Minimum Layout of Circulant Graphs into Certain Height Balanced Trees

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Abstract. A graph embedding comprises of an ordered pair of injective maps $\prec f, p \succ$ from a guest graph $G = (V(G), E(G))$ to a host graph $H = (V(H), E(H))$ which is formulated as follows: f is a mapping from $V(G)$ to $V(H)$ and p assigns to each edge (a, b) of G, a shortest path $p(a, b)$ in H. The minimum layout problem is to find an embedding $\prec f, p \succ$ from a graph G into a graph H such that $\sum_{e\in E(H)} EC_{\prec f,p\succ}(e) = \sum |(a, b) \in E(G) : e \in E(p(a, b))|$ is minimized. In this paper we develop an algorithm to find the minimum layout of embedding the circulant graph into certain height balanced trees like Fibonacci tree and wounded lobster.

Keywords: Height balanced tree · Layout · Circulant graph · Fibonacci tree

1 Introduction

Graph embedding has been an integral tool in efficient implementation of parallel algorithms on parallel computers with minimal communication overhead. A graph embedding comprises of an ordered pair of injective maps $\prec f, p \succ$ from a guest graph $G = (V(G), E(G))$ to a host graph $H = (V(H), E(H))$ which is formulated as follows: f is a mapping from $V(G)$ to $V(H)$ and p assigns to each edge (a, b) of G, a shortest path $p(a, b)$ in H [1,7]. Figure 1 illustrates a graph embedding. The edge congestion of an embedding is defined by $EC_{\prec f,p\succ}(e) = |(a, b) \in E(G) : e \in E(p(a, b))|$ [6].

The layout $L_{\prec f,p\succ}(G, H)$ of an embedding is defined as the sum of edge congestion of all the edges of H [3,5]. The minimum layout of G into H is given by $L(G, H) = \min L_{\prec f,p\succ}(G, H)$. The minimum layout problem is to find the embedding that induces $L(G, H)$. When the host graph is a tree, the layout problem finds application in graph drawing, data structures and representations and networks for parallel systems [5,10].

Maximum Induced Subgraph Problem [3]: Let $G = (V(G), E(G))$ and $S \subseteq V(G)$. Let $I_G(S) = \{(u, v) \in E(G) : u \in S \text{ and } v \in S\}$ and for $1 \le k \le |V(G)|$, let $I_G(k) = \max\limits_{S\subseteq V,\ |S|=k} |I_G(S)|$. Then the problem is to find $S \subseteq V(G)$ with

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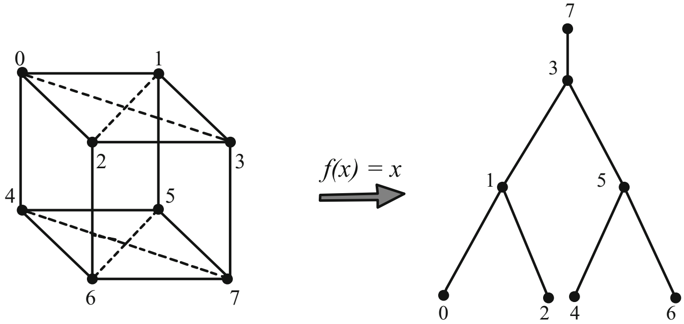

Fig. 1. Embedding of an enhanced hypercube G into rooted complete binary tree H

$|S| = k$ such that $I_G(k) = |I_G(S)|$. Such a set S is called an optimal set with respect to the maximum induced subgraph problem.

Min-cut Problem [3]: Let $G = (V(G), E(G))$ and $S \subseteq V(G)$. Let $\Theta_G(S) = \{(u, v) \in E : u \in S \text{ and } v \notin S\}$ and for $1 \leq k \leq |V(G)|$, let $\Theta_G(k) = \min_{S \subseteq V,\ |S|=k} |\Theta_G(S)|$. Then the problem is to find $S \subseteq V(G)$ with $|S| = k$ such that $\Theta_G(k) = |\Theta_G(S)|$. Such a set S is said to be optimal with respect to the min-cut problem. For any graph G, $\Theta_G(V - S) = \Theta_G(S)$ for all $S \subseteq V(G)$. If G is an r-regular graph, then $\Theta_G(k) = rk - 2I_G(k)$ for every $k \in \{1, 2, \ldots, |V(G)|\}$.

The following results provide a method for partitioning the edges of the host graph which in turn can be effectively used to solve the minimum layout problem.

