The Single Parameter Family of Gini Bonferroni Welfare Functions and the Binomial Decomposition, Transfer Sensitivity and Positional Transfer Sensitivity

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Abstract We consider the binomial decomposition of generalized Gini welfare functions in terms of the binomial welfare functions $C_j, j = 1, \dots, n$ and we examine the weighting structure of the latter, which progressively focus on the poorest part of the population. In relation with the generalized Gini welfare functions, we introduce measures of transfer sensitivity and positional transfer sensitivity and we illustrate the behaviour of the binomial welfare functions $C_j, j = 1, \dots, n$ with respect to these measures. We investigate the binomial decomposition of the Gini Bonferroni welfare functions and we illustrate the dependence of the binomial decomposition coefficients in relation with the single parameter which describes the family. Moreover we examine the family of Gini Bonferroni welfare functions with respect to the transfer sensitivity and positional transfer sensitivity principles.

Keywords Generalized Gini welfare functions · Binomial decomposition · Single parameter family of Gini Bonferroni welfare functions · Principle of transfer sensitivity · Principle of positional transfer sensitivity

1 Introduction

The study of welfare and inequality has been the research interest of many economical and social scientists, and has been understood as an investigation on the departure from the ideal situation of economic equalitarianism, where each individual of the population has an equal share of the total income. In this sense, different welfare and inequality measures, with different characteristics, have been introduced in the literature in order to express the fairness of the income distribution in society.

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The generalized Gini welfare functions introduced by Weymark [60], and the associated inequality indices in Atkinson-Kolm-Sen's (AKS) framework, see Atkinson [5], Kolm [48, 49], and Sen [55], are related by Blackorby and Donaldson's correspondence formula [13, 15], $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, where $A(\boldsymbol{x})$ denotes a generalized Gini welfare function, $G(\boldsymbol{x})$ is the associated absolute inequality index, and $\bar{x}$ is the plain mean of the income distribution $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{D}^n$ of a population of $n \geq 2$ individuals, with income domain $\mathbb{D} = [0, \infty)$.

The generalized Gini welfare functions [60] have the form $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)}$ where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and, as required by the principle of inequality aversion, $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to a particular class of the ordered weighted averaging (OWA) functions introduced by Yager [63], which in turn correspond [34] to the Choquet integrals associated with symmetric capacities.

In this paper we recall the binomial decomposition of generalized Gini welfare functions due to Calvo and De Baets [22], see also Bortot and Pereira [20]. The binomial decomposition is formulated in terms of the functional basis formed by the binomial welfare functions.

The binomial welfare functions, denoted C_j with $j = 1, \dots, n$, have null weights associated with the $j - 1$ richest individuals in the population and therefore they are progressively focused on the poorest sector of the population.

The paper is organized as follows. In Sect. 2 we review the notions of generalized Gini welfare function and associated generalized Gini inequality index, and we introduce general measures of transfer sensitivity and positional transfer sensitivity.

In Sect. 3 we briefly review the Gini and Bonferroni welfare functions and inequality indices, and we examine them with respect to the principles of transfer sensitivity and positional transfer sensitivity.

In Sect. 4 we consider the binomial decomposition of generalized Gini welfare functions in terms of the binomial welfare functions $C_j, j = 1, \dots, n$. We illustrate the weights of the binomial welfare functions $C_j, j = 1, \dots, n$, which progressively focus on the poorest sector of the population, and we examine their transfer sensitivity and positional transfer sensitivity properties.

Finally, in Sect. 5 we investigate the Gini Bonferroni welfare functions with parameter $\gamma \in [0, 1]$, particularly in the context of the binomial decomposition. Moreover, we illustrate the weighting structure of the Gini Bonferroni welfare functions and we study their measures of transfer sensitivity and positional transfer sensitivity in terms of the parameter $\gamma \in [0, 1]$. Section 6 contains some conclusive remarks.

2 Generalized Gini Welfare Functions and Inequality Indices

In this section we consider populations of $n \geq 2$ individuals and we briefly review the notions of generalized Gini welfare function and generalized Gini inequality index over the income domain $\mathbb{D} = [0, \infty)$. The income distributions in this framework are represented by points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. We introduce general measures of transfer sensitivity and positional transfer sensitivity.

We begin by presenting notation and basic definitions regarding averaging functions on the domain $\mathbb{D}^n$, with $n \geq 2$ throughout the text. Comprehensive reviews of averaging functions can be found in Chisini [27], Fodor and Roubens [35], Calvo et al. [23], Beliakov et al. [10], Grabisch et al. [46], and Beliakov et al. [9].

Notation. Points in $\mathbb{D}^n$ are denoted $\boldsymbol{x} = (x_1, \dots, x_n)$, with $\mathbf{1} = (1, \dots, 1)$, $\mathbf{0} = (0, \dots, 0)$. Accordingly, for every $x \in \mathbb{D}$, we have $x \cdot \mathbf{1} = (x, \dots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \dots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in \mathbb{D}^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \dots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \dots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \dots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

Definition 1 Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function. We say that

1. A is *monotonic* if $\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is *strictly monotonic* if $\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.
2. A is *idempotent* if $A(x \cdot \mathbf{1}) = x$, for all $x \in \mathbb{D}$. On the other hand, A is *nilpotent* if $A(x \cdot \mathbf{1}) = 0$, for all $x \in \mathbb{D}$.
3. A is *symmetric* if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1, \dots, n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.
4. A is *invariant for translations* if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is *stable for translations* if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.
5. A is *invariant for dilations* if $A(t \cdot \boldsymbol{x}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is *stable for dilations* if $A(t \cdot \boldsymbol{x}) = t A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.

The terms positive (negative), increasing (decreasing), and monotonic are used in the weak sense. Otherwise these properties are said to be strict.

We introduce the majorization relation on $\mathbb{D}^n$ and we discuss the concept of income transfer following the approach in Marshall and Olkin [51], focusing on the classical results relating majorization, income transfers, see Marshall and Olkin [51, Chap. 4, Proposition A.1].

