Chapter 9
Vanishing Dissipation Limits

The behavior of fluids in the vanishing dissipation regime, meaning when both the Reynolds number and the Péclet number are large, plays an important role in the study of turbulence. In this chapter, we examine the situation when

$$\mathrm{Sr} = 1,\ \mathrm{Ma} = \varepsilon,\ \text{and}\ \mathrm{Re} = \frac{1}{\nu},\ \mathrm{Pe} = \frac{1}{d},\ \text{with suitably chosen}\ \varepsilon,\ \nu,\ d > 0.$$

Such a choice of scaling parameters gives rise to qualitatively new difficulties in the study of the singular limit as we lose compactness in the space variable of both velocity and temperature. As a result, the singular limit is no longer a problem of convergence of solutions of the primitive system to those of the target system but rather a problem of *stability* of the target solution with respect to singular perturbations. Accordingly, we have to assume that the target system admits a regular solution at least on a certain maximal time interval $(0, T)$. Thus the *existence* of solutions to the target problem is no longer a byproduct of the singular limit analysis but a necessary hypothesis for the singular limit process to converge.

Stability of the target solution will be evaluated in the "norm" induced by a new quantity called *relative energy*, the analogue of which—the so-called relative entropy—has been introduced in the context of hyperbolic systems of conservation laws by Dafermos [67]. Formally, the relative energy reads

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \mathcal{T}, \mathbf{V}\right) \tag{9.1}$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{V}|^2 + H_{\mathcal{T}}(\varrho, \vartheta) - (\varrho - r)\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} - H_{\mathcal{T}}(r, \mathcal{T})\right] \mathrm{d}x,$$

where $[\varrho, \vartheta, \mathbf{u}]$ is a weak solution of the (unscaled) Navier-Stokes-Fourier system,

$$(\varrho, \vartheta) \mapsto H_{\mathcal{T}}(\varrho, \vartheta)$$

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is the Helmholtz function introduced in Sect. 2.2.3, and $[r, \mathcal{T}, \mathbf{V}]$ a trio of admissible smooth "test" functions. Formally, the relative energy is reminiscent of the quantity appearing in the total dissipation balance (2.52), where the arguments r, θ, and $\mathbf{V}$ are now functions of the independent variables (t, x). The relative energy $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \mathcal{T}, \mathbf{V}\right)$ can be seen as a kind of distance between the quantities $[\varrho, \vartheta, \mathbf{u}]$ and $[r, \mathcal{T}, \mathbf{V}]$. Indeed the hypothesis of thermodynamics stability (1.44) implies that

$$\left\{ \begin{array}{c} \mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \mathcal{T}, \mathbf{V}\right) \geq 0; \\ \\ \text{if } r > 0, \text{ then} \\ \\ \mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \mathcal{T}, \mathbf{V}\right) = 0 \text{ only if } [\varrho, \vartheta, \mathbf{u}] = [r, \mathcal{T}, \mathbf{V}]. \end{array} \right\}$$

Remark Note however that $\mathcal{E}$ is not a metric, in particular it is not symmetric with respect to $[\varrho, \vartheta, \mathbf{u}]$ and $[r, \mathcal{T}, \mathbf{V}]$.

The strength of the existence theory of weak solutions based on the entropy balance developed in Chap. 3 will be demonstrated by the fact that the time evolution of $\mathcal{E}$ can be controlled by means of the weak formulation introduced in Chap. 2, Sect. 2.1.

9.1 Problem Formulation

To simplify the presentation, we consider the primitive NAVIER-STOKES-FOURIER SYSTEM in the absence of external driving forces:

■ PRIMITIVE SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{9.2}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \nu \mathrm{div}_x \mathbb{S}, \tag{9.3}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x\Big(\varrho s(\varrho, \vartheta) \mathbf{u}\Big) + d \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma_\varepsilon, \tag{9.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right) \mathrm{d}x = 0. \tag{9.5}$$

In accordance with the general framework of fluid motions considered in this book, the viscous stress tensor is determined by *Newton's law*

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{9.6}$$

the heat flux by *Fourier's law*

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta, \tag{9.7}$$

and the *entropy production rate* is a non-negative measure σ_ε satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta}\Big(\varepsilon^2 \nu\, \mathbb{S} : \nabla_x \mathbf{u} - d\frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\Big). \tag{9.8}$$

9.1.1 Physical Space and Boundary Conditions

Similarly to Chap. 8, we consider an expanding family of spatial domains $\{\Omega_R\}_{R>0}$, specifically

- $\Omega_R \subset \mathbb{R}^3$ are simply connected, bounded, $C^{2+\nu}$ domains, uniformly for $R \to \infty$;
- $$\Big\{x \in \mathbb{R}^3 \ \Big|\ |x| < R\Big\} \subset \Omega_R. \tag{9.9}$$
- there exists $D > 0$ such that

$$\partial\Omega_R \subset \Big\{x \in \mathbb{R}^3 \ \Big|\ R < |x| < R + D\Big\} \tag{9.10}$$

Remark A typical example of such domains is, of course, a family of balls of radius R,

$$\Omega_R = \Big\{x \in \mathbb{R}^3 \ \Big|\ |x| < R + \delta\Big\}, \delta > 0.$$

We impose the *no-slip* boundary conditions for the velocity field

$$\mathbf{u}|_{\partial\Omega_R} = 0, \tag{9.11}$$

together with the no-flux conditions

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_R} = 0. \tag{9.12}$$

9.1.2 Initial Data

Similarly to the low Mach number limit problems considered in this book, we suppose that the initial data can be written in the form

$$\varrho(0,\cdot) \equiv \varrho_0 = \overline{\varrho} + \varepsilon\varrho_0^{(1)},\ \vartheta(0,\cdot) \equiv \vartheta_0 = \overline{\vartheta} + \varepsilon\vartheta_0^{(1)},\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{9.13}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants,

$$\left\{ \begin{array}{c} 0 < D^{-1} < \overline{\varrho}, \overline{\vartheta} < D, \\ \\ \|\varrho_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\vartheta_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\mathbf{u}_0\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} < D. \end{array} \right\} \tag{9.14}$$

Remark The parameter $D > 0$ measures the size of the data and may be chosen large enough to comply also with (9.10). Of course, the initial data perturbations $\varrho_0^{(1)}$, $\vartheta_0^{(1)}$, $\mathbf{u}_0$ as well as the corresponding weak solutions to the Navier-Stokes-Fourier system may depend on the scaling parameters ε, ν, d and also on the total mass

$$M = \int_{\Omega_R} \varrho_0 \mathrm{d}x.$$

9.1.3 Target Problem

As the family of expanding domains will eventually fill up the whole space $\mathbb{R}^3$, it makes sense to consider the limit problem with this geometry, supplemented with the far field boundary condition for the limit velocity

$$\mathbf{U} \to 0 \text{ as } |x| \to \infty.$$

Given our previous experience with the low Mach number limit and since we intend to let the diffusion coefficients ν and ω vanish, we may anticipate the following form of the target problem.

■ INCOMPRESSIBLE EULER SYSTEM WITH TEMPERATURE TRANSPORT:

$$\mathrm{div}_x \mathbf{U} = 0 \text{ in } (0,T) \times \mathbb{R}^3, \tag{9.15}$$

$$\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U}) + \nabla_x \Pi = 0 \text{ in } (0,T) \times \mathbb{R}^3, \tag{9.16}$$

$$\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta = 0 \text{ in } (0,T) \times \mathbb{R}^3. \tag{9.17}$$

Here, as observed many times in the previous chapters, the transported quantity Θ is related to the temperature deviation

$$\Theta \approx \frac{\vartheta - \overline{\vartheta}}{\varepsilon}.$$

System (9.15), (9.16)—called (incompressible) *Euler system*—decouples from (9.17) and may be solved independently. A nowadays classical result of Tosio Kato [164, 165, 167] asserts the existence of a unique classical solution $\mathbf{U}$ of the initial-value problem associated to (9.15), (9.16) in the class

$$\mathbf{U} \in C([0,T_{\max}); W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)), \ \partial_t \mathbf{U} \in C([0,T_{\max}); W^{k-1,2}(\mathbb{R}^3;\mathbb{R}^3)), \tag{9.18}$$

defined on a maximal time interval $[0,T_{\max})$, $T_{\max} > 0$ for any initial data

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0 \in W^{k,2}(\mathbb{R}^3;\mathbb{R}^3) \text{ as soon as } k \geq 3. \tag{9.19}$$

To avoid technicalities, we have taken k to be an integer. More general results can be shown, see e.g. Constantin et al. [64], Chemin [55], Danchin [73]. Note that regularity of the pressure Π can be deduced from (9.16), (9.18).

Any field $\mathbf{U}$ belonging to the regularity class (9.18) possesses a continuous gradient $\nabla_x \mathbf{U}$, in particular, the transport equation (9.17) can be uniquely solved for any initial data

$$\Theta(0,\cdot) = \Theta_0 \tag{9.20}$$

by the method of characteristics. Specifically, the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t) = \mathbf{U}(t,\mathbf{X}), \ \mathbf{X}(0) = \mathbf{X}_0, \tag{9.21}$$

admits a unique solution $\mathbf{X} = \mathbf{X}(t, \mathbf{X}_0)$ for any $\mathbf{X}_0$ in $\mathbb{R}^3$ and we set

$$\Theta(t, \mathbf{X}(t, \mathbf{X}_0)) = \Theta_0(\mathbf{X}_0), \ t \in [0, T_{\max}).$$

9.1.4 Strategy of the Proof of Stability of Smooth Solutions to the Target Problem

Our goal in this chapter is to show that solutions of the primitive Navier-Stokes-Fourier system remain close to a smooth solution of the target problem provided

$$\varepsilon, \ \nu, \ d \to 0, \ R \to \infty$$

and the initial data of the two systems are close. As we shall see, the result will be *path dependent*, meaning the rates of convergence of the singular parameters to their limit values must be interrelated in a certain specific fashion. Here, the "distance" between the data will be measured in terms of the relative energy $\mathcal{E}$.

Our strategy leans on the following steps.

- Derive a relation between the values of the relative energy $\mathcal{E}$ at the times $t = 0, \tau$.
- Take the strong solution of the target system as a test function in the relative entropy.
- Use a Gronwall lemma type argument to evaluate the distance between the two solutions by means of $\mathcal{E}$.

9.2 Relative Energy Inequality

The relative energy inequality may be seen as a refined version of the total dissipation balance (2.52), where the constants $\overline{\varrho}$, $\overline{\vartheta}$ are replaced by functions r, $\mathcal{T}$, and the velocity $\mathbf{u} = \mathbf{u} - 0$ by $\mathbf{u} - \mathbf{V}$. It is of independent interest so we formulate it for the unscaled version of the Navier-Stokes-Fourier system where we set, for a moment,

$$\varepsilon = \nu = d = R = 1, \ \Omega \equiv \Omega_1.$$

We consider a weak solution $[\varrho, \vartheta, \mathbf{u}]$ of problem (9.2)–(9.8), (9.11), (9.12) in the sense specified in Chap. 2, Sect. 2.1. The crucial observation is that the relative energy can be decomposed as the sum

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \mathcal{T}, \mathbf{V}\right) = \sum_{j=1}^{6} \mathcal{E}_j,$$

where

$$\mathcal{E}_1 = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right] \mathrm{d}x,$$

$$\mathcal{E}_2 = -\int_\Omega \varrho\mathbf{u}\cdot\mathbf{V} \, \mathrm{d}x,$$

$$\mathcal{E}_3 = \int_\Omega \frac{1}{2}\varrho|\mathbf{V}|^2 \, \mathrm{d}x,$$

$$\mathcal{E}_4 = -\int_\Omega \varrho s(\varrho,\vartheta)\mathcal{T} \, \mathrm{d}x,$$

$$\mathcal{E}_5 = -\int_\Omega \varrho\frac{\partial H_{\mathcal{T}}(r,\mathcal{T})}{\partial\varrho} \, \mathrm{d}x,$$

$$\mathcal{E}_6 = \int_\Omega \left[r\frac{\partial H_{\mathcal{T}}(r,\mathcal{T})}{\partial\varrho} - H_{\mathcal{T}}(r,\mathcal{T})\right] \mathrm{d}x,$$

where each integral can be evaluated by means of the weak formulation as long as the functions r, $\mathcal{T}$, and $\mathbf{V}$ are smooth enough, $r > 0$, $\mathcal{T} > 0$, and $\mathbf{V}$ satisfies the relevant boundary conditions, here

$$\mathbf{V}|_{\partial\Omega} = 0.$$

Our goal is to compute

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)\right]_{t=0}^{t=\tau} \equiv \mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)(\tau) - \mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)(0)$$

using only the weak formulation of the Navier-Stokes-Fourier system

Step 1 The total energy balance (2.22) yields

$$\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right] \mathrm{d}x\right]_{t=0}^{t=\tau} \tag{9.22}$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right](\tau,\cdot) \, \mathrm{d}x - \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right] \mathrm{d}x = 0$$

for a.a. $\tau \in [0,T]$.

