_k([a, b]) := \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [a, b])$ and show that, up to a subsequence, they converge to a measure $\eta \geq \nu_u^{\mathsf{d}}$. It then remains to deduce that $\eta(\{t\}) \geq \mathsf{c}(t, u(t-), u(t+)) \quad$ for all $t \in \mathrm{J}_u$, as well as the analogue of (5.4), to conclude (5.26). With this aim we adapt the proof of Proposition 3 to show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) \geq \mathsf{c}(t, u(t-), u(t+))$$

at every point $t \in \mathrm{J}_u$, and for every pair of sequences $(\alpha_k)_k$, $(\beta_k)_k$ converging to t and fulfilling (4.27c). Hence, we reparameterize the curves u_k in the very same way as in Step 1 of the proof of Proposition 3. By virtue of the a priori estimates from Step 2, the compactness arguments in Step 3 yield the existence of a Lipschitz continuous limit curve $\mathfrak{u} : C \to X$, with $C \Subset [0, \infty)$ and $\mathfrak{u}(C^-) = u(t-)$, $\mathfrak{u}(C^+) = u(t+)$. Here, we can no longer replace C with the interval $[0, C^+]$ as in the proof of Proposition 3, but we can still observe property (5.14), based on [19, Thm. 5.3]. We now show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d},\mathsf{c}_k}(u_k, [\alpha_k, \beta_k]) \geq \mathrm{Trc}_{\mathrm{VE}}(t, \mathfrak{u}, C) \geq \mathsf{c}(t, u(t-), u(t+))\,. \qquad (5.27)$$

The lim inf-inequality for the $\mathrm{Var}_{\mathsf{d}}$ contribution to $\mathrm{Var}_{\mathsf{d},\mathsf{c}_k}$ easily follows from the aforementioned compactness arguments. For the $\mathrm{GapVar}_{\mathsf{d}}^{\mu_k}$-contribution (which depends on the parameter μ_k via the viscous correction $\frac{\mu_k}{2}\mathsf{d}^2$), it is essential to use property (5.14). For the $\mathscr{R}^{\mu_k}$ contribution, we can adapt the arguments from the discussion of Case (2) in Step 5 of the proof of Proposition 3, also exploiting the lim inf-estimate

$$(t_k \to t,\ x_k \to x) \ \Rightarrow \ \liminf_{k\to\infty} \mathscr{R}^{\mu_k}(t_k, x_k) \geq \mathscr{R}^{\mu}(t,x).$$

This concludes the proof of (5.26).

- **The lower energy estimate** $\geq$ **in** ($\mathrm{E}_{\mathsf{d},\mathsf{c}_\mu}$): It follows from [19, Thm. 6.5]. Again, the energy convergence $\mathscr{E}(t, u_k(t)) \to \mathscr{E}(t, u(t))$ for every $t \in [0, T]$ follows from the limit passage in the energy balance. ■

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Appendix

Auxiliary Results

We start with the proof of the representation formula (5.16) for the transition cost $\mathrm{Trc}_{\mathrm{BV}}(t, \mathrm{u}, [0, C^+])$. In the upcoming statement, we replace the functional $u \mapsto |\mathrm{D}\mathscr{E}|(t,u) \vee 1$ by a general

$$g : X \to \mathbb{R} \qquad \text{positive and lower semicontinuous.}$$

Lemma 1 *Let* $v \in \mathrm{AC}([a,b]; X)$. *Then, there holds*

$$\begin{aligned} &\int_a^b |v'|(s) g(v(s))\,\mathrm{d}s \\ &= \sup\left\{ \sum_{i=1}^N \mathsf{d}(v(\sigma_{i-1}), v(\sigma_i)) \inf_{\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\sigma)) \ :\ (\sigma_i)_{i=1}^N \in \mathfrak{P}_f([a,b]) \right\} \qquad \text{(A.1)} \\ &=: \mathsf{S}\,. \end{aligned}$$

In particular, the map $s \mapsto |v'|(s)g(v(s))$ *is integrable on* $[a,b]$ *if and only if* $\mathsf{S} < \infty$.

Proof Let us fix $(\sigma_i)_{i=1}^N \in \mathfrak{P}_f([a,b])$. Observe that

$$\begin{aligned} \mathsf{d}(v(\sigma_{i-1}), v(\sigma_i)) \inf_{\tilde\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\tilde\sigma)) &\overset{(1)}{\leq} \int_{\sigma_{i-1}}^{\sigma_i} |v'|(\sigma) \inf_{\tilde\sigma\in[\sigma_{i-1},\sigma_i]} g(v(\tilde\sigma))\,\mathrm{d}\sigma \\ &\leq \int_{\sigma_{i-1}}^{\sigma_i} |v'|(\sigma) g(v(\sigma))\,\mathrm{d}\sigma \end{aligned}$$

with (1) due to (2.14). Therefore, upon summing up over the index $i = 1, \ldots, N$ and using that $(\sigma_i)_{i=1}^N$ is arbitrary, we conclude

$$\int_a^b |v'|(s) g(v(s))\,\mathrm{d}s \geq \mathsf{S}\,.$$

As for the converse inequality, we now consider a partition $a = \sigma_1 < \ldots < \sigma_i < \ldots = \sigma_N = b$ with fineness $\tau := \max_{i=1,\ldots,N}(\sigma_i - \sigma_{i-1})$ and introduce the functions

$$\underline{\sigma}_\tau,\, \overline{\sigma}_\tau : [a,b] \to [a,b] \quad \text{defined by} \quad \begin{cases} \overline{\sigma}_\tau(s) := \sigma_i & \text{if } s \in (\sigma_{i-1}, \sigma_i], \\ \underline{\sigma}_\tau(s) := \sigma_{i-1} & \text{if } s \in [\sigma_{i-1}, \sigma_i), \end{cases}$$

with $\underline{\sigma}_\tau(b) := b$ and $\overline{\sigma}_\tau(a) := a$. Taking into account the definition (2.15) of the metric derivative $|v'|$, it is a standard matter to check that, on the one hand,

$$\lim_{\tau \downarrow 0} \frac{1}{(\overline{\sigma}_\tau(s) - \underline{\sigma}_\tau(s))} \mathsf{d}(v(\underline{\sigma}_\tau(s)), v(\overline{\sigma}_\tau(s))) \to |v'|(s) \qquad \text{for a.a. } s \in (a,b)\,. \tag{A.2}$$

On the other hand, exploiting the lower semicontinuity of g, we observe that for every $s \in [a,b]$ there exists $\sigma_{\min,\tau}(s) \in [\underline{\sigma}_\tau(s), \overline{\sigma}_\tau(s)]$ such that

$$\inf_{\sigma \in [\underline{\sigma}_\tau(s), \overline{\sigma}_\tau(s)]} g(v(\sigma)) = g(v(\sigma_{\min,\tau}(s))).$$

Since $\sigma_{\min,\tau}(s) \to s$ as $\tau \downarrow 0$, by the continuity of v and the lower semicontinuity of g we then have

$$\liminf_{\tau \downarrow 0} g(v(\sigma_{\min,\tau}(s))) \geq g(v(s)) \qquad \text{for all } s \in [a,b].$$

Therefore, by the Fatou Lemma we have

$$\begin{aligned} \mathsf{S} &\geq \liminf_{\tau \downarrow 0} \sum_{i=1}^N \mathsf{d}(v(\sigma_{i-1}), v(\sigma_i)) \inf_{\sigma \in [\sigma_{i-1}, \sigma_i]} g(v(\sigma)) \\ &= \liminf_{\tau \downarrow 0} \int_a^b \frac{1}{(\overline{\sigma}_\tau(s) - \underline{\sigma}_\tau(s))} \mathsf{d}(v(\underline{\sigma}_\tau(s), v(\overline{\sigma}_\tau(s)))\, g(v(\sigma_{\min,\tau}(s)))\,\mathrm{d}s \\ &\geq \int_a^b |v'|(s) g(v(s))\,\mathrm{d}s \end{aligned}$$

and we then conclude (A.1). □

We conclude this Appendix by extending the *duality formula* from [2, Lemma 3.1.5] for the (squared) metric slope $|\mathrm{D}\mathscr{E}|^2(t,\cdot)$, $t \in [0,T]$ fixed, namely

$$\begin{aligned} &\frac{1}{2}|\mathrm{D}\mathscr{E}|^2(t,u) = \limsup_{\tau\downarrow 0} \frac{\mathscr{E}(t,u) - \mathscr{E}_\tau(t,u)}{\tau} \quad \text{with} \\ &\mathscr{E}_\tau(t,u) := \inf_{v\in X}\left\{\frac{1}{2\tau}\mathsf{d}^2(u,v) + \mathscr{E}(t,v)\right\} \\ &\text{the Moreau-Yosida approximation of } \mathscr{E}(t,\cdot) \end{aligned} \tag{A.3}$$

(with slight abuse of notation). We consider the case in which the dissipation potential underlying the definition of Moreau-Yosida approximation is no longer the quadratic $\psi(r) := \frac{1}{2}r^2$, but a general function

$$\psi : [0,\infty) \to [0,\infty) \text{ convex, l.s.c., with } \psi(0) = 0 \text{ and } \lim_{r\uparrow\infty}\frac{\psi(r)}{r} = \infty. \tag{A.4}$$

With ψ we may associate the *generalized Moreau-Yosida* approximation of the functional $\mathscr{E}(t,\cdot) : X \to \mathbb{R}$, via the formula (again, with slight abuse of notation, we write $\mathscr{Y}_\tau^\psi(\mathscr{E})(t,u)$ in place of $\mathscr{Y}_\tau^\psi(\mathscr{E}(t,\cdot))(u)$)

$$\mathscr{Y}_\tau^\psi(\mathscr{E})(t,u) := \inf_{v\in X}\left(\tau\psi\left(\frac{\mathsf{d}(u,v)}{\tau}\right) + \mathscr{E}(t,v)\right) \tag{A.5}$$

for $(t,u) \in [0,T] \times X$, $\tau > 0$. Combining the coercivity condition (E$_1$) with the superlinear growth of ψ, it is straightforward to check that for all $(t,u) \in [0,T] \times X$ and for all $\tau > 0$

$$M_\tau^\psi(\mathscr{E})(t,u) := \mathrm{Argmin}_{v\in X}\left(\tau\psi\left(\frac{\mathsf{d}(u,v)}{\tau}\right) + \mathscr{E}(t,v)\right) \neq \emptyset\,.$$

We have the following counterpart to [2, Lemma 3.1.5].

Lemma 2 *There holds for all* $(t,u) \in [0,T] \times X$

$$\psi^*\left(|\mathrm{D}\mathscr{E}|(t,u)\right) = \limsup_{\tau\to 0}\frac{\mathscr{E}(t,u) - \mathscr{Y}_\tau^\psi(\mathscr{E})(t,u)}{\tau}\,. \tag{A.6}$$

The *proof* follows by trivially adapting the argument for [2, Lemma 3.1.5]. We conclude this Appendix with the following lower semicontinuity result, which is crucially used in the proof of Proposition 3.

Lemma 3 *Assume* (E$_1$), (E$'_3$), *and* (A.4). *Let* $(\tau_k)_k \subset (0,\infty)$, $(t_k)_k \subset [0,T]$, *and* $(u_k)_k \subset X$ *fulfill* $\tau_k \downarrow 0$, $t_k \to t$, *and* $u_k \to u$ *for some* $(t,u) \in [0,T] \times X$, *with*

$\sup_{k\in\mathbb{N}} \mathscr{E}(t_k, u_k) \leq C$. *Then,*

$$\liminf_{k\to\infty} \frac{\mathscr{E}(t_k, u_k) - \mathscr{Y}^{\psi}_{\tau_k}(\mathscr{E})(t_k, u_k)}{\tau_k} \geq \psi^* \left(|\mathrm{D}\mathscr{E}|(t, u)\right). \tag{A.7}$$

Proof For every $k \in \mathbb{N}$, let $u^k_{\tau_k} \in M^{\psi}_{\tau_k}(\mathscr{E})(t_k, u_k)$. We have that

$$\frac{\mathscr{E}(t, u_k) - \mathscr{Y}^{\psi}_{\tau_k}(\mathscr{E})(t_k, u_k)}{\tau_k} = \frac{\mathscr{E}(t_k, u_k) - \mathscr{E}(t_k, u^k_{\tau_k}) - \tau_k \psi\left(\frac{\mathsf{d}(u_k, u^k_{\tau_k})}{\tau_k}\right)}{\tau_k}$$

$$\geq \frac{1}{\tau_k}\int_0^{\tau_k} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k, u^k_r)\right) \mathrm{d}r,$$

where the latter estimate follows from [21, Lemma 4.5], with u^k_r is a (measurable) selection in $M^{\psi}_r(\mathscr{E})(t_k, u_k)$ for $r \in (0, \tau_k)$. Observe that $\liminf_{k\to\infty} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k, u^k_r)\right) \geq \psi^*\left(|\mathrm{D}\mathscr{E}|(t, u)\right)$ taking into account that $u^k_r \to u$ as $k \to \infty$ for every $r \in (0, \tau_k)$, cf. the proof of [21, Lemma 4.5], and using the lower semicontinuity of $|\mathrm{D}\mathscr{E}|$ granted by (E'_3). Then, by Fatou's lemma we have

$$\liminf_{k\to\infty} \frac{1}{\tau_k}\int_0^{\tau_k} \psi^*\left(|\mathrm{D}\mathscr{E}|(t_k, u^k_r)\right) \mathrm{d}r \geq \psi^*\left(|\mathrm{D}\mathscr{E}|(t, u)\right),$$

which concludes the proof of (A.7).

