Chapter 2
Models

2.1 Continuum Mechanics

In this chapter, we derive the equations that describe the dynamics of fluids and solids. Matter is composed of molecules, atoms and smaller particles that all interact with each other. A description of the dynamics of these micro-structure is possible by fundamental physical laws. Such a particle centered view-point is however not feasible, if large physical objects are considered that consist of many atoms. To describe every particle in one liter of water, more than 10^{25} molecules must be considered. A description of every single molecule–or even every atom or subatomic particle–in a large scale hydrodynamical problem like the flow of water around a ship is completely out of bounds.

Instead, we consider a *continuum approach* for the description of the large scale dynamics. By a continuum, we denote a volume $V(t) \subset \mathbb{R}^3$ of (different) particles. Instead of describing every single particle, we only observe some few averaged properties of the complete volume. These properties are all considered as local density distributions. As example, we will denote by $\mathbf{v}(x, t)$ the average velocity of whatever particle may be in position $x \in V(t)$ at a given time t. Usually we assume that all physical quantities possess some smoothness. Depending on the situation, we will ask for integrability, continuity or differentiability.

In the following we will derive fundamental equations that describe the interplay of these averaged quantities. We will distinguish between basic physical principles, the *conservation principles* and *material laws*. While the conservation principles are based on *first principles* and we think of them as exact, *material laws* are usually simplifications, idealizations and derived by observation and measurements.

T. Richter, *Fluid-structure Interactions*, Lecture Notes in Computational Science and Engineering 118, DOI 10.1007/978-3-319-63970-3_2

2.1.1 Coordinate Systems

In the following, by $V(t) \subset \mathbb{R}^3$ we denote a *material volume*. We assume that $V(t)$ is entirely occupied by some material. This material has physical properties like density $\rho : V(t) \to \mathbb{R}$, velocity $\mathbf{v} : V(t) \to \mathbb{R}^3$, which is a three dimensional vector field, temperature $T : V(t) \to \mathbb{R}$ or pressure $p : V(t) \to \mathbb{R}$. We assume that the volume is moving. By $t_0 \in \mathbb{R}$ we denote the *initial time* and we observe the volume for $t \geq t_0$. By $V_0 := V(t_0)$ we denote the *reference configuration* of the volume. Often, t_0 is set arbitrarily, but we usually think of a system that is at rest and unstressed, e.g. a container filled with resting fluid or an elastic obstacle that is not deformed and where no stresses act. At time $t \geq t_0$, we denote by $V(t)$ the *current configuration*.

The volume $V(t)$ consists of particles, and we call $\hat{V} := V_0$ the material domain. For every particle $\hat{x} \in \hat{V}$, we denote by $x(\hat{x}, t) \in V(t)$ the location of the particle at time $t \geq t_0$. We assume that the path $\{x(\hat{x}, t), t \geq t_0\} \subset \mathbb{R}^3$ is continuous and that no two different particles $\hat{x}, \hat{x}' \in \hat{V}$ have the same position at any time $t \geq t_0$:

$$x(\hat{x}, t) = x(\hat{x}', t) \quad \Leftrightarrow \quad \hat{x} = \hat{x}'.$$

The mapping $\hat{T}(\hat{x}, t) := x(\hat{x}, t)$ is therefore invertible and we define the inverse mapping as $\hat{T}^{-1}(x, t) := \hat{x}(x, t)$. By $\hat{x}(x, t)$ we denote that particle $\hat{x} \in \hat{V}$ that at time $t \geq t_0$ takes position $x \in V(t)$.

In a continuum, we assume that no particles are destroyed or created such that the moving volume $V(t)$ is given by all coordinates $x \in \mathbb{R}^3$ that are occupied by a particle $\hat{x} \in \hat{V}$:

$$V(t) = \{x(\hat{x}, t) \in \mathbb{R}^3,\ \hat{x} \in \hat{V}\}.$$

Figure 2.1 shows this fundamental configuration.

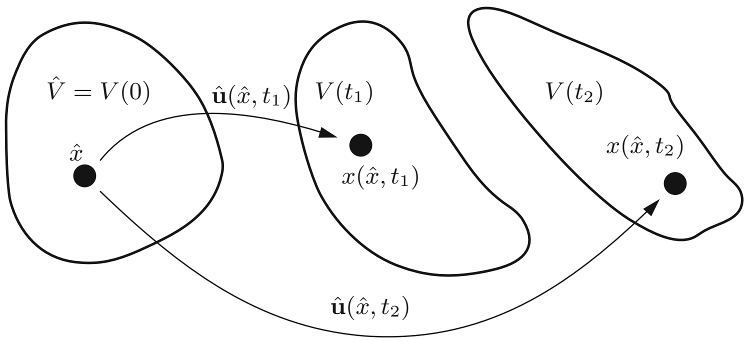

Fig. 2.1 The Lagrangian reference system. We describe the path of particles $\hat{x} \in \hat{V}$ over time. The reference volume $\hat{V}$ takes different current configurations $V(t)$ at different times. The particles within $V(t_1)$ are the same particles as in $V(t_2)$ or in $\hat{V} = V(t_0)$

We study the motion of volumes and the first fundamental property is the *deformation* of a particle $\hat{x} \in \hat{V}$. We define the *deformation* $\hat{\mathbf{u}}(\hat{x}, t)$ as

$$\hat{\mathbf{u}}(\hat{x}, t) = x(\hat{x}, t) - \hat{x}, \tag{2.1}$$

and its *material velocity* $\hat{\mathbf{v}}(\hat{x}, t)$ as

$$\hat{\mathbf{v}}(\hat{x}, t) := d_t x(\hat{x}, t) = d_t \hat{\mathbf{u}}(\hat{x}, t).$$

This particle system centered viewpoint for describing the dynamics of a continuum $V(t)$ is denoted as *Lagrangian coordinate system* or *Lagrangian framework*. In the Lagrangian system, we observe particles $\hat{x} \in \hat{V}$ and follow their paths $x(\hat{x}, t) = \hat{x} + \hat{\mathbf{u}}(\hat{x}, t)$ over time. A Lagrangian viewpoint is the natural approach for problems in solid mechanics, where the particles in the reference system are closely linked to each other and where forces are related to the relative deformation of particles to each other (think of a spring). Considering the dynamics of elastic solids, a volume comes back to the reference configuration, if the system is free of external forces

$$\begin{matrix} \hat{V} = V_0 \\ \hat{x} = x(\hat{x}, 0) \end{matrix} \xrightarrow{\text{external forces act}} \begin{matrix} V(t) \\ x(\hat{x}, t) \end{matrix} \xrightarrow{\text{absence of external forces}} \begin{matrix} V(t_\infty) = \hat{V} \\ \hat{x} = x(\hat{x}, t_\infty) \end{matrix}$$

Deformation and velocity can also be defined in the current configuration $V(t)$. By

$$x = \hat{x} + \hat{\mathbf{u}}(\hat{x}, t) \quad \Leftrightarrow \quad \mathbf{u}(x, t) := \hat{\mathbf{u}}(\hat{x}, t) = x - \hat{x}$$

we have an expression $\mathbf{u}(x, t)$ for the deformation at the spatial location $x \in V(t)$. By $\mathbf{u}(x, t)$ we describe the deformation of a particle in location $x \in \mathbb{R}^3$ at time t, we however do not know or determine which individual particle $\hat{x}$ we have in mind. If we describe all quantities in the current configuration $V(t)$ and if we are not interested in single particles at all we do not even need the concept of a reference domain.

The difference between both approaches is the viewpoint: where $\hat{\mathbf{u}}(\hat{x}, t)$ denotes the deformation of the particle $\hat{x}$ at time t, by $\mathbf{u}(x, t)$ we denote the deformation of whatever particle $\hat{x}$ happens to be at location x at time t. If at time t it holds $x = x(\hat{x}, t)$, both concepts of deformation describe the same configuration. If we base the description of the continuum on the spatial coordinates $x \in V(t)$, we speak of the *Eulerian framework*, where the focus is set on a spatial domain $V \subset \mathbb{R}^3$ and all points $x \in V$, see Fig. 2.2. This viewpoint is natural for fluid-dynamical problems. We consider the estimation of the drag-coefficient of a car. Here, the attention is on the flow around the car and we measure forces on the surface of the car, irrespective of the actual particle that at time $t \geq 0$ interacts with the car. In fluid dynamics, we want to describe velocity and pressure at spatial points $x \in V$.

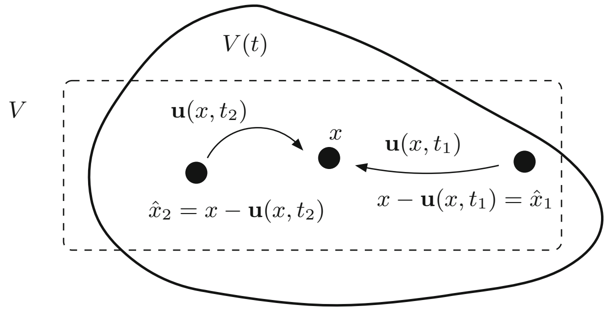

Fig. 2.2 The Eulerian reference system. We observe spatial coordinates $x \in V$, where $V \subset \mathbb{R}^3$ is a fixed view. Particles $\hat{x}$ may enter the domain V at a given time and leave it at another time. We observe properties of particles at certain times and locations, we however do not describe and follow the course of individual particles

Usually, we are not interested in what particle interacts with the car and where this particle comes from. Fluids like air or water do not have a *memory*. They behave in the same way regardless of their history. This of course is not true for all liquids. Material like polymers or rubber (which can be described as a fluid, if it is hot) actually do have a memory. Such viscoelastic fluids however are out of the scope of this book.

The Eulerian velocity $\mathbf{v}(x, t)$ is defined as the velocity in position $x \in \mathbb{R}^3$ at time t and given as

$$\mathbf{v}(x, t) = \partial_t u(x, t) = \partial_t \hat{\mathbf{u}}(\hat{x}, t) = \hat{\mathbf{v}}(\hat{x}, t).$$

In the Eulerian viewpoint, we do not describe, which particle $\hat{x}$ takes this position.

2.1.2 Deformation Gradient

In continuum mechanics, we study the behavior of moving and deforming continua $V(t)$ over time. In the following we describe the relative change of positions $x(\hat{x}, t)$ and $x(\hat{y}, t)$ of two particles $\hat{x}, \hat{y} \in \hat{V}$ in a moving continuum. Relative change of location is called strain, and strain will show to be the most fundamental quantity that causes stress within the material. By stress, we denote the internal forces between the neighboring particles in a continuum.

Let $\hat{x} \in \hat{V}$ and $\hat{y} \in \hat{V}$ be two particles that are infinitesimally close to each other, i.e. $|\hat{y} - \hat{x}| \to 0$. Under deformation, these two particles have the position

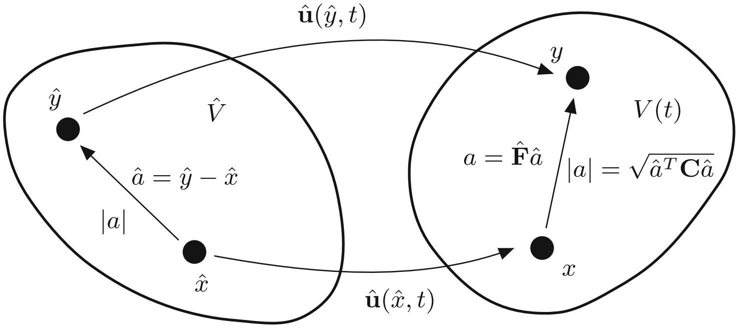

Fig. 2.3 Transformation of infinitesimal line segment $\hat{a}$ to a with $|\hat{a}| \to 0$. Deformation gradient $\hat{\mathbf{F}} = I + \hat{\nabla}\hat{\mathbf{u}}$ and squared length change $|a|^2 = \hat{a}^T\hat{\mathbf{C}}\hat{a}$ indicated by the right Cauchy-Green tensor $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}}$

$x = \hat{x} + \hat{\mathbf{u}}(\hat{x}) \in V$ and $y = \hat{y} + \hat{\mathbf{u}}(\hat{y}) \in V$. We measure the change in position $y - x$ in V with respect to $\hat{y} - \hat{x}$ in $\hat{V}$, see Fig. 2.3. By first order Taylor expansion we deduce

$$\begin{aligned} y - x &= \hat{y} + \hat{\mathbf{u}}(\hat{y}) - \hat{x} - \hat{\mathbf{u}}(\hat{x}) \\ &= \hat{y} - \hat{x} + \sum_{i=1}^{d} \hat{\partial}_i \mathbf{u}(\hat{x}) \cdot (\hat{y} - \hat{x}) + O(|\hat{y} - \hat{x}|^2) \\ &= \hat{y} - \hat{x} + \hat{\nabla}\hat{\mathbf{u}}(\hat{x})(\hat{y} - \hat{x}) + O(|\hat{y} - \hat{x}|^2), \end{aligned} \tag{2.2}$$

where by $|\hat{x}| = \sqrt{\sum_{i=1}^{d} \hat{x}_i^2}$ we denote the Euclidean norm, by $\hat{x} \cdot \hat{y} = \sum_{i=1}^{d} \hat{x}_i \hat{y}_i$ the Euclidean scalar product and by $\hat{\partial}_i$ the partial derivative with respect to $\hat{x}_i$ in the Lagrangian coordinate system. Considering the relative change in position, it holds

$$\frac{y - x}{|\hat{y} - \hat{x}|} = [I + \hat{\nabla}\hat{\mathbf{u}}(\hat{x})] \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|} + O(|\hat{y} - \hat{x}|). \tag{2.3}$$

We define

Definition 2.1 (Deformation Gradient) Let $\hat{\mathbf{u}}$ be a differentiable deformation field in the material volume $\hat{V}$. The *deformation gradient*

$$\hat{\mathbf{F}}(\hat{x}, t) := I + \hat{\nabla}\hat{\mathbf{u}}(\hat{x}, t),$$

denotes the local change of relative position under deformation.
The deformation gradient is the fundamental measure in structure dynamics.

Lemma 2.2 (Determinant of the Deformation Gradient) *Let $\hat{V}$ be a reference volume and $\hat{\mathbf{u}} : \hat{V} \to \mathbb{R}^d$ be a differentiable deformation field. The determinant of* *the deformation gradient* $\hat{J} := \det(\hat{\mathbf{F}})$ *denotes the local change of volume:*

$$|V(t)| = \int_{\hat{V}} \hat{J}\,\mathrm{d}x.$$

Proof It holds by the transformation theorem

$$|V(t)| = \int_{V(t)} 1\,\mathrm{d}x = \int_{\hat{V}} \det(I + \hat{\nabla}\hat{\mathbf{u}})\,\mathrm{d}\hat{x} = \int_{\hat{V}} \hat{J}\,\mathrm{d}\hat{x}.$$

□

The deformation gradient $\hat{\mathbf{F}}$ applies to the Lagrangian viewpoint. For an Eulerian description in $V(t)$, we can define the inverse deformation gradient $\mathbf{F}$ in a similar way. For two spatial coordinates $x, y \in V$ belonging to particles $\hat{x}$ and $\hat{y}$ in $\hat{V}$ it holds

$$\frac{\hat{y}-\hat{x}}{|y-x|} = \mathbf{F}(x)\frac{y-x}{|y-x|} + O(|y-x|),$$

with the *inverse deformation gradient* $\mathbf{F}(x,t) = I - \nabla\mathbf{u}(x,t)$. It holds $\mathbf{F} = \hat{\mathbf{F}}^{-1}$.

Very often, it will be necessary to rapidly switch between different viewpoints on the same physical problem. Sometimes, it is appropriate to consider the material centered reference domain $\hat{V}$, while sometimes the Eulerian viewpoint of the current configuration $V(t)$ is better suited. Usually, we denote all entities in the material system with a hat "$\hat{\cdot}$" and use the same notation without the hat for the Eulerian notation. Every basic property like velocity and deformation has a Eulerian counterpart, e.g. $\mathbf{v}(x,t) = \hat{\mathbf{v}}(\hat{x},t)$ and $\mathbf{u}(x,t) = \hat{\mathbf{u}}(\hat{x},t)$, where for $\hat{x}$ and x at a given time $t \geq t_0$ it always holds $x = \hat{x} + \hat{\mathbf{u}}(\hat{x},t)$. When referring to derivatives of these basic quantities, a simple "$\nabla\mathbf{u} = \hat{\nabla}\hat{\mathbf{u}}$" is usually wrong. Instead, we need to derive rules to map between both coordinate frames:

Lemma 2.3 (Transformation Between the Reference and the Current Configuration) *Let* $I = [0,T]$ *be a time interval,* $\hat{V}$ *be a reference domain and* $\hat{\mathbf{u}} \in C^1(I \times \hat{V})^3$*. We assume that* $T := \mathrm{id} + \hat{\mathbf{u}}$ *defines a* C^1*-diffeomorphism between* $\hat{V}$ *and*

$$V(t) = \{\hat{x} + \hat{\mathbf{u}}(\hat{x},t),\ \hat{x} \in \hat{V}\}.$$

Let $\hat{f} \in C^1(I \times \hat{V})$ *and* $f(x,t) = f(x(\hat{x},t),t) = \hat{f}(\hat{x},t)$ *be its counterpart in the current configuration. It holds*

$$\hat{\nabla}\hat{f} = \hat{\mathbf{F}}^T\nabla f \tag{2.4}$$

and

$$d_t f = d_t\hat{f}, \quad \partial_t f = \partial_t\hat{f} - \hat{\mathbf{F}}^{-T}\hat{\nabla}\hat{f}\cdot\hat{\mathbf{v}}. \tag{2.5}$$

Let $\hat{\mathbf{w}} \in C^1(I \times \hat{V})^3$ *be given with counterpart* $\mathbf{w}(x,t) = \hat{\mathbf{w}}(\hat{x},t)$. *It holds*

$$\hat{\nabla}\hat{\mathbf{w}} = \nabla\mathbf{w}\hat{\mathbf{F}}. \tag{2.6}$$

Proof For the spatial derivative of $f(x,t)$ it holds with $x(\hat{x},t) = \hat{x} + \hat{\mathbf{u}}(\hat{x},t)$:

$$\hat{\partial}_i \hat{f}(\hat{x},t) = \hat{\partial}_i f(x(\hat{x},t),t) = \sum_j \partial_j f(x,t)\hat{\partial}_i x^j(\hat{x},t) = \sum_j \partial_j f(x,t)\hat{\mathbf{F}}_{ji}.$$

Hence

$$\hat{\nabla}\hat{f} = \hat{\mathbf{F}}^T \nabla f.$$

Then, for a vector field $\mathbf{w} = (\mathbf{w}_i)_i$ it follows

$$(\hat{\nabla}\hat{\mathbf{w}})_{ij} = \hat{\partial}_j \hat{\mathbf{w}}_i = \sum_k \partial_k \mathbf{w}_i \hat{\partial}_j x^k(x,t) = (\nabla\mathbf{w})_{ik}\hat{\mathbf{F}}_{kj} = (\nabla\mathbf{w}\hat{\mathbf{F}})_{ij}$$

For the total time derivative it holds with $\partial_t x(\hat{x},t) = \hat{\mathbf{v}}(\hat{x}) = \hat{\mathbf{v}}(x)$

$$d_t f(x,t) = \partial_t f + \nabla f \cdot \mathbf{v}. \tag{2.7}$$

Then, with $\hat{x} = \hat{x}(x,t) = x - \mathbf{u}(x,t)$ and using (2.4):

$$d_t \hat{f}(\hat{x},t) = d_t f(x(\hat{x},t),t) = \partial_t f + \nabla f \cdot \partial_t x(\hat{x},t) = \partial_t f + \hat{F}^{-T}\hat{\nabla}\hat{f} \cdot \hat{\mathbf{v}}.$$

Finally, the last results follows with (2.7). □

2.1.3 Strain

Strain is defined as the deformation within a body relative to a reference length. Fixed body rotations or translation undergo no strain, as the relative positions of all particles is kept constant. Strain will be the basic quantity used to describe stresses in solid mechanics. A simple model is a spring, where change of length—the strain—will be proportional to a force.

Let $\hat{a} = \hat{y} - \hat{x}$ be the vector of a line-segment between the two points $\hat{x}, \hat{y} \in \hat{V}$. Then, given a deformation field $\hat{\mathbf{u}} : \hat{V} \to \mathbb{R}^3$, let $x = \hat{x} + \hat{\mathbf{u}}(\hat{x})$ and $y = \hat{y} + \hat{\mathbf{u}}(\hat{y})$ and set $a := y - x$. It holds with (2.3) that

$$a = y - x = \hat{\mathbf{F}}(\hat{x})\hat{a} + O(|\hat{a}|^2),$$

and the length of $|a|$ is given as

$$|a| = \sqrt{(\hat{\mathbf{F}}\hat{a}, \hat{\mathbf{F}}\hat{a}) + O(|\hat{a}|^3)} = \sqrt{(\hat{a}^T, \hat{\mathbf{F}}^T\hat{\mathbf{F}}\hat{a})} + O(|\hat{a}|^2).$$

For an illustration, see Fig. 2.3. By $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}}$ we denote the *right Cauchy-Green tensor* which is also denoted as the *Green deformation tensor*. This tensor is symmetric and positive definite, as

$$(\hat{\mathbf{C}}\hat{a}, \hat{a}) = (\hat{\mathbf{F}}\hat{a}, \hat{\mathbf{F}}\hat{a}) = \|\hat{\mathbf{F}}\hat{a}\|^2 > 0 \quad \forall \hat{x} \neq 0,$$

and it describes the (squared) length scaling of a line-segment in direction $\hat{a} = \hat{y} - \hat{x}$. A further commonly used strain measure is the *Green-Lagrange strain tensor* $\hat{\mathbf{E}} := \frac{1}{2}(\hat{\mathbf{C}} - I) = \frac{1}{2}(\hat{\mathbf{F}}^T\hat{\mathbf{F}} - I)$ that measures the (squared) length change of a line-segment $\hat{a} = \hat{y} - \hat{x}$ under deformation $a = y - x$:

$$\begin{aligned} \frac{1}{2}\left(|a|^2 - |\hat{a}|^2\right) &= \frac{1}{2}\left(\hat{a}^T\hat{\mathbf{C}}\hat{a} - \hat{a}^T\hat{a}\right) + O(|\hat{a}|^3) \\ &= \hat{a}^T\left(\frac{1}{2}(\hat{\mathbf{F}}^T\hat{\mathbf{F}} - I)\right)\hat{a} + O(|\hat{a}|^3). \end{aligned} \tag{2.8}$$

The tensors $\hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}}$ and $\hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{C}} - I)$ are nonlinear functions in the deformation $\hat{\mathbf{u}}$:

$$\hat{\mathbf{C}} = I + \hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T + \hat{\nabla}\hat{\mathbf{u}}^T\hat{\nabla}\hat{\mathbf{u}}, \quad \hat{\mathbf{E}} = \frac{1}{2}\left(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T + \hat{\nabla}\hat{\mathbf{u}}^T\hat{\nabla}\hat{\mathbf{u}}\right).$$

Given a very small variation in deformation, i.e. $|\hat{\nabla}\hat{\mathbf{u}}| \ll 1$, one sometimes uses linearization of the strain tensors as an approximation:

$$\mathbf{c} = I + \hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T\right).$$

These approximations can be good approximations under certain conditions. One however has to be careful, as having a small deformation $\hat{\mathbf{u}}$ is not a sufficient condition for this linearization.

The tensors $\hat{\mathbf{F}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{E}}$ and the linearized strain tensor $\boldsymbol{\varepsilon}$ all refer to the Lagrangian material coordinate system. They are called *material strain tensors*. Sometimes, we need to express strain in the spatial coordinate system, directly on the current frame $V(t)$. Hence let $x, y \in V(t)$ be two spatial coordinates at time $t \geq t_0$, spanning the line-segment $a = y - x$. By $\hat{x}, \hat{y} \in \hat{V}$ we denote the material points corresponding to this line-segment. These span the material line-segment $\hat{a} = \hat{y} - \hat{x}$. Similar to (2.3), but using the Eulerian notation $\mathbf{u}(x, t) = \hat{\mathbf{u}}(\hat{x}, t)$ we get

$$\hat{y} - \hat{x} = y - \mathbf{u}(y) - (x - \mathbf{u}(x)) = [I - \nabla\mathbf{u}(x)](y - x) + O(|y - x|^2).$$

By $\mathbf{F}(x) = I - \nabla\mathbf{u}(x)$ we denote the *inverse deformation tensor*. It holds $\mathbf{F}(x) = \hat{\mathbf{F}}(\hat{x})^{-1}$ for $x = \hat{x} + \hat{\mathbf{u}}(\hat{x})$. $\mathbf{F}(x)$ is the deformation gradient in the current configuration and it acts on the spatial coordinate system. With help of $\mathbf{F} = I - \nabla\mathbf{u}$ we can immediately analyze length changes in the spatial system. Let $a = y - x$ and $\hat{a} = \hat{y} - \hat{x}$. It holds

$$|\hat{a}|^2 = (\mathbf{F}a, \mathbf{F}a) + O(|a|^3) = a^T\mathbf{F}^T\mathbf{F}a + O(|a|^3) = a^T\hat{\mathbf{F}}^{-T}\hat{\mathbf{F}}^{-1}a + O(|a|^3).$$

The tensor $\mathbf{b}^{-1} := \hat{\mathbf{F}}^{-T}\hat{\mathbf{F}}^{-1} = \mathbf{F}^T\mathbf{F}$ is the inverse of the *left Cauchy-Green tensor* $\mathbf{b}$

$$\mathbf{b} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T.$$

As $\hat{\mathbf{C}}$, $\mathbf{b}$ is symmetric positive definite. Finally, we can define the spatial Eulerian counterpart $\mathbf{e} = \frac{1}{2}(I - \mathbf{F}^{-T}\mathbf{F}^{-1})$ to the Cauchy-Green strain tensor $\hat{\mathbf{E}}$. By (2.8), it holds

$$\frac{1}{2}\left(|a|^2 - |\hat{a}|^2\right) = \hat{a}^T\hat{\mathbf{E}}\hat{a} + O(|\hat{a}|^3),$$

and with

$$\hat{a}^T\hat{\mathbf{E}}\hat{a} = a^T\hat{\mathbf{F}}^{-1}\hat{\mathbf{E}}\hat{\mathbf{F}}^{-T}a + O(|\hat{a}|^3),$$

we introduce

$$\mathbf{e} := \frac{1}{2}\left(I - \mathbf{F}\mathbf{F}^T\right) = \frac{1}{2}\left(I - \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-T}\right) = \hat{\mathbf{F}}^{-1}\hat{\mathbf{E}}\hat{\mathbf{F}}^{-T}$$

the symmetric *Euler-Almansi strain tensor* $\mathbf{e}$ that enables us to relate length changes to the Eulerian line segment a:

$$\frac{1}{2}\left(|a|^2 - |\hat{a}|^2\right) = a^T\mathbf{e}a + O(|a|^3).$$

If for a body $\hat{V}$ it holds $\hat{\mathbf{C}} = I$ it follows that $\hat{\mathbf{E}} = 0$, and no relative changes in the position of material points $\hat{x}$ and $\hat{y}$ occur. Lengths and angles are maintained. A material body that can only undergo motion with $\hat{\mathbf{E}} = 0$ is called a *rigid body*.