Step 2 Taking $\mathbf{V}$ as a test function in the weak formulation of the momentum balance (2.9) gives rise to

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \mathbf{V} \, \mathrm{d}x\right]_{t=0}^{t=\tau} \tag{9.23}$$

$$= \int_0^\tau \int_\Omega \Big[\varrho \mathbf{u} \cdot \partial_t \mathbf{V} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \mathbf{V} + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{V} - \mathbb{S} : \nabla_x \mathbf{V}\Big] \, \mathrm{d}x \, \mathrm{d}t$$

for any $\tau \in [0, T]$.

Step 3 Taking $|\mathbf{V}|^2$ as a test function in the equation of continuity (2.2) we get

$$\left[\int_\Omega \frac{1}{2} \varrho |\mathbf{V}|^2 \, \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \mathbf{V} \cdot \partial_t \mathbf{V} + \varrho \mathbf{u} \cdot \mathbf{V} \cdot \nabla_x \mathbf{V}\Big] \, \mathrm{d}x \, \mathrm{d}t \tag{9.24}$$

for any $\tau \in [0, T]$.

Step 4 Taking $\mathcal{T}$ as a test function in the entropy balance (2.27) yields

$$-\left[\int_\Omega \varrho s(\varrho, \vartheta) \mathcal{T} \, \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \frac{\mathcal{T}}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \mathrm{d}x \, \mathrm{d}t \tag{9.25}$$

$$\leq - \int_0^\tau \int_\Omega \Big[\varrho s(\varrho, \vartheta) \partial_t \mathcal{T} + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \mathcal{T} + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \mathcal{T}\Big] \, \mathrm{d}x \, \mathrm{d}t$$

for a.a. $\tau \in [0, T]$.

Step 5 Taking $\partial_\varrho H_{\mathcal{T}}(r, \mathcal{T})$ as a test function in the equation of continuity (2.2) we obtain

$$\left[\int_\Omega \varrho \frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} \, \mathrm{d}x\right]_{t=0}^{t=\tau} \tag{9.26}$$

$$= \int_0^\tau \int_\Omega \left[\varrho \partial_t \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho}\right) + \varrho \mathbf{u} \cdot \nabla_x \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho}\right)\right] \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}t$$

for any $\tau \in [0, T]$.

Step 6 Summing up the previous identities we obtain

$$\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \mathcal{T}, \mathbf{V}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \frac{\mathcal{T}}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \mathrm{d}x \, \mathrm{d}t \tag{9.27}$$

$$\leq \int_0^\tau \int_\Omega \Big[\varrho \left(\partial_t \mathbf{V} + \mathbf{u} \cdot \nabla_x \mathbf{V}\right) \cdot (\mathbf{V} - \mathbf{u}) - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{V} + \mathbb{S} : \nabla_x \mathbf{V}\Big] \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\varrho s(\varrho, \vartheta) \partial_t \mathcal{T} + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \mathcal{T} + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \mathcal{T} \right] \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\varrho \partial_t \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} \right) + \varrho \mathbf{u} \cdot \nabla_x \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} \right) \right] \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \partial_t \left(r \frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} - H_{\mathcal{T}}(r, \mathcal{T}) \right) \mathrm{d}x \, \mathrm{d}t$$

for a.a. $\tau \in [0, T]$.

Conclusion Finally, making of use of Gibbs' equation (1.2), we compute

$$\partial_t \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} \right) = -s(r, \mathcal{T}) \partial_t \mathcal{T} - r \frac{\partial s(r, \mathcal{T})}{\partial \varrho} \partial_t \mathcal{T} + \frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho^2} \partial_t r + \frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho \partial \vartheta} \partial_t \mathcal{T},$$

$$\nabla_x \left(\frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} \right) = -s(r, \mathcal{T}) \nabla_x \mathcal{T} - r \frac{\partial s(r, \mathcal{T})}{\partial \varrho} \nabla_x \mathcal{T} + \frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho^2} \nabla_x r + \frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho \partial \vartheta} \nabla_x \mathcal{T},$$

together with the relations

$$\frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho^2} = \frac{1}{r} \frac{\partial p(r, \mathcal{T})}{\partial \varrho}, \; r \frac{\partial s(r, \mathcal{T})}{\partial \varrho} = -\frac{1}{r} \frac{\partial p(r, \mathcal{T})}{\partial \vartheta}$$

$$\frac{\partial^2 H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho \partial \vartheta} = \frac{\partial}{\partial \varrho} \left(\varrho (\vartheta - \mathcal{T}) \frac{\partial s}{\partial \vartheta} \right) (r, \mathcal{T}) = \left((\vartheta - \mathcal{T}) \frac{\partial}{\partial \varrho} \left(\varrho \frac{\partial s}{\partial \vartheta} \right) \right) (r, \mathcal{T}) = 0,$$

and

$$\left(r \frac{\partial H_{\mathcal{T}}(r, \mathcal{T})}{\partial \varrho} - H_{\mathcal{T}}(r, \mathcal{T}) \right) = p(r, \mathcal{T}).$$

Thus inequality (9.27) can be written in a more concise form as

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■ RELATIVE ENERGY INEQUALITY:

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \mathcal{T}, \mathbf{V} \right) \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \frac{\mathcal{T}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x \, \mathrm{d}t \tag{9.28}$$

$$\leq \int_0^\tau \int_\Omega \varrho (\mathbf{u} - \mathbf{V}) \cdot \nabla_x \mathbf{V} \cdot (\mathbf{V} - \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) (\mathbf{V} - \mathbf{u}) \cdot \nabla_x \mathcal{T} \, \mathrm{d}x \, \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \Big[\varrho \left(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V} \right) \cdot (\mathbf{V} - \mathbf{u}) - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{V} + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{V} \Big] \, \mathrm{d}x \, \mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \Big[\varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \partial_t \mathcal{T} + \varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \mathbf{V} \cdot \nabla_x \mathcal{T}$$

$$+ \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \mathcal{T} \Big] \, \mathrm{d}x \, \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \Big[\Big(1 - \frac{\varrho}{r} \Big) \partial_t p(r, \mathcal{T}) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \mathcal{T}) \Big] \, \mathrm{d}x \, \mathrm{d}t$$

for a.a. $\tau \in [0, T]$ and any trio of continuously differentiable test functions $[r, \mathcal{T}, \mathbf{V}]$ satisfying

$$r > 0, \ \mathcal{T} > 0, \ \mathbf{V}|_{\partial\Omega} = 0. \tag{9.29}$$

Remark Note that the requirement on smoothness of the test functions may be relaxed by a density argument if the weak solution enjoys certain regularity. Similar inequality may be derived also for the slip boundary conditions (1.19), (1.27), for which $\mathbf{V}$ must satisfy $\mathbf{V} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

9.3 Uniform Estimates

To derive suitable uniform bounds on the family of solutions to the scaled Navie-Stokes-Fourier system, certain restrictions must be imposed on the constitutive relations. These are basically the same as in Chap. 5 and we list them here for convenience:

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\vartheta), \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \Big), \ p_R = \frac{a}{3} \vartheta^4, \ a > 0; \tag{9.30}$$

$$e(\varrho, \vartheta) = e_M(\varrho, \vartheta) + e_R(\varrho, \vartheta), \ e_M = \frac{3}{2} \frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \Big), \ e_R = a \frac{\vartheta^4}{\varrho}, \tag{9.31}$$

and

$$s(\varrho, \vartheta) = s_M(\varrho, \vartheta) + s_R(\varrho, \vartheta), \ s_M(\varrho, \vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \Big), \ s_R = \frac{4}{3} a \frac{\vartheta^3}{\varrho}, \tag{9.32}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0, \text{ and } \lim_{Z\to\infty} S(Z) = 0. \tag{9.33}$$

Remark The last stipulation in (9.33) reflects the Third law of thermodynamics discussed in Chap. 1, Sect. 1.4.2. It implies, in particular, that

$$0 \le S(Z) \le S(1) \text{ for } Z \ge 1,\ 0 \le S(Z) \le S(1) - c\log(Z),\ c > 0,\ \text{for } Z < 1;$$

whence

$$\varrho s(\varrho, \vartheta) \le c\left(1 + \varrho\log(\varrho) + \varrho[\log(\vartheta)]^+ + \vartheta^3\right) \text{ for all } \varrho, \vartheta \ge 0. \tag{9.34}$$

This condition plays a technical role in our analysis and may be probably omitted.

Furthermore, the *hypothesis of thermodynamic stability* (1.44) requires $P \in C^1[0,\infty) \cap C^2(0,\infty)$,

$$P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \ge 0, \tag{9.35}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \le \sup_{z>0}\frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty, \tag{9.36}$$

and, in addition,

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{9.37}$$

The viscosity coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$ are (globally) Lipschitz continuous in $[0,\infty)$, and

$$\left.\begin{aligned} 0 < \underline{\mu}(1+\vartheta) \le \mu(\vartheta) \le \overline{\mu}(1+\vartheta), \\ 0 \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta) \end{aligned}\right\} \text{ for all } \vartheta \ge 0, \tag{9.38}$$

where $\underline{\mu}$, $\overline{\mu}$, $\overline{\eta}$ are positive constants. Similarly, $\kappa = \kappa(\vartheta)$ is a continuously differentiable function satisfying

$$0 < \underline{\kappa}(1+\vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \ge 0, \tag{9.39}$$

with positive constants $\underline{\kappa}$, $\overline{\kappa}$.

The basic uniform estimates will be derived by means of the rescaled version of the relative energy inequality associated to system (9.2)–(9.5). For

$$\mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)$$

$$= \int_{\Omega_R}\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{V}|^2 + \frac{1}{\varepsilon^2}\left(H_{\mathcal{T}}(\varrho,\vartheta) - (\varrho - r)\frac{\partial H_{\mathcal{T}}(r,\mathcal{T})}{\partial\varrho} - H_{\mathcal{T}}(r,\mathcal{T})\right)\right]\,\mathrm{d}x,$$

we have

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)\right]_{t=0}^{t=\tau} \tag{9.40}$$

$$+\int_0^\tau\int_{\Omega_R}\frac{\mathcal{T}}{\vartheta}\left(\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u} - \frac{d}{\varepsilon^2}\frac{\mathbf{q}(\varrho,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^\tau\int_{\Omega_R}\varrho(\mathbf{u}-\mathbf{V})\cdot\nabla_x\mathbf{V}\cdot(\mathbf{V}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)(\mathbf{V}-\mathbf{u})\cdot\nabla_x\mathcal{T}\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_{\Omega_R}\left[\varrho\left(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_x\mathbf{V}\right)\cdot(\mathbf{V}-\mathbf{u}) - \frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathrm{div}_x\mathbf{V} + \nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{V}\right]\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\Big[\varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)\partial_t\mathcal{T} + \varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)\mathbf{V}\cdot\nabla_x\mathcal{T}$$

$$+d\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\mathcal{T}\Big]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\left[\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\mathcal{T}) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\mathcal{T})\right]\,\mathrm{d}x\,\mathrm{d}t$$

The necessary uniform bounds can be derived in exactly the same way as in Chap. 8, Sect. 8.3. Introducing $[h]_{\mathrm{ess}}$, $[h]_{\mathrm{res}}$ as in (4.39)–(4.45), we take

$$r = \overline{\varrho},\ \mathcal{T} = \overline{\vartheta},\ \mathbf{V} = 0$$

in the relative energy inequality (9.40) to deduce the estimates:

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega_R;\mathbb{R}^3)} \leq c(D), \tag{9.41}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)}\left\|\left[\frac{\varrho-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}}\right\|_{L^2(\Omega_R)} \leq c(D), \tag{9.42}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_R)} \leq c(D), \tag{9.43}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| \, [\varrho e(\varrho,\vartheta)]_{\rm res} \|_{L^1(\Omega_R)} \leq \varepsilon^2 c(D), \tag{9.44}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| \, [\varrho s(\varrho,\vartheta)]_{\rm res} \|_{L^1(\Omega_R)} \leq \varepsilon^2 c(D), \tag{9.45}$$

along with the estimate on the measure of the residual set (cf. (4.43) and (8.37))

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}_{\rm res}[t]| \leq \varepsilon^2 c(D), \tag{9.46}$$

where the bounds depend solely on the norm of the initial data through (9.14).

