Degree Associated Reconstruction Number of Biregular Bipartite Graphs Whose Degrees Differ by at Least Two

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Abstract. A vertex-deleted subgraph of a graph G is called a *card* of G. A card of G with which the degree of the deleted vertex is also given is called a *degree associated card* (or *dacard*) of G. The *degree associated reconstruction number* of a graph G (or $drn(G)$) is the size of the smallest collection of dacards of G that uniquely determines G. It is shown that $drn(G) = 1$ or 2 for all biregular bipartite graphs with degrees d and $d+k$, $k \geq 2$ except the bistar $B_{2,2}$ on 6 vertices and that $drn(B_{2,2}) = 3$.

Keywords: Reconstruction · Reconstruction number · Isomorphism

1 Introduction

All graphs considered in this paper are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [7]. A *vertex-deleted subgraph* or *card* $G - v$ of a graph (digraph) G is the unlabeled graph (digraph) obtained from G by deleting the vertex v and all edges (arcs) incident with v. The *deck* of a graph (digraph) G is its collection of cards. Following the formulation in [2], a graph (digraph) G is *reconstructible* if it can be uniquely determined from its deck. The well-known Reconstruction Conjecture (RC) due Kelly [10] and Ulam [16] asserts that every graph with at least three vertices is reconstructible. The conjecture has been proved for many special classes, and many properties of G may be deduced from its deck. Nevertheless, the full conjecture remains open [6]. Harary and Plantholt [9] defined the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \ncong G$, these cards thus uniquely identifying G. Reconstruction numbers are known for only few classes of graphs [4].

An extension of the RC to digraphs is the *Digraph Reconstruction Conjecture* (DRC) proposed by Harary [8]. It was disproved by Stockmeyer [15] by exhibiting several infinite families of counter-examples and this made people doubt the

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RC itself. To overcome this, Ramachandran [12] introduced degree associated reconstruction for digraphs and proposed a new conjecture in 1981. It was proved [12] that the digraphs in all these counterexamples to the DRC obey the new conjecture, thereby protecting the RC from the threat posed by these digraph counterexamples.

The ordered triple (a, b, c) where a, b and c are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pair of arcs incident with v in a digraph D is called the *degree triple of* v. The *degree associated card* or *dacard* of a digraph (graph) is a pair (d, C) consisting of a card C and the degree triple (degree) d of the deleted vertex. The *dadeck* of a digraph is the multiset of all its dacards. A digraph is said to be *N-reconstructible* if it can be uniquely determined from its dadeck. The *new digraph reconstruction conjecture* [12] (NDRC) asserts that all digraphs are N-reconstructible. Ramachandran [13] then studied the degree associated reconstruction number of graphs and digraphs in 2000. The *degree* (*degree triple*) *associated reconstruction number* of a graph (digraph) D is the size of the smallest collection of dacards of D that uniquely determines D. Articles [1–3, 5, 11, 14] are recent papers on this parameter.

A graph G is *bipartite* if its vertex set is the union of two disjoint independent sets, called *partite sets* of G. A graph whose vertices all have one of two possible degrees is called a *biregular* graph. We show that if G is a biregular bipartite graph, other than the bistar $B_{2,2}$ on 6 vertices, with degrees d and $d + k$, $k \geq 2$, then $drn(G) = 1$ or 2 and that $drn(B_{2,2}) = 3$.

2 *Drn* of Biregular Bipartite Graphs

The degree of a vertex v in G is denoted by $deg_G\ v$ or simply $deg\ v$. A vertex of degree m is called an *m-vertex*. The *neighbourhood* of a vertex v in G, written $N_G(v)$ or simply $N(v)$, is the set of all vertices adjacent to v in G.

Notation. By x, x' with or without subscripts, we mean respectively a d-vertex and a $(d+k)$-vertex in the partition X. Such vertices but in the partition Y are denoted by y and y' respectively.

An *extension* of a dacard $(d(x), G - x)$ of G is a graph obtained from the dacard by adding a new vertex v and joining it to $d(x)$ vertices of the dacard and it is denoted by $H(d(x), G-x)$ (or simply by H). Throughout this paper, H and v are used in the sense of this definition. For a graph G, to prove $drn(G) = k$, we show that every extension (other than G) of at least one dacard has at most $k - 1$ dacards in common with those of G (thus $drn(G) \leq k$), and that at least one extension has precisely $k - 1$ dacards in common with those of G (thus $drn(G) \geq k$).

