Chapter 8
Problems on Large Domains

Many theoretical problems in continuum fluid mechanics are formulated on unbounded physical domains, most frequently on the whole Euclidean space $\mathbb{R}^3$. Although, arguably, any physical but also numerical space is necessarily bounded, the concept of unbounded domain offers a useful approximation in the situations when the influence of the boundary or at least its part on the behavior of the system can be neglected. The acoustic waves examined in the previous chapters are often ignored in meteorological models, where the underlying ambient space is large when compared with the characteristic speed of the fluid as well as the speed of sound. However, as we have seen in Chap. 5, the way the acoustic waves "disappear" in the asymptotic limit may include fast oscillations in the time variable caused by the reflection of acoustic waves by the physical boundary that may produce undesirable numerical instabilities. In this chapter, we examine the incompressible limit of the NAVIER-STOKES-FOURIER SYSTEM in the situation when the spatial domain is large with respect to the characteristic speed of sound in the fluid. Remarkably, although very large, our physical space is still bounded exactly in the spirit of the leading idea of this book that the notions of "large" and "small" depend on the chosen scale.

8.1 Primitive System

Similarly to the previous chapters, our starting point is the full NAVIER-STOKES-FOURIER SYSTEM, where the *Mach number* is proportional to a small parameter ε, while the remaining characteristic numbers are kept of order unity.

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■ SCALED NAVIER-STOKES-FOURIER SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{8.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p = \mathrm{div}_x \mathbb{S} + \frac{1}{\varepsilon}\nabla_x F, \tag{8.2}$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{8.3}$$

with

$$\sigma \geq \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right), \tag{8.4}$$

where the inequality sign in (8.4) is motivated by the existence theory developed in Chap. 3. The viscous stress tensor $\mathbb{S}$ satisfies the standard *Newton's rheological law*

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{8.5}$$

where the effect of the bulk viscosity may be omitted, while the heat flux $\mathbf{q}$ obeys *Fourier's law*

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta. \tag{8.6}$$

System (8.1)–(8.3) is considered on a family of spatial domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ "large" enough in order to eliminate the effect of the boundary on the local behavior of acoustic waves. Seeing that the speed of sound in (8.1)–(8.3) is proportional to $1/\varepsilon$ we shall assume that the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ has the following property.

■ PROPERTY (L):

The boundary $\partial\Omega_\varepsilon$ consists of two disjoint parts

$$\partial\Omega_\varepsilon = \Gamma \cup \Gamma_\varepsilon,$$

where Γ is a fixed compact subset of $\mathbb{R}^3$ and, for any $x \in \Omega_\varepsilon$,

$$\varepsilon \mathrm{dist}[x, \Gamma_\varepsilon] \to \infty \text{ for } \varepsilon \to 0. \tag{8.7}$$

In other words, given a fixed bounded cavity $B \subset \Omega_\varepsilon$ in the ambient space, the acoustic waves initiated in B cannot reach the boundary, reflect, and come back during a finite time interval $(0, T)$. Typically, we may consider $\Omega \subset \mathbb{R}^3$ an *exterior domain*—an unbounded domain with a compact boundary Γ—and define

$$\Omega_\varepsilon = \Omega \cap \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| < \frac{1}{\varepsilon^m} \right\}, \; m > 1.$$

Similarly to Chap. 5, we suppose that the initial distribution of the density and the temperature are close to a spatially homogeneous state, specifically,

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \tag{8.8}$$

$$\vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{8.9}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants, and

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}. \tag{8.10}$$

The analysis in this chapter will heavily lean on the assumption that both the perturbations $\varrho^{(1)}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ and the velocity field $\mathbf{u}_{0,\varepsilon}$ are *spatially localized*, specifically they satisfy the far field boundary conditions

$$\varrho^{(1)}_{0,\varepsilon} \to 0, \; \vartheta^{(1)}_{0,\varepsilon} \to 0, \; \mathbf{u}_{0,\varepsilon} \to 0 \text{ as } |x| \to \infty$$

in some sense, and the solutions we look for are supposed to enjoy a similar property.

Finally, we impose the complete slip boundary conditions and the no flux condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \; \mathbb{S}\mathbf{n} \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \; \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{8.11}$$

Problem Formulation We consider a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of (weak) solutions to problem (8.1)–(8.6), (8.11) on a compact time interval $[0, T]$ emanating from the initial state satisfying (8.8)–(8.10) on a family of spatial domains Ω_ε enjoying Property (L). Our main goal formulated in Theorem 8.3 below is to show that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ in } L^2((0, T) \times B; \mathbb{R}^3) \text{ for any compact set } B \subset \Omega_\varepsilon, \tag{8.12}$$

at least for a suitable subsequence $\varepsilon \to 0$, where the limit velocity field complies with the standard incompressibility constraint

$$\mathrm{div}_x \mathbf{U} = 0. \tag{8.13}$$

Thus, in contrast with the case of a bounded domain examined in Chap. 5, we recover *strong (pointwise) convergence* of the velocity field regardless the specific shape of the "far field" boundary Γ_ε, and, in fact, the boundary conditions imposed on Γ_ε.

The strong convergence of the velocity is a consequence of the dispersive properties of the *acoustic equation*—the energy of acoustic waves decays on any compact set. Mathematically this can be formulated in terms of *Strichartz's estimates* or their local variant discovered by Smith and Sogge [249]. Here we use probably the most general result in this direction—the celebrated *RAGE theorem*.

As already pointed out, these considerations should be independent of the behavior of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ on the far-field boundary Γ_ε, in particular, we may impose there any boundary conditions, not just (8.11). On the other hand, certain restrictions have to be made in order to prevent the energy to be "pumped" into the system at infinity. Specifically, the following hypotheses are required.

(i) The total mass of the fluid contained in Ω_ε is proportional to $|\Omega_\varepsilon|$, in particular the average density is constant.
(ii) The system dissipates energy, specifically, the total energy of the fluid contained in Ω_ε is non-increasing in time.
(iii) The system produces entropy, the total entropy is non-decreasing in time.

Typical examples of fluid motion on unbounded (exterior) domains arise in meteorology or astrophysics, where the complement of the physical space often plays a role of a rigid core (a star) around which the fluid evolves. Since the effect of gravitation is essential in these problems, it is natural to ask if the Oberbeck–Boussinesq approximation introduced in Chap. 5 can be adapted to unbounded domains.

The matter in this chapter is organized as follows. The Oberbeck–Boussinesq approximation considered on an exterior domain is introduced in Sect. 8.2. Similarly to the preceding part of this book, our analysis is based on the uniform estimates of the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ resulting from the dissipation inequality deduced in the same way as in Chap. 5 (see Sect. 8.3 and the first part of convergence proof in Sect. 8.4). The time evolution of the velocity field, specifically its gradient component, is governed by a wave equation (acoustic equation) introduced in Sect. 5.4.3 and here revisited in Sect. 8.5. Since the acoustic waves propagate with a finite speed proportional to $1/\varepsilon$, the acoustic equation may be handled as if defined on an unbounded exterior domain, where efficient tools for estimating the rate of local decay of acoustic waves as RAGE theorem are available, see Sects. 8.6 and 8.7. In particular, the desired conclusion on strong (pointwise) convergence of the velocity fields is proved and rigorously formulated in Theorem 8.2. The proof of convergence towards the limit system is then completed in Sect. 8.8 and formulated in Theorem 8.3. We finish by discussing possible extensions and refinements of these techniques in Sects. 8.9 and 8.10.

8.2 Oberbeck–Boussinesq Approximation in Exterior Domains

The OBERBECK–BOUSSINESQ APPROXIMATION has been introduced in Sect. 4.2. The fluid velocity $\mathbf{U}$ and the temperature deviation Θ satisfy

■ OBERBECK–BOUSSINESQ APPROXIMATION:

$$\mathrm{div}_x \mathbf{U} = 0, \tag{8.14}$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U}\Big) + \nabla_x \Pi = \mu(\overline{\vartheta})\Delta \mathbf{U} + r\nabla_x F, \tag{8.15}$$

$$\overline{\varrho}c_p\Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})\Big) - \kappa(\overline{\vartheta})\Delta\Theta - \overline{\varrho}\overline{\vartheta}\alpha \mathrm{div}_x(F\mathbf{U}) = 0, \tag{8.16}$$

$$r + \overline{\varrho}\alpha\Theta = 0, \tag{8.17}$$

where Π is the pressure and the quantities $c_p = c_p(\overline{\varrho}, \overline{\vartheta})$, $\alpha = \alpha(\overline{\varrho}, \overline{\vartheta})$ are defined through (4.17), (4.18).

