# Indicated Coloring of Cartesian Product of Graphs

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**Abstract.** Indicated coloring of a graph $G$ is a coloring in which there are two players Ann and Ben, Ann picks a vertex and Ben chooses a color for this vertex. The aim of Ann is to achieve a proper coloring of the whole graph $G$, while Ben tries to block the same. The smallest number of colors required for Ann to win the game on a graph $G$ is called the indicated chromatic number of $G$ and is denoted by $\chi_i(G)$. In this paper, we prove that $T\square C_n, T\square K_{n_1,n_2,\ldots,n_m}$ and $K_{n_1,n_2,\ldots,n_m}\square C_m$ are $k$-indicated colorable for all $k$ greater than or equal to the indicated chromatic number of their corresponding Cartesian product, where $T$ is any tree. Also we prove that $\chi_i(K_{k_1,k_2,\ldots,k_m}\square K_{l_1,l_2,\ldots,l_n}) = \chi(K_{k_1,k_2,\ldots,k_m}\square K_{l_1,l_2,\ldots,l_n})$. Finally we have given non-trivial examples of graphs $G$ and $H$ for which $\chi_i(G\square H) > \chi(G\square H)$.

**Keywords:** Game chromatic number · Indicated chromatic number · Cartesian product

## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. A game coloring of a graph is a coloring in which two players Ann and Ben are jointly coloring the graph $G$ by using a fixed set of colors $C$. The motive of Ann is to get a proper coloring of the whole graph, where as Ben is trying to prevent the realization of this project. The minimum number of colors required for Ann to win the game on a graph $G$ irrespective of Ben's strategy is called the game chromatic number of the graph $G$ and it is denoted by $\chi_g(G)$. The idea of indicated coloring was introduced by A. Grzesik in [3] as a slight variant of the game coloring in the following way: in each round Ann is only picking a vertex while Ben is choosing a color for this vertex. The aim of Ann as in indicated coloring is to achieve a proper coloring of the whole graph $G$, while Ben tries to "block" some vertex. A *block* vertex means an uncolored vertex which has all colors from $C$ on its neighbors. The smallest number of colors required for Ann to win the game on a graph $G$ is called the indicated chromatic number of $G$ and is denoted by $\chi_i(G)$. Clearly from the definition we see that $\omega(G) \le \chi(G) \le \chi_i(G) \le \Delta(G) + 1$. If Ann has a winning strategy using $k$ colors for a graph $G$ then we say that $G$ is $k$-indicated colorable. Let $st_k(G)$ denote a winning strategy of Ann while using $k$ colors. The coloring number of a graph $G$, denoted by col$(G)$ is defined by $\mathrm{col}(G) = 1 + \max_{H\subseteq G} \delta(H)$. By Szekeres-Wilf inequality [6], $\chi(G) \le \mathrm{col}(G)$.

Zhu in [9] has asked the following question for game coloring. Whether increasing the number of colors will favor Ann? That is, if Ann has a winning strategy using


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$k$ colors, will Ann have a winning strategy using $k+1$ colors? The same question was asked by Grzesik for indicated coloring. Also he showed by an example that the increase in number of colors does make life simple for Ann rather it makes it much harder. There has been already some partial answers to this question. For instance, Pandiya Raj et al. [2,5] showed that chordal graphs, cographs, complement of bipartite graphs, $\{P_5, K_3\}$-free graphs, $\{P_5, \text{paw}\}$-free graphs, $\{P_5, C_5, K_4-e\}$-free graphs and $\{P_5, K_4-e\}$-free graphs having induced $C_5$ are $k$-indicated colorable for all $k \geq \chi(G)$. In addition Lason in [4] has obtained the indicated chromatic number of matroids. In this paper, we obtain $T\square C_n$, $T\square K_{n_1,n_2,\ldots,n_m}$ and $K_{n_1,n_2,\ldots,n_m}\square C_m$ are $k$-indicated colorable for all $k$ greater than or equal to the indicated chromatic number of their corresponding Cartesian product, where $T$ is any tree. In addition, we have prove that $\chi_i(K_{k_1,k_2,\ldots,k_m}\square K_{l_1,l_2,\ldots,l_n}) = \chi(K_{k_1,k_2,\ldots,k_m}\square K_{l_1,l_2,\ldots,l_n})$. Finally we have given non-trivial examples of graphs $G$ and $H$ for which $\chi_i(G\square H) > \chi(G\square H)$.

Notations and terminologies not mentioned here are as in [8].