In their paper [5], Barrus and West have characterized (Theorem A) graphs G with $drn(G) = 1$.

Theorem A. The dacard $(d, G-v)$ belongs to the dadeck of only one graph (up to isomorphism) if and only if one of the following holds:

(i) $d = 0$ or $d = |V(G-v)|$;
(ii) $d = 1$ or $d = |V(G-v)| - 1$, and $G-v$ is vertex-transitive; or
(iii) $G-v$ is complete or edgeless.

Ramachandran [13] has verified that $drn(G) = 1,\ 2$ or 3 for all graphs G on at most 6 vertices, and in particular, $drn(G) = 1$ or 2 for all biregular bipartite graphs G on at most 6 vertices with degrees d and $d+k$ such that $k \geq 2$, except the bistar $B_{2,2}$ and that $drn(B_{2,2}) = 3$. So, *we assume that all biregular bipartite graphs G considered here onwards have order at least 7 and no dacard of G satisfies the conditions of Theorem A and so $drn(G) \geq 2$.*

Theorem 1. *If G is a biregular bipartite graph with a vertex adjacent to all the vertices in the other partite set, then $drn(G) = 2$.*

Proof. The graph G is clearly connected. Let z be a vertex adjacent to all the vertices in the other partite set of G. If $deg\ z = d$, then G is a complete bipartite graph and $drn(G) = 2$, since it is known [13] that $drn(K_{m,n}) = 2$ for $2 \leq m < n$. So, let $deg\ z = d+k,\ k \geq 2$.

Suppose that z is adjacent to a d-vertex. Consider the two dacards $(d+k, G-z)$ and $(d, G-w)$, where w is a d-vertex in $N(z)$. It is clear that the dacard $G-w$ is connected and so it has exactly one partite set such that every vertex in the partite set has degree d or $d+k$. To get an extension $H(d+k, G-z)$, add a new vertex v to the dacard $G-z$ and join it with precisely $d+k$ vertices. Clearly $G-z$ contains exactly one partite set (say Z_1) having a $(d-1)$-vertex and a $(d+k-1)$-vertex. If v were joined to all the vertices in Z_1, then the resulting extension H would be isomorphic to G. Otherwise, in every extension $H(d+k, G-z)$, the newly added vertex v is joined to at least one vertex not in Z. But then any d-vertex deleted dacard of $H(d+k, G-z)$ contains a vertex of degree different from d and $d+k$ from each partite set and so it is not isomorphic to $G-w$.

Suppose that no d-vertex is adjacent to z. Now consider the two dacards $(d+k, G-z)$ and $(d+k, G-w)$, where $w \in N(z)$. In $G-w$, exactly one partite set is $(d+k)$-regular. In the extension $H(d+k, G-z)$, if the newly added vertex v were joined only to the $(d+k-1)$-vertices, then H would be isomorphic to G. Otherwise, any $(d+k)$-vertex deleted dacard of $H(d+k, G-z)$ must contain irregular partite sets or a partite set is $(d+k-1)$-regular and hence it is not isomorphic to $(d+k, G-w)$. Thus no graph other than G contains both the two dacards $(d+k, G-z)$ and $(d+k, G-w)$ in its dadeck and hence $drn(G) = 2$.

Theorem 2. *Let G be a biregular bipartite graph. If G has a vertex adjacent to no d-vertices and has a vertex adjacent to no $(d+k)$-vertices, then $drn(G) = 2$.*

Proof. Let z be a vertex adjacent to no $(d+k)$-vertices and let z' be that adjacent to no d-vertices. Here we use the two dacards $(d(z), G-z)$ and $(d(z'), G-z')$. Clearly $\delta(G-z') > d-1$. In the extension of $(d(z), G-z)$, if the newly added vertex v were joined to all the $(d-1)$-vertices, then the resulting extension H would be isomorphic to G. Thus, in every extension of $(d(z), G-z)$, vertex v is not joined to a $(d-1)$-vertex of $(d(z), G-z)$. But then each dacard of $H(d(z), G-z)$ contains at least one vertex of degree at most $d-1$ and so it is not isomorphic to $(d(z'), G-z')$. Thus no graph other than G has both the dacards $(d(z), G-z)$ and $(d(z'), G-z')$ in its dadeck and $drn(G) = 2$.