The function $F = F(x)$ represents a given gravitational potential acting on the fluid. In real world applications, it is customary to take the x_3-coordinate to be vertical parallel to the gravitational force $\nabla_x F = g[0, 0, -1]$. This is indeed a reasonable approximation provided the fluid occupies a *bounded* domain $\Omega \subset \mathbb{R}^3$, where the gravitational field can be taken constant. Thus one may be tempted to study system (8.14)–(8.17) with $\nabla_x F = g[0, 0, -1]$ also un an unbounded physical space (cf. Brandolese and Schonbek [32], Danchin and Paicu [74–76]). Although such an "extrapolation" of the model is quite natural from the mathematical viewpoint, it seems a bit awkward physically. Indeed, if the self-gravitation of the fluid is neglected, the origin of the gravitational force must be an object placed *outside* the fluid domain Ω therefore a more natural setting is

$$F(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|} m(y)\, \mathrm{d}y, \ \text{with } m \geq 0, \ \mathrm{supp}[m] \subset \mathbb{R}^3 \setminus \Omega, \tag{8.18}$$

where m denotes the mass density of the object acting on the fluid by means of gravitation. In other words, F is a harmonic function in Ω, $F(x) \approx 1/|x|$ as $|x| \to \infty$.

Accordingly, we consider the Oberbeck-Boussinesq system on a domain $\Omega = R^3 \setminus K$ exterior to a compact set K, $\partial K = \Gamma$, where, in accordance with (8.18), F satisfies

$$-\Delta F = m \text{ in } \mathbb{R}^3, \ \nabla_x F \in L^2(\mathbb{R}^3; \mathbb{R}^3), \ \mathrm{supp}[m] \subset K. \tag{8.19}$$

In particular, introducing a new variable $\theta = \Theta - \overline{\vartheta}\alpha F/c_p$ we can rewrite the system (8.14)–(8.17) in the more frequently used form

$$\mathrm{div}_x \mathbf{U} = 0,$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U}\Big) + \nabla_x P = \mu\Delta\mathbf{U} - \overline{\varrho}\alpha\theta\nabla_x G,$$

$$\overline{\varrho}c_p\Big(\partial_t\theta + \mathrm{div}_x(\theta\mathbf{U})\Big) - \kappa\Delta\theta = 0,$$

where we have set $P = \Pi - F^2\overline{\varrho}\overline{\vartheta}\alpha^2/2c_p$.

8.3 Uniform Estimates

The uniform estimates derived below follow immediately from the general axioms **(i)–(iii)** stated in the introductory section, combined with the *hypothesis of thermodynamic stability* (see (1.44))

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} > 0, \tag{8.20}$$

where $e = e(\varrho,\vartheta)$ is the specific internal energy interrelated to p and s through *Gibbs' equation* (1.2). We recall that the first condition in (8.20) asserts that the compressibility of the fluid is always positive, while the second one says that the specific heat at constant volume is positive.

Although the estimates deduced below are formally the same as in Chap. 5, we have to pay special attention to the fact that the volume of the ambient space expands for $\varepsilon \to 0$. In particular, the constants associated to various embedding relations may depend on ε. Note that the existence theory developed in Chap. 3 relies essentially on boundedness of the underlying physical domain.

8.3.1 Static Solutions

Similarly to Sect. 6.3.1, we introduce the static solutions $\tilde{\varrho} = \tilde{\varrho}_\varepsilon$ satisfying

$$\nabla_x p(\tilde{\varrho},\overline{\vartheta}) = \varepsilon\tilde{\varrho}\nabla_x F. \tag{8.21}$$

Note that solutions of (8.21) depend on ε. More specifically, fixing two positive constants $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, we look for a solution to (8.21) in the whole space $\mathbb{R}^3$ satisfying the far field condition

$$\tilde{\varrho}(x) \to \overline{\varrho} \text{ as } |x| \to \infty. \tag{8.22}$$

Anticipating that $\tilde{\varrho}$ is positive, it is not difficult to integrate (8.21) to obtain

$$P(\tilde{\varrho}) = \varepsilon F + P(\overline{\varrho}), \text{ where } P'(\varrho) = \frac{1}{\varrho}\partial_\varrho p(\varrho, \overline{\vartheta}).$$

Thus, if p is twice continuously differentiable in a neighborhood of $(\overline{\varrho}, \overline{\vartheta})$, the unique solution $\tilde{\varrho}_\varepsilon$ of (8.21), (8.22) satisfies

$$\tilde{\varrho}_\varepsilon - \overline{\varrho} = \frac{\varepsilon}{P'(\overline{\varrho})}F + \varepsilon^2 h_\varepsilon F, \ \|h_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq c, \tag{8.23}$$

$$|\nabla_x \tilde{\varrho}_\varepsilon(x)| \leq \varepsilon c |\nabla_x F(x)| \text{ for all } x \in \mathbb{R}^3, \tag{8.24}$$

uniformly for $\varepsilon \to 0$.

8.3.2 Estimates Based on the Hypothesis of Thermodynamic Stability

To derive the uniform bounds, it is convenient to introduce the total dissipation inequality based on the static solutions, similar to (6.56) derived in the context of stratified fluids.

■ TOTAL DISSIPATION INEQUALITY:

$$\int_{\Omega_\varepsilon} \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})(\varrho_\varepsilon - \tilde{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})\Big)\right](t)\ \mathrm{d}x \tag{8.25}$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\left[[0,t]\times\overline{\Omega}_\varepsilon\right]$$

$$= \int_{\Omega_\varepsilon} \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \partial_\varrho H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})(\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\tilde{\varrho}_\varepsilon, \overline{\vartheta})\Big)\right] \mathrm{d}x$$

for a.a. $t \in [0, T]$,

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where

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta}\varrho s(\varrho, \vartheta)$$

is the *Helmholtz function* introduced in (2.48), and

$$\sigma_\varepsilon\Big[[0,t]\times\overline{\Omega}_\varepsilon\Big] = \int_{\Omega_\varepsilon}\Big[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(t) - \varrho_{0,\varepsilon}s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\Big]\ \mathrm{d}x \tag{8.26}$$

is the total entropy production,

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\Big),\ \mathbb{S}_\varepsilon = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon),\ \mathbf{q}_\varepsilon = \mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon). \tag{8.27}$$

Relation (8.25) reflects the general principles **(i)–(iii)** introduced in Sect. 8.1 and has bees rigorously verified in the present form in Sect. 6.4.1 as long as Ω_ε is a bounded domain. We recall that, by virtue of Gibbs' relation (1.2),

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho},\ \ \frac{\partial H_{\overline{\vartheta}}(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta}(\vartheta - \overline{\vartheta})\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta};$$

whence the hypothesis of thermodynamic stability (8.20) implies that

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \text{ is strictly convex on } (0, \infty),$$

and

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \overline{\vartheta} \text{ and increasing for } \vartheta > \overline{\vartheta}$$

(see Sect. 2.2.3).

As observed several times in this book, the total dissipation inequality (8.25) is the only source of uniform bounds available in the limit process. The minimal assumption in this respect is the expression on the right hand side, controlled exclusively by the initial data, to be bounded uniformly for $\varepsilon \to 0$. To this end, we take

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon \tilde{\varrho}^{(1)}_{0,\varepsilon},\ \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{8.28}$$

where

$$\|\tilde{\varrho}^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon)} \leq c,\ \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon)} \leq c, \tag{8.29}$$

$$\int_{\Omega_\varepsilon} \tilde{\varrho}^{(1)}_{0,\varepsilon}\ \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon}\ \mathrm{d}x = 0; \tag{8.30}$$

and

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon; \mathbb{R}^3)} \leq c, \tag{8.31}$$

where all constants are independent of ε. As a matter of fact, boundedness in L^∞ is never used and may be relaxed to weaker integrability properties, the bound in L^2, independent of ε and the size of Ω_ε, is however essential.

Remark Comparing (8.28) with (8.8) we observe that

$$\varrho_{0,\varepsilon}^{(1)} = \tilde{\varrho}_{0,\varepsilon}^{(1)} + \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon},$$

where, by virtue of (8.23),

$$\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} = \frac{1}{P'(\overline{\varrho})} F + \varepsilon h_\varepsilon F.$$

As F is the gravitational potential determined by (8.18), the initial distribution of the density $\varrho_{0,\varepsilon}$ *cannot* be taken a square integrable perturbation of the constant state $\overline{\varrho}$ on $\mathbb{R}^3$.

As a direct consequence of the structural properties of the Helmholtz function observed in Lemma 5.1, boundedness of the left-hand side of (8.25) gives rise to a number of useful uniform estimates. Similarly to Sect. 6.4, we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \leq c, \tag{8.32}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{8.33}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{8.34}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| [\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{8.35}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{8.36}$$

where the "essential" and "residual" components have been introduced through (4.39)–(4.45).

Remark We point out that, by virtue of (8.23),

$$\|\tilde{\varrho}_\varepsilon - \overline{\varrho}\|_{L^\infty(\mathbb{R}^3)} \leq \varepsilon c;$$

whence the essential and residual sets may be defined using $\overline{\varrho}$ exactly as in (4.39).

In addition to the above estimates, we control the measure of the "residual set", specifically,

$$\operatorname*{ess\ sup}_{t\in(0,T)} |\mathcal{M}^{\varepsilon}_{\rm res}[t]| \leq \varepsilon^2 c, \tag{8.37}$$

where $\mathcal{M}^{\varepsilon}_{\rm res}[t] \subset \Omega$ was introduced in (4.43). Note that estimate (8.37) is particularly important as it says that the measure of the "residual" set, on which the density and the temperature are far away from the equilibrium state $(\tilde{\varrho}_\varepsilon, \overline{\vartheta})$ (or, equivalently $(\overline{\varrho}, \overline{\vartheta})$), is small, and, in addition, independent of the measure of the whole set Ω_ε.

