(1, 2)-Domination in Graphs

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Abstract. A $(1, 2)$-dominating set in a graph $G = (V, E)$ is a set having the property that for every vertex $v \in V - S$, there is at least one vertex in S at a distance 1 from v and a second vertex in S at a distance at most 2 from v. The $(1, 2)-$domination number of G, denoted by $\gamma_{1,2}(G)$, is the minimum cardinality of a $(1, 2)-$dominating set of G. In this paper, we have derived bounds of $\gamma_{1,2}$ in terms of the order and the maximum degree. For trees, we get the bounds in terms of the number of pendant vertices. We have also characterized the graphs G of order n, for which $\gamma_{1,2}(G) = n, n - 1, n - 2$.

Keywords: Domination · (1, 2)-dominating set

1 Introduction

Hedetniemi et al. [3] introduced the concept of $(1, k)$-domination in graphs. Let k be a positive integer. A subset S of vertices is called a $(1, k)$-dominating set in G if for every vertex $v \in V - S$, there are two distinct vertices $u, w \in S$ such that u is adjacent to v, and w is within distance k of v (*i.e.* $d_G(v, w) \leq k$). Hedetniemi et al. [4,5] examined $(1, k)$-domination along with the internal distances in $(1, k)$-dominating sets. Factor and Langley [1,2] studied $(1, 2)$-domination of digraphs.

In this paper, we study $(1, 2)$-domination in graphs. All our graphs are finite and simple.

2 Bounds of $\gamma_{1,2}$ in terms of Δ

We start with the following observations.

Observation 1. *For any two graphs G and H, $\gamma_{1,2}(G \cup H) = \gamma_{1,2}(G) + \gamma_{1,2}(H)$.*

Observation 2. *If H is a spanning supergraph of G, then $\gamma_{1,2}(H) \leq \gamma_{1,2}(G)$.*

Theorem 1. *If G is a graph of order $n \geq 4$ with $\Delta(G) \geq n - 2$, then*

$$\gamma_{1,2}(G) = \begin{cases} 2 & \text{if } G \text{ is connected} \\ 3 & \text{if } G \text{ is disconnected.} \end{cases}$$

S. Arumugam et al. (Eds.): ICTCSDM 2016, LNCS 10398, pp. 128–133, 2017.
DOI: 10.1007/978-3-319-64419-6_17

Proof. When $\Delta(G) = n - 1$, let u be a full-degree vertex; and v be any other vertex in G. Then $\{u, v\}$ is a $(1, 2)$-dominating set and so $\gamma_{1,2}$(G) =2.

When $\Delta(G) = n - 2$, let u be a vertex of degree $n - 2$; and v be the vertex which is not adjacent to u.

Case 1. G is connected.
Let w be a neighbour of v. Then $\{u, w\}$ is a $(1, 2)$-dominating set and so $\gamma_{1,2}(G) =$ 2.

Case 2. G is disconnected.
Then v is an isolated vertex. Let S be any $(1, 2)$-dominating set of G. Since every isolated vertex must lie in S, $\gamma_{1,2}(G) \geq 3$. Clearly $\{u, v, x\}$ is a $(1, 2)$-dominating set for every $x \in N(u)$ and $\gamma_{1,2}(G) = 3$.

Corollary 1. $\gamma_{1,2}(G) = 2$ *for the graphs* $G = K_n, K_{1,n}, W_n, F_n$ *and* $H + K_1$ *where* H *is any graph.*

Theorem 2. *Let* G *be a connected graph of order* $n \geq 5$ *with* $2 \leq \Delta(G) \leq n-3$. *Then* $\gamma_{1,2}(G) \leq n - \Delta(G)$.