Definition 2 The *majorization relation* $\preceq$ on $\mathbb{D}^n$ is defined as follows: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that

$$\boldsymbol{x} \preceq \boldsymbol{y} \quad \text{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \dots, n \tag{1}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. As usual, we write $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and not $\boldsymbol{y} \preceq \boldsymbol{x}$, and we write $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$. We say that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are indifferent if $\boldsymbol{x} \sim \boldsymbol{y}$.

The majorization relation is a partial preorder, in the sense that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ are comparable only when $\bar{x} = \bar{y}$, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by a permutation. Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$, with mean income $\bar{x}$, it holds that $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ since $k\bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \dots, n$. The majorization is strict, $\bar{x} \cdot \mathbf{1} \prec \boldsymbol{x}$, when $\boldsymbol{x}$ is not a uniform income distribution.

Definition 3 Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that $\boldsymbol{x}$ is derived from $\boldsymbol{y}$ by means of an *income transfer* if, for some pair $i, j = 1, \dots, n$ with $y_i \leq y_j$, we have

$$x_i = (1-\varepsilon)\, y_i + \varepsilon y_j \qquad x_j = \varepsilon y_i + (1-\varepsilon)\, y_j \qquad \varepsilon \in [0, 1] \tag{2}$$

and $x_k = y_k$ for $k \neq i, j$. These formulas express an income transfer, from a richer to a poorer individual, of an income amount $\varepsilon(y_j - y_i)$. The income transfer obtains $\boldsymbol{x} = \boldsymbol{y}$ if $\varepsilon = 0$, and exchanges the relative positions of donor and recipient in the income distribution if $\varepsilon = 1$, in which case $\boldsymbol{x} \sim \boldsymbol{y}$. In the intermediate cases $\varepsilon \in (0, 1)$ the income transfer produces an income distribution $\boldsymbol{x}$ which is majorized by the original $\boldsymbol{y}$, that is $\boldsymbol{x} \prec \boldsymbol{y}$.

In general, for the majorization relation $\preceq$ and income distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{x}$ can be derived from $\boldsymbol{y}$ by means of a finite sequence of income transfers. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if any of the income transfers is not a permutation.

Definition 4 A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an *averaging function* if it is monotonic and idempotent. An averaging function is said to be *strict* if it is strictly monotonic. Note that monotonicity and idempotency implies that $\min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq \max(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{D}^n$.

Particular instances of averaging functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals. The former two are special cases of Choquet integration.

Definition 5 Given a weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0, 1]^n$, with $\sum_{i=1}^{n} w_i = 1$, the *Weighted Averaging (WA) function* associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i. \tag{3}$$

Definition 6 Given a weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0, 1]^n$, with $\sum_{i=1}^{n} w_i = 1$, the *Ordered Weighted Averaging (OWA) function* associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$A(\boldsymbol{w}) = \sum_{i=1}^{n} w_i x_{(i)}. \tag{4}$$

The traditional form of OWA functions as introduced by Yager [63] is as follows, $A(\boldsymbol{x}) = \sum_{i=1}^{n} \tilde{w}_i x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [64, 65] the theory and applications of OWA functions are discussed in detail. The following is a classical result particularly relevant in our framework. This result regards a form of dominance relation between OWA functions and the associated weighting structures, see for instance Bortot and Pereira [20] and references therein.

A class of welfare functions which plays a central role in this paper is that of the generalized Gini welfare functions introduced by Weymark [60], see also Mehran [52], Donaldson and Weymark [30, 31], Yaari [61, 62], Ebert [33], Quiggin [54], Ben-Porath and Gilboa [11].

Definition 7 Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$, with $w_1 \geq \cdots \geq w_n \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, the *generalized Gini welfare function* associated with $\boldsymbol{w}$ is the function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)} \tag{5}$$

and, in the AKS framework, the associated *generalized Gini inequality index* is defined as

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) = -\sum_{i=1}^{n} (w_i - \frac{1}{n}) x_{(i)} \,. \tag{6}$$

Generalized Gini welfare functions are *strict* if and only if $w_1 > \cdots > w_n > 0$. Moreover, generalized Gini welfare functions are stable for translations and the associated generalized Gini inequality indices are invariant for translations. Both are stable for dilations.

In relation with generalized Gini welfare functions, the principles of transfer sensitivity (TS) and positional transfer sensitivity (PTS) are based on the central notion of a progressive income transfer. Given an income distribution

$$\boldsymbol{x} = \left(x_{(1)}, \ldots, x_{(i)}, \ldots, x_{(j)}, \ldots, x_{(n)}\right)$$

and $i < j$ and $x_{(i)} \leq x_{(j)}$, we consider the progressive transfer of an income amount δ from $x_{(j)}$ to $x_{(i)}$, such that $x_{(i)} + \delta \leq x_{(j)} - \delta$. This progressive transfer results in a new income distribution

$$\boldsymbol{x}' = \left(x_{(1)}, \ldots, x_{(i)} + \delta, \ldots, x_{(j)} - \delta, \ldots, x_{(n)}\right) .$$

We consider thus a progressive income transfer δ from $x_{(j)}$ to $x_{(i)}$ with $i < j$. This transfer results in a new income distribution in which $x'_{(i)} = x_{(i)} + \delta$, $x'_{(j)} = x_{(j)} - \delta$, and $x'_{(k)} = x_{(k)}$ for $k \neq i, j$. From the definition (5) of generalized Gini welfare functions, we obtain

$$
\begin{aligned}
A(\boldsymbol{x}') - A(\boldsymbol{x}) &= \sum_{k=1}^{n} w_k x'_{(k)} - \sum_{k=1}^{n} w_k x_{(k)} \\
&= \Big[\big(w_1 x_{(1)} + \cdots + w_i (x_{(i)} + \delta) + \cdots + w_j (x_{(j)} - \delta) + \cdots + w_n x_{(n)} \big) \\
&\quad - \big(w_1 x_{(1)} + \cdots + w_i x_{(i)} + \cdots + w_j x_{(j)} + \cdots + w_n x_{(n)} \big) \Big] \\
&= \big(w_i - w_j \big) \delta \,. \qquad (7)
\end{aligned}
$$

Given that the weight difference $w_i - w_j$ is non negative, the generalized Gini welfare of the distribution $\boldsymbol{x}'$ is greater or equal than that of the original distribution $\boldsymbol{x}$. This means that the generalized Gini welfare function A satisfies the transfer sensitivity (TS) principle, or Pigou-Dalton principle, which states that welfare (inequality) measures should be non decreasing (non increasing) under progressive income transfers.