Theorem 3. *If G is a biregular bipartite graph such that all the vertices in a partite set have the same degree in G, then $drn(G) = 2$.*

Proof. Let Y be such a partite set. Let G have exactly a vertices of degree d, and b vertices of degree $d+k$, where $a, b \geq 1$ and $k \geq 2$. We consider the two dacards $(d, G-x)$ (or $(d+k, G-x')$) and $(d+k, G-y')$ (or $(d, G-y)$). In $G-y'$ (or $G-y$), exactly one partite set is $(d+k)$-regular (or d-regular). In the extension of $(d, G-x)$ (or $(d+k, G-x')$), if the newly added vertex v were joined to all $(d+k-1)$-vertices (or $(d-1)$-vertices), then the resulting extension H would be isomorphic to G. Thus, in every extension of $(d, G-x)$ (or $(d+k, G-x')$), vertex v is not joined to at least one $(d+k-1)$-vertex (or $(d-1)$-vertex). But then any $(d+k)$-vertex (or d-vertex) deleted dacard would contain either at least one vertex of different degree in each partite set or exactly one d-regular (or $(d+k)$-regular) partite set. Hence no dacard of H is isomorphic to $G-y'$ (or $G-y$). Thus no graph other than G has both the dacards $(d, G-x)$ (or $(d+k, G-x')$) and $(d+k, G-y')$ (or $(d, G-y)$) in its dadeck and $drn(G) = 2$.

Theorem 4. *If G is a biregular bipartite graph such that all but one vertex in a partite set have degree $d+k$, $(k \geq 2)$ then $drn(G) = 2$.*

Proof. Let Y be such a partite set and let y be the unique d-vertex in Y. If the other partite set X contains at least two d-vertices, then we choose the two dacards $(d+k, G-x')$ and $(d, G-y)$, where $x' \in N(y)$. In $G-y$, exactly one partite set is $(d+k)$-regular. Now we consider the extension of $(d+k, G-x')$. If the newly added vertex v were not joined to any vertex of degree d or $d+k$, then the resulting extension H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H contains at least one $(d+k+1)$- vertex, at least one vertex of different degree in both the partite sets, or a $(d+1)$-regular partite set. Hence such a dacard is not isomorphic to $G-y$. Therefore no graph other than G contains both these two dacards in its dadeck, we have $drn(G) = 2$. So, we assume that X contains a unique d-vertex.

Let us first consider the case that G is disconnected. Suppose that the two d-vertices of G are belonging to the same component of G. Then consider the two dacards $(d+k, G-x')$ and $(d, G-x)$, where x' and x are belonging to different components. Clearly $\Delta(G-x) \leq d+k$. Consider the extension of $(d+k, G-x')$. If the newly added vertex v were joined to all the $(d+k-1)$-vertices, then H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H contains at least one vertex of degree $d+k+1$ and so it is not isomorphic to $G-x$. Suppose that the two d-vertices of G are belonging to different components of G. Then consider the two dacards $(d+k, G-x')$ and $(d, G-x)$ where x' and x are in different components and $y \in N(x')$. Clearly $\Delta(G-x) \leq d+k$. In the extension $H(d+k, G-x')$, if the newly added vertex v were not joined to any d-vertex and to any $(d+k)$-vertex, then H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H contains at least one vertex of degree $d+k+1$ and so it is not isomorphic to $G-x$, we have $drn(G) = 2$.

Assume now that G is connected and if the two d-vertices are adjacent, then consider the dacards $(d, G-x)$ and $(d, G-y)$. In $G-y$, exactly one partite set is $(d+k)$-regular. In $H(d, G-x)$, if v were not joined to any $(d+k)$-vertex, then H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H contains a $(d+k+1)$-vertex and so it is not isomorphic to $G-y$. Hence $drn(G) = 2$. So, *let us assume that the two d-vertices are nonadjacent in G*. Let X have m $(d+k)$-vertices. Suppose that $|Y| \neq m+1$. Consider the two dacards $(d, G-y)$ and $(d, G-x)$. In $G-x$, exactly one partite set is $(d+k)$-regular and it has size m. In $H(d, G-y)$, if v were joined to all the $(d+k-1)$-vertices, then H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H contains at least one $(d+k+1)$-vertex, at least one vertex of different degree in both the partite sets, or there is a unique $(d+k)$-regular partite set of size not equal to m. Therefore H has no dacard isomorphic to $G-x$. Suppose that $|Y| = m+1$ and $m = d+k$. Consider the two dacards $(d, G-x)$ and $(d+k, G-x')$, where $x' \notin N(y)$. In $G-x'$, exactly one d-vertex in each partite set. In $H(d, G-x)$, if v were joined to all the $(d+k-1)$-vertices, then H would be isomorphic to G. Otherwise, any d-vertex deleted dacard of H has at least one vertex of degree at most $d+k-1$, or at least two d-vertices in the same partite set. Therefore H has no dacard isomorphic to $G-x'$.

