Degree Associated Reconstruction Number of Split Graphs with Regular Independent Set

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Abstract. A vertex-deleted subgraph of a graph G with which the degree of the deleted vertex is given is called a *degree associated card* of G. The *degree associated reconstruction number* (or *drn*) of a graph G is the size of the smallest collection of the degree associated cards of G that uniquely determines G. A *split graph* G is a graph in which the vertices can be partitioned into an independent set and a clique. We prove that the *drn* is 1 or 2 for all split graphs such that all the vertices in the independent set have equal degree, except four graphs on six vertices and for these exceptional graphs, the *drn* is 3.

Keywords: Isomorphism · Reconstruction number · Split graph

1 Introduction

All graphs considered are simple and finite. We shall mostly follow the graph theoretic terminology of [8]. A *vertex-deleted subgraph* or *card* $G - v$ of a graph (digraph) G is the unlabeled graph obtained from G by deleting the vertex v and all edges incident with v. The *deck* of a graph (digraph) G is the collection of all its cards. Following the formulation in [7], a graph (digraph) G is *reconstructible* if it can be uniquely determined from its deck. The well-known Reconstruction Conjecture (RC) of Kelly [11] and Ulam [20] has been open for more than 50 years. It asserts that every graph G with at least three vertices is reconstructible. The conjecture has been proved for many special classes, and many properties of G may be deduced from its deck. Nevertheless, the full conjecture remains open. Surveys of results on RC and related problems include [7,16]. For a reconstructible graph G, Harary and Plantholt [10] defined the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \ncong G$, these cards thus uniquely identifying G. Reconstruction number is known for only few classes of graphs [5].

An extension of RC to digraphs is the *Digraph Reconstruction Conjecture* (DRC), proposed by Harary [9]. The DRC was disproved by Stockmeyer [19]

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by exhibiting several infinite families of counter-examples. Ramachandran then proposed a variation in the DRC and introduced the degree associated reconstruction [14] and the degree associated reconstruction number [15] of graphs (digraphs).

The ordered triple (a, b, c) where a, b and c are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pair of arcs incident with v in a digraph D is called the *degree triple of* v. The *degree associated card* or *dacard* of a digraph (graph) is a pair (d, C) consisting of a card C and the degree triple (degree) d of the deleted vertex. The *degree associated deck* (or *dadeck*) of a graph (digraph) is the collection of all its dacards. A digraph is said to be *N-reconstructible* if it can be uniquely determined from its dadeck. The *new digraph reconstruction conjecture* (NDRC) asserts that all digraphs are N-reconstructible. The *degree* (*degree triple*) *associated reconstruction number* of a graph (digraph) G is the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other graph H, $H \ncong G$, this subcollection of dacards thus uniquely identifying G. Articles [1–4, 6, 12, 13, 18] are recent papers on this parameter.

A *split graph* G is a graph in which the vertices can be partitioned into an independent set (say X) and a clique (say Y). Throughout this paper, we use the notation G, X and Y in the sense of this definition. The independent set X is said to regular if all the vertices in it have equal degree in G. Ramachandran and Monikandan proved [17] that the validity of the RC for all graphs is equivalent to the validity of the RC for all 2-connected graphs G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$. As many split graphs belong to this class of 2-connected graphs, to determine any reconstruction parameter for split graphs assumes important. In this paper, we prove that $drn(G) = 1$ or 2 for all split graphs G with regular independent set except four graphs on six vertices (Fig. 1) and for these exceptional four graphs, the drn is 3.

2 Drn of Split Graphs

The next theorem, due to Barrus and West [6], characterizes all graphs G with $drn(G) = 1$.

Theorem 1. *The dacard* (C, d) *belongs to the dadeck of only one graph* (*up to isomorphism*) *if and only if one of the following holds:*

(1) $d = 0$ *or* $d = |V(C)|$;
(2) $d = 1$ *or* $d = |V(C) - 1|$, *and* C *is vertex-transitive;*
(3) C *is complete or edgeless.*

In a graph G of order ν, a vertex with degree d is called a *d-vertex* and a $(\nu - 1)$-vertex is called a *complete vertex*. By $m(d(v), G - v)$, we mean m dacards each isomorphic to $(d(v), G - v)$. The *bistar* $B_{m,n}$ is the tree with $m + n + 2$ vertices whose central vertices have m and n leaf neighbours respectively. An *s-blocking set* of a graph G is a family $\mathscr{F}$ of graphs not isomorphic to G such

