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Categories of Gestures over Topological Categories

Summary. We generalize the topological approach to gestures, and culminate in the construction of a gesture bicategory, which enriches the classical Yoneda embedding and could be a valid candidate for the conjectured space X in the diamond conjecture [720]; see also Section 61.12. We discuss first applications thereof for topological groups, and then more concretely gestures in modulation processes in Beethoven's Hammerklavier sonata. The latter offers a first concretization of answers to Lewin's big question from [605] concerning characteristic gestures. This research is a first step towards a replacement of Fregean functional abstraction by gestural dynamics.

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In Chapter 61, we presented a mathematical model for gestures in music. In that model, a gesture γ is built from two components: a combinatorial "skeleton" represented by a digraph Γ, and a "body", represented by a configuration of continuous curves $\gamma(a) : I \to X$ on the real unit interval I with values in a topological space X, one for each arrow a of the skeleton, and connected according to the digraph's vertex configuration. Given two gestures δ, γ, a morphism $f : \delta \to \gamma$ is a digraph morphism $f : \Delta \to \Gamma$ between the skeleta Δ, Γ of δ, γ, respectively, which "extends" to a morphism of the respective bodies by a continuous map defined on the respective topological spaces. See [719] or Section 61.5 for the formal setup. This defines the category *Gesture* of gestures, which shares the two crucial properties:

- The set of gestures with skeleton Γ and with body in the topological space X is canonically provided with a topology deduced from the compact-open topology on the set $I@X$ of continuous maps from I to X; this topological space is denoted by $\Gamma \overrightarrow{@} X$. We therefore are capable of defining gestures of gestures, namely gestures with values in a topological space $\Gamma \overrightarrow{@} X$. Such gestures are called hypergestures.
- The hypergesture construction entails spaces of iterated hypergestures in the sense that for a sequence $\Gamma_1, \Gamma_2, \ldots \Gamma_n$ of skeleta and a topological space X, we have the space $\Gamma_1 \overrightarrow{@} \Gamma_2 \overrightarrow{@} \ldots \Gamma_n \overrightarrow{@} X$ of n-fold hypergestures over X. We then have the theorem that this iterated construction yields homeomorphic topological spaces if we permute the order of these skeleta; see Proposition 61 (First Escher Theorem) and Corollary 1. This result is of primordial significance in the creative gestural interaction in free jazz, see [721] for a detailed discussion.

Despite these promising first results, gesture theory is still "adolescent": Here are some questions, which we have encountered after a first critical analysis of the state of the art:

1. In the definition of a gesture, no allusions to transformations are made. We only deal with continuous curve systems. However, many examples from practice are more specific, they also involve transformations generating such curves. The classical and trivial example is a shift from a note x to a note y in a parameter space X, such as $X = \mathbb{R}^m$, to fix the ideas. This shift can be seen as a curve $c : I \to \mathbb{R}^m$ with $c(t) = T^{t(y-x)}(x)$, where $T^d : \mathbb{R}^m \to \mathbb{R}^m$ is the shift operation by d. This defines a special curve, not just any continuous data.

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2. A very important point, also related to the previous one, refers to that famous gesture-theoretical question posed by David Lewin [605, p.159]: “If I am at s and wish to get to t, what characteristic gesture should I perform in order to arrive there?” A first observation must be made with respect to the general theoretical background of the question. Lewin poses it in a thoroughly transformational context. His book deals with transformations, with very classical affine functions on musical parameter spaces (mainly pitch class spaces). It is however not the only occasion where he opens up a gestural connotation of his transformational text. In many spots, he uses the word “gesture” and its paradigm, such as dancing and other motional and emotional metaphors. Lewin's gestural subtext is manifestly more than intuitive rhetorics, he means gesture and not just a fancy description of transformational configurations. This is a deep conflict in Lewin's musical thinking: He unfolds a valid transformational theory, but the subtext of gestures is not reflected in this theory. It remains a “dream of continuity while sleeping in the hard ‘cartesian’ bed of abstract algebra”.
   In view of this observations, the question reveals its full power: How could we merge the transformational reality with the dream of continuity? The immediate mathematical response is: “by continuous transformations!” But the question is not solved with this immediate reply, since it is not clear what Lewin means by “characteristic”. What is a characteristic gesture in contrast to any gesture? What is the character that has to be grasped?
3. The domain of continuous curves, I, is not only a topological space, but intrinsically has its topology derived from the linear ordering among real numbers. This information was not exploited in the previous theory. In other words: What is the reflection of this ordering relation within the topological space X that embodies a gesture? Is there any rationale to introduce “directions” in X?
4. Relating more specifically to the existent theory of Chapter 61, we see this general picture: The theory unfolds in two branches from the basic category **Digraph** of digraphs. The topological branch unfolds gestures as embodiments of digraphs within topological spaces, whereas the algebraic branch realizes digraphs as special linear categories, namely spectroids, enabeling formulaic structures typically related to commutative diagrams defined by algebraic relations. These ramifications are radically different interpretations of digraphs as basic constructors of mathematical theories. It was conjectured in [720] that one may construct a universal category X above the two categories of the two branches, *Gestoid* and *Formoid*, which would enable one to embed them as special cases of a comprising big structure.
   There are several indications that such a category might exist. The first one is the possibility to rebuild algebraic structures from gestures, more precisely: to rebuild groups through homotopy theory. It is in fact well known that any group is isomorphic to a fundamental group of a topological space [993], and we have more specifically given examples of such spaces for finitely generated abelian groups in [720], including some musical interpretation; see also Section 78.2.10.2. Intuitively speaking, since hypergestures generalize homotopy classes, one may say that *every group is realized by a group of hypergestures of loop gestures.*
   To date, the reverse direction looks less promising. No reasonable way is known to step over to gestures from abstract categories. A universal space as conjectured in the diamond conjecture should deal with this problem. Philosophically speaking, it is the problem of reconstructing gestural instances from general abstract categories.
5. A seemingly different, but in fact very relevant question arises from the deeper understanding of Yoneda's Lemma [637]. The lemma states (among other things) that in any category $\mathcal{C}$, the canonical functor $@ : X \mapsto @X$ sending an object X to its presheaf $@X$ is fully faithful, which means that the morphisms $f : X \to Y$ in $\mathcal{C}$ are in one-to-one correspondence with the morphisms (natural transformations) $F : @X \to @Y$. This sounds abstract, but it means that we may look at abstract morphisms $f : X \to Y$ in terms of ordinary “old-fashioned” Fregean functions on point sets $A@F : A@X \to A@Y$ for each given argument A. This technique is a big help for reconstructing intuitive human manipulation of mathematical objects when dealing with abstract categories. It however does not help us reconstruct the motion, which is intuitively happening, when moving from an argument x of a function f to the value $f(x)$. This deficiency is exactly what lies behind Lewin's question: The transformations are Fregean

functions and do not automatically involve any kind of motion as suggested by gestural utterances. So the question would be whether there is a way to embody Fregean functions within the realm of gestures.

The plan of this chapter is this. We first generalize the idea of a gesture from a purely topological setup to a functorial one, namely the setup of *topological categories*, i.e., categories internal to the category **Top** of topological spaces and continuous maps, replacing the unit interval I by a topological category (see Appendix Section J.4), the simplex category ∇, and continuous functions by continuous functors. This will be used in the second step, where we construct gestures from morphisms in abstract categories. This step is a decisive one towards the incorporation of abstract categories in the framework of gesture theory. It culminates in the construction of a bicategory of gestures for any category and leads to a first answer to the diamond conjecture. In a third step we apply these constructions to the important case of the category canonically associated with a topological group. We also discuss technical tools for overcoming the core problem of the mirror operation, which does not as such offer a gestural interpretation. In the fourth step, we discuss two modulations in Beethoven's Hammerklavier sonata op. 106 in order to apply the gestural approach for a deeper understanding of these modulations. We shall discuss Lewin's question about characteristic gestures. This last discussion reveals the intrinsically dramatic character of gestural interpretations of given scores.

62.1 Gestures over Topological Categories

In this section we set up the framework for a gesture theory that is based upon categories instead of plain topological spaces. In our setup, a category $\mathcal{C}$ is thought of as being a collection of morphisms, together with two maps $d, c : \mathcal{C} \to \mathcal{C}$ (d for "domain", c for "codomain"), and we write $f : d(f) \to c(f)$ to make these maps evident. In what follows, we shall start from a given *topological category* K (see Appendix Section J.4). This means that the collection of morphisms K is a topological space, and that domain and codomain, as well as the composition of morphisms (on the morphism sets with the relative topologies), are continuous.

Here are two basic examples of such categories: (1) The *simplex category* ∇ associated with the unit interval I: Its morphism set is $\nabla = \{(x,y)|x,y \in I \text{ and } x \leqslant y\}$, $d(x,y) = (x,x)$, $c(x,y) = (y,y)$, the composition of morphisms is obvious, and the topology on ∇ is the relative topology inherited from the usual product topology on $I \times I \subset \mathbb{R} \times \mathbb{R}$. (2) The *graph category* associated with any topological space X: Its morphism set is $X \times X$, equipped with the product topology, while we set $d(x,y) = (x,x), c(x,y) = (y,y)$, and again, the composition of morphisms is the obvious one. If no confusion is likely, we denote the graph category of X by X. Clearly, a graph category is a topological groupoid. In particular, the simplex category ∇ is just the subcategory of the graph category I on the pairs $(x,y), x \leqslant y$.

If K, L are two topological categories, a *topological functor* $F : K \to L$ is a functor, which is also continuous as a map between morphism sets. This defines the category **TopCat** (in fact a 2-category, see [131, Proposition 8.1.4]) of topological categories. In order to distinguish the set of topological functors $F : K \to L$ from the larger set $K@L = \mathbf{Cat}(K,L)$ of all possible functors, we write $K©L$ for $\mathbf{TopCat}(K,L)$. If X, Y are topological spaces, then the map which associates with a continuous map $f : X \to Y$ the synonymous continuous functor is fully faithful, so the category of topological spaces is a full subcategory of the category of continuous categories. Therefore we shall henceforth tacitly identify the category **Top** of topological spaces and continous maps with the associated subcategory of topological categories and continuous functors embedded in **TopCat** via the graph category associated with a topological space.

With this in mind, if K is a topological category, the set of continuous curves with values in K is by definition the set $\nabla©K$. Evidently, if K is a topological space, then $\nabla©K \xrightarrow{\sim} I@K$, where $I@K$ is the set $\mathbf{Top}(I,K)$ of continuous I-parametrized curves $c : I \to K$ in the topological space K, the bijection being induced by the restriction of a functor $F : \nabla \to K$ to the canonical diagonal embedding $I \to \nabla$ of the objects in ∇. This set $\nabla©K$ is the object set of a category also denoted by $\nabla©K$ if we take as morphisms between two curves $f, g : \nabla \to K$ the continuous natural transformations $\nu : f \to g$, which means that the defining maps $\nu : I \to K$ are continuous and satisfy the defining commutative squares for natural transformations. We do however want it to become a topological category, and this works as follows: We take the morphism set as being composed by the triples (f, g, ν) as above. The topology is defined by the following construction. The set of objects of $\nabla©K$ is given the compact-open topology induced by the topologies of ∇ and K, the subset of continuous natural transformations $\nu : I \to K$ within $I@K$ is given the topology induced by the compact-open topology on $I@K$. The triples are viewed as points in the product topology on $\nabla©K \times \nabla©K \times I@K$. Clearly, this is a topological category. Also observe that in the case of a topological space K, the compact-open topology of $I@K$ coincides with the topology induced by the isomorphism $\nabla©K \xrightarrow{\sim} I@K$ and the compact-open topology on $\nabla©K$.

Example 70 The set $\nabla©K$ can also be enforced for a not a priori topological category K as follows. Take any set $C \subset \nabla@K$ of functors $F : \nabla \to K$ into an abstract category (suppose K small, if set theory matters) and then select the finest topology on K such that all functors of C become continuous. For this construction one writes $\nabla©_C K$ to indicate that K is made a topological category via C, and that this is the set of all continuous curves with respect to this topology.

62.1.1 The Categorical Digraph of a Topological Category

In order to obtain gestures in topological categories, we need to mimic the construction of a spatial digraph [720], see also Section 61.5. To this end, we consider the two continuous tail and head functors $t, h : \nabla©K \to K$, which are defined as follows. If $\nu : f \to g$ is a natural transformation between $f, g : \nabla \to K$, then $t(\nu) = \nu(0) : f(0) \to g(0)$, and $h(\nu) = \nu(1) : f(1) \to g(1)$. So the tail and head maps are not only set maps but functors. Call this diagram of topological categories and continuous functors the *categorical digraph*[1] $\overrightarrow{K}$ *of* K. If we forget about the category and just retain the objects of this configuration, we call it the *(underlying) spatial digraph of* K. In particular, if Γ is a digraph, the set of morphisms $\Gamma@\overrightarrow{K}$ is the set of digraph morphisms into the underlying spatial digraph of K. In other words, such a morphism assigns an object of K to every vertex of Γ and a continuous curve (topological functor) $\nabla \to K$ to every arrow of Γ, with matching sources and targets. We call then, by definition, a *gesture with skeleton* Γ *and body in* K a morphism of digraphs $g : \Gamma \to \overrightarrow{K}$.

If the topological category is a topological groupoid, then we have an easy proposition which guarantees that one may reverse all arrows, in other words: the categorical digraphs of topological groupoids are self-dual.

Proposition 1. *Let K be a topological groupoid. Then we have a duality automorphism $?^* : \overrightarrow{K} \xrightarrow{\sim} \overrightarrow{K}^*$ onto the dual digraph $\overrightarrow{K}^*$ (tail and head functors exchanged), which maps a curve $g : \nabla \to K$ to its inverse curve $g^* : \nabla \to K$ defined by $g^*(x, y) = g(1-y, 1-x)^{-1}$.*

Therefore, for a topological groupoid K, the set $\Gamma@\overrightarrow{K}$ is in bijection with its dual set $\Gamma^*@\overrightarrow{K}^*$, and then with the set $\Gamma^*@\overrightarrow{K}$ associated by the duality $?^*$. Call the gesture $g* : \Gamma^* \to \overrightarrow{K}$ associated by this bijection with a given gesture $g : \Gamma \to \overrightarrow{K}$ the *dual gesture*. Intuitively it reverses the arrows of the skeleton and the morphisms of the body's curves.

62.1.2 Gestures with Body in a Topological Category

We have constructed the set $\Gamma@\overrightarrow{K}$ of gestures with skeleton Γ and body in a topological category K. In the previous theory described in Section 61.6, this set was enriched to yield a topological space in order to enable the iterative construction of hypergestures. In our present setup, we have to construct a topological category out of the above set. To do so, recall that the special case $\Gamma = \uparrow$ (one arrow between two different vertices) means that we have the topological category $\uparrow @\overrightarrow{K} \xrightarrow{\sim} \nabla©K$ of continuous curves $c : \nabla \to K$ (with the above mentioned compact-open topology).