Finally, we deduce

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega}_\varepsilon)} \leq \varepsilon^2 c, \tag{8.38}$$

therefore,

$$\int_0^T \int_{\Omega_\varepsilon} \frac{1}{\vartheta_\varepsilon} \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon \, {\rm d}x \, {\rm d}t \leq c, \tag{8.39}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} -\frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon^{\ 2}} \, {\rm d}x \, {\rm d}t \leq \varepsilon^2 c. \tag{8.40}$$

8.3.3 Estimates Based on the Specific Form of Constitutive Relations

The uniform bounds obtained in the previous section may be viewed as a consequence of the general physical principles postulated through axioms **(i)–(iii)** in the introductory section combined with the hypothesis of thermodynamic stability (8.20). In order to convert them to a more convenient language of the standard function spaces, structural properties of the thermodynamic functions as well as of the transport coefficients must be specified.

Motivated by the general hypotheses of the existence theory developed in Sect. 3, exactly as in Sect. 5, we consider the *state equation* for the pressure in the form

$$p(\varrho, \vartheta) = \underbrace{p_M(\varrho, \vartheta)}_{\text{molecular pressure}} + \underbrace{p_R(\vartheta)}_{\text{radiation pressure}}, \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ p_R = \frac{a}{3}\vartheta^4, \ a > 0, \tag{8.41}$$

while the internal energy reads

$$e(\varrho, \vartheta) = e_M(\varrho, \vartheta) + e_R(\varrho, \vartheta), \ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho}P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ e_R = a\frac{\vartheta^4}{\varrho}, \tag{8.42}$$

and, in accordance with Gibbs' relation (1.2),

$$s(\varrho, \vartheta) = s_M(\varrho, \vartheta) + s_R(\varrho, \vartheta), \ s_M(\varrho, \vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big), \ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{8.43}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{8.44}$$

The *hypothesis of thermodynamic stability* (8.20) reformulated in terms of the structural properties of P requires

$$P \in C^1[0, \infty) \cap C^2(0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{8.45}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty. \tag{8.46}$$

Furthermore, it follows from (8.46) that $P(Z)/Z^{5/3}$ is a decreasing function of Z, and we assume that

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{8.47}$$

The transport coefficients μ and κ will be continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left\{ \begin{array}{r} 0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta), \\ 0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta \geq 0, \end{array} \right\} \tag{8.48}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

To facilitate future considerations and basically without loss of generality we focus on the class of domains satisfying a slightly stronger version of Property (L), namely

$$\Omega_\varepsilon = \Omega \cap \left\{ x \in \mathbb{R}^3 \ \middle| \ |x| < d(\varepsilon) \right\}, \ \lim_{\varepsilon \to 0} \varepsilon d(\varepsilon) = \infty, \tag{8.49}$$

where Ω is an exterior domain with a regular (Lipschitz) boundary.

Now, observe that (8.48), together with estimate (8.39), and Newton's rheological law expressed in terms of (8.5), give rise to

$$\int_0^T \| \nabla_x \mathbf{u}_\varepsilon + \nabla_x{}^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times3})} \, \mathrm{d}t \leq c, \tag{8.50}$$

with c independent of $\varepsilon \to 0$.

At this stage, we apply *Korn's inequality* in the form stated in Proposition 2.1 to $r = [\varrho_\varepsilon]_{\mathrm{ess}}$, $\mathbf{v} = \mathbf{u}_\varepsilon$ and use the bounds established in (8.33), (8.37), (8.50) in order to conclude that

$$\int_0^T \| \mathbf{u}_\varepsilon \|^2_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \, \mathrm{d}t \leq c \text{ uniformly for } \varepsilon \to 0 \tag{8.51}$$

This can be seen writing

$$\Omega_\varepsilon = \Omega \cap \left\{ x \in \mathbb{R}^3 \,\middle|\, |x| < r \right\} \cup \left\{ x \in \mathbb{R}^3 \,\middle|\, r \leq |x| < d(\varepsilon) \right\}$$

for a suitable r so large that the ball $\{|x| < r\}$ contains $\partial\Omega$ in its interior. Now, writing

$$\left\{ x \in \mathbb{R}^3 \,\middle|\, r \leq |x| < d(\varepsilon) \right\} = \cup_{i=1}^{m(\varepsilon)} Q_i$$

as a union of equi-Lipschitz sets Q_i with mutually disjoint interiors, we can apply Korn's inequality on

$$\Omega \cap \left\{ x \in \mathbb{R}^3 \,\middle|\, |x| < r \right\}$$

and on each Q_i separately to obtain the desired conclusion.

In a similar fashion, we can use Fourier's law (8.6) together with (8.40) to obtain

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon{}^2} |\nabla_x \vartheta_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \varepsilon^2 c, \tag{8.52}$$

which, combined with the structural hypotheses (8.48), the uniform bounds established in (8.34), (8.37), and the *Poincaré inequality* stated in Proposition 2.2, yields

$$\int_0^T \|\vartheta_\varepsilon - \overline{\vartheta}\|^2_{W^{1,2}(\Omega_\varepsilon)} \, \mathrm{d}t + \int_0^T \| \log(\vartheta_\varepsilon) - \log(\overline{\vartheta})\|^2_{W^{1,2}(\Omega_\varepsilon)} \, \mathrm{d}t \leq \varepsilon^2 c. \tag{8.53}$$

Finally, a combination of (8.35), (8.41), and (8.47) yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]^{5/3}_{\mathrm{res}} \, \mathrm{d}x \leq \varepsilon^2 c. \tag{8.54}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^4_{\mathrm{res}} \, \mathrm{d}x \leq \varepsilon^2 c. \tag{8.55}$$

8.4 Convergence, Part I

The uniform bounds established in the previous section allow us to pass to the limit in the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$. To begin, we deduce from (8.33), (8.54) that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(\varrho_\varepsilon - \tilde{\varrho}_\varepsilon)(t,\cdot)\|_{(L^2+L^{5/3})(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0, \tag{8.56}$$

which, together with (8.23), yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(\varrho_\varepsilon - \overline{\varrho})(t,\cdot)\|_{L^{5/3}(K)} \to 0 \text{ as } \varepsilon \to 0 \text{ for any compact } K \subset \Omega. \tag{8.57}$$

Thus, at least for a suitable subsequence, ϱ_ε converges a.a. to the constant equilibrium state $\overline{\varrho}$.

Similarly, relations (8.34), (8.37), and (8.55) imply that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(\vartheta_\varepsilon - \overline{\vartheta})(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{8.58}$$

Finally, extending suitably ϑ_ε, $\mathbf{u}_\varepsilon$ outside Ω_ε (cf. Theorem 8) we may assume, in view of (8.51), (8.53) that

$$\Theta_\varepsilon \equiv \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{8.59}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),\ \operatorname{div}_x \mathbf{U} = 0, \tag{8.60}$$

passing to subsequences as the case may be.

Our next goal will be to establish pointwise (a.a.) convergence of the sequence of velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$. More specifically, we show that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ (strongly) in } L^2((0,T)\times K;\mathbb{R}^3) \text{ for any compact } K \subset \Omega. \tag{8.61}$$

Observe that for (8.61) to hold, it is enough to show that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ in } L^2(0,T;W^{-1,2}(K;\mathbb{R}^3)). \tag{8.62}$$

Indeed, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\overline{\varrho}\int_0^T \int_\Omega \varphi|\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \varphi(\overline{\varrho}-\varrho_\varepsilon)|\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varphi\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t,$$

where, in accordance with (8.57), (8.60), and the embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\int_0^T \int_\Omega \varphi(\overline{\varrho} - \varrho_\varepsilon)|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0,$$

while, as a consequence of (8.60), (8.62),

$$\int_0^T \int_\Omega \varphi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \to \overline{\varrho} \int_0^T \int_\Omega \varphi |\mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Remark As the function φ is compactly supported in Ω, its support is contained in Ω_ε for all $\varepsilon > 0$ small enough and all the above integrals are therefore well defined.

The final observation is that, by virtue of (8.32), (8.33), and (8.54),

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{5}{4}}(K;\mathbb{R}^3)} \le c(K) \text{ for any compact } K \subset \Omega.$$

As the embedding $L^{5/4}(K) \hookrightarrow W^{-1,2}(K)$ is compact, we infer that the desired relation (8.62) follows as soon as we are able to show that the family of functions

$$\left\{ t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t, \cdot) \cdot \varphi \, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \tag{8.63}$$

for any fixed $\varphi \in C_c^\infty(\Omega)$. Relation (8.63) will be shown in the following part of this chapter as a consequence of the local decay of acoustic waves. Note that (8.63) is very weak with respect to regularity in the space variable. This is because compactness in space is already guaranteed by the gradient estimate (8.51).

