Non-uniqueness of Solutions of a Semilinear Heat Equation with Singular Initial Data

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Abstract. We construct new examples of non-uniqueness of positive solutions of the Cauchy problem for the Fujita equation. The solutions we find are not self-similar and some of them blow up in finite time. Heteroclinic connections and ancient solutions of a rescaled equation play the key role in our construction.

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Non-uniqueness · Heteroclinic connections

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1 Introduction

We study non-uniqueness of positive solutions of the Fujita equation

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N, \quad p > 1, \tag{1}$$

with the initial condition

$$u(x, 0) = \ell |x|^{-m}, \qquad x \in \mathbb{R}^N \setminus \{0\}, \quad \ell \geq 0, \quad m := \frac{2}{p-1}. \tag{2}$$

By a solution of (1) in $\mathbb{R}^N \times (0, T)$, $0 < T \leq \infty$, we mean a function $u \in C^{2,1}(\mathbb{R}^N \times (0, T))$ which satisfies (1) in the classical sense in $\mathbb{R}^N \times (0, T)$. We call a solution global if $T = \infty$. By a solution of (1), (2) we mean a solution of (1) which is continuous in $\mathbb{R}^N \times [0, T) \setminus \{0, 0\}$ and $u(\cdot, t) \to u(\cdot, 0)$ in $L^1_{loc}(\mathbb{R}^N)$ as $t \to 0$.

Dedicated to Bernold Fiedler on the occasion of his 60th birthday.

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Concerning the existence of positive global solutions of (1), the Fujita exponent

$$p_F := \frac{N+2}{N}$$

is critical. In fact, if $1 < p \le p_F$ then there is no positive global solution of (1).

We introduce three more critical exponents which play an important role in the sequel. The exponent

$$p_{sg} := \begin{cases} \frac{N}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N \le 2, \end{cases}$$

is related to the existence of a singular steady state explicitly given by

$$\varphi(x) := L|x|^{-m}, \qquad x \in \mathbb{R}^N \setminus \{0\},$$

where

$$L := \{m\,(N-2-m)\}^{1/(p-1)}, \qquad m := \frac{2}{p-1}.$$

Namely, φ exists if and only if $p > p_{sg}$. The role of the Sobolev exponent

$$p_S := \begin{cases} \frac{N+2}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N \le 2, \end{cases}$$

and the Joseph-Lundgren exponent

$$p_{JL} := \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N > 10, \\ \infty & \text{for } N \le 10, \end{cases}$$

will be explained below.

The uniqueness problem for (1), (2) is of particular interest from the viewpoint of continuation beyond blow-up. The case $\ell = L$ was considered in [11] where it was shown that φ is the unique solution if $p \ge p_{JL}$ but not if $p_{sg} < p < p_{JL}$.

For $0 < \ell < L$, the following conjectures were formulated in [11] (page 41):

(C1) If $p_{sg} < p \le p_S$ then problem (1), (2) has exactly two positive solutions.
(C2) If $p_S < p < p_{JL}$ then there exist an arbitrarily large finite number of solutions when $L - \ell$ is small.

It was shown later that there are at least two positive solutions if $p_F < p < p_S$ (see [18]) or if $p = p_S$, $2 < N < 6$ (see [20]). The existence of an arbitrarily large finite number of solutions when $L - \ell$ is small was established in [25] for $p_S < p < p_{JL}$. The solutions found in [18,20,25] are self-similar. This means that they are of the form

$$u(x,t) := t^{-1/(p-1)} f(\rho), \qquad \rho := t^{-1/2}|x|, \quad x \in \mathbb{R}^N,\ t > 0,$$

where the function f satisfies

$$\begin{cases} f_{\rho\rho} + \dfrac{N-1}{\rho} f_\rho + \dfrac{\rho}{2} f_\rho + \dfrac{1}{p-1} f + f^p = 0, & \rho > 0, \\ f_\rho(0) = 0, \qquad \lim\limits_{\rho\to\infty} \rho^m f(\rho) = \ell, \qquad f(\rho) > 0, & \rho > 0. \end{cases} \tag{3}$$

In this paper we disprove conjecture (C1) from [11] by showing that if (3) has at least two solutions then there are infinitely many positive solutions of (1), (2) which are not self-similar. In particular, under the assumptions from (C2) the initial value problem (1), (2) possesses infinitely many positive solutions.