## 2 Indicated Coloring on Cartesian Product of Graphs

The Cartesian product of two graphs $G$ and $H$, denoted by $G\square H$, is a graph whose vertex set $V(G)\times V(H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ of $G\square H$ are adjacent if and only if either $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $y_1 = y_2$ and $x_1x_2 \in E(G)$. Vizing [7] proved that $\chi(G\square H) = \max\{\chi(G), \chi(H)\}$. Note that while considering the cartesian product $G\square H$, for each $v \in V(G)$, $\langle v \times V(H)\rangle$ (for $S \subseteq V(G)$, $\langle S\rangle$ denotes the induced subgraph of $S$ in $G$) is a copy of $H$ and for each $u \in V(H)$, $\langle V(G) \times u\rangle$ is a copy of $G$. Also if $S$ is an independent set in $G$ and $T$ is an independent set in $H$, then $S\square T$ is an independent set in $G\square H$.

Our main focus in Sect. 2 is to see whether the following is true. If $G$ is $k$-indicated colorable for all $k \geq \chi_i(G)$ and $H$ is $k$-indicated colorable for all $k \geq \chi_i(H)$, will $G\square H$ be $k$-indicated colorable for all $k \geq \chi_i(G\square H)$? As a first step, we have considered a few families for which this works out. In fact this also gives some partial answer to the question raised by Grzesik in [3]. Let us recall a few results done in [3,5].

**Theorem 1.** [5] *Any graph $G$ is $k$-indicated colorable for all $k \geq col(G)$.*

**Theorem 2.** [3] *Every bipartite graphs is $k$-indicated colorable for every $k \geq 2$.*

An immediate consequence of Theorem 2 is the following.

**Corollary 1.** *Let $G$ and $H$ be two non-trivial graphs. Then $G$ and $H$ are bipartite if and only if $G\square H$ is $k$-indicated colorable for all $k \geq 2$.*

*Proof.* We know that $G\square H$ is bipartite if and only if $G$ and $H$ are bipartite. Suppose $G$ and $H$ are bipartite, by using Theorem 2, $G\square H$ is $k$-indicated colorable for all $k \geq 2$. Suppose $G\square H$ is $k$-indicated colorable for all $k \geq 2$, then $2 \leq \chi(G\square H)) \leq \chi_i(G\square H) = 2$. Hence $G\square H$ is bipartite.

By using Corollary 1, we see that if both $m$ and $n$ are even, then $C_m \square C_n$ is $k$-indicated colorable for all $k \geq 2$. While considering the case when either $m$ or $n$ (or both) is odd, the $\text{col}(C_m \square C_n) = 5$. Thus for showing that $C_m \square C_n$ is $k$-indicated colorable for all $k \geq 3$, it is enough to prove that $C_m \square C_n$ is 3 and 4-indicated colorable. This still remains an open problem.

Let us next recall a strategy used in [1].

**Definition 1.** *While coloring a graph $G$ by using $k$ colors, let $\mathcal{N}_{un}(v)$ denote the number of uncolored neighbors of $v$ in $G$ and $\mathcal{C}(v)$ denote the number of available colors for $v$ in $G$. A vertex $v$ is said to be of type1 if $\mathcal{C}(v) > \mathcal{N}_{un}(v)$ and of type2 if $\mathcal{C}(v) = \mathcal{N}_{un}(v)$.*

**Lemma 1.** *Let Ann and Ben plays an indicated coloring game on graph $G$ with $k \geq \chi(G)$ colors. In certain stage, if all the uncolored vertices in $G$ can be partitioned into disjoint paths such that one end of each path is of type1 and all the other vertices are of type2, then Ann has a winning strategy.*

*Proof.* Let the color set be $\{1, 2, \ldots, k \geq \chi(G)\}$. By our assumption, let $P_1, P_2, \ldots, P_l$ be a partition of the uncolored vertices with the property that one end of each $P_i$, $1 \leq i \leq l$, is of type1 and all the other vertices in $P_i$ are of type2. Let $P_1 = v_{11}, v_{12}, \ldots, v_{1j}$ for some $j \geq 1$ and let $v_{1j}$ be of type1 and $v_{1i}$, $1 \leq i \leq j-1$ be of type2. Clearly there is always an available color for $v_{1j}$. Now let Ann present the vertices of $P_1$ in the order $v_{11}, v_{12}, \ldots, v_{1j}$ (same order of the path $P_1$). Since $\mathcal{C}(v_{1i}) = \mathcal{N}_{un}(v_{1i})$, for every $i$, $1 \leq i \leq j-1$ and one of the neighbor of $v_{1i}$, namely $v_{1(i+1)}$ is presented after $v_{1i}$. Thus Ben has an available color for each $v_{1i}$, $1 \leq i \leq j-1$. Since all the uncolored vertices where only of type1 or type2, Ben cannot create a block vertex in any of the paths. Thus a similarly technique can be applied by Ann for all the other paths to yield an indicated coloring using $k$ colors.

