

Electrical Circuits of Non-integer Order: Introduction to an Emerging Interdisciplinary Area with Examples

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Abstract The chapter is an attempt to collate the basics of fractional electric circuits involving fractional time derivatives in the sense of Riemann–Liouville, Caputo and Caputo–Fabrizio. The examples analysed use mainly Caputo time-fractional derivative but comparative analyses with derivative based on different relaxation kernels are provided, too.

Keywords Time-fractional derivative · Electric circuits · Transient regimes

1 Introduction

Fractional calculus (FC), involving integrals and derivatives of non-integer order, is the natural generalization of the classical calculus allowing better modelling and control of processes in various areas of science and engineering [1–6]. A broad range of physical phenomena can be deeply analysed by applications of models involving fractional integral and derivatives [6] thus providing accurate information of the physical systems employing the memory mechanisms and hereditary effects in various materials and processes [7] where dissipations take place.

In this chapter, we focus on fractional calculus applications on simple electric circuits involving resistors, capacitors and inductors under transient conditions. The non-local character of the transient processes in the electric circuits is directly related to the dissipative processes in their elements such as the Ohmic resistance, the capacitor charge–discharge process, dissipation of charge transfer in dielectrics [8] or energy accumulation by inductors.

Fractional calculus is a well-known tool for the investigation of nonlinear time-dependent process in electrochemistry for surface concentration determination [1] and impedance spectroscopy [9].

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The purpose of this chapter is to demonstrate the mathematical formalism in analyses of transient process of simple RLC circuits by the tools of fractional calculus and to focus on the fact that the real resistors, capacitors and inductors are nonlinear by nature with sensible dissipative processes in their work. The text collates research results from various sources and tries to present them in a straightforward manner, thus allowing an easy step from the conventional integer-order models to one with fractional derivatives.

2 Time-Fractional Derivatives to Transient Electric Circuit Analysis: The Reason to Use

The linear approach to model RLC circuits by integer-order derivatives and integrals are idealizations which do not take into account the fractality in time and the inherent nonlinearities of the electric components. The principle questions prior to some detailed analyses of fractional electric circuits are as follows: Why this modelling technique should be applied and what are the advantages of it beyond the well-known integer-order models?

We will try to answer these principle questions in view of the basic knowledge of relaxation phenomena and how this philosophy works when transients in electric circuits should be analysed.

Let us start with the well-known constitutive equations associated with the three basic elements of RLC electric circuits as follows[10]:

The voltage drop across an inductor

$$U_L(t) = L \frac{d}{dt} I(t). \quad (1)$$

The voltage drop across a resistor

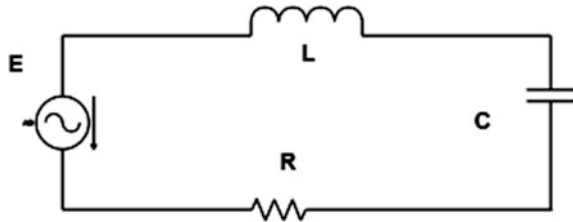
$$U_R = RI(t). \quad (2)$$

The voltage drop across a capacitor

$$U_c = \frac{1}{C} \int_0^t I(z) dz. \quad (3)$$

Applying the Kirchhoff voltage law and the above constitutive equations we can express the homogeneous integer-order differential equation expressed through the voltage drop across the capacitor (3) or alternatively non-homogenous equation about the current $i_C(t)$ (2) corresponding to RLC circuit (see Fig. 1):

Fig. 1 Basic RLC circuit used in the analysis



$$L \frac{d^2}{dt^2} U_c(t) + R \frac{d}{dt} U_c(t) + \frac{1}{C} U_c(t) = \frac{d}{dt} u(t), \tag{4}$$

where $u(t)$ is the unit step of the voltage (Heaviside function).

Let consider only a resistive medium (only Ohmic resistance) and that the current between points with a gradient of the electric potential $\partial\varphi/\partial x$ depends on its history over time, namely

$$I(t) = -\frac{1}{R} \int_0^t S_z(t-z) \frac{\partial\varphi}{\partial x} dt. \tag{5a}$$

We may simplify the problem and consider a homogeneous resistor as element of a circuit where accordingly $\partial\varphi/\partial x$ is equal to the voltage drop applied to its ends. Let us consider the voltage drop as a unit step $u(t)$ and therefore the memory integral (5a) can be expressed as (omitting the negative sign as unnecessary since the equation becomes expressed in scalar values) in contrast to (5a) where $\partial\varphi/\partial x$ is a vector.

$$I(t) = \frac{1}{R} \int_0^t S_z(t-z) u(t) dt. \tag{5b}$$

In these equations z is a dummy variable, while the negative sign in front of the history integral simply indicates that the current flow direction is opposite to the change in the potential gradient.

Moreover, (5a) and (5b) are constitutive equations which simply state *natural processes with infinite speed of change do not exist and therefore the relaxation in time should be taken into account*. In the heat conduction area, this is the well-known Cattaneo–Maxwell postulate of heat transfer across a homogeneous rigid conductor [11–15], namely

$$q_H(x, t) = - \int_{-\infty}^t S_H(x, t) \nabla T(x, t-z) dz. \tag{6}$$

In terms of the present chapter the heat flux $q_H(x, t)$ is analogue of the electric $i(x, t)$ current, while $\nabla T(x, t)$ corresponds to the electric tension (potential $\partial\varphi/\partial x$) applied to the resistor. As in the spatially homogeneous heat conductors, the current passing through a simple resistor, the relaxation kernel, should be space-independent, that is $S_R(t)$ is a function only of the time.