8.5 Acoustic Equation

The *acoustic equation*, introduced in Chap. 4 and thoroughly investigated in various parts of this book, governs the time evolution of the acoustic waves and as such represents a key tool for studying the time oscillations of the velocity field in the incompressible limits for problems endowed with ill-prepared data. It can be viewed as a linearization of system (8.1)–(8.3) around the static state $\{\overline{\varrho}, 0, \overline{\vartheta}\}$.

If $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon > 0}$ satisfy (8.1)–(8.3) in the weak sense specified in Chap. 1, we get, exactly as in Sect. 5.4.3,

$$\int_0^T \int_{\Omega_\varepsilon} \left[\varepsilon \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t = 0 \tag{8.64}$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$;

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big) \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{8.65}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon \Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t - < \sigma_\varepsilon; \varphi >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$

for any test function $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$; and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot\partial_t \boldsymbol{\varphi} + \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F \Big) \mathrm{div}_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.66}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon \Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \Big) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} (\overline{\varrho} - \varrho_\varepsilon)\nabla_x F \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\boldsymbol{\varphi} \in C_c^\infty((0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$.

Thus, after a simple manipulation, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon\omega r_\varepsilon \partial_t \varphi + \omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.67}$$

$$= A \int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon \Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ A \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t - A < \sigma_\varepsilon; \varphi >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$

for all $\varphi \in C_c^\infty((0,T)\times\Omega_\varepsilon)$, and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot\partial_t \boldsymbol{\varphi} + \omega r_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{8.68}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \Big[\omega r_\varepsilon - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \Big) \Big] \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \varepsilon \Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \Big) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} (\overline{\varrho} - \varrho_\varepsilon)\nabla_x F \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\boldsymbol{\varphi} \in C_c^\infty((0,T) \times \Omega_\varepsilon; \mathbb{R}^3)$, where we have set

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega}\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big) - \frac{\overline{\varrho}}{\omega}F, \tag{8.69}$$

with the constants ω, A determined through

$$A\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega + A\overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}. \tag{8.70}$$

As a direct consequence of *Gibbs' equation* (1.2), we have

$$\frac{\partial s}{\partial \varrho} = -\frac{1}{\varrho^2}\frac{\partial p}{\partial \vartheta},$$

in particular,

$$\omega = p_\varrho(\overline{\varrho}, \overline{\vartheta}) + \frac{|p_\vartheta(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 s_\vartheta(\overline{\varrho}, \overline{\vartheta})}$$

as soon as e, p comply with the *hypothesis of thermodynamic stability* stated in (8.20).

Finally, exactly as in Sect. 5.4.7, we introduce the "time lifting" Σ_ε of the measure σ_ε as

$$\Sigma_\varepsilon \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}_\varepsilon)) \cap C_{\text{weak}-*}([0,T], \mathcal{M}^+(\overline{\Omega}_\varepsilon))$$

$$< \Sigma_\varepsilon; \psi >_{[L^\infty(0,T;\mathcal{M}(\overline{\Omega}_\varepsilon));L^1(0,T;C(\overline{\Omega}))]} := < \sigma_\varepsilon; I[\varphi] >_{[\mathcal{M};C]([0,T]\times\overline{\Omega}_\varepsilon)}, \tag{8.71}$$

where

$$I[\varphi](t,x) = \int_0^t \varphi(s,x)\, \mathrm{d}s.$$

Consequently, system (8.67), (8.68) can be written in a concise form as

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■ ACOUSTIC EQUATION:

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon Z_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi\Big]\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} \varepsilon \mathbf{F}_\varepsilon^1 \cdot \nabla_x \varphi \, \mathrm{d}x\, \mathrm{d}t \tag{8.72}$$

for all $\varphi \in C_c^\infty((0,T) \times \overline{\Omega}_\varepsilon)$,

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \omega Z_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \Big] \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon \mathbb{F}^2_\varepsilon : \nabla_x \boldsymbol{\varphi} + \varepsilon F^3_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{8.73}$$

$$+ \frac{A}{\varepsilon\omega} < \Sigma_\varepsilon; \mathrm{div}_x \boldsymbol{\varphi} >_{[L^\infty(0,T;\mathcal{M}(\overline{\Omega}_\varepsilon));L^1(0,T;C(\overline{\Omega}_\varepsilon))]} + \int_0^T \int_{\Omega_\varepsilon} \varepsilon \mathbf{F}^4_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

$$\text{for all } \boldsymbol{\varphi} \in C^\infty_c((0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

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where we have set

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big) + \frac{A}{\varepsilon\omega} \Sigma_\varepsilon - \frac{\overline{\varrho}}{\omega} F, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon, \tag{8.74}$$

$$\mathbf{F}^1_\varepsilon = \frac{A}{\omega} \varrho_\varepsilon \Big(\frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon + \frac{A}{\omega} \frac{\kappa \nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon}, \tag{8.75}$$

$$\mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{8.76}$$

$$F^3_\varepsilon = \omega \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2} \right) + A \varrho_\varepsilon \Big(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \Big) - \Big(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \Big), \tag{8.77}$$

and

$$\mathbf{F}^4_\varepsilon = \frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \nabla_x F. \tag{8.78}$$

Here, similarly to Chap. 5, we have identified the "lifted measure"

$$\int_{\Omega_\varepsilon} \Sigma_\varepsilon \varphi \, \mathrm{d}x := < \Sigma_\varepsilon; \varphi >_{[\mathcal{M};C](\overline{\Omega}_\varepsilon)} .$$

8.5.1 Boundedness of the Data

Our next goal is to examine the integrability properties of the quantities appearing in the weak formulation of the acoustic equation (8.72), (8.73). We start by writing

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} = \frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} + \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} = \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{ess}} + \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{res}} + \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon},$$

where, in accordance with the uniform bounds (8.33), (8.37), and (8.54),

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{ess}} \right\|_{L^2(\Omega_\varepsilon)} \le c, \ \mathrm{ess} \sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \right]_{\mathrm{res}} \right\|_{L^1(\Omega_\varepsilon)} \le \varepsilon c. \tag{8.79}$$

Remark It is worth noting that the measure of the "residual set" is uniformly small as stated in (8.37). In particular, unlike on the unbounded domain Ω, the L^p norms on the residual set are comparable.

Next, by virtue of (8.23), (8.24),

$$\left\| \frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{(L^\infty \cap L^q)(\mathbb{R}^3)} \leq c \text{ for any } q > 3, \tag{8.80}$$

$$\left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq c. \tag{8.81}$$

Remark The previous computations reveal one of the main difficulties in obtaining uniform bounds, namely the terms proportional to the difference $(\tilde{\varrho} - \overline{\varrho})/\varepsilon \approx F$ that are not (uniformly) square integrable in Ω_ε.

Next, we have

$$\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} = \frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} + \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}$$

$$= \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm ess} + \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm res}$$

$$+ \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon},$$

where, by virtue of (8.33), (8.34), (8.36), (8.37),

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{8.82}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon} \right]_{\rm res} \right\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon c, \tag{8.83}$$

and, in accordance with (8.23), (8.24),

$$\left\| \frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right\|_{(L^\infty \cap L^q)(\mathbb{R}^3)} \leq c \text{ for all } q > 3, \tag{8.84}$$

$$\left\| \nabla_x \left(\frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq c. \tag{8.85}$$

Finally, as a consequence of (8.38),

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \frac{\Sigma_\varepsilon(t, \cdot)}{\varepsilon} \right\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \leq \varepsilon c, \tag{8.86}$$

and we may infer that Z_ε introduced in (8.74) can be written in the form

$$Z_\varepsilon(t, \cdot) = Z_\varepsilon^1(t, \cdot) + Z_\varepsilon^2(t, \cdot) + Z^3(t, \cdot) + Z^4, \tag{8.87}$$

where

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|Z_\varepsilon^1\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \leq \varepsilon c, \ \operatorname*{ess\ sup}_{t \in (0,T)} \|Z_\varepsilon^2\|_{L^1(\Omega_\varepsilon)}, \tag{8.88}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|Z_\varepsilon^3\|_{L^2(\Omega_\varepsilon)} \leq c, \ Z^4 = -\frac{\overline{\varrho}}{\omega} \tilde{F} \in D^{1,2}(\Omega),$$

with

$$\tilde{F} \in C^\infty(\Omega), \ \tilde{F}(x) = 0 \text{ for } |x| < r_1, \ \tilde{F}(x) = F(x) \text{ for } |x| > r_2, \tag{8.89}$$

and where $\partial\Omega \subset B(0, r_1/2)$.

