Distance Antimagic Labelings of Graphs

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Abstract. Let $G = (V, E)$ be a graph of order n. Let $f : V(G) \to \{1, 2, \ldots, n\}$ be a bijection. For any vertex $v \in V$, the neighbor sum $\sum_{u \in N(v)} f(u)$ is called the weight of the vertex v and is denoted by $w(v)$. If $w(x) \neq w(y)$ for any two distinct vertices x and y, then f is called a distance antimagic labeling. A graph which admits a distance antimagic labeling is called a distance antimagic graph. If the weights form an arithmetic progression with first term a and common difference d, then the graph is called an (a, d)-distance antimagic graph.

In this paper we prove that the hypercube Q_n is an (a, d)-distance antimagic graph. Also, we present several families of disconnected distance antimagic graphs.

Keywords: (a, d)-distance antimagic graph · Distance antimagic graph

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. We further assume that G has no isolated vertices. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

A *distance magic labeling* of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k such that $\sum_{y \in N(x)} f(y) = k$ for every $x \in V$. The constant k is called the *magic constant* of the labeling f.

The sum $\sum_{y \in N(x)} f(y)$ is called the *weight* of the vertex x and is denoted by $w(x)$.

Let G be a distance magic graph of order n with labeling f and magic constant k. Then $\sum_{u \in N_{G^c}(v)} f(u) = \frac{n(n+1)}{2} - k - f(v)$, and hence the set of all

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vertex weights in G^c is $\{\frac{n(n+1)}{2} - k - i : 1 \leq i \leq n\}$, which is an arithmetic progression with first term $a = \frac{n(n+1)}{2} - k - n$ and common difference $d = 1$.

Motivated by this observation, in [1] we introduced the following concept of (a, d)-distance antimagic graph.

Definition 1. [1] *A graph G is said to be (a, d)-distance antimagic if there exists a bijection $f : V \to \{1, 2, \ldots, n\}$ such that the set of all vertex weights is $\{a, a + d, a + 2d, ..., a + (n-1)d\}$ and any graph which admits such a labeling is called an (a, d)-distance antimagic graph.*

Thus the complement of every distance magic graph is an $(a, 1)$-distance antimagic graph.

We observe that if a graph G is (a, d)-distance antimagic with $d > 0$, then for any two distinct vertices u and v we have $w(u) \neq w(v)$. This observation naturally leads to the concept of distance antimagic labeling.

Definition 2. [3] *Let $G = (V, E)$ be a graph of order n. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection. If $w(x) \neq w(y)$ for any two distinct vertices x and y in V, then f is called a distance antimagic labeling. Any graph G which admits a distance antimagic labeling is called a distance antimagic graph.*

Definition 3. *The K_2-bistar graph $K_2(m, n)$ is the graph obtained by joining m copies of K_2 to a vertex of K_2 and n copies of K_2 to the other vertex of K_2.*

In this paper we prove that the hypercube Q_n is an (a, d)-distance antimagic graph. Also, we present several families of disconnected distance antimagic graphs.

2 Main Results

The following theorem gives an (a, d)-distance antimagic labeling of hypercubes.

Theorem 1. *For every $n \geq 3$, the hypercube Q_n is (a, d)-distance antimagic, where $a = 2^n + 2$ and $d = n - 2$. Moreover there exists an (a, d)-distance antimagic labeling $f_n : V(Q_n) \to \{1, 2, \ldots, 2^n\}$ such that if $f_n(v) = j$, then $w_{f_n}(v) = 2^n + 1 + (n-2)j, 1 \leq j \leq 2^n$.*

Proof. We prove this result by induction on n. For Q_3, the labeling f_3 given in Fig. 1 is a $(10, 1)$-distance antimagic labeling satisfying the condition that $w_{f_3}(j) = 9 + j = 2^n + 1 + j, 1 \leq j \leq 8$. We now assume that the theorem is true for Q_n. Let $f_n : V(Q_n) \to \{1, 2, 3, \ldots, 2^n\}$ be a $(2^n + 2, n - 2)$-distance antimagic labeling of Q_n such that if $f_n(v) = j$, then $w_{f_n}(v) = 2^n + 1 + j$ for all $j, 1 \leq j \leq 2^n$. Let $Q_n^{(1)}$ and $Q_n^{(2)}$ be two copies of Q_n in Q_{n+1}, with a perfect matching M consisting of edges joining a vertex of $Q_n^{(1)}$ with the corresponding vertex of $Q_n^{(2)}$. Now (See Fig. 2) define $f_{n+1} : V(Q_{n+1}) \to \{1, 2, \ldots, 2^{n+1}\}$ by

$$f_{n+1}(v) = \begin{cases} f_n(v) & \text{if } v \in V(Q_n^{(1)} \\ f_n(v_1) + 2^n & \text{if } v_1 \in V(Q_n^{(2)} \text{ and } vv_1 \in M \end{cases}$$