If we consider the relaxation kernel as an exponential function $S_H(t) = \exp(-(t - z)/\tau_z)$, where the relaxation time τ_z is finite, i.e. $\tau_z = \text{const.}$; then for $\tau_z \rightarrow 0$ the limit of Eq. (5a) is the Ohm law (2). However, in the first-order approximation of the current $i(t)$, in τ_z we have

$$i(t + \tau) \approx i(t) + \tau_z \frac{\partial i(t)}{\partial t}. \quad (7)$$

This leads to a first-order differential equation

$$\frac{1}{\tau_z} i(t) + \frac{\partial i(t)}{\partial t} = -\frac{k_1}{\tau_z} \frac{\partial \theta(x, t)}{\partial t}. \quad (8)$$

The integration of (8) leads to the constitutive Eq. (5a) which is an analogue of the Cattaneo constitutive equation and considers only *linear elastic effects* in the transient of the passing electric current.

However, if the relaxation kernel is presented as a linear combination [16]

$$S_{\text{ex}} = k_1 \delta(z) + (k_2/\tau_z) \exp(-z/\tau_z), \quad (9)$$

where $k_1 = 1/R$ and $k_2 = 1/R_2$ are the *effective resistance* and the *elastic resistance* of the resistor. $\delta(z)$ is the Dirac delta function. If the relaxation kernel is presented only by $\delta(z)$, that is $S_z(t) = \delta(t)$, then Eq. (5a) reduces to the classic Ohm law.

In case the relaxation function is expressed as S_{ex} , then the modified Ohm law leads to a current defined as

$$i(t) = \frac{1}{R_1} U(t) - \frac{1}{R_2} \frac{1}{\tau_z} \int_{-\infty}^t e^{-(\frac{t-z}{\tau_z})} \frac{\partial U(z)}{\partial z} dz. \quad (10)$$

Hence, when τ_z is infinite, the history of the current evolution process presented by the memory integral [the second term in (10)] is zero and we get the classical Ohm law. The consequent step in integration of (10), with a finite value of τ_z , leads to the well-known Kirchhoff's telegraph equation [16] (see [16] and the reference therein for more details).

Now, the natural question is: what happens if the relation function in (5a) is defined not by an exponential (Jeffrey's kernel $S_H(t)$) but with a power-law function $S_s(t, \mu) = t^{-\mu}$? In this case, we get a new form of the history integral

$$I(t) = \frac{1}{R} \int_0^t (t-z)^{-\mu} U(z) dz, \quad 0 < \mu < 1. \quad (11)$$

As we will see in the next section, the fractional integral with singular power-law kernel $S_s(t, \mu) = t^{-\mu}$ is the basis of the widely used Riemann–Liouville and Caputo fractional derivatives. Otherwise, when the relaxation kernel is of the Jeffrey’s type we get a history integral with non-singular kernel [17, 18] resulting in the Caputo–Fabrizio time-fractional derivative [15, 17, 18].

With these introductory notes, we try to explain why fractional integrals and derivatives are used to describe relaxation processes in electric circuits. The reason is simple and physically motivated: *There are no physical phenomena with infinite speed of change and the correct description should take into account relaxation processes.* If the relaxation process is rapid and the relaxation time is negligible with respect to the entire observation time at issue, then we get the well-known constitutive equation (5a). However, when the processes at issue are with time-scales comparable to the relaxation times, the histories (the memories) should be taken into account that leads to use of the memory formalism (history integrals) and the fractional calculus approach.

3 Preliminaries: Necessary Mathematical Background

We start with the mathematical background necessary to demonstrate the solution and modelling approaches. The fractional integral and derivatives of Riemann–Liouville and Caputo with singular (power-law) kernels are the principles ones used in the literature devoted to the problems discussed in this chapter. In addition, the basis of the newly defined Caputo–Fabrizio derivative with a non-singular (Jeffrey) kernel is briefly outlined.

3.1 Time-Fractional Integral and Derivatives with Singular Kernels

3.1.1 Fractional Integral

In accordance with the Riemann–Liouville approach, the fractional integral of order $\mu > 0$ is a natural result of the Cauchy’s formula reducing calculations of the m -fold primitive of a function $f(t)$: the result is a single integral of convolution type [19] for arbitrary positive number $\mu > 0$, namely

$${}_0I^\mu f(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t - \tau)^{\mu-1} f(\tau) d\tau, \quad t > 0, \quad n \in \mathbb{R}^+ \tag{12}$$

where $\Gamma(\cdot)$ is the gamma function.

For the sake of convenience, we will use also the notation ${}_0D^{-\mu}f(t)$ for ${}_0I^\mu f(t)$. Further, the law of exponents for fractional integrals means ${}_0D^{-\mu}{}_0D^{-\nu}f(t) = {}_0D^{-\mu-\nu}f(t) = {}_0D^{-\nu}{}_0D^{-\mu}f(t)$.

The Laplace transform of the fractional integral is defined by the convolution theorem as

$$\text{Laplace} [{}_0D_t^{-\mu}f(t)] = \text{Laplace} \left[\frac{t^{\mu-1}}{\Gamma(\mu)} \right] \text{Laplace}[f(t); s] = s^{-\mu}F(s), \tag{13}$$

where $\Re(s) > 0$, $\Re(\mu) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.