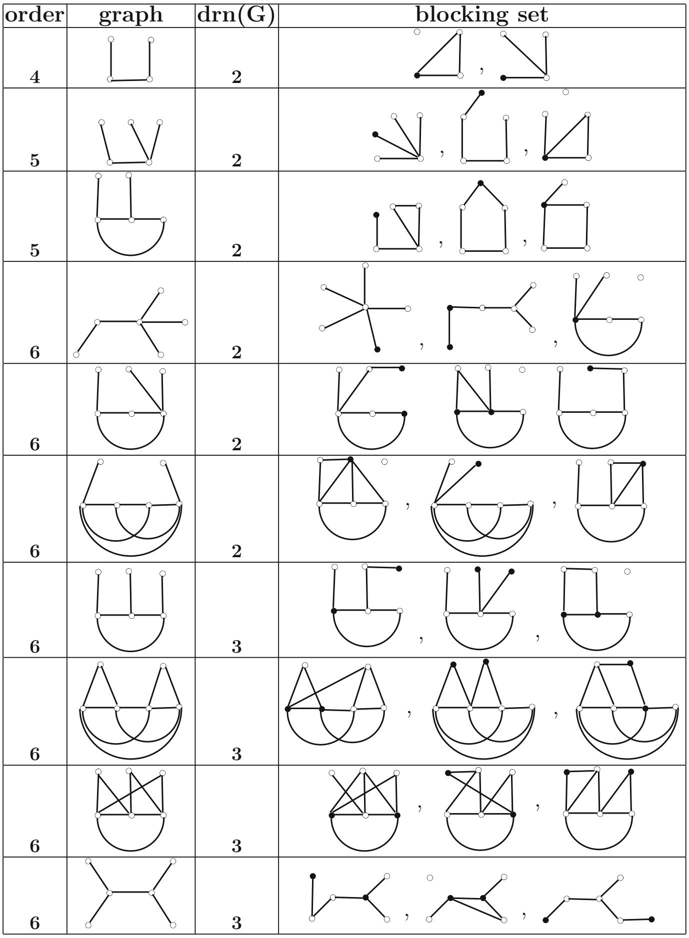

Fig. 1. Split graphs of order at most 6 with regular independent set and having *drn* 2 or 3.

that every collection of s dacards of G will appear in the dadeck of some graph of $\mathscr{F}$ and every graph in $\mathscr{F}$ will have s dacards in common with G.

Let $|X| = m > 0$, $|Y| = n > 0$ and let X be r-regular. Then clearly $0 \leq r \leq n$ and if r were 0 or n, then G would contain an isolated vertex or a complete vertex, which implies $drn(G) = 1$ by Theorem 1. Thus $1 \leq r \leq n - 1$. All split graphs G on at most six vertices with regular independent set, except the ten graphs given in the table in Fig. 1, must contain a complete vertex or an isolated vertex and so $drn(G) = 1$. The *drn* of these ten graphs is two or three (dark vertex of graphs given in Fig. 1 denotes the vertex whose removal results in a dacard common with G). So, we assume that *all split graphs G consider hereafter have order at least seven and, by Theorem 1, no isolated as well as complete vertices.*

Let Y_i denote the set of vertices in Y that are adjacent to exactly i vertices in X for $i = 0, 1, ..., m$. Then, in G, the degree of a vertex $v \in Y_i$ is $n - 1 + i$ for $i = 0, 1, ..., m$. Let $k_1, k_2, ..., k_t$ be integers, where $0 \leq k_1 < k_2 < ... < k_t \leq m$, such that $Y_{k_i} \neq \phi$ for all $i = 1, 2, ..., t$. Thus Y can be written as $\cup_{i=1}^{t} Y_{k_i}$.

An *extension* of a dacard $(d(v), G - v)$ of G is a graph obtained from the dacard by adding a new vertex w and joining it to $d(v)$ vertices of the dacard and it is denoted by $H(d(v), G - v)$ (or simply by H). Throughout this paper, H and w are used in the sense of this definition.

Theorem 2. *If G is a split graph with $r = n - 1$, then $drn(G) = 2$.*

Proof. We proceed on the value of k_1, which is the smallest integer such that Y_{k_1} is non empty.

If k_1 were equal to 0, then $|Y_0|$ would be equal to 1 (because Y_0 can contain at most only one vertex as $r = n - 1$) and since $n \geq 2$ and $r = n - 1$, it follows that Y_m would be nonempty, so G would contain a complete vertex, which is excluded.

Case 1. $k_1 = 1$.
If $n > 2$, then the vertex in Y_1 is adjacent to exactly one vertex, say s in X. Also, since $r = n - 1$, every other vertex in X is adjacent to all the vertices in $Y \setminus Y_1$. Moreover, the vertex s is non-adjacent to exactly one vertex in Y. Thus Y can be written as $Y = Y_1 \cup Y_{m-1} \cup Y_m$, where $|Y_1| = |Y_{m-1}| = 1$, which implies that $Y_m \neq \phi$ as $n > 2$. Hence G has a complete vertex, which is excluded.