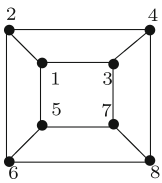

Fig. 1. Q_3 with $(10,1)$-distance antimagic labeling

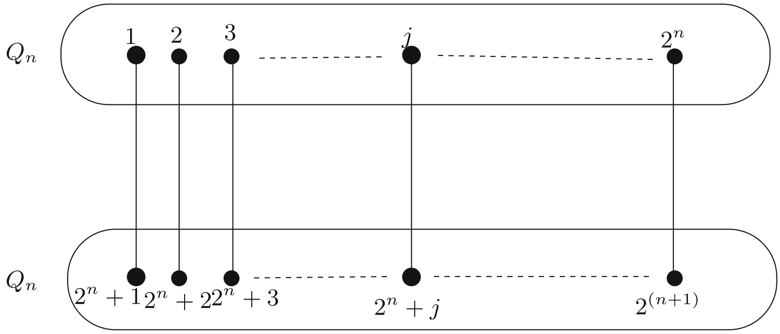

Fig. 2. Q_{n+1} with (a,d)-distance antimagic labeling

If $f_{n+1}(v) = j, 1 \leq j \leq 2^n$, then

$$\begin{aligned} w_{f_{n+1}}(v) &= w_{f_n}(v) + 2^n + j \\ &= 2^n + 1 + (n-2)j + 2^n + j \\ &= (2^{n+1} + 1) + (n-1)j. \end{aligned}$$

If $f_{n+1}(v_1) = j$, where $2^n + 1 \leq j \leq 2^{n+1}$ and $vv_1 \in M$, then

$$\begin{aligned} w_{f_{n+1}}(v_1) &= w_{f_{n+1}}(v) + n2^n + j \\ &= (1 + 2^n) + (n-2)j + n2^n + j \\ &= (1 + 2^{n+1}) + (n-1)(2^n + j). \end{aligned}$$

Thus, $w_f^{(n+1)}(j) = (1 + 2^{n+1}) + (n-1)j, j = 1, 2, 3, \ldots, 2^{n+1}$ and by induction the proof is complete.

Theorem 2. *The bistar $G = K_2(n,n)$ is distance antimagic.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u, v\}$ and $E(G) = \{u_i u : 1 \leq i \leq n\} \cup \{v_i v : 1 \leq i \leq n\} \cup \{uv\}$. Define $f : V(G) \to \{1, 2, \ldots, 2n+2\}$, by

$$f(x) = \begin{cases} 2i & \text{if } x = u_i,\ i \leq i \leq n \\ 2i+1 & \text{if } x = v_i,\ i \leq i \leq n \\ 1 & \text{if } x = v \\ 2n+2 & \text{if } x = u \end{cases}$$

Then

$$w(x) = \begin{cases} 2i & \text{if } v = u_i,\ i \le i \le n \\ 2i+1 & \text{if } v = v_i,\ i \le i \le n \\ 1 & \text{if } v = x \\ 2n+2 & \text{if } v = y \end{cases}$$

Hence f is a distance antimagic labeling of G.

Theorem 3. *Let G be an r-regular graph of order n. If G is distance antimagic, then $2G$ is also distance antimagic.*

Proof. Let f be a distance antimagic labeling of G. Let G_1 and G_2 be the two copies of G in $2G$.

Define $g : V(2G) \to \{1, 2, \ldots, 2n\}$ by

$$g(u) = \begin{cases} f(u) & \text{if } u \in V(G_1) \\ f(u) + n & \text{if } u \in V(G_2) \end{cases}$$

Let $u, v \in V(G_1 \cup G_2)$. Then

$$w_g(u) = \begin{cases} w_f(u) & \text{if } u \in V(G_1) \\ w_f(u) + rn & \text{if } u \in V(G_2) \end{cases}$$

Hence it follows that $w_g(u) \neq w_g(v)$ if $u, v \in V(G_1)$ or $u, v \in V(G_2)$.