The general case follows from the observation that Γ is the colimit of the following diagram $\mathcal{D}$ of digraphs: We take one arrow digraph $\uparrow_a = \uparrow$ for each arrow $a \in A_\Gamma$ (A_Γ is the set of arrows of Γ) and one

[1] We have chosen this wording as an analogy with the spatial digraph, where the topological *space* is now replaced by the topological *category*. Although this is a diagram of topological categories, and not just of sets, we believe that the intuitive wording is not confusing.

bullet digraph $\bullet_x = \bullet$ for each vertex $x \in V_\Gamma$ (V_Γ is the set of vertices of Γ). We take as morphisms the tail or head injections $\bullet_x \rightarrow \uparrow_a$ whenever $x = t(a)$ or $x = h(a)$. Then evidently, $\Gamma \xrightarrow{\sim} \mathrm{colim}\mathcal{D}$. Therefore, the set of gestures $\Gamma @ \overrightarrow{K}$ is bijective with the limit $\lim \mathcal{D} @ \overrightarrow{K}$ of a diagram of *the objects of* topological categories $\uparrow @ \overrightarrow{K} \xrightarrow{\sim} \nabla © K$ (for the digraph's arrows) and $\bullet @ \overrightarrow{K} \xrightarrow{\sim} K$ (for the digraph's vertices). But the maps between these objects of categories stem in fact from functors (those from the categorical graph $\overrightarrow{K}$). Therefore the limit can be taken as one of a diagram of topological categories. This yields a category, whose topology is defined as the limit topology of this diagram. This topological category is denoted by $\Gamma \overrightarrow{@} K$. In this category, a morphism is the limit of natural transformations between continuous curves and morphisms between objects of K, the latter representing the end points of the continuous curves.

Example 71 If the topological category K is a topological space, we recover the topological category $\Gamma \overrightarrow{@} K$ associated with the topological space $\Gamma \overrightarrow{@} K$ in the previous theory of Section 61.6.

The construction of the topological category $\Gamma \overrightarrow{@} K$ automatically enables the machinery of hypergestures known from the previous topological space setup. And again, we have the Escher Theorem for topological categories of hypergestures:

Proposition 2. (Escher Theorem for Topological Categories) *If Γ, Δ are digraphs and K is a topological category, then we have a canonical isomorphism of topological categories,*

$$\Gamma \overrightarrow{@} \Delta \overrightarrow{@} K \xrightarrow{\sim} \Delta \overrightarrow{@} \Gamma \overrightarrow{@} K.$$

Corollary 1. *The action*

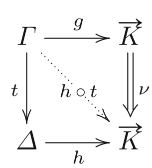
$$\overrightarrow{@} : \mathbf{Digraph} \times \mathbf{TopCat} \rightarrow \mathbf{TopCat} : (\Gamma, K) \mapsto \Gamma \overrightarrow{@} K$$

canonically extends to an action (denoted by the same symbol)

$$\overrightarrow{@} : [\mathbf{Digraph}] \times \mathbf{TopCat} \rightarrow \mathbf{TopCat} : (W, K) \mapsto W \overrightarrow{@} K$$

of the free commutative monoid [**Digraph**], *i.e., the monoid of commutative words $W = \Gamma_1 \Gamma_2 \ldots \Gamma_k$ over the alphabet* **Digraph** *of digraphs (the objects only). It is defined inductively by $\Gamma_1 \Gamma_2 \ldots \Gamma_k \overrightarrow{@} K = \Gamma_1 \overrightarrow{@} (\Gamma_2 \ldots \Gamma_k \overrightarrow{@} K)$ and*[2] *$\emptyset \overrightarrow{@} K = K$.*

With this hypergestural construction, we define the category of gestures with body in K, now also including the morphisms between their skeleta. It is denoted by *Gesture*(K). Its objects are the objects of $\Gamma \overrightarrow{@} K$ for any digraph Γ. Given two such gestures $g : \Gamma \rightarrow \overrightarrow{K}, h : \Delta \rightarrow \overrightarrow{K}$, a *morphism* $a : g \rightarrow h$ is a pair $a = (t, \nu)$, consisting of a digraph morphism $t : \Gamma \rightarrow \Delta$, and a morphism $\nu : g \rightarrow h \circ t$ in $\Gamma \overrightarrow{@} K$, which we also write as a diagram, but with the natural transformation being denoted by a double arrow in order to prevent a wrong intuition about a commutative square:

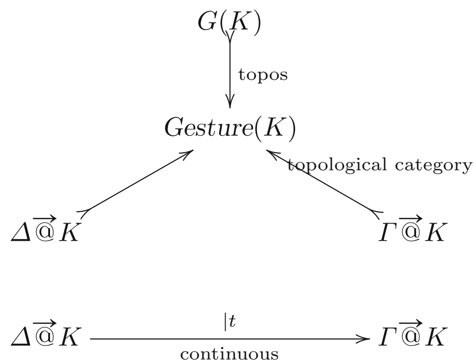


If we are given a second morphism $b : h \rightarrow k$, $b = (s, \mu)$, with codomain $k : \Sigma \rightarrow \overrightarrow{K}$, then the composition $b \circ a : g \rightarrow k$ is defined by $b \circ a = (s \circ t, \mu | t \circ \nu)$, where $\mu | t$ means that the natural transformation μ from h to $k \circ s$ is "restricted" by the digraph morphism t.

The category *Gesture*(K) therefore contains two types of subcategories: On the one hand the (comma category) topos $G(K) = \mathbf{Digraph}/\overrightarrow{K} \subset$ *Gesture*(K) of gestures with body in K, the morphism being the digraph morphisms of gesture skeleta commuting with the domain and codomain gestures. On the other

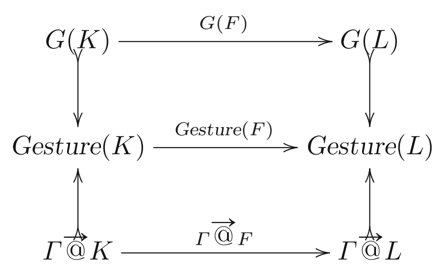
[2] $\emptyset$ denotes the empty word.

hand, we have, for each skeleton Γ, the topological category $\Gamma\overrightarrow{@}K \subset Gesture(K)$. Finally, for each digraph morphism $t : \Gamma \to \Delta$, we have a canonical continuous restriction functor $|t : \Delta\overrightarrow{@}K \to \Gamma\overrightarrow{@}K$. Here is the overall picture:



62.1.3 Varying the Underlying Topological Category

For a continuous functor $F : K \to L$ between topological categories, we have a canonical morphism of categorical digraphs $\overrightarrow{K} \to \overrightarrow{L}$, which sends vertices to vertices, namely by the given functor $F : K \to L$, and sends curves $f : \nabla \to K$ to curves $F \circ f$, whereas continuous natural transformations $\nu : f \to g$ are sent to the continuous natural transformations $F \circ \nu : F \circ f \to F \circ g$. Call this morphism a *spatial (categorical) digraph morphism* and denote it by $\overrightarrow{F}$. This morphism canonically induces a functor $Gesture(F) : Gesture(K) \to Gesture(L)$, which is compatible with the above subcategories as shown by the following commutative diagram:

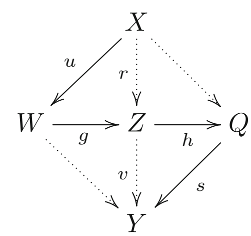


While the functor $\Gamma\overrightarrow{@}F$ is continuous, the functor $G(F)$ has a number of well-known properties of functors between topoi [639, Ch. IV.7]. The first of these properties is that $G(F)$ is right adjoint to the base change functor $\times\overrightarrow{F} : G(L) \to G(K)$ which associates with a gesture $g : \Gamma \to \overrightarrow{L}$ the fibre product gesture $g \times \overrightarrow{F} : \Gamma \times_{\overrightarrow{L}} \overrightarrow{K} \to \overrightarrow{K}$. Furthermore, the base change $\times\overrightarrow{F}$ is a logical functor (i.e., it preserves all topos-theoretical constructs, such as sub-object classifiers, finite limits and colimits, and exponentials, and has also a right adjoint). Paired with its right adjoint $\times\overrightarrow{F}_*$, the base-change functor defines a geometric morphism $G(K) \to G(L)$ [639, Ch. VII.1]. We shall come back to these facts later in Section 62.2.4, when discussing the gestural part of Yoneda's Lemma.

62.2 From Morphisms to Gestures

To conceive of a general method for generating gestures from morphisms $f : X \to Y$ in abstract categories, we start with a heuristic consideration. Suppose that we are working in a musical parameter space $\mathbb{R}^2$, which we endow with the structure of the Gaussian plane of complex numbers. Take a rotation $e^{i\theta} : x \mapsto x.e^{i\theta}$ on $\mathbb{R}^2$. In linear algebra, this morphism $f = e^{i\theta}$ is an encapsulated function, which has no relation to a gesture, but acts by Fregean "teleportation" on x. We are stressing this fact since in contradiction to the algebraic reality, our intuition of a rotation by angle θ is different in that we imagine a continuous rotational movement of x around the space origin until it reaches the final position $x.e^{i\theta}$. This process is visualized by the trace of x while rotating, i.e., by a continuous curve $c_x : I \to \mathbb{R}^2 : t \mapsto x.e^{i\theta t}$ on a circle of radius $|x|$. Each intermediate position $x.e^{i\theta t}$ corresponds to a factorization $f = e^{i\theta(1-t)} \circ e^{i\theta t} = f_{1-t} \circ f_t$ of f. In other words, the curve $c : I \to \mathrm{GL}_2(\mathbb{R})$ is a curve of factorizations of the given morphism f. This restatement of the gesture c in terms of factorizations means that c is viewed as being an "infinite" factorization insofar as the factors are parametrized by the curve parameter $t \in I$.

This enables us to rethink the basic elements of a gestural interpretation of morphisms in abstract categories. To this end, we fix a morphism $f : X \to Y$ in a category $\mathcal{C}$. The *category* $[f]$ *of factorizations of* f is defined as follows. Its morphism are the triples (u, g, v) of morphism $u : X \to W, g : W \to Z, v : Z \to Y$ such that $v \circ g \circ u = f$. The domain map is $d(u, g, v) = (u, Id_W, v \circ g)$, while the codomain map is $c(u, g, v) = (g \circ u, Id_Z, v)$. Suppose we have two morphisms $(u, g, v), (r, h, s)$ such that $c(u, g, v) = d(r, h, s)$, $h : Z \to Q$, then their composition is the morphism $(u, h \circ g, s)$, as shown in the following commutative diagram:



This construction entails a number of evident facts: To begin with, the category $[f]$ has the initial object (Id_X, Id_X, f) and the final object (f, Id_Y, Id_Y). Moreover, if $k : Y \to E$ and $l : A \to X$ are morphisms, then there are two functors $[k\circ] : [f] \to [k \circ f]$ and $[\circ l] : [f] \to [f \circ l]$, respectively, sending (u, g, v) to $(u, g, k \circ v)$ and to $(u \circ l, g, v)$, respectively (keeping the above notations). If $\mathcal{C}$ is a topological category, then so is $[f]$, if it is viewed as a subset of $\mathcal{C}^3$. Also, the two functors $k\circ, \circ l$ are continuous.

For any two objects X, Y in $\mathcal{C}$ we now build the disjoint sum $[X, Y] = \coprod_{f \in X@Y} [f]$ of the factorization categories $[f]$ (including the coproduct of topologies on the $[f]$). Therefore $\nabla@[X, Y] = \coprod_{f \in X@Y} \nabla@[f]$, and, if we endow $[X, Y]$ with the coproduct topology, also $\nabla©[X, Y] = \coprod_{f \in X@Y} \nabla©[f]$. The above construction of functors from morphisms also works in this coproduct situation, and also mutatis mutandis for topologies on these categories, i.e., conserving the above notations, we have two continuous functors $[k\circ], [X, Y] \to [X, E]$ and $[\circ l] : [X, Y] \to [A, Y]$, and their associated curve functors $\nabla©[k\circ] : \nabla©[X, Y] \to \nabla©[X, E]$ and $\nabla©[\circ l] : \nabla©[X, Y] \to \nabla©[A, Y]$.

Example 72 If $f = Id_X$, then $[f]$ is the category of sections and retractions of X, since its objects are the triples (u, Id, v) such that $v \circ u = Id_X$.

Example 73 The category $\mathcal{C}$ is defined by a topological group G, i.e., as a category, has one single object and the group elements as morphisms, then $[f] \xrightarrow{\sim} G$, where G is the graph category of the topological space G. More explicitly, the morphisms of $[f]$ are the triples (u, g, v) of elements of G such that $v \circ g \circ u = f$. Since any two of them are free and determine the third, we take the morphisms as being the pairs $(d, c) \in G \times G$, where we have $u = d, g = c \circ u^{-1}, v = f \circ u^{-1} \circ g^{-1}$. The topology is the product topology of $G \times G$.

For example, if $\mathcal{C}$ is defined by the cartesian product group $G = \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$ of the additive group $\mathbb{R}^n$ and the general affine group $\overrightarrow{GL}_n(\mathbb{R})$, $[f] \xrightarrow{\sim} \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$, the topological space category of pairs (x, g) of points x in $\mathbb{R}^n$ and affine transformations $g : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$. We then have a continuous (group action) functor $\epsilon : \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R}) \to \mathbb{R}^n : (x, g) \mapsto g(x)$ into the topological category $\mathbb{R}^n$ deduced from the group $\mathbb{R}^n$. Therefore, from a gesture $g : \Gamma \to \mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$, we deduce a gesture $\epsilon \circ g : \Gamma \to \mathbb{R}^n$. The latter is a gesture whose curves are just continuous curves in real n-space, but they are not arbitrary, since they are induced by curves of points and linear transformations. This very special case reveals the power of our construction of factor categories: They include the concept of gestures of transformations of points, and not only abstract topological gestures. But they are much more powerful since a gesture in $\mathbb{R}^n \times \overrightarrow{GL}_n(\mathbb{R})$ might well specify curves that are more general than just curves of transformations, but let us make this more precise.

As in our initial example in $\mathbb{R}^2$ of a rotational curve $c(t) = (x, e^{i\theta t})$, we may vary the transformation and fix the point x, but we may as well just take an arbitrary continuous curve $d(t) = (x(t), Id_{\mathbb{R}^2})$ in $\mathbb{R}^2$ and let the transformation remain the identity. More generally, we may vary both, the point and the curve, and consider a curve $e(t) = (x(t), g(t))$ in $\mathbb{R}^2 \times \overrightarrow{GL}_2(\mathbb{R})$. This opens the concept of a gesture, whose curves are characteristic in that they may pertain either to transformational constructs, to purely topological rationales, or to both. Such a setup works for any (topological) group action on a given module, such as, for example, the musically relevant action of the general affine group $\overrightarrow{GL}(\mathbb{Z}_{12})$ on the pitch class group $\mathbb{Z}_{12}$ (with the discrete topology, for example).