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A Duality Approach in Some Boundary Value Problems

Dan Tiba

Abstract We describe several results from the literature concerning approximation procedures for variational boundary value problems, via duality techniques. Applications in shape optimization are also indicated. Some properties are quite unexpected and this is an argument that the present duality approach may be of interest in a large class of problems.

Keywords Dual problem • Fenchel theorem • Optimal design • The control variational method

AMS (MOS) Subject Classification 65L10, 65N22, 49J45

1 Introduction

One of the most applied discretization methods for boundary value problems of different types, is FEM with its numerous variants. There are difficulties that may hinder the efficiency of this approach since the finite element grid may degenerate and certain regularity hypothesis have to be imposed in order to obtain good results, Ciarlet [7]. Starting with dimension three, the grid generation may involve a high degree of complexity (see for instance [14]).

In front tracking problems appearing in free boundary applications, such conditions may be difficult to preserve during the iterations of the algorithm. Alternative approaches are the recent virtual element method [3] or meshless methods [20]. In the second case, the used finite dimensional bases (for instance RBF) are quite complex, while the VEM has still to be developed to its full potential.

In this article, we review another approach (based on duality theory) that has already ensured powerful results [13, 21], in its variant known as the control

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variational method and can be applied to large classes of linear or nonlinear boundary value problems admitting a variational formulation. A recent development is due to Machalova and Netuka [15] for a beam model governed by a nonlinear fourth-order differential equation introduced by Gao [9].

Duality is an important principle in mathematics or physics. In the context of differential equations, dual problems may be obtained via the variational formulations (or otherwise) and are useful in optimal control, numerical approximation or theoretical advances [2, 4, 8, 11, 14]. We limit ourselves to the stationary case, characterized by the minimization of certain energies, but extensions to evolution equations are also possible [10, 23]. The dual optimization problem associated to it or to its approximation, is finite dimensional.

We discuss here three basic types of results associated to the minimization of energy, according to the constraints that are involved. In the next section, we describe problems governed by ordinary differential equations and the constraints are defined by some of the boundary conditions. Their number is finite and the dual problem is finite dimensional and provides the explicit solution to the model since no approximation is used here. Usual FEM techniques produce just approximate solutions and special techniques are needed to avoid critical situations, [5, 6].

In Sect. 3, we consider variational inequalities associated to the biharmonic equation. The constraints are given by the convex set characterizing the restriction (for instance, the obstacle problem). It is discretized by using a countable dense subset of points (no finite element) and the dual optimization problem is again finite dimensional and provides the desired approximation. This is an important advantage, removing any geometric regularity condition.

We also investigate the p-Laplacian problem and some fourth order problem by discretizing the constraints expressed via the boundary conditions. Here, we use recent papers [17, 18, 24], where numerical experiments are also reported. Partial results for the general linear elasticity system are discussed in [22].

In the last section, we briefly recall an application in shape optimization of the control variational method [21].

The variational problems that we discuss in this short review are in classical form or can be reformulated as an optimal control problem. The duality theory, based on the Fenchel theorem, has a wide range of applications to many classes of boundary value problems.

2 Ordinary Differential Equations

The classical Kirchhoff–Love model [7] in dimension two describes the deformation of a clamped cylindrical shell, under forcing acting in the normal plane, constantly along the shell. It consists in finding the displacement with components

$v_1 \in H_0^1(0,1)$, $v_2 \in H_0^2(0,1)$ such that

$$\int_0^1 \left[\frac{1}{\varepsilon}(v_1' - cv_2)(s)(u_1' - cu_2)(s) + (v_2' + cv_1)'(s)(u_2' + cu_1)'(s)\right] ds$$
$$= \int_0^1 (f_1 u_1 + f_2 u_2)(s) ds, \ \forall \ u_1 \in H_0^1(0,1), \ \forall \ u_2 \in H_0^2(0,1), \tag{2.1}$$

where (f_1, f_2) are the normal, respectively the tangential component of the force acting on the arch, $c : [0,1] \to \mathbb{R}$ denotes its curvature and $\sqrt{\varepsilon}$ is its constant thickness. If $\varphi = (\varphi_1, \varphi_2) : [0,1] \to \mathbb{R}^2$ is the parametrization of the arch, with respect to its arclength, then $c(s) = \varphi_2''(s)\varphi_1'(s) - \varphi_1''(s)\varphi_2'(s)$ is its curvature and $\theta(s) = \arctan\left(\frac{\varphi_2'(s)}{\varphi_1'(s)}\right)$ is the angle between the horizontal axis and the tangent vector $\varphi' = (\varphi_1', \varphi_2')$. Consider the orthogonal matrix

$$W(s) = \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix} \tag{2.2}$$

and define the functions

$$\begin{bmatrix} l \\ h \end{bmatrix}(t) = -\int_0^t W(t)W^{-1}(s)\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds, \tag{2.3}$$

$$g_1 = \varepsilon l, \ -g_2'' = h, \ g_2(0) = g_2(1) = 0. \tag{2.4}$$

The control variational method allows the reformulation of (2.1) as an optimal control problem. We use the notations introduced in (2.2)–(2.4):

$$\text{Min}\left\{L(u,z) = \frac{1}{2\varepsilon}\int_0^1 u^2(s)ds + \frac{1}{2}\int_0^1 [z'(s)]^2 ds\right\}, \tag{2.5}$$

subject to the state equation

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int_0^t W(t)W^{-1}(s)\begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds, \tag{2.6}$$

with restriction

$$\int_0^1 W^{-1}(s)\begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.7}$$

Relation (2.6) ensures the zero initial condition for v_1, v_2, while (2.7) expresses the zero final condition, i.e. the boundary conditions in (2.1). The formulation (2.5)–(2.7) (together with (2.2)–(2.4)) needs just $\theta \in L^\infty(0,1)$, i.e. $\varphi \in W^{1,\infty}(0,1)^2$. Under more regularity conditions, we have (according to [13]):

Theorem 2.1 *If $\varphi \in W^{3,\infty}(0,1)^2$ and $[v_1^*, v_2^*]$ is the optimal state of (2.5)–(2.7), then it satisfies (2.1).*

The existence and uniqueness of $[v_1^*, v_2^*] \in L^\infty(0,1)^2$ follows by the coercivity and strict convexity of (2.5). The above result shows that indeed (2.5)–(2.7) is a generalization of (2.1), under very weak regularity conditions on φ. Notice that the optimal control problem (2.5)–(2.7) has just two state (or equivalently, control) constraints given by (2.7). One can compute the dual problem [13], which is two dimensional (two Lagrange multipliers) and can be solved explicitly:

$$\begin{aligned}\underset{\lambda_1,\lambda_2\in R}{\mathrm{Min}}\{\frac{1}{2\varepsilon}\int_0^1[\lambda_1\varepsilon\cos\theta(s)+\lambda_2\varepsilon\sin\theta(s)+\varepsilon l(s)]^2ds+\\+\frac{1}{2}\int_0^1[(\lambda_1 w_1+\lambda_2 w_2+g_2)'(s)]^2ds\},\end{aligned} \tag{2.8}$$

where we have denoted $w_1, w_2 \in H_0^1(0,1) \cap H^2(0,1)$:

$$w_1''(s) = \sin\theta(s),\ \ w_2''(s) = -\cos\theta(s).$$

Theorem 2.2 *If $\theta \in L^\infty(0,1)$, then the unique solution to (2.5)–(2.7) is given by (2.6) and*

$$[u^*, z^*] = \lambda_1^*[\varepsilon\cos\theta, w_1] + \lambda_2^*[\varepsilon\sin\theta, w_2],$$

where $[\lambda_1^*, \lambda_2^*]$ solve (2.8).

Note that the solution to (2.8) can be obtained by a 2×2 linear algebraic system that expresses the corresponding necessary optimality conditions. The results from Theorems 2.1, 2.2 show the efficiency of the duality arguments. In [5, 6] very special FEM schemes are used to approximate (2.1), due to its singular and stiff character (given by the small parameter $\varepsilon > 0$) and the so-called “locking problem” that affects standard FEM approaches, in such cases.

Applying a similar method, variational inequalities for arches, with unilateral conditions on the boundary, are discussed in [19]. In the recent paper [15], a nonlinear beam model is discussed by the control variational approach, but the obtained solution is no more explicit as in the case of the Kirchhoff-Love model for arches. The admissible control set has a complex structure and discretization procedures have to be used.

3 Partial Differential Equations

We start with an example related to the p-Laplacian, following [24]:

$$\underset{y\in W_0^{1,p}(\Omega)}{\text{Min}} \left\{ \frac{1}{p} \int_\Omega [\, |\nabla y|^p + |y|^p]\, dx - \int_\Omega fy dx \right\}, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $p > d \geq 2, f \in L^q(\Omega), p^{-1} + q^{-1} = 1$.

The existence of a unique solution $y \in W_0^{1,p}(\Omega)$ is well known, due to the coercivity and strict convexity of (3.1). It may be interpreted as solving in a weak sense the p-Laplacian equation in Ω, with Dirichlet boundary conditions.

Let $\{x_i\}_{i\in N} \subset \partial\Omega$ be a dense subset. We approximate (3.1) by an optimization problem with a finite number of constraints (on the boundary $\partial\Omega$):

$$\underset{\substack{y\in W^{1,p}(\Omega)\\ y(x_i)=0,\ i=\overline{1,n}}}{\text{Min}} \left\{ \frac{1}{p} \int_\Omega [\, |\nabla y|^p + |y|^p]\, dx - \int_\Omega fy dx \right\}. \tag{3.2}$$

It makes sense due to the Sobolev embedding $W^{1,p}(\Omega) \subset C(\overline{\Omega})$ since $p > d$.

By the same argument as above, the minimization problem (3.2) has a unique solution $y_n \in W^{1,p}(\Omega)$.

Formally, it may be interpreted as solving the p-Laplacian equation with mixed boundary conditions: Dirichlet conditions in $\{x_i\}_{i=\overline{1,n}}$ and Neumann conditions in the remaining of $\partial\Omega$.

One can prove via convex analysis techniques the following approximation result:

Theorem 3.1 *We have $y_n \to y$, the solution to (3.1), strongly in $W^{1,p}(\Omega)$.*

This allow to replace the study of (3.1), by (3.2).

We discuss the dual problem for (3.2) which is a finite dimensional optimization problem. We define $g_n : W^{1,p}(\Omega) \to]-\infty, +\infty]$ by

$$g_n(y) = \begin{cases} 0 & y(x_i) = 0,\ i = \overline{1,n}, \\ +\infty & \quad \text{otherwise} \end{cases} \tag{3.3}$$

and $h : W^{1,p}(\Omega) \to \mathbb{R}$, given by

$$h(y) = -\frac{1}{p} \int_\Omega [|\nabla y|^p + |y|^p]\, dx + \int_\Omega fy dx. \tag{3.4}$$

Clearly, the problem (3.2) can be reexpressed via (3.3), (3.4) as

$$\underset{y\in W^{1,p}(\Omega)}{\text{Min}} \{g_n(y) - h(y)\}. \tag{3.5}$$

Since g_n is convex, proper, lower semicontinuous and h is concave, continuous on $W^{1,p}(\Omega)$, the Fenchel theorem [2] can be applied and the dual of (3.5) has the form

$$\max_{z \in W^{1,p}(\Omega)^*} \{h^*(z) - g_n^*(z)\}, \tag{3.6}$$

where h^*, g_n^* are the conjugate mappings. One can compute them and obtain

$$h^*(z) = -\frac{1}{p}|z - f|^q_{W^{1,p}(\Omega)^*}, \tag{3.7}$$

$$g_n^*(z) = \begin{cases} 0 & z = \sum_{i=1}^n \alpha_i \delta_{x_i}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.8}$$

where $\alpha_i \in \mathbb{R}$ and $\delta_{x_i} \in W^{1,p}(\Omega)^*$ is a Dirac-type functional concentrated in $x_i \in \partial\Omega$ (but it is not a distribution).

By (3.6)–(3.8), we can state

Theorem 3.2 *The dual problem is given by:*

$$\text{Min} \left\{ \frac{1}{q}|f - z|^q_{W^{1,p}(\Omega)^*};\ z = \sum_{i=1}^n \alpha_i \delta_{x_i},\ \alpha_i \in \mathbb{R} \right\}$$

and it is a finite dimensional optimization problem.

It is known that, from the solution to the dual problem one can find the solution to the primal problem as well, [2, p. 188]. If $p = 2$, the involved equations become linear. The continuity may be obtained if $f \in L^s(\Omega)$, with s sufficiently big, depending on dimension.

Similar ideas can be applied to fourth order elliptic variational inequalities. An example is given by

$$\underset{y \in K}{\text{Min}} \left\{ \frac{1}{2} \int_\Omega |\Delta y|^2 - \int_\Omega hy dx \right\}, \tag{3.9}$$

$$K = \left\{ z \in H^2(\Omega) \cap H_0^1(\Omega);\ \int_\Omega hz dx \geq -1 \right\}, \tag{3.10}$$

which can be interpreted as a simply supported plate. The unilateral condition (3.10) is related to the mechanical work performed by the force $h \in L^2(\Omega)$.

The problem (3.9), (3.10) has a unique solution $y \in K$, due to the coercivity and strict convexity of the functional (3.9).