Remark 2.4 (Right Cauchy-Green or Green-Lagrange Strain Tensor) We have two different strain measures at hand. The right Cauchy-Green strain tensor $\hat{\mathbf{C}}$ and the Green-Lagrange strain tensor $\hat{\mathbf{E}}$. Both are firmly linked and can be used to describe strains caused by deformation. For describing material laws, we will derive models, that characterize the materials reaction on strain. Most simple models will assume a linear dependency between strain and stress: if no strain is given, no stress is induced. Here, the Green-Lagrange strain tensor $\hat{\mathbf{E}}$ is the better basis, as $\hat{\mathbf{E}} = 0$ denotes a no-strain condition and a linear function $f(\hat{\mathbf{E}})$ can be consulted to model the strain-stress relationship.

2.1.4 Rate of Deformation and Strain Rate

The strain tensor is a fundamental quantity in solid mechanics, where we assume that a finite force will cause a finite deformation. An ideal spring will linearly react on external forces by some finite extension, which directly refers to strain. In fluid-mechanics however finite forces can lead to infinite deformation. A river, which is driven by the constant gravity force causes infinite strain, although the force is bounded. Here it is not the deformation and the deformation gradient that is of interest; but it is its temporal variation that serves as key quantity to model the internal forces (stresses) of the material. We already discussed that for fluid-dynamical observations, the Eulerian viewpoint is more meaningful. Hence we will derive a measure for the rate of strain in the current system $V(t)$.

By $\hat{x}, \hat{y} \in \hat{V}$ we denote two material points spanning the line-segment $\hat{a} = \hat{y} - \hat{x}$. We follow their positions $x(t) = \hat{x} + \hat{\mathbf{u}}(\hat{x}, t) \in V(t)$, $y(t) = \hat{y} + \hat{\mathbf{u}}(\hat{y}, t) \in V(t)$ and the resulting line-segment $a(t) = y(t) - x(t)$ in the current configuration $V(t)$. With $a(t) = \hat{\mathbf{F}}(t)\hat{a}$ it holds

$$\partial_t a(t) = \partial_t \hat{\mathbf{F}}(t)\hat{a}, \tag{2.9}$$

and for the deformation gradient $\hat{\mathbf{F}}(t) = I + \hat{\nabla}\hat{\mathbf{u}}(t)$ we get

$$\partial_t \hat{\mathbf{F}} = \partial_t \hat{\nabla}\hat{\mathbf{u}} = \hat{\nabla}\hat{\mathbf{v}}.$$

where we assumed sufficient regularity to change the order of derivatives. By $\hat{\nabla}\hat{\mathbf{v}}$ we denote the *material velocity gradient*. The material velocity gradient $\hat{\nabla}\hat{\mathbf{v}}(\hat{x}, t)$ denotes the spatial change of the velocity as given in the Lagrangian material system. The *spatial velocity gradient* $\nabla\mathbf{v}(x, t)$ refers to the spatial change of the velocity of whatever particles are at location x at time t. For $\hat{\mathbf{v}}(\hat{x}) = \mathbf{v}(x)$ with $x = x(\hat{x}) = \hat{x} + \hat{\mathbf{u}}(\hat{x})$ it further holds

$$\partial_t \hat{\mathbf{F}} = \nabla\mathbf{v}\hat{\nabla}x = \nabla\mathbf{v}\hat{\mathbf{F}}.$$

Then, to continue with (2.9)

$$\partial_t a(t) = \nabla\mathbf{v}\hat{\mathbf{F}}\hat{a} = \nabla\mathbf{v}a(t),$$

and the rate of length change is given by

$$\partial_t |a(t)|^2 = (\nabla\mathbf{v}a(t), a(t)) + (a(t), \nabla\mathbf{v}a(t)) = 2\left(\frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)a(t), a(t)\right).$$

Definition 2.5 (Strain Rate Tensor) By

$$\dot{\boldsymbol{\varepsilon}}(x, t) = \frac{1}{2}\left\{\nabla\mathbf{v}(x, t) + \nabla\mathbf{v}(x, t)^T\right\}.$$

we denote the *strain rate* tensor or the *rate of strain* tensor. It denotes the local change of velocity in the current system.

2.1.5 Stress

Deformation, strain and strain rate are kinematic properties. They simply describe the relative motion of particles within a volume. As such, they are pure observations of the situations and do not depend on the model under consideration. We assume that a material will react on strain or the strain rate. For expanding a spring, a certain force will be necessary.

By *stress* we denote the internal force that is acting on an imaginary surface within the volume $V(t)$. The unit of stress is force per area.

In Fig. 2.4 we show a volume $V(t)$ that is cut at an inner surface $S \subset V(t)$. By $x \in S \subset V(t)$ we denote a point on this surface with normal $\mathbf{n}$. The average forces acting on a neighborhood of $x \in S$ is denoted by the *Cauchy traction vector* $\mathbf{t}$. The right sketch of the figure shows this setting in the reference system, where by $\hat{x} \in \hat{S} \subset \hat{V}$ we denote point, surface and volume in reference state. Here, the normal vector is indicated by $\hat{\mathbf{n}}$ and the resulting *first Piola-Kirchhoff traction vector* $\hat{\mathbf{t}}$:

$$\mathbf{t} = \mathbf{t}(x, t, \mathbf{n}), \quad \hat{\mathbf{t}} = \hat{\mathbf{t}}(\hat{x}, t, \hat{\mathbf{n}}).$$

By $\mathrm{d}s$ we denote an infinitesimal neighborhood of x on the surface $S \subset V(t)$ and by $\mathrm{d}\hat{s}$ the corresponding infinitesimal neighborhood of $\hat{x}$ on $\hat{S} \subset \hat{V}$. Then, it holds

$$\mathbf{t}\mathrm{d}s = \hat{\mathbf{t}}\mathrm{d}\hat{s},$$

such that both traction vectors refer to forces in the current configuration $V(t)$. While $\mathbf{t}$ is a function in variables x and $\mathbf{n}$ of the current configuration, the first Piola-

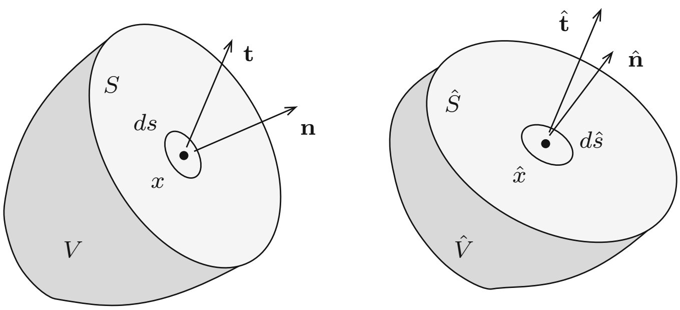

Fig. 2.4 Traction vectors on a imaginary surface in the current system (*left*) and the reference system (*right*). Cauchy's stress theorem postulates a linear dependency of the traction vectors on the normals $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ and $\hat{\mathbf{t}} = \hat{\mathbf{P}}\hat{\mathbf{n}}$

Kirchhoff traction vector is a function of $\hat{x}$ and $\hat{\mathbf{n}}$ in the Lagrangian reference system. Usually, it does not hold $|\mathbf{t}| = |\hat{\mathbf{t}}|$. The unit of stress is *force by area* and $\mathbf{t}$ refers to the area of a domain surface ds while $\hat{\mathbf{t}}$ refers to the area of the undeformed reference surface d$\hat{s}$.

The traction vectors describe a *surface tension*. Such surface tensions arise from friction or contact. Another example for a surface tension is the pressure in a liquid or gas that pushes the particle to each other (or apart from each other).

The surface tensions depend on the normal vector $\mathbf{n}$ of the imaginary surface. It holds

Theorem 2.6 (Cauchy's Stress Theorem) *There exist unique second order tensors $\boldsymbol{\sigma}$ and $\hat{\mathbf{P}}$, such that*

$$\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t)\mathbf{n}, \quad \hat{\mathbf{t}}(\hat{x}, t, \hat{\mathbf{n}}) = \hat{\mathbf{P}}(\hat{x}, t)\hat{\mathbf{n}}.$$

The tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ is symmetric and called the Cauchy stress tensor*, the tensor field $\hat{\mathbf{P}}$ is called the* first Piola-Kirchhoff stress tensor. *$\hat{\mathbf{P}}$ is usually not symmetric.*

Proof For the proof, we refer to the literature [150]. □

One immediate consequence of Cauchy's stress theorem is that traction vectors for opposite normal vectors annihilate each other, Newton's law of *actio* = *reactio*

$$\mathbf{t}(x, t, -\mathbf{n}) = \boldsymbol{\sigma}(x, t)(-\mathbf{n}) = -\boldsymbol{\sigma}(x, t)\mathbf{n} = -\mathbf{t}(x, t, \mathbf{n}).$$

As the Cauchy stress tensor must be symmetric, it consists of six independent components

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{12} & \boldsymbol{\sigma}_{13} \\ \boldsymbol{\sigma}_{12} & \boldsymbol{\sigma}_{22} & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{13} & \boldsymbol{\sigma}_{23} & \boldsymbol{\sigma}_{33} \end{pmatrix}.$$

The second order tensor $\hat{\mathbf{P}}$ is usually not symmetric and consists of nine independent entries. For the relation of $\boldsymbol{\sigma}$ and $\hat{\mathbf{P}}$ it holds

$$\boldsymbol{\sigma}\mathbf{n}\mathrm{d}s = \hat{\mathbf{P}}\hat{\mathbf{n}}\mathrm{d}\hat{s},$$

such that the two different traction vectors describe the transformation of a surface integral. We will get back to this relation in Sect. 2.1.7.

The components of the stress tensor are best understood by a decomposition of stresses into *normal stress* $\sigma \in \mathbb{R}$ and *shear stress* $\tau \in \mathbb{R}$. Let $\mathbf{t}$ be a stress vector in $x \in V(t)$ on a imaginary surface S with normal vector $\mathbf{n}$. The normal-stress σ is defined as the projection of the traction vector in normal direction

$$\sigma = \mathbf{t}^T\mathbf{n} = (\mathbf{n}, \boldsymbol{\sigma}\mathbf{n}),$$

while the shear stress is defined as the tangential part of the stress

$$\tau = \mathbf{t}^T \mathbf{t}_1 = (\mathbf{t}_1, \boldsymbol{\sigma}\mathbf{n}),$$

where $\mathbf{t}_1$ is the tangential vector that arises from projection of $\mathbf{t}$ onto the surface

$$\mathbf{t}_1 = \frac{\mathbf{t} - \sigma\mathbf{n}}{\|\mathbf{t} - \sigma\mathbf{n}\|}.$$

Then, the stress vector can be decomposed into the normal stress σ and shear stress τ by

$$\mathbf{t} = (\mathbf{t}, \mathbf{n})\mathbf{n} + (\mathbf{t}, \mathbf{t}_1)\mathbf{t}_1 = \sigma\mathbf{n} + \tau\mathbf{t}_1.$$

Here $\sigma, \tau \in \mathbb{R}$ are the lengths of the stress vectors in normal direction and tangential direction. Given the Cauchy stress tensor $\boldsymbol{\sigma}$, it holds

$$\sigma = (\mathbf{n}, \boldsymbol{\sigma}\mathbf{n}), \quad \tau = (\mathbf{t}, \boldsymbol{\sigma}\mathbf{n}).$$

If for the normal stress it holds $\sigma < 0$, the material undergoes a compression, while for $\sigma > 0$ an expansion is given. Further, it holds

$$|\boldsymbol{\sigma}\mathbf{n}|^2 = |\mathbf{t}|^2 = |\mathbf{n} \cdot \boldsymbol{\sigma}|^2 = \tau^2 + \sigma^2.$$

Next, let us assume that the imaginary surface has normal vector $\mathbf{n} = \mathbf{e}_i$ with $(\mathbf{e}_i)_j = \delta_{ij}$. The normal stress is given

$$\sigma = (\mathbf{e}_i, \boldsymbol{\sigma}\mathbf{e}_i) = \boldsymbol{\sigma}_{ii},$$

by the diagonal entry of the Cauchy-stress tensor, while the shear stress in $\mathbf{e}_k$ direction for $k \neq i$ gets

$$\tau = (\mathbf{e}_k, \boldsymbol{\sigma}\mathbf{e}_i) = \boldsymbol{\sigma}_{ki} = \boldsymbol{\sigma}_{ki}.$$

Hence the diagonal entries of $\boldsymbol{\sigma}$ refer to the normal stresses, while all off-diagonals refer to tangential shear stresses.

Remark 2.7 (Stress in the Reference System) Usually only static stresses act in the initial reference state of a system at reference time t_0. In case of a resting fluid, this stress can be caused by the hydrostatic pressure. Sometimes however, initial configurations cannot be considered to be stress-free. An example could be organic material like wood, where the undeformed reference system may be subject to stress caused by growth, see [209].

2.1.6 Conservation Principles

The most important physical conservation principles in the context of fluid-mechanics and structure-mechanics are conservation of mass, which says that

> mass is neither created nor destroyed,

conservation of momentum that says that

> the change in momentum is equivalent to the external forces

and conservation of angular momentum, saying that

> the change in angular momentum is equal to the torque.

Using the notation derived in the previous section, conservation of mass reads

$$d_t m(V(t)) = 0, \tag{2.10}$$

where the volume's mass $m(V(t))$ is given by

$$m(V(t)) = \int_{V(t)} \rho(x, t)\, \mathrm{d}\, x,$$

with a density ρ. Conservation of momentum gets

$$d_t I(V(t)) = K(V(t)) + K(\partial V(t)), \tag{2.11}$$

with the momentum $I(V(t))$

$$I(V(t)) = \int_{V(t)} \rho(x, t)\mathbf{v}(x, t), \mathrm{d}\, x,$$

and volume and surface forces $K(V(t))$ and $K(\partial V(t))$ given by:

$$K(V(t)) = \int_{V(t)} \rho(x, t)\mathbf{f}(x, t)\, \mathrm{d}\, x, \quad K(\partial V(t)) = \int_{\partial V(t)} \mathbf{t}\, \mathrm{d}s.$$

Here, $\mathbf{f}$ is a prescribed volume force density and $\mathbf{t}$ denotes the surface stress in direction $\mathbf{n}$. As discussed, it holds by Cauchy's Stress Theorem 2.6 that this surface force linearly depends on the normal direction such that it can be expressed with help of a stress tensor $\boldsymbol{\sigma} \in \mathbb{R}^{n \times n}$ as $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$. The surface allows for a transformation to a volume integral via the divergence theorem

$$K(\partial V(t)) = \int_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\sigma}\, \mathrm{d}s = \int_{V(t)} \mathrm{div}\,(\boldsymbol{\sigma})\, \mathrm{d}\, x.$$

Finally, conservation of angular momentum is given by

$$d_t L(V(t)) = D(V(t)), \tag{2.12}$$

where the angular momentum $L(V(t))$ with respect to the origin is given as

$$L(V(t)) = \int_{V(t)} x \times (\rho \mathbf{v})\, \mathrm{d}x,$$

and the torque $D(V(t))$ is defined by

$$D(V(t)) = \int_{V(t)} x \times (\rho \mathbf{f})\, \mathrm{d}x + \int_{\partial V(t)} x \times (\mathbf{n} \cdot \boldsymbol{\sigma})\, \mathrm{d}s.$$

Since the integration domain $V(t)$ in (2.10)–(2.12) depends on time t, evaluation of derivatives like $d_t m(V(t))$ is not straightforward and will be accomplished with help of the essential *Reynolds' Transport Theorem*

Lemma 2.8 (Reynolds' Transport Theorem) *Let $V(t) \subset \mathbb{R}^d$ be a material volume. Further, let $\Phi(x,t)$ be a differentiable scalar function defined on $V(t)$. Then, it holds*

$$d_t \int_{V(t)} \Phi(x,t)\, \mathrm{d}x = \int_{V(t)} \left(\partial_t \Phi(x,t) + \operatorname{div}(\Phi \mathbf{v})\right)\, \mathrm{d}x.$$

Proof The formula can be shown by elementary calculations using the transformation of $T(t) : \hat{V} \to V(t)$ to a fixed reference domain and expressing the derivatives of the functional determinant $\det(\hat{\nabla} T(t))$ with respect to its entries $\hat{\nabla}\hat{T}_{ij}$. See also Lemma 2.60. □

Applying this theorem to the scalar value $\Phi(x,t) := \rho(x,t)$ we derive the *Law of Mass Conservation*:

$$\int_{V(t)} \partial_t \rho + \operatorname{div}(\rho \mathbf{v})\, \mathrm{d}x = 0.$$

This equation is valid for every volume $V(t)$. Assuming that the expression $\partial_t \rho + \operatorname{div}(\rho \mathbf{v})$ is continuous (which is an assumption on the physical properties of the material), the equation of mass-conservation holds in a point-wise manner

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0. \tag{2.13}$$

The second basic rule is *conservation of momentum*, derived by the scalar values $\Phi(x,t) := \rho(x,t)\mathbf{v}_i(x,t)$ for every component of the velocity field. With a column-wise representation of the stress-tensor $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_d)$ Reynolds transport theorem yields:

$$\int_{V(t)} \partial_t(\rho \mathbf{v}_i) + \operatorname{div}(\rho \mathbf{v}_i \mathbf{v})\, \mathrm{d}x = \int_{V(t)} \rho \mathbf{f}_i + \operatorname{div}(\boldsymbol{\sigma}_i)\, \mathrm{d}x, \quad i = 1, \dots, d.$$

Given continuity of the integrand we can again deduce a point-wise equation

$$\partial_t(\rho \mathbf{v}_i) + \operatorname{div}(\rho \mathbf{v}_i \mathbf{v}) = \rho \mathbf{f}_i + \operatorname{div}(\boldsymbol{\sigma}_i), \quad i = 1, \dots, d.$$

By introducing the external product of two vectors

$$\mathbf{v} \otimes \mathbf{w} \in \mathbb{R}^{d \times d}, \quad (\mathbf{v} \otimes \mathbf{w})_{ij} := \mathbf{v}_i \mathbf{w}_j,$$

we can formulate the equation for the conservation of momentum in *conservative formulation*

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{f} + \operatorname{div}(\boldsymbol{\sigma}).$$

Combining this equation with the mass-conservation, we can further deduce the equation for conservation of momentum in the *non-conservative formulation*

$$\rho \partial_t \mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \rho \mathbf{f} + \operatorname{div}(\boldsymbol{\sigma}). \tag{2.14}$$

The equation for the *conservation of angular momentum* is given by

$$d_t \int_{V(t)} x \times (\rho \mathbf{v})\, \mathrm{d}x = \int_{V(t)} x \times (\rho \mathbf{f})\, \mathrm{d}x + \int_{\partial V(t)} x \times (\mathbf{n} \cdot \boldsymbol{\sigma})\, \mathrm{d}s.$$

Applying Reynolds transport theorem we can deduce the following three equations

$$\begin{aligned} i = 1 \quad & \boldsymbol{\sigma}_{23} - \boldsymbol{\sigma}_{32} = 0 \\ i = 2 \quad & \boldsymbol{\sigma}_{13} - \boldsymbol{\sigma}_{31} = 0 \\ i = 3 \quad & \boldsymbol{\sigma}_{12} - \boldsymbol{\sigma}_{21} = 0, \end{aligned}$$

that impose the symmetry of the Cauchy stress tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.15}$$

Further conservation principles are important if physical properties like entropy, energy and temperature are taken into consideration. Since we will deal with isentropic materials only, where all dynamical processes will take place without change of entropy, the three fundamental principles of mass-, momentum- and angular momentum-conservation will be sufficient to describe all desired behavior.

It remains to describe the tensor of surface-forces $\boldsymbol{\sigma}$. This tensor will heavily depend on the material under consideration, whether it is a fluid or a solid, whether the fluid is water, air or blood, the solid may be elastic or plastic or have properties of both. Here, physical modeling comes into place, exact laws for the dependence of this tensor on quantities like velocity and density usually do not exist. Since we know that $\boldsymbol{\sigma}$ is symmetric, six additional equations are required for its description. Stress models will be discussed in Sect. 2.2.

2.1.7 Conservation Principles in Different Coordinate Systems

In this section, we discuss the transformation of the conservation equations, which have been derived in the Eulerian framework, to different coordinate frameworks. Introducing the basic concepts for solid mechanics we already argued that a Lagrangian viewpoint is more natural.

Let $V(t)$ be the moving Eulerian framework and let $\hat{V}$ be the Lagrangian reference system. Further, by $\hat{W}$ we denote an arbitrary second fixed reference system, see Fig. 2.5. While the case $\hat{W} = \hat{V}$ is possible, we will allow for arbitrary systems without physical meaning. However, we assume that $\hat{W}$ is fixed in time and that there exists an invertible mapping $\hat{T}_W(t) : \hat{W} \to V(t)$ with gradient $\hat{\mathbf{F}}_W := \hat{\nabla}\hat{T}_W$ and determinant $\hat{J}_W := \det(\hat{\mathbf{F}}_W) > 0$. If we talk about the gradient $\hat{\mathbf{F}}_W$ we request that the map $\hat{T}_W$ is differentiable with respect to the spatial variables. We further assume that $\hat{T}_W$ is differentiable with respect to the temporal variable and that the inverse of the mapping $\hat{T}_W^{-1}$ is also differentiable. In other words, $\hat{T}_W$ is assumed to be a C^1-diffeomorphism on $I \times \hat{W}$.

By introducing an arbitrary reference systems $\hat{W}$ we have to deal with three different systems: the Lagrangian particles, $\hat{x} \in \hat{V}$, their Eulerian path $x(\hat{x}, t) \in V(t)$ and further the arbitrary framework with $\hat{x}_W \in \hat{W}$ with $\hat{T}_W(\hat{x}_W, t) = x = \hat{T}(\hat{x}, t)$. Note that it does not hold $\partial_t \hat{T}_W = \hat{\mathbf{v}}$, as we have to distinguish between the physical velocity $\hat{\mathbf{v}}$ of the particles and the velocity $\partial_t \hat{T}_w$ of the arbitrary coordinate system motion.

We start by describing basic properties used to map between the two systems $\hat{W}$ and $V(t)$. First, we introduce the inverse mapping $T_W(t) : V(t) \mapsto \hat{W}$.