62.2.1 Diagrams as Gestures

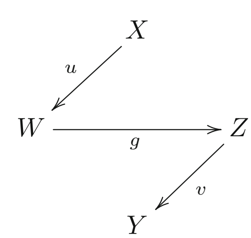
Example 72 suggests that one should take a closer look at the category of factorizations for module categories, since sections and retractions define direct summands in the abelian categories. To this end, we first construct certain standard gestures. To begin with, let $g : W \to Z$ be any morphism in a category $\mathcal{C}$. Then there is a functor $\searrow (g) : \nabla \to \mathcal{C}$ with

$$\searrow (g)(x, y) = \begin{cases} Id_W & \text{if } x = y = 0, \\ g & \text{if } 0 = x < y, \\ Id_Z & \text{if } 0 < x. \end{cases} \tag{62.1}$$

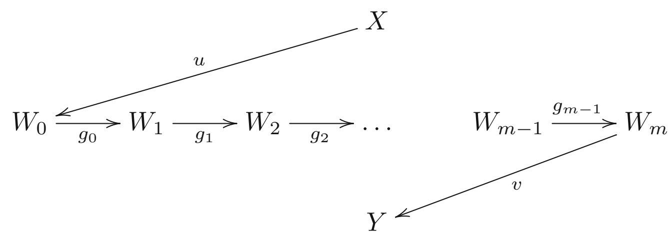
This construction method enables the construction of gestures from diagrams in categories as follows. Suppose that a category K is small. Then take the topology on K such that all functors $c : \nabla \to K$ are curves, i.e., we take $\nabla © _{\nabla @ K} K$. Consider K as a digraph with the two maps $d, c : Mor(K) \to Ob(K)$ from the morphism set $Mor(K)$ to the object set $Ob(K)$. Then we have the following morphism of digraphs $\searrow : K \to \overrightarrow{K}$ which sends a morphism $f : X \to Y$ to the curve $\searrow (f)$ with tail X and head Y. Therefore, if we have any diagram $\delta : \Delta \to K$ in the category K, we may compose it with $\searrow$ and obtain a gesture $\searrow \circ\, \delta : \Delta \to \overrightarrow{K}$, which we denote by $\overrightarrow{\delta}$ and call the *discrete gesture associated with the diagram* δ. This evidently extends to a *discrete gesture functor* $\overrightarrow{?} : \Delta @ K \to \Delta @ \overrightarrow{K}$ from the category of diagrams and natural transformations to the category of gestures of these spaces.

62.2.2 Gestures in Factorization Categories

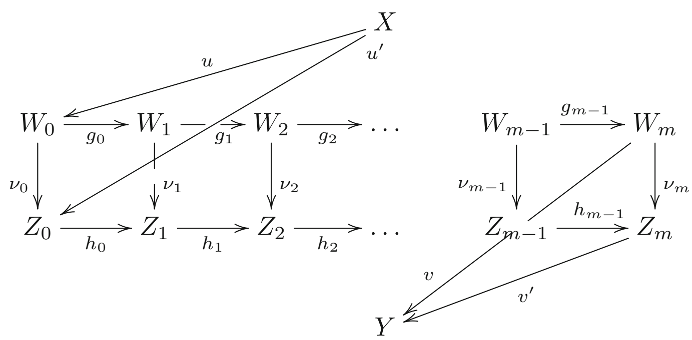
In our context, a morphism (u, g, v) in a factorization category $[f]$ with



yields a curve $\searrow (u, g, v) : \nabla \to [f]$. This construction can be iterated in the sense that for any sequence $g_. = (g_i : W_i \to W_{i+1})_{i=0,1,\ldots m-1}$ of length m of morphisms in $\mathcal{C}$, there is a functor $\searrow (g_.) : \nabla \to \mathcal{C}$ where the restriction $\searrow (g_.)|\nabla_{[i/m,(i+1)/m]} \to \mathcal{C}$ to the full subcategory $\nabla_{[i/m,(i+1)/m]} = \{(x, y)| i/m \leqslant x \leqslant y \leqslant (i+1)/m\}$ of ∇ is the above one-step construction for $g_i : W_i \to W_{i+1}$. This entails that we may also consider curves $\searrow (u, g_., v)$ associated with the chain $(u, g_., v)$ of morphisms in $[f]$:



Given two such curves $\searrow (u, g., v), \searrow (u, h., v)$ of the same length m, the second one involving the morphisms $h_i : Z_i \to Z_{i+1}$ and $u' : X \to Z_0, v' : Z_m \to Y$, a morphism $\nu_. : \searrow (u, g., v) \to \searrow (u, h., v)$ is a natural transformation consisting of a chain of morphisms $\nu_. = (\nu_0 : W_0 \to Z_0, \dots \nu_m : W_m \to Z_m)$ such that we have this commutative diagram:



62.2.3 Extensions from Homological Algebra Are Gestures

Now, if we consider the special case where $\mathcal{C} = {}_R\mathbf{Mod}$, the category of left R-modules and linear homomorphisms over a commutative ring R, then we may take the factorization category $[0]$ of the zero homomorphism $0 : 0 \to 0$ on the zero module. We may further consider two exact sequences, one $g_.$ of modules $W.$, and one $h_.$ of modules $Z.$, to generate curves, which we should call *exact curves*. Then the morphism $\nu.$ is just a morphism between exact sequences, which means that the category of exact sequences is a canonical subcategory of the category of curves in $[0]$. In particular, if we look at such short exact sequences (length 2), and we restrict ourselvesto morphisms between sequences of common initial module W and terminal module Z, we obtain the groupoid of exact sequences, and the isomorphism classes define the classical set $Ext_R(Z, W)$ of congruence classes of extension of Z by W [635]. Consequently, we have this fact:

Fact 22 *The categories of factorization are a natural extension of structures from homological algebra encountered, for example, in the construction of* $Ext^n_R(Z, W)$.

62.2.4 The Bicategory of Gestures

Suppose that for two morphisms $f : X \to Y, g : Y \to Z$ in a category $\mathcal{C}$ we are given topologies on the factorization categories $[f], [g]$ such that the two functors $[g\circ] : [f] \to [g \circ f], [\circ f] : [g] \to [g \circ f]$ are continuous (e.g. if $\mathcal{C}$ is topological). Write $G[f]$ for the gesture topos $G([f])$. For such morphisms, we denote the categorical digraphs $\overrightarrow{[f]}$ by $\overrightarrow{f}$, and the corresponding morphisms of categorical digraphs, such as the ones derived from $g\circ$ and $\circ f$, are denoted by $\overrightarrow{g\circ}$ and $\overrightarrow{\circ f}$. We therefore have two morphisms of categorical digraphs $\overrightarrow{g\circ} : \overrightarrow{f} \to \overrightarrow{g \circ f}$ and $\overrightarrow{\circ f} : \overrightarrow{g} \to \overrightarrow{g \circ f}$, which induce the two canonical functors between topoi, $[g\circ] : G[f] \to G[g \circ f]$ and $[\circ f] : G[g] \to G[g \circ f]$. Taking coproducts on the skeleta, this induces a functor between topoi,

$$b_* : G[g] \times G[f] \to G[g \circ f],$$

which maps a pair $\gamma : \Gamma \to \overrightarrow{g}, \phi : \Phi \to \overrightarrow{f}$ of gestures to the gesture

$$\overrightarrow{\circ f}\gamma \sqcup \overrightarrow{g\circ}\phi : \Gamma \sqcup \Phi \to \overrightarrow{g \circ f}.$$

On the other hand, the two base changes $\overrightarrow{g\circ}$ and $\overrightarrow{\circ f}$ induce[3] base change functors $\times\overrightarrow{g\circ} : G[g \circ f] \to G[f]$ and $\times\overrightarrow{\circ f} : G[g \circ f] \to G[g]$, which we combine to get the base change functor

$$b^* : G[g \circ f] \to G[g] \times G[f].$$

By use of routine topos-theoretical arguments, we have this result (recall from [639] the definition of a geometric morphism of topoi):

Theorem 1. *Given the above conditions and notations,*

(*i*) *the base-change functor b^* is a logical functor, i.e., it conserves all topos-theoretical structures, subobject classifier, finite limits, colimits, and exponentials.*
(*ii*) *The coproduct functor b_* is left adjoint to b^*, and*
(*iii*) *there is functor $a : G[g] \times G[f] \to G[g \circ f]$, which is right adjoint to b^* such that the pair (a, b^*) is a geometric morphism of topoi.*
(*iv*) *If f or g is the identity, then b_* is isomorphic to the identical functor. If $h : Z \to W$ is a third morphism, also sharing the above properties of f, g, then the functor b_* is associative up to isomorphisms.*

Adding up all the factorization categories relating to morphisms $f : X \to Y$, we define the coproduct category $X☞Y = \coprod_{f \in X@Y} G[f]$. If we are given a second morphism $g : Y \to Z$ with the above conditions still holding, then we have a functor deduced from the above functor b_*, notated with capital letters:

$$B_* : Y☞Z \times X☞Y \to X☞Z \tag{62.2}$$

It is associative up to isomorphisms and has the identity gesture $\emptyset \to \overrightarrow{Id_X}$ for each object X. If these constructions work for all objects and morphisms (e.g. if $\mathcal{C}$ is topological), then the composition functors (62.2) define a bicategory [637], the *gesture bicategory of* $\mathcal{C}$ denoted by $\mathcal{C}^{☞}$. This is nearly a 2-category, except that composition is only associative up to isomorphisms. This being so, the "morphic" half of Yoneda's Lemma would consist in characterizing the functors (62.2)—or else the geometric functors between the topoi $G[g] \times G[f]$ and $G[g \circ f]$—which stem from composing morphisms in the original category $\mathcal{C}$. This would enable us to think of morphisms as being represented by gestures and to calculate all of the category's operations on the level of gestures. Given that the classical "objective" Yoneda Lemma already takes care of the reconstruction of point sets from abstract objects by the transition from $\mathcal{C}$ to $\mathcal{C}^{@}$, this hypothetical "morphic" Yoneda Lemma would give us back the full gestural intuition on the level of $(\mathcal{C}^{@})^{☞}$ while working in abstract categories.

62.2.5 Entering the Diamond Space

In view of the preceding results, we have set up a concept space, as made explicit in the gesture bicategory construction $\mathcal{C}^{☞}$, which embraces the topological gesture theory of our former work [719] as well as the diagram theory backing the network approach from Lewin's and Klumpenhouwer's transformational theory, but also basic constructions from homological algebra, such as congruence morphisms between extensions in abelian categories. The philosophy of this approach is that the concept of a categorical gesture, although completely in the vein of gestural reflections fostered by musical requirements, is flexible enough to include the extremal cases of "discrete" and properly "continuous" gestures as well. The relation between these two cases being that continuous gestures are a kind of limit of factorization when the factors are becoming more and more "fine grained" until they are parametrized by continuously varying real parameters; see Figure 62.1 for an intuitive image of factorization granularity. We therefore argue that this space construction is a good candidate for our conjectural space X as described in the Diamond Conjecture [719, Section 9], see also Section 61.12. At this stage, we do not however yet state that this space X has been found since a number of tests have to be performed in order to learn about power vs. deficiencies of the present approach.

[3] Recall that the product in a comma category is the fiber product in the original category.

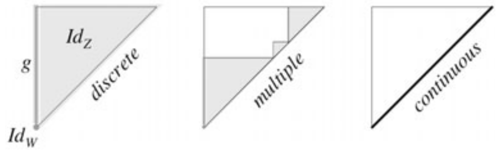


Fig. 62.1. Factorization on ∇ from one "discrete" step $\searrow$ (g) through a finite series of (discrete) factors to the "limit" of a continuous curve of factors.

62.3 Diagrams in Topological Groups for Gestures

In this section, we want to make explicit the transition from the type of diagrams used in transformational theory to gestures. Of course, we are not dealing here with the above embedding of diagram categories into gesture categories, but want to transform discrete gesture curves into continuous curves that enable an infinity of intermediate stages between the starting and the ending position of diagrammatic arrows.

To this end we first discuss the gestures with values in the factor category discussed above, namely starting from the topological group $G = \overrightarrow{GL}_n(\mathbb{R})$, so that $[f] \overset{\sim}{\to} \overrightarrow{GL}_n(\mathbb{R})$, the topological space category of affine transformations $g : \mathbb{R}^n \overset{\sim}{\to} \mathbb{R}^n$, whose morphisms are parametrized by pairs $(d, c) \in G \times G$ of group elements, the transition morphism $g : d \to c$ representing the transformation $g = c \circ d^{-1}$. We then know that the data of a curve $\delta : \nabla \to G \times G$ is equivalent to its diagonal restriction to the objects $I \rightarrowtail \nabla$, i.e., to a continuous map $\delta_I : I \to G$, where $G \rightarrowtail G \times G$ identifies G with the diagonal in $G \times G$ by the diagonal embedding (much as I is in ∇). Since I is connected, the image of δ_I must be either in the connected component $G^+ = \overrightarrow{GL}_n^+(\mathbb{R})$ or in the complementary connected component $G^- = \overrightarrow{GL}_n^-(\mathbb{R})$ of G, where $\overrightarrow{GL}_n^+(\mathbb{R})$ ($\overrightarrow{GL}_n^-(\mathbb{R})$) is the group (coset) of affine transformations with linear part in the subgroup $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ (in the coset $\mathrm{GL}_n^-(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$) of transformations with positive (negative) determinant. Therefore any gesture with body in $G \times G$ with connected skeleton must have all its object curves either in G^+ or in G^-.

Therefore connectedness of I implies that we cannot connect transformations of different determinant signatures, e.g., the identity Id_G for $G = \overrightarrow{GL}_2(\mathbb{R})$ and the mirror transformation $m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This is a major problem for the continuous gestualization of discrete gestures. In fact, if we take the morphism $Id_G \to m$, there is no continuous curve starting in Id_G and ending in m. Why should this be required? If we had such a curve, $\alpha_I : I \to G$, with $\alpha_I(0) = Id_G, \alpha_I(1) = m$, we could use it to generate a continuous curve of points $\alpha_I.x(t) = \alpha_I(t)(x) \in \mathbb{R}^2$ by evaluation of the curve at a given initial point $x \in \mathbb{R}^2$ and parameter $t \in I$ as explained in Example 73.

