64

Stokes' Theorem for Hypergestures

Summary. As singular homology is strongly related to de Rham cohomology, in particular by Stokes' classical theorem, it is natural to ask for such a theorem in our context of hypergestures. But there is a deeper reason for such a project, namely the idea that music theory of hypergestures could provide us with models of energy exchange in gestural interaction. In such a (still hypothetical) theory, Stokes' theorem would play a crucial role regarding questions of energy conservation (integral invariants).

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64.1 The Need for Stokes' Theorem for Hypergestures

Stokes' classical theorem states

$$\int_C d\omega = \int_{\partial C} \omega,$$

where C is a compact oriented k-dimensional manifold with boundary and ω is a $k-1$-form on C. The operator $d\omega$ is the exterior derivative of ω, and ∂C is the boundary of C, see Appendix Section J.8. It is well known that this formula is valid for slightly more general situations, namely, where the boundary is not a

Stokes' theorem is of primordial importance in many fields of physics, e.g. in mechanics (integral invariants, see [2]) or in electrodynamics (relating differential and integral forms of Maxwell's equations [497]). The reason why we are interested in such a theorem for mathematical music theory is twofold: On the one hand, we have initiated a homological study of hypergestural structures [727] (see Chapter 63) which has also provided us with applications to counterpoint theory [16] (see Chapter 79). As singular homology is strongly related to de Rham cohomology, in particular by Stokes' theorem, it is natural to ask for such a theorem in our context of hypergestures. But there is a deeper reason for such a project, namely the idea that music theory of hypergestures could provide us with models of energy exchange in gestural interaction. In such a (still hypothetical) theory, Stokes' theorem would play a crucial role regarding questions of energy conservation (integral invariants).

64.2 Almost Regular Manifolds, Differential Forms, and Integration for Hypergestures

We first need to specify the basic concepts that contribute to the Stokes statement. We are aware of the somewhat sloppy style in this quite standard part of the chapter; the reader is kindly asked to fill out the standard technical details.

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64.2.1 Locally Almost Regular Manifolds

In the present context we need hypergestures in manifolds since we are dealing with differentiable structures. We however need quite general manifolds in the sense of what are called "almost regular manifolds" in [617] or even more singular manifolds, where the boundaries have corners. To understand our requirement we look at typical manifolds in the context of hypergestures. In [719], we have introduced a standard topological space $|\Sigma|$ associated with a digraph Σ, see Section 61.7. It is the colimit of the digraph's arrow set A_Σ, the gluing operation being performed on the digraph vertices set V_Σ. This topological space specifies one line chart $|a| \xrightarrow{\sim} I = [0, 1]$ per arrow a and a point chart $|x|$ for each isolated vertex x. The specification of this atlas is mandatory since we don't want to glue two consecutive arrows $x \xrightarrow{a} y \xrightarrow{b} z$ to one line. The differentiability in the connecting vertex y is suspended. Or it may also happen that three or more arrows share a vertex, and then the differentiability in such a vertex would not make sense. We call *skeletal space* the manifold $|\Sigma|$ associated with skeleton Σ.

The best conceptual approach to this situation is to embed such a manifold in a differentiable manifold M as a subset whose charts are manifolds with boundary isomorphic to the unit interval I or to a zero-dimensional point manifold 0. We next need cartesian products of such manifolds when hypergestures are discussed. This means that we have to consider products of type $|\Sigma_1| \times |\Sigma_1| \times \ldots |\Sigma_n|$. These manifolds are living in cartesian products of their carrier manifolds $M_1, M_2, \ldots M_n$, and the typical boundary of a product $|\Sigma_1| \times |\Sigma_2|$ is $\partial(|\Sigma_1| \times |\Sigma_2|) = \partial|\Sigma_1| \times |\Sigma_2| \cup |\Sigma_1| \times \partial|\Sigma_2|$, see Figure 64.1 for an example.

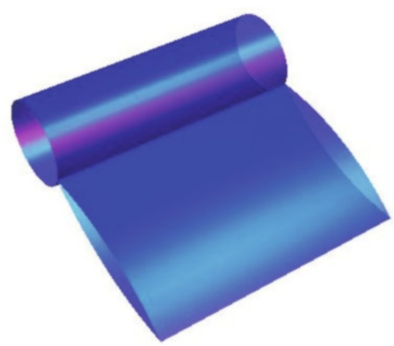

Fig. 64.1. A skeletal space.