Finally, exactly as in (8.50)–(8.55), Chap. 8, we conclude

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_R} [\varrho]^{5/3}_{\rm res} \ {\rm d}x \leq \varepsilon^2 c(D), \tag{9.47}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_R} [\vartheta]^{4}_{\rm res} \ {\rm d}x \leq \varepsilon^2 c(D), \tag{9.48}$$

and

$$\int_0^T \nu \, \|\mathbf{u}\|^2_{W^{1,2}(\Omega_R;\mathbb{R}^3)} \ {\rm d}t \leq c(D), \tag{9.49}$$

$$d\int_0^T \|\vartheta - \overline{\vartheta}\|^2_{W^{1,2}(\Omega_R)} \ {\rm d}t + d\int_0^T \|\log(\vartheta) - \log(\overline{\vartheta})\|^2_{W^{1,2}(\Omega_R)} \ {\rm d}t \leq \varepsilon^2 c(D). \tag{9.50}$$

Remark We tacitly anticipate d and ν to be small, in particular, the above estimates hold (independently of ν, d) on condition that $d < 1$, $\nu < 1$.

9.4 Well-Prepared Initial Data

To illuminate the method based on the relative entropy inequality, we first consider the well-prepared initial data. Accordingly, we consider

$$r = \overline{\varrho}, \ \mathcal{T} = \overline{\vartheta}, \ \mathbf{V} = \mathbf{U}$$

in (9.40), where $\mathbf{U}$ is a solution of the Euler system (9.15), (9.16). Unfortunately, the function $\mathbf{U}$ does not vanish on $\partial\Omega_R$ and therefore cannot be used as a test function in (9.40). Instead we take a suitable cut-off of the Euler solution.

First, we fix

$$\mathbf{v}_0 \in C_c^m(\mathbb{R}^3) \text{ and write } \mathbf{v}_0 = \mathbf{H}[\mathbf{v}_0] + \nabla_x \Psi_0, \tag{9.51}$$

$$\mathrm{supp}[\mathbf{v}_0] \subset B(0,D),\ \|\mathbf{v}_0\|_{C^m(\mathbb{R}^3)} \le D,\ m > 4, \tag{9.52}$$

where $\mathbf{H}$ denotes the Helmholtz projection defined on the whole space $\mathbb{R}^3$, and consider $\mathbf{U}$—the solution of the Euler system (9.15), (9.16) defined on a time interval $(0, T_{\max})$—satisfying

$$\mathbf{U}(0,\cdot) = \mathbf{w}_0 \equiv \mathbf{H}[\mathbf{v}_0]. \tag{9.53}$$

Remark Similarly to (9.10), (9.14), the quantity D measures the size of the initial data. Obviously, D may be chosen large enough so that both (9.10), (9.14) and (9.52) hold.

The solenoidal function $\mathbf{U}$ can be expressed by means of the Biot–Savart law

$$\mathbf{U} = -\mathbf{curl}_x \Delta_x^{-1} \mathbf{curl}_x[\mathbf{U}],$$

where

$$\Delta_x^{-1}[h](x) = \int_{\mathbb{R}^3} \frac{h(y)}{|x-y|}\, \mathrm{d}y.$$

Consequently, as

$$\mathbf{curl}_x[\mathbf{w}_0] = \mathbf{curl}_x[\mathbf{v}_0] \in C_c^{m-1}(\mathbb{R}^3),$$

and the Euler system (9.15), (9.16) propagates $\mathbf{curl}_x\mathbf{U}$ with a finite speed (see Sect. 11.20 in Appendix), we infer that

$$\begin{cases} \mathbf{U} = \mathbf{curl}_x[\mathbf{h}], \\ \\ \text{where } \Delta\mathbf{h} = 0 \text{ } (\mathbf{h} \text{ is a harmonic function) outside a bounded ball in } \mathbb{R}^3. \end{cases}$$

By the same token, $\Delta\Psi_0$ is compactly supported and we conclude that

$$|\nabla_x \Psi_0(x)| + |\mathbf{U}(t,x)| \le \frac{c(D)}{|x|^2} \le \frac{c(D)}{R^2} \text{ whenever } x \in \partial\Omega_R. \tag{9.54}$$

We introduce a cut-off function $\chi_R = \chi_R(x)$,

$$\chi_R(x) = \chi(|x| - R),\ \chi \in C_c^\infty(\mathbb{R}),\ 0 \le \chi \le 1,\ \chi(z) = 1 \text{ for } z \in [0, D].$$

It follows from (9.54) and hypothesis (9.10) that

$$\left\{ \begin{array}{c} \|\partial_t(\chi_R\mathbf{U})(t,\cdot)\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} + \|(\chi_R\mathbf{U})(t,\cdot)\|_{W^{2,p}(\mathbb{R}^3;\mathbb{R}^3)} \le c(D)R^{2\left(\frac{1}{p}-1\right)} \\ \\ \|\chi_R\nabla_x\Psi_0\|_{W^{2,p}(\mathbb{R}^3;\mathbb{R}^3)} \le c(D)R^{2\left(\frac{1}{p}-1\right)} \text{ for any } 1 \le p \le \infty,\ t \in [0, T_{\max}). \end{array} \right\} \tag{9.55}$$

The function $\mathbf{V} = (1 - \chi_R)\mathbf{U}$ vanishes on $\partial\Omega_R$, therefore can be taken, together with $r = \overline{\varrho}$, $\mathcal{T} = \overline{\vartheta}$ as a test function in the relative energy inequality (9.40) to obtain

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U}\right)\right]_{t=0}^{t=\tau} \tag{9.56}$$

$$+ \int_0^\tau \int_{\Omega_R} \frac{\overline{\vartheta}}{\vartheta}\left(\nu\mathbb{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{d}{\varepsilon^2}\frac{\mathbf{q}(\varrho, \nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \mathrm{d}x\,\mathrm{d}t$$

$$\le \int_0^\tau \int_{\Omega_R} \varrho(\mathbf{u} - \mathbf{V})\cdot\nabla_x\mathbf{V}\cdot(\mathbf{V} - \mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^\tau \int_{\Omega_R} \left[\varrho\,(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_x\mathbf{V})\cdot(\mathbf{V}-\mathbf{u}) - \frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathrm{div}_x\mathbf{V} + \nu\mathbb{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{V}\right] \mathrm{d}x\,\mathrm{d}t,$$

where, in view of (9.55),

$$\left|\int_{\Omega_R} \varrho(\mathbf{u} - \mathbf{V})\cdot\nabla_x\mathbf{V}\cdot(\mathbf{V} - \mathbf{u})\,\mathrm{d}x\right| \le \int_{\Omega_R} \varrho|\mathbf{u} - (1-\chi_R)\mathbf{U}|^2|\nabla_x\mathbf{V}|\,\mathrm{d}x \tag{9.57}$$

$$\le c(D)\int_{\Omega_R} \varrho|\mathbf{u} - (1-\chi_R)\mathbf{U}|^2\,\mathrm{d}x \le c(D)\mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U}\right).$$

Next, we compute

$$\left|\int_{\Omega_R} \nu\mathbb{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{V}\,\mathrm{d}x\right| \le \left|\int_\Omega \nu\mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x{}^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) : \nabla_x\mathbf{V}\,\mathrm{d}x\right| \tag{9.58}$$

$$+ \left|\int_{\Omega_R} \nu\eta(\vartheta)\mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{V}\,\mathrm{d}x\right|$$

$$\le \delta\int_{\Omega_R} \frac{\nu}{\vartheta}\mathbb{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,\mathrm{d}x + c(\delta)\nu\int_{\Omega_R} \vartheta(\mu(\vartheta) + \eta(\vartheta))|\nabla_x\mathbf{V}|^2\,\mathrm{d}x$$

for any $\delta > 0$, where the former integral on the right-hand side my be absorbed by the left-hand side of (9.56) for $\delta = \delta(\overline{\vartheta}) > 0$, while in accordance with hypothesis (9.38) and the bounds established in (9.48), (9.55),

$$\int_{\Omega_R} \vartheta(\mu(\vartheta) + \eta(\vartheta))|\nabla_x \mathbf{V}|^2 \, \mathrm{d}x \tag{9.59}$$

$$= \int_{\Omega_R} [\vartheta(\mu(\vartheta) + \eta(\vartheta))]_{\mathrm{ess}} |\nabla_x \mathbf{V}|^2 \, \mathrm{d}x + \int_{\Omega_R} [\vartheta(\mu(\vartheta) + \eta(\vartheta))]_{\mathrm{res}} |\nabla_x \mathbf{V}|^2 \, \mathrm{d}x$$

$$\leq c(\overline{\vartheta}) \|\mathbf{U}\|^2_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} + c \|[\vartheta]^2_{\mathrm{res}}\|_{L^2(\Omega_R)} \|\mathbf{U}\|^2_{W^{1,4}(\mathbb{R}^3;\mathbb{R}^3)} \leq c(D).$$

In view of (9.57), (9.58), inequality (9.56) reduces to

$$\left[\mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U} \right) \right]_{t=0}^{t=\tau} \tag{9.60}$$

$$\leq c(D) \int_0^\tau \left[\nu + \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U} \right) \right] \mathrm{d}t$$

$$+ \int_0^\tau \int_{\Omega_R} \left[\varrho \left(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V} \right) \cdot (\mathbf{V} - \mathbf{u}) - \frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathrm{div}_x \mathbf{V} \right] \mathrm{d}x \, \mathrm{d}t.$$

Next, we write

$$\int_{\Omega_R} \frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathrm{div}_x \mathbf{V} \, \mathrm{d}x = \int_{\Omega_R} \frac{1}{\varepsilon^2} \left(p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right) \mathrm{div}_x \mathbf{V} \, \mathrm{d}x$$

$$= \int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\mathrm{ess}} \mathrm{div}_x \mathbf{V} \, \mathrm{d}x + \int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\mathrm{res}} \mathrm{div}_x \mathbf{V} \, \mathrm{d}x,$$

where, by virtue of hypotheses (9.30) and the coercivity properties of $H_{\mathcal{T}}$ established in Chap. 5, Lemma 5.1,

$$\left| \int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\mathrm{res}} \mathrm{div}_x \mathbf{V} \, \mathrm{d}x \right| \leq c(D) \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U} \right). \tag{9.61}$$

Moreover,

$$\int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm ess} {\rm div}_x \mathbf{V} \, {\rm d}x$$

$$= \int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \overline{\varrho}) - \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} (\vartheta - \overline{\vartheta}) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm ess} {\rm div}_x \mathbf{V} \, {\rm d}x$$

$$- \int_{\Omega_R} \left[\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \frac{\varrho - \overline{\varrho}}{\varepsilon} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \frac{\vartheta - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \frac{\mathbf{U} \cdot \nabla_x \chi_R}{\varepsilon} \, {\rm d}x,$$

where, similarly to (9.61),

$$\left| \int_{\Omega_R} \frac{1}{\varepsilon^2} \left[p(\varrho, \vartheta) - \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \overline{\varrho}) - \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} (\vartheta - \overline{\vartheta}) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm ess} {\rm div}_x \mathbf{V} \, {\rm d}x \right| \tag{9.62}$$

$$\leq c(D) \mathcal{E}_{\varepsilon, R} \left(\varrho, \vartheta, \mathbf{u} \, \middle| \overline{\varrho}, \overline{\vartheta}, (1 - \chi_R) \mathbf{U} \right).$$

Finally, in accordance with (9.55),

$$\left| \int_{\Omega_R} \left[\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \frac{\varrho - \overline{\varrho}}{\varepsilon} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \frac{\vartheta - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \frac{\mathbf{U} \cdot \nabla_x \chi_R}{\varepsilon} \, {\rm d}x \right| \tag{9.63}$$

$$\leq c(D) \mathcal{E}_{\varepsilon, R} \left(\varrho, \vartheta, \mathbf{u} \, \middle| \overline{\varrho}, \overline{\vartheta}, (1 - \chi_R) \mathbf{U} \right) + \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\mathbf{U} \cdot \nabla_x \chi_R|^2}{\varepsilon^2} \, {\rm d}x$$

$$\leq c(D) \left[\mathcal{E}_{\varepsilon, R} \left(\varrho, \vartheta, \mathbf{u} \, \middle| \overline{\varrho}, \overline{\vartheta}, (1 - \chi_R) \mathbf{U} \right) + \frac{1}{\varepsilon^2 R^2} \right].$$

Thus (9.60) gives rise to

$$\left[\mathcal{E}_{\varepsilon, R} \left(\varrho, \vartheta, \mathbf{u} \, \middle| \overline{\varrho}, \overline{\vartheta}, (1 - \chi_R) \mathbf{U} \right) \right]_{t=0}^{t=\tau} \tag{9.64}$$

$$\leq c(D) \int_0^\tau \left[\nu + \frac{1}{\varepsilon^2 R^2} + \mathcal{E}_{\varepsilon, R} \left(\varrho, \vartheta, \mathbf{u} \, \middle| \overline{\varrho}, \overline{\vartheta}, (1 - \chi_R) \mathbf{U} \right) \right] {\rm d}t$$

$$+ \int_0^\tau \int_{\Omega_R} \varrho \left(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V} \right) \cdot (\mathbf{V} - \mathbf{u}) \, {\rm d}x \, {\rm d}t.$$