In order to understand the specific problem which appears with mirroring, let us look at the generators of $G = \overrightarrow{GL}_2(\mathbb{R})$ and their musical meaning (see Section 8.3 for a detailed discussion). They are (1) translations $T^{(1,0)}$ by one unit in horizontal direction, (2) all positive dilations of the first coordinate, (3) the above mirror m, (4) the horizontal transvection $t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and the 180^o rotation $R = -Id$. For all these generating transformations g_i, except for the mirror m, there is a continuous curve $\delta_i : I \to G^+$ starting at Id_G and ending at g_i. This means that all transformations $g \in G$, which can be written as products of these generators without m, have a continuous curve $\delta : I \to G^+$ such that $\delta(0) = Id_G, \delta(1) = g$. Therefore, evaluating at a point $x \in \mathbb{R}^2$ yields curves $\delta.x(t) = \delta(t)(x) \in \mathbb{R}^2$ that are induced by curves of transformations. This was also used in the component *BigBang Rubette* for composition in the Rubato Composer software environment [729]; see also Chapter 69. But the case of m does not work as is; in other words, mirroring is a non-gestural operation. In order to pass to the mirror of an object, one has to traverse the singular state of a flattened object in the mirror. The change of determinant sign is the hard point, so we are not in a state of overcoming this problem within the given space. We do not want to delve into the deep and metaphorically loaded topic of the mirror, but it is clear that the mystery of the mirror transformation must relate to the fact that there is no gesture, no continuous transition from the original to the mirror image. Vampires have no reflection in mirrors, and superstition is abundant with mirrors. Some "imaginary process" must be happening when we switch to the mirror world.

There is a well-known intuitive solution of the mirror problem, which you may find whenever you ask a person to describe what movement is the reflection of a plane figure at a line in the plane: He would immediately make that movement the one of leafing a book's page. Leafing turns the original figure to its mirrored version. The point is that instead of mirroring x to $-x$, it lifts it into a new dimension and rotates the point in this dimension until it comes down to $-x$. This procedure is more accurately described by complexification of real vector spaces. In the one-dimensional case $\mathbb{R}$, the mirroring $m(x) = -x$ is embedded in the Gaussian complex number plane $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$. Here, we have the rotation defined by multiplying x by the complex unitary number $e^{i\theta}$. The image $x.e^{i\theta}$ is the vector x looking in direction of $e^{i\theta}$. If we consider the curve $\gamma(t) = e^{i\pi t} \in \mathrm{GL}_2(\mathbb{R}), t \in I$, then the evaluated curve $\gamma.x(t) = \gamma(t)(x) = x.e^{i\pi t}$ rotates x in a half circle to $-x$. This means that the mysterious mirroring has been demystified by an inoffensive gestural curve through complex numbers. And halfway on that curve we have its imaginary position $\gamma.x(1/2) = i.x$, the purely imaginary position of the curve, where its real projection vanishes.

This means that complex numbers solve the problem of the real singularity by lifting the mirror movement orthogonally to the real axis in an imaginary realm. It might be that one reads our description as a mystification of complex numbers, but the resolution of the negation $x \mapsto -x$ by a rotation in a new dimension is no overinterpretation of complex numbers. A strong argument for this "gestural" reinterpretation of negation is in fact provided by the proof of the fundamental theorem of algebra using fundamental groups in the Gaussian plane, see [569], for example. The fundamental theorem of algebra is the most important single thorem of algebra whose proof can be based upon the thoroughly gestural toolbox of algebraic topology.

This being so, if we are given any transformation $h \in \overrightarrow{GL}_n(\mathbb{R})$, then we may complexify it, which means that we write $h = T^s \circ \eta, s \in \mathbb{R}^n, \eta \in \mathrm{GL}_n(\mathbb{R})$ and then tensorize it with the complex number $e^{i\theta} \in \mathrm{GL}_2(\mathbb{R})$ as above and obtain the transformation $h \otimes e^{i\theta} = T^s \circ (\eta \otimes e^{i\theta}) : \mathbb{R}^n \otimes \mathbb{C} \to \mathbb{R}^n \otimes \mathbb{C} : x \otimes y \mapsto s \otimes 1 + \eta(x) \otimes y.e^{i\theta}$. The determinant of a tensor product $\eta \otimes \kappa$ of linear maps $\eta \in \mathrm{GL}_u(\mathbb{R}), \kappa \in \mathrm{GL}_v(\mathbb{R})$ being $det(\eta)^v det(\kappa)^u$, we have $det(h \otimes e^{i\theta}) = det(\eta)^2 det(e^{i\theta})^n = det(\eta)^2 (1)^n > 0$. This means that complexification of any transformation h with the rotation $e^{i\theta}$ in $\mathbb{C}$ turns it into a transformation $h \otimes e^{i\theta} \in \overrightarrow{GL}^+_{2n}(\mathbb{R})$. In particular, if $\gamma : I \to \overrightarrow{GL}^+_n(\mathbb{R})$ is a positive curve, then we obtain a curve $\gamma \otimes e^{i\pi ?} : I \to \overrightarrow{GL}^+_{2n}(\mathbb{R}) : t \mapsto \gamma(t) \otimes e^{i\pi t}$ such that its value starts at $t = 0$ with $\gamma(0) \otimes 1$ and ends at $t = 1$ with $\gamma(1) \otimes -1 = -1\gamma(1) \otimes 1$. If n is odd, this yields a negative determinant transformation $-1\gamma(1)$.

For even n, this does not work directly, but one may then select a direct decomposition $\mathbb{R}^n = V \oplus W$ with odd dimension $dim(W)$. Then we take again $\mathbb{R}^n \otimes \mathbb{C} = V \otimes \mathbb{C} \oplus W \otimes \mathbb{C}$, but this time apply the complex rotation only to the second summand $W \otimes \mathbb{C}$. Denote this restricted rotation by $e^{i\pi ?}|W$. Then if $\gamma : I \to \overrightarrow{GL}^+_n(\mathbb{R})$ is a positive curve (positive determinants), we obtain a curve $\gamma \otimes e^{i\pi ?}|W : I \to \overrightarrow{GL}^+_{2n}(\mathbb{R}) : t \mapsto \gamma(t) \otimes e^{i\pi t}|W$ such that its value starts at $t = 0$ with $\gamma(0) \otimes 1$ and ends at $t = 1$ with $\gamma(1) \otimes -1|W = -1|W\gamma(1) \otimes 1$, where the latter means that -1 is only applied to the subspace W coordinates, and thus yields a negative determinant transformation. This second case is not as invariant as the first one for odd n since one has to select a direct decomposition $\mathbb{R}^n = V \oplus W$, but the variety of choices offers a strong tool for turning general curves of transformations with positive determinants into gestural curves of opposite determinant signature. To terminate these constructions on the curve level, we add the reversed construction, which takes the positive curve $\gamma : I \to \overrightarrow{GL}^+_n(\mathbb{R})$, but tensorizes it with the reversed complex rotation, i.e., $\gamma \otimes e^{i\pi(1-?)}$ in the odd-dimensional case, and $\gamma \otimes e^{i\pi(1-?)}|W$ in the even-dimensional case.

This construction is easily generalized to the level of general gestures. To do so, we have to display the signatures of all the vertices of the gesture's skeleton Γ, since we want to define curves that change the determinant sign via the preceding complexification technique. More precisely, we suppose that we are given a signature for each vertex, which means that we have a digraph morphism $\sigma : \Gamma \to Sig$, where Sig is the complete digraph

$$Sig = \; -- \circlearrowleft \ominus \underset{+-}{\overset{-+}{\rightleftarrows}} \oplus \circlearrowright ++$$

with two vertices $\oplus, \ominus$ (positive, negative). The fiber of $\ominus$ is the set of vertices with negative determinant, the complementary fiber of $\oplus$ is the set of vertices with positive determinant. This means that the arrows mapping to $\ominus \xrightarrow{-+} \oplus$ are those from a transformation with negative determinant to a transformation with positive determinant, etc. The complexification tools are only needed for the $+-$ and $-+$ arrows. The $+-$ case refers to the maps $\gamma \otimes e^{i\pi ?}$ or $\gamma \otimes e^{i\pi ?}|W$ according to the dimension being odd or even, whereas the $-+$ case refers to the maps $\gamma \otimes e^{i\pi(1-?)}$ or $\gamma \otimes e^{i\pi(1-?)}|W$. The case $++$ takes just the positive curves as is, tensored with the identity on $\mathbb{C}$, while the case $--$ takes a negative determinant copy of the positive curve. It is not relevant which negative copy we take, as long as the bijection $\mathrm{GL}_n^+(\mathbb{R}) \sim \mathrm{GL}_n^-(\mathbb{R})$ of negation is fixed once for all. The essential point is that all these complexified transformations start and end on transformations that leave the real subspace $\mathbb{R}^n \otimes 1$ invariant. The intermediate transformations however map $\mathbb{R}^n \otimes 1$ to a more general subspace of $\mathbb{R}^n \otimes \mathbb{C}$ as shown in Figure 62.2:

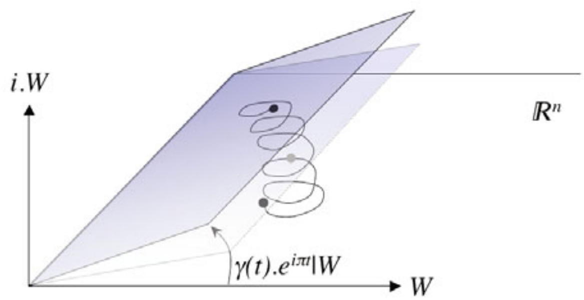


Fig. 62.2. The "leafing" transformation of a point in the original real space into complexified space as a function of the complex factor, eventually producing a change of the determinant's signature.

With this general method in mind, we now have to deal with the transformation of a discrete curve corresponding to a transformation (u, g, v) in $[f]$. In view of the above complexification method, we may concentrate on the case of $u = Id, det(v) > 0$. Our plan is to construct a continuous curve of transformations $\gamma : I \to \overrightarrow{GL}_n^+(\mathbb{R})$ from Id to v. The shifting part being trivial, we may focus on $v \in \mathrm{GL}_n^+(\mathbb{R})$. From matrix theory it is well known that $\mathrm{GL}_n(\mathbb{R})$ is generated by subgroups isomorphic to $\mathrm{GL}_2(\mathbb{R})$. So we can write v as a product of transformations v_i affecting only two coordinates. This reduces the problem to $n = 2$. We also have $v = \begin{pmatrix} det(v) & 0 \\ 0 & 1 \end{pmatrix}.s, s \in \mathrm{SL}_2(\mathbb{R})$. But $\mathrm{SL}_2(\mathbb{R})$ is generated by transvections $u(b)$, $b \in \mathbb{R}$, and the 180^o rotation w [578, XI, §2]:

$$u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which both are endpoints at $t = 1$ of countinuous curves $\delta(t) = u(tb), \rho(t) = e^{i\pi t}$, the latter being the rotation as defined by the Gaussian plane. Therefore the curve

$$v(t) = \begin{pmatrix} (1-t) + t.det(v) & 0 \\ 0 & 1 \end{pmatrix}.\delta(t).\rho(t)$$

does the job in two dimensions. In short: $\mathrm{GL}_n^+(\mathbb{R})$ is arcwise connected, and we have just given a constructive proof thereof involving standard generators. In short, given any $v \in \mathrm{GL}_n^+(\mathbb{R})$, we find a continuous curve $v(?) : I \to \mathrm{GL}_n^+(\mathbb{R})$ with $v(0) = Id, v(1) = v$. Together with the signature changing tools discussed above (using the signature morphism σ), we have this theorem:

Theorem 2. *If $\delta : \Gamma \to \overrightarrow{GL}_n(\mathbb{R})$ is a diagram of (non-singular) affine transformations on $\mathbb{R}^n$, then there is a gesture $\delta \otimes \mathbb{C} : \Gamma \to [Id_{\mathbb{R}^n \otimes \mathbb{C}}]$, whose morphisms for the extremal ∇-morphisms $(0, 1)$ of the curves $\delta \otimes \mathbb{C}(a)$*

of arrows a in Γ are the transformations $(Id_{\mathbb{R}^n \otimes \mathbb{C}}, \delta(a))$. The morphisms between curve parameters s and t, $s \leqslant t$, are the factorization quotients

$$\delta \otimes \mathbb{C}(a)(t) \circ (\delta \otimes \mathbb{C}(a)(s))^{-1}.$$

This is not a deep theorem, but it enables the extension of discrete gestures associated with diagrams in real n-space gestures to continuous gestures that may have to traverse complexification but start and end at the given real transformations. This is a general theorem, which just guarantees the said extensibility and stresses the fundamental role of complex numbers. But when thinking about Lewin's question concerning the characteristic gestures, although this might be the generic, it is not necessarily the characteristic one in n-space. We shall discuss this issue in Section 62.4.3.

Coming back to the evaluation map ϵ from Example 73, we also have an evaluation functor $\epsilon : \mathbb{R}^n \otimes \mathbb{C} \times \overrightarrow{GL}_n(\mathbb{R}) \otimes \mathbb{C} \to \mathbb{R}^n \otimes \mathbb{C}$ on topological categories. When we take a gesture $\gamma : \Gamma \to \mathbb{R}^n \otimes \mathbb{C} \times \overrightarrow{GL}_n(\mathbb{R}) \otimes \mathbb{C}$, ϵ yields a gesture of points in $\mathbb{R}^n \otimes \mathbb{C}$, and if the initial and terminal values of the curves of transformations of γ leave $\mathbb{R}^n$ invariant, the gestures in $\mathbb{R}^n \otimes \mathbb{C}$ also start and end in real points.

Example 74 A prototypical example would consist of a network of points in $\mathbb{R}^n$ connected by affine transformations, i.e., a diagram of points and transformations $\delta : \Delta \to \mathbb{R}^n$, such that for an arrow $x \xrightarrow{a} y$ in Δ, we have a non-singular affine transformation $\delta(a)$ of points $\delta(x), \delta(y) \in \mathbb{R}^n$ with $\delta(a)(\delta(x)) = \delta(y)$. Focusing on the transformations, one therefore has a diagram as in Theorem 2, which has all its curves starting at the identity and ending on the different $\delta(a)$ for arrows a of δ. So on transformations, the diagram is a star-shaped one with the identity as center and radiating to each of the $\delta(a)$. However, on the different starting points $\delta(x)$, the star is uncoupled in order to be able to transform all the $\delta(x)$ in the particular curves that traverse complex spaces when determinant signs are changed to between 1 and the signature of $det(\delta(a))$. This means that we are given pairs $(\delta(x), \delta \otimes \mathbb{C}(a))$ with variable transformations (i.e., curves) and fixed points that are transformed according to the transformation curves by $t \mapsto \delta \otimes \mathbb{C}(a)(t)(\delta(x))$. The straightforward generalization is to define non-constant curves in the points, too, i.e., $t \mapsto \delta \otimes \mathbb{C}(a)(t)(\delta(x)(t))$, which comprises the two extremal cases of the purely transformational curves and the purely "topological" point curves with constant transformation, usually the identity.