3.1.2 Riemann–Liouville Fractional Derivative

Therefore, we may define the fractional derivative $D^\mu f(t)$ with $n \in \mathbb{N}^+$ by the relations [19] as ${}_0D^\mu{}_0I^\mu = I$ but ${}_0I^\mu{}_0D^\mu \neq I$. Therefore, D^μ is the left-inverse, but not right-inverse, to the integral operator I^μ . Hence, introducing a positive integer m such that $m - 1 < \mu \leq m$ the natural definition of the Riemann–Liouville (left-sided) fractional derivative of order $\mu > 0$ is

$${}_0D^\mu f(t) := {}_0D^m{}_0I^{m-\mu}f(t) = \frac{1}{\Gamma(m - \mu)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t - \tau)^{\mu+1-m}} d\tau, \tag{14}$$

$$m - 1 < \mu \leq m, \quad m \in \mathbb{N}$$

$${}_0D^\mu f(t) = \frac{d}{dt} f(t), \quad \text{for } \mu = 1. \tag{15}$$

Thus, we have $D^0 = I^0 = I$, that is $D^\mu I^\mu = I$ for $\mu \geq 0$. Additionally, the fractional derivative of power-law function and a constant, frequently used in this chapter, is

$${}_0D^\mu t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \mu)} t^{\beta-\mu} \quad \text{and} \quad {}_0D^\mu C = C \frac{t^{-\mu}}{\Gamma(1 - \mu)} \mu > 0, \quad \beta > -1, \tag{16}$$

$t > 0$.

Similarly, the fractional integrals of the power-law function and a constant are

$${}_0D^{-\mu}t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 + \mu)}t^{\mu + \beta}, \quad {}_0D^{-\mu}C = \frac{C}{\Gamma(1 + \mu)}t^\mu, \quad \mu \neq 1, 2, \dots \quad (17a, b)$$

The Laplace transform of the Riemann–Liouville fractional derivative for $m \in N$ is

$$\begin{aligned} \text{Laplace} \left[\frac{d^m}{dt^m} f(t); s \right] &= s^m F(s) - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0+) \\ &= s^m F(s) - \sum_{k=1}^m s^{k-1} D_t^{\mu-k} f(0+), \end{aligned} \quad (18)$$

where $\Re(s) > 0$, $\Re(\mu) > 0$ and $m - 1 \leq \mu < m$.

3.1.3 Caputo Fractional Derivative

The Caputo derivative of a casual function $f(t)$, i.e. $f(t) = 0$ for $t < 0$, is defined [19, 20] as

$$\begin{aligned} {}_cD_t^\mu f(t) &= {}_0I^{m-\mu} \frac{d^m}{dt^m} f(t) = {}_0D_t^{-(m-\mu)} f^{(m)}(t) \\ &= \frac{1}{\Gamma(m - \mu)} \int_0^t \frac{f^{(m)}(\tau)}{(\tau - t)^{\mu+1-m}} d\tau, \end{aligned} \quad (19a)$$

where $m \in N$ and $m - 1 < \mu < m$

$${}_cD_t^\mu f(t) = \frac{d}{dt} f(t), \quad \text{for } \mu = 1. \quad (19b)$$

The Laplace transform of Caputo derivative is

$$\text{Laplace} [{}_cD_t^\mu f(t); s] = s^\mu F(s) - \sum_{k=0}^{m-1} f^{(k)}(0) s^{\mu-k-1}. \quad (20)$$

The Caputo derivative of a constant is zero, i.e. ${}_cD_t^\mu C = 0$ that matches the common knowledge we have from the integer-order calculus; and because of that the Riemann–Liouville derivative is the preferred fractional derivative among mathematicians, while Caputo fractional derivative is the preferred one among engineers [3].

If $f(0) = f'(0) = f''(0) = \dots = f^{(m-1)}(0) = 0$, then both Riemann–Liouville and Caputo derivatives coincide. In particular for $\mu \in (0, 1)$ and $f(0) = 0$ one has

${}_C D_t^\mu f(t) = {}_{RL} D_t^\mu f(t)$. Further, in this chapter we will use the notations ${}_{RL} D_t^\mu f(t)$ and ${}_C D_t^\mu f(t)$ to discriminate the effects of the solutions when both derivatives are used. In addition in some situations, we will use also the notation $\partial^\mu f(t)/\partial t^\mu$ meaning a time-fractional derivative without specification of the type.

Further, for simplicity, we will assume that all expressions are with the Caputo time-fractional derivative. For the sake of coherence with the existing practice describing transient regimes in electrical circuits, we will use the notation ${}_C D_t^\mu = d^\mu/dt^\mu$.

3.1.4 Mittag–Leffler Function

Generally, the fractional-order differential equation has the form

$$\sum_{k=0}^n a_k D_t^{\mu_k} f(t) = g(t). \quad (21)$$

After application of the Laplace transform and successful solution on the s -space, the inverse transform with $0 < \mu < 1$ requires a special function, namely the Mittag–Leffler function defined as [19, 21]

$$E_\mu(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\mu k + 1)}, \quad \mu > 0. \quad (22)$$

For example with $\mu = 1$ we have $E_1(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^t$ since the Mittag–Leffler function is a generalization of the exponential function and widely used for describing dissipative processes.

