Chapter 6
Linear Versus Nonlinear Stability in Hamiltonian Systems

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Abstract The stability of periodic solutions of time-independent Hamiltonian systems is often studied by linearization techniques. In the case of two degrees of freedom near stable equilibrium this is a correct procedure, in the case of three or more degrees of freedom we present some counterexamples. The case of the classical Fermi-Pasta-Ulam chain with cubic and quartic interactions illustrates the instability phenomenon.

6.1 Introduction

It is well-known that linearizing procedures in dissipative systems produce no conclusive evidence regarding stability if the eigenvalues are purely imaginary. An example is given in [6] ex. 3.2 where a second order autonomous equation with a centre equilibrium point is perturbed by nonlinear terms. For various choices of the nonlinear terms we may obtain asymptotic stability or instability of the equilibrium.

For Hamiltonian systems the stability question is more complicated. Suppose we have a time-independent Hamiltonian $H(p, q)$ with $p, q \in \mathbb{R}^n$ so that we have n degrees of freedom and a $2n$ dimensional system of differential equations. Suppose that the system has a nontrivial periodic solution $\phi(t)$ (in fact there will be many in general). We want to establish its stability by small perturbations in a neighborhood of $\phi(t)$. The usual practice is to linearize the perturbed system and consider the characteristic exponents.

From now on we will also assume that we consider the system near stable equilibrium so that we have a family of compact energy manifolds. This will enable us to apply known theorems and, if necessary, normalization techiques. If we have one or more positive Lyapunov exponents, the periodic solution will be unstable. To have only negative parts in the Hamiltonian case is impossible because of the symmetry of the spectrum in Hamiltonian systems.

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A periodic solution corresponds with a fixed point of a suitable Poincaré map of the phase-flow. Suppose now that the spectrum of the linearized flow near this fixed point has purely imaginary parts only. In many papers it is assumed then that the periodic solution is stable. We will argue that this is correct in the case of two degrees of freedom but not necessarily if $n \geq 3$. There can be various causes for instability, for instance higher order resonance or diffusion processes in phase-space. Our analysis may also have consequences for conservative, nonlinear wave equations where Galerkin projection leads to finite-dimensional but large Hamiltonian systems.

6.2 Two Degrees of Freedom

The system of equations of motion is four-dimensional, the energy manifolds near stable equilibrium are three-dimensional and compact. Apart from degenerate cases one can quite generally apply the KAM theorem that guarantees a foliation of tori of the energy manifolds around stable periodic solutions, see for instance [1] or [2] and further references there. The Weinstein theorem [7] guarantees the existence of at least two periodic solutions on an energy manifold near stable equilibrium. One can construct a transversal to the flow on the energy manifold that results in an area-preserving map, a Poincaré map, of the transversal into itself. We can choose the map so that the periodic solution produces a fixed point of the map. Because of the area-preserving character of the map, the eigenvalues associated with the fixed point will generically be real (positive and negative) or purely imaginary. The KAM tori around the stable periodic solutions are two-dimensional, the tori separate the three-dimensional energy manifold; the solutions between the tori can not escape. This means that purely imaginary eigenvalues imply stability of the solution in the nonlinear system.

6.3 Counter-Examples for More Degrees of Freedom

In the case of three or more degrees of freedom we can also apply the KAM theorem quite generally. However, the energy manifolds are $2n - 1$ dimensional, the tori at most n-dimensional. The tori do not separate the energy manifolds for $n \geq 3$. We will discuss examples showing various causes of instability but with common feature resonance.

6.3.1 The Influence of Quartic Terms

This example shows that higher order Hamiltonian perturbations may introduce instability. Indicating the quadratic, cubic and quartic parts of the Hamiltonian by

H_2, H_3, H_4 respectively we have:

$$H_2 = \frac{1}{2}(\dot{x}_1^2 + x_1^2 + \dot{x}_2^2 + 2x_2^2 + \dot{x}_3^2 + x_3^2),\ H_3 = -x_1x_2x_3,\ H_4 = -(\frac{1}{4}x_1^4 + x_1^2x_3^2 + \frac{1}{4}x_3^4).$$