Lemma 1 (Congestion Lemma) [6]**.** *Let G be an r-regular graph and $\prec f, p \succ$ be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S splits H into 2 components H_1 and H_2 and $EC_{\prec f,p\succ}(S)$ denote the sum of edge congestion over all the edges in S. Let $G_1 = G[f^{-1}(H_1)]$ and $G_2 = G[f^{-1}(H_2)]$. Suppose the following conditions hold.*

1. *For every edge $(a, b) \in G_i$, $i = 1, 2$, $p(a, b)$ has no edges in S.*
2. *For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $p(a, b)$ has exactly one edge in S.*
3. *G_1 is optimal with respect to the maximum induced subgraph problem.*

Then $EC_{\prec f,p\succ}(S)$ is minimum and $EC_{\prec f,p\succ}(S) = \Theta_G(|V(G_1)|) = \Theta_G(|V(G_2)|)$.

Lemma 2 [6]**.** *Let $\prec f, p \succ$ be an embedding from G into H. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that $EC_{\prec f,p\succ}(S_i)$ is minimum for all i. Then $L_{\prec f,p\succ}(G, H)$ is minimum and $L_{\prec f,p\succ}(G, H) = \sum_{i=1}^{p} EC_{\prec f,p\succ}(S_i)$.*

Definition 1 [10]**.** *A circulant undirected graph $G(n; \pm S)$, $S \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}$, $n \geq 3$ is defined as a graph consisting of the node set $V = \{0, 1, \ldots, n-1\}$ and the edge set $E = \{(i, j) : |j - i| \equiv s(mod\ n),\ s \in \pm S\}$.*

In this paper, we confine our work to the circulant graph $G(n;\pm S)$, where $S=\{1,2,...,j\}$, $1\le j<\lfloor n/2\rfloor$. For $n\ge 3$, $1\le j<\lfloor n/2\rfloor$, $G(n;\pm\{1,2,...,j\})$ is a $2j$-regular graph. Figure 2(a) illustrates a circulant graph.

Lemma 3 [8]**.** *A set of k consecutive nodes induces an optimal set with respect to the maximum induced subgraph problem in $G(n;\pm S)$ on k nodes.*

Lemma 4 [8]**.** *Let G be the circulant graph $G(n;\pm S)$, $n\ge 3$. Then for $1\le k\le n$,*

$$I_G(k)=\begin{cases} k(k-1)/2 & ;k\le j+1\\ kj-j(j+1)/2 & ;j+1<k\le n-j\\ \frac{1}{2}\{(n-k)^2+(4j+1)k-(2j+1)n\} & ;n-j<k\le n\ . \end{cases}$$

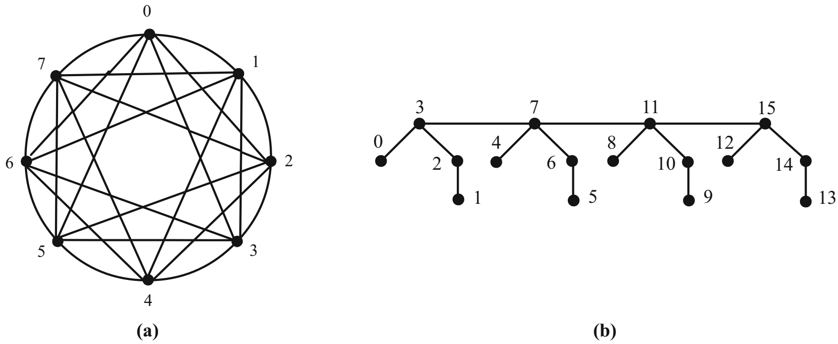

Fig. 2. (a) Circulant graph $G(8;\pm\{1,2,3\})$ (b) Wounded lobster L_4

A height balanced tree T is a rooted binary tree in which for every node v, the difference between the heights of the left and right child denoted as v_1 and v_2 respectively is at most one [2].

Fibonacci trees are a type of height balanced trees which are built recursively in one of the following two ways.

Fibonacci Tree f_h [4]: The trees f_1 and f_2 consists of only the root node. For $h\ge 3$, f_h is constructed by taking a new root node and attaching f_{h-1} on the left side and f_{h-2} on the right side of the root node by an edge as shown in Fig. 3(a).

Fibonacci Tree $f_h^{'}$ [2]: The tree f_1^1 consists of only the root node and $f_2^{'}$ is formed by attaching a pendant node to the root node. For $h\ge 3$, the left subtree of $f_h^{'}$ is $f_{h-1}^{'}$ and its right subtree is $f_{h-2}^{'}$. Figure 3(b) illustrates $f_h^{'}$ for $h=1,2,\ldots 5$.