On the other hand, the principle of positional transfer sensitivity (PTS) states that the effect of an income transfer generates higher welfare values when it occurs at lower income levels. In fact, the non negative weight difference $w_i - w_j$ can vary with the position indicated by the indices i, j. In particular, with $j = i + 1$, we may have constant weight differences (the classical Gini case) or decreasing weight differences (the classical Bonferroni case), as we will see below.

We can measure the transfer sensitivity of generalized Gini welfare functions $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$ by means of

$$
TS(A) = \sum_{i=1}^{n-1} w_i - w_{i+1} = w_1 - w_n \quad \in [0, 1] \qquad (8)
$$

where w_i are the weights of the generalized Gini welfare function, with $i = 1, \ldots, n$.

The TS measure takes values in the unit interval $[0, 1]$. A TS value further away from zero indicates a higher level of transfer sensitivity. More specifically, as the value of the TS measure increases, transfer sensitivity increases too.

We can measure the positional transfer sensitivity of generalized Gini welfare functions $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)} \neq \bar{x}$ by means of

$$
PTS(A) = 1 - \sum_{i=1}^{n-1} \frac{\omega_i \ln \omega_i}{\ln(1/(n-1))} \quad \in [0, 1] \qquad (9)
$$

where ω_i with $i = 1, \ldots, n-1$ is given by

$$
\omega_i = \frac{w_i - w_{i+1}}{w_1 - w_n} \qquad i = 1, \ldots, n-1 \qquad (10)
$$

with $\omega_1, \ldots, \omega_{n-1} \geq 0$ and $\omega_1 + \cdots + \omega_{n-1} = 1$. In the case $\omega = 0$ we conventionally take $\omega \ln \omega = 0$.

This measure takes values in the unit interval $[0, 1]$. In fact, the summation term in (9), corresponding to the Shannon entropy of the $\omega_1, \ldots, \omega_{n-1}$ distribution, takes values in [0, 1] and reaches the maximum value 1 when such distribution is uniform, $\omega_1 = \cdots = \omega_{n-1} = 1/(n-1)$. Therefore, the higher the value of the $PTS(A)$ measure, the greater the positional transfer sensitivity of generalized Gini welfare function A in relation with income transfers from individual $j+1$ to individual j, with $j = 1, \ldots, n$.

3 Gini and Bonferroni Welfare Functions and the Associated inequality Indices

The classical Gini [37–39], Bonferroni [18, 19], and De Vergottini [28, 29] welfare functions and the associated inequality indices are classical instances of the AKS generalized Gini framework. In this section we recall the basic facts about the Gini and Bonferroni welfare functions and inequality indices and we examine their properties regarding transfer sensitivity and positional transfer sensitivity.

The classical Gini welfare function $A_G(\boldsymbol{x})$ and the associated classical Gini inequality index $G(\boldsymbol{x}) = \bar{x} - A_G(\boldsymbol{x})$ are defined as

$$A_G(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^G x_{(i)} \qquad w_i^G = \frac{2(n-i)+1}{n^2} \tag{11}$$

where the weights of $A_G(\boldsymbol{x})$ are positive and strictly decreasing with unit sum, $\sum_{i=1}^{n} w_i^G = 1$, and

$$G(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\frac{1}{n} - w_i^G\right) x_{(i)} = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2} x_{(i)} \tag{12}$$

where the coefficients of $G(\boldsymbol{x})$ have zero sum.

The classical absolute Gini inequality index G is traditionally defined as

$$G(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j| = -\frac{1}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(x_{(i)} - x_{(j)}\right) \tag{13}$$

where the double summation expression for $-n^2 G(\boldsymbol{x})$ in (13) can be written as

$$(-(n-1))x_{(1)} + (1-(n-2))x_{(2)} + \cdots + ((n-2)-1)x_{(n-1)} + (n-1)x_{(n)} \tag{14}$$

which corresponds to (12).

The classical Bonferroni welfare function $A_B(\boldsymbol{x})$ and the associated classical Bonferroni inequality index $B(\boldsymbol{x}) = \bar{x} - A_B(\boldsymbol{x})$ are defined as

$$A_B(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^B x_{(i)} \qquad w_i^B = \sum_{j=i}^{n} \frac{1}{jn} \tag{15}$$

where the weights of $A_B(\boldsymbol{x})$ are positive and strictly decreasing with unit sum, $\sum_{i=1}^{n} w_i^B = 1$, and

$$B(\boldsymbol{x}) = \sum_{i=1}^{n} \left(\frac{1}{n} - w_i^B \right) x_{(i)} \tag{16}$$

where the coefficients of $B(\boldsymbol{x})$ have zero sum.

The classical absolute Bonferroni inequality index B is traditionally defined as

$$B(\boldsymbol{x}) = \bar{x} - \frac{1}{n} \sum_{i=1}^{n} m_i(\boldsymbol{x}) \tag{17}$$

where the mean income of the i poorest individuals in the population is given by

$$m_i(\boldsymbol{x}) = \frac{1}{i} \sum_{j=1}^{i} x_{(j)} \qquad \text{for} \quad i = 1, \dots, n\,. \tag{18}$$

Therefore we have

$$A_B(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} m_i(\boldsymbol{x}) \tag{19}$$

$$= \frac{1}{n} \Big[\Big(x_{(1)} \Big) + \frac{1}{2} \Big(x_{(1)} + x_{(2)} \Big) + \cdots + \frac{1}{n} \Big(x_{(1)} + \cdots + x_{(n)} \Big) \Big] \tag{20}$$

$$= \frac{1}{n} \Big[\sum_{j=1}^{n} \frac{1}{j} x_{(1)} + \sum_{j=2}^{n} \frac{1}{j} x_{(2)} + \cdots \sum_{j=n}^{n} \frac{1}{j} x_{(n)} \Big] \tag{21}$$

which corresponds to (15).