Now, let $u \in V(G_1)$ and $v \in V(G_2)$. Since $w_f(u) \neq w_f(v)$, without loss of generality we assume that $w_f(u) < w_f(v)$. Now, $w_g(u) = w_f(u) < w_f(v) < w_f(v) + rn = w_g(v)$. Thus g is a distance antimagic labeling of $2G$.

Theorem 4. *Let H be the graph obtained from the cycle C_3 by attaching a pendent vertex at one vertex. Let G be the union of r copies of H. Then G is distance antimagic.*

Proof. Let H_i be the i^{th} copy of H in G. Let $V(H_i) = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}\}$ and $E(H_i) = \{(u_{i1}, u_{i2}), (u_{i1}, u_{i3}), (u_{i2}, u_{i3}), (u_{i2}, u_{i4})\}$.

Define $f : V(G) \to \{1, 2, \ldots, 4r\}$, by

$$f(u_{ij}) = \begin{cases} 4(i-1)+1, & \text{if } j = 1 \\ 4(i-1)+2, & \text{if } j = 2 \\ 4(i-1)+3, & \text{if } j = 3 \\ 4(i-1)+4, & \text{if } j = 4 \end{cases}$$

where $1 \le i \le r$.

The vertex weights are given by

$$w(u_{ij}) = \begin{cases} 8i-3, & \text{if } j = 1 \\ 12i-4, & \text{if } j = 2 \\ 8i-5, & \text{if } j = 3 \\ 4i-2, & \text{if } j = 4 \end{cases}$$

Clearly the vertex weights are distinct.

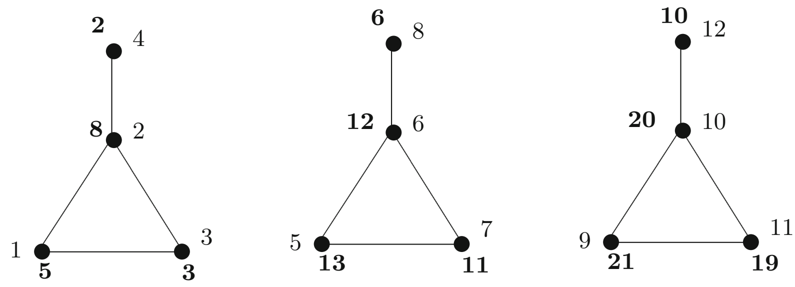

Fig. 3. Distance antimagic labeling of union of 3-pan

Example 1. The distance antimagic labeling of 3 copies of H is given in Fig. 3.

Theorem 5. *For $n = 2k+1$, let H_k be the graph obtained from the path $(u_1, u_2, \ldots, u_{2k+1})$ by adding the edges (u_i, u_{i+2}) where i is odd. Let G be the union of r copies of H_3 where $n \geq 1$. Then G is distance antimagic.*

Proof. Let G_i be the i^{th} copy of H_3 in G, given in Fig. 4.

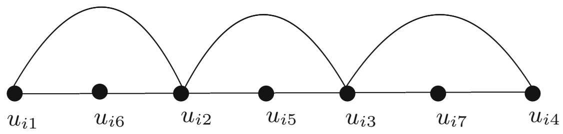

Fig. 4. The graph H_3

Define $f : V(G) \to \{1, 2, \ldots, 4r\}$ by

$$f(u_{ij}) = \begin{cases} 7(i-1)+1 & \text{if } j = 1 \\ 7(i-1)+2 & \text{if } j = 2 \\ 7(i-1)+3 & \text{if } j = 3 \\ 7(i-1)+4 & \text{if } j = 4 \\ 7(i-1)+5 & \text{if } j = 5 \\ 7(i-1)+6 & \text{if } j = 6 \\ 7(i-1)+7 & \text{if } j = 7 \end{cases}$$

The vertex weights are given by

$$w(u_{ij}) = \begin{cases} 14i-6 & \text{if } j = 1 \\ 28i-13 & \text{if } j = 2 \\ 28i-10 & \text{if } j = 3 \\ 14i-4 & \text{if } j = 4 \\ 14i-9 & \text{if } j = 5 \\ 14i-11 & \text{if } j = 6 \\ 14i-7 & \text{if } j = 7 \end{cases}$$

Clearly the vertex weights are distinct.

3 Conclusion and Scope

We have proved the existence of distance antimagic labeling of some families of disconnected graphs and the hypercube Q_n. The existence of distance antimagic labelings for various graph products and other families of disconnected graphs are problems for further investigation.

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