**Theorem 3.** *Let $T$ be any tree. Then*

*(i) $T \square C_m$ is $k$-indicated colorable for all $k \geq \chi_i(T \square C_m) = \chi(T \square C_m)$*
*(ii) $T \square K_{n_1,n_2,\ldots,n_m}$ is $k$-indicated colorable for all $k \geq \chi_i(T \square K_{n_1,n_2,\ldots,n_m}) = m$.*

*Proof.* Let $v_0$ be a center of $T$. Let $V_i$ be the set of all vertices of $T$ which are at a distance $i$ from $v_0$, $1 \leq i \leq r$ where $r$ is the radius of the tree. Let us label the vertices of $T$ as $v_0, v_1, v_2, \ldots, v_{n-1}$ such that the vertices of $V_i$ are to the left of the vertices of $V_j$ for every $i, j$ such that $1 \leq i < j \leq r$. Let $v_{ij} = (v_i, u_j)$ be the vertex of $T \square G$ where $v_i \in T$, $u_j \in G$, $0 \leq i \leq n-1$ and $0 \leq j \leq |V(G)| - 1$. Let $c(v)$ denote the color given by Ben to the vertex $v$ and if $v$ is uncolored then assume that $c(v) = \emptyset$. In $T \square G$, let $H_0, H_1, \ldots, H_{n-1}$ be the copies of $G$ corresponding to the vertices $v_0, v_1, \ldots, v_{n-1}$ of $T$ respectively. If $v_i$ and $v_j$ are non-adjacent vertices in $T$ then $\langle V(H_i), V(H_j) \rangle = \emptyset$ in $T \square G$. If $v_i$ and $v_j$ are adjacent vertices in $T$ then $\langle V(H_i), V(H_j) \rangle = \{v_{il} v_{jl} : 0 \leq l \leq |V(G)| - 1\}$ in $T \square G$.

(i) Let us consider the graph $G = C_m$. Suppose $m$ is even, $T \square C_m$ is bipartite and by using Theorem 2, $T \square C_m$ is $k$-indicated colorable for all $k \geq 2 = \chi_i(T \square C_m)$. Now let us consider $m$ to be odd. It is easy to observe that $\chi(T \square C_m) = 3$ and $\text{col}(T \square C_m) = 4$. Hence by using Theorem 1, it is enough to show that $T \square C_m$ is 3-indicated colorable. Let the color set be $\{1, 2, 3\}$. Let Ann present the vertices of $H_0$ in any order. Since the

$\mathrm{col}(H_0) = 3$, Ben always has an available color for each vertex of $H_0$. Irrespective of Ben's strategy, there exist a vertex $v_{0j}$ of $H_0$ having two different colors in its neighbor, namely $v_{0(j-1)}$ and $v_{0(j+1)}$ where $0 \leq j \leq m-1$ and $j$ is taken mod $n$. Now consider the subgraph $G_1 = \langle\{v_0 \cup V_1\}\rangle \square C_m$ in $T \square C_m$. Let $H_i$ be the $C_m$ copy of the vertex $v_i \in V_1$, $1 \leq i \leq |V_1|$. Since $\langle V(H_0), V(H_i)\rangle = \{v_{0j}v_{ij} : 0 \leq j \leq m-1\}$ for all $1 \leq i \leq |V_1|$ and there are 3 colors, the uncolored vertices of $G_1$ are the vertices of $H_i$ which are the vertices of type2. Now Ann will present the vertex $v_{ij}$ of $H_i$ for all $1 \leq i \leq |V_1|$. Suppose Ben color $v_{ij}$ with the color of $v_{0(j-1)}$ then the vertex $v_{i(j-1)}$ is a vertex of type1, otherwise $v_{i(j+1)}$ is a vertex of type1 where $1 \leq i \leq |V_1|$, $0 \leq j \leq m-1$ and $j$ is taken mod $n$. By using Lemma 1, Ann have an winning strategy on $G_1$.

Let us consider the subgraph $G_i = \langle\{V_{i-1} \cup V_i\}\rangle \square C_m$ where $2 \leq i \leq r$. Similarly Ann follow the same procedure to presents the uncolored vertices of $G_i$, and thus Ann has a winning strategy for $G_i$, $2 \leq i \leq r$. This yields a winning strategy for Ann on the graph $T \square C_m$ with 3 colors.