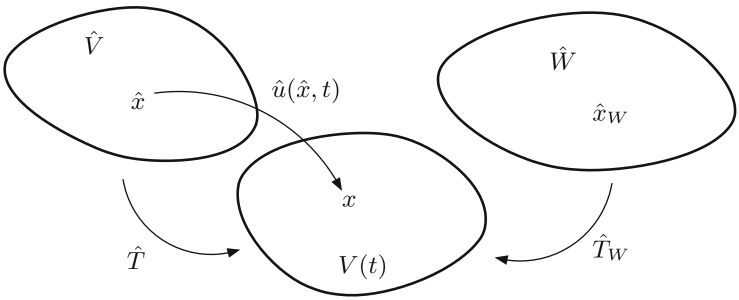

Fig. 2.5 Moving Eulerian volume $V(t)$ with Lagrangian reference $\hat{V}$ and third arbitrary reference volume $\hat{W}$

Lemma 2.9 (Inverse Mapping) *By* $T_W(t) : V(t) \to \hat{W}$ *we denote the* inverse mapping, *by* $\mathbf{F}_W := \nabla T_W$ *its gradient and by* $J_W := \det(\mathbf{F}_W)$ *its determinant. Given sufficient regularity, It holds*

$$\mathbf{F}_W = \hat{\mathbf{F}}_W^{-1}, \quad J_W := \hat{J}_W^{-1}, \quad \partial_t T_W = -\hat{\mathbf{F}}_W^{-1} \partial_t \hat{T}_W.$$

Proof It holds

$$T_W \circ \hat{T}_W = \widehat{\mathrm{id}} \quad \Rightarrow \quad \mathbf{F}_W \hat{\mathbf{F}}_W = I \quad \Rightarrow \quad \mathbf{F}_W = \hat{\mathbf{F}}_W^{-1}.$$

By taking the determinant of both sides, we immediately get $J_W = \hat{J}_W^{-1}$. Finally,

$$T_W \circ \hat{T}_W = \widehat{\mathrm{id}} \quad \Rightarrow \quad 0 = d_t T_W(\hat{T}_W(\hat{x}, t), t) = \partial_t T_W + \nabla T_W \partial_t \hat{T}_W.$$

Using $\nabla T_W = \mathbf{F}_W = \hat{\mathbf{F}}_W^{-1}$ we obtain the relation $\partial_t T_W = -\hat{\mathbf{F}}_W^{-1} \partial_t \hat{T}_W$. □

In Lemma 2.3 we already considered the transformation of spatial and temporal derivatives between the Eulerian and the Lagrangian coordinate system. Similarly it holds for a scalar function $f : V(t) \to \mathbb{R}$ and a vector field $\mathbf{w} : V(t) \to \mathbb{R}^d$ with counterparts $\hat{f}$ and $\hat{\mathbf{w}}$ on $\hat{W}$:

$$\nabla f = \hat{\mathbf{F}}_W^{-T} \hat{\nabla} \hat{f}, \quad \nabla \mathbf{w} = \hat{\nabla} \hat{\mathbf{w}} \hat{\mathbf{F}}_W^{-1}. \tag{2.16}$$

For temporal derivatives transformed to general coordinate systems $\hat{W}$ we must take care of two different velocities: the particle velocity $\hat{\mathbf{v}}$ and the domain velocity $\partial_t \hat{T}_W$, which do not coincide if $\hat{W} \neq \hat{V}$:

Lemma 2.10 (Transformation of Temporal Derivatives) *Let* $f : V(t) \to \mathbb{R}$ *with counterpart* $\hat{f}(\hat{x}_W, t) = f(x, t)$. *Given sufficient regularity, it holds*

$$\partial_t f = \partial_t \hat{f} - (\hat{\mathbf{F}}_W^{-1} \partial_t \hat{T}_W \cdot \hat{\nabla}) \hat{f}, \quad d_t f = \partial_t \hat{f} + (\hat{\mathbf{F}}_W^{-1} (\hat{\mathbf{v}} - \partial_t \hat{T}_W) \cdot \hat{\nabla}) \hat{f}.$$

Proof With $\hat{x}_W = T_W(x, t)$ it holds

$$\partial_t f(x, t) = d_t \hat{f}(\hat{x}_W, t) = d_t \hat{f}(T_W(x, t), t) = \partial_t \hat{f} + \hat{\nabla} \hat{f} \cdot \partial_t T_W.$$

The first result follows with help of Lemma 2.9. The relation for the material derivative is given by

$$d_t f(x, t) = \partial_t f(x, t) + \nabla f \cdot \partial_t x.$$

Here, $\partial_t x = \mathbf{v} = \hat{\mathbf{v}}$ refers to the trace of particles, where $\mathbf{v} = \hat{\mathbf{v}}$ is the velocity of the particle and not the velocity of the mapping $\hat{T}_W$. Together with (2.16) and the transformation of the partial time derivative we get

$$d_t f = \partial_t \hat{f} - (\hat{\mathbf{F}}_W^{-1} \partial_t \hat{T}_W \cdot \hat{\nabla})\hat{f} + \hat{\mathbf{F}}_W^{-T} \hat{\nabla}\hat{f} \cdot \hat{\mathbf{v}}.$$

□

Remark 2.11 (Transformation Between Lagrangian and Eulerian Coordinates) If $\hat{V} = \hat{W}$ it holds $\hat{T}_W = \hat{T}$ as well as $\hat{\mathbf{F}}_W = \hat{\mathbf{F}}$ and $\hat{J}_W = \hat{J}$. The statements of Lemma 2.10 simplify to

$$\hat{W} = \hat{V} \quad \Rightarrow \quad \partial_t f = \partial_t \hat{f} - (\hat{\mathbf{F}}^{-1} \hat{\mathbf{v}} \cdot \hat{\nabla})\hat{f}, \quad d_t f = \partial_t \hat{f}.$$

This results explains, why the convective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ will not appear in Lagrangian coordinates. See also Lemma 2.3.

In the following we discuss the transformation of the conservation principles to arbitrary coordinate reference systems $\hat{W}$. This transformation will be fundamental for solid mechanics, where the natural view-point is the Lagrangian one with $\hat{W} = \hat{V}$. Further, one of the standard approaches for coupling fluid-structure interactions relies on the mapping of the fluid problem onto a fixed reference system. Since this reference system will not be the Lagrangian one, we proceed without specifying the connotation of $\hat{W}$. The equation for conservation of momentum (2.14) is given by

$$\rho \partial_t \mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \rho \mathbf{f} + \operatorname{div}(\boldsymbol{\sigma}) \text{ in } V(t),$$

with a density ρ, velocity $\mathbf{v}$, volume force $\mathbf{f}$ and the Eulerian stress-tensor $\boldsymbol{\sigma}$. The specific form of this stress-tensor will be discussed in later sections. Here, we only assume that this stress tensor is symmetric $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. By $\hat{\mathbf{v}}(\hat{x}_W, t) = \mathbf{v}(x, t)$, $\hat{\rho}(\hat{x}_W, t) = \rho(x, t)$, $\hat{\mathbf{f}}(\hat{x}_W, t) = \mathbf{f}(x, t)$ as well as $\hat{\boldsymbol{\sigma}}(\hat{x}_W, t) = \boldsymbol{\sigma}(x, t)$ we denote the counterparts of these quantities in the reference system $\hat{W}$. By (2.16) and 2.10 it holds:

$$\partial_t \mathbf{v} = \partial_t \hat{\mathbf{v}} - (\hat{\mathbf{F}}_W^{-1} \partial_t \hat{T}_W \cdot \hat{\nabla})\hat{\mathbf{v}},$$
$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \mathbf{v} \mathbf{v} = \hat{\nabla} \hat{\mathbf{v}} \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{v}} = (\hat{\mathbf{F}}_W^{-1} \hat{\mathbf{v}} \cdot \hat{\nabla})\hat{\mathbf{v}},$$

and combined, we get:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \partial_t \hat{\mathbf{v}} + (\hat{\mathbf{F}}_W^{-1} (\hat{\mathbf{v}} - \partial_t \hat{T}_W) \cdot \hat{\nabla})\hat{\mathbf{v}}. \tag{2.17}$$

As discussed above, in the case of a mapping to the Lagrangian reference system, the mapping's temporal derivative is the velocity $\partial_t \hat{T} = \hat{\mathbf{v}}$ and the momentum terms simplify to

$$\hat{V} = \hat{W} \quad \Rightarrow \quad \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \partial_t \hat{\mathbf{v}}. \tag{2.18}$$

It remains to transform the divergence of the stresses to the reference domain. Here, a simple transformation of $\operatorname{div}(\boldsymbol{\sigma})$ to the reference system is not sufficient. We need to keep the meaning of this stress-term in mind, indicating surface-forces in normal-direction. The normal vectors are transformed, if the underlying domain $\hat{V} \to V(t)$ is deformed. Therefore we must base the mapping process on the correct representation of these surface forces. We need to find a representation of the first Piola-Kirchhoff stress tensor $\hat{\mathbf{P}}$ in the reference system, such that it holds:

$$\int_{\partial\hat{W}} \hat{\mathbf{P}}\hat{\mathbf{n}}\,\mathrm{d}\hat{s} = \int_{\partial V(t)} \boldsymbol{\sigma}\mathbf{n}\,\mathrm{d}s.$$

$\hat{\mathbf{P}}$ will be called the *Piola transformation* of $\boldsymbol{\sigma}$. For the derivation of this transformation we first regard vector fields $\mathbf{w} : V(t) \to \mathbb{R}^d$ with reference counterpart $\hat{\mathbf{w}} : \hat{W} \to \mathbb{R}^d$.

Lemma 2.12 (Piola Transformation) *Let* $\mathbf{w} : V(t) \to \mathbb{R}^d$ *be a differentiable vector field and* $\hat{\mathbf{w}}$ *its representation in the reference system* $\hat{W}$*. The* Piola transformation *of* $\mathbf{w}$ *is given by*

$$\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}.$$

On every volume $V(t)$ *with corresponding reference volume* $\hat{W}$ *it holds*

$$\int_{\partial V(t)} \mathbf{n}\cdot\mathbf{w}\,\mathrm{d}s \int_{\partial\hat{W}} \hat{\mathbf{n}}\cdot(\hat{J}_W\hat{\mathbf{F}}_W^{-1}\hat{\mathbf{w}})\,\mathrm{d}\hat{s},$$

$$\int_{V(t)} \operatorname{div}(\mathbf{w})\,\mathrm{d}x = \int_{\hat{W}} \widehat{\operatorname{div}}\,(\hat{J}_W\hat{\mathbf{F}}_W^{-1}\hat{\mathbf{w}})\,\mathrm{d}\hat{x}.$$

Further, in a point-wise sense it holds

$$\hat{J}_W \operatorname{div}(\mathbf{w}) = \widehat{\operatorname{div}}\,(\hat{J}_W\hat{\mathbf{F}}_W^{-1}\hat{\mathbf{w}}).$$

Proof We use a variational argument. Let ξ be differentiable on $V(t)$ with reference counterpart $\hat{\xi} \in \hat{W}$, such that

$$\int_{\partial V(t)} \mathbf{n}\cdot\mathbf{w}\xi\,\mathrm{d}s = \int_{V(t)} \operatorname{div}(\mathbf{w}\xi)\,\mathrm{d}x = \int_{\hat{W}} \hat{J}_W \operatorname{div}(\mathbf{w}\xi)\,\mathrm{d}\hat{x}. \tag{2.19}$$

Next, with (2.16) we get for $\hat{\xi} = \xi$:

$$\int_{\hat{W}} \hat{J}_W \operatorname{div}(\mathbf{w}\xi)\,\mathrm{d}\hat{x} = \int_{\hat{W}} \hat{J}_W \operatorname{div}(\mathbf{w})\hat{\xi}\,\mathrm{d}\hat{x} + \int_{\hat{W}} \hat{J}_W\hat{\mathbf{w}}\cdot\hat{\mathbf{F}}_W^{-T}\hat{\nabla}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{2.20}$$

With Green's formula, the second integral is transformed to

$$\int_{\hat{W}} \hat{J}_W \hat{\mathbf{w}} \cdot \hat{\mathbf{F}}_W^{-T} \hat{\nabla}\hat{\xi} \, \mathrm{d}\hat{x} = \int_{\hat{W}} \hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}} \cdot \hat{\nabla}\hat{\xi} \, \mathrm{d}\hat{x}$$
$$= -\int_{\hat{W}} \widehat{\mathrm{div}}\, (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}) \hat{\xi} \, \mathrm{d}\hat{x} + \int_{\partial\hat{W}} \hat{\mathbf{n}} \cdot (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}) \, \hat{\xi} \, \mathrm{d}\hat{s}. \tag{2.21}$$

Combining (2.19)–(2.21) gives

$$\int_{\partial V(t)} n \cdot \mathbf{w} \, \xi \, \mathrm{d}s - \int_{\hat{W}} \hat{J}_W \, \mathrm{div}\, (\mathbf{w}) \hat{\xi} \, \mathrm{d}\hat{x}$$
$$= -\int_{\hat{W}} \widehat{\mathrm{div}}\, (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}) \hat{\xi} \, \mathrm{d}\hat{x} + \int_{\partial\hat{W}} \hat{\mathbf{n}} \cdot (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}) \hat{\xi} \, \mathrm{d}\hat{s}.$$

By picking a Dirac sequence $\{\hat{\xi}_\varepsilon^y\}_{\varepsilon>0}$ where $\hat{\xi}_\varepsilon^y \in C_0^\infty(\hat{W})$ with

$$\int_{\hat{W}} \hat{\xi}_\varepsilon^y(\hat{x}) \hat{f}(\hat{x}) \, \mathrm{d}\hat{x} \xrightarrow[\varepsilon \to 0]{} \hat{f}(\hat{y}) \quad \forall \hat{f} \in C(\hat{W}),$$

we conclude for all inner points

$$\hat{J}_W \, \mathrm{div}\, (\mathbf{w}) = \widehat{\mathrm{div}}\, (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}).$$

Hence

$$\int_{V(t)} \mathrm{div}\, (\mathbf{w}) \, \mathrm{d}\, x = \int_{\hat{W}} \hat{J}_W \, \mathrm{div}\, (\hat{\mathbf{w}}) \, \mathrm{d}\hat{x} = \int_{\hat{W}} \widehat{\mathrm{div}}\, (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{w}}) \, \mathrm{d}\hat{x}.$$

The relation for the surface integral follows by Gauss' divergence theorem. □

This important result is used to transform the surface forces to the reference system. Let $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_i)_{i=1}^d$ be the row-vectors (or the column-vectors since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ by the conservation of angular momentum). It holds:

$$F_i(\partial V(t)) := \int_{\partial V(t)} \mathbf{n} \cdot \boldsymbol{\sigma}_i \, \mathrm{d}s = \int_{V(t)} \mathrm{div}\, (\boldsymbol{\sigma}_i) \, \mathrm{d}\, x$$

and with the just proven lemma we conclude

$$F_i(\partial V(t)) = \int_{\hat{W}} \widehat{\mathrm{div}}(\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\boldsymbol{\sigma}}_i) \, \mathrm{d}\hat{x} = \int_{\partial\hat{W}} \hat{\mathbf{n}} \cdot (\hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\boldsymbol{\sigma}}_i) \, \mathrm{d}\hat{s}.$$

Reassembling the stress-tensor $\hat{\boldsymbol{\sigma}} = (\hat{\boldsymbol{\sigma}}_i)$ we get the reference presentation of the surface forces:

$$F(\partial V(t)) = \int_{\partial\hat{W}} (\hat{J}_W \hat{\boldsymbol{\sigma}} \hat{\mathbf{F}}_W^{-T}) \hat{\mathbf{n}} \, \mathrm{d}\hat{s} = \int_{\hat{W}} \widehat{\mathrm{div}} \, (\hat{J}_W \hat{\boldsymbol{\sigma}} \hat{\mathbf{F}}_W^{-T}) \, \mathrm{d}\hat{x}.$$

We define

Definition 2.13 (Piola Kirchhoff Stress Tensors) The *First Piola Kirchhoff stress tensor* given by

$$\hat{\mathbf{P}} := \hat{J}_W \hat{\boldsymbol{\sigma}} \hat{\mathbf{F}}_W^{-T}.$$

It relates forces in the Eulerian coordinate framework with coordinates in a reference framework $\hat{W}$. The *Second Piola Kirchhoff stress tensor* given by

$$\hat{\boldsymbol{\Sigma}} := \hat{\mathbf{F}}_W^{-1} \hat{\mathbf{P}} = \hat{J}_W \hat{\mathbf{F}}_W^{-1} \hat{\boldsymbol{\sigma}} \hat{\mathbf{F}}_W^{-T}.$$

Unlike the Eulerian stress tensor $\boldsymbol{\sigma}$, the 1st Piola Kirchhoff stress tensor $\hat{\mathbf{P}}$ is not symmetric. The 2nd Piola Kirchhoff stress tensor is symmetric but it does not have an immediate physical explanation.

Using the first Piola Kirchhoff stress tensor and Relation (2.17) the momentum equation on arbitrary reference systems $\hat{W}$ is given by:

$$\hat{J}_W \hat{\rho} \big(\partial_t \hat{\mathbf{v}} + (\hat{\mathbf{F}}_W^{-1} (\hat{\mathbf{v}} - \partial_t \hat{T}_W) \cdot \hat{\nabla}) \hat{\mathbf{v}} \big) = \hat{J}_W \hat{\rho} \hat{f} + \widehat{\mathrm{div}} \, (\hat{J}_W \hat{\boldsymbol{\sigma}} \hat{\mathbf{F}}_W^{-T}). \tag{2.22}$$

2.2 Material Laws

The basic concepts of continuum mechanics introduced in the previous section are exact in a way that they are based on fundamental physical principles. The conservation principles for mass, momentum and angular momentum constitute a systems of four partial differential equations for ten unknowns: density ρ, velocity field $\mathbf{v}$ and the six unknowns of the symmetric stress tensor $\boldsymbol{\sigma}$. This system is underdetermined. To close it, additional equations are required that connect the values of the stress tensor to computable fundamental quantities like velocity, density or deformation.

In the following sections, we will derive such *material laws* that describe the properties of the stress tensors in the different formulations like $\boldsymbol{\sigma}$, $\hat{\boldsymbol{\Sigma}}$ or $\hat{\mathbf{P}}$. We assume that these stress tensors will depend on strain or strain rate given as deformation gradient $\hat{\mathbf{F}}$, its inverse $\mathbf{F}$, or tensors like $\hat{\mathbf{C}}$, $\hat{\mathbf{E}}$, $\mathbf{b}$, $\mathbf{e}$ or $\dot{\boldsymbol{\varepsilon}}$. We denote this relation by tensor-valued functions

$$\boldsymbol{\sigma} = f(\dot{\boldsymbol{\varepsilon}}), \quad \hat{\mathbf{P}} = \hat{f}(\hat{\mathbf{F}}), \quad \hat{\boldsymbol{\Sigma}} = \hat{f}(\hat{\mathbf{E}}),$$

or by similar expressions in $\hat{\mathbf{E}}$ or $\mathbf{b}$. We assume that all materials are homogenous and do not explicitly depend on the location $x \in V(t)$.

We are not considering arbitrary material laws but postulate several assumption on the material's properties:

1. *Objectivity:* The material law is independent of the spectators viewpoint. This property will hold for every physical material.
2. *Homogeneity:* We assume that the material is homogenous, i.e. the strain-stress relation will not explicitly depend on the location $x \in V(t)$.
3. *Isentropic and isothermal processes:* We assume that entropy and temperature do not play a role. There is no conversion between heat and kinetic energy. The temperature stays constant and does not affect the material law. This assumption is a simplification, as most elastic materials and also some fluids show a strong dependency on the temperature.
4. *Isotropy:* There is no distinct direction in the material. The response to strain or strain rate is the same in all directions. This assumption rules out anisotropic materials like fiber-reinforced composites or also biological tissue, where layers are usually directed anisotropically. Most fluids however are isotropic.

These assumptions lead to a strong simplification of possible material laws. The following *Rivlin-Ericksen Theorem* shows that all such possible material laws depend on symmetric strain tensors $\mathbf{C}$, $\mathbf{E}$ or $\dot{\boldsymbol{\varepsilon}}$ only and that all material laws are quadratic polynomials in the invariants of these tensors.

Theorem 2.14 (Rivlin-Ericksen Theorem) *A stress response function $\tilde{f}(\hat{\mathbf{F}})$ is isotropic and indifferent with respect to the coordinate system, if and only if it depends on the symmetric strain tensors only*

$$\tilde{f}(\hat{\mathbf{F}}) = \hat{f}(\hat{\mathbf{F}}^T\hat{\mathbf{F}}) = \hat{f}(\hat{\mathbf{C}}).$$

Further it is given as a quadratic polynomial

$$\hat{f}(\hat{\mathbf{C}}) = \beta_0(i(\hat{\mathbf{C}}))I + \beta_1(i(\hat{\mathbf{C}}))\hat{\mathbf{C}} + \beta_2(i(\hat{\mathbf{C}}))\hat{\mathbf{C}}^2, \qquad (2.23)$$

with scalar coefficients β_i that depend on the invariants (under orthogonal transformation) of the symmetric tensors $\mathbf{C}$*:*

$$I_1(\mathbf{C}) = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2(\mathbf{C}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad I_3(\mathbf{C}) = \lambda_1\lambda_2\lambda_3,$$

where λ_1, λ_2 and λ_3 are the three eigenvalues of $\mathbf{C}$.

Proof For a proof, we refer to the original contribution by Rivlin and Ericksen [290] or to a modern presentation by Turesdell and Noll [327]. □

As a symmetric positive definite tensor, $\mathbf{C}$ has three positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and a system of orthogonal eigenvectors. We know that eigenvalues are invariant to orthogonal transformation. To derive these invariants, we further cite the following Lemma:

Lemma 2.15 *Given a tensor* $\mathbf{A} \in \mathbb{R}^{3\times 3}$ *it holds for every* $\lambda \in \mathbb{R}$

$$\det(\mathbf{A} - \lambda I) = -\lambda^3 + I_1(\mathbf{A})\lambda^2 + I_2(\mathbf{A})\lambda + I_3(\mathbf{A}),$$

with

$$I_1(\mathbf{A}) = \mathrm{tr}(\mathbf{A}), \quad I_2(\mathbf{A}) = \frac{1}{2}\left(\mathrm{tr}(\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right), \quad I_3(\mathbf{A}) = \det(\mathbf{A}).$$

If $\mathbf{A}$ *is symmetric positive definite with eigenvalues* $\lambda_1, \lambda_2, \lambda_3$*, it further holds*

$$I_1(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2(\mathbf{A}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad I_3(\mathbf{A}) = \lambda_1\lambda_2\lambda_3.$$

Proof See [195]. □

The Rivlin-Ericksen Theorem 2.14 strongly limits possible material laws for homogenous and isotropic materials. All material laws—including fluids and solids—considered in the context of this book will fall under this theorem.

As every matrix satisfies its own characteristic polynomial, it holds for $\hat{\mathbf{C}} \in \mathbb{R}^{3\times 3}$ that

$$\hat{\mathbf{C}}^3 = I_1(\hat{\mathbf{C}})\hat{\mathbf{C}}^2 + I_2(\hat{\mathbf{C}})\hat{\mathbf{C}} + I_3(\hat{\mathbf{C}}). \tag{2.24}$$

Using this relation, the material law (2.23) is equivalent to a second representation

$$\hat{f}(\hat{\mathbf{C}}) = \gamma_0(i(\hat{\mathbf{C}}))I + \gamma_1(i(\hat{\mathbf{C}}))\hat{\mathbf{C}} + \gamma_2(i(\hat{\mathbf{C}}))\hat{\mathbf{C}}^{-1}.$$

Remark 2.16 As the two tensors $\hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{C}} - I)$ are directly connected, every material law in $\hat{\mathbf{C}}$ can also be expressed in $\hat{\mathbf{E}}$, as

$$\alpha_0 I + \alpha_1\hat{\mathbf{C}} + \alpha_2\hat{\mathbf{C}}^2 = (\alpha_0 + \alpha_1 + \alpha_2)I + (2\alpha_1 + 4\alpha_2)\hat{\mathbf{E}} + 4\alpha_2\hat{\mathbf{E}}^2.$$

Further, for the eigenvalues of $\hat{\mathbf{E}}$ and $\hat{\mathbf{C}}$ there holds a linear relation

$$\lambda_i w_i = \hat{\mathbf{C}}w_i = 2\hat{\mathbf{E}}w_i + w_i \quad \Leftrightarrow \quad \frac{1}{2}(\lambda_i - 1)w_i = \hat{\mathbf{E}}w_i.$$

2.2.1 Hyperelastic Materials

A solid is called *hyperelastic* if the relation between strain and stress comes from an energy density function

$$\hat{\boldsymbol{\Sigma}} = \frac{\partial W(\hat{\mathbf{E}})}{\partial \hat{\mathbf{E}}},$$

or

$$\hat{\mathbf{P}} = \frac{\partial W(\hat{\mathbf{F}})}{\partial \hat{\mathbf{F}}}.$$

This constitutes a relation between the second Piola-Kirchhoff stress tensor and the strain or between the deformation gradient and the first Piola-Kirchhoff stress, respectively. Many of the commonly used materials like the *St. Venant Kirchhoff* model or the *Mooney-Rivlin solid* are of this type. Stress tensors for incompressible materials can be derived by energy functions of the type

$$W = W(\mathbf{F}) - p\,(\det(\mathbf{F}) - 1)$$

that penalize the change of volume $J = \det(\mathbf{F})$.

As the derivation of the models is not in the focus of this book, we just refer to the literature for more reading on this very important concept, see Holzapfel [195] for a very comprehensive exposure.

2.2.2 Linearizations

For simplicity, we sometimes consider linear models. Two different types of nonlinearities must be considered: first, the material nonlinearity which denotes a nonlinear relation between stress and strain. Second, the geometric nonlinearity, which comes from the discrepancy between reference coordinate system and current system and which is expressed by the deformation gradient $\mathbf{e} = \hat{\mathbf{F}}\hat{\mathbf{e}}$.

Regarding the Rivlin-Ericksen Theorem 2.14, linearity of a material means that only the first invariant $I_1(\mathbf{E}) = \text{tr}(\mathbf{E})$ may enter the law and that no higher order terms may appear. Further, in geometrically linearized situations, the symmetric strain tensor $\hat{\mathbf{E}}$ is approximated and linearized

$$\hat{\mathbf{E}} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u}) \approx \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) =: \hat{\boldsymbol{\varepsilon}},$$

assuming that $|\nabla\mathbf{u}| \ll 1$ is small.

Lemma 2.17 (Linear Material Law) *A stress response function $f(\cdot)$ for a linear, homogenous and isotropic material depends on the linearized strain $\hat{\boldsymbol{\varepsilon}} = \hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T$ or on the strain rate tensor $\dot{\boldsymbol{\varepsilon}} = \nabla\mathbf{v} + \nabla\mathbf{v}^T$ and its first invariant only*

$$\hat{f}(\hat{\boldsymbol{\varepsilon}}) = \beta_0 \operatorname{tr}(\hat{\boldsymbol{\varepsilon}})I + \beta_1\hat{\boldsymbol{\varepsilon}}.$$

In fluid mechanics, the Navier-Stokes equations follow such a linear material law and in structure mechanics, the Navier-Lamé problem considers these simplifications. While in fluid mechanics a fully linear material law—the Navier-Stokes model—is a very accurate model for many relevant fluids, linearization in solid mechanics is usually not feasible. Here, linear models only apply to very small deformations $|\hat{\mathbf{u}}| \ll 1$ and very small changes in deformation $|\hat{\nabla}\hat{\mathbf{u}}| \ll 1$. In particular, linearized solid models are no longer invariant with respect to fixed body rotations. In the context of fluid-structure interactions, use of linear models can significantly damp the dynamics.