The approximating problem is

$$\operatorname*{Inf}_{y\in K_n}\left\{\frac{1}{2}\int_\Omega |\Delta y|^2 dx - \int_\Omega hy dx\right\}, \tag{3.11}$$

$$K_n = \left\{z \in H^2(\Omega);\ z(x_i) = 0,\ i = \overline{1,n},\ \int_\Omega hz dx \geq -1\right\}, \tag{3.12}$$

where $\{x_i\}_{i\in N} \subset \partial\Omega$ is, as before, a dense subset.

Notice that (3.11), (3.12) may have no solution due to the possible lack of coercivity. One can use minimizing sequences in (3.11), (3.12). However, the dual problem has solutions and the Fenchel theorem can be applied.

Theorem 3.3 *The dual problem is given by*

$$\operatorname{Min}\frac{1}{2}\int_\Omega |z|^2 dx$$

subject to

$$D^* z - h \in \overline{\operatorname{conv}}\,\{\{0\} \cup A_n\},$$

$$A_n = \left\{-h + \sum_{i=1}^{n} \alpha_i \delta_{x_i},\ \alpha_i \in \mathbb{R}\right\},$$

where $D^* : L^2(\Omega) \to H^2(\Omega)^*$ *is the adjoint of the linear continuous operator* $D : H^2(\Omega) \to L^2(\Omega)$, $Dy = \Delta y$.

The finite dimensional character of A_n is the key point in this approach. Other applications of the duality approach to second and fourth order variational inequalities of obstacle type are due to [16–18]. Different duality concepts, for high order nonlinear elliptic problems and for systems are discussed in [11, 12, 14].

4 An Application in Shape Optimization

The arguments from the previous sections have a variational character and are strongly related to optimization problems. We argue here via a shape optimization example [1], that they may have further consequences in optimization theory:

$$\operatorname{Min}\int_\Omega u(x)dx, \tag{4.1}$$

$$\Delta(u^3\Delta y) = f \quad \text{in } \Omega, \tag{4.2}$$

$$y = \Delta y = 0 \quad \text{in } \partial\Omega, \tag{4.3}$$

$$0 < m \le u(x) \le M \text{ a.e. in } \Omega, \tag{4.4}$$

$$y \in C, \tag{4.5}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $C \subset L^2(\Omega)$ is a given nonempty closed subset, $m, M \in \mathbb{R}, f \in L^2(\Omega)$.

In dimension two, relations (4.2), (4.3) model the equilibrium state of a simply supported plate with thickness u satisfying (4.4) and deflection y, under the vertical load f. The geometric optimization problem (4.1)–(4.5) consists in finding the plate of minimal volume, such that the deflection remains in the prescribed set C. For instance, we may take ($\tau \in \mathbb{R}_+$ given):

$$C = \{z \in L^2(\Omega);\ z(x) \ge -\tau \text{ a.e. in } \Omega\}, \tag{4.6}$$

which is a safety condition (the deflection should not overpass some limit). In this example (4.6), C is even convex. However, the optimization problem (4.1)–(4.6) remains strongly nonconvex, even for C convex, due to the nonlinear character of the dependence $u \to y$ defined by (4.2).

It enters the category of control by coefficients problems. Notice that the boundary value problem (4.2), (4.3) has a unique weak solution $y \in H^2(\Omega) \cap H_0^1(\Omega)$.

Denote by $w \in H^2(\Omega) \cap H_0^1(\Omega)$ the unique solution to the Dirichlet problem $\Delta w = f$ in Ω. Then, (4.2), (4.3) is equivalent with

$$\Delta y = wl \quad \text{in } \Omega, \tag{4.7}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{4.8}$$

where $l = u^{-3} \in L^\infty(\Omega)$. Equations (4.7), (4.8) together with the above definition of w may be interpreted as the optimality conditions for a linear-quadratic control problem and is one of the simplest examples of the application of the control variational method, [1].

The shape optimization problem (4.1)–(4.5) becomes

$$\text{Min} \int_\Omega l^{-\frac{1}{3}}(x)dx \tag{4.9}$$

subject to (4.7), (4.8) and the constraints $l \in [M^{-3}, m^{-3}]$ and (4.5). Due to the linearity of the dependence $l \to y$ defined by (4.7) and the strict convexity of the functional (4.9), we infer

Theorem 4.1 *The problem (4.1)–(4.5) has at least one optimal pair* $[y^*, u^*] \in H^2(\Omega) \times L^2(\Omega)$. *If C is convex, the optimal pair is unique.*

The existence is a consequence of usual weak lower semicontinuity arguments and the boundedness of the set of admissible thicknesses. The problem (4.1)–(4.5) may have many local optimal pairs since it is nonconvex, but the global optimal pair is unique if C is convex. Uniqueness is a very unusual property in optimal design. Such results may be extended to clamped plates [1, 21]. If M is big enough, then one can prove that the set of admissible pairs is nonvoid. A general presentation of shape optimization problems can be found in [19].

5 Conclusion

We have performed a short review of the control variational approach and some of its applications. An important ingredient is the Fenchel duality theorem and the analysis of the corresponding dual problems. The literature on duality methods in differential equations is very rich and includes a large variety of arguments and results. Obtaining the exact solution in certain non autonomous boundary value problems, proving the uniqueness of the minimizer in some shape optimization examples or developing new numerical discretization procedures via dense subsets of points (in the considered domain) are useful properties that show the applicability of such ideas in many directions of interest.

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On the Structural Properties of Nonlinear Flows

Augusto Visintin

Abstract This work deals with the structural stability of quasilinear first-order flows w.r.t. arbitrary perturbations not only of data but also of operators. This rests upon a variational formulation based on works of Brezis, Ekeland, Nayroles and Fitzpatrick, and on the use of *evolutionary* Γ*-convergence* w.r.t. a nonlinear topology of weak type. This approach is extended to flows of a class of nonmonotone operators. A theory in progress is outlined, and is also used to prove the structural compactness and stability of doubly-nonlinear parabolic flows of the form

$$\alpha(D_t u) + \partial\gamma(u) \ni h,$$

α being a maximal monotone operator, and γ a lower semicontinuous convex function on a Hilbert space.

Keywords Doubly-nonlinear parabolic flow • Evolutionary Γ-convergence • Fitzpatrick theory • Nonlinear weak convergence • Structural compactness and stability

AMS (MOS) Subject Classification 35K60, 47H05, 49J40, 58E

Foreword Gianni Gilardi was one of my teachers at the University of Pavia. I remember how accurate his lessons were, and the answers he gave to the many questions I set to him when I was entering research. After leaving Pavia, I could no longer profit of his suggestions; but I visited his books several times, and here I wish to mention a couple of them.

First I would consider *Analisi Tre* (McGraw-Hill, Milano, 1994), the third volume of a "trilogy" that Gianni devoted to mathematical analysis, which was recently followed by a volume on functional analysis. Analisi Tre is a remarkable account of several issues of advanced analysis: Banach and Hilbert spaces, distributions,

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complex analysis, integral transforms and their use for the analysis of PDEs, and so on; it also includes several examples and solved exercises. This book grew from the notes of the course of *Mathematical methods for engineering* that Gianni held for several years; I still preserve the hand-written mimeographed copy of the pre-TEX era, that he had produced more than 30 years ago.

It is pity that Analisi Tre has not been translated into English, since some of its results miss in the current textbooks. It may also be noticed that this book has no bibliography; but that is understandable, since as far as I remember it was not customary of the Author to visit the literature: most often he just did not need that, and reproduced arguments by himself.

I will also mention a monograph, that Gianni wrote jointly with Franco Brezzi: *Functional Analysis, Functional Spaces, Partial Differential Equations.* This was published as an introductory part of the *Finite Element Handbook* (Chaps. 1, 2, and 3, pp. 1–121 of Part 1, Kardestuncer and Norrie eds., McGraw-Hill, New York, 1987). This is a rich account of basic notions and results on Sobolev spaces, that are used to study boundary-value problems for linear PDEs. This text was rather popular among us, young researchers at the *Istituto di Analisi Numerica del C.N.R.,* directed by Enrico Magenes. We were used to refer to it as the *Manuale delle Giovani Marmotte* (i.e., the Junior Woodchucks Guidebook).

1 Introduction

The purpose of this work is to illustrate some aspects of an ongoing research on the *structural compactness and stability* of evolutionary PDEs.

Structural Compactness and Stability The term *stability* is usually referred to robustness of a system as some data are perturbed. By *structural stability* of a differential problem we shall mean robustness under perturbations in the data and in the operator(s) that govern the problem. This has an obvious applicative motivation: in reality not only data but also differential operators are accessible just with some approximation; moreover perturbations often occur, e.g. in the coefficients of quasilinear operators. Structural stability may thus be regarded as a basic requisite for the applicative feasibility of mathematical models.

Let us illustrate these concepts in the set-up of abstract function spaces, for a model problem of the form $Au \ni h$, A being a multi-valued operator acting between Banach spaces, and h a datum. Given bounded families of data and of operators, we shall formulate the issue of stability in terms of two properties:

(i) *structural compactness:* existence of convergent sequences of data $\{h_n\}$ and of operators $\{A_n\}$ w.r.t. topologies that must be specified;
(ii) *structural stability:* if $A_n u_n \ni h_n$ for any n, $A_n \to A$, $h_n \to h$ and $u_n \to u$, then u is a solution of the asymptotic problem: $Au \ni h$.

Here the selection of the relevant topologies is the crucial issue.

For models that are formulated as a minimization principle, structural compactness and stability can be studied via De Giorgi's theory of Γ-convergence. Indirectly, this provides analogous structural properties of compactness and stability for the associated Euler-Lagrange equation. In particular this applies to equations that are governed by a cyclically maximal monotone operator; this typically occurs in stationary models. Here we extend that approach to noncyclically maximal monotone operators and to associated first-order flows.

First-Order Flows In 1976 the pioneering articles [10] of Brezis and Ekeland and [36] of Nayroles provided a new formulation for first-order flows of cyclically maximal monotone operators. In 1988 in the seminal paper [21] Fitzpatrick characterized maximal monotone operators in terms of a nonstandard minimization principle. Only recently it has been realized that an analogous approach in finite-dimension had already been pointed out in 1982 by Krylov in [27]. Fitzpatrick's article prompted the generalization [45] of [10] and [36].

In the present work we deal with the extension of that theory to nonmonotone flows of the form

$$D_t u + \alpha(u) \ni h \qquad \text{in } V', \text{a.e. in time} (D_t := \partial/\partial t); \tag{1.1}$$

here V is a Hilbert space, and $\alpha : V \to \mathscr{P}(V')$ is a nonmonotone operator. The structural stability of the initial-value problem for (1.1) was studied in [47] and in the parallel work [51] for a maximal monotone α.

After discussing the general theory, here we address doubly-nonlinear equations of the form

$$\alpha(D_t u) + \partial\gamma(u) \ni h \qquad \text{in } V', \text{a.e. in time}, \tag{1.2}$$

α being a (possibly noncyclically) maximal monotone operator, and γ a lower semicontinuous convex function(al). Incidentally, the reader will notice that this equation misses a monotone structure, despite of the presence of two monotone operators. This equation applies to rate-independent processes, whenever α is positively homogeneous of degree zero (the sign function is an example).

This method can also be used to prove the structural stability of another class of nonlinear flows that we do not address in this work, of the form

$$\begin{cases} D_t w + \alpha(u) \ni h \\ w \in \partial\varphi(u) \end{cases} \qquad \text{in } V', \text{a.e. in time}, \tag{1.3}$$

φ being also a lower semicontinuous convex function(al); see [51]. It is known that several phenomena of mathematical physics are modeled by PDEs of the form (1.1)–(1.3). For instance, if the function φ is positively homogeneous of degree one, then (1.3) represents a rate-independent flow; this is thus a model of hysteresis.

Plan of Work In Sect. 2 we outline some elements of the Fitzpatrick theory of [21], define semi-monotone operators, and extend that theory to these operators. In Sect. 4 we introduce the notion of semi-monotonicity, see e.g. [11] and [26]. On this basis, we then generalize the *Brezis-Ekeland-Nayroles principle* ("BEN principle" for short) to the first-order flow of those operators; see Theorems 8 and 9. For these results arguments and details may be found in [51].

In Sect. 5 we define the notion of *evolutionary Γ-convergence of weak type* after [50], and state a compactness theorem that is used ahead. In Sect. 6 we define what we call the *nonlinear weak topology* of $V \times V'$, and illustrate the structural compactness and stability of first-order flows associated to representable operators, see Theorems 11 and 12. The two latter sections mainly illustrate results of [50] and [51]. In Sects. 7 we use Theorem 12 to show the structural compactness and stability of the pseudo-monotone flow (1.2), see Theorem 13.

A Look at the Literature As we pointed out, Krylov's paper [27] is of 1982,[1] and Fitzpatrick's Theorem 1 appeared in 1988 in the proceedings [21]. Both papers were not noticed for several years; some of the results of Fitzpatrick were eventually rediscovered by Martinez-Legaz and Théra [30] and (independently) by Burachik and Svaiter [13]. Since then a rapidly expanding literature has been devoted to this theory, see e.g. [4–6, 14, 28, 29, 37, 38], besides many others. As we already pointed out, Fitzpatrick's formulation of maximal monotone operators via a null-minimization principle casts a new light upon a formulation of first-order subdifferential flows, that Brezis and Ekeland [10] and by Nayroles [36] had proposed in 1976. Brezis and Ekeland [10] and Nayroles [36] assumed α to be cyclically maximal monotone; on the basis of the Fitzpatrick theory the extension to general maximal monotone operators was then rather obvious, see [45]. This was then further extended to nonmonotone flows in [48].