The rich literature on the three classical cases of generalized Gini welfare functions—Gini, Bonferroni and De Vergottini—includes, for instance, Kolm [47], Atkinson [5], Sen [55, 56], Mehran [52], Blackorby and Donaldson [13–16], Lorenzen [50], Donaldson and Weymark [30, 31], Nygård and Sandström [53], Blackorby et al. [17], Weymark [60], Yitzhaki [66], Giorgi [40, 41], Benedetti [12], Ebert [32], Shorrocks and Foster [57], Yaari [62], Silber [58], Bossert [21], Tarsitano [59], Ben Porath and Gilboa [11], Zoli [68], Gajdos [36], Aaberge [1–3], Giorgi and Crescenzi [42, 43], Chakravarty and Muliere [26], Chakravarty [24, 25], Bárcena and Imedio [6], Giorgi and Nadarajah [44], Bárcena and Silber [7, 8], Aristondo et al. [4], and Zenga [67].

We now consider a progressive transfer δ from $x_{(j)}$ to $x_{(i)}$ with $i < j$. This transfer results in a new income distribution in which $x'_{(i)} = x_{(i)} + \delta$, $x'_{(j)} = x_{(j)} - \delta$, and $x'_{(k)} = x_{(k)}$ for $k \neq i, j$. From (11) and (15) we obtain

$$\begin{aligned} A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) &= \sum_{k=1}^{n} w_k^G x'_{(k)} - \sum_{k=1}^{n} w_k^G x_{(k)} \\ &= \Big[\big(w_1^G x_{(1)} + \cdots + w_i^G (x_{(i)} + \delta) + \cdots + w_j^G (x_{(j)} - \delta) + \cdots + w_n^G x_{(n)} \big) \\ &\quad - \big(w_1^G x_{(1)} + \cdots + w_i^G x_{(i)} + \cdots + w_j^G x_{(j)} + \cdots + w_n^G x_{(n)} \big) \Big] \\ &= \big(w_i^G \delta - w_j^G \delta \big) = \Big(\frac{2(n-i)+1}{n^2} - \frac{2(n-j)+1}{n^2} \Big) \delta \\ &= \frac{2}{n^2} (j - i)\, \delta \end{aligned} \tag{22}$$

$$\begin{aligned} A_B(\boldsymbol{x}') - A_B(\boldsymbol{x}) &= \sum_{k=1}^{n} w_k^B x'_{(k)} - \sum_{k=1}^{n} w_k^B x_{(k)} \\ &= \Big[\big(w_1^B x_{(1)} + \cdots + w_i^B (x_{(i)} + \delta) + \cdots + w_j^B (x_{(j)} - \delta) + \cdots + w_n^B x_{(n)} \big) \\ &\quad - \big(w_1^B x_{(1)} + \cdots + w_i^B x_{(i)} + \cdots + w_j^B x_{(j)} + \cdots + w_n^B x_{(n)} \big) \Big] \\ &= \big(w_i^B \delta - w_j^B \delta \big) = \Big(\sum_{k=i}^{n} \frac{1}{nk} - \sum_{k=j}^{n} \frac{1}{nk} \Big) \delta \\ &= \frac{1}{n} \Big(\frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{j-1} \Big) \delta = \Big(\frac{1}{n} \sum_{k=i}^{j-1} \frac{1}{k} \Big) \delta. \end{aligned} \tag{23}$$

Since $A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) > 0$ and $A_B(\boldsymbol{x}') - A_B(\boldsymbol{x}) > 0$, both welfare functions satisfy the principle of transfer sensitivity. Expression (22) implies that the increase in welfare, in the Gini case, depends on the difference $(j-i)$, irrespectively of the actual positions i, j. The Bonferroni welfare function, on the other hand, does depend on the actual positions i, j. Expression (23) indicates that the increase in welfare is greater if the transfer occurs at lower income levels and therefore the Bonferroni welfare function satisfies the principle of positional transfer sensitivity.

4 The Binomial Decomposition

In this section we review the binomial decomposition of generalized Gini welfare functions due to Calvo and De Baets [22] and Bortot and Pereira [20]. We examine the weighting structures of the binomial welfare functions C_j, with $j = 1, \ldots, n$, and we illustrate their properties regarding transfer sensitivity and positional transfer sensitivity.

Definition 8 The *binomial welfare functions* $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $j = 1, \dots, n$, are defined as

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}}\, x_{(i)} \qquad j = 1, \dots, n \tag{24}$$

where the binomial weights w_{ji}, $i, j = 1, \dots, n$ are null when $i + j > n + 1$, according to the usual convention that $\binom{p}{q} = 0$ when $p < q$, with $p, q = 0, 1, \dots$ Given that the binomial weights are decreasing, $w_{j1} \geq w_{j2} \geq \dots \geq w_{jn}$ for $j = 1, \dots, n$, the binomial welfare functions are generalized Gini welfare functions.

With the exception of $C_1(\boldsymbol{x}) = \bar{x}$, the binomial welfare functions C_j, $j = 2, \dots, n$ have an increasing number of null weights, in correspondence with $x_{(n-j+2)}, \dots, x_{(n)}$. The weight normalization of the binomial welfare functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \dots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \dots, n\,. \tag{25}$$

The binomial welfare functions C_j, $j = 1, \dots, n$ are continuous, idempotent, and stable for translations, where the latter two properties follow immediately from the unit sum normalization of the binomial weights. Moreover, due to the cumulative property of the binomial weights, see Calvo and De Baets [22], see also Bortot and Pereira [20], the binomial welfare functions satisfy the relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \dots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$.