Finally, assume that $m \geq d+k+1$. Consider the two dacards $(d+k, G-x')$ and $(d, G-y)$, where $y \notin N(x')$. Exactly one partite set of $G-x'$, say Y_1 contains a $(d+k-1)$-vertex. Similarly, exactly one partite set of $G-y$, say X_1 contains a $(d+k-1)$-vertex. Now we construct two new dacards of a supergraph, say G_1 obtained from G by adding a new vertex z to the partite set X and joining it with all the vertices in the partite set Y and therefore the new vertex attains the degree at least $d+k+1$. By adding a new vertex w_1 and joining it to all the vertices in Y_1 of $G-x'$ gives a new dacard $(d+k, G_1-x')$. Similarly, by adding a new vertex w_2 and joining it to the neighbours of every vertex in X_1 of $G-y$ gives a new dacard $(d+1, G_1-y)$. Clearly, $deg_{G_1-x'}\ w_1 = m+1$ and $deg_{G_1-y}\ w_2 = m$.

In the extension $H_1(d+1, G_1-y)$, if v were not joined to the pair of vertices of degrees d and $d+k$, then H_1 would be isomorphic to G_1. Otherwise, any $(d+k)$-vertex deleted dacard of the extension H_1 is connected and it contains a unique $(m+1)$-vertex and a $(d+k-1)$-vertex (or $(d+1)$-vertex) in the same partite set. Hence such a dacard of H_1 is not isomorphic to $(d+k, G_1-x')$. Therefore $drn(G_1) = 2$. This means that G_1 can be obtained uniquely from the new dacards $(d+k, G_1-x')$ and $(d+1, G_1-y)$. Now the vertex z in G_1 is identifiable as the unique vertex of degree at least $d + k + 1$. Consequently, G can be obtained uniquely from G_1 by deleting the vertex z. In other words, G can be obtained uniquely from the two dacards $(d + k, G - x')$ and $(d, G - y)$ and hence $drn(G) = 2$.

Theorem 5. *If G is a biregular bipartite graph such that all but one vertex in a partite set have degree d, then $drn(G) = 2$.*

Proof. Let Y be such a partite set. We proceed by two cases.

Case 1. X contains at least two $(d + k)$-vertices.
Here we consider the dacards $(d, G-x)$ (or $(d+k, G-x')$) and $(d+k, G-y')$, where x (or x') $\notin N(y')$. In $G - y'$, exactly one partite set is regular. In $H(d, G - x)$, if v were joined to all the $(d-1)$-vertices, H would be isomorphic to G; otherwise, any $(d + k)$-vertex deleted dacard of H contains irregular partite sets and so it is not isomorphic to $G - y'$.

Case 2. X contains exactly one $(d + k)$-vertex.
Case 2.1. Both "$|X| = |Y| = d + k + 1$" and $y' \notin N(x')$ hold.
In $(d + k, G - x')$, exactly one partite set is regular and the unique $(d + k)$-vertex is adjacent to all the d-vertices. In $H(d, G - x)$, if v were joined to all the $(d-1)$-vertices, then H would be isomorphic to G; otherwise, any $(d+k)$-vertex deleted dacard of H contains at least one vertex of degree $d - 2$, no vertex of degree $d + k$, or the dacard has irregular partite sets. Therefore no dacard of H is isomorphic to $G - x'$.

Case 2.2. Either $|X| = |Y| = d + k + 1$ or $y' \notin N(x')$ does not hold.
If G is disconnected, then we consider the dacards $(d, G - x)$ and $(d + k, G - y')$, where x and y' belong to different components of G. In $G - y'$, there is a unique component such that one of the partite sets of the component must contain $(d - 1)$-vertices. In $H(d, G - x)$, if v were joined to all the $(d - 1)$-vertices, then H would be isomorphic to G; otherwise, any $(d + k)$-vertex deleted dacard of H contains at least two components containing $(d - 1)$-vertices or at least two partite sets containing $(d - 1)$-vertices and so it is not isomorphic to $G - y'$.