Example 75 If S^1 denotes the unit circle group, which we may view as an extension of the pitch class group, consider the topological group $G = T^{S^1} \rtimes \mathbb{Z}_2$ generated by translations $T^t, t \in S^1$, and the reflection $-Id$ (the generator of $\mathbb{Z}_2$). This group has two connected components, namely $G^+ = T^{S^1}$ and $G^- = T^{S^1}. - Id$. As for the general affine group, this group defines a category with one object, and we also have $[Id] \xrightarrow{\sim} G$, the topological space category with morphisms $(d, c) \in G \times G$ representing the quotient $g = c \circ d^{-1}$. As for the general affine group, the morphisms $Id \to T^t$ can easily be extended to continuous curves, while for $Id \to T^t. - Id$ this is not possible for the same reason as before. We may however resolve the conflict by again adding a dimension and embedding S^1 in the sphere S^2 as one of its meridian circles. What was previously done by the rotation in $\mathbb{C}$ now works by the half circle rotation around the polar axis or another axis through the embedded S^1.

62.4 Modulations in Beethoven's "Hammerklavier" Sonata op.106/Allegro: A Gestural Interpretation

The following section is not the first occasion where gestural aspects of Beethoven's compositions have been discussed; see Robert S. Hatten's study [446], or Jürgen Uhde's and Renate Wieland's books [1068, 1067], for example. Our discussion however differs from earlier investigations in these two points:

- To begin with, we are applying the previous categorical theory of gestures and do not stick to the more metaphorical and intuitive usage of the term "gesture" in previous studies. Of course, this is also a risky enterprise since many statements, which may be acceptable or valid on those more intuitive levels of conceptualization, might become questionable, dubious or even untenable when made mathematically explicit.

- Second, we focus on very delicate modulatory processes in the Allegro movement of Beethoven's op.106, see Section 28.2 for a detailed discussion. These are known to have quite non-standard appearances, partly breaking with standard combinations, such as the modulation $B_\flat$-major $\rightarrow$ G-major to the secondary theme in the exposition, where the modulation to the dominant is expected by standard sonata theory [659], and partly because some so-called "catastrophe" modulations deviate from standard modulation processes altogether.

We shall not reconsider a basic discussion of these modulatory processes, but rely on the established methods and results as exposed in [714, ch.28.2]. However, the gestural aspects of these modulations will open considerations of dynamical nature that do not rely solely on these methods and results. We therefore hope that the following discussion is also useful for a basic discourse on gestures in modulatory processes.

Why is this a desirable topic? The argument is that a purely structural analysis of modulatory processes (among others) may fail to capture the energetic understanding, the dramatic tension of the musical deployment. We are not claiming here that gestural analysis is comprising all such aspects, but it seems worthwhile to approach those energetic and dramatic tensions by gestural dynamics since the theory of gestures is an ideal mediator between static structures and energetic processes without having to recur to psychological, narrative, or other extra-musical categories. Our hope is nevertheless that gestural considerations might *eventually* converge to a fairly complete understanding of what is called the "dramatic content" of absolute music, such as Beethoven's late works.

One word about the *intrinsic usage of gestures* in the following analysis, as opposed to transformational structures. Is it really necessary to work with gestures? Couldn't one as well restate most if not all those gestural reflections in terms of transformational (hyper)networks? It is true that some of the following gestures (e.g. the gestures α_i and α'_j shown in Figure 62.5) seem to be "overdressed" versions of transformational networks. There are (at least) three arguments against such a suspicion:

- Gestures are completely different objects from networks. Intuitively speaking, networks only deal with start and end points of gestures. Also, hypergestures are generalized homotopies, which networks are not. Many of the following hypergestures are intrinsically continuous constructions, which require different, namely topological, technical tools than transformational networks, which are essentially built upon affine algebraic transformations. This is particularly dramatic in the context of the complexification gestures, which move *as* curves out of the real spaces into complex superspaces.
- Lewin's own unsolved dilemma is that he imagined continuous movements (his dancers!), but worked with agebraic (Fregean) transformations. What we offering here (and in the paper [720] with Moreno Andreatta) is nothing less than the one-to-one construction of Lewin's dreams in terms of precise mathematics.
- The very language of gestures opens a style of thoughts and a paradigm of understanding that the transformational paradigm would not have offered. In mathematics, the modern conceptual linear algebra opened so many new ideas that would never have been conceived of in terms of old-fashioned matrix calculus. Of course, once you have the idea, it is possible to translate it back into the old language, but this restatement is only possible *ex post*. Or, to put it into Lewinian dance language: How can you understand the dancer's touch point configuration with the dance floor if you are not told how he or she is connected by his or her real movements?

62.4.1 Recapitulation of the Results from Section 28.2

The modulation architecture in the Allegro movement of op. 106 is derived from a model of tonal modulation that uses inversional and transpositional symmetries on pitch class sets as "modulators", i.e., as operators, which transform pitch class sets (in 12-temperament) into each other. The tonalities in this model are the triadic interpretations $X^{(3)}$, i.e. coverings by the seven standard triadic degree chords $I_X, II_X, \ldots VII_X$ of the twelve diatonic scales $X = C, D_\flat, D, E_\flat, E, F, \ldots B_\flat, B$. According to a fundamental hypothesis on this composition, a hypothesis that is derived from classical analyses by Erwin Ratz and Jürgen Uhde (see Section 28.2), the system of possible modulators is the automorphism group $Aut(\mathfrak{M}_0)$ of the diminished seventh chord $\mathfrak{M}_0 = \{d_\flat, e, g, b_\flat\}$. This group partitions the set of tonalities into two orbits, the eight element orbit $W = Aut(\mathfrak{M}_0).B_\flat^{(3)}$ (the "world") consisting of the tonalities of scale set $\{B_\flat, D_\flat, E, G, A, C, E_\flat, G_\flat\}$, and the four element orbit $W^* = Aut(\mathfrak{M}_0).D^{(3)}$ (the "antiworld") consisting of the tonalities of scale set $\{D, F, A_\flat, B\}$.

This implies that modulations according to the given model are only admitted within W or within W^*, since no modulators are available in order to switch between these worlds. It turns out that what Uhde has coined catastrophe modulations are exactly those when Beethoven switches between world W and antiworld W^*. And here, the "responsible" diminished seventh chord appears with a nearly obsessive density, annihilating any melodic or tonal framework. All other modulations, within the world or within the antiworld, obey strictly the modulation rules provided by our model. Moreover, the modulators in these cases not only act as hidden symmetries but are also visible as symmetries between note groups that are within the modulating score segment, see Section 28.2.

Although the above results are describing the abstract modulatory structures and also the modulator symmetries in a strikingly precise way, which by far exceeds the predictive power of general modulation theories, the dramatic character of these modulations is not represented. In fact, much more is happening here than a verifiable instantiation of the model's abstract characteristics. The richness of the modulation dynamics has an impact that cannot be comprised by transformational diagrams connecting groups of notes. And this precisely, since diagrams incorporate no real movement, because their arrows are just as "cartesian" as plain set theory. We have discussed this topic in detail in the theoretical part of this chapter and in previous work [719]. This is the reason why we propose drawing a gestural picture of these modulation processes, which transcends the results as described in Section 28.2. So the following discussion is not about the previous analysis and its model, but about the added value that gestural reflections can contribute to the understanding of the note-wise embodiment of the composer's ideas following his famous statement[4]: "Was der Geist sinnlich von der Musik empfindet, das ist die Verkörperung geistiger Erkenntnis."

62.4.2 The Modulation $B_\flat$-major $\rightsquigarrow$ G-major Between Measure 31 and Measure 44

The first modulation, $B_\flat$-major $\rightsquigarrow$ G-major, in the transition (measures 39-46) to the second subject could in principle be performed by use of a "pedal modulation" [948]. We do however not encounter this modulation, but 'merely' a sequence of $VII_{G\text{-major}}$-degrees whose top notes are shifted by minor thirds from each other, i.e., exactly the situation of the pivot VII and the third translation, as predicted by the modulation with restricted modulators (Section 28.2.2).

This compact description from Section 28.2.4 however does not grasp the elaborate note process around that abstract fact of the $VII_{G\text{-major}}$ degree. This process consists of four groups, (A) measures 31 to 34, (B) measures 35-36, (C) measures 37-38, (D) measures 39-44. We do not discuss the concluding figure in measures 45-46, where the modulation is already terminated, and refer to [718, 9.2.1] for that matter. The entire process is typical for many of the modulations in this movement: It seems as if there were obstructions to a fast and easy modulation, which have to be surmounted. In the present case, the fanfare of part (B) is repeated in the subsequent part (C), but the second appearance first neutralizes $I_{B_\flat\text{-major}}$ to the simple note d on the third beat of measure 37, which is the third of $B_\flat$-major and the fifth of G-major. The next chord then replaces the $e_\flat$ from the original fanfare by $f_\sharp$ and creates $V_{G\text{-major}} = IV_{A\text{-major}} = I_{D\text{-major}}$. The movement $e_\flat \mapsto f_\sharp$ is a minor third (in terms of chromatic pitches and pitch classes, the present model is not based upon tonal alterations). This short formula is ambiguous in terms of which symmetry might have caused it. We have two candidates: $f_\sharp = T^3(e_\flat) = T^9.-1(e_\flat)$, transposition or inversion. The general modulation model with unrestricted modulators would yield the inversion as modulator, in fact, $T^9.-1.B_\flat^{(3)} = G^{(3)}$. The fanfare of part (C) could therefore also result from the inversional modulator acting upon $e_\flat$. But $T^9.-1 \notin Aut(\mathfrak{M}_0)$. Therefore only T^3 can transform $B_\flat^{(3)}$ into $G^{(3)}$. But this is not clear in part (C). A modulation process

[4] What the spirit perceives through the senses from music, is the embodiment of spiritual insight.

Fig. 62.3. The modulation process $B_\flat$-major $\rightsquigarrow$ G-major between measure 31 and measure 45.

has happened, but neither is it evident which symmetry was applied, nor is it terminated since the cadence part in the target tonality is not achieved in part (C). This state of ambiguity is expressed by the fermata in measure 38. It is a moment of hesitation of uncertainty: What happened, where are we? Could we really go on in G-major and step over directly to the last quarter of measure 46? Playing this shortened version sounds like not having digested the process, like stepping into a new tonality in a haphazard way without having made clear how we left the old one.

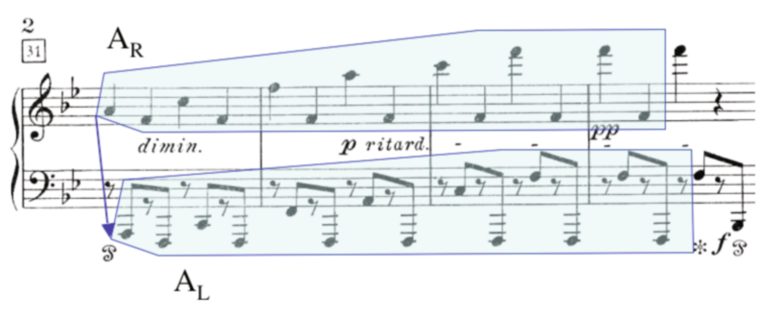


Fig. 62.4. The echo hypergesture preceding the modulation $B_\flat$-major $\rightsquigarrow$ G-major.

Of course, the plain appearance of degree $VII_{G\text{-major}} \subset V^7_{G\text{-major}}$ makes the target clear and cadences the new tonality. But again, this would also be true if we made that brute connection to the last quarter of measure 46, since that one initiates an arpeggiated $VII_{G\text{-major}}$. The point is that the stopping movement terminated by the fermata in measure 38 was not only defined there, but started much earlier in part (A). Harmonically, this part is a repeated arpeggio of $V_{B_\flat\text{-major}}$, terminating on the descending fifth step $f \mapsto b_\flat$ at the end of measure 34. This is by no means remarkable. But the shape of the arpeggio is! To begin with, part (A) splits into two subsets A_R and A_L, whose onset and pitch relate by a downward shifting,

$A_L = T^{(1/8,-12)}.A_R$, which creates a deeper and weaker (eighth notes) echo A_L of the right hand part A_R (time in 1/8, pitch in semitone units). We shall henceforth focus on the parameters onset and pitch and position these two parameters in $\mathbb{R}^2$, onset for the first, pitch for the second coordinate. Supposing for the moment that A_R and A_L are gestures, this echo turns out to be the endpoint of a hypergesture curve $A(t) = T^{t(1/8,-12)}.A_R, t \in I$ that is the evaluation of the curve $T^{t(1/8,-12)}, t \in I$, of transformations at A_R, see Figure 62.4.

This descending echo is the outer shape of a movement that becomes already visible in the initial gesture framed by A_R. How is this gesture constructed? Refer to Figure 62.5 for the following discussion. To begin with, we have seven small descending interval gestures $\alpha_1, \alpha_2, \alpha_3, \alpha'_1, \alpha'_2, \alpha'_3, \alpha_4 \in \uparrow \overrightarrow{@}\mathbb{R}^2$, which are induced by descending translation curves T_i along the vectors $v_1 = (1/4,-4), v_2 = (1/4,-7), v_3 = (1/4,-12), v'_1 = (1/4,-16), v'_2 = (1/4,-19), v'_3 = v_4 = (1/4,-24)$ to the same periodically repeated f, the fifth of the given tonality. Since we have this note as a fixed reference point, we view the translations as being the dual curves $T_i = S_i^*$ to the translation curves S_i associated with $-v_i$, i.e., $S_i(0) = Id, S_i(1) = T^{-v_i}$. All T_i are then evaluated at the seven instances of f, the evaluation starting at the higher interval note and ending at f.

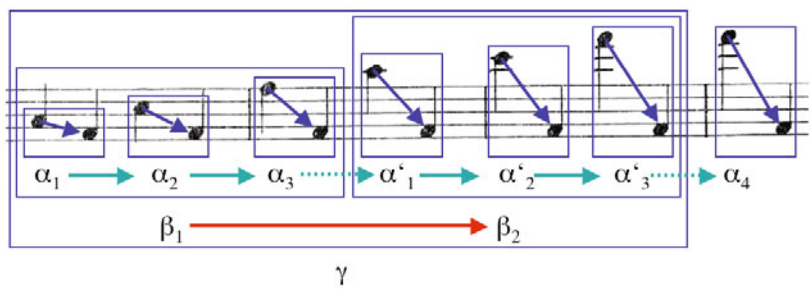


Fig. 62.5. The right hand hypergesture underlying A_R.