3.1.5 Alternative Representation of the Fractional Derivative with Singular Kernels

In order to facilitate the understanding of the transition from the classical integer-order models to the ones involving time-fractional derivatives, we will present this step as [22]

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\mu}} \frac{d^\mu}{dt^\mu}. \quad (23)$$

Here the parameter σ (with a dimension in seconds) plays a role of normalizing function, and when $\mu = 1$ we get the ordinary fractional derivative. This presentation is more intuitive rather than mathematically correct but for people not involved in fractional calculus, it would be the more acceptable approach.

3.2 Fractional Derivatives with Non-singular Kernel

The hot topic in modelling of dissipative phenomena involves fractional derivatives by the application of fractional derivatives. As it is stated in the seminal work of Caputo and Fabrizio [17] many classical constitutive equations (see the comments in [15] and the references therein) cannot model some transport dissipative process with advanced characteristics. To satisfy these requirements a new time-fractional derivative with a non-singular smooth exponential kernel was conceived by Caputo and Fabrizio [17], namely

$${}_{CF}D_t^\mu f(t) = \frac{M(\mu)}{1-\mu} \int_0^t \exp\left[-\frac{\mu(t-z)}{1-\mu}\right] \frac{df(z)}{dz} dz \quad (24)$$

$${}_{CF}D_t^\mu f(t) = \frac{1}{1-\mu} \int_0^t \exp\left[-\frac{\mu(t-z)}{1-\mu}\right] \frac{df(z)}{dz} dz, \quad (25)$$

where $M(\mu)$ in Eq. (24) is a normalization function such that $M(0) = M(1) = 1$. With $M(\mu) = 1$ suggested for convenience in [17] we get the final definition of the Caputo–Fabrizio time-fractional derivative in the form of Eq. (25). The derivative of a constant is zero as in the classical Caputo derivative [19, 20], but now the exponential (Jeffrey’s) kernel has no singularity. An application to transients in RC circuit is demonstrated in this chapter.

The Laplace transform of Caputo–Fabrizio derivative is [17]

$$\begin{aligned} \text{Laplace} [{}_{CF}D_t^\mu f(t)] &= \frac{1}{1-\mu} \text{Laplace}[f(t)] \text{Laplace}\left(\exp - \frac{\mu}{1-\mu} t\right) \\ &= \frac{s \text{Laplace}[f(t)] - f(0)}{s + \mu(1-s)}. \end{aligned} \quad (26)$$

Similar to the alternative and intuitive presentation of the transition from fractional to integer-order derivative we have the following [24, 25]:

- Fractional in time

$$\frac{d}{dt} \rightarrow \frac{1}{\exp\left[-\frac{(1-\mu)}{2-\mu}\right] \sigma_t} {}_{CF}D_t^\mu f(t) \quad (27)$$

- Fractional in space

$$\frac{d}{dt} \rightarrow \frac{1}{\exp\left[-\frac{(1-\mu)}{2-\mu}\right] \sigma_x} {}_{\text{CF}}D_x^\mu f(t), \quad (28)$$

where for $\mu = 1$ we recover the ordinary (integer-order) derivatives

4 Transients in Fractional Electric Circuits: Analyses by Examples

4.1 Fractional RC Circuit

The conventional integer-order differential equation for the RC circuit is given by

$$E(t) = R \frac{dq(t)}{dt} + \frac{q(t)}{C}. \quad (29)$$

We will investigate two simple cases: capacitor discharge and transient due to unit step of the voltage.

4.1.1 Capacitor Discharge

In case of capacitor discharge through the resistor R we have $E(t) = 0$

$$C \frac{dU_c}{dt} + \frac{1}{R} U_c(t) = 0 \quad (30)$$

and in this case the solution of (30) classic solution is [25]

$$U_c = U_0 \exp\left(-\frac{t}{RC}\right), \quad U_c(t=0) = U_0. \quad (31)$$

With Riemann–Liouville time-fractional derivative the analogue of (31) is

$${}_0D^\mu U_c(t) + \frac{1}{R} U_c(t) = 0. \quad (32)$$

Then, the solution is [26]

$$U_c^{\text{RL}}(t) = U_0 E_\mu\left(-\frac{1}{RC} t^\mu\right) - U_0 \frac{\mu}{\Gamma(1-\mu)} \int_0^t (t-z)^{-\mu} z^{\mu-1} E_\mu\left(-\frac{1}{RC} z^\mu\right) dz \quad (33)$$

or equivalently in a dimensionless form

$$\frac{U_c^{\text{RL}}(t)}{U_0} = E_\mu\left(-\frac{1}{\text{RC}}t^\mu\right) - \frac{\mu}{\Gamma(1-\mu)} \int_0^t (t-z)^{-\mu} z^{\mu-1} E_\mu\left(-\frac{1}{\text{RC}}z^\mu\right) dz. \quad (34)$$

With Caputo derivative the alternative equation of the capacitor discharge is

$${}_c D_t^\mu U_c(t) + \frac{1}{R} U_c(t) = 0'. \quad (35)$$

Then, the solution is

$$U_c^C(t) = U_0 E_\mu\left(-\frac{1}{\text{RC}}t^\mu\right) \quad (36)$$

or equivalently in a dimensionless form

$$\frac{U_c^C(t)}{U_0} = E_\mu\left(-\frac{1}{\text{RC}}t^\mu\right). \quad (37)$$

We can see that the only difference between (33) and (36) is the last term in (RC-1d) due to the fact that the Riemann–Liouville derivative of a constant is not zero (see 17b).