Now we assume that G is connected. Consider the dacards $(d+k, G-y')$ and $(d, G - x)$, where $x \notin N(y')$. Exactly one partite set of $G - y'$, say Y_1 is regular. Similarly, the dacard $G - x$ contains exactly one partite set, say X_1 such that every vertex in X_1 has degree at least d. Now we construct two new dacards of a supergraph say G_1 obtained from G by adding a new vertex z to the partite set Y and joining it to all the vertices in the partite set X and therefore the new vertex attains the degree at least $d + k + 1$. By adding a new vertex w_1 and joining it to the neighbours of every vertex in Y_1 of $G - y'$ gives the new dacard $(d + k, G_1 - y')$. Note that degree of w_1 is at least $d + k + 1$ in the supergraph G_1. Similarly, by adding a new vertex w_2 and joining it to all the vertices in X_1 of $G - x$ gives the new dacard $(d + 1, G_1 - x)$.

In the extension $H_1(d+1, G_1-x)$, if v were joined to all the $(d-1)$-vertices and to w_2, then the resulting extension H_1 would be isomorphic to G_1. Otherwise, any $(d+k)$-vertex deleted dacard of the extension H_1 is connected and containing no vertex that is adjacent to all the vertices of the other partite set, or the dacard has a unique partite set having a $(d-1)$-vertex and a vertex (say w) such that w is adjacent to all the vertices of the other partite set. Hence no $(d+k)$-vertex deleted dacard of H_1 is isomorphic to $G_1 - y'$ and $drn(G_1) = 2$, which means that G_1 can be obtained uniquely from the new dacards $(d+1, G_1 - x)$ and $(d+k, G_1 - y')$. Now the vertex z in G_1 is identifiable as the unique vertex of degree at least $d+k+1$. Consequently, G can be obtained uniquely from G_1 by deleting the vertex z. In other words, G can be obtained uniquely from the two dacards $(d+1, G_1 - x)$ and $(d+k, G_1 - y')$ and so $drn(G) = 2$.

Theorem 6. *If G is a biregular bipartite graph such that every partite set contains at least two vertices of degree d as well as $d+k$, where $k \geq 2$, then $drn(G) = 2$.*

Proof. Assume first that G is connected. Let X have m_1 vertices of degree d and n_1 vertices of degree $d+k$. Let that in Y be m_2 and n_2 respectively.

Case 1. $m_1 \neq m_2$.
Without loss of generality, let us take that $m_1 > m_2$. Here we use the two dacards $(d, G-y)$ and $(d+k, G-x')$. In $G-x'$, exactly one partite set contains m_1 vertices of degree d and the rest of them in the partite set have degree $d+k$. In $H(d, G-y)$, if v were joined to all the vertices of degree $d-1$ or $d+k-1$ then H would be isomorphic to G; otherwise, any $(d+k)$-vertex deleted dacard of H contains at most $m_1 - 1$ vertices of degree d in each partite set, or exactly one partite set has m_1 vertices of degree d and has at least one vertex of degree $d-1$ or $d+k-1$. Thus no dacard of H would be isomorphic to $G - x'$.

Case 2. $n_1 \neq n_2$.
Without loss of generality, let us take that $n_1 > n_2$. Here we use the two dacards $(d+k, G-y')$ and $(d, G-x)$. In $G-x$, exactly one partite set contains n_1 vertices of degree $d+k$ and the rest of them in the partite set have of degree d. In $H(d+k, G-y')$, if v were joined to all the vertices of degree $d-1$ or $d+k-1$ then H would be isomorphic to G; otherwise, any d-vertex deleted dacard of H contains at most $n_1 - 1$ vertices of degree $d+k$ in each partite set, or exactly one partite set has n_1 vertices of degree $d+k$ and has at least one vertex of degree $d-1$ or $d+k-1$. Thus no dacard of H would be isomorphic to $G - x$.