The equations of motion can be written as:

$$\begin{cases} \ddot{x}_1 + x_1 &= x_2x_3 + x_1^3 + 2x_1x_3^2, \\ \ddot{x}_2 + 2x_2 &= x_1x_3, \\ \ddot{x}_3 + x_3 &= x_1x_2 + 2x_1^2x_3 + x_3^3. \end{cases} \tag{6.1}$$

The origin of phase-space corresponds with stable equilibrium. Localizing in a neighborhood of this equilibrium we can rescale $\dot{x}_i, x_i \to \varepsilon\dot{x}_i, \varepsilon x_i,\ i = 1, \ldots, 3$ resulting in:

$$\begin{cases} \ddot{x}_1 + x_1 &= \varepsilon x_2x_3 + \varepsilon^2(x_1^3 + 2x_1x_3^2), \\ \ddot{x}_2 + 2x_2 &= \varepsilon x_1x_3, \\ \ddot{x}_3 + x_3 &= \varepsilon x_1x_2 + \varepsilon^2(2x_1^2x_3 + x_3^3). \end{cases} \tag{6.2}$$

The system induced by Hamiltonian $H_2 + \varepsilon H_3 + \varepsilon^2 H_4$ admits the three normal modes in the coordinate planes. Consider the x_1 normal mode to $O(\varepsilon)$, the solution is harmonic:

$$x_1(t) = \phi(t) = r_0 \cos(t + \theta_0).$$

Puttting $x_1 = y + \phi(t)$ and linearizing near the normal mode in system (6.2) to $O(\varepsilon)$ we obtain:

$$\begin{cases} \ddot{y} + y &= 0, \\ \ddot{x}_2 + 2x_2 &= \varepsilon\phi(t)x_3, \\ \ddot{x}_3 + x_3 &= \varepsilon\phi(t)x_2. \end{cases} \tag{6.3}$$

We have kept the notation x_2, x_3 to avoid too many new symbols. The righthand sides of the last two equations contain non-resonant quasi-periodic terms that keep the inhomogeneous solutions $O(|x_3|), O(|x_2|)$ respectively. Put in a different way, normalizing the equations for x_2, x_3 involves non-resonant terms to any order. We conclude to linear stability of the x_1 normal mode. The higher order terms $O(\varepsilon^2)$ destroy this picture as was shown in [5] that the system induced by $H_2 + H_4$ contains two unstable normal modes; for an illustration see Fig. 6.1.

We can also linearize around the normal mode including the cubic terms of the equations. The x_1 normal mode satisfies to $O(\varepsilon^2)$ the equation

$$\ddot{x}_1 + x_1 = \varepsilon^2 x_1^3.$$

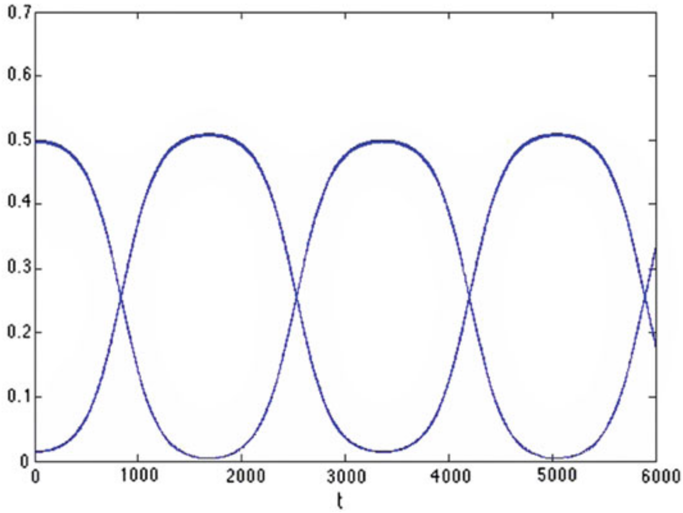

Fig. 6.1 The actions of system (6.2) with $x_1(0) = 1$, $x_2(0) = x_3(0) = 0.1$, velocities zero; $\varepsilon = 0.1$ and 6000 time-steps. The action $I_1 = \frac{1}{2}(\dot{x}_1^2 + x_1^2)$ associated with the x_1 normal mode starts in 0.5 and shows recurrence on around 3400 time-steps. The sum of the actions $I_2 + I_3$ associated with the x_2, x_3 modes starts near zero and shows similar recurrence