(ii) It is easy to observe that $\chi(T \square K_{n_1,n_2,\ldots,n_m}) = m$ and $\mathrm{col}(T \square K_{n_1,n_2,\ldots,n_m}) = m + 1$. Hence by using Theorem 1, it is enough to show that $T \square K_{n_1,n_2,\ldots,n_m}$ is $m$-indicated colorable. Let the color set be $\{1, 2, \ldots, m\}$. Let $U_i$, $1 \leq i \leq m$ be the $m$-partites of the graph $K_{n_1,n_2,\ldots,n_m}$. Let us consider the subgraph $G_0 = \langle\{u_1, u_2, \ldots, u_m\}\rangle$ in $K_{n_1,n_2,\ldots,n_m}$, where $u_i \in U_i$, $1 \leq i \leq m$. Clearly $G_0 \cong K_m$ and $\omega(K_{n_1,n_2,\ldots,n_m}) = m$. Now consider the graph $T \square G_0$. Let $J_0, J_1, \ldots, J_{n-1}$ be the copies of $G_0$ corresponding to the vertices $v_0, v_1, \ldots, v_{n-1}$ of $T$ respectively.

Ann starts presenting the vertices of $J_0$ in any order. Since $\mathrm{col}(J_0) = m$, Ben have an available color for each vertex of $J_0$. Now consider the subgraph $G_1 = \langle\{v_0 \cup V_1\}\rangle \square G_0$ in $T \square G_0$. Since $\langle V(J_0), V(J_i)\rangle = \{v_{0j}v_{ij} : 1 \leq j \leq m\}$ for all $1 \leq i \leq |V_1|$ and there are $m$ colors, the uncolored vertices of $G_1$ are the vertices of $J_i$ which are the vertices of type2. Now Ann will present the vertex $v_{i1}$ of $J_i$ for all $1 \leq i \leq |V_1|$. Ben should color $v_{i1}$ with one of the color from $\{1, 2, \ldots, m\} \backslash \{c(v_{01})\}$ and let it be $c_i$, $1 \leq i \leq |V_1|$. For each $J_i$, $1 \leq i \leq |V_1|$ there is a vertex $v_{ii'}$ which is adjacent to the color $c_i$ of $J_0$ where $1 \leq i' \leq m$ and thus the vertex $v_{ii'}$ is a vertex of type1. By using Lemma 1, Ann has a winning strategy on $G_1$.

Let us next consider the subgraph $G_i = \langle\{V_{i-1} \cup V_i\}\rangle \square G_0$ where $2 \leq i \leq r$ in $T \square G_0$. Let Ann follow a similar procedure as done in $G_1$, for presenting the uncolored vertices of $G_i$. This will give Ann a winning strategy for $G_i$, $2 \leq i \leq r$, and hence a winning strategy for $T \square G_0$ with $m$ colors. Let it be $st_m(T \square G_0)$. Now consider the graph $T \square K_{n_1,n_2,\ldots,n_m}$. The strategy $st_m(T \square G_0)$ makes Ben to color exactly one vertex of $U_i$, $1 \leq i \leq m$ in each of the copies of $K_{n_1,n_2,\ldots,n_m}$ corresponding to the vertices of $T$. Hence for the remaining vertices in each of the $U_i$ in the copies of $K_{n_1,n_2,\ldots,n_m}$ corresponding to the vertices of $T$, Ben will be forced to give the color given to vertex in $U_i$ that is already colored. Thus Ann can presents the remaining vertices in any order and this will yield an $m$-indicated coloring for $T \square K_{n_1,n_2,\ldots,n_m}$.

An immediate consequence of Theorem 3 is the following.

**Corollary 2.** *For all $m \geq 2$, the graph $T \square K_m$ is $k$-indicated colorable for all $k \geq m$.*

In a similar fashion but with a little more involved arguments, we have showed Theorems 4 and 5.

**Theorem 4.** *For all $m \geq 3$ and $n \geq 3$, the graph $K_{n_1,n_2,\ldots,n_m} \square C_n$ is k-indicated colorable for all $k \geq m$.*

**Theorem 5.** *For all $m \geq 2$ and $n \geq 2$, $\chi_i(K_{k_1,k_2,\ldots,k_m} \square K_{l_1,l_2,\ldots,l_n}) = \chi(K_{k_1,k_2,\ldots,k_m} \square K_{l_1,l_2,\ldots,l_n})$.*

An immediate consequence of Theorems 4 and 5 is the following.