With the Caputo–Fabrizio derivative, the same problem has a model resembling (35), namely

$${}_{\text{CF}} D_t^\mu U_c(t) + \frac{1}{R} U_c(t) = 0. \quad (38)$$

The simple solutions through the Laplace transform yield

$$U_c^{\text{CF}}(t) = U_0 \exp\left(-\frac{2\mu}{2\text{RC} + 2(1-\mu)}t\right) \quad (39)$$

or equivalently in a dimensionless form

$$\frac{U_c^{\text{CF}}(t)}{U_0} = \exp\left(-\frac{2\mu}{2\text{RC} + 2(1-\mu)}t\right). \quad (40)$$

4.1.2 Transient Due to Unit Step of the Voltage

Assuming the transient analysis, that $E(t) = u(t)$ is a unit step function, we may express (29) in a dimensionless form [23]

$$\frac{d\eta_c(\tau)}{d\tau} + \eta_c(\tau) = u(\tau), \quad \eta_c(\tau) = \frac{q(t)}{C}, \quad \tau = \frac{t}{RC}. \quad (41a, b)$$

The time-fractional counterpart of (41a) (assuming Caputo derivatives) is

$$\frac{d^\mu \eta_c(\tau)}{d\tau^\mu} + \eta_c(\tau) = u(\tau), \quad 0 < \mu < 1. \quad (42)$$

Solution in the Time Domain

With the Laplace transform of the Caputo derivative (in the alternative form expressed by Eq. 20) we have from (42) that

$$V_c(\bar{s}) = \frac{1}{\bar{s}(\bar{s}^\mu + 1)}, V_c(\bar{s}) = L[\eta_c(\tau)] \quad \text{and} \quad \bar{s} = (RC)s. \quad (43a, b, c)$$

The inverse Laplace transform of (43a) yields a solution in the time domain [23] :

$$\eta_c(t) = u(t) \left[1 - \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{t}{RC}\right)^{k\mu}}{\Gamma(k\mu + 1)} \right] = u(t) [1 - E_{\mu,1}(\tau)] \quad (44a, b)$$

or equivalently in a dimensionless form

$$\frac{\eta_c(t)}{u(t)} = 1 - E_{\mu,1}(\tau). \quad (44c)$$

For $\mu = 1$ we have $E_{1,1} = e^{-\tau} = e^{-\frac{t}{RC}}$ and $\eta_c(t)/u(t) = 1 - e^{-t/RC}$.

The solution was tested by a simple example [23] calculating the times required the response to attain 10% of its initial value and the 90% value of the final (saturation) level. The answers are straightforward because from (44a) we have [23], for instance

$$0.99 = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{(\tau_{ss})^{k\mu}}{\Gamma(k\mu + 1)} = 1 - E_{\mu,1}(\tau_{ss}). \quad (45a)$$

This allows to find a solution with respect to the dimensionless settling time defined as $\tau_{ss} = t_{ss}/RC$. From (45a) we have

$$f(\tau_{ss}) = \sum_{k=0}^{\infty} (-1)^k \frac{(\tau_{ss})^{k\mu}}{\Gamma(k\mu + 1)} - 0.01 = 0. \quad (45b)$$

The dimensionless settling time t_{ss}/RC can be determined by the Newton-Raphson method, where the m th iteration of solution of (45b) can be expressed as [23]

$$\tau_{ss}^{(m+1)} = \tau_{ss}^{(m)} - \frac{f[\tau_{ss}^{(m)}]}{f'[\tau_{ss}^{(m)}]}. \tag{46}$$

The plot of $\tau_{ss} = t_{ss}/RC$ as a function of the fractional-order μ demonstrates strong decaying behaviour (see Fig. 2).

Solution in the Frequency Domain

Starting from the Laplace transform (44) and multiplying both sides by $\bar{s} = (RC)s$ we get [23]

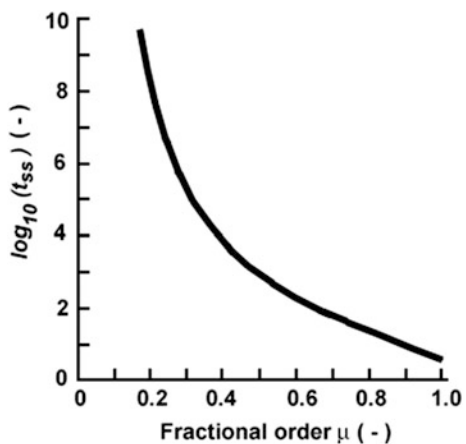
$$H(\bar{s}) = \frac{1}{\bar{s}^\mu + 1}. \tag{47}$$

Now we may construct the Bode plot from the relation $H(j\omega)|_{\bar{s}=j\omega}$ resulting in two equations:

$$|H(j\omega)|_{dB} = 20 \log_{10}(|H(j\omega)|) \tag{48a}$$

$$\arg H(j\omega) = \frac{180 \arg H(j\omega)}{\pi}. \tag{48b}$$

Fig. 2 Semi-logarithmic relationship of the settling time and the fractional order μ
Adapted from [23]



4.2 Fractional RL Circuit

With only resistor and inductor charged by a voltage source $U(t)$, the integer-order model is

$$L \frac{d}{dt} I(t) + RI(t) = E(t). \quad (49)$$

We will consider some sub-examples of (49) and the solutions of their time-fractional analogues.