Proposition 1 *Generalized Gini welfare functions* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \dots + \alpha_n C_n(\boldsymbol{x}) \tag{26}$$

where the coefficients α_j, $j = 1, \dots, n$ *are subject to the following conditions,*

$$\alpha_1 = 1 - \sum_{j=2}^{n} \alpha_j \geq 0 \tag{27}$$

$$\sum_{j=2}^{n} \left[1 - n \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \right] \alpha_j \leq 1 \qquad i = 2, \dots, n \tag{28}$$

$$\sum_{j=2}^{n} \frac{\binom{n-i}{j-2}}{\binom{n}{j}}\, \alpha_j \geq 0 \qquad i = 2, \dots, n\,. \tag{29}$$

The binomial welfare functions constitute therefore a functional basis for the generalized Gini welfare functions, which can be uniquely expressed as

$A(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j \, C_j(\mathbf{x})$ where the coefficients α_j, $j = 1, \dots, n$ satisfy the constraints (27)–(29), one of which is $\sum_{j=1}^{n} \alpha_j = 1$. However, the binomial decomposition does not express a simple convex combination of the binomial welfare functions, as the condition $\alpha_1 + \cdots + \alpha_n = 1$ might suggest. In fact, condition (27) ensures $\alpha_1 \geq 0$ but conditions (28) and (29) allow for negative $\alpha_2, \dots, \alpha_n$ values.

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $C_2(\boldsymbol{x})$, which has $n-1$ positive linearly decreasing weights and one null last weight, is the only strict binomial welfare function. In terms of the classical Gini welfare function we have that

$$A^c(\boldsymbol{x}) = \frac{1}{n}\, C_1(\boldsymbol{x}) + \frac{n-1}{n}\, C_2(\boldsymbol{x})\,. \tag{30}$$

The remaining binomial welfare functions $C_j(\boldsymbol{x})$, $j = 3, \dots, n$, have $n - j + 1$ positive non-linear decreasing weights and $j - 1$ null last weights.

In dimensions $n = 4, 8$ the weights $w_{ij} \in [0, 1]$, $i, j = 1, \dots, n$ of the binomial welfare functions C_j, $j = 1, \dots, n$ are as follows,

$$\begin{aligned}
n = 4 \quad C_1 &: (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \\
C_2 &: (\tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}, 0) \\
C_3 &: (\tfrac{3}{4}, \tfrac{1}{4}, 0, 0) \\
C_4 &: (1, 0, 0, 0)
\end{aligned}$$

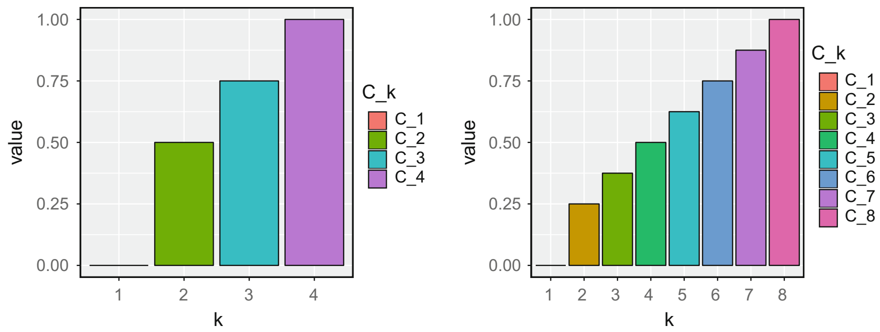
$$\begin{aligned}
n = 8 \quad C_1 &: (\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}) \\
C_2 &: (\tfrac{7}{28}, \tfrac{6}{28}, \tfrac{5}{28}, \tfrac{4}{28}, \tfrac{3}{28}, \tfrac{2}{28}, \tfrac{1}{28}, 0) \\
C_3 &: (\tfrac{21}{56}, \tfrac{15}{56}, \tfrac{10}{56}, \tfrac{6}{56}, \tfrac{3}{56}, \tfrac{1}{56}, 0, 0) \\
C_4 &: (\tfrac{35}{70}, \tfrac{20}{70}, \tfrac{10}{70}, \tfrac{4}{70}, \tfrac{1}{70}, 0, 0, 0) \\
C_5 &: (\tfrac{35}{56}, \tfrac{15}{56}, \tfrac{5}{56}, \tfrac{1}{56}, 0, 0, 0, 0) \\
C_6 &: (\tfrac{21}{28}, \tfrac{6}{28}, \tfrac{1}{28}, 0, 0, 0, 0, 0) \\
C_7 &: (\tfrac{7}{8}, \tfrac{1}{8}, 0, 0, 0, 0, 0, 0) \\
C_8 &: (1, 0, 0, 0, 0, 0, 0, 0)
\end{aligned}$$

The binomial welfare functions C_j, $j = 1, \dots, n$ have null weights associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population.

In order to measure the transfer sensitivity of the binomial welfare functions C_j, with $j = 1, \dots, n$ we consider a transfer from the richest to the poorest individual. To measure the transfer sensitivity of the binomial welfare functions C_j, with $j = 1, \dots, n$, we use expression (8).

In Fig. 1 we can see the values of the $TS(C_j)$ measure of the binomial welfare functions C_j, with $j = 1, \dots, n$ for the cases $n = 4, 8$.