Remark Note that F being determined by (8.19) admits a decomposition

$$F = \tilde{F} + G, \ G \in L^2(\mathbb{R}^3).$$

We recall that the space $D^{1,2}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|v\|_{D^{1,2}(\Omega)}^2 = \int_\Omega |\nabla_x v|^2 \, dx.$$

Now, similarly,

$$\mathbf{V}_\varepsilon = [\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess} + [\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res},$$

where, by virtue of (8.32), (8.37), and (8.54),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|[\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess}\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \operatorname*{ess\ sup}_{t\in(0,T)} \|[\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res}\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le \varepsilon c. \tag{8.90}$$

The "forcing terms" $\mathbf{F}^1_\varepsilon$, $\mathbb{F}^2_\varepsilon$, F^3_ε, and $\mathbf{F}^4_\varepsilon$ can be treated in a similar manner. We focus only on the most complicated term:

$$\omega\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + A\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2}\right)$$

$$= \omega\left(\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^2}\right) + A\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right)$$

$$+\omega\left(\frac{\tilde{\varrho}_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + A\left(\frac{\tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2}\right).$$

Seeing that ω and A have been chosen to satisfy

$$\omega + A\partial_\varrho(\varrho s)(\overline{\varrho}, \overline{\vartheta}) - \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) = 0,$$

the quantity

$$\omega\left(\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^2}\right) + A\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right)$$

contains only quadratic terms proportional to $\varrho_\varepsilon - \tilde{\varrho}_\varepsilon$, $\vartheta - \overline{\vartheta}$ and as such may be handled by means of the estimates (8.33)–(8.37), (8.53)–(8.55). Moreover, by the same token, we may use (8.23), (8.24) to deduce

$$\left\| \omega\left(\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon^2}\right) + A\left(\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \tilde{\varrho}_\varepsilon s(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right) - \right. \tag{8.91}$$

$$\left. \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\tilde{\varrho}_\varepsilon, \overline{\vartheta})}{\varepsilon^2}\right) \right\|_{(L^\infty \cap L^q)(\mathbb{R}^3)} \le c \text{ for all } q > 3/2.$$

8.5.2 Acoustic Equation Revisited

Summing up the previous considerations, we may rewrite the acoustic equation (8.72), (8.73) in a more concise form.

■ ACOUSTIC EQUATION (REVISITED):

$$\varepsilon \int_0^T < Z_\varepsilon(t,\cdot), \partial_t \varphi > \ \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{8.92}$$

$$= -\varepsilon < Z_{0,\varepsilon}, \varphi(0,\cdot) > + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \left(\mathbf{H}_\varepsilon^1 \cdot \nabla_x \varphi + \mathbf{H}_\varepsilon^2 \cdot \nabla_x \varphi \right) \mathrm{d}x \ \mathrm{d}t,$$

for any $\varphi \in C_c^1([0,T) \times \overline{\Omega}_\varepsilon)$,

$$\varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \omega \int_0^T < Z_\varepsilon(t,\cdot), \mathrm{div}_x \boldsymbol{\varphi} > \ \mathrm{d}t \tag{8.93}$$

$$= -\varepsilon \int_{\Omega_\varepsilon} \mathbf{V}_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x$$

$$+ \varepsilon \int_0^T < G_\varepsilon^1(t,\cdot), \mathrm{div}_x \boldsymbol{\varphi} > + \varepsilon \int_0^T \int_\Omega \mathbb{G}_\varepsilon^2 : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}t$$

$$+ \varepsilon \int_0^T \int_\Omega \mathbb{G}_\varepsilon^3 : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \mathbf{G}_\varepsilon^4 \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t,$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$.

Remark Note that, unlike (8.72), (8.73), the weak formulation (8.92), (8.93) already incorporates the satisfaction of the initial conditions.

We have

$$Z_\varepsilon \in C_{\mathrm{weak}-(*)}([0,T]; \mathcal{M}(\overline{\Omega}_\varepsilon)),$$

and

$$Z_\varepsilon = Z_\varepsilon^1 + Z_\varepsilon^2 + Z_\varepsilon^3 + Z^{1,2},$$

where

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|Z_\varepsilon^1(t,\cdot)\|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)} \le \varepsilon c,\ \operatorname*{ess\,sup}_{t\in(0,T)} \|Z_\varepsilon^2(t,\cdot)\|_{L^1(\Omega_\varepsilon)} \le c, \tag{8.94}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|Z_\varepsilon^3(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \le c,\ Z^{1,2} = -\frac{\overline{\varrho}}{\omega}\tilde{F} \in D^{1,2}(\Omega),$$

and

$$Z_{0,\varepsilon} = Z_{0,\varepsilon}^1 + Z_{0,\varepsilon}^2 + Z_{0,\varepsilon}^3 + Z^{1,2}, \tag{8.95}$$

where

$$\|Z_{0,\varepsilon}^1\|_{\mathcal{M}(\overline{\Omega}_\varepsilon)} \le \varepsilon c,\ \|Z_{0,\varepsilon}^2\|_{L^1(\Omega_\varepsilon)} \le c,\ \|Z_{0,\varepsilon}^3\|_{L^2(\Omega_\varepsilon)} \le c. \tag{8.96}$$

Furthermore,

$$\mathbf{V}_\varepsilon = \mathbf{V}_\varepsilon^1 + \mathbf{V}_\varepsilon^2,$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{V}_\varepsilon^1\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} \le \varepsilon c,\ \operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{V}_\varepsilon^2\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \tag{8.97}$$

$$\|\mathbf{V}_{0,\varepsilon}\|_{(L^\infty\cap L^2)(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \tag{8.98}$$

and

$$\mathbf{V}_\varepsilon \in C_{\text{weak}}([0,T];L^1(\Omega_\varepsilon)).$$

Finally,

$$\int_0^T \left(\|\mathbf{H}_\varepsilon^1\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^3)} + \|\mathbf{H}_\varepsilon^2\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \right) \mathrm{d}t \le c, \tag{8.99}$$

$$G^1 \in C_{\text{weak}-(*)}([0,T];\mathcal{M}^+(\overline{\Omega}_\varepsilon)),\ \sup_{t\in(0,T)} \|G^1(t,\cdot)\|_{\mathcal{M}(\overline{\Omega}_\varepsilon)} \le c, \tag{8.100}$$

$$\int_0^T \left(\|\mathbb{G}_\varepsilon^2\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^{3\times3})} + \|\mathbb{G}_\varepsilon^3\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times3})} \right) \mathrm{d}t \le c, \tag{8.101}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{G}_\varepsilon^4(t,\cdot)\|_{(L^{5/3})(\Omega_\varepsilon;\mathbb{R}^3)} \le c, \tag{8.102}$$

where all constants are independent of ε.

8.6 Regularization and Extension to Ω

As already observed and used in several parts of this book, the acoustic equation (8.92), (8.93) provides a suitable platform for studying the time evolution of the gradient component of the velocity field, and, in particular, for establishing the desired property (8.63) that guarantees strong (pointwise) convergence of the velocity fields.

To facilitate the forthcoming discussion it is more convenient

- to deal with classical (strong) solutions to the acoustic system (8.92), (8.93);
- to consider the problem on the limit domain Ω rather than Ω_ε.

8.6.1 Regularization

A standard regularization of generalized functions is provided by a spatial convolution with a family of regularizing kernels $\{\zeta_\delta\}_{\delta>0}$, namely

$$[v]^\delta(x) = \int_{\mathbb{R}^3} \zeta_\delta(x-y)v(y)\,\mathrm{d}y,$$

where the kernels ζ_δ are specified in Sect. 11.2 in Appendix. Note that this can be applied to a general *distribution* $v \in \mathcal{D}'\mathbb{R}^3$, setting

$$[v]^\delta(x) = \langle v; \zeta_\delta(x-\cdot)\rangle \text{ for any } x \in \mathbb{R}^3.$$

Regularization of vector valued functions (distributions) is performed componentwise.

For $\varepsilon > 0$, Ω_ε fixed for a moment, we proceed by regularizing the initial data and the driving forces in (8.92), (8.93).

Regularizing the Initial Data As for $Z^2_{0,\varepsilon}$, we take

$$Z^2_{0,\varepsilon,\delta} = [\chi_\delta Z^2_{0,\varepsilon}]^\gamma,$$

where χ_δ is a cut-off function

$$\chi_\delta(x) = \begin{cases} 1 \text{ for } x \in \Omega_\varepsilon,\ \operatorname{dist}[x, \partial\Omega_\varepsilon] > 1, \\ \\ 0 \text{ otherwise} \end{cases}.$$

It is straightforward to see that

$$\left\|Z^2_{0,\varepsilon,\delta}\right\|_{L^1(\Omega_\varepsilon)} \le \|Z^2_{0,\varepsilon}\|_{L^1(\Omega_\varepsilon)}, \tag{8.103}$$

and that δ, $\gamma(\delta)$ can be adjusted in such a way that

$$Z^2_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon),\ Z^2_{0,\varepsilon,\delta} \to Z^2_{0,\varepsilon} \ \text{ in } L^1(\Omega_\varepsilon) \text{ as } \delta \to 0 \tag{8.104}$$

for any fixed ε.

Applying the same treatment to $Z^3_{0,\varepsilon}$ we obtain $Z^3_{0,\varepsilon,\delta}$,

$$\left\| Z^3_{0,\varepsilon,\delta} \right\|_{L^2(\Omega_\varepsilon)} \leq \| Z^3_{0,\varepsilon} \|_{L^2(\Omega_\varepsilon)}, \tag{8.105}$$

$$Z^3_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon),\ Z^3_{0,\varepsilon,\delta} \to Z^3_{0,\varepsilon} \ \text{ in } L^2(\Omega_\varepsilon) \text{ as } \delta \to 0 \tag{8.106}$$

for any fixed ε.