Recall from Section 61.6.1 that we denote by $\uparrow^n$ the digraph consisting of $n+1$ vertices, and having one arrow from vertex i to vertex $i+1$ for all $i = 0, 1, 2, \ldots n-1$. Then we have a hypergesture $\beta \in \uparrow^6 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$ that deforms α_1 into α_2, etc., and α'_3 into α_4. We leave it to the reader as an exercise to describe these deformations explicitly in terms of homotopies of translation curves over the category of affine transformations on $\mathbb{R}^2$. The hypergesture's β projections $p_1, p_2 : \uparrow^6 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2 \to \uparrow^6 \overrightarrow{@}\mathbb{R}^2$ via the head and tail maps on $\uparrow$ yield the periodic gesture of successively shifted f's on the one hand, and the ascending upper voice gesture on the other. But there is more: We look at the shorter hypergestures $\beta_1, \beta_2 \in \uparrow^2 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$, restrictions of β to the three vertices $\alpha_1, \alpha_2, \alpha_3$ and $\alpha'_1, \alpha'_2, \alpha'_3$, respectively. Then they are endpoints of a hypergesture $\gamma \in \uparrow \overrightarrow{@} \uparrow^2 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$. It projects to the f hypergesture and shifts the subgesture of β_1 built from the three f's in time (by 3/4) to the corresponding subgesture of β_2. The subgesture of β_1 connecting a, c, f is shifted by an octave and time (by 3/4) to the corresponding subgesture of β_2 connecting the octave shift of a, c, f. Putting everything together, including the echo, we obtain a hypergesture $\gamma^+ \in \uparrow \overrightarrow{@} \uparrow \overrightarrow{@} \uparrow^2 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$. Observe that such hypergesture constructions go beyond hypernetwork constructions since they are intrinsically topological.

Fig. 62.6. The beginning of part (D) shows an ascending twofold octave echo gesture on the fifth d.

So we are facing a hypergesture γ^+ that is traced on the "dance floor" of the score by part (A), and which is a strong and multilayered expression of the descending movement towards the fifth (for the dominant) of $B_\flat$-major. Everything is stopping at this point, also supported by the "ritardando" and "diminuendo" performance directions. The process's period, expressed in the fanfare part (B), opens the transformed fanfare in part (C) as discussed above. But the mentioned transformational ambiguity needs to be resolved, and this is achieved in part (D) by use of a counter-gesture corresponding to the hypergesture of part (A). How is this one structured, and how does it respond to the halting hypergesture of part (A)? The first two measures, 39 and 40, show a quadruple appearance of degree $V_{G\text{-major}}$ as initiated in (C). After the fermata, a new movement is initiated, as shown in Figure 62.6. This is similar to the octave descending echo gesture in part (A). However, now it is ascending in two octave steps $X_1 \to X_2 \to X_3$ (we refrain from a more precise gesture description, since this is straightforward here). This opening gesture towards the new tonality indicates that the moment of halting and cadential termination is over: the movement is now reversed, ascending towards new horizons, new skies.

It is not surprising that the following configuration of gestures appearing in part (D) is a complete mirror of the configuration encountered in part (A). The hypergestural anatomy is shown in Figure 62.7. Let us

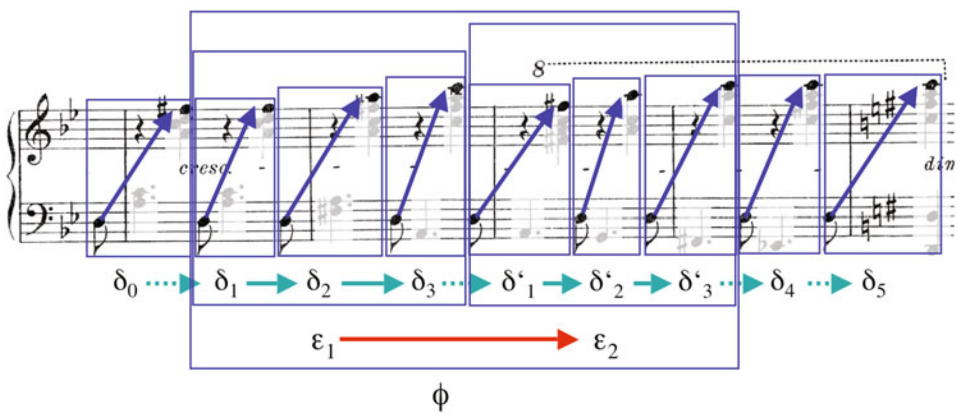


Fig. 62.7. The hypergestural configuration in part (D) mirrors the one from part (A).

describe what this "mirroring" looks like in detail. To begin with, we have a strong similarity of hypergestural configurations. We first have nine small ascending interval gestures $\delta_0, \delta_1, \delta_2, \delta_3, \delta'_1, \delta'_2, \delta'_3, \delta_4, \delta_5 \in \uparrow \overrightarrow{@}\mathbb{R}^2$, which again are anchored in the periodically repeated fifth d of the given tonality G-major and add up like β in (A) previously to a hypergesture $\epsilon \in \uparrow^8 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$. The three gestures $\delta_1, \delta_2, \delta_3$ and $\delta'_1, \delta'_2, \delta'_3$ define hypergestures $\epsilon_1, \epsilon_2 \in \uparrow^2 \overrightarrow{@} \uparrow \overrightarrow{@}\mathbb{R}^2$, much as β_1, β_2 did in the previous situation. And these two hypergestures also are connected to a big hypergesture ϕ, similarly to the above hypergesture γ. And again, the gestures δ_i, δ'_i in ϕ are anchored on the fifth and have the upper voice showing a characteristic chord. While in part (A) this was the arpeggiated $V_{B_\flat\text{-major}}$, here it is the arpeggiated modulation pivot $VII_{G\text{-major}}$ (which is the upper triad in the score's $V^7_{G\text{-major}}$). Moreover, as already mentioned previously in this section, the pivotal $VII_{G\text{-major}}$ appears arpeggiated in steps of two ascending minor thirds, which again stresses the third transposition symmetry against the inversion that would have modulated this configuration without restricted modulators.

62.4.3 Lewin's Characteristic Gestures Identified?

The situation is full of similarities above the level of the elementary interval gestures in (A) and in (D). But these are not similar. They are the carriers of what had been called "mirroring" above. What does this look like? Let us take the paradigmatic example of the two gestures α_1 in (A) and δ_1 in (D). We had associated gesture α_1 with the descending onset-time shift transformation $t_A = T^{(1/4,-4)}$, and δ_1 can be associated with the ascending onset-time shift transformation $t_B = T^{(1/4,28)}$. The straightforward curve of transformations is then $T_1 = S_1^*$ for α_1 and $T_B(t) = T^{t.t_B}, t \in I$, for δ_1. Evaluating these transformational gestures (curves) at the anchor points $f, d \in \mathbb{R}^2$ yields two curves $T_1.f, T_B.d \in \uparrow \overrightarrow{@}\mathbb{R}^2$. Although these curves of transformations and of evaluated points are mathematically straightforward, they are not evident per se. Why should the curve connecting start and end be a linear trajectory? One could also select $T'_B(t) = T^{t^2.t_B}$, for example. The trace would be the same. This simple example shows that the gestures on the score's dance floor are multiple. Only the points of contact are unchanged. The difference on the gestural level pertains to the interpretative effort (the "aesthesic" position in Valéry's wording), not to the work's neutral level, see Section 2.2.2 or [704]. On the gestural interpretation level, there is a manifold of solutions beyond the generic one.

A priori, there are essentially two levels where a solution may be sought out: the evaluated points level, i.e., the curves in $\uparrow \overrightarrow{@}\mathbb{R}^2$, or the transformational level, i.e., the curves of transformations in $\uparrow \overrightarrow{@}[Id_{\overrightarrow{GL}(\mathbb{R}^2)}]$. (We put aside the "mixed" solutions with variable points and transformations in $\uparrow \overrightarrow{@}[Id_{\mathbb{R}^2 \times \overrightarrow{GL}(\mathbb{R}^2)}]$ as already described in Example 74.) The difference of these approaches lies in their semantic power, which is expressed in the mathematical manifold which they describe. The simple curve level $\uparrow \overrightarrow{@}\mathbb{R}^2$ offers a big topological space, but no a priori semantics. Any curve would do, be it induced by a physical hand movement rationale as developed in [772], or by any poetic phantasy of spatial curves. In contrast, the transformational level $\uparrow \overrightarrow{@}[Id_{\overrightarrow{GL}(\mathbb{R}^2)}]$ defines a repertory, which is more expressive as it refers to curves of transformations, such as rotational curves, or curves of transvections, which, for example, may be loaded by musical meaning. This situation is much the same as for transformational theory, where two determined notes are connected by an affine transformation out of a set of transformations, which is essentially the stabilizer subgroup of the point of departure, and where the selection of an element of that stabilizer expresses a semantic choice—except that the manifold of curves is much larger than it is for that theory.

Let us look at the expressive richness on the hypergesture level, which defines curve gestures between gesture α_1 and gesture δ_1. One first interpretative action is the earlier defined dual gesture construction $T_1 = S_1^*$ for α_1. It exchanges the start and the end of the gesture and so doing means that the perspective is taken from the higher note $a = T_1(0)$ towards the anchor note $f = T_1(1)$, but the underlying logic stems from the dual gesture S_1, evaluated to the dual curve α_1^* that views a as the endpoint of a movement starting in f. Duality is interpreted as going backwards, coming back to the root f, although we are moving forwards in terms of the onset parameter. This is the first part of the mirroring operations. It is not a gesture, but a reinterpretation of the given gesture's curve parameter. Next, we want to compare α_1^* to δ_1. They are both ascending gestures, and they do so from the same scale degree, the fifth (call it dominant, if that matters) of the given tonality. The first looks backwards, the second forwards in time. Looking into the past and then into the future is a simple dramatic logic of the gestural trajectory. *We view this duality argument as a first example of thinking about Lewin's "characteristic gesture" in the sense that the gestural operation is a characteristic for the expression of a specific musico-logical thought.*

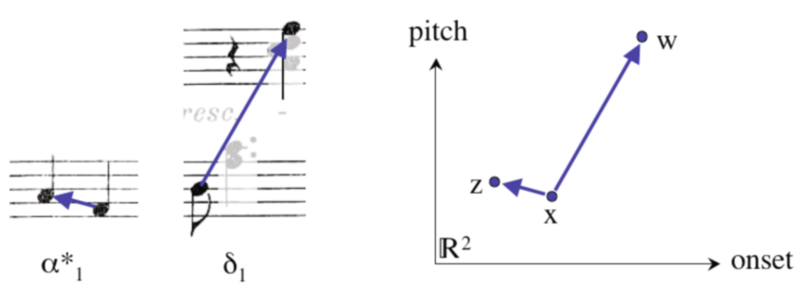


Fig. 62.8. The elementary gestures and their abstract representation for the mirroring operations.

There are several options to connect α_1^* to δ_1 in a hypergestural curve in order to construct a further differentiation of Lewin's "characteristic" movements. To ease the formal setup, let us think about both gestures starting from the same point $x \in \mathbb{R}^2$ and ending in z for α_1 and in w for δ_1. So this prototypical representation has $\alpha_1^*(t) = T^{t(z-x)}.x$ and $\delta_1(t) = T^{t(w-x)}.x$, see also Figure 62.8.

On the level of transformations, we have the prototypical gestures $T_\alpha(t) = T^{t(z-x)}, t \in I$, and $T_\delta(t) = T^{t(w-x)}, t \in I$, representing α_1^* and δ_1. In order to connect them by a curve (in fact a homotopy), one may set $T_\alpha(s)(t) = T^{(Q(s)(z-x))t}\phi(s)$ where $\phi : I \to \mathrm{GL}_2(\mathbb{R})$ is a loop starting and ending at Id, and where $Q(s) = T^{s(w-z)}$. The left part of Figure 62.9 shows this construction when evaluated at x and with trivial

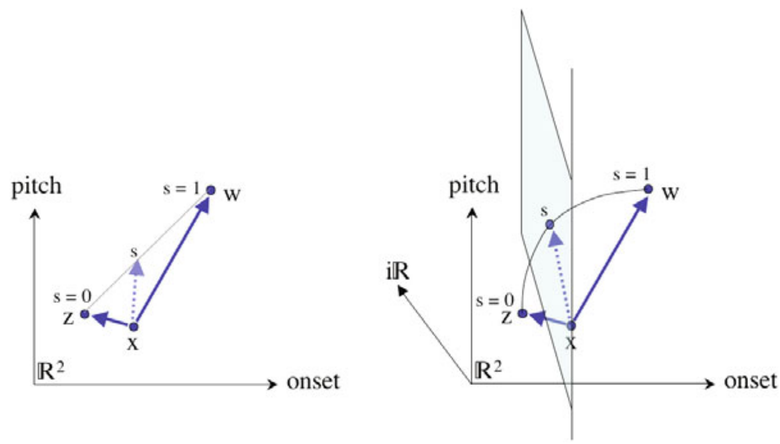


Fig. 62.9. The elementary gestures and their abstract representation for two mirroring operations.

ϕ. One may also refer to the complexification technique described above in Section 62.3 and set

$$Q(s) = \begin{pmatrix} e^{-i\pi s}((1+s) - s\Delta_1) & 0 \\ 0 & ((1-s) + s\Delta_2) \end{pmatrix}$$

with $\Delta_i = (w_i - x_i)/(z_i - x_i), i = 1, 2$, where $Q(s)$ rotates the onset part by $e^{-i\pi s}$ (clockwise) and produces an imaginary onset for $s \neq 0, 1$; see Figure 62.9, right part. It is fundamentally different from the first solution in that it *means* mirroring time instead of just pointing to continuously changing interval directions in the first choice. The difference in these gestures is that they express in characteristic ways the mirroring operation from the backwards oriented interval in (A) to the forwards oriented one in (D). These operations represent totally different musical thoughts. We would adhere more to the second choice in the sense that it is dramatic and coincides with the fermata: For a short moment, time becomes imaginary (here also imaginary in terms of complex numbers), and when we have transgressed that "higher sphere of pure imagination", we are heading for a new tonal region.

62.4.4 Modulation $E_\flat$-major ⇝ D-major/B-minor from W to W^*

This modulation is a catastrophe in the sense of Uhde since it leads to the antiworld W^*. As we may recognize already from the score in Figure 62.10, measures 189-197 are of a dramatic shape. Any elaborate motivic or harmonic effort is postponed in favor of a pertinent rhythmical presentation of diminished seventh chords. An approach to modulation via the inversion between e and f (provided by the modulation Theorem 30 in Section 27.1.4), $U_{e/f} = T^9.-1$, $VI_{E_\flat\text{-major}} \mapsto I_{D\text{-major}} = U_{e/f}.VI_{E_\flat\text{-major}}$ (measures 189-192), fails; the resolution of all alteration signs indicates the exit from tonal space. We hear the "generator" of the catastrophe, the diminished seventh chord as such. The situation before the modulation is similar to that in the previous modulation, where we also moved down and stayed on the fifth until the tonic was reached at the end of measure 34. In measures 187-188, the dominant degree $V_{E_\flat\text{-major}}$ appears four times, each time

Fig. 62.10. The modulation process $E_\flat$-major $\rightsquigarrow$ D-major/B-minor between measure 189 and measure 197.

initiating a downward four-note motif $b_\flat, a_\flat, g, f$ that finally reaches the tonic and degree $I_{E_\flat\text{-major}}$ at the beginning of measure 189. In this measure a two-measure-periodic rhythmic duration sequence of multiples of eights, namely $1/8, 3/8, 1/8, 3/8, 1/8, 3/8, 1/8, 2/8$ (followed by a 1/8 rest in the left hand), which concatenates twice the rhythm of the first four notes of the fanfare, is established and repeated without exception five times until measure 198, where the $I_{D\text{-major}}$ is reached. The rhythmical energy then breaks down to an even shorter rhythm, namely the very beginning rhythm $1/8, 3/8$ of the fanfare.