In both cases $n = 4, 8$ we observe that TS increases linearly for $j = 3, \dots, n$, which means that the TS difference $C_j - C_{j-1}$ between any 2 consecutive binomial welfare functions is the same. This can be proved as follows,



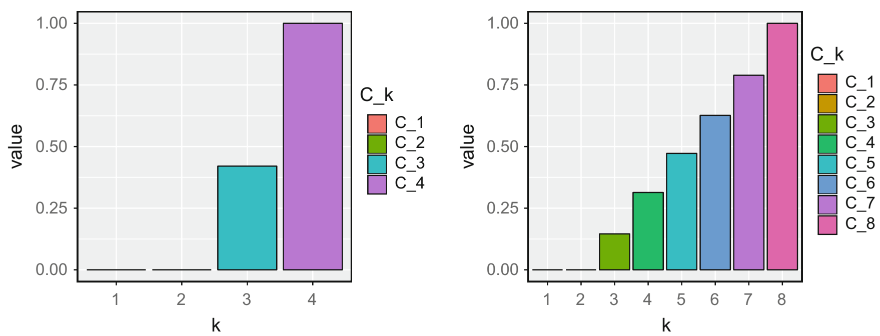
(a) Case $n = 4$ (b) Case $n = 8$

Fig. 1 Transfer sensitivity of C_j, for $j = 1, \dots, n$

$$(w_{j1} - w_{jn}) - (w_{j-1,1} - w_{j-1,n}) = \frac{1}{\binom{n}{j}}\left[\binom{n-1}{j-1} - \binom{n-n}{j-1}\right] - \frac{1}{\binom{n}{j-1}}\left[\binom{n-1}{j-2} - \binom{n-n}{j-2}\right]$$
$$= \frac{(n-1)!\, j!}{(j-1)!\, n!} - \frac{(n-1)!(j-1)!}{n!(j-2)!}$$
$$= \frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$$

where w_{ji} are the binomial weights in (24) with $i, j = 1, \dots, n$.

In order to measure the positional transfer sensitivity of the binomial welfare functions C_j, with $j = 1, \dots, n$ we consider $n - 1$ income transfers, each time from an individual in position j to the individual in position $j - 1$, with $j = 1, \dots, n$. To measure the positional transfer sensitivity of the binomial welfare functions C_j, with $j = 1, \dots, n$ we use expression (9)



(a) Case $n = 4$ (b) Case $n = 8$

Fig. 2 Positional transfer sensitivity of C_j, for $j = 1, \dots, n$

In Fig. 2 we illustrate the *PTS* values of the binomial welfare functions C_j, with $j = 1, \ldots, n$ in the cases $n = 4, 8$. We observe in both cases that *PTS* is null for $j = 1, 2$ while for $j = 3, \ldots, n$ it increases monotonically, not linearly.

5 The Single Parameter Gini Bonferroni Welfare Functions

The single parameter family of Gini Bonferroni (GB) welfare functions, which interpolates between the classical Gini and Bonferroni cases, has been introduced by Bárcena and Silber [8]. We recall the definition of the single parameter GB welfare functions and we examine their binomial decomposition. Moreover, we study the measures of transfer sensitivity and positional transfer sensitivity in terms of the parameter $\gamma \in [0, 1]$.

The welfare functions of the GB family are of the form

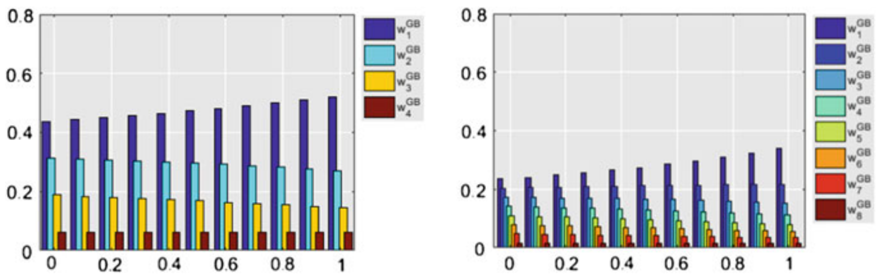
$$A_{GB}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^{GB} x_{(i)} \tag{31}$$

with

$$w_i^{GB} = (1/n^2)\left[n - i(n/i)^\gamma + \sum_{j=i}^{n}(n/j)^\gamma\right] \qquad \gamma \in [0, 1] \tag{32}$$

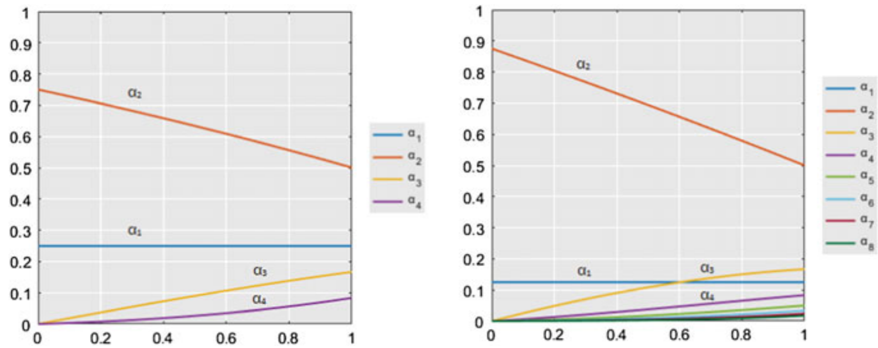
where the classical Gini and Bonferroni welfare functions are special cases with $\gamma = 0, 1$. Note that when $\gamma = 0$ we obtain the "equally distributed equivalent level of income" corresponding to the Gini welfare function, while when $\gamma = 1$ we obtain the "equally distributed equivalent level of income" corresponding to the Bonferroni welfare function.

Given that the weights of the GB welfare functions are strictly decreasing, $w_1^{GB} > w_2^{GB} > \cdots > w_n^{GB} = 1/n^2$, the GB welfare functions are generalized Gini welfare functions. The weighting structure of the GB welfare functions is illustrated in Fig. 3 in the cases $n = 4, 8$.



(a) Case $n = 4$ (b) Case $n = 8$

Fig. 3 Weights of the GB welfare functions for parameter values $\gamma = 0, 0.1, \ldots, 1$



(a) $\alpha_1, \ldots, \alpha_4$ with $\gamma \in [0, 1]$. (b) $\alpha_1, \ldots, \alpha_8$ with $\gamma \in [0, 1]$.