The "measure-valued" component $Z^1_{0,\varepsilon} \in \mathcal{M}^1(\overline{\Omega}_\varepsilon)$ is slightly more delicate. First, we use the approximation theorem (Theorem 12 in Notation, Definitions, and Function Spaces, Sect. 7) to construct a sequence $\tilde{Z}^1_{0,\varepsilon,\delta}$ such that

$$\tilde{Z}^1_{0,\varepsilon,\delta} \in L^1(\Omega_\varepsilon),\ \tilde{Z}^1_{0,\varepsilon,\delta} \in L^1(\Omega_\varepsilon) \geq 0,\ \left\| \tilde{Z}^1_{0,\varepsilon,\delta} \right\|_{L^1(\Omega_\varepsilon)} \leq \| Z^1_{0,\varepsilon} \|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)},$$
$$\tilde{Z}^1_{0,\varepsilon,\delta} \to Z^1_{0,\varepsilon} \text{ weakly - (*) in } \mathcal{M}(\overline{\Omega_\varepsilon}).$$

(cf. Theorem 12). Next, similarly to the above, we cut-off and regularize the functions $\tilde{Z}^1_{0,\varepsilon,\delta}$ to obtain $Z^1_{0,\varepsilon,\delta}$ such that

$$Z^1_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon),\ Z^1_{0,\varepsilon,\delta} \in L^1(\Omega_\varepsilon) \geq 0,\ \left\| Z^1_{0,\varepsilon,\delta} \right\|_{L^1(\Omega_\varepsilon)} \leq \| Z^1_{0,\varepsilon} \|_{\mathcal{M}^+(\overline{\Omega}_\varepsilon)}, \tag{8.107}$$

$$Z^1_{0,\varepsilon,\delta} \to Z^1_{0,\varepsilon} \text{ weakly - (*) in } \mathcal{M}(\overline{\Omega_\varepsilon}) \text{ for any fixed } \varepsilon > 0, \tag{8.108}$$

specifically,

$$\int_{\Omega_\varepsilon} Z^1_{0,\varepsilon,\delta} \varphi \ \mathrm{d}x \to \left\langle Z^1_{0,\varepsilon}; \varphi \right\rangle_{\mathcal{M}(\overline{\Omega}_\varepsilon), C(\overline{\Omega}_\varepsilon)} \text{ for any } \varphi \in C(\overline{\Omega_\varepsilon}).$$

Finally, with (8.98) in mind, we may construct $\mathbf{V}_{0,\varepsilon,\delta}$,

$$\mathbf{V}_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon; \mathbb{R}^3),\ \| \mathbf{V}_{0,\varepsilon,\delta} \|_{(L^\infty \cap L^2)(\Omega_\varepsilon;\mathbb{R}^3)} \leq \| \mathbf{V}_{0,\varepsilon} \|_{(L^\infty \cap L^2)(\Omega_\varepsilon;\mathbb{R}^3)} \tag{8.109}$$

$$\mathbf{V}_{0,\varepsilon,\delta} \to \mathbf{V}_{0,\varepsilon} \text{ in } L^2(\Omega_\varepsilon; \mathbb{R}^3) \text{ as } \delta \to 0 \tag{8.110}$$

for any fixed ε.

Regularizing the Forcing Terms The forces $\mathbf{H}^j_\varepsilon$, $\mathbb{G}^j_\varepsilon$, $j = 1, 2$, $\mathbf{G}^3_\varepsilon$ can be regularized by means of the following procedure.

- Extend a given function $H \in L^2(0, T; X)$, $X = L^1(\Omega),\ L^2(\Omega),\ \mathcal{M}^+(\Omega)$ to be zero for $t \leq 0$, $t \geq T$.

- Use the regularization *in time* by means of the convolution

$$[H]^{\delta}(\tau) = \int_{-\infty}^{\infty} \zeta_{\delta}(\tau - t)H(t)\, \mathrm{d}t$$

to produce an approximate sequence

$$H^{\delta} \in C^{\infty}(\mathbb{R}; X),\ \|H^{\delta}\|_{L^2(R;X)} \le \|H\|_{L^2(0,T;X)},\ H^{\delta} \to H \text{ in } L^2(0,T;X)$$

cf. Sect. 11.2 in Appendix.

- Approximate H^{δ} by piece-wise constant functions, specifically by H^{δ}_N,

$$H^{\delta}_N = \sum_{j=0}^{N-1} \chi_{[(Tj)/N, T(j+1)/N]} h_j,\ h_j \in X.$$

- Similarly to the preceding section, approximate each function $h_j \in X$ by $\tilde{h}_j \in C^{\infty}_c(\Omega_{\varepsilon})$ producing

$$\tilde{H}^{\delta}_N = \sum_{j=0}^{N-1} \chi_{[(Tj)/N, T(j+1)/N]} \tilde{h}_j.$$

- Regularize the functions $\tilde{H}^{\delta}_N$ performing once more the time convolution

$$\left[\tilde{H}^{\delta}_N\right]^{\delta}(\tau) = \int_{-\infty}^{\infty} \zeta_{\delta}(\tau - t)\, \tilde{H}^{\delta}_N(t) \mathrm{d}t.$$

Going back to the acoustic equation (8.92), (8.93), we may regularize the forcing terms as follows:

$$\mathbf{H}^{j}_{\varepsilon,\delta} \in C^{\infty}_c([0,T] \times \Omega_{\varepsilon}; \mathbb{R}^3),\ j = 1,2,$$

$$\int_0^T \left(\|\mathbf{H}^1_{\varepsilon,\delta}\|^2_{L^1(\Omega_{\varepsilon};\mathbb{R}^3)} + \|\mathbf{H}^2_{\varepsilon,\delta}\|^2_{L^2(\Omega_{\varepsilon};\mathbb{R}^3)} \right) \mathrm{d}t \tag{8.111}$$

$$\le \int_0^T \left(\|\mathbf{H}^1_{\varepsilon}\|^2_{L^1(\Omega_{\varepsilon};\mathbb{R}^3)} + \|\mathbf{H}^2_{\varepsilon}\|^2_{L^2(\Omega_{\varepsilon};\mathbb{R}^3)} \right) \mathrm{d}t,$$

$$\mathbf{H}^{j}_{\varepsilon,\delta} \to \mathbf{H}^{j}_{\varepsilon} \text{ in } L^2(0,T; L^j(\Omega_{\varepsilon}; \mathbb{R}^3)) \text{ as } \delta \to 0,\ j = 1,2 \tag{8.112}$$

for any fixed $\varepsilon > 0$;

$$\mathbb{G}^j_{\varepsilon,\delta} \in C^\infty_c([0,T] \times \Omega_\varepsilon; \mathbb{R}^{3\times 3}),\ j = 2, 3,$$

$$\int_0^T \left(\|\mathbb{G}^1_{\varepsilon,\delta}\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} + \|\mathbb{G}^2_{\varepsilon,\delta}\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} \right) \mathrm{d}t, \tag{8.113}$$

$$\leq \int_0^T \left(\|\mathbb{G}^1_{\varepsilon}\|^2_{L^1(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} + \|\mathbb{G}^2_{\varepsilon}\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^{3\times 3})} \right) \mathrm{d}t$$

$$\mathbb{G}^2_{\varepsilon,\delta} \to \mathbb{G}^2_\varepsilon \text{ in } L^2(0,T; L^1(\Omega_\varepsilon; \mathbb{R}^{3\times 3})) \text{ as } \delta \to 0, \tag{8.114}$$

$$\mathbb{G}^3_{\varepsilon,\delta} \to \mathbb{G}^3_\varepsilon \text{ in } L^2(0,T; L^2(\Omega_\varepsilon; \mathbb{R}^{3\times 3})) \text{ as } \delta \to 0, \tag{8.115}$$

and

$$\sup_{t\in[0,T]} \left\| \mathbf{G}^4_{\varepsilon,\delta}(t,\cdot) \right\|_{L^{5/3}(\Omega;\mathbb{R}^3)} \leq \operatorname{ess} \sup_{t\in[0,T]} \left\| \mathbf{G}^4_{\varepsilon}(t,\cdot) \right\|_{L^{5/3}(\Omega;\mathbb{R}^3)}, \tag{8.116}$$

$$\mathbf{G}^4_{\varepsilon,\delta} \to \mathbf{G}^3_\varepsilon \text{ in } L^p(0,T; L^{5/3}(\Omega_\varepsilon; \mathbb{R}^{3\times 3})),\ 1 \leq p < \infty \text{ as } \delta \to 0 \tag{8.117}$$

for any fixed $\varepsilon > 0$.