Before we state our results more precisely, we recall some known facts about problem (3). Set

$$\ell^* := \sup\{\ell > 0 \ : \ (3) \text{ has a solution}\},$$
$$\ell_* := \inf\{\ell > 0 \ : \ (3) \text{ has at least two solutions}\}.$$

The results of [18–20] and [25] imply:

Proposition 1. *Let $p_F < p < p_{JL}$. Then*

$$0 \le \ell_* < \ell^* < \infty.$$

Moreover, we have:

(i) $\ell_* = 0$ *if* $p_F < p < p_S$ *or if* $p = p_S$ *and* $2 < N < 6$*;*
(ii) $\ell_* \in (0, L]$ *if* $p_S < p < p_{JL}$ *or if* $p = p_S$ *and* $N \ge 6$*.*

These statements are contained in Propositions A and B in [21] and Remark 1.4 (iv) in [20].

We remark here that if $p \ge p_{JL}$ then $\ell^* = L$ and (3) has a unique solution for $\ell \in (0, L)$, see [19]. Concerning ordering and intersection properties of solutions of (3), the following was established in [18,19,21].

Proposition 2. *Let $p_F < p < p_{JL}$ and $\ell \in (\ell_*, \ell^*)$. Then there is a solution f_0 of (3) with the property that if f is a different solution then $f(\rho) > f_0(\rho)$ for all $\rho > 0$. If f_1 and f_2 are two solutions of (3), $f_1, f_2 \not\equiv f_0$, then there is $\rho_0 > 0$ such that $f_1(\rho_0) = f_2(\rho_0)$.*

The first statement follows from Lemma 3.1 (i) in [19] and the second from Proposition 4.1 in [21].

Now we can state our result on the non-uniqueness.

Theorem 1. *Assume that $p_F < p < p_{JL}$ and $\ell \in (\ell_*, \ell^*)$. Let f be a solution of (3), $f \not\equiv f_0$.*

(i) *There is a solution u of (1), (2) in $\mathbb{R}^N \times (0, \infty)$ such that*

$$f_0(\rho) < t^{1/(p-1)} u(x,t) < f(\rho), \qquad x \in \mathbb{R}^N, \quad t > 0.$$

(ii) *For every* $T > 0$ *there is a solution* u *of* (1), (2) *in* $\mathbb{R}^N \times (0, T)$ *such that*

$$t^{1/(p-1)} u(x,t) > f(\rho), \qquad x \in \mathbb{R}^N, \quad 0 < t < T,$$

and

$$\lim_{t \to T} u(0,t) = \infty.$$

It is clear from Proposition 2 that the solution in Theorem 1 (i) cannot be self-similar. Obviously, the solution in Theorem 1 (ii) is not self-similar either. For a more detailed description of these solutions see Propositions 3 and 4. Since problem (1), (2) possesses a scaling invariance, Theorem 1 in fact yields two different one-parameter families of solutions. Namely, if u is a solution which is not self-similar then

$$u_\lambda(x,t) := \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \qquad \lambda > 0,$$

is also a solution and it is different from u if $\lambda \neq 1$.

As a consequence of Theorem 1 we obtain the existence of infinitely many non-selfsimilar continuations of some backward self-similar solutions of (1) beyond their blow-up time. Namely, it was shown [6] that there is a sequence $\{\ell_n\}_{n=1}^{\infty}$ such that $\ell_n \to L$ and $\ell_n |x|^{-m}$ is the blow-up profile of a backward self-similar solution. Theorem 1 then yields a continuum of continuations which remain regular for all t after blow-up and converge to zero as $t \to \infty$, and another continuum of continuations which blow up at the origin again.

A radial solution which blows up twice was first found in [15] for $p > p_{JL}$. That result was later extended in [16] by finding radial solutions (for the same range of p) which blow up k-times, where $k > 1$ is an arbitrary integer. The solutions from [15,16] blow-up at the origin at each blow-up time.