Proof. Let G be a connected graph with the given hypothesis. Let $\Delta(G) = n-1-k$. Then $2 \leq k \leq n-3$ and $n - \Delta(G) = k+1$. Let $V(G) = \{u, v_i | 1 \leq i \leq n-1\}$, where u is a vertex of degree $\Delta(G)$, and $N(u) = \{v_{k+1}, v_{k+2}, ..., v_{n-1}\}$. Then $V(G) = N[u] \cup V_1$, where $V_1 = \{v_1, v_2, ..., v_k\}$. Since G is connected, at least one vertex in V_1 has a neighbour in $N(u)$.

Case 1. Every vertex in V_1 has some neighbour in $N(u)$.
Without loss of generality, assume that v_i is adjacent to v_{j_i} in $N(u)$, for $1 \leq i \leq k$. The vertices $v_{j_1}, v_{j_2}, ..., v_{j_k}$ need not be distinct. Let $V_2 = \{v_{j_i} | 1 \leq i \leq k\} \subseteq N(u)$. Let $S = \{u, v_{j_1}, v_{j_2}, ..., v_{j_k}\} (= V_2 \cup \{u\})$.

Every vertex $v_i \in N(u) - S$ is adjacent to u and at a distance at most 2 from v_{j_1}. Every $v_i \in V_1$ is adjacent to v_{j_i} and at a distance 2 from u. Hence S is a $(1, 2)$-dominating set and so $\gamma_{1,2}(G) \leq k+1$.

Case 2. Some vertex in V_1 has no neighbour in $N(u)$.
Without loss of generality, let $V_1^{'} = \{v_1, v_2, ..., v_r\} \subseteq V_1$ be the set of vertices that have no neighbours in $N(u)$. Let $V_1^{''} = V_1 - V_1^{'} = \{v_{r+1}, v_{r+2}, ..., v_k\}$. Then $V(G) = N[u] \cup V_1^{'} \cup V_1^{''}$. Since G is connected, at least one vertex in $V_1^{'}$ is adjacent to some vertex in $V_1^{''}$. Without loss of generality, let v_1 be adjacent to v_{r+1}. Without loss of generality, assume that v_i is adjacent to v_{j_i} in $N(u)$, for $r+1 \leq i \leq k$. The vertices $v_{j_{r+1}}, v_{j_{r+2}}, ..., v_{j_k}$ need not be distinct. Let $V_2 = \{v_{j_i} | r+1 \leq i \leq k\} \subseteq N(u)$.

Let $S = \{u, v_{j_{r+1}}, v_{j_{r+2}}, ..., v_{j_k}, v_1, v_2, ..., v_r\} (= V_2 \cup V_1^{'} \cup \{u\})$. Every $v_i \in N(u) - V_2$ is adjacent to u and at a distance at most 2 from $v_{j_{r+1}}$. Every $v_i \in V_1^{''}$ is adjacent to v_{j_i} and at a distance 2 from u. Hence S is a $(1, 2)$-dominating set and so $\gamma_{1,2}(G) \leq k+1$.

A *wounded spider* is the graph formed by subdividing at most $n-1$ of the edges of a star $K_{1,n}$ for $n \geq 2$. Let $WS_{n,t}$ denote the wounded spider formed by subdividing t edges of $K_{1,n}$, $1 \leq t \leq n-1$.

Corollary 2. $\gamma_{1,2}(WS_{n,t}) = t + 1$.

Proof. Let $V[WS_{n,t}] = \{u, v_1, v_2, ..., v_n, v_1^{'}, v_2^{'}, ..., v_t^{'}\}$ and $E[WS_{n,t}] = \{uv_j, v_iv_i^{'}| 1 \leq j \leq n, 1 \leq i \leq t\}$. Let S be any $(1, 2)$-dominating set of $WS_{n,t}$. For $1 \leq i \leq t$, to dominate v_i, either $v_i \in S$ or $v_i^{'} \in S$. Moreover, for $t+1 \leq j \leq n$, to dominate v_j, *either* $u \in S$ *or* $v_j \in S$. Therefore, $|S| \geq t + 1$.

Note that $t = n - \Delta - 1$. When $t \geq 2$, then $\Delta(WS_{n,t}) \leq n - 3$; and by Theorem 7, $\gamma_{1,2}(WS_{n,t}) \leq t + 1$. Hence $\gamma_{1,2}(WS_{n,t}) = t + 1$.