Since the original formulation of [10] and [36] of 1976, the BEN principle was applied in several works. For the study of doubly-nonlinear evolutionary PDEs, this principle was used e.g. in [39] and [43]; see also Sect. 8.10 of [40]. In [47] the dependence of the solution of quasilinear maximal monotone equations on data and operators was studied, by applying Γ-convergence to the null-minimization problem.

It is well-known that Γ-convergence is due to De Giorgi; see [20] and e.g. the monographs [2, 7, 8, 18]. The notion of *evolutionary Γ-convergence* w.r.t. a weak topology, that we review in Sect. 5, is weaker than that formulated in [41], see also [19, 31, 32], where Γ-convergence was required for almost every instant. Our choice for a weaker notion is due to the scarcity of uniform estimates for the problems that we deal with here.

Doubly nonlinear equations of the form (1.2) were studied in several papers, see e.g. [1, 16, 17, 33, 34, 42]. Structural compactness and stability of equations (1.1)

[1]This author learned of this paper only recently from Ulisse Stefanelli.

((1.3), resp.) were addressed in [47] (and [51], resp.). See also [46], where structural compactness was not addressed, and [49], where the Fitzpatrick theory was not used.

2 Fitzpatrick Theory

In this section we outline the tenets of the theory that was originated by the pioneering paper [21] of Fitzpatrick.

The Fitzpatrick Theorem Let V be a real Banach space, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V. Let $\alpha : V \to \mathscr{P}(V')$ be a (possibly multi-valued) measurable operator, that is, such that $g^{-1}(A) := \{v \in V : g(v) \cap A \neq \emptyset\}$ is measurable, for any open subset A of V' (see e.g. [3, 15, 25]). For instance, maximal monotone operators have this property. We shall always assume that α is proper, i.e., the image set $\alpha(V)$ is not reduced to the empty set.

In 1988 in [21] Fitzpatrick defined what is now called the *Fitzpatrick function:*

$$\begin{aligned} f_\alpha(v, v^*) &:= \langle v^*, v\rangle + \sup\left\{\langle v^* - \tilde{v}^*, \tilde{v} - v\rangle : \tilde{v} \in V, \tilde{v}^* \in \alpha(\tilde{v})\right\} \\ &= \sup\left\{\langle v^*, \tilde{v}\rangle - \langle \tilde{v}^*, \tilde{v} - v\rangle : \tilde{v} \in V, \tilde{v}^* \in \alpha(\tilde{v})\right\} \quad \forall (v, v^*) \in V \times V'. \end{aligned} \tag{2.1}$$

This function is convex and lower semicontinuous, since it is the supremum of a family of affine and continuous functions. Fitzpatrick proved the following result.[2]

Theorem 1 ([21]) *A proper measurable operator $\alpha : V \to \mathscr{P}(V')$ is maximal monotone if and only if, defining f_α as in (2.1),*

$$f_\alpha(v, v^*) \geq \langle v^*, v\rangle \qquad \forall (v, v^*) \in V \times V', \tag{2.2}$$

$$f_\alpha(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.3}$$

This theorem extends a classical result of convex analysis. Let $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ be any proper function, and denote its convex conjugate function and its subdifferential respectively by $\varphi^* : V' \to \mathbf{R} \cup \{+\infty\}$ and $\partial\varphi : V \to \mathscr{P}(V')$. Then

$$\varphi(v) + \varphi^*(v^*) \geq \langle v^*, v\rangle \qquad \forall (v, v^*) \in V \times V', \tag{2.4}$$

$$\varphi(v) + \varphi^*(v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \partial\varphi(v). \tag{2.5}$$

We shall refer to (2.2), (2.3) ((2.4), (2.5), resp.) as the *Fitzpatrick system* (the *Fenchel system,* resp.), and to the mapping $(v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$ as the *Fenchel function* of the operator $\partial\varphi$. Most often the Fitzpatrick function and the Fenchel function do not coincide.

[2]The proof was included in [21]. As those proceedings may not be easily available to the Reader, the argument was displayed also in [47].

Representative Functions The Fitzpatrick theorem suggests the following generalization of the notion of Fitzpatrick function. One says that a function f (variationally) *represents* a proper measurable operator $\alpha : V \to \mathscr{P}(V')$ if

$$\begin{aligned} & f : V\times V' \to \mathbf{R} \cup \{+\infty\} \text{ is convex and lower semicontinuous,} \\ & f(v, v^*) \geq \langle v^*, v\rangle \qquad \forall (v, v^*) \in V\times V', \\ & f(v, v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \end{aligned} \tag{2.6}$$

One accordingly says that f is a *representative function,* and that α is *representable.* We shall denote by $\mathscr{F}(V)$ the class of the functions that fulfill $(2.6)_1$ and $(2.6)_2$. In [21] it was proved that representable operators are monotone. They need not be either cyclically monotone or maximal monotone; a counterexample is provided ahead. However, by Theorem 1 if α is represented by its Fitzpatrick function then it is maximal monotone.

Let us next assume that the Banach space V is reflexive. This assumption simplifies a lot this theory, although for some of the assertions that follow it is not needed. (A large community is engaged in the development of the Fitzpatrick theory, and a part of that research deals with the extension of known results to nonreflexive spaces.)

Besides the duality between V an V', we shall consider the duality between the product space $V\times V'$ and its dual $V'\times V$, and the corresponding convex conjugation. In this set-up, the convex conjugate of any function $g : V\times V' \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\begin{aligned} g^*(w^*, w) := \sup\big\{\langle w^*, v\rangle + \langle v^*, w\rangle - g(v, v^*) : (v, v^*) \in V\times V'\big\} \\ \forall (w^*, w) \in V'\times V. \end{aligned} \tag{2.7}$$

Let us denote by $I_\alpha : V\times V' \to \mathbf{R} \cup \{+\infty\}$ the indicator function of the graph (denoted graph(α)) of any operator $\alpha : V \to \mathscr{P}(V')$; i.e., for any $(v, v^*) \in V\times V'$,

$$I_\alpha(v, v^*) = 0 \quad \text{if } v^* \in \alpha(v), \qquad I_\alpha(v, v^*) = +\infty \quad \text{if } v^* \notin \alpha(v).$$

Let us also denote by π the duality pairing:

$$\pi : V\times V' \to \mathbf{R} : (v, v^*) \mapsto \langle v^*, v\rangle, \tag{2.8}$$

and by $\mathscr{I}$ the permutation operator $V \times V' \to V' \times V : (v, v^*) \mapsto (v^*, v)$. The definition (2.1) thus also reads $f_\alpha = (\pi + I_\alpha)^* \circ \mathscr{I}$, that is,

$$f_\alpha(v, v^*) = (\pi + I_\alpha)^*(v^*, v) \qquad \forall (v, v^*) \in V\times V'. \tag{2.9}$$

Some Results Next we review some relevant properties of representative functions.

Theorem 2 ([13, 14, 44]) *Let V be a reflexive Banach space. A function $g \in \mathscr{F}(V)$ represents a maximal monotone operator $\alpha : V \to \mathscr{P}(V')$ if and only if $g^* \in \mathscr{F}(V')$.*

If this holds, then g^* represents the inverse operator $\alpha^{-1} : V' \to \mathscr{P}(V)$. Therefore $g^* \circ \mathscr{I}$ $(\in \mathscr{F}(V))$ represents α.

The convex conjugate function of f_α, denoted by $(f_\alpha)^*$, thus also represents α, whenever the operator α is maximal monotone. By the next statement, these are respectively the smallest and the largest representative function of α.

Theorem 3 ([13, 21, 29, 37]) *Let V be a reflexive Banach space, $\alpha : V \to \mathscr{P}(V')$ be a maximal monotone operator, f_α be its Fitzpatrick function, and $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. Then*

$$\begin{aligned} & g \in \mathscr{F}(V), \quad g \text{ represents } \alpha \quad \Leftrightarrow \\ & f_\alpha \le g \le (f_\alpha)^* \quad \text{pointwise in } V \times V'. \end{aligned} \tag{2.10}$$

Theorem 4 ([44]) *Let V be a reflexive Banach space. Any maximal monotone operator $\alpha : V \to \mathscr{P}(V')$ can be represented by a function $g \in \mathscr{F}(V)$ such that $g^* = g \circ \mathscr{I}^{-1}$.*

Any function $g \in \mathscr{F}(V)$ that fulfills the latter condition is called a *self-dual* representative. Next we display a simple example.

Proposition 1 *If a function $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semicontinuous, then its Fenchel function $\Phi : (v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$ is self-dual.*

Proof For any $(v, v^*), (w, w^*) \in V \times V'$,

$$\Phi(v, v^*) + \Phi(w, w^*) = \varphi(v) + \varphi^*(v^*) + \varphi(w) + \varphi^*(w^*) \ge \langle w^*, v\rangle + \langle v^*, w\rangle, \tag{2.11}$$

and equality holds iff $w^* \in \partial\varphi(v)$ and $v^* \in \partial\varphi(w)$. Thus $\Phi^* = \Phi \circ \mathscr{I}^{-1}$; more precisely,

$$\begin{aligned} & \Phi^*(v^*, v) = \Phi(v, v^*) \qquad \forall (v, v^*) \in V \times V', \\ & (w, w^*) \in \partial\Phi^*(v^*, v) \quad \Leftrightarrow \quad w^* \in \partial\varphi(v), \quad w \in \partial\varphi^*(v^*). \end{aligned} \tag{2.12}$$

□

For a generic maximal monotone operator to display a self-dual representative is nontrivial. The next result provides such a representative.

Theorem 5 ([4]) *Let $\alpha : V \to \mathscr{P}(V')$ be a maximal monotone operator, f_α be its Fitzpatrick function, and set*

$$\begin{aligned} & F_\alpha(v, v^*, w, w^*) := f_\alpha(v + w, v^* + w^*) + f_\alpha(v - w, v^* - w^*) + \|w\|_V^2 + \|w^*\|_{V'}^2 \\ & \forall (v, v^*), (w, w^*) \in V \times V', \end{aligned} \tag{2.13}$$

$$\phi_\alpha(v, v^*) := \frac{1}{2} \inf \left\{F_\alpha(v, v^*, w, w^*) : (w, w^*) \in V \times V'\right\} \qquad \forall (v, v^*) \in V \times V'. \tag{2.14}$$

Then ϕ_α is a self-dual representative of α.

Self-duality may be of much interest for variational problems. In this respect see e.g. [22–24], where however no use is made of the Fitzpatrick theory.

Further results of the theory of variational representation of operators are briefly reviewed e.g. in [39, 47, 51].

Examples (i) For any proper function $\varphi : V \to \mathbf{R} \cup \{+\infty\}$, the Fenchel function $f : (v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$ represents the cyclically monotone operator $\partial\varphi$. (As it is well-known, if φ is convex and lower-semicontinuous, then $\partial\varphi$ is also maximal monotone.) Incidentally note that the Fenchel function coincides with the Fitzpatrick function $f_{\partial\varphi}$ only exceptionally.

(ii) Let $A : V \to V'$ be a linear, bounded and invertible monotone operator, and define the convex and continuous mapping

$$f_b : V \times V' \to \mathbf{R} : (v, v^*) \mapsto b[\langle Av, v\rangle + \langle v^*, A^{-1}v^*\rangle] \qquad \forall b > 0. \tag{2.15}$$

For $b = 1/2$, f_b represents the operator A; this is actually the Fenchel function of A. For any $b > 1/2$, f_b represents the monotone (but not maximal monotone) operator $\alpha(0) = \{0\}$, $\alpha(v) = \emptyset$ for any $v \neq 0$. For $0 < b < 1/2$, f_b does not represents any proper operator.

(iii) Other examples of interest for the theory of PDEs were provided e.g. in [47]; they include for instance a representative function for quasilinear elliptic operators of the form

$$H_0^1(\Omega) \to \mathscr{P}(H^{-1}(\Omega)) : v \mapsto -\nabla \cdot \boldsymbol{\beta}(\nabla v) \quad (\Omega \text{ being a domain of } \mathbf{R}^N), \tag{2.16}$$

for any maximal monotone mapping $\boldsymbol{\beta} : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$.

See also the representation of nonmonotone operators in the next section.

Minimization Principles Let a function $g \in \mathscr{F}(V)$ represent a proper operator $\alpha : V \to \mathscr{P}(V')$, and let us define the function

$$J(v, v^*) := f(v, v^*) - \langle v^*, v\rangle \qquad \forall (v, v^*) \in V \times V'. \tag{2.17}$$

Minimizing J w.r.t. both variables v, v^*, the system (2.6) yields

$$J(v, v^*) = \inf J \quad \Leftrightarrow \quad J(v, v^*) = 0 \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.18}$$

If instead we minimize $J(v, v^*)$ just w.r.t. v for a fixed $v^* \in V'$, then

$$J(v, v^*) = 0 \;\big(= \inf J(\cdot, v^*)\big) \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.19}$$

In this case the implication "$J(v, v^*) = \inf J(\cdot, v^*) \Rightarrow v^* \in \alpha(v)$" may fail, since for some $v^* \in V'$ a priori one might have $\inf J(\cdot, v^*) = +\infty$; this would mean that $v^* \notin \alpha(V)$. However, α is surjective under suitable assumptions, e.g. maximal monotonicity and coerciveness; in this respect see e.g. Sect. 8.10 of [40], which deals with a cyclically monotone operator α.