Later, radial solutions of (1) which blow up twice were constructed in [17] for $p > p_S$. There, the new features are that the two blow-up times can be controlled (but not prescribed precisely) and the second blow-up is on a sphere.

We can prescribe both blow-up times precisely but our continuation beyond the first blow-up time is not minimal while the continuations in [15–17] are minimal.

Let us mention here that the issue of non-uniqueness of continuations beyond singularity is relevant for many other parabolic equations, such as the heat flow for harmonic maps between spheres [3], the Yang-Mills heat flow [12], the mean curvature flow [1], fourth order equations [10], to give just a few examples.

Before we introduce our second non-uniqueness result, we recall that if $p_F < p < p_S$ then there is a unique positive solution f^* of (3) with $\ell = 0$, see [5,13,29]. The function f^* satisfies

$$f^*(\rho) = O\left(\rho^{m-N} \exp(-\rho^2/4)\right) \qquad \text{as} \quad \rho \to \infty, \tag{4}$$

see [23], and $u(x,t) = t^{-1/(p-1)} f^*(\rho)$ is a solution of (1), (2) with $\ell = 0$, see [13].

Our second result on non-uniqueness is the following:

Theorem 2. *Let* $p_F < p < p_S$ *and* $\ell = 0$.

(i) *There is a solution u of* (1), (2) *in* $\mathbb{R}^N \times (0,\infty)$ *such that*

$$0 < t^{1/(p-1)}u(x,t) < f^*(\rho), \qquad x \in \mathbb{R}^N, \quad t > 0.$$

(ii) *For every $T > 0$ there is a solution u of* (1) (2) *in* $\mathbb{R}^N \times (0,T)$ *such that*

$$t^{1/(p-1)}u(x,t) > f^*(\rho), \qquad x \in \mathbb{R}^N, \quad 0 < t < T,$$

and

$$\lim_{t\to T} u(0,t) = \infty.$$

As before, the solutions in Theorem 2 are not self-similar and they yield two different one-parameter families of solutions. A more detailed description of these solutions is given in Propositions 5 and 6.

For other previous examples of non-uniqueness of solutions of (1) with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{5}$$

we refer to [14,26] where the case $p = p_{sg}$, $u_0 \in L^p(\mathbb{R}^N)$ was treated. In the case when the domain is a ball and the homogeneous Dirichlet boundary condition is imposed, examples of non-uniqueness were given in [2,22].

On the other hand, it is well known that for $q > N(p-1)/2$, $q \geq 1$ or $q = N(p-1)/2 > 1$ and $u_0 \in L^q(\mathbb{R}^N)$ there exists a unique solution u of (1), (5) in the class $C([0,T), L^q(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T), L^\infty(\mathbb{R}^N))$ for some $T \in (0,\infty]$, see [4,27,28].

2 Results for a Transformed Equation

For a solution u of (1) defined for $t \in (0,T)$, we set

$$v(y,s) := t^{1/(p-1)}u(x,t), \qquad y := t^{-1/2}x, \quad s := \log t.$$

Then we obtain the following equation for v:

$$v_s = \Delta v + \frac{1}{2} y \cdot \nabla v + \frac{1}{p-1} v + v^p, \qquad y \in \mathbb{R}^N, \tag{6}$$

where $s \in (-\infty, \log T)$.

If f is a solution of (3) then it is a radial steady state of (6).

The first two propositions give a more precise description of the solutions from Theorem 1.

Proposition 3. *Assume that $p_F < p < p_{JL}$ and $\ell \in (\ell_*, \ell^*)$. Let f be a solution of* (3), *$f \not\equiv f_0$. Then there exists a solution v of* (6) *in* $\mathbb{R}^N \times \mathbb{R}$ *with the following properties:*

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in $\rho = |y|$ and in s, and satisfies*

$$f_0(|y|) < v(y,s) < f(|y|), \qquad y \in \mathbb{R}^N, \quad s \in \mathbb{R},$$

(ii) $\|v(\cdot,s)-f(|\cdot|)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to-\infty$,
(iii) $\|v(\cdot,s)-f_0(|\cdot|)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to\infty$.