2.2.3 Incompressible Materials

Some materials have an incompressible behavior which means that the volume

$$|V(t)| = \int_{V(t)} 1 \,\mathrm{d}x \equiv \text{“const”}$$

does not change. For an incompressible material, there is no expansion or compression. Many fluids—like water—can be considered incompressible. Incompressibility further applies to many biological structures. We can describe change of volume in the current system by Reynolds transport theorem

$$0 = d_t|V(t)| = d_t \int_{V(t)} 1 \,\mathrm{d}x = \int_{V(t)} \nabla\cdot\mathbf{v} \,\mathrm{d}x = \int_{\partial V(t)} n\cdot\mathbf{v} \,\mathrm{d}s, \tag{2.25}$$

but also in the reference configuration by transformation

$$0 = d_t|V(t)| = d_t \int_{V(t)} 1 \,\mathrm{d}x = \int_{\hat{V}} d_t\hat{J} \,\mathrm{d}\hat{x}. \tag{2.26}$$

For a fluid, modeled in the current configuration, (2.25) says that the flow is “divergence-free” with $\operatorname{div} \mathbf{v} = 0$ and also that the total normal flow over the volume’s boundary is zero. For a divergence free velocity field it holds

$$\operatorname{tr}(\dot{\boldsymbol{\varepsilon}}) = 0,$$

and in light of Lemma 2.17, the material law is further simplified to

$$f(\dot{\boldsymbol{\varepsilon}}) = \beta_1 \dot{\boldsymbol{\varepsilon}}.$$

To cope with isotropic expansion and compression forces, we introduce a pressure variable as part of the material law:

$$f(\dot{\boldsymbol{\varepsilon}}, p) = -pI + \beta_1 \dot{\boldsymbol{\varepsilon}}.$$

This pressure is required to enforce the incompressibility of the velocity field, see also Sect. 2.3.

Considering solid's, incompressibility in terms of (2.26) means that the determinant of the deformation gradient will be constant $d_t \hat{J} = 0$. As $\hat{\mathbf{F}} = I$ in the reference system, incompressibility simply says $\hat{J} = 1$ for all times $t \geq t_0$. Further, it then holds that

$$\det(\hat{\mathbf{C}}) = \det(\hat{\mathbf{F}})^2 = 1.$$

For the Green-Lagrange strain tensor $\hat{\mathbf{E}}$ it follows that third and second invariant fall together, see Lemma 2.15 and Remark 2.16.

2.3 The Solid Problem

As discussed, we usually describe the dynamics of elastic structures in the Lagrangian reference system. Hence considering the conservation law (2.22) we choose $\hat{W} = \hat{V}$ as reference system. In light of Remark 2.11, the momentum equation is given by

$$\hat{J}\hat{\rho}\partial_{tt}\hat{\mathbf{u}} = \hat{J}\hat{\rho}\hat{\mathbf{f}} + \widehat{\operatorname{div}}(\hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}),$$

where we eliminated the velocity using $\partial_t \hat{\mathbf{u}} = \hat{\mathbf{v}}$. Considering material laws as introduced in the previous section, stresses will depend on strain, and hence on the displacement $\hat{\mathbf{u}}$. The density is known at initial time $\rho(x, 0) = \hat{\rho}^0(\hat{x})$. For $t \geq 0$ conservation of mass yields

$$m(\hat{V}) := \int_{\hat{V}} \hat{\rho}^0(\hat{x})\, \mathrm{d}\hat{x} \stackrel{!}{=} \int_{V(t)} \rho(x, t)\, \mathrm{d}x = \int_{\hat{V}} \hat{J}\hat{\rho}(\hat{x}, t)\, \mathrm{d}\hat{x} =: m(V(t)).$$

At time $t \geq 0$, the relation

$$\hat{\rho}(\hat{x}, t) = \hat{J}^{-1}(\hat{x}, t)\hat{\rho}^0(\hat{x}) \tag{2.27}$$

describes the density in every point $\hat{x}$ of the reference system. The full problem of elastic structures formulated in the Lagrangian reference system $\hat{V}$ is given by:

$$\hat{\rho}^0 \partial_{tt}\hat{\mathbf{u}} - \widehat{\mathrm{div}}\,(\mathbf{F}\hat{\boldsymbol{\Sigma}}) = \hat{\rho}^0\hat{\mathbf{f}} \tag{2.28}$$

It remains to complete this partial differential equation by appropriate boundary conditions and initial conditions. Let $\hat{\mathcal{S}} \subset \mathbb{R}^d$ be the solid domain in reference configuration. At time $t = 0$, we specify initial conditions for density, deformation and velocity

$$\hat{\rho}(\cdot, 0) = \hat{\rho}^0(\cdot), \quad \hat{\mathbf{u}}(\cdot, 0) = \hat{\mathbf{u}}^0(\cdot), \quad \partial_t\hat{\mathbf{u}}(\cdot, 0) = \hat{\mathbf{v}}^0(\cdot), \quad t = 0. \tag{2.29}$$

For all times $t \geq 0$, by $\hat{\mathbf{f}}(\hat{x}, t)$ we denote the acting volume force field. Note that this force field is directed in the Eulerian framework, such that for example the gravity is given by $\mathbf{f} = -9.81 e_3 \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$, with $e_3 = (0, 0, 1)^T$, independent of the reference framework. The boundary of the domain $\hat{\Gamma}_s := \partial\hat{\mathcal{S}}$ is split into a Dirichlet boundary part $\hat{\Gamma}_s^D$ and into a Neumann part $\hat{\Gamma}_s^N$. On the Dirichlet boundary, we specify boundary conditions for the deformation

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^D \text{ on } \hat{\Gamma}_s^D \times [0, T]. \tag{2.30}$$

Note that by $\hat{\mathbf{v}} = \partial_t\hat{\mathbf{u}}$ we also uniquely define the velocity on the boundary. The usual Neumann condition on Γ_s^N specifies the boundary stresses by

$$\mathbf{n} \cdot \hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}} = \mathbf{n} \cdot \hat{J}\hat{\boldsymbol{\sigma}}_s\hat{\mathbf{F}}^{-T} = \hat{\mathbf{g}}_s^{(\hat{n})} \text{ on } \hat{\Gamma}_s^N \times [0, T]. \tag{2.31}$$

If the external forces $\mathbf{f}$ and the boundary data $\hat{\mathbf{g}}_s^{(\hat{n})}$ and $\hat{\mathbf{u}}^D$ do not explicitly depend on time, the solution can run into a stationary limit $\hat{\mathbf{u}}(\cdot, t) \to \hat{\mathbf{u}}(\cdot)$ that does not depend on time. In this case, it holds $\partial_t\hat{\mathbf{v}} = 0$ and hence $\partial_{tt}\hat{\mathbf{u}} = 0$. If such a stationary solution exists, we can directly consider the stationary system of equations:

$$-\widehat{\mathrm{div}}\,(\hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}) = \hat{\rho}^0\hat{\mathbf{f}}. \tag{2.32}$$

Finally, it remains to provide material laws for specific solids. One of the most simple model is the *St. Venant Kirchhoff material* that postulates a linear dependency between strain tensor $\hat{\mathbf{E}}$ and stresses:

Definition 2.18 (St. Venant Kirchhoff Material) The St. Venant Kirchhoff material follows the material law

$$\hat{\boldsymbol{\Sigma}} = 2\mu_s\hat{\mathbf{E}} + \lambda_s \operatorname{tr}(\hat{\mathbf{E}})I,$$

with the first λ_s and second μ_s Lamé parameters. (μ_s is also called the *shear modulus*.) These two parameters are related to the Poisson ratio ν_s that describes the compressibility and Young's modulus E_s that describes the stiffness:

$$\nu_s = \frac{\lambda_s}{2(\lambda_s + \mu_s)}, \quad E_s = \frac{\mu_s(3\lambda_s + 2\mu_s)}{\lambda_s + \mu_s}.$$

The linear relation between strain and stress is called *Hooke's Law*. The Poisson ratio ν_s describes the compressibility of the system. It holds

$$\nu_s = \frac{1}{2}\left(\frac{1}{1 + \frac{\mu_s}{\lambda_s}}\right) < \frac{1}{2}.$$

The Poisson ratio $\nu_s = \frac{1}{2}$ refers to $\lambda_s \to \infty$ hence to incompressible materials. The Poisson ratio describes the reaction of the material on directional compression, see Fig. 2.6. For a Poisson ratio $\nu_s = \frac{1}{2}$, the volume will stay constant, for $\nu_s < \frac{1}{2}$ the volume will decrease. There are some materials with negative Poisson ratio. Here, the material will react to the compression in one direction with compression in the orthogonal directions. Such materials play some role for computational means in the context of fluid-structure interactions, see Sect. 3.5.1. The St. Venant Kirchhoff model is a suitable approximation for metals at small deformations. Steel has a Poisson ratio of about $\nu_s \approx 0.3$ and a Young modulus $E_s \approx 200 \cdot 10^9\,\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}$.

Hooke's Law applied to an incompressible material leads to the incompressible Neo Hookean material law.

Definition 2.19 (Incompressible Neo-Hookean Material) The incompressible Neo-Hookean material law is given by

$$\hat{\mathbf{P}} = \hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}} = -p\hat{\mathbf{F}}^{-T} + 2\mu_s\hat{\mathbf{F}}^{-T}\hat{\mathbf{E}},$$

with the shear modulus μ_s and the Poisson ratio $\nu_s = \frac{1}{2}$. By p we denote the undetermined pressure.

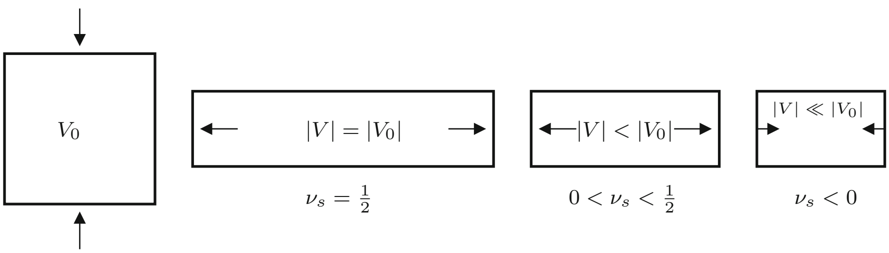

Fig. 2.6 Material behavior under compression for different Poisson ratios. *Left*: incompressible material $\nu_s = \frac{1}{2}$. *Middle*: compressible material $0 < \nu_s < \frac{1}{2}$. *Right*: auxetic material with $\nu_s < 0$

We conclude and formulate the following often used systems of equations

Problem 2.20 (Conservation Laws for a St. Venant Kirchhoff Material) *Let $\Omega \subset \mathbb{R}^d$ be a domain with boundary $\Gamma = \partial\Omega$ with $\Gamma = \Gamma^D \cup \Gamma^N$. Further, let $\hat\rho^0 : \Omega \to \mathbb{R}_+$ be the materials density, $\hat{\mathbf{f}} \in C(\Omega)^d$ be a given right hand side, $\hat{\mathbf{u}}^D, \hat{\mathbf{v}}^D \in C(\Gamma^D)$ be Dirichlet boundary data, $\hat{\mathbf{g}}^{(\mathbf{n})} \in C(\Gamma^N)$ be the Neumann data. With initial deformation and velocity $\hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0 \in C(\Omega)^d$ find deformation and velocity*

$$\hat{\mathbf{u}}(t) \in C^2(\Omega)^d \cap C(\Omega \cup \Gamma^D)^d \cup C^1(\Omega \cup \Gamma^N)^d,$$

such that

$$\hat\rho^0 \partial_{tt}\hat{\mathbf{u}} - \widehat{\operatorname{div}}\left(\hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}\right) = \hat\rho^0 \hat{\mathbf{f}} \quad t \geq 0,$$

where

$$\hat{\boldsymbol{\Sigma}} = 2\mu_s \hat{\mathbf{E}} + \lambda_s \operatorname{tr}(\hat{\mathbf{E}}) I,$$

and

$$\hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0,\ d_t\hat{\mathbf{u}}(0) = \hat{\mathbf{v}}^0 \text{ in } \Omega,$$

with the boundary conditions

$$\hat{\mathbf{u}}(t) = \hat{\mathbf{u}}^D \text{ on } \Gamma^N, \quad \hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}\hat{\mathbf{n}} = \hat{\mathbf{g}}^{(\mathbf{n})}.$$

and, for the incompressible materials we define:

Problem 2.21 (Conservation Laws for the Incompressible Neo-Hookean Material) *Let $\Omega \subset \mathbb{R}^d$ be a domain with boundary $\Gamma = \partial\Omega$ with $\Gamma = \Gamma^D \cup \Gamma^N$. Further, let $\hat\rho^0 : \Omega \to \mathbb{R}_+$ be the materials density, $\hat{\mathbf{f}} \in C(\Omega)^d$ be a given right hand side, $\hat{\mathbf{u}}^D, \hat{\mathbf{v}}^D \in C(\Gamma^D)$ be Dirichlet boundary data, $\hat{\mathbf{g}}^{(\mathbf{n})} \in C(\Gamma^N)$ be the Neumann data. With initial deformation and velocity $\hat{\mathbf{u}}^0, \hat{\mathbf{v}}^0 \in C(\Omega)^d$ find deformation, velocity and pressure*

$$\hat{\mathbf{u}}(t) \in C^2(\Omega)^d \cap C(\Omega \cup \Gamma^D)^d \cap C^1(\Omega \cup \Gamma^N)^d, \quad \hat p(t) \in C^1(\Omega) \cap C(\Omega \cup \Gamma^N),$$

such that

$$\hat J = 0, \quad \hat\rho^0 \partial_{tt}\hat{\mathbf{u}} - \widehat{\operatorname{div}}\left(\hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}\right) = \hat\rho^0 \hat{\mathbf{f}} \quad t \geq 0,$$

where

$$\hat{\boldsymbol{\Sigma}} = -\hat p \hat{\mathbf{F}}^{-T} + 2\mu_s \hat{\mathbf{F}}^{-T}\hat{\mathbf{E}}$$

and

$$\hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0,\ d_t\hat{\mathbf{u}}(0) = \hat{\mathbf{v}}^0 \ in\ \Omega,$$

with the boundary conditions

$$\hat{\mathbf{u}}(t) = \hat{\mathbf{u}}^D \ on\ \Gamma^N, \quad \hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}\hat{\mathbf{n}} = \hat{\mathbf{g}}^{(\mathbf{n})}.$$

2.3.1 The Navier-Lamé Equations

The model for an elastic solid governed by one of the material laws is a system of nonlinear partial differential equations. Its analysis is difficult and theoretical results exist for small deformation only. As a nonlinear set of equations, uniqueness cannot be expected in the general case.

To get better insight into the problem, we will simplify the problem with the following assumptions:

- The deformation gradient $\hat{\mathbf{F}}$ is so small that we can approximate $\hat{\mathbf{F}} = I$ and $\hat{J} = 1$. By this simplification, the concept of Eulerian and Lagrangian coordinates fall together. We will therefore also skip all hat's that indicate reference variables.
- Further the strains are so small that we can linearize the Green-Lagrange strain tensor

$$\hat{\mathbf{E}} = \frac{1}{2}(\hat{\nabla}\mathbf{u} + \hat{\nabla}\mathbf{u}^T + \hat{\nabla}\hat{\mathbf{u}}^T\hat{\nabla}\hat{\mathbf{u}}) \approx \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}) =: \boldsymbol{\varepsilon}.$$

  This simplification not only rules out very large elastic deformations, it also penalizes rigid body rotations.
- Just for simplicity (this will not change the character of the equation) we set $\hat{\rho}^0 = 1$.

Considering the linear St. Venant Kirchhoff material (with these simplifications) the resulting set of equations are the

Problem 2.22 (Navier-Lamé Equations) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary split into Dirichlet- and Neumann-part $\partial\Omega = \Gamma^D \cup \Gamma^N$. On the time interval $I = [0, T]$ we search for solutions $\mathbf{u} : I \times \Omega \to \mathbb{R}^3$ such that*

$$\begin{aligned} \partial_{tt}\mathbf{u} - \operatorname{div}\boldsymbol{\sigma} &= \mathbf{f} && in\ I \times \Omega \\ \mathbf{u} = \mathbf{u}^0, \quad d_t\mathbf{u} &= \mathbf{v}^0 && for\ \{0\} \times \Omega \\ \mathbf{u} &= \mathbf{u}^D && on\ I \times \Gamma^D \\ \boldsymbol{\sigma}\,\mathbf{n} &= \mathbf{u}^{\sigma} && on\ I \times \Gamma^N, \end{aligned} \tag{2.33}$$

with the linearized material law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}\,(\boldsymbol{\varepsilon})I, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

As a further simplification, we also consider the stationary limit of the Navier-Lamé equations:

Problem 2.23 (Stationary Navier-Lamé Equations) *Find* $u \in C^2(\Omega)^3 \cap C(\Omega \cup \Gamma^D)^3 \cap C^1(\Omega \cup \Gamma^N)^3$ *such that*

$$\begin{aligned} -\,\mathrm{div}\,\boldsymbol{\sigma} &= \mathbf{f} && \text{in } \Omega \\ \mathbf{u} &= \mathbf{u}^D && \text{on } \Gamma^D \\ \boldsymbol{\sigma}\,\mathbf{n} &= \mathbf{u}^\sigma && \text{on } \Gamma^N, \end{aligned} \tag{2.34}$$

with the linearized material law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}\,(\boldsymbol{\varepsilon})I, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

As usual, analysis of classical solutions is difficult. This is partly to the fact that the solution $\mathbf{u}$ often exhibits singularities in boundary nodes at the transit between Dirichlet and Neumann parts. The well known *Theorem of Cosserat* states that classical solutions to the stationary problem, Problem 2.23, are unique if the Dirichlet boundary Γ^D contains at least three independent points and that—in the general case—they can differ by a rigid body motion only

$$\mathbf{u}_1(x) - \mathbf{u}_2(x) = \mathbf{b} + \mathbf{B}x,$$

where $\mathbf{b} \in \mathbb{R}^3$ is a translation vector and $\mathbf{B} \in \mathbb{R}^{3\times 3}$ is a skew-symmetric matrix, see e.g. [97].

For the following, we will introduce a weak formulation of the Navier-Lamé equations that will offer an easy access to show existence and uniqueness of solutions:

Lemma 2.24 (Variational Formulation) *Every classical solution to Problem 2.23 is also solution to the variational formulation*

$$\begin{aligned} &\mathbf{u} \in \bar{\mathbf{u}}^D + H_0^1(\Omega;\Gamma^D)^3 \\ &\qquad (\boldsymbol{\sigma}, \nabla\phi) = (\mathbf{f}, \phi) + \langle \mathbf{u}^\sigma, \phi\rangle_{\Gamma^N} \quad \forall \phi \in H_0^1(\Omega;\Gamma^D)^3, \end{aligned} \tag{2.35}$$

where $\bar{\mathbf{u}}^D \in H^1(\Omega)^d$ *is an extension of the Dirichlet data* $\mathbf{u}^D$ *into the domain.*

Existence and uniqueness of solutions can be shown by standard arguments of elliptic equations. The difficulty however is to show ellipticity, i.e.

$$\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T, \nabla \mathbf{u}) + \lambda\big(\operatorname{tr}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)I, \nabla \mathbf{u}\big) \geq c\|\nabla \mathbf{u}\|^2,$$

as $\nabla \mathbf{u} + \nabla \mathbf{u}^T = 0$ does not necessarily impose $\nabla \mathbf{u} = 0$. This is a consequence of *Korn's inequality*:

Theorem 2.25 (1st Korn's Inequality) *Let $\Omega \subset \mathbb{R}^3$ be a domain. Then, it holds*

$$\|\nabla \mathbf{v}\| \leq c_{korn}\|\boldsymbol{\varepsilon}(\mathbf{v})\| \quad \forall \mathbf{v} \in H_0^1(\Omega)^3$$

with a constant $c_{korn} > 0$. This inequality corresponds to the case of Dirichlet boundary values on the complete boundary $\Gamma^D = \partial\Omega$.

Korn's first inequality deals with the case of homogenous Dirichlet conditions on the complete boundary $\partial\Omega$. In the context of structural mechanics, this limitation is severe, as no free boundary motion and deformation would be allowed. The case of general boundary conditions, with a Neumann part $\Gamma_N \subset \partial\Omega$ is less trivial and handled by Korn's second inequality:

Theorem 2.26 (2nd Korn's Inequality) *Let $\Omega \subset \mathbb{R}^3$ be a domain with Lipschitz-boundary. Then, it holds*

$$\|\nabla \mathbf{v}\| \leq c_{korn}\,(\|\varepsilon(\mathbf{v})\| + \|\mathbf{v}\|) \quad \forall \mathbf{v} \in H^1(\Omega)^3.$$

with a constant $c_{korn} > 0$.

Proof The simple proof of 1st Korn's inequality is based on integration by parts and vanishing traces of $\mathbf{v}$ on the complete boundary $\partial\Omega$. The proof of Korn's 2nd inequality is more involved and we refer to the literature, see e.g. [98, 196]. □

Continuity and ellipticity of the bilinear form allows to apply the standard theory for linear elliptic problems to the Navier-Lamé equations.

Lemma 2.27 (Existence of Unique Solutions) *Let $\mathbf{f} \in L^2(\Omega)^3$, $\bar{\mathbf{u}}^D \in H^1(\Omega)^3$ be an extension of the Dirichlet data into the domain and $\mathbf{u}^\sigma \in H^1(\partial\Omega)^3$. There exists a unique solution $\mathbf{u} \in \bar{\mathbf{u}}^D + H_0^1(\Omega;\Gamma^D)^3$ to the linear Navier-Lamé equations and it holds*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq c\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}^D\|_{L^2(\Gamma^D)} + \|\mathbf{u}^\sigma\|_{H^1(\Gamma^N)}\right),$$

with a constant $c > 0$.

Proof We must show that the variational formulation is bilinear, symmetric, continuous and elliptic. Further, the right hand side is continuous, such that existence of a unique solution follows by the Theorem of Lax-Milgram, see [293]. □

Concerning the regularity of the solution, we cite the following lemma, see [97], which gives conditions that lead to classical solutions.

Lemma 2.28 (Strong Regularity of the Navier-Lamé Problem) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\alpha}$ for $\alpha > 0$. Given that the problem data has the regularity*

$$\mathbf{f} \in C^{\alpha}(\bar{\Omega})^3, \quad \bar{\mathbf{u}}^{\sigma} \in C^{1+\alpha}(\Omega)^{3\times 3}_{sym}, \quad \bar{\mathbf{u}}^{D} \in C^{2+\alpha}(\bar{\Omega})^3,$$

the weak solution $\mathbf{u} \in H^1_0(\Omega; \Gamma^D)^3$ of (2.35) is also a classical solution

$$\mathbf{u} \in C^2(\Omega)^3 \cap C^1(\Omega \cup \Gamma^N)^3 \cap C(\Omega \cup \Gamma^D)^3.$$

A further regularity result with less strict assumption on the regularity of the domain and the problem data is given by Shi and Wright [308]:

Lemma 2.29 (Weak Regularity of the Navier-Lamé Problem) *Let $\Omega \subset \mathbb{R}^3$ be a domain with $W^{2,3}$ boundary. Further, let $\mathbf{f} \in L^2(\Omega)^d$. Then, for the solution of the stationary Navier-Lamé problem with homogenous Dirichlet data $\mathbf{u}^D = 0$ it holds*

$$\|\mathbf{u}\|_{H^2(\Omega)^3 \cap H^1_0(\Omega)^3} \leq c\|\mathbf{f}\|_{L^2(\Omega)^3}.$$

Regularity of solutions is usually restricted at points, where Neumann and Dirichlet parts of the boundary come together. Here, we usually have singularities in the gradient of the solution and the stress tensor.

2.3.1.1 The Incompressible Navier-Lamé Equations

For incompressible linear materials with $\nu = \frac{1}{2}$, the stress tensor is reduced to

$$\boldsymbol{\sigma} = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T),$$

as $\mathrm{tr}\,(\boldsymbol{\varepsilon}) = \mathrm{div}\,\mathbf{u} = 0$. The material is no longer able to react on purely isotropic stresses. To formulate the incompressible Navier-Lamé equations, we consider a minimization problem in the space of divergence free functions

$$\mathbf{u} \in V_0: \quad E(\mathbf{u}) \leq E(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - l(\mathbf{v}) \quad \forall \mathbf{v} \in V_0,$$

where $a(v, v) := (\boldsymbol{\sigma}, \nabla\mathbf{v})$, $l(\mathbf{v}) := (\mathbf{f}, \mathbf{v}) + \langle u^{\sigma}, \mathbf{v}\rangle_{\Gamma^N}$ and where V_0 is the space of weakly divergence free functions

$$V_0 = \{\phi \in H^1_0(\Omega; \Gamma^D)^3,\ (\mathrm{div}\ \phi, \xi) = 0\ \forall \xi \in L^2(\Omega)\}. \tag{2.36}$$

The Hilbert space V_0 is a closed subspace of $H^1_0(\Omega; \Gamma^D)^3$, such that the existence of a unique solution follows as shown in Lemma 2.27. To derive a variational formulation, we use the Euler-Lagrange approach for constraint minimization problems and define the Lagrange functional

$$\mathcal{L}(\mathbf{u},p) = \frac{1}{2}a(\mathbf{u},\mathbf{u}) - l(\mathbf{u}) - (p, \text{div } \mathbf{u}),$$

with a Lagrange multiplier $p \in L^2(\Omega)$. A possible solution is given as stationary point of $\mathcal{L}(\mathbf{u},p)$:

$$\begin{aligned} d_{\mathbf{u}}\mathcal{L}(\mathbf{u},p)(\phi) &= a(\mathbf{u},\phi) - l(\phi) - (p, \text{div } \phi) &\stackrel{!}{=} 0 \quad &\forall \phi \in H_0^1(\Omega;\Gamma^D)^3 \\ d_p\mathcal{L}(\mathbf{u},p)(\xi) &= -(\xi, \text{div } \mathbf{u}) &\stackrel{!}{=} 0 \quad &\forall \xi \in L^2(\Omega). \end{aligned}$$

We include the Lagrange multiplier into the stress tensor and define

$$\boldsymbol{\sigma}_I(\mathbf{u},p) = -pI + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T),$$

where we identify $p \in L^2(\Omega)$ with a *pressure function*. This identification is reasonable, as $-pI$ acts as isotropic stress in all directions. The problem is now to find $\{\mathbf{u},p\} \in H_0^1(\Omega;\Gamma^D)^3 \times L^2(\Omega)$ such that

$$\big(\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - pI, \boldsymbol{\varepsilon}(\phi)\big) + (\text{div } \mathbf{u}, \xi) = (\mathbf{f},\phi) + \langle u^{\sigma}, \phi\rangle_{\Gamma^N} \qquad (2.37)$$

for all $\phi \in H_0^1(\Omega;\Gamma^D)^3$ and $\xi \in L^2(\Omega)$.