4.2.1 Constant Electromotive Force

With initial condition $I(t = 0) = I_0 \neq 0$ and $E(t) = E_0$, the solution of (49) is [27]

$$I(t) = \left[I_0 - \frac{E_0 L}{R} \right] \exp\left(-\frac{R}{L} t\right) + \frac{E_0 L}{R} \quad (50a)$$

or equivalently

$$\frac{I(t)}{I_0} = \left[1 - L \frac{I_{R0}}{I_0} \right] \exp\left(-\frac{R}{L} t\right) + L \frac{I_{R0}}{I_0}, \quad I_{R0} = \frac{E_0}{R}. \quad (50b)$$

With Riemann–Liouville time-fractional derivative the analogue of (49) is [26, 27]

$${}_0 D^\mu I(t) + \frac{R}{L} I(t) = \frac{E_0}{L}. \quad (51a)$$

With a dimensionless solution

$$\frac{I(t)}{I_0} = E_\mu\left(-\frac{R}{L} t^\mu\right) + \mu \int_0^t \left[\frac{U_0}{I_0 L} - \frac{(t-z)^\mu}{\Gamma(1-\mu)} \right] z^{\mu-1} E'_\mu\left(-\frac{R}{L} z^\mu\right) dz. \quad (51b)$$

Now, with Caputo derivative the equivalent of (49) and (51a) is [26]

$${}_C D_t^\mu I(t) + \frac{R}{L} I(t) = \frac{E_0}{L}. \quad (52a)$$

Then, the dimensionless solution is

$$\frac{I(t)}{I_0} = \left(1 - \mu \frac{I_{R0}}{I_0} \right) E_\mu\left(-\frac{R}{L} t^\mu\right) + \mu \frac{I_{R0}}{I_0}, \quad I_{R0} = \frac{E_0}{R}. \quad (52b)$$

With the Caputo–Fabrizio derivative the same transient process is modelled as

$${}_{CF}D_t^\mu I(t) + \frac{R}{L}I(t) = \frac{E_0}{L}, \tag{53a}$$

and therefore the solution is

$$\frac{I(t)}{I_0} = \frac{I_{R0}}{I_0} - \left(\frac{I_{R0}}{I_0} - 1\right) \exp\left[-2\mu \frac{1}{L/R + (1 - \mu)t}\right], \quad I_{R0} = \frac{E_0}{R}. \tag{53b}$$

The plots in Fig. 3 show the graphical representations of the developed solutions, as it was reported by [26] for $\mu = 0.5$; there is no similarity between the developed solutions. However, when $\mu \rightarrow 1$, particularly for $\mu = 0.999$, solutions with the classical ($\mu = 1$) and the Caputo–Fabrizio derivative coincide matching in the exponential current increase and the steady-state value that is easy to check from the developed dimensionless solutions (see Fig. 4 in the original work).

4.2.2 Absence of Electromotive Force

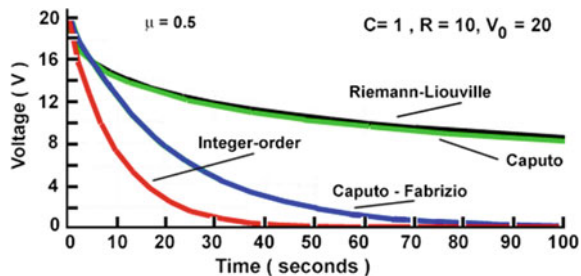
Hence with $E(t) = 0$ we may rewrite (49) in terms of Caputo time-fractional derivative as

$${}_CD_t^\mu I(t) + \frac{R}{L}I(t) = 0. \tag{54}$$

Then, the solution through the Laplace transform of (54) yields [27]

$$I(s) = I_0 \frac{s^{\mu-1}}{(s^\mu + R/L)} \Rightarrow I(t) = I_0 E_\mu\left(-\frac{R}{L}t^\mu\right). \tag{55}$$

Fig. 3 Time evolution of the voltage calculated by different fractional derivatives in the case $\mu = 0.5$. Adapted from [26]



4.2.3 Electromotive Force as a Unit Step Function

With an electromotive force as a unit step, i.e. $E(t) = u(t) = 1$ we have with the Caputo derivative

$${}_c D_t^\mu I(t) + \frac{R}{L} I(t) = \frac{u(t)}{L}. \quad (56)$$

With the initial conditions $I(t=0) = I_0 > 0$ taking into account that $\mu \rightarrow 1$ ${}_c D_t^\mu I(t) \rightarrow dI/dt$ we may express (56) as

$${}_c D_t^\mu I(t) = \frac{1}{L} u(t) - \frac{R}{L} I(t). \quad (57)$$

The Laplace transform of (57) is

$$I(s) = \frac{1}{L s(s^\mu + R/L)} + I_0 \frac{s^{\mu-1}}{(s^\mu + R/L)}. \quad (58)$$

Then, the time domain solution is

$$I(t) = \left(I_0 - \frac{1}{R} \right) E_\mu \left(-\frac{R}{L} t^\mu \right) + \frac{1}{R}. \quad (59)$$

4.3 Fractional RLC Circuit

The integer-order equation of an RLC circuit is

$$L \frac{d^2}{dt^2} q(t) + R \frac{d}{dt} q(t) + \frac{q(t)}{C} = 0. \quad (60)$$

The last term $q(t)/C$ is crucial for the oscillatory behaviour of the circuit since the RL circuit only exhibits a decaying transient behaviour.