3 Extension of the Fitzpatrick Theory to Nonmonotone Operators

In this section we extend the notion of representative function to a class of nonmonotone operators.

First we need some definitions. Let V be a linear space, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V and its dual V'. We shall say that a (possibly nonconvex) function f *algebraically represents* a (possibly nonmonotone) operator $\alpha : V \to \mathscr{P}(V')$ if

$$\begin{aligned} & f : V \times V' \to \mathbf{R} \cup \{+\infty\}, \\ & f(v, v^*) \geq \langle v^*, v \rangle \qquad \forall (v, v^*) \in V \times V', \\ & f(v, v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \end{aligned} \tag{3.1}$$

Henceforth we shall assume that V is a Banach space. Extending (3.1) to the topological set-up, it is necessary to distinguish among topologies that are intermediate between the strong and the weak topology of $V \times V'$. Henceforth we fix such a topology and denote it by σ.

We shall say that f *topologically represents* α w.r.t. σ (or more briefly, σ-*represents* α) if

$$f \text{ fulfills (3.1) and is } \sigma\text{-lower semicontinuous.} \tag{3.2}$$

(By an alternative nonequivalent definition, one might assume that f is sequentially σ-lower semicontinuous.) We shall display a nontrivial example in the next section. We shall denote by $\mathscr{E}(V)$ ($\mathscr{E}_\sigma(V)$, resp.) the class of the functions that fulfill (3.1) ((3.1) and (3.2), resp.). In particular we shall denote by $\mathscr{E}_s(V)$ ($\mathscr{E}_w(V)$, resp.) the class corresponding to the strong (weak, resp.) topology of $V \times V'$. Thus

$$\mathscr{F}(V) \subset \mathscr{E}_w(V) \subset \mathscr{E}_\sigma(V) \subset \mathscr{E}_s(V) \subset \mathscr{E}(V). \tag{3.3}$$

Next we display some simple results, and refer to [51] for some of the arguments.

Proposition 2 *Let $f, g \in \mathscr{E}_\sigma(V)$, and denote by α_f, α_g the respective represented operators. If $f \leq g$ in $V \times V'$ then* $\text{graph}(\alpha_g) \subset \text{graph}(\alpha_f)$.

Proposition 3 *Let J be a nonempty index set, and let $\psi_j \in \mathscr{E}_\sigma(V)$ represent an operator $\alpha_j : V \to \mathscr{P}(V')$ with graph A_j, for any $j \in J$. Then $\psi = \sup_{j \in J} \psi_j \in \mathscr{E}_\sigma(V)$ represents the operator $\alpha : V \to \mathscr{P}(V')$ with graph $A = \bigcap_{j \in J} A_j$.*

If $\psi_j \in \mathscr{F}(V)$ for any $j \in J$, then $\psi \in \mathscr{F}(V)$.

Proof By (3.1)$_2$, (3.1)$_3$, and by the definition of ψ,

$$\begin{aligned} &\psi(v,v^*) \ge \langle v^*, v\rangle \qquad \forall (v,v^*) \in V\times V', \\ &\psi(v,v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad \forall j\in J, \quad \psi_j(v,v^*) = \langle v^*, v\rangle. \end{aligned} \tag{3.4}$$

As the pointwise supremum of any family of lower semi-continuous functions is lower semi-continuous, this yields the thesis for $\mathscr{E}_\sigma(V)$. If $\mathscr{E}_\sigma(V)$ is replaced by $\mathscr{F}(V)$, it suffices to recall that the pointwise supremum of any family of convex functions is convex. □

Proposition 4 *Let $\psi_j \in \mathscr{E}_\sigma(V)$ represent an operator $a_j \in V \to \mathscr{P}(V')$ for any $j = 1,\dots,N$ (for a finite N). Then $\psi = \min_{j=1,\dots,N} \psi_j \in \mathscr{E}_\sigma(V)$, and it represents the operator $\alpha : V \to \mathscr{P}(V')$ with graph $A = \bigcup_{j\in J} A_j$.*

Obviously no analogous property holds for $\mathscr{F}(V)$, since the minimization does not preserve convexity.

Proof As any ψ_j is lower σ-semi-continuous, the same holds for ψ, since for any topology the pointwise minimum of any *finite* family of sequentially lower semi-continuous functions has the same property. By (3.1)$_2$, (3.1)$_3$, and by the definition of ψ, for any $(v,v^*) \in V\times V'$,

$$\psi(v,v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad \exists j\in J: \quad \psi_j(v,v^*) = \langle v^*, v\rangle.$$

This yields the thesis. □

Representation of the Sum of Operators Next we display two results for equations of the form

$$\alpha_1(v) + \alpha_2(v) \ni h \qquad \text{in } V', \tag{3.5}$$

α_1, α_2 being two representable operators $V \to \mathscr{P}(V')$.

Theorem 6 ([51]) *Let V be a real reflexive Banach space, $\alpha_i : V \to \mathscr{P}(V')$ for $i = 1, 2$, and assume that*

$$\exists \tilde v \in V : \alpha_1(\tilde v) \ne \emptyset \text{ and } \alpha_2(\tilde v) \ne \emptyset, \tag{3.6}$$

$$g_i \in \mathscr{E}_w(V) \text{ and } g_i \text{ represents } \alpha_i \ (i = 1,2), \tag{3.7}$$

$$\inf_{(v,v^*)\in V\times V'} \{g_1(v, v^* - z^*) + g_2(v, z^*)\} \to +\infty \quad \text{as } \|z^*\|_{V'} \to +\infty. \tag{3.8}$$

Then: (i) The partial inf-convolution

$$(g_1 \oplus g_2)(v,v^*) := \inf_{z^*\in V'} \{g_1(v, v^* - z^*) + g_2(v, z^*)\} \qquad \forall (v,v^*) \in V\times V' \tag{3.9}$$

is an element of $\mathscr{E}_w(V)$, and represents the operator $\alpha_1 + \alpha_2$.

(ii) If also $g_1, g_2 \in \mathscr{F}(V)$, then $g_1 \oplus g_2 \in \mathscr{F}(V)$.

By setting $\alpha_2 = D_t$ in (3.5) and integrating in time, one can extend the BEN principle to nonmonotone flows; see Sect. 4.

This theorem can be applied e.g. to doubly nonlinear PDEs of the form

$$a(u) - \nabla \cdot \gamma(\nabla u) \ni h \qquad \text{in } \mathscr{D}'(\Omega),$$

with $a : \mathbf{R} \to \mathscr{P}(\mathbf{R})$, $\gamma : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$ maximal monotone functions, and Ω an open subset of $\mathbf{R}^N$.

4 Flow of Semi-Monotone Operators

In this section we introduce a class of nonmonotone operators, deal with their representation (in the sense that we defined in the previous section), and extend the Brezis-Ekeland-Nayroles (BEN) principle to the corresponding flow. We just state some results and refer to [51] for the arguments.

Functional Set-Up Let us assume that we are given a triplet of real Hilbert spaces

$$V \subset H = H' \subset V' \text{ with continuous, dense and compact injections,} \tag{4.1}$$

whence

$$L^2(0,T;V) \subset L^2(0,T;H) = L^2(0,T;H)' \subset L^2(0,T;V)' \simeq L^2(0,T;V') \\ \text{with continuous and dense injections.} \tag{4.2}$$

Let us also set

$$\mathscr{V} := \{v \in L^2(0,T;V) : D_t v \in L^2(0,T;V')\}. \tag{4.3}$$

Equipped with the graph norm, this is a Hilbert space. Notice that $\mathscr{V} \subset C^0([0,T];H)$ by a standard identification. We shall also identify the spaces $L^2(0,T;V) \times L^2(0,T;V')$ and $L^2(0,T;V \times V')$.

Semi-Monotone Operators If X is a Banach space, by X_s (X_w, resp.) we shall denote the topological vector space that is obtained by equipping X with the strong (weak, resp.) topology. Let us assume that

$$\begin{aligned} &b : V \to \mathscr{P}(V') \text{ is maximal monotone,} \\ &\beta_v : V \to V' \text{ is maximal monotone, } \forall v \in V, \\ &V_w \to (V')_s : v \mapsto \beta_v(u) \text{ is continuous, } \forall u \in V, \end{aligned} \tag{4.4}$$

and define the operator

$$\alpha(v) := b(v) + \beta_v(v) \qquad \forall v \in V. \tag{4.5}$$

This operator is measurable, but obviously need not be monotone. Following Browder [11], we shall call *semi-monotone* operators like α. This class is comprised in that of (multi-valued) generalized pseudo-monotone operators of Browder and Hess [12], which extends that of (single-valued) pseudo-monotone operators of Brezis [9]. This class is also included in that addressed by Kenmochi in [26]. A more general class is dealt with in [51].

Let us next assume that

$$\exists A, B > 0 : \forall u, v \in V, \qquad \|b(v) + \beta_v(u)\|_{V'} \le A \max\{\|v\|_V, \|u\|_V\} + B, \tag{4.6}$$

so that

$$\|\alpha(v)\|_{V'} \le A\|v\|_V + B \qquad \forall v \in V. \tag{4.7}$$

These operators then canonically determine the global-in-time operators

$$\begin{aligned} &\widehat{\beta}_v : L^2(0,T;V) \to \mathscr{P}(L^2(0,T;V')) : \\ &[\widehat{\beta}_v(u)](t) = \beta_{v(t)}(u(t)) \quad \text{for a.e. } t \in]0,T[, \forall u, v \in L^2(0,T;V), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\widehat{\alpha} : L^2(0,T;V) \to \mathscr{P}(L^2(0,T;V')) : \\ &[\widehat{\alpha}(v)](t) = \beta_{v(t)}(v(t)) \quad \text{for a.e. } t \in]0,T[, \forall v \in L^2(0,T;V). \end{aligned} \tag{4.9}$$

Theorem 7 ([51]) *Let (4.1), (4.4), (4.5) hold, and equip $V \times V'$ with a topology σ that is intermediate between the strong and the weak topology of $V \times V'$. Let a function $g_z \in \mathscr{E}_\sigma(V)$ represent the operator $b + \beta_z$ for any $z \in V$, and be such that the mapping*

$$V \times V \times V' \to \mathbf{R} \cup \{+\infty\} : (z, v, v^*) \mapsto g_z(v, v^*) \text{ is measurable.} \tag{4.10}$$

Let us set

$$\varphi(v, v^*) := g_v(v, v^*) \qquad \forall (v, v^*) \in V \times V', \tag{4.11}$$

and denote by $\tilde{\varphi}$ the lower semicontinuous regularized function of φ with respect to σ (i.e., the pointwise supremum of its σ-lower-semicontinuous minorants). Then $\tilde{\varphi} \in \mathscr{E}_\sigma(V)$, and it represents the operator α.

A Differential Example Next we display a differential semi-monotone operator in a Sobolev space, and exhibit a representative function, here in the Banach set-up.

Let Ω be a bounded Lipschitz domain of $\mathbf{R}^N$ ($N \geq 1$), $p \in [2, +\infty[$, and set $V := W_0^{1,p}(\Omega)$, whence $V' := W^{-1,p'}(\Omega)$. Let us assume that, setting $p' = p/(p-1)$,

$$\begin{aligned} &b : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N) \text{ is maximal monotone,} \\ &\beta_z : \mathbf{R}^N \to \mathbf{R}^N \text{ is maximal monotone, } \forall z \in \mathbf{R}, \\ &\exists a_1, a_2 \in \mathbf{R}^+ : \forall z \in \mathbf{R}, \forall v \in \mathbf{R}^N \qquad |\beta_z(v)| \leq a_1 |v|^{p-1} + a_2, \end{aligned} \tag{4.12}$$

For any $z \in \mathbf{R}$, let $b + \beta_z$ be represented by a function $g_z \in \mathscr{F}(\mathbf{R}^N)$.

As it is shown in [47], for any $z \in V$ the maximal monotone operator

$$\gamma_z : V \to \mathscr{P}(V') : v \mapsto -\nabla \cdot [b + \beta_z](\nabla v)$$

can then be represented by the following function $\varphi_z \in \mathscr{F}(V)$:

$$\varphi_z(v, v^*) = \inf \left\{ \int_\Omega g_z(\nabla v, \boldsymbol{\eta}_{v^*})\, dx : \boldsymbol{\eta}_{v^*} \in L^{p'}(\Omega)^N, -\nabla \cdot \boldsymbol{\eta}_{v^*} = v^* \text{ in } \mathscr{D}'(\Omega) \right\}, \tag{4.13}$$

for any $(v, v^*) \in V \times V'$. By Theorem 7, the function

$$\psi(v, v') = \varphi_v(v, v') \qquad \forall (v, v') \in V \times V' \tag{4.14}$$

then represents the semi-monotone operator

$$\alpha : V \to \mathscr{P}(V') : v \mapsto \gamma_v(v) = -\nabla \cdot [b + \beta_v](\nabla v)$$

in any topology σ that is intermediate between the strong and the weak topology of $V \times V'$.