The beginning of this rhythmical pattern also parallels the ambiguous situation in measures 37-38. There, it was shown that the transition from $e_\flat$ to $f_\sharp$ was ambiguous, being either an inversion under $T^9.-1$ or a transposition T^3. Since the admitted symmetries exclude the inversion, the transposition was left and also showed its pivotal seventh degree in the following measures. But now, the situation is significantly different. The same inversion $T^9.-1$ does in fact transform $E_\flat$-major to D-major (or B-minor for that matter). But this is forbidden as above. Moreover—and this is different here—there is no other symmetry in the group of modulators that would do the job. We are confronting two tonalities in different worlds, $E_\flat$-major $\in W$ and D-major $\in W^*$. As can be expected from our previous modulation, the transformation $e_\flat \to f_\sharp$ also appears in the transition from the third of $VI_{E_\flat\text{-major}}$ to the third of $I_{D\text{-major}}$ in the action $T^9.-1(VI_{E_\flat\text{-major}}) = I_{D\text{-major}}$ on the neigboring degrees in measures 190-191. The impossibility of applying this modulator coincides with the above rhythmic pattern. This time there is no modulator. And as opposed to the fanfare that works in our previous case, it cannot be completed, i.e., it is blocked in its salient initial stage.

62.4.5 The Fanfare

In order to discuss the gestural interpretation of this process, we have to investigate the fanfare in more detail. We shall lead this discussion in a less technical style regarding the gestural formalisms, because technicalities can easily be filled out by the attentive reader, and because the point here is less the technical than the semantic level enabled by our gestural toolbox.

As is evident from the previous description, we have to focus on the rhythmic structure of this gesturality. To do so, we look at the time coordinates of the fanfare: onset and duration, see Figure 62.11. They show a bipartite gestural anatomy. We recognize two groups of gestures: the two ascending arrows to the left, and the two horizontal arrows to the right. These groups correspond to the pitch-ascending part and the pitch-descending part in the fanfare.

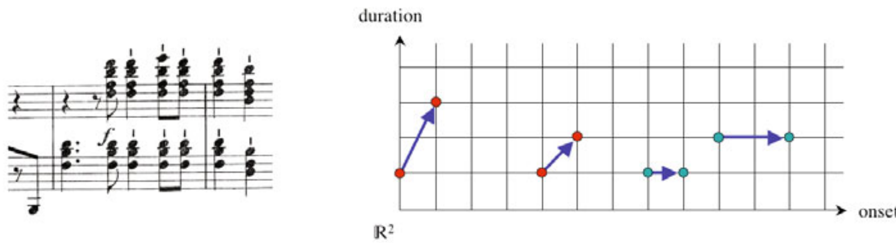


Fig. 62.11. The representation of the fanfare on the time-related plane of onset and duration.

So we view these structures as a gestural construction, which starts with the first ascending curve and deforms to the second ascending curve. This defines a hypergesture in the rhythm-space of onset and duration. In our understanding, the ascending character means that we address a downbeat, a halting energy. This elementary gesture (first arrow) is deformed to a second appearance (second ascending arrow). This deformation is shown as hypergesture ρ in the left lower part of Figure 62.12. This interpretation is ontologically non-trivial since it creates a continuous transition from the initial note to the second longer one, which amounts to imagining an entire curve of intermediate notes that succeed each other in infinitely near onset times and durations. This enrichment in fact fills out the empty time-space that is not denoted on the score by what in our musical imagination takes place while the first note is being heard/played. The hypergesture connecting the first and second arrow gestures is the connection of this first rhythmic step to the second in the same way, but conceptually and in the perceptive/performative level at a higher stage of imaginative coherence.

The first hypergesture ρ is followed by a second hypergesture σ, which deforms an arrow connecting two eighth notes to the arrow between two quarter notes. This time, the deformation of these arrows is not the hypergesture connecting a repeated halting movement, but expresses the halting movement of a regular succession of notes of same duration. It is not the repetition of a halting movement, but the halting of a repetitive movement: the roles of repetition and halting are exchanged. In order to connect these two hypergestures ρ and σ, we give two gestural transitions: First, the initial hypergesture ρ is rotated (in a rotative curve gesture) into the intermediate hypergesture ρ' (upper left corner in Figure 62.12). Then, ρ' is deformed into σ, but also replaced by its dual ρ'^* (not shown explicitly in the figure, since it just reverses the homotopy direction). This guarantees that the hypergesture moves from a lower duration to a higher one as required in σ. So we have a hypergesture of skeleton $\uparrow^2$ connecting ρ and σ. The other variant avoids duality, but uses a diagonal mirror operation to flip ρ into ρ'' in the upper right corner of Figure 62.12. Of course, this requires a complexification of the real 2-space, which we do not draw to keep the visualization simple. Again, from the intermediate ρ'', we deform down to σ by a similar transformation curve as that for the preceding case between ρ' and σ. And again, we have a hypergesture of skeleton $\uparrow^2$, this time generating σ from ρ via ρ''.

Again, we have different characteristics, which are addressed in these two paths from ρ to σ: The first keeps orientation (by a rotation in the onset-duration plane), but has to reverse curve time in the duality switch, while the second reverses onset and duration (through an imaginary rotation in complex numbers). According to the above statement that the second σ is not the repetition of a halting movement (which ρ was), but the halting of a repetitive movement. The exchange of onset and duration in the imaginary mirror path seems more to the point of this rhythmical construction. The dialectic pairing of ρ and σ in this interpretation resolves the repeated attempt to halt time in ρ by its hypergestural deformation to a completed

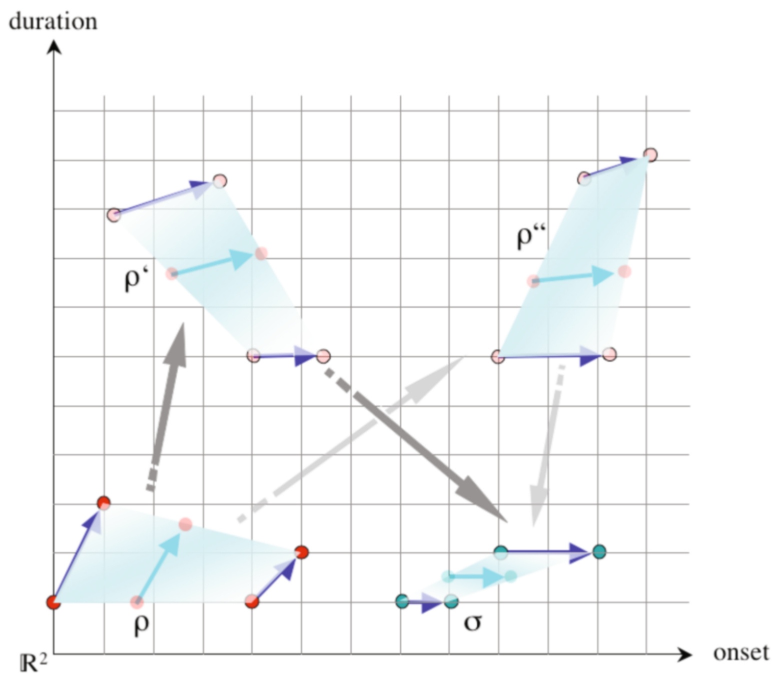


Fig. 62.12. The first hypergesture ρ is followed by a second hypergesture σ, which deforms an arrow connecting two eighth notes to the arrow between two quarter notes. We show the hypergestural connection between these two hypergestures.

repetition, and then into its halting slow-down. From this point of view, the repeated presentation of the fanfare's first part ρ in the catastrophe modulation perpetuates that internally already prototyped repeated halting and thereby expresses in an unfolding of the "idea in a nutshell" the failure to release the tension and to modulate in a well-structured way into the antiworld. The final reduction to the initial arrow of ρ in the ritardando measures 199-200 completes this failure and brings the energies to their exhaustion, the gesture dissolves.

62.5 Conclusion for the Categorical Gesture Approach

In this chapter, we have constructed a categorical framework of gestures, generalizing the topological approach from Chapter 61, and culminating in the construction of a gesture bicategory, which enriches the classical Yoneda embedding. This framework could be a valid candidate for the conjectured space X in the diamond conjecture (see Section 61.12). Future research will have to investigate typical algebraic categories of modules or topoi above module categories, or word monoids (in particular regarding scale theory [274], which deals a lot with factorization!), which are representative of algebraic music theories. We have discussed first applications of this framework for topological groups, and then more concretely gestures in modulation processes in Beethoven's Hammerklavier sonata. The latter have offered a first concretization of answers to Lewin's big question from [605] concerning characteristic gestures. The characters in our setup have been provided by well-chosen transformational gestures and their semantic interpretation in terms of dramatic instances. Despite these concrete examples, the present research has not solved the morphic half of Yoneda's idea, namely the fully gestural reconstruction of arrows in abstract categories, but it is a first step towards a replacement of Fregean functional abstraction by gestural dynamics. The fact that core constructions in homological algebra, such as extensions, are naturally incorporated in this approach is a sign of having embarked in the right direction.

62.6 Functorial Gestures: General Addresses

To this point, mathematical gesture theory has been developed under the tacit assumption that the classical addressed approach in functorial mathematical music theory, based upon the category $\mathbf{Mod}^{@}$ of modules, would not apply to gestures. However, such a restriction is superfluous, and we will develop here the functorial extension of the theory of local gestures in topological categories. We shall later include this extension into the global gesture theory, too, see Section 66.8.

In the categorial context, a continuous curve is a continuous functor $g : \nabla \to X$ with values in a topological category X. Its "points" are the values at the objects $x \in \nabla$. To generalize this situation to more functorially conceived points, we take the domain $\nabla \times A$ for a general address, i.e., a topological category $A \in \mathbf{TopCat}$. An A-addressed curve in X is a continuous functor $g : \nabla \times A \to X$. Evidently, if $f : B \to A$ is an address change in **TopCat**, we get a new curve $g.f := g \times (Id_\nabla \times f)$ from g and f. Also, if $x \in I$ is an object of ∇, we have the injection $f : 1 \to \nabla : 0 \mapsto x$ and the associated address change curve $g|x : A \to I \times A \to X : a \mapsto g(x, a)$; we shall need this latter construction especially for $g|0$ and $g|1$.

As in the zero-address case, we have a topological category (with object set) denoted by $\nabla©_A X := (\nabla \times A)©X$, and continuous natural transformations $\nu : g \to h$ of curves $g, h : \nabla \times A \to X$ as morphisms. The definition of the topology on this set is completely analogous to the construction in the zero-addressed case, see Section 62.1. Clearly, the address change $f : B \to A$ now induces a continuous functor $©f : \nabla©_A X \to \nabla©_B X$. Also, if $x \in I$ is an object (a point) of ∇, we have a continuous functor $©|x : \nabla©_A X \to A©X$ into the topological category $A©X$ with the compact-open topology. In particular, we have the tail and head functors $t_A = ©|0, h_A = ©|1 : \nabla©_A X \to A©X$, which define the *$A$-addressed categorical digraph $A@\overrightarrow{X}$ of X*. The digraph of the object maps of $A@\overrightarrow{X}$ is called the *spatial A-addressed digraph of X*. Again, similarly to the zero-addressed case, an *A-addressed gestures with skeleton digraph Γ and body X* is a digraph morphism $g : \Gamma \to A@\overrightarrow{X}$ into (the spatial digraph of) $A@\overrightarrow{X}$. The set of these gestures is denoted by $\Gamma@_A\overrightarrow{X}$.

We have the following functorial maps: Given an address change $f : B \to A$, a continuous functor of topological categories $m : X \to Y$, and a digraph morphism $t : \Delta \to \Gamma$, we have

$$\Gamma@_f X : \Gamma@_A\overrightarrow{X} \to \Gamma@_B\overrightarrow{X},$$
$$\Gamma@_A m : \Gamma@_A\overrightarrow{X} \to \Gamma@_A\overrightarrow{Y},$$
$$t@_A X : \Gamma@_A\overrightarrow{X} \to \Delta@_A\overrightarrow{X}.$$

To turn $\Gamma@_A\overrightarrow{X}$ into a topological category, denoted as earlier by $\Gamma\overrightarrow{@}_A X$, observe that Γ is the colimit of the diagram $\mathcal{D}$ of digraphs described in Section 62.1.2. Then we have $\Gamma@_A\overrightarrow{X} \xrightarrow{\sim} \lim_{\mathcal{D}} A@\overrightarrow{X}$ since $\uparrow @_A\overrightarrow{X} \xrightarrow{\sim} A@\overrightarrow{X}$, which turns $\Gamma@_A\overrightarrow{X}$ into a topological category since **TopCat** is finitely complete.

One expects the Escher Theorem 61 to be true for general addresses, too. We have this extension, which is valid for locally compact Hausdorff addresses:

Proposition 3. (Functorial Escher Theorem) *If Γ, Δ are digraphs, X is a topological category, and A, B are two locally compact Hausdorff topological categories, then we have a canonical isomorphism of topological categories,*

$$\Gamma\overrightarrow{@}_A\Delta\overrightarrow{@}_B X \xrightarrow{\sim} \Delta\overrightarrow{@}_B\Gamma\overrightarrow{@}_A X.$$

Modulo taking limits, the proof boils down to the special case where $\Gamma = \Delta = \uparrow$. Then we have to prove that there is a (canonical) isomorphism $\uparrow \overrightarrow{@}_A \uparrow \overrightarrow{@}_B X \xrightarrow{\sim} \uparrow \overrightarrow{@}_B \uparrow \overrightarrow{@}_A X$ of topological categories. Because of $\uparrow @_A\overrightarrow{X} \xrightarrow{\sim} A@\overrightarrow{X}$, this means that $A@\overrightarrow{B@\overrightarrow{X}} \xrightarrow{\sim} B@\overrightarrow{A@\overrightarrow{X}}$, i.e.,

$$(\nabla \times A)©((\nabla \times B)©X) \xrightarrow{\sim} (\nabla \times B)©((\nabla \times A)©X).$$

This isomorphism follows from the following series of isomorphisms that are all due either to the universal property of the compact-open topology (Mathematical Appendix, Section J.4.1.2, Theorem on Exponential Correspondence) or to the isomorphism $X \times Y \xrightarrow{\sim} Y \times X$ of cartesian products. We start with the right expression above.

$$
\begin{aligned}
(\nabla \times B)©((\nabla \times A)©X) &\xrightarrow{\sim} (\nabla \times B)©(\nabla©(A©X)) \xrightarrow{\sim}\\
\nabla©(B©(\nabla©(A©X))) &\xrightarrow{\sim} \nabla©(B \times \nabla©(A©X)) \xrightarrow{\sim}\\
\nabla©(\nabla \times B©(A©X)) &\xrightarrow{\sim} \nabla©(\nabla©(B©(A©X))) \xrightarrow{\sim}\\
\nabla©(\nabla©((B \times A)©X)) &\xrightarrow{\sim} \nabla©(\nabla©((A \times B)©X)) \xrightarrow{\sim}\\
(\nabla \times \nabla)©((A \times B)©X) &\xrightarrow{\sim} (\nabla \times \nabla)©((A \times B)©X) \text{ permute the two copies of } \nabla.
\end{aligned}
$$

The last expression is the result we get from the same procedure, but starting from the left expression above, and we are done.