Finally, we find

$$G^1_{\varepsilon,\delta} \in C^\infty_c([0,T] \times \Omega)$$

such that

$$\sup_{t\in[0,T]} \|G^1_{\varepsilon,\delta}\|_{L^1(\Omega_\varepsilon)} \leq \sup_{t\in[0,T]} \|G^1_\varepsilon\|_{\mathcal{M}(\overline{\Omega}_\varepsilon)}, \tag{8.118}$$

$$\int_{\Omega_\varepsilon} G^1_{\varepsilon,\delta}(t,\cdot)\varphi \, \mathrm{d}x \to \left\langle G^1_\varepsilon(t,\cdot), \varphi \right\rangle \text{ for any } \varphi \in C(\overline{\Omega}_\varepsilon),\ t \in [0,T] \text{ as } \delta \to 0 \tag{8.119}$$

for any fixed $\varepsilon > 0$.

8.6.2 Reduction to Smooth Data

We recall that our ultimate goal is to show (8.63), or, in terms of the present notation,

$$\left\{ t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T), \tag{8.120}$$

for any fixed $\boldsymbol{\varphi} \in C_c^\infty(\Omega; \mathbb{R}^3)$. For the rest of this section we therefore fix $\boldsymbol{\varphi}$ and suppose its support is contained in a ball $B \subset \Omega$.

As it is definitely more convenient to replace the abstract weak formulation of the acoustic equation by a classical one, meaning to consider the regularized data constructed in the previous section, we show that the error in (8.120) resulting from such a simplification can be made arbitrarily small.

Step 1: Eliminating the Initial Data $Z^{1,2}$ We start by the term $Z^{1,2}$ appearing in (8.95). For a given (small) constant $\zeta > 0$, we find a function $Z_\zeta^{1,2}$,

$$Z_\zeta^{1,2} \in C_c^\infty(\Omega),\ \|\nabla_x Z_\zeta^{1,2} - \nabla_x Z^{1,2}\|^2_{L^2(\Omega;\mathbb{R}^3)} < \zeta.$$

In view of (8.49),

$$Z_\zeta^{1,2} \in C_c^\infty(\Omega_\varepsilon)$$

as soon as $0 < \varepsilon < \varepsilon_0(\zeta)$.

We estimate the error resulting from replacing $Z^{1,2}$ by $Z_\zeta^{1,2}$ in the acoustic equation. More specifically, we look for (weak) solutions to the problem

$$\varepsilon \partial_t Z_\zeta + \mathrm{div}_x \mathbf{V}_\zeta = 0,\ \varepsilon \partial_t \mathbf{V}_\zeta + \omega \nabla_x Z_\zeta = 0 \text{ in } (0,T) \times \Omega_\varepsilon,\ \mathbf{V}_\zeta \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

with that initial data

$$Z_\zeta(0,\cdot) = Z^{1,2} - Z_\zeta^{1,2},\ \mathbf{V}_\zeta(0,\cdot) = 0,$$

or, more precisely, in its weak formulation

$$\varepsilon \int_0^T \int_{\Omega_\varepsilon} Z_\zeta(t,\cdot) \cdot \partial_t \varphi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t \tag{8.121}$$

$$= - \int_{\Omega_\varepsilon} \varepsilon \left(Z^{1,2} - Z_\eta^{1,2} \right) \varphi(0,\cdot)\,\mathrm{d}x \text{ for any } \varphi \in C_c^1([0,T) \times \overline{\Omega}_\varepsilon),$$

$$\varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \omega \int_0^T \int_{\Omega_\varepsilon} Z_\zeta(t,\cdot) \mathrm{div}_x \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = 0 \tag{8.122}$$

$$\text{for any } \boldsymbol{\varphi} \in C_c^1([0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3),\ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

System (8.121), (8.122) can be seen as a weak formulation of the standard *acoustic wave equation* with the initial data

$$Z_\zeta(0,\cdot) \in W^{1,2}(\Omega_\varepsilon),\ \mathbf{V}_\zeta(0,\cdot) = 0$$

belonging to the associated energy space $W^{1,2} \times W^{1,2}_{\rm n}$. Consequently, the problem admits a unique solution

$$Z_\zeta \in C([0,T]; W^{1,2}(\Omega_\varepsilon)), \tag{8.123}$$

$$\mathbf{V}_\zeta = \nabla_x \Psi_\zeta \in W^{1,2}(\Omega_\varepsilon), \ \int_{\Omega_\varepsilon} \Psi_\zeta \ {\rm d}x = 0, \ \nabla_x \Psi_\zeta \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

satisfying the energy balance

$$\begin{aligned}\int_{\Omega_\varepsilon} \omega|\nabla_x Z_\zeta(t,\cdot)|^2 + |{\rm div}_x \mathbf{V}_\zeta(t,\cdot)|^2 \ {\rm d}x &= \int_{\Omega_\varepsilon} \omega|\nabla_x Z_\zeta(0,\cdot)|^2 + \omega|{\rm div}_x \mathbf{V}_\zeta(0,\cdot)|^2 \ {\rm d}x \\ &= \omega\|\nabla_x Z^{1,2}_\eta - \nabla_x Z^{1,2}\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le \omega\zeta,\end{aligned} \tag{8.124}$$

cf. Sect. 11.1 in Appendix.

To proceed, we need to show that solutions of system (8.121), (8.122) admits a finite speed of propagation proportional to $\sqrt{\omega}/\varepsilon$. This can be seen by "integrating" (8.121), (8.122) over the space-time cone

$$\mathcal{C} = \left\{ (t,x) \ \Big| \ t \in (0,\tau), \ x \in B \cap \Omega, \ {\rm dist}[\partial B] > \frac{\sqrt{\omega}}{\varepsilon} t \right\}$$

where $B = B(r,0)$ is a ball (centered at zero) containing $\partial\Omega$ in its interior. As Z_η, $\mathbf{V}_\eta$ belong to $W^{1,2}(C)$ (the time derivatives being computed from the equations), the Gauss-Green theorem can be used to obtain

$$\begin{aligned}0 &= \int_{\mathcal{C}} \left[\omega\partial_t Z_\zeta Z_\zeta + \omega{\rm div}_x \mathbf{V}_\zeta Z_\zeta + \partial_t \mathbf{V}_\zeta \cdot \mathbf{V}_\zeta + \frac{\omega}{\varepsilon}\nabla_x Z_\zeta \cdot \mathbf{V}_\zeta \right] {\rm d}x \ {\rm d}t \\ &= \int_{\mathcal{C}} \left[\frac{1}{2}\partial_t \left(\omega|Z_\zeta|^2 + |\mathbf{V}|^2\right) + \frac{\omega}{\varepsilon}{\rm div}_x \left(Z_\zeta \mathbf{V}_\zeta\right) \right] {\rm d}x \ {\rm d}t \\ &= \left[\int_{\left\{|x| < r - \frac{\sqrt{\omega}}{\varepsilon}t\right\} \cap \Omega} \frac{1}{2}\left(\omega|Z_\zeta|^2 + |\mathbf{V}|^2\right) {\rm d}x \right]_{t=0}^{t=\tau} \\ &\quad + \int_{\left\{t \in (0,\tau), \ x = r - \frac{\sqrt{\omega}}{\varepsilon}t\right\}} \left[\frac{1}{2}\left(\omega|Z_\zeta|^2 + |\mathbf{V}|^2\right) n_t + \frac{\omega}{\varepsilon} Z_\zeta \mathbf{V}_\zeta \mathbf{n}_x \right] {\rm dS}_{t,x},\end{aligned}$$

where

$$\frac{1}{2}\left(\omega|Z_\zeta|^2 + |\mathbf{V}|^2\right) n_t + \frac{\omega}{\varepsilon} Z_\zeta \mathbf{V}_\zeta n_x \ge \frac{1}{2}\omega \left(|Z_\zeta|^2 + \left| \frac{\mathbf{V}}{\sqrt{\omega}} \right|^2 \right) \left(n_t - \frac{\sqrt{\omega}}{\varepsilon}|\mathbf{n}_x| \right) \ge 0$$

yielding the desired conclusion

$$\int_{\left\{|x|<r-\frac{\sqrt{\omega}}{\varepsilon}\tau\right\}\cap\Omega} \frac{1}{2}\left(\omega|Z_\zeta|^2+|\mathbf{V}|^2\right)\,\mathrm{d}x \le \int_{\{|x|<r\}\cap\Omega} \frac{1}{2}\left(\omega|Z_\zeta|^2+|\mathbf{V}|^2\right)\,\mathrm{d}x \tag{8.125}$$

for any $0 \le \tau \le T$.