Proposition 4. *Assume that* $p_F<p<p_{JL}$ *and* $\ell\in(\ell_*,\ell^*)$. *Let* f *be a solution of* (3), $f\not\equiv f_0$. *Then for every* $s^*\in\mathbb{R}$ *there is a solution* v *of* (6) *in* $\mathbb{R}^N\times(-\infty,s^*)$ *such that:*

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in* $\rho=|y|$ *and increasing in* s, *and satisfies*

$$v(y,s)>f(|y|),\qquad y\in\mathbb{R}^N,\quad s\in(-\infty,s^*),$$

(ii) $\|v(\cdot,s)-f(|\cdot|)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to-\infty$,
(iii) $v(0,s)\to\infty$ *as* $s\to s^*$.

The next two propositions describe in more detail the solutions from Theorem 2.

Proposition 5. *Assume that* $p_F<p<p_S$. *Then there exists a solution* v *of* (6) *in* $\mathbb{R}^N\times\mathbb{R}$ *with the following properties:*

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in* $\rho=|y|$ *and in* s, *and satisfies*

$$0<v(y,s)<f^*(|y|),\qquad y\in\mathbb{R}^N,\quad s\in\mathbb{R},$$

(ii) $\|v(\cdot,s)-f^*(|\cdot|)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to-\infty$,
(iii) $\|v(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to\infty$.

Proposition 6. *Assume that* $p_F<p<p_S$. *Then for every* $s^*\in\mathbb{R}$ *there is a solution* v *of* (6) *in* $\mathbb{R}^N\times(-\infty,s^*)$ *such that:*

(i) *The solution is positive, radially symmetric in space with respect to the origin, decreasing in* $\rho=|y|$ *and increasing in* s, *and satisfies*

$$v(y,s)>f^*(|y|),\qquad y\in\mathbb{R}^N,\quad s\in(-\infty,s^*),$$

(ii) $\|v(\cdot,s)-f^*(|\cdot|)\|_{L^\infty(\mathbb{R}^N)}\to 0$ *as* $s\to-\infty$,
(iii) $v(0,s)\to\infty$ *as* $s\to s^*$.

3 Proofs of the Main Results

Proof of Proposition 3. For each $\theta\in(0,1)$, let $v^\theta(y,s)$ denote the solution of (6) with the following initial data:

$$v^\theta(y,0)=f^\theta(y):=(1-\theta)f_0(|y|)+\theta f(|y|),\qquad y\in\mathbb{R}^N.$$

Then, since $f_0(|y|)<f(|y|)$ for $y\in\mathbb{R}^N$ and since the function $g(v):=\frac{1}{p-1}v+v^p$ is strictly convex, one easily finds that

$$\Delta f^\theta+\frac{1}{2}y\cdot\nabla f^\theta+\frac{1}{p-1}f^\theta+(f^\theta)^p<0,\qquad y\in\mathbb{R}^N. \tag{7}$$

In other words, f^θ is a time-independent strict super-solution of (6). Consequently, $v^\theta(y,s)$ is decreasing in s and satisfies $f^\theta(|y|) \geq v^\theta(y,s) > f_0(|y|)$ for all $s \geq 0$, $y \in \mathbb{R}^N$. Hence v^θ is defined for all $s \geq 0$ and converges as $s \to \infty$ to a stationary solution that lies between f^θ and f_0. Since there is no stationary solution that lies between f and f_0 by Proposition 2, we have $v^\theta(y,s) \to f_0(|y|)$ as $s \to \infty$ uniformly in $y \in \mathbb{R}^N$. Now, for each $\theta \in [\frac{1}{2}, 1)$, let s_θ be such that

$$v^\theta(0, s_\theta) = \frac{f_0(0) + f(0)}{2}.$$

Since v^θ is decreasing in s, the above quantity s_θ is uniquely determined, and we have

$$s_\theta = 0 \quad \text{for} \quad \theta = \frac{1}{2}, \qquad s_\theta \to \infty \quad \text{as} \quad \theta \nearrow 1.$$

Let us define

$$\hat{v}^\theta(y,s) := v^\theta(y, s + s_\theta). \tag{8}$$

Then $\hat{v}^\theta$ is a solution of (6) on the time interval $[-s_\theta, \infty)$ and it satisfies

$$\hat{v}^\theta(0,0) = \frac{f_0(0) + f(0)}{2}, \qquad \hat{v}^\theta(y,s) \searrow f_0(|y|) \quad \text{as} \quad s \to \infty.$$