But observe that due to singular points in digraphs, such products can be inhomogeneous in their dimension. A product may be a disjoint union of submanifolds of different dimensions.

To get a reasonable category of such manifolds, we consider differentiable morphism $L \to M$ of the carrier manifolds L, M of $\mathcal{L}, \mathcal{M}$, respectively, that restrict to atlas-compatible maps $f : \mathcal{L}^I \to \mathcal{M}^J$, where I, J designate the atlases of $\mathcal{L}, \mathcal{M}$, respectively. Atlas-compatibility means that, as in mathematical music theory of global compositions, we are also given a map $g : I \to J$ such that f sends I-chart $\mathcal{L}_i$ to J-chart $\mathcal{M}_{g(i)}$. We denote this *category of locally almost regular manifolds* by $LARM$. Such a manifold need not have a determined dimension, but may have several dimensions according to connected components and charts. In what follows, we shall call dimension $dim(\mathcal{L})$ of an almost regular manifold $\mathcal{L}$ the maximal dimension of such components. The submanifold of $\mathcal{L}$ of a determined dimension k will be denoted by $\mathcal{L}^k$.

The most important application of $LARM$ for the Stokes theory lies in a reinterpretation of hypergestures. Suppose we are given a hypergesture $c \in \Sigma_1\Sigma_2 \ldots \Sigma_n \overrightarrow{@} \mathcal{L}$ over n skeleta $\Sigma_1, \Sigma_2, \ldots \Sigma_n$ with values in a locally almost regular manifold $\mathcal{L}$. By the very definition of hypergestures, and by the adjointness property of the manifold $|\Sigma|$ associated with skeleton Σ (Proposition 62), as well as the adjointness of the cartesian product and repeated function spaces (also known as currying in computer science), $\Sigma_1\Sigma_2 \ldots \Sigma_n \overrightarrow{@} \mathcal{L} \xrightarrow{\sim} |\Sigma_1| \times |\Sigma_2| \times \ldots |\Sigma_n| © \mathcal{L}$, the set of continuous functions from the cartesian product of the skeletal manifolds to $\mathcal{L}$. Within this function set, we exhibit the differentiable morphisms and denote their set by $|\Sigma_1| \times |\Sigma_2| \times \ldots |\Sigma_n| ⓓ \mathcal{L}$. The morphisms in the latter (more precisely: their corresponding hypergestures) are called *differentiable hypergestures*; the set of these hypergestures is also denoted by $\Sigma_1 \Sigma_2 \ldots \Sigma_n ⓓ \mathcal{L}$. In the context of the Stokes theorem, we need differentiable singular n-cubes. Their generalization to hypergestures are differentiable gestural n-cubes, namely the elements of $\Sigma_1 \Sigma_2 \ldots \Sigma_n ⓓ \mathcal{L}$. The free module $\mathbb{R}\Sigma_1 \Sigma_2 \ldots \Sigma_n ⓓ \mathcal{L}$ of $\mathbb{R}$-linear combinations of differentiable gestural n-cubes (the module basis) defines the *(differentiable) n-chains over* $\Sigma_1, \Sigma_2, \ldots \Sigma_n$ with values in $\mathcal{L}$.

64.2.2 Differential Forms

On a locally almost regular manifold $\mathcal{L}$ (we omit the atlas if possible to ease notation), differential forms can be considered in the sense that they are defined on each chart as usual. If such a chart $\mathcal{L}_i$ has dimension n, the differential forms of dimension $k \leqslant n$ define non-trivial real vector spaces $\bigwedge^k \mathcal{L}_{i,x}$ at each point x of $\mathcal{L}_i$. A differential k-form ω on $\mathcal{L}$ is a differentiable section in each chart $\bigwedge^k \mathcal{L}_i$. Since our manifolds are of different dimensions locally, we will have to deal with forms that don't have the same dimension everywhere, they are not homogeneous. We therefore consider the direct sum $\bigwedge^{\oplus k} \mathcal{L} = \bigoplus_{l \leqslant k} \bigwedge^l \mathcal{L}$. If we take a differential form $\omega \in \bigwedge^{\oplus k} \mathcal{L}$, its l-component will be denoted by ω_l. As in the classical case, for a morphism $f : \mathcal{L} \to \mathcal{M}$ of locally almost regular manifolds, one has the canonical inverse image $f^*\omega \in \bigwedge^k \mathcal{L}$ for $\omega \in \bigwedge^k \mathcal{M}$.