Let $|V(f_h)|=m_h$ and $|V(f_h^{'})|=m_h^{'}$. Then, $m_h=2F_h-1$ and $m_h^{'}=F_{h+2}-1$, where F_h denotes the Fibonacci number.

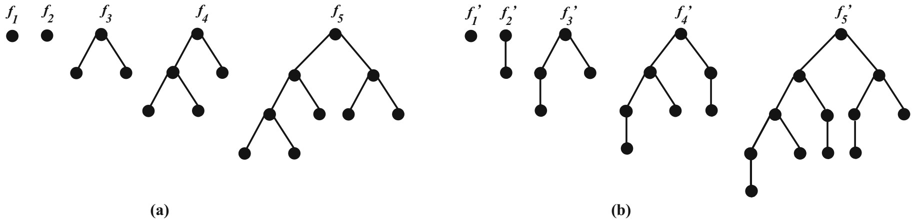

Fig. 3. (a) f_h type Fibonacci trees (b) $f_h^{'}$ type Fibonacci trees

Definition 2 [9]**.** *A lobster is a tree with the property that the removal of pendant nodes leaves a caterpillar. A wounded lobster L_n is a lobster satisfying the following conditions:*

(i) There are 2^{n-2} spine nodes and every spine node is adjacent to exactly one node of degree 2 and one node of degree 1.
(ii) Removal of pendant nodes incident at nodes of degree 2 leaves a caterpillar.

Figure 2(b) illustrates a wounded lobster.

There are several techniques for traversing the nodes of a tree according to the order in which the nodes are visited. In this paper we confine our study to postorder.

Algorithm 1. Postorder Tree Traversal Algorithm

Do the following recursively until all nodes are traversed:
Step 1 - Traverse left subtree.
Step 2 - Traverse right subtree.
Step 3 - Visit root node.

2 Main Results

In this section we embed the circulant graph into Fibonacci trees and wounded lobster to minimize their layouts.

Theorem 1. *The minimum layout of circulant graphs $G = G(m_h; \pm S)$ and $G^{'} = G(m_h^{'}; \pm S)$ into the Fibonacci trees is given by (a) $L(G, f_h) = F_{h-2}.\Theta_G(m_3) + F_{h-3}.\Theta_G(m_4) + \ldots + F_2.\Theta_G(m_{h-1}) + 2|S|$ and (b)$L(G', f_h^{'}) = F_{h-1}.\Theta_{G'}(m_2^{'}) + F_{h-2}.\Theta_{G'}(m_3^{'}) + \ldots + 2\Theta_{G'}(m_{h-2}^{'}) + \Theta_{G'}(m_{h-1}^{'}) + 2|S|$.*

Proof. We split the proof into three parts comprising of labeling the guest and host graphs, followed by the proposal of embedding and layout computation.

Guest and Host Labeling: Label the circulant graph and the two types of Fibonacci trees as in the pattern given in Table 1.

Table 1. Labeling algorithm

Labeling I	Labeling II
Guest Graph: Label the consecutive nodes of $G(m_h; \pm S)$ as $0, 1, 2, \ldots, m_h - 1$ in the clockwise direction	**Guest Graph:** Label the consecutive nodes of $G(m_h^{'}; \pm S)$ as $0, 1, \ldots, m_h^{'} - 1$ in the clockwise direction
Host Graph: Label the nodes of f_h by postorder tree traversal from 0 to $m_h - 1$	**Host Graph:** Label the nodes of $f_h^{'}$ by postorder tree traversal from 0 to $m_h^{'} - 1$

Proposed Embedding: Define an embedding $\prec f, p \succ$ from $G(m_h; \pm S)$ into f_h and $G(m_h^{'}; \pm S)$ into $f_h^{'}$ such that $f(x) = x$.

Layout Computation: We split the proof into two cases.

Proof for (a): For $1 \leq i \leq m_h - 1$, let S_i be an edge cut of f_h such that its removal disengages f_h into two components X_i and $\overline{X}_i$ as shown in Fig. 4(a), with the node set V_i of X_i being as follows.