The incompressible Navier-Lamé equations, as a minimization problem with side condition is a *saddle-point system*. Existence and uniqueness theory cannot be based on ellipticity (in p). Instead, we split the proof for the existence of a well defined solution in two parts. We start by finding a suitable deformation field. Therefore, we restrict the space of admissible functions to those that already fulfill the divergence condition in the space V_0, see (2.36). Then, it holds

Lemma 2.30 (Incompressible Navier-Lamé—Existence of Unique Solutions (Displacement)) *Let* $\mathbf{f} \in L^2(\Omega)^3$, $\bar{\mathbf{u}}^D \in H^1(\Omega)^3$ *be an extension of the Dirichlet data into the domain and* $\mathbf{u}^{\sigma} \in H^1(\Gamma^N)^3$. *There exists a unique solution* $u \in \bar{u}^D + H_0^1(\Omega;\Gamma^D)^d$ *to the variational problem*

$$(2\mu\varepsilon(\mathbf{u}), \varepsilon(\phi)) = (\mathbf{f},\phi) + \langle \mathbf{u}^{\sigma}, \phi\rangle_{\Gamma^N} \quad \forall \phi \in H_0^1(\Omega;\Gamma^D)^3.$$

For this solution it holds

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq c\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{u}^D\|_{L^2(\Gamma^D)} + \|\mathbf{u}^{\sigma}\|_{H^1(\Gamma^N)}\right).$$

Finally, $\mathbf{u} \in V_0$ *minimizes the energy function in the space* V_0

$$E(\mathbf{u}) \leq E(\mathbf{v}) \quad \forall \mathbf{v} \in V_0.$$

Proof The subspace $V_0 \subset H_0^1(\Omega;\Gamma_D)^3$ is a Hilbert-space. The variational formulation is V_0-elliptic and the existence of a unique solution as well as the a priori estimate follow in the same way as shown in Lemma 2.27. □

Next, given a deformation field $u \in V_0$ we find a corresponding pressure by analyzing the equation

$$p \in L^2(\Omega):$$
$$-(p,\nabla\phi) = (\mathbf{f},\phi) + \langle \mathbf{u}^\sigma,\phi\rangle_{\Gamma^N} - (2\mu\boldsymbol{\varepsilon}(\phi),\nabla\phi) \quad \forall \phi \in H_0^1(\Omega;\Gamma^D)^3.$$

Existence of solutions to this problem cannot be shown by simple variational arguments. Instead, we will define by

$$\langle \operatorname{grad} p,\phi\rangle := -(p,\nabla\cdot\phi) \quad \forall \phi \in H_0^1(\Omega;\Gamma^D)^3,$$

the weak gradient operator $-\operatorname{grad} = \operatorname{div}^* : L^2(\Omega) \to H^{-1}(\Omega)$ and show existence by proving surjectivity of $-\operatorname{grad}$ in appropriate function spaces. We postpone this discussion to Sect. 2.4.5, where we will come across the same pressure problem concerning the incompressible Stokes equations.

2.3.1.2 The Non-stationary Navier-Lamé Equations

The non-stationary system of Navier-Lamé equations as given in Definition 2.22 is a hyperbolic problem

$$\partial_{tt}\mathbf{u} - \operatorname{div}(\boldsymbol{\sigma}) = 0, \quad \mathbf{u}(0) = \mathbf{u}^0, \quad \partial_t\mathbf{u}(0) = \mathbf{v}^0.$$

For simplicity we will consider the case of homogenous Dirichlet data only and we will further assume that $\mathbf{f} = 0$. We multiply the differential equation by $\phi = \partial_t\mathbf{u}$ and integrate over the spatial domain to get

$$0 = (\partial_{tt}\mathbf{u},\partial_t\mathbf{u}) + (\boldsymbol{\sigma}(\mathbf{u}),\boldsymbol{\varepsilon}(\partial_t\mathbf{u})) = \frac{d}{dt}\left(\underbrace{\frac{1}{2}\|\partial_t\mathbf{u}\|^2 + \frac{1}{2}(\boldsymbol{\sigma}(\mathbf{u}),\boldsymbol{\varepsilon})}_{=:E(t)}\right),$$

where by $E(t)$ we denote the energy of the system. This energy does not change over time (remember that we consider the homogenous problem only). Integration over the temporal domain $I = [0,T]$ yields the relation

$$E(t) = E(0) \quad t \geq 0, \quad E(0) = \frac{1}{2}\|\mathbf{v}^0\|^2 + \frac{1}{2}\left(\boldsymbol{\sigma}(\mathbf{u}^0),\boldsymbol{\varepsilon}(\mathbf{u}^0)\right),$$

with the initial velocity $\mathbf{v}^0 = \partial_t \mathbf{u}^0$. Hence a solution must by unique and it is bounded by the initial data.

Conservation of energy $d_t E(t) = 0$ shows the close relation to the wave equation. Existence of solutions to this simple (linear, symmetric and positive) problem can be shown by the Fourier approach. The operator

$$\langle \mathcal{L}\mathbf{u}, \mathbf{v}\rangle := \Big(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}\big(\boldsymbol{\varepsilon}(\mathbf{u})\big), \boldsymbol{\varepsilon}(\mathbf{v})\Big)$$

is symmetric, positive definite, selfadjoint and a bijection. Its inverse is bound and considered as operator $\mathcal{L}^{-1} : L^2(\Omega)^d \to L^2(\Omega)^d$ it is compact. Hence $\mathcal{L}$ has a spectrum of positive eigenvalues, with no finite accumulation point. Further, an orthonormal basis of eigenvectors exists. This allows to diagonalize the system of equations, such that it decomposes into a sequence of scalar initial values problems that have a solution that can be constructed by elementary principles. For the details on this construction, we refer to the literature [268].

A recent result on the regularity of the non-stationary Navier-Lamé problem with homogenous Dirichlet data is given by Mitrea and Monniaux [243]. They basically show that given sufficient regularity of the domain's boundary (Lipschitz), the solution of the non-stationary Navier-Lamé problem with zero initial data and zero Dirichlet data satisfies $\mathbf{u} \in H^1(I; L^2(\Omega)^3)$ for every right hand side $\mathbf{f} \in L^2(I; L^2(\Omega)^3)$.

In the upcoming chapters, we will see that the coupling of the solid equation to the fluid equations brings along further challenges for the analysis of the partial differential equations. The *kinematic coupling condition*, see Sect. 3.1 will ask for continuity of solid- and fluid-velocities on a common interface $\mathcal{I}(t) = \partial\mathcal{S}(t) \cap \partial\mathcal{F}(t)$

$$\mathbf{v}_f = \mathbf{v}_s \text{ on } \mathcal{I}(t).$$

In the case of stationary problems, this kinematic coupling condition is just a usual no-slip boundary condition $\mathbf{v}_f = 0$ for the fluid's velocity. For fully non-stationary problems, a real coupling between the two velocities is introduced. The solution of the Navier-Stokes equations is well defined for velocities with traces in

$$\mathbf{v}_f\Big|_{\partial\mathcal{F}} \in H^{\frac{1}{2}}(\partial\mathcal{F}),$$

which—as seen from the solid problem—will require

$$\mathbf{v}_f\Big|_{\mathcal{I}} = \mathbf{v}_s\Big|_{\mathcal{I}} \quad \Rightarrow \quad \mathbf{v}_s \in H^1(\mathcal{S}).$$

However, the previous analysis only gives

$$\mathbf{v}_s = \partial_t \mathbf{u}_s \in L^2(I; L^2(\Omega)^3).$$

This is not sufficient to define a $H^{1/2}$-trace on $\mathcal{I}$. This problem has two possible solutions. First—and this will be our usual procedure—we can simply assume additional a priori knowledge on the regularity of $\mathbf{u}_s$ and therefore $\mathbf{v}_s$. This can be guaranteed for small and regular problem data, if the boundaries of the coupled problem have very high regularity. Coutand and Shkoller [106] show the existence of solutions for the coupling of elastic solids with the Navier-Stokes equations, if the solid with boundary of class H^4 is completely embedded in a fluid-domain with boundary of class H^3, given sufficient regularity of the right hand side and the boundary data, see [106]. A second approach to enforce sufficient regularity it to add damping terms to the solid equation. Gazzola and Squassina show the following result, see [162].

Theorem 2.31 (Damped Wave Equation) *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. The strongly damped wave equation*

$$\partial_{tt}u - \Delta u - \omega\Delta\partial_t u + \mu\partial_t u = 0 \text{ in } [0,T] \times \Omega,$$

with initial values

$$u(0,\cdot) = u_0 \in H^1(\Omega), \quad \partial_t u(0,\cdot) = u_1 \in L^2(\Omega),$$

and homogenous Dirichlet values on $\partial\Omega$ and the damping parameters

$$\omega > 0, \quad \mu > -\omega\lambda_1,$$

where λ_1 is the first eigenvalue of $-\Delta$ has a unique solution satisfying

$$u \in L^\infty([0,T], H_0^1(\Omega)) \cap W^{1,\infty}([0,T], L^2(\Omega)), \quad \partial_t u \in L^2([0,T], H_0^1(\Omega)).$$

For the proof, see Gazolla and Squassina [162].

By adding strong damping terms, we are able to assure sufficient regularity to realize the kinematic coupling condition between solid problem and fluid problem.

2.3.2 Theory of Nonlinear Hyper-Elastic Material

Tackling the existence and uniqueness problem of the full elastic structure equation (using the St. Venant Kirchhoff material law) is complicated by the nonlinearity of the problem. Here, we will not give details on the complex proofs, but will simply cite some important results. A good overview on the theory of nonlinear elastic materials is given in the textbook of Ciarlet [97].

All approaches for the nonlinear problem will at some time use a linearization of the problem and will consult the theory that has been derived for the linear Navier-Lamé problem. Further, most approaches use variational techniques, such that the starting point for every analysis is the following weak formulation of the problem:

Lemma 2.32 (Weak Formulation of the Hyper-Elastic Structures) *Let* $\bar{\mathbf{u}}^D \in H^1(\hat{\mathcal{S}})^d$ *be an extension of the Dirichlet data on* Γ^D *into the domain* Ω. *If the solution*

$$\hat{\mathbf{u}}_f \in \bar{\mathbf{u}}_f^D + H_0^1(\hat{\Omega};\Gamma^D)^d$$

of the variational formulation

$$(\hat{\mathbf{F}}\hat{\boldsymbol{\Sigma}}_s, \hat{\nabla}\hat{\phi})_{\hat{\mathcal{S}}} = (\rho_s^0 \hat{\mathbf{f}}_s, \hat{\phi}), \quad \forall \hat{\phi} \in H_0^1(\hat{\Omega};\Gamma^D)^d, \tag{2.38}$$

has sufficient regularity $\hat{\mathbf{u}} \in C^2(\hat{\Omega}) \cap C(\hat{\Omega} \cup \Gamma^D) \cap C^1(\hat{\Omega} \cup \Gamma^D)$, *it is also a solution to the classical formulation of the elastic structure equations (2.32) with Dirichlet data on* Γ_s^D.

Using the implicit function theorem, Ciarlet [97] proofs the following result for weak solutions of the elastic structure equation governed by the St. Venant Kirchhoff material:

Lemma 2.33 (Stationary St. Venant Kirchhoff Material) *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with* C^2*-boundary. Then, for every* $p > 3$ *there exists a constant* α *such that for every* $\mathbf{f} \in L^p(\Omega)^d$ *with* $\|\mathbf{f}\|_{L^p} \le \alpha$ *there exists a unique solution* $\mathbf{u} \in W^{2,p}(\Omega)$ *to the stationary elastic structure equation governed by the St. Venant Kirchhoff material.*

For the proof, we refer to the literature [97].

2.4 The Fluid Problem

In fluid-dynamics, we describe the flow of particles in the Eulerian framework. Looking at a fixed coordinate $x \in \mathbb{R}^d$ we observe a particle $\hat{x}(x,t)$ that at time t is in position x. The fate of a single particle is of no interest.

We will only consider incompressible fluids, i.e. a given moving volume $V(t)$ will not change its size under motion:

$$d_t|V(t)| = 0, \quad t \ge 0.$$

Applying Reynolds' Transport theorem, Lemma 2.8 to the scalar $\Phi \equiv 1$ yields:

$$d_t|V(t)| = d_t \int_{V(t)} 1 \,\mathrm{d}x = \int_{V(t)} \operatorname{div} \mathbf{v} \,\mathrm{d}x.$$

Hence as $V(t)$ can be chosen arbitrarily, we deduce the point-wise equation for the incompressibility of a fluid, see also Sect. 2.2.3:

$$\operatorname{div}\,\mathbf{v} = 0. \tag{2.39}$$

Using this condition, conservation of mass (2.13) reduces to a transport equation for the fluid's density:

$$\partial_t \rho_f + (\mathbf{v}\cdot\nabla)\rho_f = 0. \tag{2.40}$$

For further simplification, we will restrict all our considerations to homogenous fluids, where the density at initial time $t = 0$ is constant in the complete volume $\rho_f(x,0) = \rho_f^0(x) \equiv \rho_f$. Given (2.40) it hereby follows that the density is homogenous at all times $t \geq 0$ and conservation of mass is reduced to the divergence condition (2.39).

To close the system of equations for incompressible fluids we must introduce material laws that model the dependency of the stress tensor $\boldsymbol{\sigma}_f$ on velocity and pressure. We are considering *Navier-Stokes* fluids only that linearly depend on the strain rate following Hooke's law

$$\boldsymbol{\sigma} = 2\mu_f\dot{\boldsymbol{\varepsilon}} + \lambda\,\operatorname{tr}(\dot{\boldsymbol{\varepsilon}})I.$$

As for an incompressible fluid it holds $\operatorname{div}\,\mathbf{v} = \operatorname{tr}(\dot{\boldsymbol{\varepsilon}}) = 0$, the stress tensor simplifies to

$$\boldsymbol{\sigma} = -pI + \mu_f(\nabla\mathbf{v} + \nabla\mathbf{v}^T), \tag{2.41}$$

where again by p we denote the undetermined pressure that will act as Lagrange multiplier to ensure the divergence condition $\operatorname{div}\,\mathbf{v} = 0$. By $\mu_f = \rho_f\nu_f$ we denote the dynamic viscosity of the fluid and by ν_f its kinematic viscosity. The complete set of the Navier-Stokes equations is given by

$$\rho_f(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \operatorname{div}\,\boldsymbol{\sigma} = \rho_f\mathbf{f}, \quad \operatorname{div}\,\mathbf{v} = 0,$$

or, using the material law for a Navier-Stokes fluid

$$\rho_f\,(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \nabla p - \rho_f\nu_f\operatorname{div}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) = \rho_f\mathbf{f}, \quad \operatorname{div}\,\mathbf{v} = 0. \tag{2.42}$$

Remark 2.34 (Symmetry of the Stress-Tensor) For an incompressible fluid, the stress-tensor allows for a further simplification. It holds:

$$\left[\operatorname{div}\,\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)\right]_i = \sum_j \partial_j\big(\partial_j\mathbf{v}_i + \partial_i\mathbf{v}_j\big) = \Delta\mathbf{v}_i + \partial_i\underbrace{\operatorname{div}\,\mathbf{v}}_{=0}, \quad \text{for } i = 1,2,3,$$

and Eq. (2.42) is equivalent to the reduced formulation

$$\rho_f\left(\partial_t \mathbf{v}_f + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - \rho_f \nu_f \Delta\mathbf{v} + \nabla p = \rho_f \mathbf{f}, \quad \text{div } \mathbf{v} = 0.$$

Usually, this simplified set of equations is considered as the Navier-Stokes equations. However, while both equations yield the same solution $(\mathbf{v}, p)$, the value of boundary stresses is altered, if the reduced tensor $\tilde{\boldsymbol{\sigma}}_f = \mu_f \nabla\mathbf{v} - pI$ would be considered:

$$\tilde{\boldsymbol{\sigma}}_f n \neq \boldsymbol{\sigma}_f n.$$

In the context of fluid-structure interactions, boundary stresses will be important to couple flow and structure problem. Out of this reason, we will always consider the full symmetric stress tensor $\boldsymbol{\sigma}_f$. One of the coupling conditions will couple normal stresses of the fluid problem and the solid problem

$$\mathbf{n}\cdot\boldsymbol{\sigma}_f = \mathbf{n}\cdot\boldsymbol{\sigma}_s,$$

where by $\boldsymbol{\sigma}_s$ we denote the Cauchy stress tensor of the solid, i.e.

$$\boldsymbol{\sigma}_s = \hat{J}^{-1}\mathbf{F}\hat{\boldsymbol{\Sigma}}_s\hat{\mathbf{F}}^T.$$

Here, it will matter, whether it holds

$$-p\mathbf{n} + \rho_f \nu_f \mathbf{n}\cdot(\nabla\mathbf{v} + \nabla\mathbf{v}^T) = \mathbf{n}\boldsymbol{\sigma}_s,$$

or

$$-p\mathbf{n} + \rho_f \nu_f \mathbf{n}\cdot\nabla\mathbf{v} = \mathbf{n}\boldsymbol{\sigma}_s,$$

as usually we have

$$\mathbf{n}\cdot\nabla\mathbf{v}^T \neq 0.$$

2.4.1 Boundary and Initial Conditions

The system of equations is completed by adequate boundary and initial conditions. Let $\mathcal{F} \subset \mathbb{R}^d$ be the fluid-domain. At time $t = 0$ we prescribe an initial condition for the velocity

$$\mathbf{v}(x, 0) = \mathbf{v}^0(x) \quad x \in \mathcal{F}.$$

As the density is constant $\rho_f(x,t) \equiv \rho_f$ for all times (and homogenous in the domain), we do not need an initial condition here, but simply consider $\rho_f \in \mathbb{R}$ as a problem parameter. The boundary $\partial\mathcal{F}$ is split into a Dirichlet part Γ_f^D and into a Neumann part Γ_f^N. On Γ_f^D we prescribe Dirichlet conditions for the velocity

$$\mathbf{v}(x,t) = \mathbf{v}^D(x,t) \quad \text{on } \Gamma_f^D \times [0,T].$$

In the case $\mathbf{v}^D = 0$, we denote this condition as the *no-slip condition*. Physical observation tells us that viscosity will cause the fluid to stick to the boundary. This condition holds for the flow of water over elastic material (at usual velocities). The importance of viscous effects is lessened at high velocities, when e.g. considering the aerodynamical flow of air around a plane. Here, one often refers to the *slip condition* that only prescribes the flow in normal direction

$$\mathbf{n} \cdot \mathbf{v}(x,t) = 0 \quad \text{on } \Gamma_f^D \times [0,T].$$

The slip boundary condition prevents the flow from entering the boundary, it however allows for tangential flow. All examples considered in this work will be in the viscous regime where no-slip condition are usually well-placed. Boundaries with non homogenous Dirichlet data are often *inflow boundaries*.

Neumann conditions model situations, where we do not know the velocity profile at the boundary, but where assumptions on the boundary stress are given:

$$\boldsymbol{\sigma}_f(x,t)n(x,t) = g^\sigma(x,t) \quad \text{on } \Gamma_f^N \times [0,T].$$

The typical application of Neumann conditions are *outflow boundaries*, where the profile of the flow is not known and a Dirichlet condition cannot be prescribed. See Fig. 2.7 for a typical configuration of a flow problem with different boundary parts. We will come back to outflow boundary conditions in Sect. 2.4.2, as the exact form will depend on the material law and the Cauchy stress tensor $\boldsymbol{\sigma}_f$.

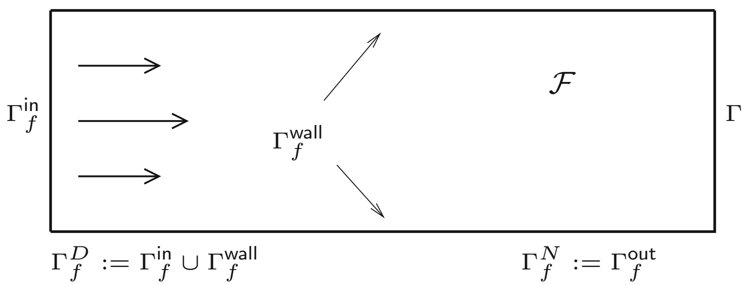

Fig. 2.7 Typical configuration of a flow problem with Dirichlet inflow boundary Γ_f^{in} and Dirichlet no-slip boundary on the walls Γ_f^{wall} as well as an outflow boundary Γ_f^{out} of Neumann type

If only no-slip and outflow boundary conditions are taken into account, the complete set of incompressible flow equations on the (fixed) domain $\mathcal{F} \subset \mathbb{R}^d$ is given by

Problem 2.35 (Incompressible Navier-Stokes Equations) *Velocity and pressure*

$$\mathbf{v}(t) \in C^2(\mathcal{F}) \cap C(\mathcal{F} \cup \Gamma_f^D) \cap C^1(\mathcal{F} \cup \Gamma_f^N), \quad p(t) \in C^1(\mathcal{F}) \cap C(\mathcal{F} \cup \Gamma_f^N),$$

are given as solution of

$$\begin{aligned} \operatorname{div} \mathbf{v} = 0, \quad \rho_f\,(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) &= \rho_f \mathbf{f} + \operatorname{div} \boldsymbol{\sigma}_f && \textit{on } \mathcal{F} \times [0,T], \\ \mathbf{v}(\cdot,0) &= \mathbf{v}^0(\cdot) && \textit{on } \mathcal{F}, \\ \mathbf{v} &= \mathbf{v}^D && \textit{on } \Gamma_f^D \times [0,T], \\ \boldsymbol{\sigma}_f \mathbf{n} &= g^{\sigma} && \textit{on } \Gamma_f^N \times [0,T]. \end{aligned} \tag{2.43}$$

If boundary data $\mathbf{v}^D$ and g^{σ} as well as volume force $\mathbf{f}$ do not explicitly depend on time, the flow configurations can tend to a stationary limit, where it holds $\partial_t \mathbf{v} = 0$. Stationary in the context of fluid dynamics stands for a flow that at all times looks the same way, it does not imply that the fluid is at rest, which would mean $\mathbf{v} = 0$. If we know that the flow will reach a stationary limit, we can immediately consider the set of stationary equations, given as a boundary value problem.

Problem 2.36 (Stationary Incompressible Navier-Stokes Equations) *Velocity and pressure*

$$\mathbf{v} \in C^2(\mathcal{F}) \cap C(\mathcal{F} \cup \Gamma_f^D) \cap C^1(\mathcal{F} \cup \Gamma_f^N), \quad p \in C^1(\mathcal{F}) \cap C(\mathcal{F} \cup \Gamma_f^N),$$

are given as solution of

$$\begin{aligned} \operatorname{div} \mathbf{v} = 0, \quad \rho_f (\mathbf{v}\cdot\nabla)\mathbf{v} &= \rho_f \mathbf{f} + \operatorname{div} \boldsymbol{\sigma}_f && \textit{on } \mathcal{F}, \\ \mathbf{v} &= \mathbf{v}^D && \textit{on } \Gamma_f^D, \\ \boldsymbol{\sigma}_f \mathbf{n} &= g^{\sigma} && \textit{on } \Gamma_f^N. \end{aligned} \tag{2.44}$$

Not all autonomous flow problems have a stationary limit. This stems from the nonlinearity of the Navier-Stokes equations and whether a flow is stationary or instationary will depend on the problem data like density, viscosity, right hand side f and inflow velocity $\mathbf{v}^D$.

2.4.2 The "do-nothing" Outflow Condition

Many problem configurations feature boundaries, where the flow has mainly an outflow-character. We will call this boundary Γ_f^{out}. Here, the solution is not known a priori and cannot be specified in terms of a Dirichlet condition. Any boundary condition that is enforced, will be a model for the flow at the outflow boundary. Hence a common practice is to not describe a condition at all, but simply use the "natural" boundary condition, that arises from integration by parts. We consider the stationary Stokes equations:

$$(\boldsymbol{\sigma}_f, \nabla\phi)_{\mathcal{F}} = -(\operatorname{div}\, \boldsymbol{\sigma}_f, \phi)_{\mathcal{F}} + \langle \boldsymbol{\sigma}_f \mathbf{n}, \phi\rangle_{\Gamma_f^{\text{out}}},$$

from where we can deduce the "outflow-condition"

$$\boldsymbol{\sigma}_f \mathbf{n} = 0 \text{ on } \Gamma_f^{\text{out}}.$$

In Fig. 2.8, we show a solution to a "channel-flow" problem using this natural outflow-condition. The domain is a channel with length L and height H

$$\mathcal{F} = (0, L) \times (0, H),$$

on the left boundary Γ_f^{in} we impose a Dirichlet inflow profile

$$\mathbf{v} = \mathbf{v}^D = \frac{4\bar{\mathbf{v}}}{H^2}\begin{pmatrix} y(H-y) \\ 0 \end{pmatrix} \text{ on } \Gamma_f^{\text{in}} = 0 \times (0, H), \tag{2.45}$$

where $\bar{\mathbf{v}}$ is the peak velocity. On the horizontal lines Γ_f^{wall} we impose homogenous Dirichlet conditions

$$\mathbf{v} = 0 \text{ on } \Gamma_f^{\text{wall}} = (0, L) \times 0 \cup (0, L) \times H.$$

The outflow boundary is given as

$$\Gamma_f^{\text{out}} = L \times (0, H).$$

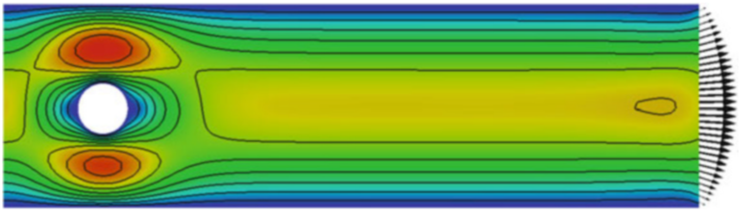

Fig. 2.8 Channel flow with natural outflow condition $\boldsymbol{\sigma}_f \mathbf{n} = 0$. The velocity field gets deflected and does not follow the Poiseuille flow

In Fig. 2.8 we see that the velocity vectors get deflected and swing out of line. Considering the outflow model $\boldsymbol{\sigma}_f \mathbf{n} = 0$, which simply states that no external stresses act, this behavior can be interpreted as a duct that ends in an open space, such that the fluid can expand in all directions.