**Corollary 3.** *For all $m \geq 3$ and $n \geq 3$, the graph $K_m \square C_n$ is k-indicated colorable for all $k \geq m$ and $\chi_i(K_m \square K_n) = \chi(K_m \square K_n)$.*

By the definition of indicated coloring, $\chi_i(G) \geq \chi(G)$ and thus $\chi_i(G \square H) \geq \chi(G \square H)$. The families of graphs considered for our discussion till now are examples of graphs for which $\chi_i(G \square H) = \chi(G \square H)$. But we do have examples of non-trivial graphs $G$ and $H$ for which $\chi_i(G \square H) > \chi(G \square H)$. This is done in Proposition 1 and Theorem 6.

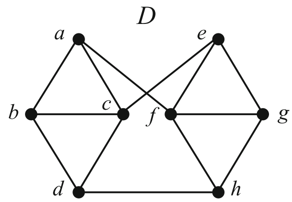


**Fig. 1.** Graph with $\chi_i(D) = \chi(D) + 1 = 4$.

**Proposition 1.** *Let $D$ be the graph given in Fig. 1. Then $\chi_i(D \square K_2) > \chi(D \square K_2)$.*

*Proof.* Let us consider the graph $D$ given in Fig. 1. Clearly $D$ is a uniquely colorable graph such that $\chi(D) = 3$ and $\chi_i(D) = 4$ (see, [3]). Let us consider the graph $D \square K_2$. Clearly $D \square K_2$ contains two copies of $D$. Let us denote these copies by $D_1$ and $D_2$. Let the vertices of $D_1$ be $a, b, \ldots, h$ as shown in Fig. 1, and its corresponding vertices of $D_2$ be denoted by $a', b', \ldots, h'$ respectively. By the definition of Cartesian product, $aa', bb', \ldots, hh' \in E(D \square K_2)$. It is clear that $\chi(D \square K_2) = 3$. Let the colors set be $\{1, 2, 3\}$. We have to show that there is no winning strategy for Ann using 3 colors. In any 3-coloring of $D \square K_2$ the vertices $\{a, d, g\}$, $\{b, e, h\}$ and $\{c, f\}$ should receive the same color $c_1, c_2$ and $c_3$ respectively and the vertices $\{a', d', g'\}$, $\{b', e', h'\}$ and $\{c', f'\}$ should receive the same color $c_2, c_3$ and $c_1$ respectively or $c_3, c_1$ and $c_2$ respectively such that $\{c_1, c_2, c_3\} = \{1, 2, 3\}$. Let Ann start by presenting the vertex $a$ in $D_1$ and let the color given by Ben be 1. Let the following be the strategy followed by Ben.

(i) color the vertex $d$ with 2 or 3, (or) color the vertex $d'$ with 1.
(ii) color any one of the vertex of $\{f, g, e', h'\}$ with 2.

If Ben is able to accomplish one of the above, then clearly Ann does not have a winning strategy. In order to avoid (i), she has to present the vertices in the order $b, c, d, d'$. But even in this case Ann cannot prevent Ben from applying (ii). Thus Ben wins the game on $D \square K_2$ with 3 colors and hence $\chi_i(D \square K_2) > \chi(D \square K_2)$.

This idea for $D \square K_2$ can be generalised to $D \square T$ where $T$ is any tree.

**Theorem 6.** *Let D be the graph given in Fig. 1 and T be any tree. Then* $\chi_i(D \square T) > \chi(G \square T)$.

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## References

1. Francis, P., Francis Raj, S.: Indicated Coloring of Graphs (Preprint)
2. Francis Raj, S., Pandiya Raj, R., Patil, H.P.: On indicated chromatic number of graphs. Graphs Combin. **33**, 203–219 (2017)
3. Grzesik, A.: Indicated coloring of graphs. Discrete Math. **312**, 3467–3472 (2012)
4. Lason, M.: Indicated coloring of matroids. Discrete Appl. Math. **179**, 241–243 (2014)
5. Pandiya Raj, R., Francis Raj, S., Patil, H.P.: On indicated coloring of graphs. Graphs Combin. **31**, 2357–2367 (2015)
6. Szekeres, G., Wilf, H.S.: An inequality for the chromatic number of a graph. J. Combin. Theory **4**, 1–3 (1968)
7. Vizing, V.G.: The Cartesian product of graphs. Vycisl. Sistemy. **9**, 30–43 (1963)
8. West, D.B.: Introduction to Graph Theory. Prentice-Hall, Englewood Cliffs (2000)
9. Zhu, X.: The game coloring number of planar graphs. J. Combin. Theory Ser. B. **75**, 245–258 (1999)