When $t = 1$, then $\Delta(WS_{n,t}) \geq n - 2$; and by Theorem 1, $\gamma_{1,2}(WS_{n,t}) = 2$.

3 Composition of Two Graphs

Theorem 3. *Let G be a non-trivial connected graph. Then for any graph H, $\gamma_{1,2}(GoH) = |V(G)|$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(H) = \{u_1, u_2, ..., u_s\}$. Let $H_1, H_2, ..., H_n$ denote the copies of H, where every vertex of H_i is adjacent to v_i, $1 \leq i \leq n$. Let $V(H_i) = \{u_1^i, u_2^i, ..., u_s^i\}$. Let S be any $(1, 2)$-dominating set of GoH. Since there is no adjacency between the vertices in H_i and H_j for $i \neq j$, every u_r^i in H_i is adjacent to either v_i or u_k^i, where $u_k^i \in N[u_r^i]$. Hence for each i, $1 \leq i \leq n$, to dominate $V(H_i)$, we need at least one vertex in S. Hence $\gamma_{1,2}(GoH) \geq n$. Let $S_1 = \{v_1, v_2, ..., v_n\}$. For every $u_r^i, 1 \leq i \leq n, 1 \leq r \leq s, d_{GoH}(u_r^i, v_i) = 1$ *and* $d_{GoH}(u_r^i, v_j) = 2$ for every $v_j \in N_G(v_i)$. Hence S_1 is a $(1, 2)$-dominating set and so $\gamma_{1,2}(GoH) = n$.

Corollary 3. *Let G be any graph having t isolates. Then for any graph H, $\gamma_{1,2}(GoH) = |V(G)| + t$, where $t \geq 0$.*

Proof. Let $G_1, G_2, ..., G_k$ be the components of G.

Then $\gamma_{1,2}(GoH) = \sum_{i=1}^{k} \gamma_{1,2}(G_ioH)$.

Case 1. $t = 0$.
Since each G_i is connected, by Theorem 3, $\gamma_{1,2}(G_ioH) = |V(G_i)|$. Hence $\gamma_{1,2}(GoH) = |V(G)|$.

Case 2: $t \neq 0$.
Without loss of generality, let $G_1, G_2, ..., G_t$ denote the components of order 1. Then G_ioH has a full - degree vertex; and so by Theorem 1, $\gamma_{1,2}(G_ioH) = 2$, for $1 \leq i \leq t$. By Theorem 3, $\gamma_{1,2}(G_ioH) = |V(G_i)|$, for $t + 1 \leq i \leq k$. Thus, we get the result.

4 Some Characterizations

Theorem 4. *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{1,2}(G) = n$ if and only if $n = 2$.*

Proof. When $G = K_2$, the result is obvious. Conversely, suppose that $n \neq 2$.

Claim. $\gamma_{1,2}(G) < n$.

We prove this result by induction on n.

When $n = 3$, a set of any two vertices of G is a $(1, 2)$-dominating set of G and so $\gamma_{1,2}(G) = 2 < n$.

Assume the result for $n = k$ with $k \geq 3$.

Next, let G be a connected graph of order $n = k + 1$. Let v be a vertex that is not a cut vertex in G. Then $G - v$ is connected, and of order $n - 1 = k$. By the induction hypothesis, $G - v$ has a $(1, 2)$-dominating set S with $|S| < k$. (i.e.) $|S| \leq k - 1$.

Let u be a neighbour of v.

Case 1. $u \in S$.
Since G is connected, $n \geq 3$ and v is not a cut-vertex in G, u has another neighbour (say) w. Then $S \cup \{w\}$ is a $(1, 2)$-dominating set in G and so $\gamma_{1,2}(G) \leq k < n$.