Often, computational domains are chosen simply as a restriction of a larger domain to an area where the interesting dynamics happen. Numerically, boundary lines often must be drawn to scale the problem down to a reasonable size. In such situations, a good outflow boundary should have as little influence on the solution as possible. Regarding Fig. 2.8, the exact location of the outflow boundary should not change the flow pattern inside the domain. The natural condition does not satisfy this request.

One of the most simple analytical solutions to a channel problem is the *Poiseuille flow*. An extension of the inflow data (2.45) into the domain

$$\mathbf{v}(x, y) = \frac{4\bar{\mathbf{v}}}{H^2} \begin{pmatrix} y(H-y) \\ 0 \end{pmatrix},$$

satisfies the Navier-Stokes equations in channels (without obstacle) together with the pressure field

$$p(x, y) = \frac{8\bar{v}}{H^2} x + c,$$

for every $c \in \mathbb{R}$. In channel-like situations as shown in Fig. 2.8, an outflow condition should allow for Poiseuille flows without deterioration.

By a small modification of this outflow condition, we allow the Poiseuille flow to leave the domain without deflection. Using the reduced stress tensor introduced in Remark 2.34

$$\tilde{\boldsymbol{\sigma}}_f = \rho_f \nu_f \nabla \mathbf{v} - pI,$$

it holds for the Poiseuille flow that

$$\tilde{\boldsymbol{\sigma}}_f \mathbf{n} = (\mathbf{n} \cdot \nabla)\mathbf{v} - p\mathbf{n} = 0 \text{ on } \Gamma_f^{\text{out}}.$$

This condition is called the *do-nothing outflow condition*, as it has as little impact on the flow as possible (or as it is the natural boundary condition, that arises without doing anything, when using the reduced tensor), see [188]. In Fig. 2.9, we show the flow around a cylinder using this do-nothing condition. Here, he streamlines leave the domain in a straight way. Compare Fig. 2.8.

Remark 2.37 (Outflow Conditions) We must stress that the *do-nothing* outflow condition is not the better condition from a physical point of view. It is simply a model that allows for some standard flow situations like Poiseuille flow or Couette flow to reduce the sensitivity of the solution on the position of artificial boundaries.

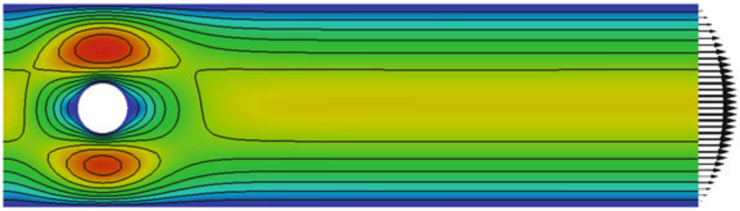

Fig. 2.9 Channel flow with the *do-nothing outflow condition* $\rho_f \nu_f \nabla \mathbf{v} \mathbf{n} - pI = 0$ on Γ_f^{out}. The streamlines are not deflected on the right outflow boundary. Compare Fig. 2.8

From a good outflow condition we expect that it has as little influence on the flow field as possible. If the outflow boundary is far away from a region of interest (e.g. from an obstacle) we expect that the flow close to the obstacle is not influenced by the position of the outflow boundary, if the outflow boundary condition does a good job. The *do-nothing* condition works excellent in several configurations. It does not only allow Poiseuille or Couette flows to leave the domain, it further allows vortices to leave the domain and has very small influence on these vortices, if the boundary is artificially cutting through them. However, many situations exist, where the analysis of outflow conditions is still not sufficiently developed: whenever the outflow boundary is not a single straight line normal to the main flow-direction, it will cause a deflection of the flow field. Further, if one considers more general material laws of non-Newtonian fluids, the *do-nothing* condition has an impact on the flow-field, see [338].

The *do-nothing* boundary condition brings along a further "hidden" boundary condition that normalizes the pressure. It can be shown [188] that on every straight outflow boundary-line segment $\Gamma_i \subset \partial \mathcal{F}$ that is enclosed by no-slip Dirichlet boundaries, it holds

$$\int_{\Gamma_i^{\text{out}}} p \, \mathrm{d}s = 0,$$

on all outflow boundaries Γ_i^{out}, such that the average outflow pressure is zero. This condition has two implications: first, whenever an outflow boundary of *do-nothing* type is given, no pressure-normalization has to be included in the trial spaces. Second, the *do-nothing* condition can be used to prescribe pressure drops on boundary segments in order to drive the flow:

$$\int_{\Gamma_i} \{\rho_f \nu_f \mathbf{n} \cdot \nabla \mathbf{v} - p\mathbf{n}\} \mathrm{d}s = \int_{\Gamma_i} P_i \, \mathrm{d}s, \quad i = 1, \dots, N^{\text{out}}, \quad P_i \in \mathbb{R}.$$

This gets important, if the flow is driven by pressure differences and not by means of Dirichlet conditions. A frequently considered situation arises in hemodynamical simulations in which a flow in a part of the channel-system (i.e., the cardiovascular system) is investigated. This small part of the overall problem can be coupled by prescribing pressure values, e.g. taken from the pressure profile as measured from the heart-beat.

2.4.3 Reynolds Number

Simulations with the incompressible Navier-Stokes equations help to gain better insight into flow configurations. They can be used to replace and complement experiments. For a better comparison of similar flow-configurations that for instance arise by scaling in wind tunnel experiments, we introduce a non-dimensional form of the incompressible Navier-Stokes equations. First, let L_f be a unit length and $\bar{V}_f$ be a unit velocity. We define the non-dimensional values (without physical units)

$$x^* := \frac{1}{L_f}x, \quad \mathbf{v}^* := \frac{1}{V_f}\mathbf{v}, \quad t^* := \frac{V_f}{L_f}t, \quad p^* := \frac{1}{V_f^2 \rho_f}p. \tag{2.46}$$

For these new values, it holds:

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*)v^* = \frac{L_f}{V_f^2}\left\{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right\},$$
$$\Delta^* \mathbf{v}^* = \frac{L_f^2}{V_f}\Delta \mathbf{v}, \quad \nabla^* p^* = \frac{L_f}{V_f^2 \rho_f}\nabla p,$$

and the Navier-Stokes equations in non-dimensional form (with homogenous right hand side) reads

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* - \frac{\nu_f}{V_f L_f}\operatorname{div}^* \left\{\nabla^* \mathbf{v}^* + (\nabla^* \mathbf{v}^*)^T\right\} - \nabla^* p^* = 0,$$
$$\nabla^* \cdot \mathbf{v}^* = 0.$$

The quantity

$$Re := \frac{V_f L_f}{\nu_f} = \frac{V_f L_f \rho_f}{\mu_f},$$

is called the *Reynolds number*. Scaled flow configurations with the same Reynolds number are equivalent. If the flow is known in the non-dimensional unit-system, it can be scaled to every equivalent configuration via (2.46). The Reynolds number is a good measure to describe the dynamical behavior of a flow configuration. Flows at low Reynolds number tend to have a stationary solution, while flows at higher Reynolds numbers have non-stationary or even turbulent solutions. The definition of the Reynolds number is somewhat arbitrary as fixing a reference velocity V_f and length L_f is usually not unique. The Reynolds number may therefore only by used to compare different flow situations for one configuration, e.g. the flow around a ship with length $L = 100\,\mathrm{m}$ compared to a down-scaled model of the same ship with length 5 m.

2.4.4 The Linear Stokes Equations

In flow situations where friction effects are very large compared to acceleration terms, the Navier-Stokes equations can be simplified by neglecting the convective term $(\mathbf{v}\cdot\nabla)\mathbf{v}$. This case is given, if the Reynolds number tends to zero $Re \to 0$. If the right hand side of the equation as well as boundary data does not depend on time, the flow field will be stationary and we end up with the stationary Stokes equations

$$-\rho_f \nu_f \Delta \mathbf{v} + \nabla p = \rho_f \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{F},$$

with the usual Dirichlet or Neumann boundary conditions on $\partial\mathcal{F}$. By renormalizing the pressure $\bar{p} = (\rho_f \nu_f)^{-1} p$ and the volume force $\bar{\mathbf{f}} = \nu_f^{-1}\mathbf{f}$ all physical parameters can be omitted and we derive the equations in non-dimensionalized form.

Problem 2.38 (Stokes Equations) *Velocity* $\mathbf{v} \in C^2(\mathcal{F}) \cap C(\bar{\mathcal{F}})$ *and pressure* $p \in C^1(\mathcal{F})$ *are given as solution of*

$$-\Delta \mathbf{v} + \nabla \bar{p} = \bar{\mathbf{f}}, \quad \operatorname{div} \mathbf{v} = 0 \textit{ in } \mathcal{F}. \tag{2.47}$$

Compared to the full incompressible Navier-Stokes equations, this equation is rather simple looking. As a saddle-point system it however still obtains one of the most important features of incompressible flows. While the physical relevance of the Stokes equations is very limited, it serves as entry-point to the mathematical analysis and the design of finite element discretizations for flow problems.

2.4.5 Theory of Incompressible Flows

If there exists a unique solution $\{\mathbf{v}, p\}$ to the incompressible Navier-Stokes equations is still not known in all configuration. The stationary case is well understood, if we only consider Dirichlet boundary conditions. Here, a solution exists for small Reynolds numbers and it is unique, if the data is sufficiently small. When we consider general outflow conditions, we have no possibility to control the nonlinearity $(\mathbf{v}\cdot\nabla)\mathbf{v}$. In the instationary configuration there exists no proof for the existence of a unique solution under reasonable data assumptions. In three dimensions, the problem of proving the existence of a global smooth solution is considered open and one of the *Millenium Prize Problems*, see [89].

We start by deriving a weak formulation of the Navier-Stokes equations:

Lemma 2.39 (Weak Formulation of the Navier-Stokes Equations) *Let* $\bar{\mathbf{v}}^D \in H^1(\mathcal{F})^d$ *be an extension of the Dirichlet data on* Γ_f^D *into the domain* $\mathcal{F}$. *If the solution*

$$\mathbf{v} \in \bar{\mathbf{v}}^D + \mathcal{V}_f, \quad \mathcal{V}_f := H_0^1(\mathcal{F}; \Gamma_f^D)^d, \quad p \in \mathcal{L}_f, \quad \mathcal{L}_f := L^2(\mathcal{F}),$$

of the variational formulation

$$\begin{aligned}(\rho_f(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}), \phi)_{\mathcal{F}} + (\boldsymbol{\sigma}_f, \nabla\phi)_{\mathcal{F}} \\ -\rho_f \nu_f \langle n \cdot \nabla \mathbf{v}^T, \phi\rangle_{\Gamma_f^{out}} &= (\rho_f \mathbf{f}, \phi)_{\mathcal{F}} \qquad \forall \phi \in \mathcal{V}_f, \\ (\operatorname{div} \mathbf{v}, \xi)_{\mathcal{F}} &= 0 \qquad \forall \xi \in \mathcal{L}_f,\end{aligned} \tag{2.48}$$

has sufficient regularity $v_f \in C^2(\mathcal{F}) \cap C(\mathcal{F} \cup \Gamma_f^D) \cap C^1(\mathcal{F} \cup \Gamma_f^{out})$ *and* $p \in C^1(\mathcal{F})$, *it also solves the classical formulation of the Navier-Stokes equations, Problem 2.35 with Dirichlet data on* Γ_f^D *and the* do-nothing *outflow condition on* Γ_f^{out}.

Proof This follows by integration by parts and with basic variational principles. The boundary term on Γ_f^{out} is required as we use the full symmetric stress-tensor such that the solution of the variational formulation fulfills the *do-nothing* condition, see Sect. 2.4.2. □

Remark 2.40 (Uniqueness of the Pressure in Dirichlet Problem) If the configuration has Dirichlet boundaries all around the boundary $\Gamma_f^D = \partial\mathcal{F}$, a solution cannot be unique: let $\{\mathbf{v}, p\} \in \mathcal{V}_f \times \mathcal{L}_f$ be a solution. Then, it holds for $\{\mathbf{v}, p+c\}$ with $c \in \mathbb{R}$:

$$\begin{aligned}(\boldsymbol{\sigma}_f, \nabla\phi)_{\mathcal{F}} &= \rho_f \nu_f (\nabla \mathbf{v} + \nabla \mathbf{v}^T, \nabla\phi)_{\mathcal{F}} - (p_f + c, \nabla\cdot\phi)_{\mathcal{F}} \\ &= \rho_f \nu_f (\nabla \mathbf{v} + \nabla \mathbf{v}^T, \nabla\phi)_{\mathcal{F}} - (p_f, \nabla\cdot\phi)_{\mathcal{F}} + (\underbrace{\nabla c}_{=0}, \phi)_{\mathcal{F}} - \langle cn, \underbrace{\phi}_{=0}\rangle_{\partial\mathcal{F}}.\end{aligned}$$

If $\Gamma_f^D = \partial\mathcal{F}$ the pressure can only be unique up to a constant. In this case, we normalize the pressure-space

$$\mathcal{L}_f = L^2(\mathcal{F}) \setminus \mathbb{R}.$$

The Navier-Stokes equations brings along two characteristic difficulties for theoretical analysis and numerical discretization, the nonlinearity $(\mathbf{v}\cdot\nabla)\mathbf{v}$ and the side-condition of divergence freeness $\operatorname{div} \mathbf{v} = 0$. We will first focus on this second difficulty and consider the linear Stokes equations.

2.4.5.1 Existence and Uniqueness of Solutions to the Stokes Equations

In the following, we consider the stationary Stokes equations

$$\mathbf{v}, p \in \mathcal{V}_f \times L_f, \quad \mathcal{V}_f := H_0^1(\mathcal{F}; \partial\mathcal{F})^d, \quad \mathcal{L}_f := L^2(\mathcal{F}) \setminus \mathbb{R} :$$
$$(\nabla \mathbf{v}, \nabla \phi)_{\mathcal{F}} - (p, \nabla \cdot \phi)_{\mathcal{F}} + (\nabla \cdot \mathbf{v}, \xi)_{\mathcal{F}} = (\mathbf{f}, \phi)_{\mathcal{F}} \quad \forall \{\phi, \xi\} \in \mathcal{V}_f \times \mathcal{L}_f.$$

Here, we assume homogenous Dirichlet conditions on the complete boundary $\partial\mathcal{F}$ and further we consider the non-symmetric form of the stress tensor. Every solution $\mathbf{v} \in \mathcal{V}_f$ will be weakly divergence free in the space

$$\mathbf{v} \in \mathcal{V}_0 := \{\phi \in \mathcal{V}_f, \ (\text{div}\ \phi, \xi)_{\mathcal{F}} = 0 \quad \forall \xi \in \mathcal{L}_f\} \subset \mathcal{V}_f.$$

By restricting the Stokes equations to this space, it remains to find

$$\mathbf{v} \in \mathcal{V}_0 : \quad (\nabla \mathbf{v}, \nabla \phi)_{\mathcal{F}} = (\mathbf{f}, \phi)_{\mathcal{F}} \quad \forall \phi \in \mathcal{V}_0. \tag{2.49}$$

Lemma 2.41 (Stokes Velocity) *For every* $\mathbf{f} \in H^{-1}(\mathcal{F})^d$ *there exists a unique velocity* $\mathbf{v} \in \mathcal{V}_0 \subset \mathcal{V}_f$ *as solution of the Stokes equations. Further, it holds*

$$\|\nabla \mathbf{v}\| \leq \|\mathbf{f}\|_{-1}.$$

Proof The space $\mathcal{V}_0 \subset \mathcal{V}_f$ is a Hilbert-space with the scalar product $(\nabla \cdot, \nabla \cdot)$. Riesz representation theorem guarantees the existence of a unique solution $\mathbf{v} \in \mathcal{V}_0$ to (2.49) and further gives the error estimate. □

2.4.5.2 Existence and Uniqueness of the Pressure

Now that we have shown the existence of a unique and divergence-free velocity field $\mathbf{v} \in \mathcal{V}_0 \subset \mathcal{V}_f$, the pressure is determined by the equation

$$p \in \mathcal{L}_f : \quad (p, \nabla \cdot \phi) = (\mathbf{f}, \phi) - (\nabla \mathbf{v}, \nabla \phi) \quad \forall \phi \in \mathcal{V}_f. \tag{2.50}$$

As this equation is not elliptic, we cannot proof existence with Riesz representation theorem or a generalization like Lax-Milgram. Instead, we reformulate this variational equation in operator notation as

$$-\operatorname{grad} p = l, \tag{2.51}$$

where $-\text{grad} : \mathcal{L}_f \to H^{-1}$ is the weak gradient

$$-\langle \operatorname{grad} p, \phi \rangle = (p, \nabla \cdot \phi) \quad \forall \phi \in \mathcal{V}_f,$$

and $l \in H^{-1}$ a linear functional defined by

$$l(\phi) = (\mathbf{f}, \phi) - (\nabla \mathbf{v}, \nabla \phi) \quad \forall \phi \in \mathcal{V}_f.$$

Whether Eqs. (2.50) or (2.51) have a solution depends on the surjectivity of the weak gradient operator. The difficulty of the analysis for this equation is the low regularity of the problem. Two important results from the literature help up to answer the questions of existence and uniqueness of solutions. It holds

Theorem 2.42 (de Rham) *Let $l \in H^{-1}$. The equation*

$$-\operatorname{grad} p = l,$$

has a unique solution $p \in \mathcal{L}_f$, if and only if

$$l \in \mathcal{V}_0^\circ,$$

where by $\mathcal{V}_0^\circ$ we denote the annihilator of $\mathcal{V}_0$ in H^{-1}

$$\mathcal{V}_0^\circ := \{f \in H^{-1},\ f(\phi) = 0 \quad \forall \phi \in \mathcal{V}_0\} \subset H^{-1}.$$

And:

Theorem 2.43 *Let $\mathcal{F}$ be a bounded domain with Lipschitz boundary and $p \in L^2(\mathcal{F})$ be such that* grad $p \in H^{-1}(\mathcal{F})$. *Then, it holds*

$$\gamma \|p\|_{L^2(\mathcal{F})\setminus\mathbb{R}} \leq \|\operatorname{grad} p\|_{-1},$$

with a constant $\gamma = \gamma(\mathcal{F})$ that depends on the domain only.

Proof For proofs of these essential theorems we refer to the literature. See Teman [321], de Rham [112] and Nečas [251]. □

We will quote yet another Theorem to show equivalence of Theorem 2.43 with further conditions that will be handy in the context of the Stokes equations; both for proofing existence and uniqueness of the pressure, as well as for numerical error analysis.

Theorem 2.44 (Nečas) *The following three properties are equivalent*

(i) The weak gradient operator $-\operatorname{grad} : \mathcal{L}_f \to \mathcal{V}_0^\circ$ *is an isomorphism.*
(ii) For every $p \in L^2(\mathcal{F})$ it holds

$$\|\operatorname{grad} p\|_{-1} \geq \gamma \|p\| \quad \forall p \in \mathcal{L}_f, \tag{2.52}$$

where $\gamma > 0$ is a constant. (This is exactly Theorem 2.43).

(ii) The inf-sup condition holds

$$\inf_{\xi\in\mathcal{L}_f}\sup_{\phi\in\mathcal{V}_f}\frac{(\xi,\nabla\cdot\phi)}{\|\xi\|\,\|\nabla\phi\|}\ge\gamma, \tag{2.53}$$

with a constant $\gamma > 0$.

Proof Again, we refer to the literature [251, 321]. □

All these preparations allow us to show the existence of a unique solution to the Stokes equations:

Lemma 2.45 (Stokes) *Let* $\mathcal{F}\subset\mathbb{R}^d$ *be a domain with Lipschitz boundary. The Stokes equation has a unique solution* $\mathbf{v}\in\mathcal{V}_f$ *and* $p\in\mathcal{L}_f$ *for every* $f\in H^{-1}$. *It holds*

$$\|\nabla\mathbf{v}\|+\gamma\|p\|\le c\|\mathbf{f}\|_{-1},$$

where $c > 0$ *is a constant.*

Proof The existence of a unique function $\mathbf{v}\in\mathcal{V}_0$ solving the velocity equation has already been shown. The functional

$$l(\phi)=(\nabla\mathbf{v},\nabla\phi)-(\mathbf{f},\phi)$$

is bound in $H^{-1}(\mathcal{F})$ and further, it holds $l\in\mathcal{V}_0^{\circ}$. Hence existence of a unique weak pressure $p\in\mathcal{L}_f$ solving $-\operatorname{grad} p = l$ follows by Lemma 2.44.

Finally, by using the inf-sup inequality we have

$$\begin{aligned}\gamma\|p\| &\le \sup_{\phi\in\mathcal{V}_f}\frac{(p,\nabla\cdot\phi)}{\|\nabla\phi\|} = \sup_{\phi\in\mathcal{V}_f}\frac{(\mathbf{f},\phi)-(\nabla\mathbf{v},\nabla\phi)}{\|\nabla\phi\|}\\ &\le \|\mathbf{f}\|_{-1}+\|\nabla\mathbf{v}\|\le 2\|\mathbf{f}\|_{-1}.\end{aligned}$$

□

During the proof of this Lemma, we have used the following useful inequality for the divergence operator

$$\|\operatorname{div}\,\mathbf{v}\|\le\sqrt{d}\|\nabla\mathbf{v}\|\quad\forall\mathbf{v}\in H^1(\mathcal{F})^d,\qquad\|\operatorname{div}\,\mathbf{v}\|\le\|\nabla\mathbf{v}\|\quad\forall\mathbf{v}\in H_0^1(\mathcal{F})^d, \tag{2.54}$$

which follows with help of Young's inequality in the general case and with help of integration by parts of the mixed terms in the case of zero trace velocity fields.

Despite the special saddle-point character of the Stokes equations it shows that we still get a unique solution that continuously depends on the right hand side **f**. We only get L^2-regularity for the pressure. The most important tool in the analysis of incompressible flows is the inf-sup condition. If the right hand side **f** and the domain is sufficiently regular, we will get higher regularity of the solution. Here, the same rule of thump holds as for the Laplace equation:

Lemma 2.46 (Regularity of the Stokes Equations) *Let $\mathcal{F}$ be a convex polygonal domain and $\mathbf{f} \in L^2(\mathcal{F})^d$. Then the solution of the Stokes equations is bounded*

$$\|\nabla^2 \mathbf{v}\| + \|\nabla p\| \le c_s \|\mathbf{f}\|,$$

with a stability constant $c_s > 0$.

If $\mathcal{F} \subset \mathbb{R}^d$ is a domain with smooth C^{k+2}-boundary for $k \ge 0$ and $\mathbf{f} \in H^k(\mathcal{F})^d$ it holds

$$\|\mathbf{v}\|_{H^{k+2}(\mathcal{F})} + \|p\|_{H^{k+1}(\mathcal{F})} \le c\|\mathbf{f}\|_{H^k(\mathcal{F})}.$$

Proof For a proof to these results, we refer to the literature [160, 321]. □

2.4.5.3 The Stationary Navier-Stokes Equations

Next, we discuss the stationary Navier-Stokes equations including the nonlinearity

$$\begin{aligned}
&\{\mathbf{v}, p\} \in \mathcal{V}_f \times L_f, \quad \mathcal{V}_f := H^1_0(\mathcal{F}; \partial\mathcal{F})^d, \quad \mathcal{L}_f := L^2(\mathcal{F}) \setminus \mathbb{R} : \\
&\qquad \frac{1}{Re}(\nabla \mathbf{v}, \nabla \phi) + (\mathbf{v} \cdot \nabla \mathbf{v}, \phi) - (p, \nabla \cdot \phi) + (\nabla \cdot \mathbf{v}, \xi) = (\mathbf{f}, \phi) \\
&\qquad\qquad \forall \{\phi, \xi\} \in \mathcal{V}_f \times \mathcal{L}_f,
\end{aligned} \tag{2.55}$$

again considering homogenous Dirichlet conditions $\mathbf{v} = 0$ only. Here, this restriction is essential not merely given for technical reasons, as the following Lemma shows:

Lemma 2.47 (Nonlinearity of the Navier-Stokes Equations) *For $\mathbf{v}, \mathbf{w} \in H^1_0(\mathcal{F})^d$ with* $\operatorname{div}\, \mathbf{v} = 0$ *it holds:*

$$(\mathbf{v} \cdot \nabla \mathbf{w}, \mathbf{w}) = 0. \tag{2.56}$$

In the case of an outflow boundary $\Gamma_f^{out} \subset \partial\mathcal{F}$ it holds for all $\mathbf{v}, \mathbf{w} \in H^1_0(\mathcal{F}; \Gamma_f^D)^d$ with $\operatorname{div}\, \mathbf{v} = 0$

$$\Big((\mathbf{v} \cdot \nabla)\mathbf{w}, \mathbf{w}\Big) = \frac{1}{2} \int_{\Gamma_f^{out}} \mathbf{n} \cdot \mathbf{v} |\mathbf{w}|^2 \, \mathrm{d}s. \tag{2.57}$$

Proof In the case of general boundary conditions it holds

$$\begin{aligned}\Big((\mathbf{v}\cdot\nabla)\mathbf{w},\mathbf{w}\Big)_{\mathcal{F}} &= \sum_{i,j}(\mathbf{v}_j\partial_j\mathbf{w}_i,\mathbf{w}_i)_{\mathcal{F}} \\ &= \sum_{i,j}\Big\{\int_{\partial\mathcal{F}}\mathbf{n}_j\mathbf{w}_i\mathbf{v}_j\mathbf{w}_i\,\mathrm{d}s - (\mathbf{w}_i,\partial_j\mathbf{v}_j\mathbf{w}_i)_{\mathcal{F}} - (\mathbf{w}_i,\mathbf{v}_j\partial_j\mathbf{w}_i)_{\mathcal{F}}\Big\} \\ &= -\underbrace{(\mathbf{w},(\operatorname{div}\,\mathbf{v})\mathbf{w})_{\mathcal{F}}}_{=0} - ((\mathbf{v}\cdot\nabla)\mathbf{w},\mathbf{w})_{\mathcal{F}} + \int_{\partial\mathcal{F}}(\mathbf{n}\cdot\mathbf{v})|\mathbf{w}|^2\,\mathrm{d}s.\end{aligned}$$

This shows the two assertions. □

This special structure of the nonlinearity will be the key to theoretical analysis of the incompressible Navier-Stokes equations.