The final step is to handle the integral

$$\int_0^\tau \int_{\Omega_R} \varrho \left(\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_x \mathbf{V}\right) \cdot (\mathbf{V} - \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}t = \sum_{j=1}^{5} \int_0^\tau I_j \mathrm{d}t,$$

where

$$I_1 = \int_{\Omega_R} \varrho \Big(\mathbf{V} - \mathbf{u}\Big) \cdot (\chi_R \mathbf{U}) \cdot \nabla_x \mathbf{U} \,\mathrm{d}x,$$

$$I_2 = \int_{\Omega_R} \varrho \Big(\mathbf{V} - \mathbf{u}\Big) \cdot \mathbf{U} \cdot \nabla_x \left(\chi_R \mathbf{U}\right) \,\mathrm{d}x,$$

$$I_3 = \int_{\Omega_R} \varrho \Big(\mathbf{u} - \mathbf{V}\Big) \cdot (\chi_R \mathbf{U}) \cdot \nabla_x \left(\chi_R \mathbf{U}\right) \,\mathrm{d}x,$$

$$I_4 = \int_{\Omega_R} \varrho \Big(\mathbf{V} - \mathbf{u}\Big) \cdot \partial_t \left(\chi_R \mathbf{U}\right) \,\mathrm{d}x,$$

$$I_5 = \int_{\Omega_R} \varrho \Big(\mathbf{V} - \mathbf{u}\Big) \cdot \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}\Big) \,\mathrm{d}x \,\mathrm{d}t = \int_{\Omega_R} \varrho \Big(\mathbf{u} - \mathbf{V}\Big) \cdot \nabla_x \Pi \,\mathrm{d}x.$$

First, writing

$$\varrho(\mathbf{u} - \mathbf{V}) = [\varrho]_{\mathrm{ess}} (\mathbf{u} - \mathbf{V}) + [\varrho]_{\mathrm{res}} (\mathbf{u} - \mathbf{V})),$$

we observe that, by virtue of (9.41),

$$\operatorname*{ess\ sup}_{t \in (0,\tau)} \| [\varrho]_{\mathrm{ess}} (\mathbf{u} - \mathbf{V})(t, \cdot) \|_{L^2(\Omega_R;\mathbb{R}^3)} \leq c(D). \tag{9.65}$$

Similarly, by virtue of (9.41), (9.47),

$$\operatorname*{ess\ sup}_{t \in (0,\tau)} \| [\varrho]_{\mathrm{res}} (\mathbf{u} - \mathbf{V})(t, \cdot) \|_{L^{5/4}(\Omega_R;\mathbb{R}^3)} \leq c(D). \tag{9.66}$$

As a consequence of (9.55) we may infer that

$$\left| \sum_{j=1}^{4} I_j \right| \leq c(D) \left(R^{-1} + R^{-8/5}\right) \leq c(D) R^{-1} \text{ provided } R \geq 1. \tag{9.67}$$

To conclude, we have

$$\int_0^\tau I_5 \mathrm{d}t = \int_0^\tau \int_{\Omega_R} \varrho \Big(\mathbf{u} - \mathbf{V} \Big) \cdot \nabla_x \Pi \ \mathrm{d}x = \int_{\Omega_R} \varrho \mathbf{u} \cdot \nabla_x \Pi \ \mathrm{d}x - \int_{\Omega_R} \varrho \mathbf{V} \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t,$$

where, in accordance with the weak formulation of the equation of continuity (2.2),

$$\int_0^\tau \int_{\Omega_R} \varrho \mathbf{u} \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t = -\varepsilon \int_0^\tau \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \partial_t \Pi \ \mathrm{d}x \ \mathrm{d}t + \varepsilon \left[\int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \Pi \ \mathrm{d}x \right]_{t=0}^{t=\tau}, \tag{9.68}$$

and, by virtue of the estimates (9.42), (9.47),

$$\left| \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \partial_t \Pi \ \mathrm{d}x \right| \leq c_1(D) \left(\|\partial_t \Pi\|_{L^2(\mathbb{R}^3)} + \|\partial_t \Pi\|_{L^\infty(\mathbb{R}^3)} \right) \leq c_2(D), \tag{9.69}$$

and, by the same token,

$$\left| \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \Pi \ \mathrm{d}x \right| \leq c_1(D) \left(\|\Pi\|_{L^2(\mathbb{R}^3)} + \|\Pi\|_{L^\infty(\mathbb{R}^3)} \right) \leq c_2(D). \tag{9.70}$$

Remark Note that the pressure in the Euler system can be "computed", namely

$$\Pi = -\Delta_x^{-1} \mathrm{div}_x \mathrm{div}_x (\mathbf{U} \times \mathbf{U}),$$

in particular,

$$\sup_{t \in [0,\tau]} \|\Pi(t, \cdot)\|_{L^p(\mathbb{R}^3)} \leq c(p, \tau, D) \text{ for any } 1 < p < \infty, \ \tau < T_{\max},$$

see Sect. 11.20 in Appendix.

Finally, the last integral to handle reads

$$\int_{\Omega_R} \varrho \mathbf{V} \cdot \nabla_x \Pi \ \mathrm{d}x = \varepsilon \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \mathbf{V} \cdot \nabla_x \Pi \ \mathrm{d}x + \overline{\varrho} \int_{\Omega_R} \mathbf{V} \cdot \nabla_x \Pi \ \mathrm{d}x \tag{9.71}$$

$$= \varepsilon \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \mathbf{V} \cdot \nabla_x \Pi \ \mathrm{d}x - \overline{\varrho} \int_{\Omega_R} \nabla_x \chi_R \cdot \mathbf{U} \ \Pi \ \mathrm{d}x,$$

where the first integral can be estimates exactly as in (9.69), (9.70), while, by virtue of (9.55),

$$\left| \int_{\Omega_R} \nabla_x \chi_R \cdot \mathbf{U}\Pi \, \mathrm{d}x \right| \leq c_1(D) R^{-1} \|\Pi\|_{L^2(\mathbb{R}^3)} \leq c_2 R^{-1}.$$

Summing up the previous estimates and going back to (9.64) we may infer that

$$\left[\mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \middle| \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U} \right) \right]_{t=0}^{t=\tau} \tag{9.72}$$

$$\leq c(D) \int_0^\tau \left[\varepsilon + \nu + \frac{1}{R} + \frac{1}{\varepsilon^2 R^2} + \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \middle| \overline{\varrho}, \overline{\vartheta}, (1-\chi_R)\mathbf{U} \right) \right] \mathrm{d}t$$

whenever $\tau < T_{\max}$, where $T_{\max}$ is the life-span for the Euler system. Consequently, a straightforward application of Gronwall's lemma yields the following result.

■ VANISHING DIFFUSION LIMIT—WELL PREPARED INITIAL DATA:

Theorem 9.1 *Let $\{\Omega_R\}_{R\geq 1}$ be a family of uniformly $C^{2,\nu}$ simply connected bounded domains in $\mathbb{R}^3$ satisfying (9.9), (9.10). Let the constitutive hypotheses (9.30)–(9.39) be satisfied.*

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (9.2)–(9.8), (9.11), (9.12) in $(0,T) \times \Omega_R$ starting from the initial data

$$\varrho(0,\cdot) \equiv \varrho_0 = \overline{\varrho} + \varepsilon \varrho_0^{(1)}, \ \vartheta(0,\cdot) \equiv \vartheta_0 = \overline{\vartheta} + \varepsilon \vartheta_0^{(1)}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0,$$

where

$$\left\{ \begin{array}{c} 0 < D^{-1} < \overline{\varrho}, \overline{\vartheta} < D, \\ \\ \|\varrho_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\vartheta_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\mathbf{u}_0\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} < D. \end{array} \right\}$$

Let $\mathbf{U}$ be a (strong) solution to the Euler system (9.15), (9.16) in $\mathbb{R}^3 \times (0, T_{\max})$ starting from the initial data

$$\mathbf{U}(0,\cdot) = \mathbf{w}_0 = \mathbf{H}[\mathbf{v}_0],$$

where

$$\mathbf{v}_0 \in C_c^m(\mathbb{R}^3), \ \mathrm{supp}[\mathbf{v}_0] \subset B(0,D), \ \|\mathbf{v}_0\|_{C^m(\mathbb{R}^3)} \leq D, \ m > 4.$$

Then for any compact $K \subset \mathbb{R}^3$ and any $T \in (0, T_{\max})$, there are $c_1 = c(T, D)$, $c_2(D)$ such that

$$\int_K \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2}\left(H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \right](\tau, \cdot)\,\mathrm{d}x \tag{9.73}$$

$$\leq c_1(T, D)\left(\varepsilon + \nu + \frac{1}{R} + \frac{1}{\varepsilon^2 R^2} \right)$$

$$+ c_2(D)\left(\left\| \frac{\varrho_0 - \overline{\varrho}}{\varepsilon} \right\|^2_{L^2(\Omega_R)} + \left\| \frac{\vartheta_0 - \overline{\vartheta}}{\varepsilon} \right\|^2_{L^2(\Omega_R)} + \|\mathbf{u}_0 - \mathbf{w}_0\|^2_{L^2(\Omega_R;\mathbb{R}^3)} \right)$$

for a.a. $\tau \in [0, T)$ provided $R = R(K)$ is large enough.

Remark Theorem 9.1 yields uniform in time convergence of $\mathbf{u}$ towards the solutions of the Euler system and asymptotic spatial homogeneity in ϱ and ϑ provided the right-hand side of (9.73) tends to zero, in particular, $\varrho_0^{(1)}$, $\vartheta_0^{(1)}$ must be small and the initial velocity close to a solenoidal (divergenceless) function $\mathbf{v}_0$. Such a situation corresponds to the so-called *well-prepared initial data*.

9.5 Ill-Prepared Initial Data

The stability result established in Theorem 9.1 is quite restrictive with respect to the initial data that must be close to the expected limit solution. This can be improved by choosing a more refined ansatz of the test functions $[r, \mathcal{T}, \mathbf{V}]$ in the relative energy inequality. The basic idea used several times in this book, is to augment the basic state $[\overline{\varrho}, \overline{\vartheta}, \mathbf{U}]$ by the oscillatory component produced by acoustic waves.

9.5.1 Acoustic Equation

The equation governing the propagation of acoustic waves is represented by the homogeneous part of acoustic system (8.141), (8.142), specifically we get

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■ ACOUSTIC WAVE EQUATION:

$$\varepsilon\partial_t Z + \Delta\Psi = 0,\ \varepsilon\partial_t\nabla_x\Psi + \omega\nabla_x Z = 0 \text{ in } (0, T) \times \mathbb{R}^3, \tag{9.74}$$

with the wave speed $\sqrt{\omega}/\varepsilon$,

$$\omega = p_\varrho(\overline{\varrho}, \overline{\vartheta}) + \frac{|p_\vartheta(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 s_\vartheta(\overline{\varrho}, \overline{\vartheta})} > 0,$$

and the initial data

$$Z(0,\cdot) = Z_0,\ \nabla_x \Psi(0,\cdot) = \nabla_x \Psi_0. \tag{9.75}$$

The potential Ψ_0 was introduced in (9.51) as the gradient component of a compactly supported vector field $\mathbf{v}_0$. As showed in (8.125), solutions of (9.74) admit the finite speed of propagation $\sqrt{\omega}/\varepsilon$. In particular, for the initial data

$$Z_0,\ \nabla_x \Psi_0 = \mathbf{H}^\perp[\mathbf{v}_0],\ \mathrm{supp}[\mathbf{v}_0], \mathrm{supp}[Z_0]\ \subset B(0,D), \tag{9.76}$$

the solution of (9.74), (9.75) satisfies

$$\nabla_x \Psi(t,x) = \nabla_x \Psi_0(x),\ \Delta \Psi(t,x) = Z(t,x) = 0 \text{ whenever } t \geq 0,\ |x| > D + t\frac{\sqrt{\omega}}{\varepsilon}. \tag{9.77}$$

To facilitate future considerations, it is convenient that the acoustic waves may not reach the boundary $\partial \Omega_R$ of the physical space within the time lap $(0,T)$. Accordingly, we suppose that

$$R > D + T\frac{\sqrt{\omega}}{\varepsilon}. \tag{9.78}$$

It is easy to see that solutions of acoustic system (9.74), (9.75) with spatially concentrated initial data conserve the total energy,

$$\int_{\mathbb{R}^3} \left[\omega |Z|^2 + |\nabla_x \Psi|^2 \right] (\tau,\cdot)\ \mathrm{d}x = \int_{\mathbb{R}^3} \left[\omega |Z_0|^2 + |\nabla_x \Psi_0|^2 \right] (\tau,\cdot)\ \mathrm{d}x \text{ for any } \tau \geq 0.$$