For $1 \leq i \leq m_{h-1}$,

$$V_i = \begin{cases} \{0, 1, \ldots, i-1\}, & if \ i = m_g, 1 \leq g \leq h-1 \\ \{m_a, m_a + 1, \ldots, i-1\}, & if \ i = m_a + m_b, 1 \leq b < a \leq h-1 \\ \{i-1\}, & otherwise. \end{cases}$$

For $m_{h-1} + 1 \leq i \leq m_h - 1$,

$$V_i = \begin{cases} \{m_{h-1}, m_{h-1} + 1, \ldots, i-1\}, if \ i = m_{h-1} + m_g, 1 \leq g \leq h-1 \\ \{m_{h-1} + m_a, m_{h-1} + m_a + 1, \ldots, i-1\}, if \ i = m_{h-1} + m_a + m_b, \\ \qquad\qquad 1 \leq b < a \leq h-1 \\ \{i-1\}, \quad otherwise. \end{cases}$$

Let G_i be the graph induced by $\{f^{-1}(u) : u \in V_i\}$. It can be noted that X_i is consecutively labeled for all i and hence by Lemma 3, V_i is an optimal set with respect to the maximum induced subgraph problem. S_i also satisfies conditions (i) and (ii) of Lemma 1. In addition, $\{S_i\}_{i=1}^{m_h - 1}$ forms a partition of $E(f_h)$. Hence by Lemma 2, $L_{\prec f,p \succ}(G, f_h)$ is minimum.

Let $m_h - 1 = F_h + F_{h-2} + F_{h-3} + F_{h-4} + \ldots F_2$, where $m_h - 1$ represents the number of edge cuts of f_h and $F_h, F_{h-2}, F_{h-3}, \ldots, F_3, F_2$ denote the number of node sets V_i of cardinality $m_2, m_3, \ldots, m_{h-2}$ and m_{h-1} respectively.

Layout: $L(G, f_h) = \sum_{i=1}^{m_h - 1} EC_{\prec f,p \succ}(S_i) = \sum_{i=1}^{m_h - 1} \Theta_G(|V_i|) = \sum_{i=1}^{F_h} \Theta_G(m_2) + \sum_{i=1}^{F_{h-2}} \Theta_G \ (m_3) + \sum_{i=1}^{F_{h-3}} \Theta_G(m_4) + \ldots + \sum_{i=1}^{F_3} \Theta_G(m_{h-2}) + \sum_{i=1}^{F_2} \Theta_G(m_{h-1}) = F_{h-2}.\Theta_G(m_3) + F_{h-3}.\Theta_G(m_4) + \ldots + F_2.\Theta_G(m_{h-1}) + 2|S|$.

Proof for (b): Let $S_i^{'}$, $1 \leq i \leq m_h - 1$ be an edge cut of $f_h^{'}$ such that removal of $S_i^{'}$ disconnects $f_h^{'}$ into two components Y_i and $\overline{Y}_i$ as depicted in Fig. 4(b) where the node set $V_i^{'}$ of Y_i is defined by replacing m_g, m_a, m_b and m_{h-1} in V_i of case (a) by $m_g^{'}, m_a^{'}, m_b^{'}$ and $m_{h-1}^{'}$ respectively.

Let $G_i^{'}$ be the graph induced by $\{f^{-1}(a) : a \in V_i^{'}\}$. Clearly X_i is labeled consecutively for all i and hence by Lemma 3, $V_i^{'}$ is an optimal set with respect to the maximum induced subgraph problem. $S_i^{'}$ also satisfies the remaining two conditions of Lemma 1. In addition, $\{S_i^{'}\}_{i=1}^{m_h^{'}-1}$ forms a partition of $E(f_h^{'})$. Hence by Lemma 2, $L_{\prec f,p \succ}(G, f_h^{'}) = L(G, f_h^{'})$.

Let $m_h^{'} - 1 = F_h + F_{h-1} + F_{h-2} + \ldots F_2$, where $F_h, F_{h-1}, F_{h-2}, \ldots, F_3, F_2$ denote the number of nodes sets $V_i^{'}$ of cardinality $m_1^{'}, m_2^{'}, \ldots, m_{h-2}^{'}$ and $m_{h-1}^{'}$ respectively.

Layout: $L(G^{'}, f_h^{'}) = \sum_{i=1}^{m_h^{'}-1} EC_{\prec f,p \succ}(S_i^{'}) = \sum_{i=1}^{m_h^{'}-1} \Theta_{G^{'}}(|V_i^{'}|) = \sum_{i=1}^{F_h} \Theta_{G^{'}}(m_1^{'}) + \sum_{i=1}^{F_{h-1}} \Theta_{G^{'}}(m_2^{'}) + \ldots + \sum_{i=1}^{F_3} \Theta_{G^{'}}(m_{h-2}^{'}) + \sum_{i=1}^{F_2} \Theta_{G^{'}}(m_{h-1}^{'}) = F_{h-1}.\Theta_{G^{'}}(m_2^{'}) + F_{h-2}.\Theta_{G^{'}}\ (m_3^{'}) + \ldots + 2\Theta_{G^{'}}(m_{h-2}^{'}) + \Theta_{G^{'}}(m_{h-1}^{'}) + 2|S|$.