By parabolic estimates, we can find a sequence $\theta_k \to 1$ such that $\hat{v}^{\theta_k}$ converges to a solution of (6) which is defined for all $s \in \mathbb{R}$, and we denote it by $\hat{v}(y,s)$. Clearly, $\hat{v}$ is non-increasing in s and satisfies

$$f_0(|y|) < \hat{v}(y,s) < f(|y|), \quad y \in \mathbb{R}^N, \quad s \in \mathbb{R}, \qquad \hat{v}(0,0) = \frac{f_0(0) + f(0)}{2}.$$

The monotonicity of $\hat{v}(y,s)$ and the parabolic estimates, along with the inequalities $f_0 < \hat{v} < f$ imply that $\hat{v}$ converges to some stationary solutions $f^\pm$ of (6) as $s \to \pm\infty$ that satisfy

$$f_0 \leq f^+ \leq f^- \leq f, \qquad f^+(0) \leq \frac{f_0(0) + f(0)}{2} \leq f^-(0).$$

By Proposition 2, we have $f^+ = f_0$ and $f^- = f$. Hence

$$\hat{v}(y,s) \to \begin{cases} f(|y|) & \text{as } s \to -\infty, \\ f_0(|y|) & \text{as } s \to \infty, \end{cases} \quad \text{uniformly in} \quad y \in \mathbb{R}^N.$$

This completes the proof of the proposition. □

Proof of Proposition 4. We use a similar argument as in the proof of Proposition 3. For each $\theta > 1$, let $v^\theta(y,s)$ denote the solution of (6) with the initial data

$$v^\theta(y,0) = f^\theta(y) := \theta f(|y|), \qquad y \in \mathbb{R}^N.$$

Since f^θ with $\theta > 1$ satisfies the inequality opposite to (7), v^θ is increasing in s. Denote by $[0, \sigma_\theta)$ the maximal time-interval for the existence of v^θ. Now, for each $\theta \in (1, 2)$, let s_θ be such that

$$v^\theta(0, s_\theta) = \frac{3}{2} f(0),$$

and define $\hat{v}^\theta$ by (8). Then, arguing as in the proof of Proposition 3, we can choose a sequence $\theta_k \searrow 1$ such that $\hat{v}^{\theta_k}$ converges to a solution $\hat{v}$ of (6) such that $\hat{v}$ is defined for $s \in (-\infty, s^*)$ for some $0 < \hat{s} \le \infty$, $\hat{v}$ is increasing in s, and satisfies

$$\hat{v}(0,0) = \frac{3}{2} f(0), \qquad \lim_{s \to -\infty} \hat{v}(y, s) = f(y) \ \text{ uniformly in } \ y \in \mathbb{R}^N.$$

Next we show that $\hat{s} < \infty$. Suppose $\hat{s} = \infty$. Then Lemma 3.1 in [21] implies that, as $s \to \infty$, $\hat{v}(\cdot, s)$ converges to a regular or singular radial steady state of (6) which is bigger than f. However, such a steady state does not exist, see Proposition 4.1 in [21]. This is a contradiction.

Any shift of $\hat{v}$ in s yields again a solution of (6), so the blow-up time s^* can be chosen arbitrarily. □

Proof of Proposition 5. One can proceed as in the proof of Proposition 3 with f_0 and f replaced by 0 and f^*, respectively. □

In the proof of Proposition 6 we shall use the following fact:

Lemma 1. *If $p_F < p < p_S$ then there is no solution f^s of the problem*

$$\begin{cases} f_{\rho\rho} + \dfrac{N-1}{\rho} f_\rho + \dfrac{\rho}{2} f_\rho + \dfrac{1}{p-1} f + f^p = 0, & \rho > 0, \\ f(0) = \infty, \qquad f(\rho) > 0, \quad \rho > 0, \end{cases} \tag{9}$$

such that $f^s(\rho) > f^*(\rho)$ for $\rho > 0$.