Moreover, differentiating (9.74) with respect to the x-variable, we deduce higher order energy balance

$$\omega \|Z(\tau,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3)} + \|\nabla_x \Psi(\tau,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3,\mathbb{R}^{3\times 3})} =\neq \tag{9.79}$$

$$\omega \|Z_0\|^2_{W^{k,2}(\mathbb{R}^3)} + \|\nabla_x \Psi_0\|^2_{W^{k,2}(\mathbb{R}^3,\mathbb{R}^{3\times 3})},\ k = 0,1,2,\ldots,\ \tau \geq 0.$$

Similarly to their counterpart investigated in Chap. 8, solutions of the acoustic equation considered on the unbounded physical space $\mathbb{R}^3$ enjoy certain dispersive decay properties that are crucial for future analysis. Here, we report the celebrated *Strichartz estimates*

$$\|Z(\tau,\cdot)\|_{L^q(\mathbb{R}^3)} + \|\nabla_x \Psi(\tau,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \tag{9.80}$$

$$\leq c(p,q)\left(1+\frac{\tau}{\varepsilon}\right)^{\frac{1}{q}-\frac{1}{p}}\left[\|Z_0\|_{W^{4,p}(\mathbb{R}^3)} + \|\nabla_x \Psi_0\|_{W^{3,p}(\mathbb{R}^3;\mathbb{R}^3)}\right]$$

for

$$\frac{1}{p}+\frac{1}{q}=1,\ 1<p\leq 2,$$

see Theorem 11.13 in Appendix Similarly to (9.80) we may differentiate the equations to obtain higher order version of (9.80), namely

$$\|Z(\tau,\cdot)\|_{W^{k,q}(\mathbb{R}^3)} + \|\nabla_x \Psi(\tau,\cdot)\|_{W^{k,q}(\mathbb{R}^3;\mathbb{R}^3)} \tag{9.81}$$

$$\leq c(p,q)\left(1+\frac{\tau}{\varepsilon}\right)^{\frac{1}{q}-\frac{1}{p}}\left[\|Z_0\|_{W^{4+k,p}(\mathbb{R}^3)} + \|\nabla_x \Psi_0\|_{W^{3+k,p}(\mathbb{R}^3)}\right],\ k=0,1,\ldots$$

for

$$\frac{1}{p}+\frac{1}{q}=1,\ 1<p\leq 2.$$

Note that, in accordance with hypothesis (9.52), the right-hand side of (9.81) remains bounded by a constant $c = c(D)$ at least for $k = 0, 1$.

9.5.2 Transport Equation

For a given solution $\mathbf{U}$ of the Euler system (9.15), (9.16), we consider the transport equation

$$\partial_t P + \mathbf{U}\cdot\nabla_x P = 0,\ \Theta(0,\cdot) = P_0 \text{ in } (0,T_{\max})\times\mathbb{R}^3. \tag{9.82}$$

As $\mathbf{U}$ is regular, problem (9.82) admits a unique solution for any given initial datum P_0 that may be computed by the method of characteristics, see Sect. 11.20 in Appendix. More precisely, solutions of (9.82) enjoy the same regularity as those of the Euler system,

$$\|P(\tau,\cdot)\|_{W^{k,2}(\mathbb{R}^3)} + \|\partial_t P(\tau,\cdot)\|_{W^{k-1,2}(\mathbb{R}^3)} \leq c(\tau,D) \tag{9.83}$$

as soon as

$$\|P_0\|_{W^{k,2}(\mathbb{R}^3)} \leq c(D),\ k \geq 3.$$

Moreover, the solution P remains compactly supported for any positive time as long as P_0 has compact support.

9.5.3 Stability via the Relative Energy Inequality

We consider a trio of test functions

$$\mathbf{V} = (1 - \chi_R)\,(\mathbf{U} + \nabla_x \Psi)\,,\ r = \overline{\varrho} + \varepsilon\Lambda,\ \mathcal{T} = \overline{\vartheta} + \varepsilon\Theta,$$

where $\mathbf{U}$ is the solution of the Euler system (9.15), (9.16) in $(0, T_{\max}) \times \mathbb{R}^3$,

$$\mathbf{U}(0, \cdot) = \mathbf{H}[\mathbf{v}_0],$$

and Λ and Θ are uniquely determined as the unique solution of the system

$$\frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda + \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta = Z, \tag{9.84}$$

$$\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda + \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta = P, \tag{9.85}$$

where $[Z, \nabla_x \Psi]$ is the solution of the acoustic system (9.74), with the initial data

$$Z_0 = \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda_0 + \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta_0,\ \nabla_x \Psi_0 = \mathbf{H}^{\perp}[\mathbf{v}_0], \tag{9.86}$$

and P solves the transport equation (9.82), with the initial data

$$P_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda_0 + \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta_0 = P_0. \tag{9.87}$$

Similarly to $\mathbf{v}_0$, the functions Λ_0, Θ_0 belong to the class

$$\Lambda_0,\ \Theta_0 \in C^m_c(\mathbb{R}^3),\ \|\Lambda_0\|_{C^m(\mathbb{R}^3)} + \|\Theta_0\|_{C^m(\mathbb{R}^3)} \leq D,\ \mathrm{supp}[\Lambda_0],\ \mathrm{supp}[\Theta_0] \subset B(0, D). \tag{9.88}$$

With this ansatz, the relative energy inequality (9.40) reads

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\Big|\overline{\varrho}+\varepsilon\Lambda,\overline{\vartheta}+\varepsilon\Theta,(1-\chi_R)(\mathbf{U}+\nabla_x\Psi)\right)\right]_{t=0}^{t=\tau} \tag{9.89}$$

$$+\int_0^\tau\int_{\Omega_R}\frac{\overline{\vartheta}+\varepsilon\Theta}{\vartheta}\left(\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{d}{\varepsilon^2}\frac{\mathbf{q}(\varrho,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq\int_0^\tau\int_{\Omega_R}\varrho(\mathbf{u}-\mathbf{V})\cdot\nabla_x\mathbf{V}\cdot(\mathbf{V}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\varrho\left(s(\varrho,\vartheta)-s(r,\mathcal{T})\right)(\mathbf{V}-\mathbf{u})\cdot\nabla_x\Theta\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_{\Omega_R}\left[\varrho\left(\partial_t\mathbf{V}+\mathbf{V}\cdot\nabla_x\mathbf{V}\right)\cdot(\mathbf{V}-\mathbf{u})-\frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}+\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{V}\right]\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\left[\varrho\left(s(\varrho,\vartheta)-s(r,\mathcal{T})\right)\partial_t\Theta+\varrho\left(s(\varrho,\vartheta)-s(r,\mathcal{T})\right)\mathbf{V}\cdot\nabla_x\Theta\right.$$

$$\left.+d\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\right]\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\left[\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\mathcal{T})-\frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\mathcal{T})\right]\,\mathrm{d}x\,\mathrm{d}t$$

Similarly to the preceding part, our goal is to "absorb" all terms on the right-hand side by means of a Gronwall type argument.

Step 1 To begin, we observe that the integrals

$$\int_0^\tau\int_{\Omega_R}\varrho(\mathbf{u}-\mathbf{V})\cdot\nabla_x\mathbf{V}\cdot(\mathbf{V}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^\tau\int_{\Omega_R}\nu\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{V}\,\mathrm{d}x\,\mathrm{d}t$$

can be handled exactly as in (9.57), (9.58).

Moreover,

$$\frac{d}{\varepsilon}\left|\int_{\Omega_R}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\,\mathrm{d}x\right|=\frac{d}{\varepsilon}\left|\int_{\Omega_R}\frac{\kappa(\vartheta)}{\vartheta}\nabla_x\vartheta\cdot\nabla_x\Theta\,\mathrm{d}x\right|$$

$$\leq\delta\frac{d}{\varepsilon^2}\int_{\Omega_R}\frac{\kappa(\vartheta)}{\vartheta^2}|\nabla_x\vartheta|^2\,\mathrm{d}x+c(\delta)d\int_{\Omega_R}\kappa(\vartheta)|\nabla_x\Theta|^2\,\mathrm{d}x$$

for any $\delta > 0$; whence the first integral can be absorbed by the left-hand side of (9.89).

Finally, we have

$$\int_{\Omega_R} \kappa(\vartheta)|\nabla_x\Theta|^2 \, \mathrm{d}x = \int_{\Omega_R} [\kappa(\vartheta)]_{\mathrm{ess}}|\nabla_x\Theta|^2 \, \mathrm{d}x + \int_{\Omega_R} [\kappa(\vartheta)]_{\mathrm{res}}|\nabla_x\Theta|^2 \, \mathrm{d}x$$

$$\leq c_1(D) \left[\|\nabla_x\Theta\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\nabla_x\Theta\|^2_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \right] \leq c_2(D).$$

Indeed the function Θ is a linear combination of P and Z; where P is compactly supported and Z admits the energy bound (9.79).

Thus (9.89) reduces to

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho} + \varepsilon\Lambda, \overline{\vartheta} + \varepsilon\Theta, (1-\chi_R)(\mathbf{U} + \nabla_x\Psi)\right)\right]_{t=0}^{t=\tau} \tag{9.90}$$

$$\leq c(D) \int_0^\tau \left[\nu + d + \mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \mathcal{T}, \mathbf{V}\right)\right] \mathrm{d}t$$

$$+\frac{1}{\varepsilon}\int_0^\tau \int_{\Omega_R} \varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)(\mathbf{V}-\mathbf{u})\cdot\nabla_x\Theta \, \mathrm{d}x\, \mathrm{d}t$$

$$+\int_0^\tau \int_{\Omega_R} \left[\varrho\left(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_x\mathbf{V}\right)\cdot(\mathbf{V}-\mathbf{u}) - \frac{1}{\varepsilon^2} p(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\right] \mathrm{d}x\, \mathrm{d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau \int_{\Omega_R} \left[\varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)\partial_t\Theta + \varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)\mathbf{V}\cdot\nabla_x\Theta\right] \mathrm{d}x\, \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau \int_{\Omega_R} \left[\left(1 - \frac{\varrho}{r}\right)\partial_t p(r,\mathcal{T}) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\mathcal{T})\right] \mathrm{d}x\, \mathrm{d}t$$

Step 2 The next observation is that the integral

$$\int_0^\tau \int_{\Omega_R} \left[\varrho\left(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_x\mathbf{V}\right)\cdot(\mathbf{V}-\mathbf{u})\right.$$

can be handled in a similar way as its counterpart in the preceding section. Indeed we have

$$\int_0^\tau \int_{\Omega_R} \varrho\left(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_x\mathbf{V}\right)\cdot(\mathbf{V}-\mathbf{u}) \, \mathrm{d}x\, \mathrm{d}t = \sum_{j=1}^{9} \int_0^\tau I_j \mathrm{d}t,$$

where

$$I_1 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot (\chi_R(\mathbf{U} + \nabla_x \psi)) \cdot \nabla_x \mathbf{U} \, \mathrm{d}x,$$

$$I_2 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \mathbf{U} \cdot \nabla_x \left(\chi_R(\mathbf{U} + \nabla_x \Psi)\right) \, \mathrm{d}x,$$

$$I_3 = \int_{\Omega_R} \varrho\Big(\mathbf{u} - \mathbf{V}\Big) \cdot (\chi_R \mathbf{U}) \cdot \nabla_x \left(\chi_R(\mathbf{U} + \nabla_x \Psi)\right) \, \mathrm{d}x,$$

$$I_4 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \partial_t \left(\chi_R(\mathbf{U} + \nabla_x \Psi)\right) \, \mathrm{d}x,$$

$$I_5 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}\Big) \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_R} \varrho\Big(\mathbf{u} - \mathbf{V}\Big) \cdot \nabla_x \Pi \, \mathrm{d}x,$$

$$I_6 = \frac{1}{2} \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x,$$

$$I_7 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \nabla_x \mathbf{U} \cdot \nabla_x \Psi \, \mathrm{d}x,$$

$$I_8 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \mathbf{U} \cdot {\nabla_x}^2 \Psi \, \mathrm{d}x,$$

and

$$I_9 = \int_{\Omega_R} \varrho\Big(\mathbf{V} - \mathbf{u}\Big) \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x$$

Now, as a consequence (9.78), the function Ψ coincides with Ψ_0 on the support of χ_R, in particular, we may apply the bounds (9.55) in the same way as when deriving (9.67) to obtain

$$\left| \sum_{j=1}^{4} I_j \right| \leq c(D) R^{-1}. \tag{9.91}$$

Moreover, I_5 is exactly the same as in the preceding section, therefore estimates (9.69)–(9.71) remain valid yielding

$$\left| \int_0^\tau I_5 \, \mathrm{d}t \right| \leq c(D) \left(\varepsilon + \frac{1}{R} \right). \tag{9.92}$$

As for I_6 it can be treated in a similar way. First, we write

$$\int_0^\tau I_6 \, \mathrm{d}t = \frac{1}{2} \int_{\Omega_R} \varrho \mathbf{u} \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_{\Omega_R} \varrho \mathbf{V} \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x \, \mathrm{d}t,$$

where the first integral on the right-hand side can be bounded exactly as in (9.69), (9.70) as, in view of the energy estimates (9.79), $|\nabla_x \Psi|^2$ enjoys the same integrability properties as Π.