If $n = 2$, then assume $m > 4$ (as otherwise $\nu \leq 6$, which is excluded). Clearly the partite set Y can be written as $Y = Y_1 \cup Y_{m-1}$, where $|Y_1| = |Y_{m-1}| = 1$. The dadeck of G consists of only the dacards $(m, K_2 \cup (m-1)K_1)$, $(2, K_{1,m-1} \cup K_1)$, $(1, K_{1,m})$ and $(m-1)(1, B_{m-2,1})$. Now consider the two dacards $(m, K_2 \cup (m-1)K_1)$ and $(1, K_{1,m})$. To get an extension $H(m, K_2 \cup (m-1)K_1)$ non-isomorphic to G, add a new vertex and join it to the two vertices of positive degree. But then every one-vertex deleted dacard of H must contain a cycle and so it is non-isomorphic to $(1, K_{1,m})$. Thus no graph ($\ncong G$) has both the dacards $(m, K_2 \cup (m-1)K_1)$ and $(1, K_{1,m})$ in its dadeck and hence $drn(G) \leq 2$.

Case 2. $k_1 = 2$.
Clearly a vertex in Y_2 is adjacent to exactly two vertices, say s, t in X. Also, since $r = n - 1$, every vertex in X, other than s and t, is adjacent to all the vertices in $Y \setminus Y_2$ and so every vertex in $Y \setminus Y_2$ gets at least $m - 2$ neighbours in X. Since $r = n - 1$, the vertex s (respectively t) is nonadjacent to exactly one vertex, say s' (respectively t') in $Y \setminus Y_2$.

If $s' \neq t'$ (this happens when $n \geq 3$), then every vertex in $Y \setminus Y_2$ gets at least $m - 1$ neighbours in X and hence $Y = Y_2 \cup Y_{m-1} \cup Y_m$, where $|Y_2| = 1$ and $|Y_{m-1}| = 2$. We can take that $n = 3$ (as otherwise n would be at least four and G would contain a complete vertex). Since G has order at least seven, we have $m \geq 4$. Now consider the dacards $(m + 1, G - v)$ and $(2, G - u)$, where $v \in Y_{m-1}, u \in X$, and $uv \in E(G)$. The dacard $G - u$ contains exactly $m - 1$ vertices of degree two. To get an extension $H(m+1, G-v)$, join the newly added vertex w to all but one vertex, say z in $G-v$. If z were the unique 2-vertex, then H would be isomorphic to G. If z is not the unique 2-vertex, then every 2-vertex deleted dacard of the resulting extension H must contain at most $m-2$ vertices of degree two and so it is not isomorphic to $G-u$. Hence $drn(G) \leq 2$.

Now assume $s^{'} = t^{'}$ (this happens when $n \geq 2$). Then $Y = Y_2 \cup Y_{m-2} \cup Y_m$ where $|Y_2| = 1$ and $|Y_{m-2}| = 1$. We can take that $n = 2$ (as otherwise n would be at least three and G would contain a complete vertex). Since G has order at least seven, we have $m \geq 5$. Hence, in this case, the graph G is isomorphic to the bistar $B_{2,m-2}$ whose drn is proved (Barrus and West [6]) to be 2.

Case 3. $k_1 > 2$.
Consider the dacards $(n-1+k_t, G-v)$ and $(n-1, G-u)$, where $v \in Y_{k_t}$, $u \in X$ and $uv \notin E(G)$. The dacard $G-u$ contains no n-vertices. To get an extension $H(n-1+k_t, G-v)$, add a new vertex w to $G-v$ and join it to some set of vertices (say $Y^{'}$) in $Y \setminus \{v\}$ and some set of vertices (say $X^{'}$) in X.

Suppose $|X^{'}| = k_t$ and $Y^{'} = Y \setminus \{v\}$. If $X^{'}$ consists of only $(n-2)$-vertices, then $H \cong G$. If every vertex in $X^{'}$ has degree $n-1$, then $m - k_t = k_t$ and the resulting extension has no $(n-1)$-vertex and so it has no dacard isomorphic to $(n-1, G-u)$. Therefore we assume that $X^{'}$ contains vertices of degree $n-1$, $n-2$ and that it contains no vertices of other degree. But then any $(n-1)$-vertex deleted dacard of the resulting extension must contain an n-vertex and so it is not isomorphic to $G-u$.

Suppose $X^{'} = X$ and $|Y^{'}| = n-1+k_t-m$. Then any $(n-1)$-vertex deleted dacard of the extension H must contain an n-vertex and so it is not isomorphic to $G-u$.