Lemma 2.48 (Stability Estimate for the Velocity) *Let* $\mathbf{v}\in\mathcal{V}_0\subset H_0^1(\mathcal{F})^d$ *be a velocity field solving the Navier-Stokes equations. It holds for* $\mathbf{f}\in L^2(\mathcal{F})^d$

$$\|\nabla\mathbf{v}\| \le \nu^{-1}\|\mathbf{f}\|_{-1}.$$

Proof This results immediately follows with Lemma 2.47. □

Remark 2.49 (Outflow Conditions and Stability Estimates) Lemma 2.47 shows that the nonlinearity of the Navier-Stokes equations is only controllable, if Dirichlet or at least no-penetration conditions

$$\mathbf{v}\cdot\mathbf{n} = 0,$$

are given on all boundaries. For the *do-nothing* conditions but also for the *no-stress* condition introduced in Sect. 2.4.2 a boundary term remains. The problem of this remaining boundary term

$$\frac{1}{2}\int_{\Gamma_f^{\text{out}}}\mathbf{n}\cdot\mathbf{n}|\mathbf{w}|^2\,\mathrm{d}o,$$

is the unknown sign. If there would be only outflow, i.e. $\mathbf{n}\cdot\mathbf{v}\ge 0$, we still get stability in the sense of Lemma 2.48. In the general setting, the boundary term however can be negative or positive. Braack and Mucha [61] introduced a modification of the do-nothing condition, denoted the *directional do-nothing condition* that cancels the negative part of the boundary term and results in

$$-p\mathbf{n} + \rho_f\nu_f\mathbf{n}\cdot\nabla\mathbf{v} - \frac{1}{2}(\mathbf{v}\cdot\mathbf{n})_-\mathbf{v} = 0 \text{ on } \Gamma_f^{\text{out}},$$

where by $(\mathbf{v}\cdot\mathbf{n})_-$ we denote

$$(\mathbf{v}\cdot\mathbf{n})_- = \begin{cases} 0 & \mathbf{v}\cdot\mathbf{n} \geq 0, \\ \mathbf{v}\cdot\mathbf{n} & \mathbf{v}\cdot\mathbf{n} < 0. \end{cases}$$

This condition is easily realized by a modification of the variational formulation

$$(\mathbf{v}\cdot\nabla\mathbf{v},\phi) + (\rho_f\nu_f\nabla\mathbf{v},\nabla\phi) - (p,\nabla\cdot\phi) - \frac{1}{2}\int_{\Gamma_f^{\text{out}}}(\mathbf{v}\cdot\mathbf{n})_-\mathbf{v}\cdot\phi\,\mathrm{d}o = (\mathbf{f},\phi).$$

Braack and Mucha can show existence and uniqueness of solutions (for small data). Furthermore, they report better numerical stability when using this directional do-nothing condition. Finally, this modified condition still allows for Poiseuille and Couette flow as well as vortices to leave the domain with little impact. See [61] for details.

Like for the Stokes equations, proofs for existence and uniqueness are split into first finding the velocity (this is a nonlinear problem now) and second, finding an appropriate pressure. While this second part is exactly as for the linear Stokes problem, showing existence and uniqueness of a velocity requires careful treatment of the nonlinearity.

$$\nu(\nabla\mathbf{v},\nabla\phi) + ((\mathbf{v}\cdot\nabla)\mathbf{v},\phi) = (\mathbf{f},\phi)\quad \forall\phi\in\mathcal{V}_0. \tag{2.58}$$

Lemma 2.50 (Solutions for the Navier-Stokes Equations) *Let $\mathcal{F}\subset\mathbb{R}^d$ be a domain with Lipschitz boundary. Further, let $\mathbf{f}\in H^{-1}(\mathcal{F})$. There exists a solution $\{\mathbf{v},p\}\in\mathcal{V}_f\times\mathcal{L}_f$ to the Navier-Stokes equations (2.55) for every Reynolds number. It holds*

$$\|\nabla\mathbf{v}\| + \|p\| \leq c\|\mathbf{f}\|_{-1}.$$

This solution is unique, if

$$c^2\nu^{-2}\|\mathbf{f}\|_{-1} \leq 1,$$

where $c > 0$ is a constant depending on the domain $\mathcal{F}$.

Proof For the proof, we again refer to the literature [251, 321]. □

The incompressible Navier-Stokes problem with homogenous Dirichlet values has a solution $\{\mathbf{v},p\}\in\mathcal{V}_f\times\mathcal{L}_f$ for all Reynolds numbers and all right hand sides $\mathbf{f}\in H^{-1}(\mathcal{F})$. This solution is unique only if the Reynolds number is very small:

$$Re \leq \sqrt{\frac{1}{c^2\|\mathbf{f}\|_{-1}}}.$$

Most application problems however deal with high Reynolds numbers $Re \gg 1000$ and a unique solution cannot be guaranteed. As we know that flows at very high Reynolds numbers get turbulent, we cannot expect a unique result for arbitrary Reynolds numbers. The gap between theory and observation however is still very large.

Nearly no theoretical results are known for different boundary conditions, in particular for outflow conditions like the *do-nothing* condition. Here, it is even unknown, whether the homogenous problem

$$-\frac{1}{Re}\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0,$$

with homogenous boundary conditions

$$\mathbf{v} = 0 \text{ on } \Gamma_f^D, \quad \frac{1}{Re}\partial_n \mathbf{v} - p\mathbf{n} = 0 \text{ on } \Gamma_f^{\text{out}}$$

only has the trivial solution $\mathbf{v} = 0$ and $p = 0$ or if other non-trivial solutions exist.

Finally, we cite a regularity result for the stationary Navier-Stokes equations which is in agreement to the expectation:

Lemma 2.51 (Regularity of the Navier-Stokes Solution) *Let $\mathcal{F} \subset \mathbb{R}^d$ be a convex polygonal or smooth domain of class $C^{2,1}$. Further, let $\bar{\mathbf{v}}^D \in H^2(\mathcal{F})^d$ be a smooth extension of the Dirichlet data $\mathbf{v}^D$ on $\partial\mathcal{F}$ into the domain. Finally, let $\mathbf{f} \in L^2(\mathcal{F})^d$. The solution to the Navier-Stokes equations has the regularity $\mathbf{v} \in H^2(\mathcal{F}) \cap \mathcal{V}_f$ and $p \in H^1(\mathcal{F}) \cap \mathcal{L}_f$ and it holds*

$$\|\nabla^2 \mathbf{v}\| + \|\nabla p\| \le c_s\{\|\mathbf{f}\| + \|\nabla^2 \bar{\mathbf{v}}^D\|\},$$

where the stability constant is related to the Reynolds number $c_s \sim Re$.

Next, let $\mathcal{F}$ be a C^{k+2}-domain and $\mathbf{f} \in H^k(\mathcal{F})^d$. Then, every solution $\mathbf{v} \in H_0^1(\mathcal{F})^d$ and $p \in L^2(\mathcal{F})$ of the stationary Navier-Stokes equations has the regularity

$$\|\mathbf{v}\|_{H^{k+2}(\mathcal{F})} + \|p\|_{H^{k+1}(\mathcal{F})} \le c\|\mathbf{f}\|_{H^k(\mathcal{F})}.$$

Proof For a proof of this result we refer to the literature, see Girault and Raviart [165] or Sohr [312]. □

2.4.5.4 The Non-stationary Navier-Stokes Equations

Finally, we discuss the non-stationary Navier-Stokes equations

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^{\text{in}} && t = 0, \\ (\partial_t \mathbf{v}, \phi) + ((\mathbf{v}\cdot\nabla)\mathbf{v}, \phi) + \nu(\nabla\mathbf{v}, \nabla\phi) - (p, \nabla\cdot\phi) &= (\mathbf{f}, \phi) && \forall \phi \in \mathcal{V}_f, \\ (\nabla\cdot\mathbf{v}, \xi) &= 0 && \forall \xi \in \mathcal{L}_f. \end{aligned}$$

Like in the stationary case, we can restrict the problem to the space of divergence free functions $\mathcal{V}_0 \subset \mathcal{V}$. Integration of the variational formulation over the time-interval $I = [0, T]$ gives

$$\int_I \{(\partial_t \mathbf{v}, \phi) + ((\mathbf{v} \cdot \nabla)\mathbf{v}, \phi) + \nu(\nabla \mathbf{v}, \nabla \phi)\}\, \mathrm{d}t = \int_I (\mathbf{f}, \phi)\, \mathrm{d}t.$$

To analyze this variational formulation, we must first specify suitable function spaces. For the velocity part, natural choices for $\mathbf{v}$ and test function ϕ are

$$\mathbf{v}, \phi \in L^2(I; \mathcal{V}_0),$$

the space of square-integrable functions in time that map into $\mathcal{V}_0$. For the time-derivative of the velocity, we further ask for

$$\partial_t \mathbf{v} \in L^2(I; H^{-1}(\mathcal{F})).$$

We denote this space by $W(0, T)$

$$W(0, T) := \{\phi \in L^2(I; \mathcal{V}_0),\ \partial_t \phi \in L^2(I; H^{-1}(\mathcal{F}))\}. \tag{2.59}$$

The spaces

$$\mathcal{V}_0 \subset H_0^1(\Omega)^d \subset L^2(\Omega)^d \cong [L^2(\Omega)^d]^* \subset H^{-1}(\Omega)$$

constitute a Gelfand triple and it holds (see [321])

$$W(0, T) \hookrightarrow C(\bar{I}; L^2(\Omega)^d).$$

Every function $\mathbf{v} \in W(0, T)$ is almost everywhere equal to a continuous function in time that maps into $L^2(\Omega)^d$. It remains to discuss the nonlinearity: does for functions $\mathbf{v}, \phi \in W(0, T)$ hold that

$$\int_I ((\mathbf{v} \cdot \nabla)\mathbf{v}, \phi)\, \mathrm{d}t < \infty?$$

An answer is given by the following result:

Lemma 2.52 *Let $\Omega \subset \mathbb{R}^d$ be an open set. For $d = 2$ it holds*

$$\|\mathbf{v}\|_{L^4(\Omega)} \le c\|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}}.$$

In the case $d = 3$ it holds

$$\|\mathbf{v}\|_{L^4(\Omega)} \le c\|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{3}{2}}.$$

Proof A proof is given by Temam [321]. □

We consider the two-dimensional case. By Hölder's inequality ($1 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$) and this Lemma we get

$$((\mathbf{v} \cdot \nabla)\mathbf{v}, \phi) \le c\|\mathbf{v}\|_{L^4} \|\nabla \mathbf{v}\| \, \|\phi\|_{L^4} \le c\|\mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{3}{2}} \|\phi\|^{\frac{1}{2}} \|\nabla \phi\|^{\frac{1}{2}}.$$

Using the embedding $W(0,T) \hookrightarrow C(\bar{I}; L^2(\Omega))$ it follows for the temporal integral by using Hölder's inequality (in time)

$$\begin{aligned} &\int_I ((\mathbf{v} \cdot \nabla)\mathbf{v}, \phi) \, \mathrm{d}t \\ &\qquad \le c\|\phi\|^{\frac{1}{2}}_{C(\bar{I};L^2(\Omega))} \|\mathbf{v}\|^{\frac{1}{2}}_{C(\bar{I};L^2(\Omega))} \int_I \|\nabla \mathbf{v}\|^{\frac{3}{2}} \|\nabla \phi\|^{\frac{1}{2}} \, \mathrm{d}t \\ &\qquad \le c\|\phi\|^{\frac{1}{2}}_{W(0,T)} \|\mathbf{v}\|^{\frac{1}{2}}_{W(0,T)} \|\mathbf{v}\|^{\frac{3}{2}}_{W(0,T)} \|\phi\|^{\frac{1}{2}}_{W(0,T)} \\ &\qquad \le c\|\mathbf{v}\|^2_{W(0,T)} \|\phi\|_{W(0,T)}. \end{aligned}$$

This is exactly the desired stability result for the variational formulation. The nonlinearity is not bound in the three-dimensional case, if we ask for $\mathbf{v}, \phi \in W(0,T)$. We cite the following results that can be found in Temam [321]:

Lemma 2.53 (Instationary Navier-Stokes Equations) *Let $\mathcal{F} \subset \mathbb{R}^d$ be a Lipschitz domain and*

$$\mathbf{f} \in L^2(I; H^{-1}(\mathcal{F})), \quad \mathbf{v}^0 \in \mathcal{V}_0.$$

Then, the instationary Navier-Stokes equation has at least one solution for arbitrary Reynolds numbers. This solution is unique in the two dimensional case (for arbitrary Reynolds numbers) and it holds

$$\mathbf{v} \in L^2(I; \mathcal{V}_0), \quad \partial_t \mathbf{v} \in L^2(I; H^{-1}(\mathcal{F})).$$

In the three-dimensional case, unity is usually not given, and the solution has the reduced regularity

$$\mathbf{v} \in L^{\frac{8}{3}}(I; L^4(\Omega)), \quad \partial_t \mathbf{v} \in L^{\frac{4}{3}}(I; H^{-1}(\Omega)).$$

It is remarkable that the non-stationary solution is unique for all Reynolds numbers, if we look at the two-dimensional problem. Working with the stationary equation, uniqueness is only guaranteed for small data assumptions.

To prove existence of global solutions, uniqueness and regularity of the three dimensional problem is one of big open problems in applied mathematics, see [89].

2.5 Flow Problems on Moving Domains

In this section, we discuss models for flows on a moving domain $\mathcal{F}(t) \subset \mathbb{R}^d$. Let $I = [0, T]$ be the temporal interval. Then, the space-time domain is given as

$$\mathcal{G} = \{(t, \mathcal{F}(t)) \subset I \times \mathbb{R}^d\} \subset \mathbb{R}^{d+1}.$$

This setting is more complex then the tensor-product design of fixed domains $I \times \mathcal{F} \subset \mathbb{R}^{d+1}$. In $\mathcal{G}$ it is difficult to formulate the proper function spaces like (2.59) with a different regularity in time and space. We define

Problem 2.54 (Incompressible Navier-Stokes Equations on a Moving Domain) *Let $\mathcal{G} = \{(t, \mathcal{F}(t)),\ t \in I = [0, T]\}$ be the moving space time domain. Velocity and pressure*

$$\mathbf{v} \in L^2(I; \mathcal{V}_f(t)), \quad \partial_t \mathbf{v} \in L^2(I; \mathcal{V}_f(t)^*), \quad p \in L^2(I; L^2(\mathcal{F}(t))),$$

are determined as solution to the incompressible Navier-Stokes equations on the moving domain

$$(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}, \phi)_{\mathcal{F}(t)} + (\boldsymbol{\sigma}_f, \nabla\phi)_{\mathcal{F}(t)} + (\text{div } \mathbf{v}, \xi)_{\mathcal{F}(t)} = (\mathbf{f}_f, \phi)_{\mathcal{F}(t)} \quad \text{a.e. } t \in [0, T] \tag{2.60}$$

for all $\phi \in \mathcal{V}_f(t)$ and $\xi \in \mathcal{L}_f(t)$.

Mostly we will assume that the domain motion is given by a mapping from a fixed reference domain $\hat{\mathcal{F}} \subset \mathbb{R}^d$

$$\hat{T}(t) : \hat{\mathcal{F}} \mapsto \mathcal{F}(t).$$

First we assume that this mapping is given as part of the problem data, such that we can prescribe properties like invertibility, regularity. Later on, when analyzing fluid-structure interactions, this mapping will be an unknown part of the solution. This will strongly complicate the analysis, as regularity will no longer be part of the problem description but must result from the system of equations.

For here and for simplicity, we assume that $\hat{\mathcal{F}} = \mathcal{F}(0)$, i.e., the reference domain is the domain at initial time. The mapping is defined as Function from the fixed space-time domain to $\mathbb{R}^d$

$$\hat{T} : I \times \hat{\mathcal{F}} \to \mathbb{R}^d.$$

We will specify further assumptions on this mapping at a later point. The time-derivative of this mapping $\partial_t \hat{T}$ denotes a velocity. This velocity is not the physical velocity of the fluid particles, but it is the *domain velocity*. In the general case it is arbitrary and, in particular, it holds $\partial_t \hat{T} \neq \mathbf{v}$.

2.5.1 Eulerian Techniques for Flow Problems on Moving Domains

Discretization of partial differential equations is difficult if the domain is in motion. Usually, every discretization consists of first discretizing the domain $\mathcal{F} \subset \mathbb{R}^d$ by a mesh $\mathcal{F}_h$. If $\mathcal{F}(t)$ is moving the meshes $\mathcal{F}_h(t)$ also cannot be fixed.

We consider time stepping methods, where the solution determined in discrete time steps only

$$0 = t_0 < t_1 < \cdots < t_M = T.$$

By $\mathbf{v}^m := \mathbf{v}(t_m)$ and by $p^m := p(t_m)$ we denote velocity and pressure at time t_m. Then, in a discrete setting, approximations $\mathbf{v}_{kh}^m$ and $\mathbf{v}_{kh}^{m-1}$ will live on different meshes—or in the context of finite elements—in different function spaces V_{kh}^m and V_{kh}^{m-1}. Usual time-discretization schemes approximate the temporal derivative by finite differences

$$\partial_t \mathbf{v}_h(t_m) \approx \frac{\mathbf{v}_{kh}^m - \mathbf{v}_{kh}^{m-1}}{t_m - t_{m-1}}.$$

Now we assume that $\mathbf{v}_{kh}^m \in V_{kh}^m$ and $\mathbf{v}_{kh}^{m-1} \in V_{kh}^{m-1}$ are element of different finite element spaces. In this case, $\mathbf{v}_{kh}^m - \mathbf{v}_{kh}^{m-1}$ will most likely neither belong to V_{kh}^m nor to V_{kh}^{m-1}.

This problem gets even more severe, if we consider a spatial coordinate $x \in \mathcal{F}(t_m)$ that is not part of the domain at the old time step $x \notin \mathcal{F}(t_{m-1})$. Here, the expression $\mathbf{v}_{kh}^m(x) - \mathbf{v}_{kh}^{m-1}(x)$ is not well defined at all.

Eulerian schemes for moving domain problems will require non-standard discretization techniques and a non-standard analysis. We will pick up this discussion at a later point in Sect. 3.6 and Chaps. 6 and 12.

2.5.2 The Arbitrary Lagrangian Eulerian (ALE) Formulation for Moving Domain Problems

Another possibility to deal with the motion of the fluid-domain is to introduce a fixed reference domain $\hat{\mathcal{F}} \subset \mathbb{R}^d$ and the mapping

$$\hat{T}_f(t) : \hat{\mathcal{F}} \to \mathcal{F}(t).$$

We can use this mapping to transform the Navier-Stokes equations onto the reference domain $\hat{\mathcal{F}}$ and to define velocity and pressure in the reference system

$$\hat{\mathbf{v}}(\hat{x}, t) := \mathbf{v}(\hat{T}_f(\hat{x}, t), t), \quad \hat{p}(\hat{x}, t) := p(\hat{T}_f(\hat{x}, t), t) \quad \forall \hat{x} \in \hat{\mathcal{F}}. \tag{2.61}$$

The mapping $\hat{T}_f$ has to be invertible, such that at time $t \in I$, every spatial point $x \in \mathcal{F}(t)$ is uniquely given by one coordinate $\hat{x} \in \hat{\mathcal{F}}$.

If the mapping $\hat{T}_f$ is a C^1-diffeomorphism, it can be used to transform the Navier-Stokes equations onto $\hat{\mathcal{F}}$ using $\hat{\mathbf{v}}$ and $\hat{p}$ as principle variables. All relations required for this transformation have already been derived in Sect. 2.1.7. By (2.22) and with Definition 2.13 it holds by (2.61)

$$
\begin{aligned}
\rho_f\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \phi\big)_{\mathcal{F}(t)} &= \rho_f\big(\hat{J}_f(\partial_t\hat{\mathbf{v}} + \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}} - \partial_t\hat{T}_f)\cdot\nabla\hat{\mathbf{v}}), \hat{\phi}\big)_{\hat{\mathcal{F}}},\\
\big(\boldsymbol{\sigma}_f, \nabla\phi\big)_{\mathcal{F}(t)} &= \big(\hat{J}_f\hat{\boldsymbol{\sigma}}_f\hat{\mathbf{F}}_f^{-T}, \hat{\nabla}\hat{\phi}\big)_{\hat{\mathcal{F}}},\\
\big(\operatorname{div}\,\mathbf{v}, \xi\big)_{\mathcal{F}(t)} &= \big(\widehat{\operatorname{div}}\,(\hat{J}_f\hat{\mathbf{F}}_f^{-1}\hat{\mathbf{v}}), \hat{\xi}\big)_{\hat{\mathcal{F}}}.
\end{aligned}
\tag{2.62}
$$

The Cauchy stress tensor $\hat{\boldsymbol{\sigma}}(\hat{x})$ expressed in the reference system is derived with help of (2.16)

$$
\hat{\boldsymbol{\sigma}}_f := -\hat{p}I + \rho_f\nu_f(\hat{\nabla}\hat{\mathbf{v}}\hat{\mathbf{F}}_f^{-1} + \hat{\mathbf{F}}^{-T}\hat{\nabla}\hat{\mathbf{v}}^T).
\tag{2.63}
$$

By these transformations we formulate the system of Navier-Stokes equations in ALE coordinates.

Problem 2.55 (Incompressible Navier-Stokes Equations in ALE) *Let $\hat{\mathcal{F}}$ be a suitable reference domain, $\hat{T}_f$ a C^1-diffeomorphism on $I\times\hat{\mathcal{F}}$ with $\hat{T}_f(t) : \hat{\mathcal{F}} \mapsto \mathcal{F}(t)$. Then, velocity and pressure*

$$
\hat{\mathbf{v}} \in L^2(I;\hat{\mathcal{V}}_f),\ \ \partial_t\hat{\mathbf{v}} \in L^2(I;\hat{\mathcal{V}}_f^*),\quad p \in L^2(I;\hat{\mathcal{L}}_f)
$$

are given as solution to

$$
\begin{aligned}
\rho_f\big(\hat{J}_f(\partial_t\hat{\mathbf{v}} + \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}} - \partial_t\hat{T}_f)\cdot\nabla\hat{\mathbf{v}}), \hat{\phi}\big)_{\hat{\mathcal{F}}}&\\
+\big(\hat{J}_f\hat{\boldsymbol{\sigma}}_f\hat{\mathbf{F}}_f^{-T}, \hat{\nabla}\hat{\phi}\big)_{\hat{\mathcal{F}}} &= (\rho_f\hat{J}_f\hat{\mathbf{f}}_f, \hat{\phi})_{\hat{\mathcal{F}}}\\
\big(\widehat{\operatorname{div}}\,(\hat{J}_f\hat{\mathbf{F}}_f^{-1}\hat{\mathbf{v}}), \hat{\xi}\big)_{\hat{\mathcal{F}}} &= 0,
\end{aligned}
\tag{2.64}
$$

for all $\hat{\phi} \in \mathcal{V}_f$ and $\hat{\xi} \in \mathcal{L}_f$.

The derivation of the system of equations is performed on a formal basis. We still need to argue that the solutions to Problems 2.55 and 2.54 are in a meaningful way equivalent.

Considering the strong formulation of the Navier-Stokes equations, equivalence of a notation on the moving Eulerian domain $\mathcal{F}(t)$ and the fixed reference domain $\hat{\mathcal{F}}$ can be shown by classical arguments. If we assume that $\hat{T}_W$ is a C^2-diffeomorphism the equation can be transformed to an equivalent expression. In the variational formulation, we must first discuss the question of equivalence of Sobolev spaces under a mapping of the domain.

Lemma 2.56 (Transformation of Sobolev-Spaces) *Let Ω and $\hat{\Omega}$ be two domains in $\mathbb{R}^d$ and let $\hat{T} \in C^{k,1}(\hat{\Omega})^d$ be a diffeomorphism with $\hat{T}(\hat{\Omega}) = \Omega$ and $\hat{T}^{-1}(\Omega) = \hat{\Omega}$. Then, the composition operators*

$$\phi := \hat{\phi} \circ \hat{T}^{-1} \quad \forall \hat{\phi} \in H^{k+1}(\hat{\Omega}) \text{ and } \hat{\phi} := \phi \circ \hat{T} \quad \forall \phi \in H^{k+1}(\Omega),$$

are continuous. Hence the Sobolev spaces $H^{k+1}(\Omega)$ and $H^{k+1}(\hat{\Omega})$ are equivalent

$$H^{k+1}(\hat{\Omega}) \cong H^{k+1}(\Omega),$$

such that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|\hat{v}\|_{H^{k+1}(\hat{\Omega})} \le \|\hat{v} \circ T\|_{H^{k+1}(\Omega)} \le c_2 \|\hat{v}\|_{H^{k+1}(\hat{\Omega})} \quad \forall \hat{v} \in H^{k+1}(\hat{\Omega}).$$

For the proof, we refer to the literature, *Satz 4.1 - Transformationssatz*, in [350].