The second integral reads

$$\int_{\Omega_R} \varrho \mathbf{V} \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x = \varepsilon \int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon} \mathbf{V} \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x + \overline{\varrho} \int_{\Omega_R} \mathbf{V} \cdot \nabla_x |\nabla_x \Psi|^2,$$

where the first term is controlled exactly as in (9.71), while

$$\int_{\Omega_R} \mathbf{V} \cdot \nabla_x |\nabla_x \Psi|^2 \, \mathrm{d}x = \int_{\Omega_R} \mathrm{div}_x (\chi_R (\mathbf{U} + \nabla_x \Psi)) |\nabla_x \Psi|^2 \, \mathrm{d}x - \int_{\Omega_R} \Delta \Psi |\nabla_x \Psi|^2 \, \mathrm{d}x,$$

where again the first term is handled as in (9.71). Now, we use the energy and decay estimates for the acoustic potential (9.79), (9.81) to conclude that

$$\int_{\Omega_R} \Delta \Psi |\nabla_x \Psi|^2 \, \mathrm{d}x$$

$$\leq c(q) \| \Delta \Psi \|_{L^q(\Omega)} \| \nabla_x \Psi \|_{L^p(\Omega)} \| \nabla_x \Psi \|_{L^2(\Omega)}, \ \leq c(q, D) \left(1 + \frac{\tau}{\varepsilon} \right)^{\left(\frac{2}{q} - 1 \right)}$$

where

$$\frac{1}{q} + \frac{1}{p} + \frac{1}{2} = 1.$$

Thus taking q close to ∞, $p > 2$ close to 2, we may infer that

$$\left| \int_{\Omega_R} \Delta \Psi |\nabla_x \Psi|^2 \, \mathrm{d}x \right| \leq c(\alpha, D, T) \varepsilon^\alpha \text{ for any } 0 \leq \alpha < 1.$$

In view of (9.65), (9.66), the integrals I_7, I_8 can be estimated in a similar fashion using again the decay estimates (9.81).

Finally, we use (9.77), (9.78) to rewrite

$$I_9 = \int_{\Omega_R} \varrho \Big(\mathbf{V} - \mathbf{u} \Big) \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x = \int_{\Omega_R} \varrho \Big(\mathbf{U} + \nabla_x \Psi - \mathbf{u} \Big) \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x$$

$$= \overline{\varrho} \int_{\mathbb{R}^3} \nabla_x \Psi \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x + \int_{\Omega_R} (\varrho - \overline{\varrho}) \nabla_x \Psi \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x$$

$$+ \int_{\Omega_R} (\varrho - \overline{\varrho}) \mathbf{U} \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x - \int_{\Omega_R} \varrho \mathbf{u} \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x,$$

where, furthermore,

$$\left|\int_{\Omega_R} (\varrho - \overline{\varrho})\nabla_x \Psi \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x\right| + \left|\int_{\Omega_R} (\varrho - \overline{\varrho})\mathbf{U} \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x\right|$$

$$\leq c(D)\left|\int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon}\nabla_x \Psi \cdot \nabla_x Z \, \mathrm{d}x\right| + c(D)\left|\int_{\Omega_R} \frac{\varrho - \overline{\varrho}}{\varepsilon}\mathbf{U} \cdot \nabla_x Z \, \mathrm{d}x\right| \leq c(\alpha, D)\varepsilon^{\alpha}, \ 0 \leq \alpha < 1,$$

where we have used the dispersive estimates (9.81).

Summing up the previous observations we can rewrite (9.90) as

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho} + \varepsilon\Lambda, \overline{\vartheta} + \varepsilon\Theta, (1 - \chi_R)(\mathbf{U} + \nabla_x \Psi)\right)\right]_{t=0}^{t=\tau} - \left[\frac{\overline{\varrho}}{2}\int_{\mathbb{R}^3} |\nabla_x \Psi|^2 \, \mathrm{d}x\right]_{t=0}^{t=\tau} \tag{9.93}$$

$$\leq c(D)\int_0^{\tau}\left[\nu + d + \varepsilon + c(\alpha)\varepsilon^{\alpha} + \frac{1}{R} + \mathcal{E}_{\varepsilon,R}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \mathcal{T}, \mathbf{V}\right)\right]\mathrm{d}t$$

$$+\frac{1}{\varepsilon}\int_0^{\tau}\int_{\Omega_R} \varrho\left(s(\varrho, \vartheta) - s(r, \mathcal{T})\right)(\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^{\tau}\int_{\Omega_R} p(\varrho, \vartheta)\mathrm{div}_x \mathbf{V} \, \mathrm{d}x \, \mathrm{d}t - \int_0^{\tau}\int_{\Omega_R} \varrho\mathbf{u} \cdot \partial_t \nabla_x \Psi \, \mathrm{d}x \, \mathrm{d}t$$

$$-\frac{1}{\varepsilon}\int_0^{\tau}\int_{\Omega_R}\left[\varrho\left(s(\varrho, \vartheta) - s(r, \mathcal{T})\right)\partial_t \Theta + \varrho\left(s(\varrho, \vartheta) - s(r, \mathcal{T})\right)\mathbf{V} \cdot \nabla_x \Theta\right]\mathrm{d}x \, \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^{\tau}\int_{\Omega_R}\left[\left(1 - \frac{\varrho}{r}\right)\partial_t p(r, \mathcal{T}) - \frac{\varrho}{r}\mathbf{u} \cdot \nabla_x p(r, \mathcal{T})\right]\mathrm{d}x \, \mathrm{d}t, \ 0 \leq \alpha < 1.$$

Step 3 We write

$$\frac{1}{\varepsilon}\int_{\Omega_R} \varrho\left(s(\varrho, \vartheta) - s(r, \mathcal{T})\right)(\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x$$

$$= \frac{1}{\varepsilon}\int_{\Omega_R} \varrho\left[s(\varrho, \vartheta) - s(r, \mathcal{T})\right]_{\mathrm{ess}}(\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x$$

$$+\frac{1}{\varepsilon}\int_{\Omega_R} \varrho\left[s(\varrho, \vartheta) - s(r, \mathcal{T})\right]_{\mathrm{res}}(\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x,$$

where

$$\left| \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(\varrho, \vartheta) - s(r, \mathcal{T}) \right]_{\rm ess} (\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \ {\rm d}x \right|$$

$$\leq c_1(D) \|\nabla_x \Theta\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \int_{\Omega_R} \left[\varrho |\mathbf{V} - \mathbf{u}|^2 + \frac{1}{\varepsilon^2} \left(|[\varrho - r]_{\rm ess}|^2 + |[\vartheta - \mathcal{T}]_{\rm ess}|^2 \right) \right] \ {\rm d}x$$

$$\leq c_2(D) \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \ \Big| r, \mathcal{T}, \mathbf{V} \right).$$

Next,

$$\frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(\varrho, \vartheta) - s(r, \mathcal{T}) \right]_{\rm res} (\mathbf{V} - \mathbf{u}) \cdot \nabla_x \Theta \ {\rm d}x,$$

$$= \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(\varrho, \vartheta) - s(r, \mathcal{T}) \right]_{\rm res} \mathbf{V} \cdot \nabla_x \Theta \ {\rm d}x + \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(r, \mathcal{T}) - s(\varrho, \vartheta) \right]_{\rm res} \mathbf{u} \cdot \nabla_x \Theta \ {\rm d}x,$$

where, by virtue of (9.45), (9.47),

$$\left| \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(\varrho, \vartheta) - s(r, \mathcal{T}) \right]_{\rm res} \mathbf{V} \cdot \nabla_x \Theta \ {\rm d}x \right| \leq \varepsilon c(D).$$

Now, using hypothesis (9.33), or, specifically (9.34), we get

$$\varrho \left| [s(\varrho, \vartheta) - s(r, \mathcal{T})]_{\rm res} \right| \leq c \left[\varrho + \varrho |\log(\varrho)| + \varrho [\log \vartheta]^+ + \vartheta^3 \right]_{\rm res} \leq c [\varrho + \varrho^{1+\delta} + \vartheta^3]_{\rm res}$$

where $\delta > 0$ can be taken arbitrarily small.

Remark This is the only point when we effectively use hypothesis (9.33) (the Third law of thermodynamics).

Consequently, by virtue of the uniform bounds (9.47)–(9.49),

$$\left| \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(r, \mathcal{T}) - s(\varrho, \vartheta) \right]_{\rm res} \mathbf{u} \cdot \nabla_x \Theta \ {\rm d}x \right|$$

$$\leq \frac{c_1(D)}{\varepsilon \nu^{1/2}} \|\nabla_x \Theta\|_{L^\infty(\Omega_R)} \|\nu^{1/2} \mathbf{u}\|_{L^6(\Omega_R;\mathbb{R}^3)} \left\| [\varrho + \varrho^{1+\delta} + \vartheta^3]_{\rm res} \right\|_{L^{6/5}(\Omega_R)}$$

$$+ \frac{c_1(D)}{\varepsilon \nu^{1/2}} \varepsilon^{5/3} \|\nu^{1/2} \mathbf{u}\|_{W^{1,2}(\Omega_R;\mathbb{R}^3)}.$$

Thus we conclude that

$$\left| \frac{1}{\varepsilon} \int_{\Omega_R} \varrho \left[s(r, \mathcal{T}) - s(\varrho, \vartheta) \right]_{\rm res} \mathbf{u} \cdot \nabla_x \Theta \ {\rm d}x \right| \leq c(D) \frac{\varepsilon^{2/3}}{\nu^{1/2}}. \tag{9.94}$$

After this step, the inequality (9.93) gives rise to

$$\left[\mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \left| \overline{\varrho} + \varepsilon \Lambda, \overline{\vartheta} + \varepsilon \Theta, (1 - \chi_R)(\mathbf{U} + \nabla_x \Psi) \right. \right) \right]_{t=0}^{t=\tau} - \left[\frac{\overline{\varrho}}{2} \int_{\mathbb{R}^3} |\nabla_x \Psi|^2 \ {\rm d}x \right]_{t=0}^{t=\tau} \tag{9.95}$$

$$\leq c(D) \int_0^\tau \left[\nu + d + \varepsilon + c(\alpha)\varepsilon^\alpha + \frac{1}{R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} + \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \middle| \, r, \mathcal{T}, \mathbf{V} \right) \right] {\rm d}t$$

$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_R} p(\varrho, \vartheta) {\rm div}_x \mathbf{V} \ {\rm d}x \ {\rm d}t - \int_0^\tau \int_{\Omega_R} \varrho \mathbf{u} \cdot \partial_t \nabla_x \Psi \ {\rm d}x \ {\rm d}t$$

$$- \frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_R} \left[\varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \partial_t \Theta + \varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \mathbf{V} \cdot \nabla_x \Theta \right] {\rm d}x \ {\rm d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_R} \left[\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \mathcal{T}) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \mathcal{T}) \right] \ {\rm d}x \ {\rm d}t, \ \ 0 \leq \alpha < 1.$$

Step 4 The next step is to observe that we can replace $\mathbf{V}$ in the remaining three integrals on the right-hand side of (9.95) by $\mathbf{U} + \nabla_x \Psi$ committing an error of order $\frac{1}{\varepsilon R}$. Indeed we have

$$\frac{1}{\varepsilon^2} \int_{\Omega_R} p(\varrho, \vartheta) {\rm div}_x (\chi_R(\mathbf{U} + \nabla_x \Psi)) \ {\rm d}x = \frac{1}{\varepsilon^2} \int_{\Omega_R} \left(p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right) {\rm div}_x (\chi_R(\mathbf{U} + \nabla_x \Psi)) \ {\rm d}x$$

$$= \frac{1}{\varepsilon} \int_{\Omega_R} \frac{\left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm ess}}{\varepsilon} {\rm div}_x (\chi_R(\mathbf{U} + \nabla_x \Psi)) \ {\rm d}x$$

$$+ \int_{\Omega_R} \frac{\left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm res}}{\varepsilon^2} {\rm div}_x (\chi_R(\mathbf{U} + \nabla_x \Psi)) \ {\rm d}x,$$

where, by virtue of (9.42), (9.43), combined with (9.55),

$$\left| \frac{1}{\varepsilon} \int_{\Omega_R} \frac{\left[p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \right]_{\rm ess}}{\varepsilon} {\rm div}_x (\chi_R(\mathbf{U} + \nabla_x \Psi)) \ {\rm d}x \right| \leq c(D) \frac{1}{\varepsilon R},$$

and, using (9.46)–(9.48), and again (9.55),

$$\left| \int_{\Omega_R} \frac{\left[p(\varrho,\vartheta) - p(\overline{\varrho},\overline{\vartheta})\right]_{\rm res}}{\varepsilon^2} {\rm div}_x(\chi_R(\mathbf{U} + \nabla_x \Psi))\ {\rm d}x \right| \le \frac{c(D)}{R^2}.$$