Cauchy Problem and Extended BEN Principle Let us fix a single-valued operator $\alpha : V \to \mathscr{P}(V')$ as above, any $u^* \in L^2(0, T; V')$, any $u^0 \in H$, and consider the Cauchy problem

$$\begin{cases} u \in \mathscr{V}, \\ D_t u + \alpha(u) \ni u^* \qquad \text{in } V', \text{ a.e. in }]0, T[, \\ u(0) = u^0. \end{cases} \tag{4.15}$$

This problem can equivalently be formulated globally-in-time as follows:

$$\begin{cases} u \in \mathscr{V}, \\ D_t u + \widehat{\alpha}(u) \ni u^* \qquad \text{in } L^2(0, T; V'), \\ u(0) = u^0. \end{cases} \tag{4.16}$$

Next we still assume that V is a Hilbert space. We deal with the representation (in the sense of Sect. 3) of the operator $D_t + \widehat{\alpha}$ in $\mathscr{E}_w(L^2(0,T;V))$, with domain $\mathscr{V}_{u^0} = \{v \in \mathscr{V} : v(0) = u^0\}$, and extend the Brezis-Ekeland-Nayroles principle of [10, 36]; see also [45].

Theorem 8 (Extended BEN Principle) *Assume that* (4.1)*,* (4.4)*,* (4.5) *and* (4.6) *are fulfilled, and define* $\widehat{\alpha}$ *and* φ *as in* (4.9) *and* (4.11)*. Fix any* $u^0 \in H$*, and set*

$$\begin{aligned}\Phi(v,v^*) &:= \int_0^T [\varphi(v, v^* - D_t v) - \langle v^* - D_t v, v\rangle]\, dt \\ &= \int_0^T [\varphi(v, v^* - D_t v) - \langle v^*, v\rangle]\, dt + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u^0\|_H^2 \qquad (4.17)\\ &\qquad\qquad \forall (v,v^*) \in \mathscr{V}_{u^0} \times L^2(0,T;V'),\\ \Phi(v,v^*) &:= +\infty \qquad \textit{for any other } (v,v^*) \in L^2(0,T;V)\times L^2(0,T;V').\end{aligned}$$

Then $\Phi \in \mathscr{E}_s(L^2(0,T;V))$*, and the Cauchy problem* (4.16) *is equivalent to the following null-minimization problem:*

$$u \in \mathscr{V}_{u^0}, \qquad \Phi(u,u^*) = 0 \ \Big(= \inf_{\mathscr{V}_{u^0}} \Phi(\cdot, u^*)\Big). \qquad (4.18)$$

Proof $\Phi \in \mathscr{E}_s(L^2(0,T;V))$ by part (ii) of Theorem 6, with $\alpha_1 = \widehat{\alpha}$, $\alpha_2 = D_t$ and the space $\mathscr{V}$ in place of V. As φ represents α,

$$\varphi(v, v^* - D_t v, t) - \langle v^* - D_t v, v\rangle \geq 0 \qquad \forall (v,v^*) \in \mathscr{V}_{u^0} \times L^2(0,T;V');$$

moreover, this function vanishes a.e. in $]0,T[$ if and only if v solves (4.15). This entails that $\Phi(v,v^*) \geq 0$ for any $(v,v^*) \in \mathscr{V} \times L^2(0,T;V')$, and that the equation $\Phi(u,u^*) = 0$ is equivalent to (4.15). □

Twice Time-Integrated Extended BEN Principle For reasons that we shall see ahead, it is convenient to amend the above set-up by applying a further time integration, as follows.

Let us still assume (4.1)–(4.3), and define the measure

$$\mu(A) = \int_A (T-t)\, dt \quad \forall A \in \mathscr{L}(0,T) \quad \text{i.e.} \quad d\mu(t) = (T-t)dt. \qquad (4.19)$$

For any Hilbert space X, let us introduce the weighted space

$$L^2_\mu(0,T;X) := \Big\{\mu\text{-measurable } v :]0,T[\to X : \int_0^T \|v(t)\|_X^2\, d\mu(t) < +\infty\Big\}, \qquad (4.20)$$

and equip it with the graph norm. This is a Hilbert spaces, and

$$L^2_\mu(0,T;V) \subset L^2_\mu(0,T;H) = L^2_\mu(0,T;H)' \subset L^2_\mu(0,T;V)' = L^2_\mu(0,T;V'), \tag{4.21}$$

with continuous and dense injections. Let us set

$$\mathscr{V}_\mu := \big\{v \in L^2_\mu(0,T;V) : D_t v \in L^2_\mu(0,T;V')\big\}, \tag{4.22}$$

fix any $u^0 \in H$, and define the affine subspace

$$\mathscr{V}_{\mu,u^0} := \big\{v \in L^2_\mu(0,T;V) : D_t v \in L^2_\mu(0,T;V'), v(0) = u^0\big\}. \tag{4.23}$$

Notice that

$$\begin{aligned} &\int_0^T \langle D_t v, v\rangle \, d\mu(t) \overset{(4.19)}{=} \int_0^T d\tau \int_0^\tau \langle D_t v, v\rangle \, dt \\ &= \frac{1}{2}\int_0^T d\tau \int_0^\tau D_t\big(\|v(t)\|_H^2\big)\, dt = \frac{1}{2}\int_0^T \|v(\tau)\|_H^2 \, d\tau - \frac{T}{2}\|v(0)\|_H^2 \qquad \forall v \in \mathscr{V}, \end{aligned} \tag{4.24}$$

so that D_t is monotone on $\mathscr{V}_{\mu,u^0}$.

By a double time-integration, let us define the nonnegative functional

$$\begin{aligned} \tilde{\Phi}(v,v^*) &:= \int_0^T d\tau \int_0^\tau \big[\varphi(v, v^* - D_t v) - \langle v^* - D_t v, v\rangle\big]\, dt \\ &\overset{(4.24)}{=} \int_0^T \big[\varphi(v, v^* - D_t v) - \langle v^*, v\rangle\big]\, d\mu(t) + \frac{1}{2}\int_0^T \|v(t)\|_H^2 \, dt - \frac{T}{2}\|u^0\|_H^2 \\ &\qquad\qquad \forall (v,v^*) \in \mathscr{V}_{\mu,u^0} \times L^2(0,T;V'), \\ \tilde{\Phi}(v,v^*) &:= +\infty \qquad \text{for any other } (v,v^*) \in L^2(0,T;V) \times L^2(0,T;V'). \end{aligned} \tag{4.25}$$

As $\mathscr{V} \cap \mathscr{V}_\mu$ is dense in $\mathscr{V}_\mu$, we get the next statement.

Theorem 9 (Time-Integrated Extended BEN Principle) *Assume that* (4.1), (4.4), (4.5) *hold, and that*

$$\mathscr{V}_w \to L^2(0,T;V')_s : z \mapsto \widehat{\beta}_z(v) \text{ is continuous, } \forall v \in L^2(0,T;V). \tag{4.26}$$

Let α be represented by a function $\varphi \in \mathscr{E}_w(L^2(0,T;V))$, and define $\widehat{\alpha}$ and $\tilde{\Phi}$ as in (4.9) *and* (4.25)*.*

Then $\tilde{\Phi} \in \mathscr{E}_s(\mathscr{V}_\mu)$, *and the Cauchy problem* (4.16) *is equivalent to the following null-minimization problem:*

$$u \in \mathscr{V}_{\mu,u^0}, \qquad \tilde{\Phi}(u, u^*) = 0 \;\Big(= \inf_{\mathscr{V}_{\mu,u^0}} \tilde{\Phi}(\cdot, u^*)\Big). \tag{4.27}$$

Proof It suffices to notice that, as $\varphi \in \mathscr{E}_s(V)$, the Cauchy problem (4.15) is equivalent to the null-minimization of the functional that is obtained by integrating the nonnegative function $\varphi(v, v^* - D_t v) - \langle v^* - D_t v, v\rangle$ any number of times in $]0, T[$. □

By comparing the functionals (4.17) and (4.25), in the latter the Reader will notice the occurrence of the term $\frac{1}{2}\int_0^T \|v(t)\|_H^2\,dt$, rather than $\frac{1}{2}\|v(T)\|_H^2$. In Sect. 6 we shall see that the functional (4.25) is more prone than (4.17) to passage to the limit by weak convergence. This is the main reason for introducing this further time integration.

We just represented the problem (4.15) w.r.t. the strong topology. We shall be concerned with the structural properties of compactness and stability of the initial-value problem, in the sense that we illustrate in the next section; for that purpose we shall represent the operator w.r.t. a weaker topology.

5 Interlude: Evolutionary Γ-Convergence of Weak Type

Here we extend De Giorgi's notion of Γ-convergence to operators that act on time-dependent functions and range in a Banach space X; we then state a related result of compactness, that will be used in the proof of Theorem 12 of the next section. We refer to [50] for details and proofs.

Evolutionary Γ-Convergence of Weak Type Let X be a real separable and reflexive Banach space, $p \in [1, +\infty[$, $T > 0$, and define the measure μ as in (4.19). Let us equip $L^p_\mu(0, T; X)$ with a topology τ that is intermediate between the weak and the strong topology.[3] For any operator

$$\psi : L^p_\mu(0, T; X) \to L^1_\mu(0, T) : w \mapsto \psi_w,$$

let us set

$$[\psi, \xi](w) = \int_0^T \psi_w(t)\,\xi(t)\,d\mu(t) \qquad \forall w \in L^p_\mu(0, T; X), \forall \xi \in L^\infty(0, T). \tag{5.1}$$

[3] We assume this in consideration of the application of the next section. A reader interested just in evolutionary Γ-convergence might go through the present section assuming that μ is the Lebesgue measure and that τ is the weak topology.

Let $\{\psi_n\}$ be a sequence of operators $L^p_\mu(0,T;X) \to L^1_\mu(0,T)$ such that

$$\begin{gathered}\forall \text{ bounded subset } A \text{ of } L^p_\mu(0,T;X),\\ \{\psi_{n,w} : w \in A, n \in \mathbf{N}\} \text{ is bounded in } L^1_\mu(0,T).\end{gathered} \tag{5.2}$$

If also $\psi : L^p_\mu(0,T;X) \to L^1_\mu(0,T)$, we shall say that

$$\begin{gathered}\psi_n \text{ sequentially } \varGamma\text{-converges to } \psi\\ \text{in the topology } \tau \text{ of } L^p_\mu(0,T;X) \text{ and}\\ \text{in the weak topology of } L^1_\mu(0,T)\end{gathered} \tag{5.3}$$

if and only if

$$\begin{gathered}[\psi_n, \xi] \text{ sequentially } \varGamma\tau\text{-converges to } [\psi, \xi] \text{ in}\\ L^p_\mu(0,T;X),\ \forall \text{ nonnegative } \xi \in L^\infty(0,T).\end{gathered} \tag{5.4}$$

By the classical definition of sequential $\varGamma$-convergence, this means that for any $\xi \in L^\infty_+(0,T)$

$$\begin{gathered}\forall w \in L^p_\mu(0,T;X), \forall \text{ sequence } \{w_n\} \text{ in } L^p_\mu(0,T;X),\\ \text{if } w_n \underset{\tau}{\to} w \text{ in } L^p_\mu(0,T;X) \text{ then } \liminf_{n\to+\infty} [\psi_n, \xi](w_n) \ge [\psi, \xi](w),\end{gathered} \tag{5.5}$$

$$\begin{gathered}\forall w \in L^p_\mu(0,T;X), \exists \text{ sequence } \{w_n\} \text{ of } L^p_\mu(0,T;X) \text{ such that}\\ w_n \underset{\tau}{\to} w \text{ in } L^p_\mu(0,T;X) \text{ and } \lim_{n\to+\infty} [\psi_n, \xi](w_n) = [\psi, \xi](w).\end{gathered} \tag{5.6}$$

This definition of *evolutionary $\varGamma$-convergence* is quite different from that of [41] as well as from that of [19, 31, 32]. By a simple transformation, the present definition fits the rather general framework of $\bar{\varGamma}$-convergence, see Chap. 16 of [18]; that monograph however does not encompass Theorem 10 ahead.

We shall be concerned with superposition operators of the form

$$\begin{gathered}\psi_w(t) = \varphi(w(t)) \qquad \forall w \in L^p_\mu(0,T;X), \text{ for a.e. } t \in]0,T[,\\ \varphi : X \to \mathbf{R}^+ \ \text{ being a } \textit{lower semicontinuous function.}\end{gathered} \tag{5.7}$$

The next result provides the compactness of evolutionary $\varGamma$-convergence, and characterizes the $\varGamma$-limit as a superposition operator.