In the classical case, one has the exterior derivative operator $d : \bigwedge^k \mathcal{L} \to \bigwedge^{k+1} \mathcal{L}$ with $d^2 = 0$. For the non-homogeneous case mentioned above, we need a derivative operator $d_\oplus$ defined by $d_\oplus \omega = (\omega_0, d\omega_0, d\omega_1, d\omega_2, \ldots)$ for $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$. For this operator, we have $d^2_\oplus \omega = (\omega_0, d\omega_0, 0, \ldots)$. And as in the classical case, the operators d and $d_\oplus$ commute with inverse images.

64.2.3 Integration

Modulo linear extensions to n-chains, we need to define $\int_c \omega$ for a gestural n-cube $c \in \Sigma_1 \Sigma_2 \ldots \Sigma_n ⓓ \mathcal{L}$. As usual, the formula is defined to mean $\int_{|\Sigma_1| \times |\Sigma_2| \times \ldots |\Sigma_n|} c^*\omega$, which amounts to restricting ourselves to the special case $\mathcal{L} = |\Sigma_1| \times |\Sigma_2| \times \ldots |\Sigma_n|$. We shall define the integral by recursion on the hypergestural parameters and recalling the Fubini theorem for iterated integration [999, Theorem 3-1]. Let $(\lambda, t) \in T|\Sigma_1|_\lambda$, the tangent space at $\lambda \in |\Sigma_1|$. This argument defines a form $c^*\omega_{\lambda,t} \in \bigwedge^{\oplus(n-1)} |\Sigma_2| \times \ldots |\Sigma_n|$, and we may suppose by recursion that $I(\lambda, t) = \int_{|\Sigma_2| \times \ldots |\Sigma_n|} c^*\omega_{\lambda,t}$ is defined, which yields an element of $\bigwedge^{\oplus 1} |\Sigma_1|$. So we are left with the definition of the integral for $n = 0, 1$. If $n = 0$, $c \in \mathcal{L}$, and $\omega \in \mathcal{F}(\mathcal{L})$ is a function. Then we set $\int_c \omega = \omega(c)$. In dimension $n = 1$, there are three cases for Σ_1:

1. If $A_{\Sigma_1} = \varnothing$, then set $\int_c \omega = \sum_{i \in V_{\Sigma_1}} \omega_0(c(i)) = \sum_{i \in V_{\Sigma_1}} \int_{c(i)} \omega_0$.
2. Recall from [727, Section 3] that for an arrow a of Σ_1, a^- denotes the subskeleton of Σ_1 after taking away the tail $t(a)$ and all arrows connected to $t(a)$, and a^+ denotes the subskeleton of Σ_1 after taking away the head $h(a)$ and all arrows connected to $h(a)$. In this second case, we suppose that there is at least one arrow a, but both A_{a^-} and A_{a^+} are empty. This means that, besides isolated vertices, there are either a number of loops on a single vertex or a number of arrows between two distinct points. This is the classical one-dimensional situation for integration on the unit interval. So we define $\int_c \omega = \sum_{a \in A_{\Sigma_1}} \int_a \omega_1 + \int_{\text{isolated vertices}} \omega$, where $\int_a \omega_1$ is the evident classical integration.
3. In the third case, there is an arrow a such that $A_{a^-} \cup A_{a^+} \neq \varnothing$. We then set the recursive formula $\int_c \omega = \sum_{a \in A_{\Sigma_1}} (\int_{c|a^-} \omega - \int_{c|a^+} \omega)$, a formula that reminds us of the definition of the face operator ?▫ given in [727, Definition 3.1].

64.3 Stokes' Theorem

For the proof of Stokes' theorem for hypergestures, we need a technical lemma. It refers to the Escher theorem operation on chains $c \in \Sigma_1\Sigma_2 \dots \Sigma_n ⓓ\mathcal{L}$ which generates a chain $c_j \in \Sigma_j\Sigma_1\Sigma_2 \dots \widehat{\Sigma_j} \dots \Sigma_n ⓓ\mathcal{L}$.