As the integral

$$\frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_R} \varrho \left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right) \mathbf{V}\cdot\nabla_x\Theta\Big]\ {\rm d}x\ {\rm d}t$$

can be handled in a similar fashion, we are allowed to rewrite (9.95) in the form

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,\overline{\varrho}+\varepsilon\Lambda,\overline{\vartheta}+\varepsilon\Theta,(1-\chi_R)(\mathbf{U}+\nabla_x\Psi)\right)\right]_{t=0}^{t=\tau} - \left[\frac{\overline{\varrho}}{2}\int_{\mathbb{R}^3} |\nabla_x\Psi|^2\ {\rm d}x\right]_{t=0}^{t=\tau} \tag{9.96}$$

$$\le c(D)\int_0^\tau \left[\nu + d + \varepsilon + c(\alpha)\varepsilon^\alpha + \frac{1}{\varepsilon R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} + \mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)\right] {\rm d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\left(p(\overline{\varrho},\overline{\vartheta}) - p(\varrho,\vartheta)\right)\Delta\Psi\ {\rm d}x\ {\rm d}t - \int_0^\tau\int_{\Omega_R}\varrho\mathbf{u}\cdot\partial_t\nabla_x\Psi\ {\rm d}x\ {\rm d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\left[\varrho\left(s(\varrho,\vartheta)-s(r,\mathcal{T})\right)\partial_t\Theta + \varrho\left(s(\varrho,\vartheta)-s(r,\mathcal{T})\right)(\mathbf{U}+\nabla_x\Psi)\cdot\nabla_x\Theta\right] {\rm d}x\ {\rm d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\left[\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\mathcal{T}) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\mathcal{T})\right] {\rm d}x\ {\rm d}t,\ 0\le\alpha<1.$$

Step 5 We rewrite the integrals containing the pressure as

$$\frac{1}{\varepsilon^2}\int_{\Omega_R}\left[\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\mathcal{T}) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\mathcal{T})\right] {\rm d}x \tag{9.97}$$

$$= \int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho}\partial_t\Lambda + \frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta}\partial_t\Theta\right) {\rm d}x$$

$$-\frac{1}{\varepsilon}\int_{\Omega_R}\varrho\mathbf{u}\cdot\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho}\nabla_x\Lambda + \frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta}\nabla_x\Theta\right) {\rm d}x$$

Assume, for a moment, that we can replace in the above expression

$$\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho} \text{ by } \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}, \text{ and } \frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta} \text{ by } \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}.$$

Consequently, in accordance with (9.84),

$$
\begin{aligned}
&\int_{\Omega_R} \frac{r-\varrho}{\varepsilon}\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho}\partial_t\Lambda + \frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta}\partial_t\Theta\right)\,\mathrm{d}x \qquad (9.98)\\
&-\frac{1}{\varepsilon}\int_{\Omega_R}\varrho\mathbf{u}\cdot\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho}\nabla_x\Lambda + \frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta}\nabla_x\Theta\right)\,\mathrm{d}x\\
&\approx \int_{\Omega_R} \frac{r-\varrho}{\varepsilon}\left(\frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\partial_t\Lambda + \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\partial_t\Theta\right)\,\mathrm{d}x\\
&-\frac{1}{\varepsilon}\int_{\Omega_R}\varrho\mathbf{u}\cdot\left(\frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\nabla_x\Lambda + \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\nabla_x\Theta\right)\,\mathrm{d}x\\
&= \omega\int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\partial_t Z\,\mathrm{d}x - \frac{\omega}{\varepsilon}\int_{\Omega_R}\varrho\mathbf{u}\cdot\nabla_x Z\,\mathrm{d}x\\
&= \omega\int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\partial_t Z\,\mathrm{d}x + \int_{\Omega_R}\varrho\mathbf{u}\cdot\partial_t\nabla_x\Psi\,\mathrm{d}x,
\end{aligned}
$$

where the last term will cancel with his counterpart in (9.96).

Finally, we check the error committed by the approximation in (9.98) in two steps. First,

$$
\begin{aligned}
&\left|\int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\left[\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\right)\partial_t\Lambda + \left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\right)\partial_t\Theta\right]\,\mathrm{d}x\right|\\
&\le c(D)\int_{\Omega_R}\left|\frac{r-\varrho}{\varepsilon}\right|\varepsilon\left(|\partial_t\Lambda| + |\partial_t\Theta|\right)\,\mathrm{d}x \le c(D,\varepsilon)(\varepsilon+\varepsilon^\alpha),\ 0\le\alpha<1,
\end{aligned}
$$

where we have used Eqs. (9.74), (9.82) to express $\partial_t\Lambda$, $\partial_t\Theta$, together with the bounds (9.81), (9.83).

The second step is to approximate

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\\
&= \varepsilon\left[\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\frac{\partial p}{\partial\varrho}\right)(\overline{\varrho},\overline{\vartheta})\Lambda + \frac{1}{\overline{\varrho}}\frac{\partial^2 p}{\partial\varrho\partial\vartheta}(\overline{\varrho},\overline{\vartheta})\Theta\right] + \varepsilon^2 r_1,\ \|r_1\|_{L^\infty(\mathbb{R}^3)} \le c(D),
\end{aligned}
$$

and

$$\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial \vartheta} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}$$

$$= \varepsilon\left[\frac{1}{\overline{\varrho}}\frac{\partial^2 p}{\partial\varrho\partial\vartheta}(\overline{\varrho},\overline{\vartheta})\Lambda + \frac{1}{\overline{\varrho}}\frac{\partial^2 p}{\partial\vartheta^2}(\overline{\varrho},\overline{\vartheta})\Theta\right] + \varepsilon^2 r_2,\ \|r_2\|_{L^\infty(\mathbb{R}^3)} \le c(D).$$

Thus we have

$$\left|\frac{1}{\varepsilon}\int_{\Omega_R}\varrho\mathbf{u}\cdot\left[\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\varrho} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\right)\nabla_x\Lambda\right|\right.$$

$$\left.+\left(\frac{1}{r}\frac{\partial p(r,\mathcal{T})}{\partial\vartheta} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\right)\nabla_x\Theta\right]\,\mathrm{d}x\right|$$

$$\le \varepsilon c(D) + \left|\frac{\partial}{\partial\varrho}\left(\frac{1}{\varrho}\frac{\partial p}{\partial\varrho}\right)(\overline{\varrho},\overline{\vartheta})\int_{\Omega_R}\varrho\mathbf{u}\cdot\nabla_x\Lambda^2\,\mathrm{d}x\right| + \left|\frac{1}{\overline{\varrho}}\frac{\partial^2 p}{\partial\varrho\partial\vartheta}(\overline{\varrho},\overline{\vartheta})\int_{\Omega_R}\varrho\mathbf{u}\cdot\nabla_x(\Lambda\Theta)\,\mathrm{d}x\right|$$

$$\left|\frac{1}{2\overline{\varrho}}\frac{\partial^2 p}{\partial\vartheta^2}(\overline{\varrho},\overline{\vartheta})\int_{\Omega_R}\varrho\mathbf{u}\cdot\nabla_x\Theta^2\,\mathrm{d}x\right|,$$

where the gradient dependent terms are of order ε, which can be shown in the same way as in (9.68)–(9.70).

Thus we may rewrite (9.96) as

$$\left[\mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,\overline{\varrho}+\varepsilon\Lambda,\overline{\vartheta}+\varepsilon\Theta,(1-\chi_R)(\mathbf{U}+\nabla_x\Psi)\right)\right]_{t=0}^{t=\tau} - \left[\frac{\overline{\varrho}}{2}\int_{\mathbb{R}^3}|\nabla_x\Psi|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} \tag{9.99}$$

$$\le c(D)\int_0^\tau\left[\nu + d + \varepsilon + c(\alpha)\varepsilon^\alpha + \frac{1}{\varepsilon R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} + \mathcal{E}_{\varepsilon,R}\left(\varrho,\vartheta,\mathbf{u}\,\middle|\,r,\mathcal{T},\mathbf{V}\right)\right]\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\left(p(\overline{\varrho},\overline{\vartheta}) - p(\varrho,\vartheta)\right)\Delta\Psi\,\mathrm{d}x\,\mathrm{d}t + \omega\int_0^\tau\int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\partial_t Z\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\left[\varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)\partial_t\Theta + \varrho\left(s(\varrho,\vartheta) - s(r,\mathcal{T})\right)(\mathbf{U}+\nabla_x\Psi)\cdot\nabla_x\Theta\right]\mathrm{d}x\,\mathrm{d}t$$

for $0 \le \alpha < 1$.

Step 6 Repeating the arguments of the previous step, we may replace

$$\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\Big(p(\overline{\varrho},\overline{\vartheta})-p(\varrho,\vartheta)\Big)\,\Delta\Psi\;\mathrm{d}x\;\mathrm{d}t$$

$$\approx\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{\overline{\varrho}-\varrho}{\varepsilon}+\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\right)\Delta\Psi\;\mathrm{d}x\;\mathrm{d}t,$$

committing an error of order ε^α, $0\le\alpha<1$, and

$$-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\Big[\varrho\,(s(\varrho,\vartheta)-s(r,\mathcal{T}))\,\partial_t\Theta+\varrho\,(s(\varrho,\vartheta)-s(r,\mathcal{T}))\,\mathbf{U}\cdot\nabla_x\Theta\Big]\;\mathrm{d}x\;\mathrm{d}t$$

$$\approx\int_0^\tau\int_{\Omega_R}\Big[\left(\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{r-\varrho}{\varepsilon}+\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\;\mathrm{d}x\frac{\mathcal{T}-\vartheta}{\varepsilon}\right)\partial_t\Theta\;\mathrm{d}x\;\mathrm{d}t$$

$$\int_0^\tau\int_{\Omega_R}\Big[\left(\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{r-\varrho}{\varepsilon}+\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\mathcal{T}-\vartheta}{\varepsilon}\right)\mathbf{U}\cdot\nabla_x\Theta\;\mathrm{d}x\;\mathrm{d}t$$

with an error of order ε.

Summing up the previous estimates and using the first equation in (9.74), we get

$$\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_R}\Big(p(\overline{\varrho},\overline{\vartheta})-p(\varrho,\vartheta)\Big)\,\Delta\Psi\;\mathrm{d}x\;\mathrm{d}t+\omega\int_0^\tau\int_{\Omega_R}\frac{r-\varrho}{\varepsilon}\partial_t Z\;\mathrm{d}x\;\mathrm{d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_R}\Big[\varrho\,(s(\varrho,\vartheta)-s(r,\mathcal{T}))\,\partial_t\Theta+\varrho\,(s(\varrho,\vartheta)-s(r,\mathcal{T}))\,\mathbf{U}\cdot\nabla_x\Theta\Big]\;\mathrm{d}x\;\mathrm{d}t$$

$$\approx-\int_0^\tau\int_{\Omega_R}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{\overline{\varrho}-\varrho}{\varepsilon}+\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\right)\partial_t Z\;\mathrm{d}x\;\mathrm{d}t$$

$$+\omega\int_0^\tau\int_{\Omega_R}\frac{\overline{\varrho}-\varrho}{\varepsilon}\partial_t Z\;\mathrm{d}x\;\mathrm{d}t$$

$$+\int_0^\tau\int_{\Omega_R}\Big[\left(\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{\overline{\varrho}-\varrho}{\varepsilon}+\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\right)\partial_t\Theta\;\mathrm{d}x\;\mathrm{d}t$$

$$+\int_0^\tau\int_{\Omega_R}\Big[\left(\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\frac{\overline{\varrho}-\varrho}{\varepsilon}+\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\right)\mathbf{U}\cdot\nabla_x\Theta\;\mathrm{d}x\;\mathrm{d}t$$

$$
+\omega \int_0^\tau \int_{\Omega_R} \Lambda \partial_t Z \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\Omega_R} \Big[\left(\overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Lambda + \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \right) \partial_t \Theta \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\Omega_R} \Big[\left(\overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Lambda + \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Theta \right) \mathbf{U} \cdot \nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t.
$$