Theorem 2. *The minimum layout of $G = G(2^n; \pm S)$ into the wounded lobster L_n is given by $L(G, L_n) = \frac{1}{3}\{2^{n-1}(12j(2^{n-4} + 1) + 2^{n-3}(3 - 2^n) - 7)\}$.*

Proof. **Guest and Host Labeling:** Label $G(2^n; \pm S)$ in the clockwise direction as described in Table 1. Label L_n using postorder tree traversal order from 0 to $2^n - 1$.

Proposed Embedding: Define an embedding $\prec f, p \succ$ from $G(2^n; \pm S)$ into L_n such that $f(x) = x$.

Layout Computation: Table 2 gives three sets of edge cuts covering $E(L_n)$ and the node set of the components obtained by the removal of these edge cuts as depicted in Fig. 4(c).

Let G_r, $G_r^{'}$ and $G_r^{''}$ be the inverse image of Y_r, $Y_r^{'}$ and $Y_r^{''}$ respectively under $\prec f, p \succ$. By Lemma 3, the node set of all the three inverse images are optimal in G with respect to the maximum induced subgraph problem. All three edge cuts S_r, $S_r^{'}$

Table 2. Edge cuts of L_n

Edge Cuts	Components	V(Component)
S_r $r = 1, 2, \ldots, 2^{n-1}$	Y_r, $\overline{Y}_r$	$V(Y_r) = \begin{cases} \{4(r-1)\} & if\ r\ is\ odd \\ \{2(r-2)+1\} & if\ r\ is\ even \end{cases}$
$S_r^{'}$ $r = 1, 2, \ldots, 2^{n-2}$	$Y_r^{'}$, $\overline{Y}_r^{'}$	$V(Y_r^{'}) = \{4(r-1)+1, 4(r-1)+2\}$
$S_r^{''}$ $r = 1, 2, \ldots, 2^{n-2} - 1$	$Y_r^{''}$, $\overline{Y}_r^{''}$	$V(Y_r^{''}) = \{4(r-1)+0, \ldots, 4(r-1)+3\}$

and $S_r^{''}$ also satisfy the remaining two conditions of Lemma 1. In addition, $\{S_r, r = 1, 2, \ldots, 2^{n-1}\} \cup \{S_r^{'}, r = 1, 2, \ldots, 2^{n-2}\} \cup \{S_r^{''}, r = 1, 2, \ldots, 2^{n-2} - 1\}$ forms a partition of $E(L_n)$. Hence by Lemma 2, the layout induced by the embedding $\prec f, p \succ$ is minimum.

From Lemmas 2 and 4, $L(G, L_n) = \sum_{r=1}^{2^{n-1}} EC_{\prec f,p\succ}(S_r) + \sum_{r=1}^{2^{n-2}} EC_{\prec f,p\succ}(S_r^{'}) + \sum_{r=1}^{2^{n-2}-1} EC_{\prec f,p\succ}(S_r^{''}) = \sum_{r=1}^{2^{n-1}} \Theta_G(|V(Y_r)|) + \sum_{r=1}^{2^{n-2}} \Theta_G(|V(Y_r^{'})|) + \left\{ \sum_{r=1}^{2^{n-3}} \Theta_G (|V(Y_r^{''})|) + \sum_{r=2^{n-3}+1}^{2^{n-2}-1} \Theta_G(|V(Y_r^{''})|) \right\} = \frac{2^{n-1}}{3} \{ 12j\,(2^{n-4}+1) + 2^{n-3}(3-2^n) - 7 \}$.

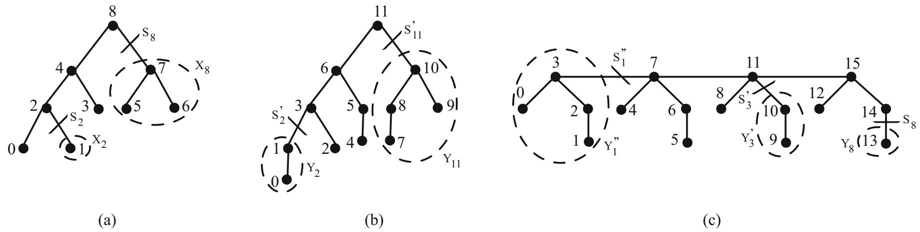

Fig. 4. Edge cuts of (a) f_5 (b) $f_5^{'}$ (c) L_4

3 Conclusion

In this paper we have embedded and found the minimum layout of the circulant graph into certain classes of height balanced trees like Fibonacci trees and wounded lobster by using edge partitioning techniques and isoperimetric methods.

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