Case 2. $u \notin S$.
Since S is a $(1, 2)$-dominating set in $G - v$, there exists a vertex $w \in S$ that is adjacent to u. Then $S \cup \{u\}$ is a $(1, 2)$-dominating set in G and so $\gamma_{1,2}(G) \leq k < n$.

Thus, by induction, the result follows.

Theorem 5. *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{1,2}(G) = n - 1$ iff $n = 3$. i.e. $\gamma_{1,2}(G) = n - 1$ iff $G = P_3$ or K_3.*

Proof. When $n = 3$, a set of any two vertices of G is a $(1, 2)$-dominating set of G and so $\gamma_{1,2}(G) = 2 = n - 1$. Conversely, suppose that $n \neq 3$.

Claim. $\gamma_{1,2}(G) < n - 1$.

We shall prove this result by induction on n.

When $n = 4$, since G is connected, $\Delta(G) \geq 2$. Now, any two adjacent vertices form a (1, 2)-dominating set and so $\gamma_{1,2}(G) = 2 < n - 1$.

Assume the result for $n = k$ with $k \geq 4$.

Next, let G be a connected graph of order $n = k + 1$. The rest of the proof is similar to the proof of Theorem 4.

Corollary 4. *Let G be any graph of order n. Then*

(i) $\gamma_{1,2}(G) = n$ *iff* $G = sK_1 \cup \frac{n-s}{2} K_2$, *with* $0 \leq s \leq n$.
(ii) $\gamma_{1,2}(G) = n - 1$ *iff* $G = sK_1 \cup \frac{n-s-3}{2} K_2 \cup H$, *where* $H \cong P_3$ *or* K_3, *with* $0 \leq s \leq n - 3$.

Theorem 6. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{1,2}(G) = n - 2$ iff $G = P_5$ or G is of order 4.*

Proof. If $G = P_5$ or G is of order 4, it is easy to verify that $\gamma_{1,2}(G) = n - 2$. Conversely, suppose that

$$\gamma_{1,2}(G) = n - 2. \tag{1}$$

Let $n = 5$. If $\Delta(G) \geq 3(= n-2)$, then $\gamma_{1,2}(G) = 2$ (by Theorem 1), contradicting (1). If $\Delta(G) = 2$, then G is either P_5 or C_5; but $\gamma_{1,2}(C_5) = 2$, and so $G = P_5$. Now, let $n \geq 6$.

Claim. $\gamma_{1,2}(G) < n - 2$.

We shall prove this result by induction on n.

When $n = 6$, if $\Delta(G) \geq n - 2$, then $\gamma_{1,2}(G) = 2$ (by Theorem 1); if $3 \leq \Delta(G) \leq n - 3$, then $\gamma_{1,2}(G) \leq n - 3$ (by Theorem 7); if $\Delta(G) = 2$, then G is either P_6 or C_6 and $\gamma_{1,2}(G) \leq 3$; and in all these cases, we get a contradiction to (1).

Assume the result for $n = k$ with $k \geq 6$. Next, let G be a connected graph of order $n = k + 1$. The rest of the proof is similar to the proof of Theorem 4.

Corollary 5. *For any graph G of order n, $\gamma_{1,2}(G) = n - 2$ iff G is one of the following graphs:*

(i) $G = sK_1 \cup \frac{n-6-s}{2} K_2 \cup H$, *where* $H = 2P_3, 2K_3$ *or* $P_3 \cup K_3$, *with* $0 \leq s \leq n-6$.
(ii) $G = sK_1 \cup P_5 \cup \frac{n-5-s}{2} K_2$, *with* $0 \leq s \leq n - 5$.
(ii) $G = sK_1 \cup \frac{n-4-s}{2} K_2 \cup H$ *where H is a connected graph of order* 4, *with* $0 \leq s \leq n - 4$.