We now assume the only remaining case that $\phi \neq X^{'} \subsetneq X$ and $\phi \neq Y^{'} \subsetneq Y \setminus \{v\}$. Then $|Y^{'}| < n-2$, which implies $|X^{'}| > k_t$ because the associated degree of $G-v$ is $n-1+k_t$. Since $G-v$ has exactly k_t vertices of degree $n-2$ (in X of it), it follows that $X^{'}$ must contain at least one $(n-1)$-vertex. Now this vertex must occur as an n-vertex in any $(n-1)$-vertex deleted dacard of the resulting extension H and so such a dacard is not isomorphic to $G-u$. Hence $drn(G) \leq 2$ and by Theorem 1, $drn(G) = 2$.

Theorem 3. *If G is a split graph with $r \leq n-2$, then $drn(G) = 2$.*

Proof. We proceed by two cases depending upon the value of r as below.

Case 1. $r \leq n-3$.
Now $n \geq 4$ and $k_t \leq m-1$. Consider the dacards $(n-1+k_t, G-v)$ and $(r, G-u)$, where $v \in Y_{k_t}$ and $u \in X$. Clearly the dacard $G-u$ contains no vertices of degree $r-1$ or $r+1$. To get an extension $H(G-v)$, add a new vertex w to $G-v$ and join it to some set of vertices (say $Y^{'}$) in $Y \setminus \{v\}$ and some set of vertices (say $X^{'}$) in X.

Suppose $Y^{'} = Y \setminus \{v\}$ and $|X^{'}| = k_t$. If every vertex in $X^{'}$ has degree $r-1$, then $H \cong G$. If every vertex in $X^{'}$ has degree r, then $m - k_t = k_t$ and the resulting extension has no r-vertex, so it has no dacard isomorphic to $(r, G-u)$. We therefore assume that $X^{'}$ contains vertices of degree $r-1, r$ and that it contains no vertices of other degree. But then any r-vertex deleted dacard of the resulting extension must contain an $(r+1)$-vertex and hence such a dacard is not isomorphic to $G-u$.

Suppose $|Y^{'}| = n-1+k_t-m$ and $X^{'} = X$. Then every r-vertex deleted dacard must contain an $(r+1)$-vertex (because $k_t \leq m-1$) and hence it is not isomorphic to $G-u$.

Now we consider the remaining case that $\phi \neq X^{'} \subsetneq X$ and $\phi \neq Y^{'} \subsetneq Y \setminus \{v\}$. Then $|Y^{'}| < n-2$, which implies $|X^{'}| > k_t$ because the associated degree of $G-v$ is $n-1+k_t$. Since $G-v$ has exactly k_t vertices of degree $r-1$ (in X of it), it follows that $X^{'}$ must contain at least one r-vertex. But then this vertex will occur as an $(r+1)$-vertex in every r-vertex deleted dacard of the resulting extension H and so such a dacard is not isomorphic to $G-u$. Hence $drn(G) \leq 2$.

Case 2. $r = n-2$.
Now $n \geq 3$ and $k_t \leq m-1$. If $|Y_0|$ were at least two, then either r would be at most $n-3$ or G would have a complete vertex, giving a contradiction. Therefore $|Y_0| = 0$ or 1. Also if $|Y_0| = 0$, then, since $r = n-2$, it follows that $|Y_1| \leq 4$. If $|Y_1|$ were 3 or 4, then the order of G would be at most six. Thus, either $|Y_0| = 0$ and $|Y_1| \leq 2$, or else $|Y_0| = 1$.

Now proceeding as in Case 1 but with the two dacards $(n-1+k_t, G-v)$ and $(n-2, G-u)$, where $v \in Y_{k_t}$, $u \in X$ and u is nonadjacent to a vertex in Y_1, we will have $drn(G) \leq 2$ and by Theorem 1, $drn(G) = 2$.

3 Conclusion

For graphs with at least three vertices, knowing the degree of the deleted vertex is equivalent to knowing the total number of edges. A simple counting argument computes the size of the graph when its entire deck is known. So the dadeck gives the same information as the deck. However, the counting argument requires the entire deck, so an individual dacard gives more information than the corresponding card.

In the above sections, we have proved that the drn is at most 3 for a split graph G with regular independent set. There is a hope to complete a proof of $drn(G) \leq 3$ for all split graphs G. With reference to our results, it seems that the drn of bipartite graphs, with a regular independent partite set, is likely to be at most 3. However, extending this result to the family of all bipartite graphs needs intensive work as because reconstructibility of the family of all bipartite graphs remains open.

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