Proof of Proposition 5. Suppose f^s is such a solution. Let

$$C := \inf\{c \in \mathbb{R} : c f^s(\rho) \ge f^*(\rho) \text{ for all } \rho \in (0, \infty)\} \in (0, 1].$$

Then there are two cases:

Case I: $C f^s(R) = f^*(R)$ and $C f^s_\rho(R) = f^*_\rho(R)$ at some $R \in (0, \infty)$.

Case II: $C f^s(\rho) > f^*(\rho)$ for all $\rho \in (0, \infty)$ and $\lim_{\rho \to \infty} \dfrac{f^s(\rho)}{f^*(\rho)} = 1$.

Case I: f^* and f^s satisfy

$$(h(\rho) f^*_\rho)_\rho + h(\rho) \Big\{ \frac{1}{p-1} f^* + (f^*)^p \Big\} = 0$$

and

$$(h(\rho) f^s_\rho)_\rho + h(\rho) \Big\{ \frac{1}{p-1} f^s + (f^s)^p \Big\} = 0,$$

respectively, where $h(\rho) := \rho^{N-1}\exp(\rho^2/4)$. Multiplying the first equation by f^s and the second by f^* then taking their difference, we obtain

$$\frac{d}{d\rho}\{h(f^*_\rho f^s - f^* f^s_\rho)\} = -h\{(f^*)^{p-1} - (f^s)^{p-1}\}f^* f^s. \tag{10}$$

Integrating this on $[\rho, R]$, we have

$$\Big[h(f^*_\rho f^s - f^* f^s_\rho)\Big]_\rho^R = -\int_\rho^R h(\sigma)\{(f^*)^{p-1}(\sigma) - (f^s)^{p-1}(\sigma)\}f^*(\sigma)f^s(\sigma)d\sigma > 0.$$

Since $f^*_\rho f^s - f^* f^s_\rho = 0$ at $\rho = R$, we obtain $f^*_\rho f^s - f^* f^s_\rho < 0$ for $\rho \in (0, R)$. This implies that f^s/f^* is increasing in $\rho \in (0, R)$. However, this contradicts the assumption that f^s is singular at $\rho = 0$.

Case II: It follows from (4) that

$$f^*(\rho) \le A\rho^{m-N}\exp(-\rho^2/4), \qquad \rho > 1,$$

for some constant $A > 0$. Then we have

$$\begin{aligned} h(1)f^*_\rho(1) - h(\rho)f^*_\rho(\rho) &= \int_1^\rho h\Big\{\frac{1}{p-1}f^* + (f^*)^p\Big\}d\sigma \le K\int_1^\rho \sigma^{m-1}d\sigma \\ &= \frac{K}{m}(\rho^m - 1), \qquad \rho > 1, \end{aligned}$$

for some constant $K > 0$. Therefore, we have

$$0 < -f^*_\rho(\rho) \le K^*\rho^{m-N}\exp(-\rho^2/4), \qquad \rho > 1,$$

for some constant $K^* > 0$ and the same holds for f^s_ρ.

Hence the right-hand side of (10) is integrable up to $\rho = \infty$, and $h(f^*_\rho f^s - f^* f^s_\rho) \to 0$ as $\rho \to \infty$, so that

$$h(f^*_\rho f^s - f^* f^s_\rho) = \int_\rho^\infty h\{(f^*)^{p-1} - (f^s)^{p-1}\}f^* f^s d\sigma < 0, \quad \rho > 0.$$

This implies that f^s/f^* is increasing in $\rho \in (0, \infty)$, a contradiction. □

Proof of Proposition 6. The proof is analogous to the proof of Proposition 4 with f replaced by f^* except that we now use Lemma 1 to show that $\hat{s} < \infty$. We again suppose $\hat{s} = \infty$. Then Lemma 3.1 in [21] guarantees that $\hat{v}(y, s)$ converges (as $s \to \infty$) to a regular or singular radial steady state of (6) which is bigger than f^*. However, by Lemma 1, such a singular steady state does not exist. On the other hand, regular steady states different from f^* satisfy (3) with $\ell > 0$ and their value at 0 is smaller than $f^*(0)$, see [13], a contradiction. □

We remark that alternative proofs of Propositions 3–6 can be given using linearizations around f, f^* and construction of suitable sub- and supersolutions, see [7,9].

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