4.3.1 Analysis Through the Caputo Derivative

Since the transformation of the integer-order equations to the fractional time counterparts could be considered as a formal replacement of the integer-order derivatives by fractional ones, which might cast some doubts what really needs this step, we will use the alternative approach to explain intuitively the requirement to use fractional derivatives. With the expression (23) we may present (60) as [22]

$$\frac{1}{\sigma^{1(1-\mu)}} L \frac{d^{2\mu}}{dt^{2\mu}} q(t) + R \frac{d^\mu}{dt^\mu} q(t) + \frac{q(t)}{C} = 0, \quad (61)$$

where $\frac{d^\mu}{dt^\mu}$ is the Caputo derivative, assumed by default.

The solution of (61) by the Laplace transform is [22]

$$q_{\mu(\text{RLC})}(t) = q_0 E_\mu \left[-\frac{R\sigma^{1-\mu}}{2L} t^\mu \right] \cdot E_{2\mu} \left\{ -\left[\frac{1}{LC} - \frac{R^2}{4L^2} \right] \sigma^{2(1-\mu)} t^{2\mu} \right\}. \quad (62)$$

In the undamped case when $R < 2\sqrt{L/C}$ and $q_{\mu(\text{RLC})}(t=0) = q_0$ the natural frequency is defined as $\omega_0 = \sqrt{1/LC}$.

In the limiting situation with $\mu = 1$ the solution (62) reduces to the classical solution defining the time constant $\tau_{RL} = 2L/R$:

$$q_{(\text{RLC})}(t) = q_0 \exp\left(-\frac{R}{2L}t\right) \cos(\Omega t), \quad \Omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}. \quad (63a, b)$$

The solution (63a) allows demonstrating the functional relationship between the normalizing parameter σ and the fractional order μ [22]

$$\mu = \sigma \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \quad 0 < \sigma \leq \frac{1}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}. \quad (64a, b)$$

When the *undamped case*, $R < 2\sqrt{L/C}$ and the damping factor is defined as $\alpha = \sqrt{R/2L}$. With the condition $\alpha < \omega_0$ solution reduces to [22]

$$q_{\mu(\text{RLC})}(t_1)_{\text{under}} = q_0 E_\mu \left[-\frac{R}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \mu^{(1-\mu)} t_1^\mu \right] \cdot E_{2\mu} \left\{ -\mu^{2(1-\mu)} t_1^{2\mu} \right\} \quad (65a)$$

$$\frac{t_1}{t} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \quad (65b)$$

When the *overdamped case* is at issue where $\alpha > \omega_0$ and $R > 2\sqrt{L/C}$ the solution (62) reduces to [22]

$$q_{\mu(\text{RLC})}(t)_{\text{over}} = \bar{q}_0 E_\mu \left[-\sigma^{1-\mu} \frac{R}{2L} t^\mu \right] \cdot E_{2\mu} \left\{ -\sigma^{(1-\mu)} t^\mu \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \right\}. \quad (66)$$

For $\mu = 1$ the solution (66) reduces to the well-known aperiodic solution of the capacitor discharge

$$q_{\mu=1(\text{RLC})}(t)_{\text{over}} = \bar{q}_0 \exp\left[-\frac{t}{t_{\mu=1}}\right], \quad t_{\mu=1} = \frac{2L}{R\left[1 + \sqrt{1 - \frac{4L}{R^2C}}\right]}, \quad (67a, b)$$

where \bar{q}_0 is the charge of the capacitor at $t = 0$

4.3.2 Analysis Through the Caputo–Fabrizio Derivative

Case with Voltage Drop Dependent on Space and Time

For this analysis, we will use the model of Riaza [28] and the results of Atangana and Nieto [24]

$$\frac{\partial^2 U(x, t)}{\partial x^2} - LC \frac{\partial^2 U(x, t)}{\partial t^2} - (RC + GL) \quad (68a)$$

$$\frac{\partial U(x, t)}{\partial x} - GRU(x, t), \quad (68b)$$

where G denotes the electric resistance of the materials connecting the resistors (conductors) in the circuits.

Their fractional analogues of (68a, b) are [24]

$$\begin{aligned} & \frac{{}_{\text{CF}}D_x^\beta U(x, t)}{\exp\left(-\frac{\beta}{1-\beta}\sigma_x\right)} - LC \frac{{}_{\text{CF}}D_t^\beta U(x, t)}{\exp\left(-\frac{\beta}{1-\beta}\sigma_t\right)} \\ & - \frac{(RC + GL)}{\exp\left(-\frac{1-\mu}{2-\beta}\sigma_t\right)} [{}_{\text{CF}}D_t^\mu U(x, t)] - GRU(x, t) = 0 \end{aligned} \quad (69)$$

with $1 < \beta < 2$ and $0 < \mu < 1$.

This model has no analytical solution and only numerical approaches are possible. Atangana and Nieto [24] used finite-difference approximations in time and space of the Caputo–Fabrizio derivatives and the Crank–Nicolson solution scheme. More details about the numerical solution and the stability analysis are available in the original work [24]. We will skip the mentioned problems which are beyond the scope of the present chapter.