The solutions are elliptic functions that are more complicated to handle. However, for ε small we can determine the solution by the Poincaré-Lindstedt (or Poincaré continuation) method; see [6] ch. 10. The solution can be written as

$$\phi(t) = r_0(\varepsilon^2)\cos(t + \varepsilon^2\eta(\varepsilon^2)t + \phi_0)$$

where $r_0(\varepsilon^2), \eta(\varepsilon^2)$ have convergent Taylor expansions with respect to their argument. In this way we find linear stability but again instability in the full, nonlinear system.

6.3.2 Instability by the Presence of Mathieu-Tongues

Consider the Hamiltonian with

$$H_2 = \frac{1}{2}(\dot{x}_1^2 + 4x_1^2 + \dot{x}_2^2 + 4x_2^2 + \dot{x}_3^2 + \omega^2 x_3^2),\ H_3 = -(x_1 + x_2)x_3^2.$$

Applying the same scaling with small, positive parameter ε as before we have:

$$\begin{cases} \ddot{x}_1 + 4x_1 &= \varepsilon x_3^2, \\ \ddot{x}_2 + 4x_2 &= \varepsilon x_3^2, \\ \ddot{x}_3 + \omega^2 x_3 &= \varepsilon 2(x_1 + x_2)x_3. \end{cases} \tag{6.4}$$

The x_1 normal mode is harmonic, we put for this solution

$$x_1(t) = \phi(t) = r_0 \cos(2t + \theta_0).$$

We assume now that ω^2 is close but not equal to 1. Puttting $x_1 = y + \phi(t)$ and linearizing near the normal mode in system (6.4) we find:

$$\begin{cases} \ddot{y} + 4y &= 0, \\ \ddot{x}_2 + 4x_2 &= 0, \\ \ddot{x}_3 + \omega^2 x_3 &= 2\varepsilon\phi(t)x_3. \end{cases} \tag{6.5}$$

Stability or instability depends now on the Mathieu instability tongues of the third equation. Given ω near 1, $x_3(0)$ can be chosen small enough to produce stability of the x_1 normal mode, see Fig. 6.2 right, so formally the x_1 normal mode is stable. However, a slightly smaller perturbation of the frequency 1 may put the solution $x_3(t)$ with the same initial conditions in the unstable Mathieu tongue, see Fig. 6.2 left. These phenomena are subtle and should be kept in mind near resonance.

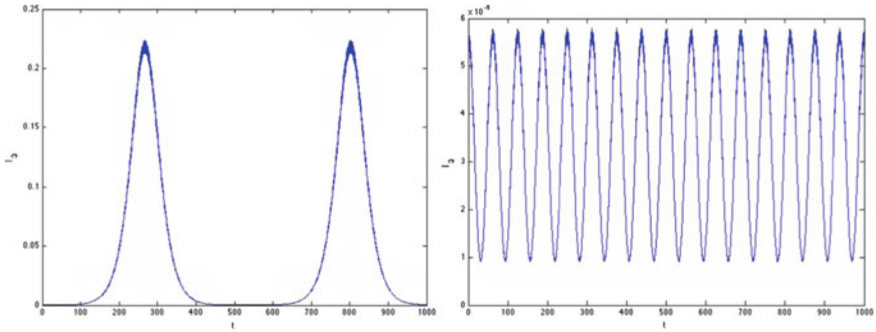

Fig. 6.2 The action $I_3 = \frac{1}{2}(\dot{x}_3^2 + \omega^2 x_3^2)$ in two cases of system (6.4) with $x_1(0) = 1, x_2(0) = 0.1, x_3(0) = 0.01$, velocities zero; $\varepsilon = 0.1$ and 1000 time-steps. *Left* the case $\omega^2 = 1.1$ leading to instability of the x_1 normal mode, $0 < I_3 < 0.23$. *Right* the slightly more detuned case $\omega^2 = 1.15$ leading to stability of the x_1 normal mode, the fluctuations of I_3 are of size 10^{-4}