Theorem 10 ([50]) *Let X be a real separable and reflexive Banach space, $p \in [1,+\infty[$, $T > 0$. Let $\{\varphi_n\}$ be a sequence of lower semicontinuous functions $X \to \mathbf{R}^+$*

that is equi-coercive and equi-bounded, in the sense that

$$\exists C_1, C_2, C_3 > 0 : \forall n, \forall w \in X,$$
$$C_1 \|w\|_X^p \le \varphi_n(w) \le C_2 \|w\|_X^p + C_3. \tag{5.8}$$

Moreover let $\varphi_n(0) = 0$ *for any* n. *Let* μ *be defined as in* (4.19)*. Let us equip* $L^p_\mu(0,T;X)$ *with a topology* τ *that is intermediate between the weak and the strong topology. Let us also assume that*

$$\begin{aligned} &\text{for any sequence } \{F_n\} \text{ of functionals } L^p_\mu(0,T;X) \to \mathbf{R}^+ \cup \{+\infty\},\\ &\text{if } \sup_{n\in\mathbf{N}} \big\{\|w\|_{L^p_\mu(0,T;X)} : w \in L^p_\mu(0,T;X), F_n(w) \le C\big\} < +\infty,\\ &\text{then } \{F_n\} \text{ has a sequentially } \Gamma\tau\text{-convergent subsequence.} \end{aligned} \tag{5.9}$$

Then there exists a lower semicontinuous function $\varphi : X \to \mathbf{R}^+$ *such that* $\varphi(0) = 0$ *and, defining the operators* $\psi, \psi_n : L^2_\mu(0,T;X) \to L^1_\mu(0,T)$ *for any* n *as in* (5.7)*, possibly extracting a subsequence,*

$$\begin{aligned} &\psi_n \text{ sequentially } \Gamma\text{-converges to } \psi\\ &\text{in the topology } \tau \text{ of } L^p_\mu(0,T;X) \text{ and}\\ &\text{in the weak topology of } L^1_\mu(0,T). \end{aligned} \tag{5.10}$$

6 Γ-Compactness and Γ-Stability of Null-Minimization

Here we illustrate the structural compactness and stability of minimization principles. We then state a result that rests on evolutionary Γ-convergence w.r.t. what we shall refer to as a *nonlinear weak topology.*

Structural Compactness and Stability Let us first illustrate a fairly general set-up. Let X be a topological space and $\mathscr{G}$ be a family of functionals $X \to \mathbf{R} \cup \{+\infty\}$, equipped with a suitable notion of variational convergence. We shall use the following terminology, which specifies what we anticipated in the Introduction.

(i) We shall say that the problem of minimizing these functionals is *structurally compact* if the family $\mathscr{G}$ is sequentially compact, and the corresponding minimizers range in a sequentially relatively compact subset of X. (We restrict ourselves to sequential concepts for the sake of simplicity.)

(ii) We shall say that this problem is *structurally stable* if

$$\begin{cases} u_n \to u & \text{in } X\\ \Phi_n \to \Phi & \text{in } \mathscr{G}\\ \Phi_n(u_n) - \inf \Phi_n \to 0 \end{cases} \quad \Rightarrow \quad \Phi(u) = \inf \Phi. \tag{6.1}$$

(The convergence $\Psi_n(u_n) - \inf\Psi_n \to 0$ obviously extends the minimization condition $\Psi_n(u_n) = \inf\Psi_n$ for any n.) Part (i) is clearly instrumental to (ii). These definitions are trivially extended to null-minimization problems.

The selection of the notion of convergence of the family of functionals $\mathscr{G}$ is crucial, and may not be an obvious choice. Structural compactness and stability are in competition: sequential compactness requires a sufficiently weak convergence, whereas stability requires this convergence to be strong enough. The need of compactness suggests one to use a weak-type topology for X. We shall see that Γ-convergence w.r.t. a suitable weak-type topology is especially appropriate, more than the Mosco-convergence (namely, the simultaneous weak and strong Γ-convergence to a same function, see e.g. [2, 35]).

A Nonlinear Topology of Weak Type Let us still use the notation (2.8). We shall name *nonlinear weak topology of* $V \times V'$, and denote by $\tilde{\pi}$, the coarsest among the topologies of this space that are finer than the product of the weak topology of V by the weak topology of V', and for which the mapping π is continuous. For any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$, thus[4]

$$\begin{aligned} &(v_n, v_n^*) \underset{\tilde{\pi}}{\Rightarrow} (v, v^*) \quad \text{in } V \times V' \quad \Leftrightarrow \\ &v_n \rightharpoonup v \quad \text{in } V, \quad v_n^* \rightharpoonup v^* \quad \text{in } V', \quad \langle v_n^*, v_n \rangle \to \langle v^*, v \rangle, \end{aligned} \tag{6.2}$$

and similarly for nets. This construction is extended to the space $L^2_\mu(0, T; V \times V')$ in an obvious way: in this case the duality product reads $L^2_\mu(0, T; V \times V') \to \mathbf{R} : (v, v^*) \mapsto \int_0^T \langle v^*, v \rangle \, d\mu(t)$. Accordingly we set

$$\begin{aligned} &(v_n, v_n^*) \underset{\tilde{\pi}}{\Rightarrow} (v, v^*) \quad \text{in } L^2_\mu(0, T; V \times V') \quad \Leftrightarrow \\ &v_n \rightharpoonup v \ \text{ in } L^2_\mu(0, T; V), \ v_n^* \rightharpoonup v^* \ \text{ in } L^2_\mu(0, T; V') \ \text{ and} \\ &\int_0^T \langle v_n^*, v_n \rangle \, d\mu(t) \to \int_0^T \langle v^*, v \rangle \, d\mu(t), \end{aligned} \tag{6.3}$$

and similarly for nets.

***$\Gamma\tilde{\pi}$* -Compactness and *$\Gamma\tilde{\pi}$*-Stability of $\mathscr{E}_w(V)$** As the weak topology and the nonlinear weak-type topology $\tilde{\pi}$ are not metrizable, in either case one must distinguish between *topological* and *sequential* Γ-convergence, see e.g. [2, 18]. If not otherwise specified, henceforth reference to the topological notion should be understood.

[4]We denote the strong, weak, and weak star convergence respectively by $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$.

It is known that bounded subsets of a separable and reflexive space equipped with the weak topology are metrizable. The same holds for the nonlinear weak topology $\tilde{\pi}$ of $V \times V'$, as it was proved in Sect. 4 of [47]. This property is at the basis of the next statement, where we define $\mathscr{E}_{\tilde{\pi}}(V)$ as in Sect. 2 (here with $\sigma = \tilde{\pi}$).

Theorem 11 ([47] Γ-Compactness, Γ-Closedness, Γ-Stability) *Let V be a separable real Banach space, and $\{\gamma_n\}$ be an equi-coercive sequence in $\mathscr{E}_{\tilde{\pi}}(V)$, in the sense that*

$$\sup_{n \in \mathbf{N}} \Big\{ \|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \gamma_n(v, v^*) \le C \Big\} < +\infty \quad \forall C \in \mathbf{R}. \tag{6.4}$$

Then: (i) There exists $\gamma : V \times V' \to \mathbf{R} \cup \{+\infty\}$ such that, possibly extracting a subsequence, γ_n $\Gamma\tilde{\pi}$-converges to γ both topologically and sequentially.

(ii) This entails that $\gamma \in \mathscr{E}_{\tilde{\pi}}(V)$ ($\gamma \in \mathscr{F}(V)$ if $\gamma_n \in \mathscr{F}(V)$ for any n).

(iii) If α_n (α, resp.) is the operator that is represented by γ_n (γ, resp.) for any n, then the superior limit of graph(α_n) *in the sense of Kuratowski is included in* graph(α), *i.e.,*

$$\forall \textit{ sequence } \{(v_n, v_n^*) \in \text{graph}(\alpha_n)\}, \quad (v_n, v_n^*) \underset{\tilde{\pi}}{\to} (v, v^*) \ \Rightarrow \ v^* \in \alpha(v). \tag{6.5}$$

(iv) The assertions above hold also if the space V and $\gamma_n \in \mathscr{E}_{\tilde{\pi}}(V)$ are respectively replaced by $L^2_\mu(0, T; V)$ and $\gamma_n \in \mathscr{E}_{\tilde{\pi}}(L^2_\mu(0, T; V))$.

Remark 1 In general the sequence $\{\text{graph}(\alpha_n)\}$ does not converge in the sense of Kuratowski. For instance, let us define f_b as in (2.15), for any $b \ge 1/2$. If $b_n = 1/2 + 1/n$, then f_{b_n} $\Gamma\tilde{\pi}$-converges to $f_{1/2}$, but the represented operators α_n do not converge in the sense of Kuratowski to the operator α that is represented by $f_{1/2}$. Actually, in this case graph$(\alpha_n) = \{(0, 0)\}$, so that the inferior limit of graph(α_n) in the sense of Kuratowski does not include graph(α). □

Superposition Operators Let $\{\varphi_n\}$ be a sequence of representative functions of $\mathscr{E}_{\tilde{\pi}}(V)$, and define the superposition (i.e., Nemytskiĭ) operators

$$\psi_n : L^2_\mu(0, T; V \times V') \to L^1(0, T) : w \mapsto \varphi_n(w) \qquad \forall n. \tag{6.6}$$

The following question arises:

$$\begin{array}{l} \text{if } \psi_n \ \Gamma\tilde{\pi}\text{-converges to some operator } \psi \text{ in the sense of (5.10),} \\ \text{is then } \psi \text{ necessarily a superposition operator, too?} \end{array} \tag{6.7}$$

This would exclude the onset of long memory in the limit. The next statement, which is essentially a particular case of Theorem 10, provides a positive answer.

Theorem 12 *Let V be a real separable Hilbert space, and μ be the measure on $]0,T[$ such that $d\mu(t) = (T-t)\,dt$. Let $\{\varphi_n\}$ be a sequence of lower semicontinuous functions $V \times V' \to \mathbf{R}^+$ such that*

$$\varphi_n \in \mathscr{E}_{\tilde{\pi}}(V) \qquad \forall n, \tag{6.8}$$

$$\begin{aligned} &\exists C_1, C_2, C_3 > 0 : \forall n, \forall w \in V \times V', \\ &C_1 \|w\|^2_{V\times V'} \le \varphi_n(w) \le C_2 \|w\|^2_{V\times V'} + C_3, \end{aligned} \tag{6.9}$$

$$\varphi_n(0) = 0 \qquad \forall n, \tag{6.10}$$

and define the operators $\psi_n : L^2_\mu(0,T; V\times V') \to L^1_\mu(0,T)$ by

$$\psi_{n,w}(t) = \varphi_n(w(t)) \qquad \forall w \in L^2_\mu(0,T; V\times V'), \text{ for a.e. } t \in]0,T[, \forall n. \tag{6.11}$$

Then there exists a lower semicontinuous function $\varphi : V\times V' \to \mathbf{R}^+$ such that

$$\varphi \in \mathscr{E}_{\tilde{\pi}}(V) \qquad (\varphi \in \mathscr{F}(V) \text{ if } \varphi_n \in \mathscr{F}(V) \text{ for any } n), \tag{6.12}$$

and such that, defining the corresponding operator $\psi : L^2_\mu(0,T; V\times V') \to L^1_\mu(0,T)$ as in (6.11), possibly extracting a subsequence,

$$\begin{aligned} &\psi_n \text{ sequentially } \Gamma\text{-converges to } \psi \\ &\text{in the topology } \tilde{\pi} \text{ of } L^2_\mu(0,T; V\times V') \text{ and} \\ &\text{in the weak topology of } L^1_\mu(0,T). \end{aligned} \tag{6.13}$$

Proof Let us apply Theorem 10 with $X = V\times V'$, $p = 2$ and the topology $\tau = \tilde{\pi}$; the hypothesis (5.9) is indeed fulfilled for this topology, because of Theorem 4.4 of [47]. It then suffices to show that (6.8) entails (6.12). To that aim let us set

$$\left.\begin{aligned} J_n(t, v, v^*) &:= \varphi_n(t, v, v^*) - \langle v^*, v\rangle \\ J(t, v, v^*) &:= \varphi(t, v, v^*) - \langle v^*, v\rangle \end{aligned}\right. \qquad \forall (v, v^*) \in V\times V', \text{ for a.e. } t, \forall n. \tag{6.14}$$

By (6.8),

$$\int_A J_n(t, v, v^*)\, d\mu(t) \ge 0 \qquad \forall (v, v^*) \in V\times V', \forall A \in \mathscr{L}(0,T), \tag{6.15}$$

and by (6.13) this inequality is preserved in the limit. Therefore $J(t, v, v^*) \ge 0$ a.e. in $]0,T[$. As $\varphi(t,\cdot)$ is $\tilde{\pi}$-lower semicontinuous for a.e. t, (6.12) follows. □

7 Structural Stability of a Class of Doubly-Nonlinear Flows

The structural compactness and stability of flows of the form (1.1) was addressed in [47] and [51]. In this section we deal with the structural compactness and stability of doubly-nonlinear equations of the form

$$\alpha(D_t u) + \partial\gamma(u) \ni h, \tag{7.1}$$

for a maximal monotone operator α and a lower semicontinuous convex function γ.