Solution in Time Domain Only

In this case the analogue of (60) is [25]

$$L \frac{{}_{\text{CF}}D_t^{2\mu} q(t)}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t\right)} + R \frac{{}_{\text{CF}}D_t^\mu q(t)}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t\right)} + \frac{q(t)}{C} = 0. \quad (70)$$

In this model the second fractional derivative is considered as a sequential one, namely

$${}_{CF}D_t^{2\mu}q(t) = {}_{CF}D_t^\mu [{}_{CF}D_t^\mu q(t)]. \tag{71}$$

The Laplace transform of (70) is [25]

$$L \frac{\text{Laplace} [{}_{CF}D_t^{2\mu}q(t)]}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t^2\right)} + R \frac{1}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t\right)} \frac{sQ(s) - q(0)}{s + \mu(1-s)} + \frac{Q(s)}{C} = 0. \tag{72}$$

With

$$\text{Laplace} [{}_{CF}D_t^{2\mu}q(t)] = s \frac{\{s\text{Laplace}[q(t) - q(0)]\}}{s + \mu(1-s)}. \tag{73}$$

The solution becomes

$$Q(s) = q(0) \frac{[a(s) + 1]}{sa(s) + b(s) + \frac{1}{C}} \tag{74a}$$

$$a(s) = \frac{1}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t^2\right)} \frac{s}{[s + \mu(1-s)]^2} \tag{74b}$$

$$b(s) = \frac{1}{\exp\left(-\frac{1-\mu}{2-\mu}\sigma_t\right)} \frac{R}{[s + \mu(1-s)]}. \tag{74c}$$

The exact solution can be obtained by the inverse Laplace transform, namely

$$q(t) = \text{Laplace}^{-1} \left\{ \frac{q(0)[a(s) + 1]}{sa(s) + b(s) + \frac{1}{C}} \right\}. \tag{75}$$

More details concerning numerical simulations are available in the original work [25].

5 Electrical Impedance Spectroscopy to RC Circuits: An Example

The electric impedance spectroscopy is wide and well-developed scientific area [9] and the fractional-order analysis allows deeper understanding of the physical problems at issue. With this short example taken from [29], we demonstrate how the fractional-order analysis of RC circuit works.

The electrical impedance spectroscopy applies a potential difference between two electrodes attached to a sample by passing a low power alternating electric current. This input is then compared to the corresponding output voltage and current. The impedance is defined by the simple relation

$$Z(s) = \frac{U(s)}{I(s)}. \quad (76)$$

From the Kirchoff law of a simple RC circuit, we have that

$$u = R_p i + U_{C_p} \quad \text{and} \quad i = i_R + i_c = \frac{U_{C_p}}{R_p} + C_p \frac{dU_{C_p}}{dt}. \quad (77a, b)$$

In term of time-fractional derivatives the fractional versions of (77a,b) are

$$i_R = \frac{U_{C_p}}{R_p} \quad \text{and} \quad i_c = C_p \frac{d^\mu U_{C_p}}{dt^\mu}. \quad (78a, b)$$

Hence, we have

$$i = \frac{U_{C_p}}{R_p} + C_p \frac{d^\mu U_{C_p}}{dt^\mu} \quad \text{and} \quad u = R_s i + U_{C_p}. \quad (79a, b)$$

In this model the time-fractional derivative will be considered in its alternative form $(1/\sigma^{1-\mu})(d^\mu/dt^\mu)$. Further, the Laplace transforms of (79a,b) are

$$I(s) = \frac{U_{C_p}(s)}{R_p} + \frac{C_p}{\sigma^{1-\mu}} s^\mu U_{C_p}(s) \quad \text{and} \quad U(s) = R_s I(s) + U_{C_p}(s). \quad (80a, b)$$

Therefore, from the definition of the impedance (76) we have [30]

$$Z(s^\mu) = R_s + \frac{R_p}{1 + \frac{R_p C_p}{\sigma^{1-\mu}} s^\mu} \Rightarrow Z(j\omega)^\mu = R_s + \frac{R_p}{1 + \frac{R_p C_p}{\sigma^{1-\mu}} (j\omega)^\mu}. \quad (81a, b)$$

Since the normalizing function, σ needs justification that the simplest approach applicable to the RC circuit is the choice $\sigma = R_p C_p$ which allows to reduce the Cole model [29]. In the case when $\mu = 1$ we obtain an ideal RC circuit.

6 Further Commentaries

Fractional calculus is attractive not only for electrical engineers but also for any scientist applying the equivalent electric circuit approach to model biomedical problems derivatives (assume the Caputo derivative); it is possible to define a generalized in [31, 32] because of the application of the Laplace transform to the fractional impedance of a fractional in time device with a proportionality to s^μ [33–35]. In this case the well-known general devices with $\mu = -1$ (capacitor), $\mu = 0$ (resistor) and $\mu = 1$ (inductor) are special cases. In this direction the Cole–Cole impedance [29, 34, 36] with fractional developed on the basis of equivalent fractional-order circuits is a powerful tool in the bioimpedance measurements [22, 23, 34, 37].

Further, the diffuse layer capacitances at micro- and nanoelectrodes are well described by simple RC circuits [38]. The anomalous diffusion in ionic solutions, especially in the low-frequency limit where the systems may be a present anomalous electrical response, are well described by anomalous diffusion and equivalent RC circuits of fractional order [39]. In this context, the development of supercapacitors [40] is more accurately described by fractional order in their transient processes of charging and discharging [37].

Systems process with equivalent RLC circuit models which allow deep analysis by the fractional calculus approach for a hot research area with a continuously number of publication on various applications which are hard to encompass in a single chapter. For more complex cases, some literature sources are highly recommended [31, 41–43].

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