Lemma 1. *If* $c \in \Sigma_1\Sigma_2 \dots \Sigma_n ⓓ\mathcal{L}$ *is a differentiable* n*-cube,* $1 \leqslant j \leqslant n$*,* $a \in A_{\Sigma_j}$*, and* $\lambda \in |\Sigma_1|$*, then we have*

$$(c_j|a^{\pm})^{\square}(\lambda) = (c(\lambda)_j|a^{\pm})^{\square},$$

and therefore also

$$(c_j)^{\square}(\lambda) = (c(\lambda)_j)^{\square}.$$

The lemma follows from the observation that (1) the face operator yields the same linear combination on both sides since it acts on the same $\Sigma_j|a^{\pm}$, and (2) the evaluation at λ is taken on the same face operator result.

Theorem 41 (Stokes' Theorem for Hypergestures) *Let* $c \in \mathbb{R}\Sigma_1\Sigma_2 \dots \Sigma_k ⓓ\mathcal{L}$ *be a* k*-chain in a* k*-dimensional locally almost regular manifold* $\mathcal{L}$*, and let* $f \in \bigwedge^{k-1}\mathcal{L}$*. Then*

$$\int_c d^{\oplus} f = \int_{\partial c} f.$$

Proof. We can of course restrict to gestural k-cubes. For $k = 1$, f is a function on $\mathcal{L}$ and $c \in \Sigma ⓓ\mathcal{L}$. Let first $A_{\Sigma} = \varnothing$. Then $\int_{\partial c} f = \sum_{i \in V_{\Sigma}} f(c(i))$, whereas $\int_c d^{\oplus} f = \sum_{i \in V_{\Sigma}} (d^{\oplus} f)_0(c(i)) = \sum_{i \in V_{\Sigma}} f(c(i))$ yields the same. For the second case, $A_{a^-} \cup A_{a^+} = \varnothing$, but since arrows exist, we may focus on the subskeleton bearing those arrows, the discrete part having been already dealt with. Here,

$$\begin{aligned}
\int_c d^{\oplus} f &= \sum_{a \in A_{\Sigma}} \int_a df \\
&= \sum_{a \in A_{\Sigma}} \int_{\partial a} f \\
&= \sum_{a \in A_{\Sigma}} f(c(h(a))) - f(c(t(a))) \\
&= \int_{\partial c} f,
\end{aligned}$$

this is the classical case. For the third case, $A_{a^-} \cup A_{a^+} \neq \varnothing$, we have

$$\begin{aligned}
\int_c d^{\oplus} f &= \sum_{a \in A_{\Sigma}} \int_{c|a^-} d^{\oplus} f - \int_{c|a^+} d^{\oplus} f \\
&= \sum_{a \in A_{\Sigma}} \int_{\partial(c|a^-)} f - \int_{\partial(c|a^+)} f \\
&= \sum_{a \in A_{\Sigma}} \int_{(c|a^-)^{\square}} f - \int_{(c|a^+)^{\square}} f \\
&= \int_{\partial c} f
\end{aligned}$$

by recursion and since ∂ and $?^{\square}$ coincide in dimension one.

The case of higher dimensions runs as follows:

$$
\begin{aligned}
\int_c d^{\oplus} f &= \int_{|\Sigma_1|} \int_{c(\lambda)} d^{\oplus} f \ (\lambda \in |\Sigma_1|) \\
&= \int_{|\Sigma_1|} \int_{\partial c(\lambda)} f \quad \text{(recursion)} \\
&= \int_{|\Sigma_1|} \sum_j (-1)^j \int_{(c(\lambda)_j)^{\square}} f \\
&= \sum_j (-1)^j \int_{|\Sigma_1|} \int_{(c(\lambda)_j)^{\square}} f \\
&= \sum_j (-1)^j \int_{|\Sigma_1|} \int_{(c_j)^{\square}(\lambda)} f \quad \text{(Lemma 1)} \\
&= \sum_j (-1)^j \int_{(c_j)^{\square}} f \\
&= \int_{\partial c} f.
\end{aligned}
$$

This concludes the proof of Stokes' theorem.