We shall give musical applications of this functorial formalism in Section 78.4.1.

62.7 Yoneda's Lemma for Gestures

The classical Yoneda Lemma (Appendix Section G.2) deals with the Yoneda functor $Y : \mathcal{C} \to \mathcal{C}^{@} : X \mapsto @X$, and in its general shape states that for any presheaf (contraviariant functor) $F \in \mathcal{C}^{@}$ and object X in $\mathcal{C}$, we have a bijection

$$Nat(@X, F) \xrightarrow{\sim} X@F.$$

In our present situation, we also have presheaves, but they are different from the classical ones. For a topological category X, we have the presheaf (denoted by the classical symbol as no confusion is likely) $@X$: **Digraph** $\times$ **TopCat** $\to$ **TopCat** : $(\Sigma, A) \mapsto \Sigma\overrightarrow{@}_A X$. It is not representable in the classical sense, but nevertheless we have a representational situation here. Denote by $\overrightarrow{\mathbf{TopCat}}^{©}$ the category of presheaves F : **Digraph** $\times$ **TopCat** $\to$ **TopCat**; its objects are also called *gestural presheaves*. Observe that the natural transformations in this category need to refer to morphisms in **TopCat**; we then also denote by $Nat_{TC}(F, G)$ the set of morphisms $f : F \to G$ in $\overrightarrow{\mathbf{TopCat}}^{©}$.

Definition 111 *A gestural presheaf F* : **Digraph** $\times$ **TopCat** $\to$ **TopCat** *is said to be* gesturally representable *iff it is isomorphic in* $\overrightarrow{\mathbf{TopCat}}^{©}$ *to* $@X$ *for a topological category X. We then also say that F* is represented by the gesture space *X.*

In what follows, we shall prove a Yoneda Lemma that identifies morphisms in $\overrightarrow{\mathbf{TopCat}}^{©}$, i.e., natural transformations $f : @X \to F$ of gestural presheaves with an evaluation of determined functors at X. To this end, we consider the category **TC-Digraph** of TC-digraphs that are internal to $\mathbf{TopCat}^{©}$, the category of presheaves on **TopCat** that have values in **TopCat** (we also call them "continuous presheaves"). A TC-digraph D is given by natural head and tail morphisms $D = \eta, \tau : C \rightrightarrows P$ between (continuous) presheaves $C, P \in \mathbf{TopCat}^{©}$, C being called the *presheaf of curves*, while P is called the *presheaf of points*. A morphism $\overrightarrow{f} = (f^{\nabla}, f^{\cdot}) : D_1 \to D_2$ for $D_1 = \eta_1, \tau_1 : C_1 \rightrightarrows P_1, D_2 = \eta_2, \tau_2 : C_2 \rightrightarrows P_2$ is the usual pair of morphisms $f^{\nabla} : C_1 \to C_2, f^{\cdot} : P_1 \to P_2$ that commutes with head and tail morphisms. Every gestural presheaf F gives rise to such a TC-digraph $F^{\nabla} \rightrightarrows F^{\cdot}$ that evaluates to $A@F^{\nabla} = (\uparrow, A)@F, A@F^{\cdot} = (., A)@F$, the evaluation at the line digraph $\uparrow$ and the singleton digraph $.$, while the head and tail morphisms are defined by the two injections $. \rightrightarrows \uparrow$. This digraph is denoted by $\overrightarrow{F}$. In particular, if $F = @X$, we have $A@X^{\nabla} = (\uparrow, A)@X = A@X^{\nabla}, A@X^{\cdot} = (., A)@X = A@X^{\cdot}$, where $X^{\nabla} := \nabla@X, X^{\cdot} := X$. This digraph is denoted by $\overrightarrow{X}$. With this digraph formalism, we have a Yoneda Lemma for gestural presheaves F that are limits of curve functors, i.e. $(\Sigma, A)@F \xrightarrow{\sim} \lim_{\mathcal{D}}(F^{\nabla}, F^{\cdot})$, where $\mathcal{D}$ is the usual diagram of digraphs whose colimit is Σ. Call such functors *limiting functors*. For example, all the gesturally representable functors are limiting.

Theorem 39 (Yoneda Lemma for Functorial Gestures) *For a topological category X and a limiting gestural presheaf F, we have a bijection*

$$Nat_{TC}(@X, F) \xrightarrow{\sim} \mathbf{TC\text{-}Digraph}(\overrightarrow{X}, \overrightarrow{F}).$$

If $\overrightarrow{f} = (f^{\nabla}, f^{\cdot})$ *is a morphism* $\overrightarrow{X} \to \overrightarrow{F}$*, the classical Yoneda Lemma states that this is equivalent to having a pair* $f^{\nabla} \in X^{\nabla}@F^{\nabla}, f^{\cdot} \in X@F^{\cdot}$ *such that the two images* $\eta f^{\nabla}, \tau f^{\nabla} : X^{\nabla} \rightrightarrows F^{\cdot}$ *defined by the functors* $\eta, \tau : F^{\nabla} \rightrightarrows F^{\cdot}$ *and the two two images* $f^{\cdot}h, f^{\cdot}t : X^{\nabla} \rightrightarrows F^{\cdot}$ *defined by head and tail morphisms* $X^{\nabla} \rightrightarrows X$ *coincide*[5]*, respectively:*

$$\eta f^{\nabla} = f^{\cdot}h,$$
$$\tau f^{\nabla} = f^{\cdot}t.$$

The proof runs as follows. For a natural transformation $f : @X \to F$, the evaluation $(\Sigma, A)@f : \Sigma\overrightarrow{@}_A@X \to (\Sigma, A)@F$ for general digraphs Σ and topological categories A commutes with its evaluations $(\uparrow, A)@f :\uparrow \overrightarrow{@}_A X \to (\uparrow, A)@F$ $(., A)@f : .\overrightarrow{@}_A X \to (., A)@F$ at the arrow $\uparrow$ and the point digraph ., which means that, in view of the limiting character of these functors, the arrow and point evaluation functors determine one-to-one the original morphism f. But these two evaluations mean the commutativity (with the left and right vertical arrows, respectively) of this functor diagram:

$$\begin{array}{ccc} A@X^{\nabla} & \xrightarrow{A@f^{\nabla}} & A@F^{\nabla} \\ {\scriptstyle h}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle \eta}\downarrow\downarrow{\scriptstyle \tau} \\ A@X^{\cdot} & \xrightarrow[A@f^{\cdot}]{} & A@F^{\cdot} \end{array}$$

But this is equivalent to the Yoneda evaluation at A of the digraph morphism $\overrightarrow{f} : \overrightarrow{X} \to \overrightarrow{F}$, and we are done.

This lemma has a deep impact on the gestural understanding of artistic utterance. While the functor F is not defined by gestures, nor has its values in gestural structures, the functors $@X$ are gestural by their very construction. The natural transformations $f : @X \to F$ define gestural perspectives on F, our understanding of F in terms of gestural functors. One could call the entire big functor $Nat_{TC}(@?, F)$ the *functorial gestural aesthetics of* F. It tells us how much we can know about F in terms of gestural constructions. This is an important tool to discuss musical constructions that are *not*, a priori, gestural in nature. Such a situation may occur typically in electronic music, but also in classical constructions of scores that are not derived from gestural aspects. In terms of the classical Yoneda Lemma, we could consider the category **TC-Digraph** and the Yoneda functor Y : **TC-Digraph** → **TC-Digraph**$^{©}$, and the above big functor would mean restricting the functorial domains to the subcategory $@TC \subset$ **TC-Digraph** of gesturally representable functors $\overrightarrow{X}$, and asking whether the functor Y : **TC-Digraph** → **TC-Digraph**$^{©} \to @TC^{©}$ is still fully faithful.

62.8 Examples from Music

In the following three examples, we shall illustrate this situation, namely for constructions of sound waves, spectral compositions, and MIDI-related ON-OFF processes.

In all three examples, we shall use non-representable functors of the same nature: powers functors, which are well-known to be non-representable as **TopCat** is not a topos.

62.8.1 Collections of Acoustical Waves

The first example considers the topological space $Z = \mathcal{C}^n([0,1], \mathbb{R})$ of n times differentiable functions on the unit interval, describing sound events in a defined time interval $[0,1]$. The functors are $A@F_Z^{\nabla} = 2^{A@Z^{\nabla}}$, whose elements are sets g of morphisms $g_i : A \to Z^{\nabla}$, or, equivalently, morphisms $g_i : \nabla \times A \to Z$, the latter being interpreted as curves with values in Z that are parametrized by values in A, i.e., for each $a \in A$, we have a curve $g_{i,a} : \nabla \to Z$, which is equivalent to a curve $[0,1] \to Z$ as we are dealing with topological

[5] This can be restated as a diagram limit condition.

spaces here. Technologically, this means that we are given a processual setup that creates a curve of sound events for each parameter choice $a \in A$. Such a situation is standard when working with Max MSP software, for example.

Next, we also need the functor $F^{\cdot}$, which we define by $A@F^{\cdot}_Z = 2^{A@Z}$. Its values are the sets of A-parametrized sound events. The head and tail transformations are the evident maps $\eta, \tau : A@F^{\nabla}_Z \to A@F^{\cdot}_Z$ that send curves to their head and tail values. These sets are given the indiscrete topology.

It is a bit tricky to find morphisms $f : @X \to F_Z$. One way of doing so is to think about the above condition $\eta f^{\nabla} = f^{\cdot} h,\ \tau f^{\nabla} = f^{\cdot} t$. We have to select two sets, $f^{\nabla} \subset X^{\nabla}@Z^{\nabla}, f^{\cdot} \subset X@Z$, such that these conditions hold, meaning that for every $g^{\nabla} \in f^{\nabla}$ there is a $g^{\cdot} \in f^{\cdot}$ such that $\eta g^{\nabla} = g^{\cdot} h$, and a $g^{\cdot}_{*} \in f^{\cdot}$ with $\tau g^{\nabla} = g^{\cdot}_{*} t$, and vice versa. For example, $\eta g^{\nabla} = g^{\cdot} h : X^{\nabla} \to F_Z$ means that for every curve $\kappa : \nabla \to X$, we have $g^{\nabla}(\kappa)(1) = g^{\cdot}(\kappa(1))$. These are quite involved conditions. But it is easy to find solutions. Take any set $f^{\cdot} \subset X@Z$. Then define f^{∇} as follows. For every $g^{\cdot} \in f^{\cdot}$, define $g^{\cdot,\nabla}(\kappa, m) = g^{\cdot}(\kappa(m))$ for any curve $\kappa : \nabla \to X$ and morphism (!) $m \in \nabla$. The set f^{∇} is built from these morphisms $g^{\cdot,\nabla}$, and it is evident that this is a solution to our problem. It is also clear that this solution holds for any topological space Z.

62.8.2 Collections of Spectral Music Data

The second example uses the same architecture as the previous one, but Z is now a different space that is related to spectral composition methods. Instead of sets of parametrized sound events in $\mathcal{C}^n([0,1], \mathbb{R})$, we now define Fourier coefficients that are time-dependent. More precisely, we define sound events as functions of time $x \in \mathbb{R}$ via the Fourier expressions

$$w(x) = \sum_{n}^{\pm\infty} c_n(x) e^{i2\pi n\nu(x)x},$$

where every member of the function sequence $c_n, \nu : \mathbb{R} \to \mathbb{C}$ has compact support, we have real values for ν, and the sequence produces a convergent sum at all times. Call Z_{spec} the topological space of these sequences with one of the usual topologies (in fact defined by scalar products). Then our digraph of presheaves $F^{\nabla}_{Z_{spec}} \rightrightarrows F^{\cdot}_{Z_{spec}}$ represents sets of parametrized time-dependent Fourier coefficients used in classical spectral compositions.

62.8.3 MIDI-Type ON-OFF Transformations

For our third example, we take $A@F^{\cdot}_{EHL} = 2^{A@\mathbb{R}^{EHL}}$ and $A@F^{\nabla}_{EHLDGC} = 2^{A@\mathbb{R}^{EHLDGC}}$, where $\mathbb{R}^{EHL}$ is the three-dimensional real vector space of note events with onset (E), pitch (H), and loudness (L), whereas $\mathbb{R}^{EHLDGC}$ is the six-dimensional real vector space of note events with onset (E), pitch (H), loudness (L), duration (D), glissando (G), and crescendo (C). Again, the powersets are given the indiscrete topology. Here, the head and tail functors are defined by the typical operators from the MIDI ON and OFF functions, namely η (for ON) is defined by the first projection $p_{EHL} : \mathbb{R}^{EHLDGC} \to \mathbb{R}^{EHL} : (x, y, z, u, v, w) \mapsto (x, y, z)$, while τ (for OFF) is defined by the alteration function $\alpha : \mathbb{R}^{EHLDGC} \to \mathbb{R}^{EHL} : (x, y, z, u, v, w) \mapsto (x+u, y+v, z+w)$. In this situation, we can construct a morphism as follows. We again start with a set $f^{\cdot} \subset X@\mathbb{R}^{EHL}$. The set $f^{\nabla} \subset X@\mathbb{R}^{EHLDGC}$ consists of these morphisms: For every $l : X \to \mathbb{R}^{EHL}$ in $f^{\cdot}$, we take its two composed morphisms $l_0 : X^{\nabla} \to \mathbb{R}^{EHL} : \kappa \mapsto l(\kappa(0))$ and $l_1 : X^{\nabla} \to \mathbb{R}^{EHL} : \kappa \mapsto l(\kappa(1))$. Then we define members of $f^{\nabla} \subset X@\mathbb{R}^{EHLDGC}$ by taking for each $l : X \to \mathbb{R}^{EHL}$ the function $l^{\nabla} : X^{\nabla} \to \mathbb{R}^{EHLDGC} : \kappa \mapsto (l_1(\kappa), l_0(\kappa) - l_1(\kappa))$. It is immediate that this defines a solution.