Let the Hilbert spaces V, H be as in (4.1), and let four sequences $\{\alpha_n\}, \{\gamma_n\}, \{u_n^0\}$ and $\{h_n\}$ be given such that

$$\forall n,\ \alpha_n : H \to \mathscr{P}(H) \text{ is maximal monotone}, \tag{7.2}$$

$$\exists C_1, C_2 > 0 : \forall n, \forall (v, v^*) \in \text{graph}(\alpha_n), \quad \langle v^*, v\rangle \geq C_1 |v\|_H^2 - C_2, \tag{7.3}$$

$$\exists C_3, C_4 > 0 : \forall n, \forall (v, v^*) \in \text{graph}(\alpha_n), \quad \|v^*\|_H \leq C_3 \|v\|_H + C_4, \tag{7.4}$$

$$\alpha_n(0) \ni 0 \qquad \forall n, \tag{7.5}$$

$$\forall n,\ \gamma_n : V \to \mathbf{R} \text{ is convex and lower semicontinuous}, \tag{7.6}$$

$$\exists \bar{C}_1, \ldots, \bar{C}_4 > 0 : \forall n, \forall v \in V, \quad \bar{C}_1 \|v\|_V^2 - \bar{C}_2 \leq \gamma_n(v) \leq \bar{C}_3 \|v\|_V^2 + \bar{C}_4, \tag{7.7}$$

$$u_n^0 \to u^0 \quad \text{in } V, \tag{7.8}$$

$$h_n \to h \quad \text{in } L^2(0,T;H). \tag{7.9}$$

For any n, we formulate the following initial-value problem

$$\begin{cases} u_n \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad z_n \in L^\infty(0,T;H), \\ \alpha_n(D_t u_n) + z_n \ni h_n \quad \text{in } H, \text{ a.e. in }]0,T[, \\ z_n \in \partial\gamma_n(u_n) \quad \text{in } H, \text{ a.e. in }]0,T[, \\ u_n(0) = u_n^0. \end{cases} \tag{7.10}$$

This abstract formulation encompasses several doubly-nonlinear parabolic PDEs. For instance, let Ω be a bounded Lipschitz domain of $\mathbf{R}^N$ ($N \geq 1$), $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Let $\alpha : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$ be maximal monotone, let $\tilde{\gamma} : \mathbf{R}^N \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous convex function, and define

$$\gamma : H_0^1(\Omega) \to \mathbf{R} \cup \{+\infty\}, \qquad \gamma(v) = \int_\Omega \tilde{\gamma}(\nabla v)\, dx \quad \forall v \in H_0^1(\Omega).$$

The inclusion (7.1) is then equivalent to the following quasilinear PDE:

$$\alpha(D_t u) - \nabla\cdot\partial\tilde{\gamma}(\nabla u) \ni h \qquad \text{in } \Omega \times]0,T[, \tag{7.11}$$

which is natural to formulate in the sense of distributions. Existence of a solution for problems like this has been studied in several works, see e.g. [1, 16, 17, 42]. In particular we have the following result.

Lemma 1 *Under the hypotheses* (4.1)*,* (7.2)–(7.9)*, for any n the initial-value problem* (7.10) *has at least one solution* $(u_n, z_n) \in [H^1(0,T;H) \cap L^\infty(0,T;V)] \times L^\infty(0,T;H)$*. Moreover, the sequence* $\{(u_n, z_n)\}$ *is bounded in this space.*

Theorem 13 ahead provides structural compactness and structural stability of this PDE.

Null-Minimization Next we reformulate the problem (7.10) in terms of null-minimization. Let us first notice that

$$\int_0^T d\tau \int_0^\tau (D_t v, z)_H \, dt = \int_0^T d\tau \int_0^\tau D_t \gamma_n(v(\tau)) dt = \int_0^T \gamma_n(v(\tau)) \, d\tau - T\gamma_n(v(0))$$
$$\forall v \in H^1(0,T;H) \cap L^\infty(0,T;V), \forall w \in \mathscr{W}, \text{ with } \forall z \in \partial\gamma_n(v) \text{ a.e. in }]0,T[. \tag{7.12}$$

For any n, let us represent the operator α_n for instance by $\varphi_n = (\pi + I_{\alpha_n})^{**}$ $(\in \mathscr{F}(H))$, define the measure μ as in (4.19) and the nonnegative twice-time-integrated functional

$$\begin{aligned}\tilde{\Phi}_n(u,z,h) := &\int_0^T d\tau \int_0^\tau \big[\gamma_n(u) + \gamma_n^*(z) - \langle z, u\rangle\big] \, dt \\ &+ \left(\int_0^T d\tau \int_0^\tau \big[\varphi_n(D_t u, h - z) - (D_t u, h - z)_H\big] \, dt\right)^+ .\end{aligned} \tag{7.13}$$

Notice that

$$\begin{aligned}\tilde{\Phi}_n(u,z,h) &\overset{(7.12)}{=} \int_0^T \big[\gamma_n(u) + \gamma_n^*(z) - \langle z, u\rangle\big] \, d\mu(t) \\ &+ \left(\int_0^T \big[\varphi_n(D_t u, h - z) - (D_t u, h)_H\big] \, d\mu(t) + \int_0^T \gamma_n(u(\tau)) \, d\tau - T\gamma_n(u_n^0)\right)^+\end{aligned} \tag{7.14}$$

for any $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$ with $u(0) = u_n^0$, any $z \in L^2_\mu(0,T;H)$, and any $h \in L^2_\mu(0,T;V')$; $\tilde{\Phi}_n(u,z,h) := +\infty$ for any other $(u,z,h) \in [H^1(0,T;H) \cap L^\infty(0,T;V)] \times \mathscr{W} \times \mathscr{V}'$.

Proposition 5 *For any n, the pair* (u_n, z_n) *solves the initial-value problem* (7.10) *if and only if*

$$\begin{cases} u_n \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad z_n \in L^\infty(0,T;H), \\ \tilde{\Phi}_n(u_n, z_n, h_n) = 0 \ \left(= \inf\limits_{[H^1(0,T;H)\cap L^\infty(0,T;V)]\times L^\infty(0,T;H)} \tilde{\Phi}_n(\cdot,\cdot,h_n)\right), \\ u_n(0) = u_n^0. \end{cases} \tag{7.15}$$

Moreover, (7.15)$_2$ *and* (7.15)$_3$ *are equivalent to*

$$\begin{cases} \displaystyle\int_0^T \big[\gamma_n(u_n) + \gamma_n^*(z_n) - \langle z_n, u_n\rangle\big]\, d\mu(t) \le 0, \\ \displaystyle\int_0^T \big[\varphi_n(D_t u_n, h_n - z_n) - (h_n, D_t u_n)_H\big]\, d\mu(t) + \int_0^T \gamma(u_n(\tau))\, d\tau - T\gamma(u_n^0) \le 0. \end{cases} \tag{7.16}$$

Proof By the Fenchel system (2.4) and (2.5), the first integrand of (7.14) is nonnegative. The null-minimization principle (7.15) is thus equivalent to the system (7.16). Incidentally, the first inequality of (7.16) may be replaced by the corresponding equation $\int_0^T \ldots\, d\mu(t) = 0$; for the second inequality the analogous equivalence is not obvious.

The first inequality of (7.16) is equivalent to (7.10)$_3$. This entails that

$$\int_0^T (D_t u_n, z_n)_H\, d\mu(t) = \int_0^T \gamma(u_n(\tau))\, d\tau - T\gamma(u_n^0);$$

(7.10)$_4$ then follows. The second inequality of (7.16) is then equivalent to

$$\int_0^T \big[\varphi_n(D_t u_n, h_n - z_n) - (D_t u_n, h_n - z_n)_H\big]\, d\mu(t) \le 0,$$

which is tantamount to (7.10)$_2$. □

Next we prove that this problem is structurally compact and stable.

Theorem 13 *Let* (4.1), (7.2)–(7.9) *be fulfilled. For any n, let* (u_n, z_n) *be a solution of problem* (7.10), *and set* $\varphi_n = (\pi + I_{\alpha_n})^{**}$ $(\in \mathscr{F}(H))$. *Then:*

(i) There exist $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$ *and* $z \in L^\infty(0,T;H)$ *such that, possibly extracting a subsequence,*

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{7.17}$$

$$z_n \overset{*}{\rightharpoonup} z \qquad \text{in } L^\infty(0,T;H). \tag{7.18}$$

(ii) There exists a function $\varphi \in \mathscr{F}(H)$ *such that, setting*

$$\begin{aligned} &\psi_{n,(v_1,v_2)}(t) = \varphi_n(v_1(t), v_2(t)), \qquad \psi_{(v_1,v_2)}(t) = \varphi(v_1(t), v_2(t)), \\ &\text{for a.e. } t \in\,]0,T[,\ \forall (v_1, v_2) \in L^2_\mu(0,T;H\times H), \forall n, \end{aligned} \tag{7.19}$$

then $\psi_n, \psi : L^2_\mu(0,T;H\times H) \to L^1_\mu(0,T)$ *and, possibly extracting a subsequence,*

$$\begin{aligned}&\psi_n \text{ sequentially } \Gamma\text{-converges to } \psi\\ &\text{in the topology } \tilde\pi \text{ of } L^2_\mu(0,T;H\times H) \text{ and}\\ &\text{in the weak topology of } L^1_\mu(0,T) \text{ (cf. (5.3)).}\end{aligned} \tag{7.20}$$

(iii) There exists a convex and lower semicontinuous function $\gamma : V \to \mathbf{R}$ *that fulfills lower and upper estimates like* (7.7)*, and such that, setting*

$$\widehat{\gamma}_n(w) = \int_0^T \gamma_n(w(t))\,dt \quad \forall n, \quad \widehat{\gamma}(w) = \int_0^T \gamma(w(t))\,dt \quad \forall w \in L^2(0,T;V), \tag{7.21}$$

possibly extracting a subsequence,

$$\begin{aligned}&\widehat{\gamma}_n \text{ strongly } \Gamma\text{-converges to} \widehat{\gamma} \text{ in } L^2(0,T;V), \text{ and}\\ &\widehat{\gamma}^*_n \text{ sequentially weakly } \Gamma\text{-converges to } \widehat{\gamma}^* \text{ in } L^2(0,T;V).\end{aligned} \tag{7.22}$$

(iv) Denoting by $\alpha : H \to \mathscr{P}(H)$ *the operator that is represented by* φ*, the pair* (u, z) *solves the corresponding initial-value problem*

$$\begin{cases} u \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad z \in L^\infty(0,T;H),\\ \alpha(D_t u) + z \ni h \quad \text{in } H, \text{ a.e. in }]0,T[,\\ z \in \partial\gamma(u) \quad \text{in } V', \text{ a.e. in }]0,T[,\\ u(0) = u^0. \end{cases} \tag{7.23}$$

Proof (i) Because of (7.3) and (7.4), we can apply Theorem 12. As the functions φ_ns do not depend explicitly on t, there exists a function $\varphi \in \mathscr{F}(H)$ such that, defining ψ as in (7.19) and possibly extracting a subsequence, (7.20) is fulfilled.

By (7.6) and (7.7), there exists a convex and lower semicontinuous function $\gamma : V \to \mathbf{R}$ such that

$$\gamma_n \xrightarrow{\Gamma} \gamma \qquad \text{strongly in } L^2(0,T;H), \tag{7.24}$$

$$\bar C_1 |v\|^2_H - \bar C_2 \le \gamma(v) \le \bar C_3 \|v\|^2_H + \bar C_4 \qquad \forall v \in H. \tag{7.25}$$

After e.g. [2, pp. 282–283], this entails that

$$\gamma^*_n \xrightarrow{\Gamma} \gamma^* \qquad \text{weakly in } L^2(0,T;H). \tag{7.26}$$

(ii) By Lemma 1, (7.17) and (7.18) hold up to extracting subsequences. This yields

$$u_n \to u \qquad \text{in } L^2(0,T;H). \tag{7.27}$$

By (7.24) and (7.26), passing to the inferior limit in $(7.16)_1$ we thus get

$$\int_0^T \big[\gamma(u) + \gamma^*(z)\big]\,dt \le \int_0^T \langle z,u\rangle\,dt, \tag{7.28}$$

which is equivalent to

$$z \in \partial\gamma(u) \quad \text{in } H, \text{ a.e. in }]0,T[. \tag{7.29}$$

(iii) Notice that

$$\begin{aligned}&\int_0^T (D_t u_n, z_n)_H\,d\mu(t) \overset{(7.10)_3}{=} \int_0^T D_t\gamma_n(u_n)\,(T-t)\,dt\\ &= \int_0^T \gamma_n(u_n)\,dt - T\gamma_n(u_n^0) \to \int_0^T \gamma(u)\,dt - T\gamma(u^0) \overset{(7.29)}{=} \int_0^T (D_t u, z)_H\,d\mu(t).\end{aligned} \tag{7.30}$$

By (7.9) then

$$(D_t u_n, h_n - z_n) \underset{\tilde{\pi}}{\rightarrow} (D_t u, h - z) \qquad \text{in } L^2_\mu(0,T;H\times H). \tag{7.31}$$

(iv) (7.20) yields

$$\begin{aligned}G_n(v,w) := \int_0^T \varphi_n(v,w)\,d\mu(t) \overset{\Gamma\tilde{\pi}}{\to} \int_0^T \varphi(v,w)\,d\mu(t) =: G(v,w)\\ \text{sequentially in } L^2_\mu(0,T;H\times H).\end{aligned} \tag{7.32}$$

Therefore

$$\begin{aligned}G(D_t u, h - z) &\overset{(7.31),(7.32)}{\le} \liminf_{n\to\infty} G_n(D_t u_n, h_n - z_n)\\ &\overset{(7.12),(7.16)_2}{\le} \liminf_{n\to\infty} \int_0^T \langle D_t u_n, h_n - z_n\rangle\,d\mu(t) \overset{(7.31)}{=} \int_0^T \langle D_t u, h - z\rangle\,d\mu(t).\end{aligned} \tag{7.33}$$

Thus u fulfills the time-integrated BEN principle, cf. Theorem 9; the inclusion $(7.23)_2$ is thus established. □

Remarks 2 (i) If instead of requiring an initial condition one prescribes u to be T-periodic in time, then $\int_0^T \langle z, D_t u\rangle\,dt = 0$ (as $z \in \partial\gamma(u)$). In this case it is not needed to introduce the weight function $\mu(t)$, and the above argument can be much simplified.

(ii) Doubly-nonlinear equations of the form (1.3) can also be formulated as null-minimization problems. Their structural compactness and stability can be proved via the techniques of this section, see [51]. □

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