Case 3. $m_1 = m_2$ and $n_1 = n_2$.
Consider the dacards $(d, G-y)$ and $(d+k, G-x')$. The dacard $G-y$ (respectively $G-x'$) contains a unique partite set, say Z such that every vertex in it has degree $d-1$ or $d+k-1$. Now we construct two new dacards of a supergraph say G_1 obtained from G by adding a new vertex z to the partite set Z and joining it to all the vertices in the partite set Y and therefore the new vertex attains the degree at least $d+k+1$ in G_1. By adding a new vertex w_1 to $G-y$ and joining it to all the neighbours of every vertex in Z of $G-y$ gives the new dacard $(d+1, G_1 - y)$. Note that the degree of w_1 is at least $d+k+1$ in the graph G_1. Similarly, by adding a new vertex w_2 to $G-x'$ and joining it to all the vertices in Z of $G-x'$ gives the other new dacard $(d+k, G_1 - x')$.

In the extension $H_1(d+1, G_1-y)$, if v were joined to w_2 and to all the vertices of degree $d-1$ or $d+k-1$, then the resulting extension H_1 would be isomorphic to G_1. Otherwise, any $(d+k)$-vertex deleted dacard of the extension H_1 is connected and it contains no vertex adjacent to all the vertices of the other partite set, or exactly one partite set has a $(d-1)$-vertex and a vertex (say w) such that w is adjacent to all the vertices of the other partite set. Hence no dacard of H_1 is isomorphic to $(d+k, G_1-x')$ and so $drn(G_1)=2$, which means that G_1 can be obtained uniquely from the new dacards $(d+1, G_1-y)$ and $(d+k, G_1-x')$. Now the vertex z in G_1 is identifiable as the unique vertex of degree at least $d+k+1$. Consequently, G can be obtained uniquely from G_1 by deleting the vertex z. In other words, G can be obtained uniquely from the two dacards $(d+1, G_1-y)$ and $(d+k, G_1-x')$ and hence $drn(G)=2$, which completes the case that G is connected.

Now G is assumed to be disconnected; let $G_1, G_2, ..., G_n$ be the components of G. Let (X_i, Y_i) be the bipartition of G_i, $i=1,2,\ldots,n$. Suppose that X_i contains m'_i (respectively n'_i) vertices of degree d (respectively $d+k$) and Y_i contains m''_i (respectively n''_i) vertices of degree d (respectively $d+k$). If $m'_i > m''_i$ (or $n'_i > n''_i$) for some i, then consider any one component of G and proceeding as in Case 1 (or Case 2), we get $drn(G)=2$. So, assume that $m'_i = m''_i$ and $n'_i = n''_i$ for all i.

Suppose that $m'_i \geq m'_j$ or $n'_i \geq n'_j$ for some $i \neq j$. If $m'_i > m'_j$ and $n'_i = n'_j$ for some $i \neq j$, then we consider the two dacards $(d+k, G-x'_{n'_i})$ and $(d, G-x_{n'_j})$. In $G-x_{n'_j}$, there is a component containing exactly one partite set with n'_i vertices of degree $d+k$ and all the vertices in the other partite set have degree d or $d+k$. In $H(d+k, G-x'_{n'_i})$, if v were joined to all the vertices of degree $d-1$ or $d+1$, then H would be isomorphic to G; otherwise any d-vertex deleted dacard of H contains a component such that one of the partite sets of the component contains m'_i vertices of degree d and all the vertices in the other partite set have degree $d-1$ or $d+k-1$. Therefore no dacard of H would be isomorphic to $G-x_{n'_j}$.

If $m'_i = m'_j$ and $n'_i > n'_j$ for some $i \neq j$, then consider the two dacards $(d, G-x_{n'_i})$ and $(d+k, G-x'_{n'_j})$. In $G-x'_{n'_j}$, there is a component containing exactly one partite set having n'_i vertices of degree $d+k$ and all the vertices in the other partite set have degree d or $d+k$. In $H(d, G-x_{n'_i})$, if v were joined to the pairs of vertices of degrees $d-1$ and $d+k-1$, then H would be isomorphic to G; otherwise, any $(d+k)$-vertex deleted dacard of H contains a component such that one of the partite sets of it contains m'_i vertices of degree d and all the vertices in other partite set have degree $d-1$ or $d+k-1$. Thus no dacard of H would be isomorphic to $G-x'_{n'_j}$. Finally, if $m'_i = m'_j$ and $n'_i = n'_j$ for all $i \neq j$; then consider any one component of G and by proceeding as in Case 3, we get $drn(G)=2$.

From Theorems 3 to 6, we have the following main result.

Theorem 7. *If G is a biregular bipartite graph with degrees d and $d+k$, where $k \geq 2$, then $drn(G)=2$.*

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