Fig. 4 Coefficients of the binomial decomposition for $n = 4, 8$

In the framework of the binomial decomposition (26), each GB welfare function $A_{GB}(\boldsymbol{x})$ can be uniquely expressed in terms of the binomial Gini welfare functions $C_1, C_2, \ldots C_n$ as follows,

$$A_{GB}(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \cdots + \alpha_n C_n(\boldsymbol{x}) \qquad \gamma \in [0, 1] \tag{33}$$

which can be written as

$$\sum_{i=1}^{n} w_i^{GB} x_{(i)} = \alpha_1 \sum_{i=1}^{n} w_{1i}\, x_{(i)} + \alpha_2 \sum_{i=1}^{n} w_{2i}\, x_{(i)} + \cdots + \alpha_n \sum_{i=1}^{n} w_{ni}\, x_{(i)} \qquad \gamma \in [0, 1]\,. \tag{34}$$

The expression of the binomial decomposition is unique and therefore, for each value of the parameter $\gamma \in [0, 1]$, we obtain a unique vector $(\alpha_1, \ldots, \alpha_n)$ by solving the linear system

$$\begin{cases} w_1^{GB} = \alpha_1 w_{11} + \alpha_2 w_{21} + \cdots + \alpha_n w_{n1} \\ w_2^{GB} = \alpha_1 w_{12} + \alpha_2 w_{22} + \cdots + \alpha_n w_{n2} \\ \qquad \cdots \\ w_n^{GB} = \alpha_1 w_{1n} + \alpha_2 w_{2n} + \cdots + \alpha_n w_{nn} \end{cases} \tag{35}$$

where the binomial weights w_{ji}, $i, j = 1, \ldots, n$ are as in (24).

In Fig. 4 we depict the vector $(\alpha_1, \ldots, \alpha_n)$ as a function of the parameter $\gamma \in [0, 1]$ in the cases $n = 4, 8$.

We observe, as expected, that $\alpha_1 = 1/n$ is independent of the parameter $\gamma \in [0, 1]$ since, in the last equation of the linear system (35), we have $w_n^{GB} = 1/n^2$ and $w_{1n} = 1/n$ and $w_{2n} = \cdots = w_{nn} = 0$.

On the other hand, we observe that only α_2 is decreasing, whereas $\alpha_3, \ldots \alpha_n$ are increasing with respect to $\gamma \in [0, 1]$.

It is well known that the classical Gini welfare function is 2-additive, see for instance Grabisch [45] and Bortot and Pereira [20] and references therein. On the other hand, the classical Bonferroni welfare function is n-additive. In fact in Fig. 4 we observe that only $\alpha_1, \alpha_2 \neq 0$ in the classical Gini case $\gamma = 0$, and $\alpha_1, \ldots, \alpha_n \neq 0$ in the classical Bonferroni case $\gamma = 1$.

In order to illustrate the PTS principle in relation with the classical Gini and Bonferroni welfare functions, corresponding to the extreme values of the parameter $\gamma = 0, 1$, consider first the classical Gini welfare function $A_G(\boldsymbol{x})$, whose weighting structure for $n = 8$ is given by (11) as follows,

$$\boldsymbol{w}^G = \left(\frac{15}{64}, \frac{13}{64}, \frac{11}{64}, \frac{9}{64}, \frac{7}{64}, \frac{5}{64}, \frac{3}{64}, \frac{1}{64}\right). \tag{36}$$

Consider now a progressive income transfer δ from $x_{(j)}$ to $x_{(i)}$ with $j = i + 1$. This transfer results in a new income distribution in which $x'_{(i)} = x_{(i)} + \delta,\ \ x'_{(j)} = x_{(j)} - \delta$, and $x'_{(k)} = x_{(k)}$ for $k \neq i, j$. According to the expression for the classical Gini welfare difference (22), we obtain

$$\begin{aligned}
&\text{for } i = 1,\ \ j = 2\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_1^G - w_2^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 2,\ \ j = 3\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_2^G - w_3^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 3,\ \ j = 4\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_3^G - w_4^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 4,\ \ j = 5\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_4^G - w_5^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 5,\ \ j = 6\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_5^G - w_6^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 6,\ \ j = 7\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_6^G - w_7^G)\delta = \frac{1}{32}\,\delta \\
&\text{for } i = 7,\ \ j = 8\ :\ A_G(\boldsymbol{x}') - A_G(\boldsymbol{x}) = (w_7^G - w_8^G)\delta = \frac{1}{32}\,\delta.
\end{aligned}$$

We can see that any progressive income transfer generates the same increase in welfare, meaning that the classical Gini welfare function does not satisfies PTS.

Consider now the classical Bonferroni welfare function $A_B(\boldsymbol{x})$, whose weighting structure for $n = 8$ is given by (15) as follows,

$$\boldsymbol{w}^B = \left(\frac{761}{2240}, \frac{481}{2240}, \frac{341}{2240}, \frac{743}{6720}, \frac{533}{6720}, \frac{73}{1344}, \frac{15}{448}, \frac{1}{64}\right). \tag{37}$$

As before, consider a progressive income transfer δ from $x_{(j)}$ to $x_{(i)}$ with $j = i+1$. This transfer results in a new income distribution in which $x'_{(i)} = x_{(i)} + \delta,\ \ x'_{(j)} = x_{(j)} - \delta$, and $x'_{(k)} = x_{(k)}$ for $k \neq i, j$. According to the expression for the classical Bonferroni welfare difference (23), we obtain

$$\begin{aligned}
&\text{for } i=1,\ \ j=2\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_1^B-w_2^B)\delta=\frac{1}{8}\,\delta\\
&\text{for } i=2,\ \ j=3\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_2^B-w_3^B)\delta=\frac{1}{16}\,\delta\\
&\text{for } i=3,\ \ j=4\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_3^B-w_4^B)\delta=\frac{1}{24}\,\delta\\
&\text{for } i=4,\ \ j=5\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_4^B-w_5^B)\delta=\frac{1}{32}\,\delta\\
&\text{for } i=5,\ \ j=6\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_5^B-w_6^B)\delta=\frac{1}{40}\,\delta\\
&\text{for } i=6,\ \ j=7\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_6^B-w_7^B)\delta=\frac{1}{48}\,\delta\\
&\text{for } i=7,\ \ j=8\ :\ A_B(\boldsymbol{x}')-A_B(\boldsymbol{x})=(w_7^B-w_8^B)\delta=\frac{1}{56}\,\delta.
\end{aligned}$$

We can see in this case that the actual position in which the progressive income transfer occurs has a differentiated impact on welfare. More specifically, the increase in welfare is greater when the transfer applies to the lowest income levels.