5 Trees

Theorem 7. *Let T be a tree of order $n \geq 2$. Then $\gamma_{1,2}(T) = 2$ if and only if T is a Star or Double Star.*

Proof. Suppose that $\gamma_{1,2}(T) = 2$. Let $S = \{u, v\}$ be a $(1, 2)$-dominating set of T. Then every vertex in T is adjacent with either u or v. Hence $V(T) = N[u] \cup N[v]$. Then for every $x \in N(u)$ and $y \in V(T)$, $d(x, y) \leq d(x, u) + d(u, y) \leq 3$; similarly, for every $x \in N(v)$ and $y \in V(T)$, $d(x, y) \leq 3$; for every $x \in V - \{u, v\}$, $d(u, x) + d(x, v) \leq 3$; and so $d(u, v) \leq 3$. Hence $diam(T) \leq 3$; and so T is a Star or a Double Star $D_{r,\ s}$ (where $r + s = n - 2$). Converse is obvious.

Theorem 7 deals with the trees of diameter 2 and 3. The next result deals with trees of diameter ≥ 4.

Theorem 8. *Let T be a tree of order n with r pendant vertices. Then*

(i) $3 \leq \gamma_{1,2}(T) \leq n - r$, *if* $diam(T) \geq 5$
(ii) $\gamma_{1,2}(T) = n - r$, *if* $diam(T) = 3$ *or* 4.

Proof. Let $diam(T) \geq 3$. Let V_1 denote the set of all pendant vertices in T. Then $|V - V_1| \geq 2$ and $V - V_1$ is a $(1, 2)$-dominating set; and so

$$\gamma_{1,2}(T) \leq n - r. \tag{2}$$

Using Theorem 7, $\gamma_{1,2}(T) \geq 3$; and so (i) follows.

When $diam(T) = 3$, T is a double star; and by Theorem 7, $\gamma_{1,2}(T) = 2 = n-r$. When $diam(T) = 4$, $diam(T - V_1) = 2$; and so $T - V_1$ is $K_{1,\ n-r-1}$, where $n-r-1 \geq 2$. Let $V(K_{1,\ n-r-1}) = \{u, u_1, u_2, ..., u_{n-r-1}\}$. For $1 \leq j \leq d_T(u_i)-1$, let v_{i_j} denote a pendant vertex adjacent to u_i. For $1 \leq t \leq d_T(u) - n - r - 1$, let w_t denote a pendant vertex adjacent to u. (If $d_T(u) = n - r - 1$, then there is no w_t's).

By (2), $\gamma_{1,2}(T) \leq n - r$. Assume the contrary that $\gamma_{1,2}(T) \neq n - r$. Then there is a $(1, 2)$-dominating set S_1 of cardinality $n - r - 1$.

If $S_1 = \{u_1, u_2, ..., u_{n-r-1}\}$, then there is no vertex at a distance 2 from v_{i_j}, for $1 \leq i \leq n - r - 1$ and $1 \leq j \leq d_T(u_i) - 1$, which is a contradiction.

Then $u_i \notin S_1$, for some i, $1 \leq i \leq n - r - 1$. Without loss of generality, let $u_1, u_2, ..., u_k \notin S_1$ and $u_{k+1}, u_{k+2}, ..., u_{n-r-1} \in S_1$, where $1 \leq k \leq n - r - 1$. For $1 \leq i \leq k$, $u_i \notin S_1$; and so all v_{i_j}'s must lie in S_1. But $|S_1| = n - r - 1$. Hence it follows that, $d(u_i) = 2$ for $i = 1, 2, 3, ..., k$, and $S_1 = \{v_{1_1}, v_{2_1}, ..., v_{k_1}, u_{k+1}, u_{k+2}, ..., u_{n-r-1}\}$.

Case 1. $k = n - r - 1$.
Now $S_1 = \{v_{1_1}, v_{2_1}, ..., v_{(n-r-1)_1}\}$; and so u is not $(1, 2)$-dominated by S_1, a contradiction.

Case 2. $k < n - r - 1$.
Now there is no vertex in S_1 at a distance at most 2 from v_{s_j}, $k+1 \leq s \leq n-r-1$, a contradiction.
Hence $\gamma_{1,2}(T) = n - r$.

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