Introducing the notation

$$
a = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}, \ b = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ d = \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega = \overline{\varrho} \left(a + \frac{b^2}{d} \right),
$$

we may write the above expression in a concise form

$$
\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_R} \Big(p(\overline{\varrho}, \overline{\vartheta}) - p(\varrho, \vartheta) \Big) \Delta \Psi \, \mathrm{d}x \, \mathrm{d}t + \omega \int_0^\tau \int_{\Omega_R} \frac{r - \varrho}{\varepsilon} \partial_t Z \, \mathrm{d}x \, \mathrm{d}t
$$

$$
-\frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_R} \Big[\varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \partial_t \Theta + \varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \mathbf{U} \cdot \nabla_x \Theta \Big] \, \mathrm{d}x \, \mathrm{d}t
$$

$$
\approx \int_0^\tau \int_{\Omega_R} \left(\frac{b^2}{b^2 + ad} \frac{\overline{\varrho} - \varrho}{\varepsilon} - \frac{bd}{b^2 + ad} \frac{\overline{\vartheta} - \vartheta}{\varepsilon} \right) \partial_t (a\Lambda + b\Theta) \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\Omega_R} \left(d \frac{\overline{\vartheta} - \vartheta}{\varepsilon} - b \frac{\overline{\varrho} - \varrho}{\varepsilon} \right) \partial_t \Theta \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\Omega_R} \Big[\left(d \frac{\overline{\vartheta} - \vartheta}{\varepsilon} - b \frac{\overline{\varrho} - \varrho}{\varepsilon} \right) \mathbf{U} \cdot \nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\Omega_R} \Lambda \partial_t (a\Lambda + b\Theta) \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda) \partial_t \Theta \, \mathrm{d}x \, \mathrm{d}t
$$

$$
+ \int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda) \mathbf{U} \cdot \nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t.
$$

Now, the rest is just a bit of simple algebra. First we write

$$
\int_{\Omega_R} \Lambda \partial_t (a\Lambda + b\Theta) + (d\Theta - b\Lambda) \partial_t \Theta \, \mathrm{d}x
$$

$$
= \frac{1}{2} \int_{\Omega_R} \left[\frac{d}{b^2 + ad} \partial_t (a\Lambda + bT)^2 + \frac{a}{b^2 + ad} \partial_t (dT - b\Lambda)^2 \right] \, \mathrm{d}x
$$

$$
= \frac{1}{2} \overline{\varrho} \omega \left[\int_{\mathbb{R}^3} Z^2 \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \frac{1}{2} \frac{a}{b^2 + ad} \left[\int_{\mathbb{R}^3} P^2 \, \mathrm{d}x \right]_{t=0}^{t=\tau}
$$

In view of the acoustic energy balance (9.79), the former integral cancels with

$$-\left[\frac{\overline{\varrho}}{2}\int_{\mathbb{R}^3} |\nabla_x \Psi|^2 \, \mathrm{d}x\right]_{t=0}^{t=\tau}$$

appearing on the left-hand side of (9.99), while

$$\frac{1}{2}\frac{a}{b^2+ad}\left[\int_{\mathbb{R}^3} P^2 \, \mathrm{d}x\right]_{t=0}^{t=\tau} = 0$$

as P satisfies the transport equation (9.82) with $\mathrm{div}_x \mathbf{U} = 0$.

Similarly,

$$\int_0^\tau \int_{\Omega_R} \left(d\frac{\overline{\vartheta}-\vartheta}{\varepsilon} - b\frac{\overline{\varrho}-\varrho}{\varepsilon}\right) \partial_t \Theta \, \mathrm{d}x \, \mathrm{d}t \tag{9.100}$$

$$+\int_0^\tau \int_{\Omega_R} \left(\frac{b^2}{b^2+ad}\frac{\overline{\varrho}-\varrho}{\varepsilon} - \frac{bd}{b^2+ad}\frac{\overline{\vartheta}-\vartheta}{\varepsilon}\right) \partial_t (a\Lambda + b\Theta) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \frac{a}{b^2+ad}\int_0^\tau \int_{\Omega_R} \left(d\frac{\overline{\vartheta}-\vartheta}{\varepsilon} - b\frac{\overline{\varrho}-\varrho}{\varepsilon}\right) \partial_t P \, \mathrm{d}x \, \mathrm{d}t,$$

while

$$\int_0^\tau \int_{\Omega_R} \Big[\left(d\frac{\overline{\vartheta}-\vartheta}{\varepsilon} - b\frac{\overline{\varrho}-\varrho}{\varepsilon}\right) \mathbf{U}\cdot\nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t \tag{9.101}$$

$$+\int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda)\mathbf{U}\cdot\nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t$$

$$= \frac{b}{b^2+ad}\int_0^\tau \int_{\Omega_R} \left(d\frac{\overline{\vartheta}-\vartheta}{\varepsilon} - b\frac{\overline{\varrho}-\varrho}{\varepsilon}\right) \mathbf{U}\cdot\nabla_x (a\Lambda + b\Theta) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\frac{a}{b^2+ad}\int_0^\tau \int_{\Omega_R} \left(d\frac{\overline{\vartheta}-\vartheta}{\varepsilon} - b\frac{\overline{\varrho}-\varrho}{\varepsilon}\right) \mathbf{U}\cdot\nabla_x (d\Theta - b\Lambda) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\frac{b}{b^2+ad}\int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda)\mathbf{U}\cdot\nabla_x (a\Lambda + b\Theta) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\frac{a}{b^2+ad}\int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda)\mathbf{U}\cdot\nabla_x (d\Theta - b\Lambda) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \frac{b\omega}{b^2 + ad} \int_0^\tau \int_{\Omega_R} \left(d \frac{\overline{\vartheta} - \vartheta}{\varepsilon} - b \frac{\overline{\varrho} - \varrho}{\varepsilon} \right) \mathbf{U} \cdot \nabla_x Z \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{a}{b^2 + ad} \int_0^\tau \int_{\Omega_R} \left(d \frac{\overline{\vartheta} - \vartheta}{\varepsilon} - b \frac{\overline{\varrho} - \varrho}{\varepsilon} \right) \mathbf{U} \cdot \nabla_x P \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{b\omega}{b^2 + ad} \int_0^\tau \int_{\Omega_R} (d\Theta - b\Lambda) \mathbf{U} \cdot \nabla_x Z \, \mathrm{d}x \, \mathrm{d}t.$$

Thus, putting together (9.100), (9.101), we observe that the P-dependent terms cancel out as P satisfies the transport equation (9.82) whereas the integrals containing $\nabla_x Z$ are of order $c(\alpha, D)\varepsilon^\alpha$, $0 \leq \alpha < 1$ as a consequence of the dispersive estimate (9.80).

Step 7 Finally, we observe that the integral

$$\frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_R} \varrho \left(s(\varrho, \vartheta) - s(r, \mathcal{T}) \right) \nabla_x \Psi \cdot \nabla_x \Theta \, \mathrm{d}x \, \mathrm{d}t$$

is small of order $c(D, \alpha)\varepsilon^\alpha$, $0 \leq \alpha < 1$ due to the dispersive estimates (9.80).

Thus the relative energy inequality finally gives rise to

$$\left[\mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \overline{\varrho} + \varepsilon\Lambda, \overline{\vartheta} + \varepsilon\Theta, (1 - \chi_R)(\mathbf{U} + \nabla_x \Psi) \right) \right]_{t=0}^{t=\tau} \tag{9.102}$$

$$\leq c(D) \int_0^\tau \left[\nu + d + \varepsilon + c(\alpha)\varepsilon^\alpha + \frac{1}{\varepsilon R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} + \mathcal{E}_{\varepsilon,R} \left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \mathcal{T}, \mathbf{V} \right) \right] \mathrm{d}t$$

for $0 \leq \alpha < 1$.

9.5.4 Conclusion

Applying Gronwall's lemma to (9.102) we obtain the following conclusion.

■ VANISHING DIFFUSION LIMIT—ILL PREPARED INITIAL DATA:

Theorem 9.2 *Let $\{\Omega_R\}_{R \geq 1}$ be a family of uniformly $C^{2,\nu}$ simply connected bounded domains in $\mathbb{R}^3$ satisfying (9.9), (9.10). Let the constitutive hypotheses (9.30)–(9.39) be satisfied.*

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system (9.2)–(9.8), (9.11), (9.12) in $(0, T) \times \Omega_R$ starting from the initial data

$$\varrho(0, \cdot) \equiv \varrho_0 = \overline{\varrho} + \varepsilon \varrho_0^{(1)}, \ \vartheta(0, \cdot) \equiv \vartheta_0 = \overline{\vartheta} + \varepsilon \vartheta_0^{(1)}, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0,$$

where

$$\left\{ \begin{array}{c} 0 < D^{-1} < \overline{\varrho}, \overline{\vartheta} < D, \\ \\ \|\varrho_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\vartheta_0^{(1)}\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} + \|\mathbf{u}_0\|_{(L^2\cap L^\infty)(\mathbb{R}^3)} < D, \end{array} \right\}$$

In addition, let

$$R > D + T\frac{\sqrt{\omega}}{\varepsilon}, \text{ where } \omega = p_\varrho(\overline{\varrho}, \overline{\vartheta}) + \frac{|p_\vartheta(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 s_\vartheta(\overline{\varrho}, \overline{\vartheta})}.$$

Let $\mathbf{U}$ *be a (strong) solution to the Euler system (9.15), (9.16) in* $\mathbb{R}^3 \times (0, T_{\max})$ *starting from the initial data*

$$\mathbf{U}(0, \cdot) = \mathbf{H}[\mathbf{v}_0],$$

where

$$\mathbf{v}_0 \in C_c^m(\mathbb{R}^3),\ \operatorname{supp}[\mathbf{v}_0] \subset B(0, D),\ \|\mathbf{v}_0\|_{C^m(\mathbb{R}^3)} \leq D,\ m > 4.$$

Let $[Z, \Psi]$ *be the solution of the acoustic system (9.74), (9.75), with the initial data*

$$Z_0 = \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda_0 + \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta_0,\ \nabla_x \Psi_0 = \mathbf{H}^\perp[\mathbf{v}_0]$$

where

$$\Lambda_0,\ \Theta_0 \in C_c^m(\mathbb{R}^3),\ \|\Lambda_0\|_{C^m(\mathbb{R}^3)} + \|\Theta_0\|_{C^m(\mathbb{R}^3)} \leq D,\ \operatorname{supp}[\Lambda_0],\ \operatorname{supp}[\Theta_0] \subset B(0, D).$$

Let P *solve the transport equation (9.82) in* $(0, T_{\max}) \times \mathbb{R}^3$*, with the initial data*

$$P_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda_0 + \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta_0.$$

Finally, let Λ *and* Θ *be determined as*

$$\frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda + \frac{1}{\overline{\varrho}\omega}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta = Z,$$

$$\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Lambda + \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Theta = P.$$

Then for any compact $K \subset \mathbb{R}^3$ and any $T \in (0, T_{\max})$, there are $c_1 = c(T, D)$, $c_2(D)$ such that

$$\int_K \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U} - \nabla_x \Psi|^2 (\tau, \cdot) \, \mathrm{d}x$$

$$+ \left\| \left[\frac{\varrho(\tau, \cdot) - \overline{\varrho}}{\varepsilon} - \Lambda(\tau, \cdot) \right]_{\mathrm{ess}} \right\|^2_{L^2(K)} + \left\| \left[\frac{\vartheta(\tau, \cdot) - \overline{\vartheta}}{\varepsilon} - \Theta(\tau, \cdot) \right]_{\mathrm{ess}} \right\|^2_{L^2(K)}$$

$$+ \frac{1}{\varepsilon^2} \int_K \left(1 + |[\varrho]_{\mathrm{res}}(\tau, \cdot)|^{5/3} + |[\vartheta]_{\mathrm{res}}(\tau, \cdot)|^4 \right) \, \mathrm{d}x$$

$$\leq c_1(\alpha, T, D) \left(\nu + d + \varepsilon + c(\alpha) \varepsilon^\alpha + \frac{1}{\varepsilon R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} \right)$$

$$+ c_2(D) \left(\left\| \varrho_0^{(1)} - \Lambda_0 \right\|^2_{L^2(\Omega_R)} + \left\| \vartheta_0^{(1)} - \Theta_0 \right\|^2_{L^2(\Omega_R)} + \| \mathbf{u}_0 - \mathbf{v}_0 \|^2_{L^2(\Omega_R; \mathbb{R}^3)} \right)$$

for any $0 \leq \alpha < 1$ *and for a.a.* $\tau \in [0, T)$.

Remark Theorem 9.2 gives an explicit rate of convergence in terms of the scaling parameters. In particular, we need the quantity

$$\left(\nu + d + \varepsilon + c(\alpha) \varepsilon^\alpha + \frac{1}{\varepsilon R} + \frac{\varepsilon^{2/3}}{\nu^{1/2}} \right)$$

to be small. Such a process is termed *path dependent*.