6.4 Application to a Chain with 4 Interacting Particles

Consider a periodic chain consisting of four particles of equal mass ($m = 1$) with quadratic and cubic nearest-neighbor interaction. With position q_j and momentum $p_j = \dot{q}_j$, $j = 1 \ldots 4$, the Hamiltonian is of the form

$$H(p,q) = \sum_{j=1}^{4} (\frac{1}{2} p_j^2 + V(q_{j+1} - q_j)) \text{ with } V(z) = \frac{1}{2}z^2 + \frac{\alpha}{3}z^3 + \frac{\beta}{4}z^4. \quad (6.6)$$

This is a low-dimensional case of the periodic Fermi-Pasta-Ulam problem; usually many more particles are considered in this classical problem. We will choose

$$\alpha = 1, \beta = -1.$$

The corresponding equations of motion were studied in [4] where the stability and instability of the short-periodic solutions was established for arbitrary α and β. The equations induced by Hamiltonian (6.6) have a second integral of motion, the momentum integral $\sum_1^4 p_j =$ constant. This enables us to reduce the 4 degrees-of-freedom equations of motion to 3 degrees-of-freedom. The symplectic transformation was carried out in [3] producing with $\alpha = 1, \beta = -1$:

$$\begin{cases} H_2 = 2x_1^2 + x_2^2 + x_3^2 + \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2), \ H_3 = -4x_1x_2x_3, \\ H_4 = -\frac{1}{4}(4x_1^4 + x_2^4 + 6x_2^2x_3^2 + x_3^4 + 12x_1^2(x_2^2 + x_3^2)). \end{cases} \quad (6.7)$$

Rescaling as before $x_i \to \varepsilon x_i, \dot{x}_i \to \varepsilon \dot{x}_i, i = 1, 2, 3$ in a neighborhood of stable equilibrium we find the equations of motion:

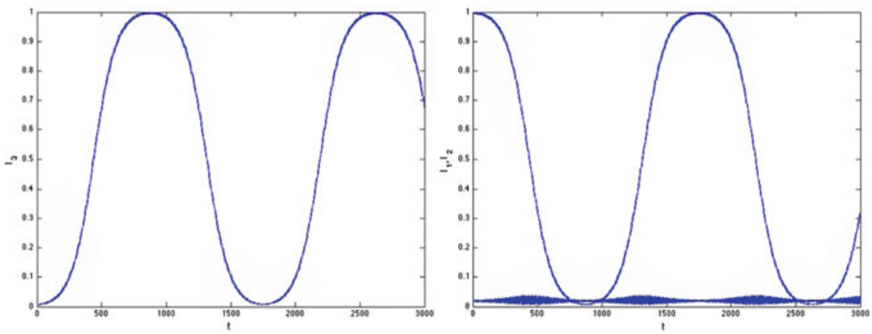

Fig. 6.3 The actions for 3000 timesteps near the unstable x_2 normal mode of system (6.8) with $\varepsilon = 0.1$, initial conditions $x_1(0) = x_3(0) = 0.1$, $x_2(0) = 1$ and initial velocities zero. *Left* the action $I_3(t) = \frac{1}{2}(\dot{x}_3^2 + 2x_3^2)$ starting near zero and increasing to values near 1. Right $I_2(t)$ starting at $I_2(0) = 1$ and $I_1(t)$ which remains small

$$\begin{cases} \ddot{x}_1 + 4x_1 = 4\varepsilon x_2 x_3 + \varepsilon^2 \left(4x_1^3 + 6x_1(x_2^2 + x_3^2)\right), \\ \ddot{x}_2 + 2x_2 = 4\varepsilon x_1 x_3 + \varepsilon^2 (x_2^3 + 3x_2 x_3^2 + 6x_1^2 x_2), \\ \ddot{x}_3 + 2x_3 = 4\varepsilon x_1 x_2 + \varepsilon^2 (x_3^3 + 3x_2^2 x_3 + 6x_1^2 x_3). \end{cases} \tag{6.8}$$

The three normal modes (in the coordinate planes) satisfy the equations of system (6.8). It was shown in [4] that the x_1 normal mode is stable, the x_2 and x_3 normal modes are unstable. Consider normal mode x_2. Linearization near the normal mode to $O(\varepsilon)$ produces stability as in the examples presented before. The instability is illustrated in Fig. 6.3.

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