In general, we can measure the transfer sensitivity and positional transfer sensitivity of the GB welfare functions in terms of the parameter $\gamma \in [0,1]$ using the measures in (8) and (9) as follows,

$$TS(\gamma)=\sum_{i=1}^{n-1} w_i^{GB}-w_{i+1}^{GB}=w_1^{GB}-w_n^{GB}\,, \tag{38}$$

where w_i^{GB} are the weights of the single parameter GB welfare functions A_{GB} associated with the parameter $\gamma \in [0,1]$, with $i=1,\dots,n$.

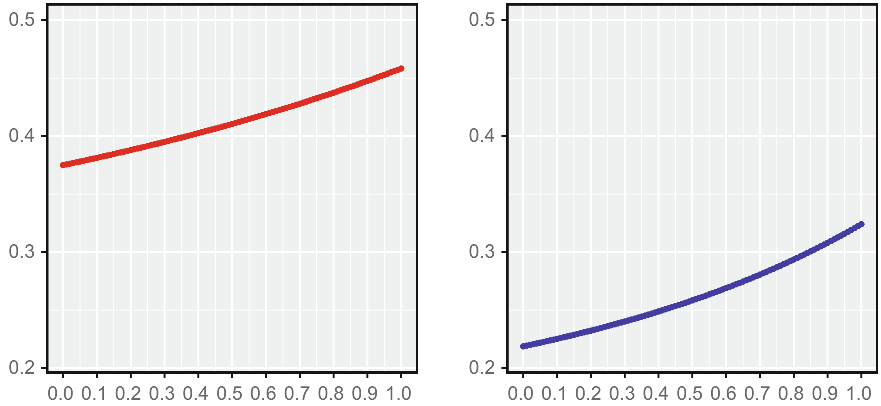
$$PTS(\gamma)=1+\sum_{i=1}^{n-1}\frac{\omega_i \ln \omega_i}{\ln(n-1)}\,, \tag{39}$$

where ω_i, with $i=1,\dots,n-1$, is given by

$$\omega_i=\frac{w_i^{GB}-w_{i+1}^{GB}}{w_1^{GB}-w_n^{GB}}$$

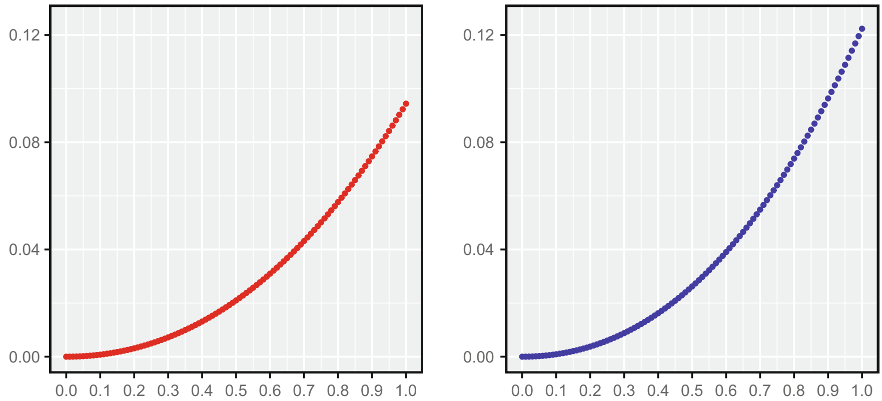
where w_i^{GB} are the weights of the GB welfare functions, with $i=1,\dots,n$.

In Figs. 5 and 6 we can see the measures of transfer sensitivity and positional transfer sensitivity of the GB welfare functions associated with the parameter $\gamma \in [0,1]$, in the cases $n=4,8$. As the parameter γ value increases, we observe that both transfer sensitivity and positional transfer sensitivity of the A_{GB} welfare function increase too. Notice the fact that transfer sensitivity is not null for $\gamma=0$, corresponding to the classical Gini case.



(a) TS in the case $n = 4$ (b) TS in the case $n = 8$

Fig. 5 Transfer sensitivity of the A_{GB} for parameter values $\gamma \in [0, 1]$, with $n = 4, 8$



(a) PTS in the case $n = 4$ (b) PTS in the case $n = 8$

Fig. 6 Positional transfer sensitivity of the A_{GB} for parameter values $\gamma \in [0, 1]$, with $n = 4, 8$

6 Conclusions

We have examined the binomial decomposition of the single parameter family of GB welfare functions and we have illustrated the dependence of the binomial decomposition coefficients in relation with the parameter which describes the GB family. We have found that $\alpha_1 = 1/n$ is independent of the parameter $\gamma \in [0, 1]$ and we have observed that only α_2 is decreasing, whereas $\alpha_3, \ldots, \alpha_n$ are increasing with respect to $\gamma \in [0, 1]$. In particular, since the binomial coefficients α_j with $j = 1, \ldots, n$ are non negative with unit sum, the decrease in α_2 compensates the increase in $\alpha_3, \ldots, \alpha_n$

with respect to $\gamma \in [0, 1]$. The Bonferroni welfare function, associated with $\gamma = 1$, is obtained by means of this compensatory mechanism.

Moreover, we have illustrated the transfer sensitivity and positional transfer sensitivity of the binomial welfare functions, and we have examined their properties with respect to these principles. For this purpose, we have introduced measures of transfer sensitivity and positional transfer sensitivity for generalized Gini welfare functions and, in particular, we have illustrated the behaviour of these measures in the case of the GB welfare functions, in relation with the values of the parameter $\gamma \in [0, 1]$.

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