Considering stationary problems the velocity is a H^1 function, given in $\mathbf{v} \in H^1(\mathcal{F})^d$, the pressure is a L^2 function, given in $p \in L^2(\mathcal{F})$. Hence for $H^1(\mathcal{F})$ and $H^1(\hat{\mathcal{F}})$ to be equivalent, which is a necessary assumption for equivalent solution concepts, the mapping $\hat{T}_W$ must be a $C^{0,1}$-diffeomorphism in space. Equivalence of Sobolev spaces on $\mathcal{F}$ and $\hat{\mathcal{F}}$ is important to have equivalent concepts of convergence and variational formulations. The ALE transformation is a mapping in space and time. Failer [133] showed the equivalence of the following spaces in space and time:

Lemma 2.57 (Transformation of Bochner-Spaces) *Let $\hat{\Omega}$ and $\Omega(t)$ for $t \in I = [0, T]$ be domains in $\mathbb{R}^d$ and let $\hat{T} : I \times \hat{\Omega} \to \Omega(t)$ with $\hat{T}(\hat{\Omega}) = \Omega(t)$ be a*

$$C(I; C^1(\hat{\Omega})) \cap C^1(I; C(\Omega))$$

diffeomorphism. Then, the composition operators

$$\begin{aligned}
\phi &:= \hat{\phi} \circ \hat{T}^{-1} && \forall \hat{\phi} \in \{\hat{\phi} : \ \hat{\phi} \in L^2(I; H^1(\hat{\Omega})), \ \partial_t \hat{\phi} \in L^2(I; L^2(\hat{\Omega}))\} \\
\hat{\phi} &:= \phi \circ \hat{T} && \forall \phi \in \{\phi : \ \phi \in L^2(I; H^1(\Omega(t))), \ \partial_t \phi \in L^2(I; L^2(\Omega(t)))\}
\end{aligned}$$

are continuous and the spaces

$$\begin{gathered}
\hat{W}(I) := \{\hat{\phi} : \ \hat{\phi} \in L^2(I; H^1(\hat{\Omega})), \ \partial_t \hat{\phi} \in L^2(I; L^2(\hat{\Omega}))\} \\
\cong \\
W(I) := \{\phi : \ \phi \in L^2(I; H^1(\Omega(t))), \ \partial_t \phi \in L^2(I; L^2(\Omega(t)))\}
\end{gathered}$$

are equivalent.

Using this result, we can claim equivalence of solutions of the Navier-Stokes equations in ALE and in Eulerian coordinates, if the solution is found in $W(I)$, i.e. with $\partial_t \mathbf{v} \in L^2(I; L^2(\mathcal{F}(t)))$.

Lemma 2.58 (Navier-Stokes in ALE Coordinates) *Let $\hat{\mathcal{F}} \subset \mathbb{R}^d$ be a smooth domain and $\hat{T}_f : \hat{\mathcal{F}} \to \mathcal{F}(t)$ be a $C(I;C^1(\hat{\mathcal{F}})) \cap C^1(I;C(\hat{\mathcal{F}}))$-diffeomorphism. Then, for every solution $(\hat{\mathbf{v}},\hat{p}) \in \hat{W}(I) \times L^2(I;L^2(\hat{\mathcal{F}}))$ of (2.64) there exists a solution $(\mathbf{v},p) \in W(I) \times L^2(I;L^2(\mathcal{F}(t)))$ of (2.60) with $\hat{\mathbf{v}}(\hat{x},t) = \mathbf{v}(\hat{T}_f(\hat{x},t),t)$ and $\hat{p}(\hat{x},t) = p(\hat{T}_f(\hat{x},t),t)$ almost everywhere.*

The equivalence of two different representations of the Navier-Stokes equations in ALE and in Eulerian coordinates also states that both formulations allow for the same solution concept. If the Eulerian formulation of the Navier-Stokes equations has a unique solution $(\mathbf{v}(t),p(t))$, for suitable mappings $\hat{T}_f$, the ALE formulation will have a corresponding unique solution $(\hat{\mathbf{v}},\hat{p})$ and it holds

$$
\begin{aligned}
c(\hat{T}_f(t))^{-1}\big\{\|\nabla\mathbf{v}(t)\|_{\mathcal{F}(t)} + \|p(t)\|_{\mathcal{F}(t)}\big\}& \\
\le \|\hat{\nabla}\hat{\mathbf{v}}\|_{\hat{\mathcal{F}}} + \|\hat{p}\|_{\hat{\mathcal{F}}} \le& \\
c(\hat{T}_f(t))\big\{\|\nabla\mathbf{v}(t)\|_{\mathcal{F}(t)} + \|p(t)\|_{\mathcal{F}(t)}\big\}.& \qquad (2.65)
\end{aligned}
$$

The constant $c(\hat{T}_f(t))$ will depend on the deformation and, if $\hat{T}_f$ looses its regularity, $c(\hat{T}_f(t)) \to \infty$ is possible.

The variational formulation (2.64) has the benefit, that the domain $\hat{\mathcal{F}}$ is fixed and that the function spaces $\hat{\mathcal{V}}_f$ and $\hat{\mathcal{L}}_f$ do not change in time. A standard finite element triangulation $\hat{\mathcal{F}}_h$ of $\hat{\mathcal{F}}$ can be constructed and used for defining discrete function spaces. The removal of the domain motion comes at the price of additional nonlinearities introduced in the equation. These nonlinearities all depend on the domain map $\hat{T}_f$.

The equivalence of the Eulerian and the ALE formulation of the Navier-Stokes equations strictly depends on the regularity of the mapping $\hat{T}_f$. If this mapping looses its regularity, the equivalence is also lost.

Remark 2.59 (Divergence in ALE Coordinates) On first sight, the divergence condition in ALE coordinates

$$
\widehat{\operatorname{div}}\left(\hat{J}\hat{\mathbf{F}}^{-1}\hat{\mathbf{v}}\right) = 0,
$$

calls for the evaluation of $\hat{\mathbf{u}}$'s second derivatives. It however turns out that all these second derivatives cancel out, if $\hat{\mathbf{u}} \in C^2(\hat{\mathcal{F}})^d$.
The following two technical lemma show this relation. First, we derive a rule for the partial derivatives of a matrices inverse and for the determinant of a matrix:

Lemma 2.60 (Partial Derivatives of Inverse and Determinant) *Let $\hat{\mathbf{F}} : \mathbb{R}^{n\times n} \to \mathbb{R}$ be differentiable and invertible, $\hat{J} = \det(\hat{\mathbf{F}})$. By $\hat{\partial}_k\hat{\mathbf{F}} = (\hat{\partial}_k\hat{\mathbf{F}}_{ij})_{ij}$ and $\hat{\partial}_k\hat{\mathbf{F}}^{-1} = (\hat{\partial}_k\hat{\mathbf{F}}^{-1}_{ij})_{ij}$ we denote matrices of partial derivatives of $\hat{\mathbf{F}}$ and its inverse. It holds*

$$
\hat{\partial}_k\hat{\mathbf{F}}^{-1} = -\hat{\mathbf{F}}^{-1}\hat{\partial}_k\hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}, \quad \hat{\partial}_k\hat{J} = \hat{J}\operatorname{tr}(\hat{\mathbf{F}}^{-1}\hat{\partial}_k\hat{\mathbf{F}}) \qquad (2.66)
$$

Proof

(i) By $\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}} = I$ we get for $k = 1, \ldots, n$

$$0 = \sum_{l=1}^{n} \hat{\partial}_k \hat{\mathbf{F}}_{il}^{-1} \hat{\mathbf{F}}_{lj} + \hat{\mathbf{F}}_{il}^{-1} \hat{\partial}_k \hat{\mathbf{F}}_{lj} \quad \Rightarrow \quad \hat{\partial}_k \hat{\mathbf{F}}^{-1} \hat{\mathbf{F}} + \hat{\mathbf{F}}^{-1} \hat{\partial}_k \hat{\mathbf{F}} = 0,$$

such that the first result follows by multiplication with $\hat{\mathbf{F}}^{-1}$. Likewise, the inverse relation holds

$$\hat{\partial}_k \hat{\mathbf{F}} = -\hat{\mathbf{F}} \hat{\partial}_k \hat{\mathbf{F}}^{-1} \hat{\mathbf{F}}. \tag{2.67}$$

(ii) We denote by Δ_{ij} the cofactor of $\hat{\mathbf{F}}$

$$\Delta_{ij} := (-1)^{i+j} \det(\hat{\mathbf{F}}_{kl})_{k \neq i, l \neq j},$$

Then, the determinant j can be given as

$$\delta_{ik} \hat{J} = \sum_{l=1}^{n} \Delta_{il} \hat{\mathbf{F}}_{kl}, \quad i = 1, \ldots, n. \tag{2.68}$$

Differentiation of this formula ($k = i$) w.r.t. the entries $\hat{\mathbf{F}}_{ij}$ gives

$$\frac{\hat{\partial} \hat{J}}{\hat{\partial} \hat{\mathbf{F}}_{ij}} = \sum_{l=1}^{n} \underbrace{\frac{\hat{\partial} \Delta_{il}}{\hat{\partial} \hat{\mathbf{F}}_{ij}}}_{=0} \hat{\mathbf{F}}_{il} + \Delta_{il} \underbrace{\frac{\hat{\partial} \hat{\mathbf{F}}_{il}}{\hat{\partial} \hat{\mathbf{F}}_{ij}}}_{=\delta_{lj}} = \Delta_{ij}, \tag{2.69}$$

as Δ_{il} does not depend on $\hat{\mathbf{F}}_{ij}$. Hereby, we get with (2.67) and (2.69) and (2.68)

$$\begin{aligned} \hat{\partial}_k \hat{J} &= \sum_{ij} \frac{\hat{\partial} \hat{J}}{\hat{\mathbf{F}}_{ij}} \hat{\partial}_k \hat{\mathbf{F}}_{ij} = -\sum_{ij} \Delta_{ij} (\hat{\mathbf{F}} \hat{\partial}_k \hat{\mathbf{F}}^{-1} \hat{\mathbf{F}})_{ij} \\ &= -\sum_{jrs} \underbrace{\left(\sum_i \Delta_{ij} \hat{\mathbf{F}}_{ir} \right)}_{=\delta_{jr} \hat{J}} \hat{\partial}_k \hat{\mathbf{F}}_{rs}^{-1} \hat{\mathbf{F}}_{sj} = -\hat{J} \sum_{rs} \hat{\partial}_k \hat{\mathbf{F}}_{rs}^{-1} \hat{\mathbf{F}}_{sr}, \end{aligned}$$

and hence using $A : B = \text{tr}(AB^T)$:

$$\hat{\partial}_k \hat{J} = -\hat{J}\hat{\partial}_k \hat{\mathbf{F}}^{-1} : \hat{\mathbf{F}}^T = \hat{J}\hat{\mathbf{F}}^{-1}\hat{\partial}_k \hat{\mathbf{F}}\hat{\mathbf{F}}^{-1} : \hat{\mathbf{F}}^T$$
$$= \hat{J}\,\text{tr}\left(\hat{\mathbf{F}}^{-1}\hat{\partial}_k \hat{\mathbf{F}}\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\right) = \hat{J}\,\text{tr}\left(\hat{\mathbf{F}}^{-1}\hat{\partial}_k \hat{\mathbf{F}}\right)$$

□

With help of these differentiation rules we can reformulate the divergence in ALE coordinates

Lemma 2.61 (Divergence in ALE Coordinates) *Let* $\hat{\mathbf{u}} \in C^2(\Omega)^d$, $\hat{\mathbf{F}} = I + \hat{\nabla}\hat{\mathbf{u}}$ *be invertible and* $\hat{J} = \det(\hat{\mathbf{F}})$. *It holds*

$$\text{div}\left(\hat{J}\hat{\mathbf{F}}^{-1}\mathbf{v}\right) = \sum_{kl} \hat{J}\hat{\mathbf{F}}^{-1}_{kl}\hat{\partial}_k \mathbf{v}_l = \hat{J}\hat{\mathbf{F}}^{-1} : \nabla \mathbf{v}^T = \hat{J}\,\text{tr}(\hat{\mathbf{F}}^{-1}\nabla \mathbf{v}).$$

Proof We start by component-wise differentiation

$$\text{div}\left(\hat{J}\hat{\mathbf{F}}^{-1}\mathbf{v}\right) = \sum_k \hat{\partial}_k (\hat{J}\hat{\mathbf{F}}^{-1}\mathbf{v})_k = \sum_{kl}\left\{\hat{\partial}_k \hat{J}\hat{\mathbf{F}}^{-1}_{kl}\mathbf{v}_l + \hat{J}\hat{\partial}_k \hat{\mathbf{F}}^{-1}_{kl}\mathbf{v}_l + \hat{J}\hat{\mathbf{F}}^{-1}\hat{\partial}_k \mathbf{v}_l\right\}.$$

While the third term already has the final form, we will show that the first two parts cancel out. Using the two parts of Lemma 2.60, we get

$$\text{div}\left(\hat{J}\hat{\mathbf{F}}^{-1}\mathbf{v}\right)$$
$$= \hat{J}\hat{\mathbf{F}}^{-1} : \nabla \mathbf{v}^T + \hat{J}\sum_l \mathbf{v}_l \left(\sum_k \left(\text{tr}(\hat{\mathbf{F}}^{-1}\hat{\partial}_k \hat{\mathbf{F}})\hat{\mathbf{F}}^{-1}_{kl} - (\hat{\mathbf{F}}^{-1}\hat{\partial}_k \hat{\mathbf{F}}\hat{\mathbf{F}}^{-1})_{kl}\right)\right)$$
$$= \hat{J}\sum_l \mathbf{v}_l \sum_{kij}\left(\hat{\mathbf{F}}^{-1}_{ij}\hat{\partial}_k \hat{\mathbf{F}}_{ji}\hat{\mathbf{F}}^{-1}_{kl} - \hat{\mathbf{F}}^{-1}_{kj}\hat{\partial}_k \hat{\mathbf{F}}_{ji}\hat{\mathbf{F}}^{-1}_{il}\right)$$

Next, we use the specific form $\hat{\mathbf{F}} = I + \hat{\nabla}\hat{\mathbf{u}}$ and the symmetry of the second derivatives $\hat{\partial}_{ij}\hat{\mathbf{u}} = \hat{\partial}_{ij}\hat{\mathbf{u}}$. Then,

$$\text{div}\left(\hat{J}\hat{\mathbf{F}}^{-1}\mathbf{v}\right)$$
$$= \hat{J}\hat{\mathbf{F}}^{-1} : \nabla \mathbf{v}^T + \hat{J}\sum_l \mathbf{v}_l \sum_{kij}\left(\hat{\mathbf{F}}^{-1}_{ij}\hat{\partial}_i \hat{\mathbf{F}}_{jk}\hat{\mathbf{F}}^{-1}_{kl} - \hat{\mathbf{F}}^{-1}_{kj}\hat{\partial}_k \hat{\mathbf{F}}_{ji}\hat{\mathbf{F}}^{-1}_{il}\right)$$
$$= \hat{J}\sum_l \mathbf{v}_l \sum_{kij}\left(\hat{\mathbf{F}}^{-1}_{kj}\hat{\partial}_k \hat{\mathbf{F}}_{ji}\hat{\mathbf{F}}^{-1}_{il} - \hat{\mathbf{F}}^{-1}_{kj}\hat{\partial}_k \hat{\mathbf{F}}_{ji}\hat{\mathbf{F}}^{-1}_{il}\right) = 0,$$

where we switched the indices i and k in the first part. □

The crucial inequality for the analysis of the Navier-Stokes and Stokes equations is the inf-sup condition (2.53). We assume that on $\hat{\mathcal{F}}$ it holds:

$$\inf_{\hat{\xi}\in L^2(\hat{\mathcal{F}})} \sup_{\hat{\phi}\in H_0^1(\hat{\mathcal{F}})^d} \frac{(\widehat{\text{div}}\,\hat{\phi},\hat{\xi})}{\|\hat{\nabla}\hat{\phi}\|_{\hat{\mathcal{F}}}\,\|\hat{\xi}\|_{\hat{\mathcal{F}}}} \geq \hat{\gamma} > 0.$$

For simplicity, $\hat{T}_f(t) : \hat{\mathcal{F}} \to \mathcal{F}(t)$ be a C^2-diffeomorphism with $\hat{T}_f(\hat{\mathcal{F}},0) = \hat{\mathcal{F}}$. In light of Lemma 2.56, the Sobolev-spaces on $\mathcal{F}(t)$ and $\hat{\mathcal{F}}$ are equivalent

$$H^1(\mathcal{F}(t)) \cong H^1(\hat{\mathcal{F}}), \quad L^2(\mathcal{F}(t)) \cong L^2(\hat{\mathcal{F}}).$$

On $\mathcal{F}(t)$ it holds

$$\frac{(\text{div}\;\phi,\xi)_{\mathcal{F}(t)}}{\|\xi\|_{\mathcal{F}(t)}\|\nabla\phi\|_{\mathcal{F}(t)}} = \frac{(\widehat{\text{div}}\,(\hat{J}_f\hat{\mathbf{F}}_f^{-1}\hat{\phi}),\hat{\xi})_{\hat{\mathcal{F}}}}{\|\hat{J}_f^{\frac{1}{2}}\hat{\xi}\|_{\hat{\mathcal{F}}}\|\hat{J}_f^{\frac{1}{2}}\hat{\nabla}\hat{\phi}\hat{\mathbf{F}}_f^{-T}\|_{\hat{\mathcal{F}}}},$$

where

$$\xi(\hat{T}_f(\hat{x},t)) = \hat{\xi}(\hat{x}), \quad \phi(\hat{T}_f(\hat{x},t)) = \hat{\phi}(\hat{x}).$$

We substitute

$$\tilde{\phi} := \hat{J}_f\hat{\mathbf{F}}_f^{-1}\hat{\phi} \quad \Rightarrow \quad \hat{\phi} = \hat{J}_f^{-1}\hat{\mathbf{F}}_f\tilde{\phi}.$$

Due to the strong regularity of $\hat{T}_f \in C^2$ it holds for every $\hat{\phi} \in H^1(\hat{\mathcal{F}})^d$

$$\|\hat{\nabla}\tilde{\phi}\|_{\hat{\mathcal{F}}} \leq \|\hat{J}_f\hat{\mathbf{F}}_f^{-1}\|_{W^{1,\infty}(\hat{\mathcal{F}})}\|\hat{\phi}\|_{H^1(\hat{\mathcal{F}})},$$

that $\tilde{\phi} \in H^1(\hat{\mathcal{F}})$. With Poincaré's inequality we get the estimate

$$\|\hat{\nabla}\hat{\phi}\|_{\hat{\mathcal{F}}} \leq \|\hat{J}_f^{-1}\hat{\mathbf{F}}_f\|_{W^{1,\infty}(\hat{\mathcal{F}})}\|\tilde{\phi}\|_{H^1(\hat{\mathcal{F}})} \leq c_P\|\hat{J}_f^{-1}\hat{\mathbf{F}}_f\|_{W^{1,\infty}(\hat{\mathcal{F}})}\|\hat{\nabla}\tilde{\phi}\|_{\hat{\mathcal{F}}}.$$

With these preparations, we can carry over the inf-sup condition from $\hat{\mathcal{F}} = \mathcal{F}(0)$ to $\mathcal{F}(t)$:

$$\inf_{\xi\in L^2(\mathcal{F}(t))} \sup_{\phi\in H^1(\mathcal{F}(t))^d} \frac{(\text{div}\;\phi,\xi)_{\mathcal{F}(t)}}{\|\xi\|_{\mathcal{F}(t)}\|\nabla\phi\|_{\mathcal{F}(t)}}$$

$$= \inf_{\hat{\xi}\in L^2(\hat{\mathcal{F}})} \sup_{\hat{\phi}\in H^1(\hat{\mathcal{F}})^d} \frac{(\widehat{\text{div}}\,(\hat{J}_f\hat{\mathbf{F}}_f^{-1}\hat{\phi}),\hat{\xi})_{\hat{\mathcal{F}}}}{\|\hat{J}_f^{\frac{1}{2}}\hat{\xi}\|_{\hat{\mathcal{F}}}\|\hat{J}_f^{\frac{1}{2}}\hat{\mathbf{F}}_f^{-T}\hat{\nabla}\hat{\phi}\|_{\hat{\mathcal{F}}}}$$

$$= \inf_{\hat\xi\in L^2(\hat{\mathcal{F}})} \sup_{\tilde\phi\in H^1(\hat{\mathcal{F}})^d} \frac{(\widehat{\operatorname{div}}\,\tilde\phi,\hat\xi)_{\mathcal{F}(t)}}{\|\hat J_f^{\frac12}\hat\xi\|_{\hat{\mathcal{F}}}\|\hat J_f^{\frac12}\hat{\mathbf{F}}_f^{-T}\hat\nabla(\hat J_f^{-1}\hat{\mathbf{F}}_f\tilde\phi)\|_{\hat{\mathcal{F}}}}$$

$$\geq c_P^{-1}\|\hat J_f^{\frac12}\|_{L^\infty}^{-1}\|\hat J_f^{\frac12}\hat{\mathbf{F}}_f^{-T}\|_{L^\infty}^{-1}\|\hat J_f^{-1}\hat{\mathbf{F}}_f\|_{W^{1,\infty}}^{-1} \inf_{\hat\xi\in L^2(\hat{\mathcal{F}})} \sup_{\tilde\phi\in H^1(\hat{\mathcal{F}})^d} \frac{(\widehat{\operatorname{div}}\,\tilde\phi,\hat\xi)_{\mathcal{F}(t)}}{\|\hat\xi\|_{\hat{\mathcal{F}}}\|\hat\nabla\tilde\phi\|_{\hat{\mathcal{F}}}}$$

$$\geq c(\hat T_f(t))\hat\gamma =: \gamma(t) \geq \gamma_0 > 0.$$

Depending on the regularity of the transformation $\hat T_f$, the inf-sup constant $\gamma(t)$ can be significantly closer to zero than $\hat\gamma$. See [247] for a study on the stability of the Stokes problem on moving and strongly deformed domains.

2.5.3 *Definition of the ALE Map*

The ALE formulation of the Navier-Stokes equations carries an arbitrariness, as for a given moving domain $\mathcal{F}(t)$ different reference domains $\hat{\mathcal{F}}$ and different mappings $\hat T_f(t) : \hat{\mathcal{F}} \to \mathcal{F}(t)$ can be taken into account. While a straightforward choice for the reference domain is $\hat{\mathcal{F}} = \mathcal{F}(0)$, other choices are still possible. However, even for one reference domain, we can still choose between different mappings $\hat T_f(t) : \hat{\mathcal{F}} \to \mathcal{F}(t)$. On complex domains these ALE-maps must be constructed with help of auxiliary problems. Assuming that the motion of the boundary $\partial\mathcal{F}(t)$ is known, and that $\hat{\mathcal{F}} = \mathcal{F}(0)$, we can construct the mapping by

$$\hat T_f(\hat x, t) := \hat x + \hat{\mathbf{u}}_f(\hat x, t),$$

where by $\hat{\mathbf{u}}_f$ we denote a *deformation* of the fluid domain. The constraint $\partial\hat{\mathcal{F}} \to \partial\mathcal{F}(t)$ can be used as boundary values for the fluid deformation $\hat{\mathbf{u}}_f$. In the interior of $\hat{\mathcal{F}}$ the deformation $\hat{\mathbf{u}}_f$ is constructed by solving a partial differential equation. The most simple approach is to define $\hat{\mathbf{u}}_f$ as the harmonic extension of the boundary values to the fluid domain

$$-\hat\Delta\hat{\mathbf{u}}_f = 0 \text{ in } \hat{\mathcal{F}}, \quad \hat{\mathbf{u}}_f(t) = \hat{\mathbf{u}}_f^D(t) \text{ on } \partial\hat{\mathcal{F}}, \tag{2.70}$$

where $\hat{\mathbf{u}}_f^D(t)$ is the deformation of the boundary points. The crucial point is the regularity of this deformation $\mathbf{u}_f$ that will define the regularity of the domain mapping. We know that for strict equivalence between the ALE formulation and the Eulerian formulation of the incompressible Navier-Stokes problem, very high regularity is required. In the interior of the fluid domain $\hat{\mathcal{F}}$, qualitative regularity is given by the smoothing property of the Laplace-operator, as the right hand side is zero in (2.70). At the boundaries however, the regularity of $\mathbf{u}_f$ is limited by the regularity of $\mathbf{u}_s$ and further by the shape of the boundary. If the solid domain imposes edges entering the fluid-domain, we must expect corner singularities. Even on convex domains, we cannot expect more than $\hat{\mathbf{u}}_f \in H^2(\hat{\mathcal{F}})$ and on concave domains we even loose H^2-regularity. Some remedy is given by choosing the biharmonic operator for extending the deformation to the fluid-domain, e.g., by the equation

$$\hat{\Delta}^2 \hat{\mathbf{u}}_f = 0 \text{ in } \hat{\mathcal{F}},$$

with the interface boundary conditions

$$\hat{\mathbf{u}}_f = \hat{\mathbf{u}}_s \text{ and } \nabla \hat{\mathbf{u}}_f = \nabla \hat{\mathbf{u}}_s \text{ on } \hat{\mathcal{I}}.$$

The biharmonic operator has better regularity properties and yields a smooth transition from fluid- to solid domain. Numerical experiments show that the case of solid domains that enter the fluid domain with sharp edges imposes strong regularity problems, if large deformation appears. To be precise, it is not a large bending of the solid domain that causes problems, but a large deformation of the fluid domain that can also be due to fixed body translation or rotation of the solid.

A drawback of the biharmonic extension is the large computational effort that is necessary to discretize fourth order equations. One either has to use finite elements with global differentiability or one has to use mixed methods that require the introduction of artificial variables, blowing up the complexity of the overall system. Yet another method for constructing the ALE map is by means of a pseudo-elasticity problem, governed by the linear Navier-Lamé problem

$$-\widehat{\operatorname{div}} \left(\mu (\hat{\nabla} \hat{\mathbf{u}}_f + \hat{\nabla} \hat{\mathbf{u}}_f^T) + \lambda_e \widehat{\operatorname{div}}\, \hat{\mathbf{u}}_f I \right) = 0 \text{ in } \hat{\mathcal{F}}, \quad \hat{\mathbf{u}}_f = \hat{\mathbf{u}}_s \text{ on } \hat{\mathcal{I}}.$$

The "material parameters" μ_e, λ_e can be chosen in such a way that a stiff mapping with little deformation is constructed close to the interface.

In Sect. 5.3.5, we will discuss the quantitative regularity properties of different extension techniques and analyze their performance on simple benchmark problems.