Recalling our goal—proving (8.120)—we realize that what matters is only the behavior of the solution $\mathbf{V}_\zeta$ on the fixed compact set containing supp$[\boldsymbol{\varphi}]$. As the family Ω_ε enjoys Property (L) specified through (8.49), and i view of the finite speed of propagation property enjoyed by solutions of (8.121), (8.122), we may therefore replace $\mathbf{V}_\zeta$ in by a weak solution $\tilde{\mathbf{V}}_\eta = \nabla_x \tilde{\Psi}_\zeta$ of the same system on the limit domain Ω. Accordingly,

$$\int_\Omega \mathbf{V}_\zeta \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \tilde{\mathbf{V}}_\zeta \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \nabla_x \tilde{\Psi}_\zeta \cdot \mathbf{H}^\perp \boldsymbol{\varphi}\,\mathrm{d}x \tag{8.126}$$

$$= \int_\Omega \nabla_x \tilde{\Psi}_\zeta \cdot \mathbf{H}^\perp \boldsymbol{\varphi}\,\mathrm{d}x = -\int_\Omega \Delta \tilde{\Psi}_\zeta\ \Phi\,\mathrm{d}x,$$

where $\mathbf{H}$ denotes the Helmholtz projection on the limit domain Ω and $\mathbf{H}^\perp[\boldsymbol{\varphi}] = \nabla_x \Phi$. Note that, similarly to $\Delta\Psi_\zeta$,

$$\sup_{t\in(0,T)} \|\Delta\tilde{\Psi}_\zeta\|^2_{L^2(\Omega)} \le \omega\zeta$$

by virtue of teh energy bounds stated in (8.124). Finally, as $\varphi \in C_c^\infty(B;\mathbb{R}^3)$, we get $\Phi \in D^{1,p}(\Omega)$,

$$\|\nabla_x \Phi\|_{L^p(\Omega;\mathbb{R}^3)} \le c(p) \text{ for all } 1 < p < \infty,$$

in particular, by virtue of Sobolev inequality, $\Phi \in L^2(\Omega)$ (cf. Theorem 7). Thus the bound (8.124) yields the desired conclusion

$$\left|\int_\Omega \mathbf{V}_\zeta \cdot \boldsymbol{\varphi}\,\mathrm{d}x\right| \le \|\Delta\tilde{\Psi}_\zeta\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)} \le c(\boldsymbol{\varphi})\sqrt{\zeta}, \tag{8.127}$$

meaning the error in (8.120) can be made small if we replace $Z^{1,2}$ by $Z^{1,2}_\eta$ in (8.95).

Step 2: Approximating Data Given by Measure The next step is to estimate the error in (8.120) if we replace $[Z_\varepsilon, \mathbf{V}_\varepsilon]$ by the solution of the same system endowed with the mollified initial data

$$Z_{\varepsilon,\delta}(0,\cdot) = Z^1_{0,\varepsilon,\delta} + Z^2_{0,\varepsilon,\delta} + Z^3_{0,\varepsilon,\delta} + Z^{1,2}_\zeta,\ \mathbf{V}_{\varepsilon,\delta}(0,\cdot) = \mathbf{V}_{0,\delta},$$

and with the driving forces determined through the regularized functions

$$\mathbf{H}^j_{\varepsilon,\delta}, \ \ \mathbb{G}^j_{\varepsilon,\delta}, \ j = 1,2, \ \mathbf{G}^3_{\varepsilon,\delta}$$

identified in Sect. 8.6.1. As the deviation between the solution of the homogeneous acoustic system emanating from the data $[Z^{1,2}, 0]$ and $[Z^{1,2}_\eta, 0]$ has been estimated in the previous part, our goal reduces to showing

$$\sup_{t \in (0,T)} \left| \int_\Omega \mathbf{V}_\zeta(t,\cdot) \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right| < o(\zeta), \ o(\zeta) \to 0 \text{ as } \zeta \to 0. \tag{8.128}$$

for a given (small) $\zeta > 0$, where $\boldsymbol{\varphi}$ is the same as in (8.120), and $[Z_\zeta, \mathbf{V}_\zeta]$ is a (weak) solution of the acoustic system

$$\varepsilon \int_0^T < Z_\zeta(t,\cdot), \partial_t \varphi > \ \mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{8.129}$$

$$= -\varepsilon < Z_{0,\zeta}, \varphi(0,\cdot) > + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbf{H}_\zeta \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t,$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega}_\varepsilon)$,

$$\varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \partial_t \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t + \omega \int_0^T < Z_\zeta(t,\cdot), \mathrm{div}_x \boldsymbol{\varphi} > \ \mathrm{d}t \tag{8.130}$$

$$= -\varepsilon \int_{\Omega_\varepsilon} \mathbf{V}_{0,\zeta} \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x$$

$$+\varepsilon \int_0^T \left\langle g_\zeta, \mathrm{div}_x \boldsymbol{\varphi} \right\rangle \ \mathrm{d}t + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbb{G}_\zeta(t,\cdot) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}t + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \mathbf{h}_\zeta \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t,$$

for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}_\varepsilon; \mathbb{R}^3)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$, with the initial data

$$Z_{0,\zeta} = \sum_{j=1}^3 \left(Z^j_{0,\varepsilon} - Z^j_{0,\varepsilon,\delta} \right), \ \mathbf{V}_{0,\zeta} = \mathbf{V}_{0,\varepsilon} - \mathbf{V}_{0,\varepsilon,\delta},$$

and the forces

$$\mathbf{H}_\zeta = \sum_{j=1}^2 \left(\mathbf{H}^j_\varepsilon - \mathbf{H}^j_{\varepsilon,\delta} \right), \ \mathbb{G}_\zeta = \sum_{j=1}^2 \left(\mathbb{G}^j_\varepsilon - \mathbb{G}^j_{\varepsilon,\delta} \right), \mathbf{g}_\zeta = \mathbf{G}^3_\varepsilon - \mathbf{G}^2_{\varepsilon,\delta}.$$

To begin, we fix $\varepsilon = \varepsilon(\zeta)$ is in **Step 1** to guarantee (8.127). With ε fixed and the approximation estimates (8.104), (8.106), (8.108), we may take $\delta = \delta(\varepsilon)$ so small that

$$D_{\mathcal{M}(\overline{\Omega}_\varepsilon)}[Z_{0,\zeta}, 0] < \zeta, \tag{8.131}$$

where D denotes the metric in the $\mathcal{M}$ weak-(*) topology on bounded sets in $\mathcal{M}(\overline{\Omega}_\varepsilon)$. Next, by virtue of (8.110),

$$\|\mathbf{V}_\zeta\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} < \zeta. \tag{8.132}$$

Similarly, evoking (8.112), (8.114), (8.115) we get

$$\|\mathbf{H}_\zeta\|_{L^2(0,T;L^1(\Omega;\mathbb{R}^{3\times 3}))} < \zeta, \ \|\mathbb{G}_\zeta\|_{L^2(0,T;L^1(\Omega;\mathbb{R}^{3\times 3}))} < \zeta, \tag{8.133}$$

and, by virtue of (8.117),

$$\|\mathbf{h}_\zeta\|_{L^p(0,T;L^{5/3}(\Omega;\mathbb{R}^3))} < c(p)\zeta, \ 1 \le p < \infty. \tag{8.134}$$

Finally, in accordance with (8.119),

$$g_\zeta \in C_{\text{weak}-(*)}([0,T]; \mathcal{M}(\overline{\Omega}_\varepsilon)),$$

$$\int_0^T \left| D_{\mathcal{M}(\overline{\Omega}_\varepsilon)}[g_\zeta(t,\cdot), 0] \right|^p \, \mathrm{d}t < c(p)\zeta, \ 1 \le p < \infty. \tag{8.135}$$

Remark Note that, as $\varepsilon > 0$ is fixed, the L^2-norm dominates the L^1-norm in Ω_ε.

Roughly speaking, we have to show that solutions of the acoustic system (8.129), (8.130) with "small" data are "small". The main difficulty is that the data are very irregular (measures) and so are the solutions. Note, however, that regularity of $[Z_\zeta, \mathbf{V}_\zeta]$ is the same as that of $[Z_\varepsilon, \mathbf{V}_\varepsilon]$ as the approximate data are regular.

Writing

$$\int_\Omega \mathbf{V}_\zeta \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \left(\mathbf{H}[\boldsymbol{\varphi}] + \mathbf{H}^\perp[\boldsymbol{\varphi}] \right) \, \mathrm{d}x,$$

where $\mathbf{H}$ is the Helmholtz projection in Ω_ε, we immediately see by taking $\psi(t)\mathbf{H}[\boldsymbol{\varphi}]$, $\psi \in C_c^\infty([0,T)$ as test function in (8.130) that

$$\left\{ t \mapsto \int_{\Omega_\varepsilon} \mathbf{V}_\zeta \cdot \mathbf{H}[\varphi] \, \mathrm{d}x \right\} \to 0 \text{ in } C[0,T] \text{ as } \zeta \to 0.$$

Thus showing (8.128) reduces to

$$\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \mathbf{V}_\zeta(t,\cdot)\cdot \mathbf{H}^\perp[\varphi]\ \mathrm{d}x \right| < o(\zeta),\ o(\zeta)\to 0 \text{ as } \zeta \to 0. \tag{8.136}$$

Our idea, similar to Sect. 5.4.6, is to regularize (8.129), (8.130) by means of the spectral projections associated to the Neumann Laplacian $\Delta_{\mathcal{N},\Omega_\varepsilon}$,

$$\Delta_{\mathcal{N},\Omega_\varepsilon} v = \Delta v$$

defined on

$$\mathcal{D}[\Delta_{\mathcal{N},\Omega_\varepsilon}] = \left\{ v \in W^{2,2}(\Omega_\varepsilon) \ \middle|\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0 \text{ (in the sense of traces)} \right\}.$$

It is well-known that if $\partial\Omega_\varepsilon$ is regular, the operator $-\Delta_{\mathcal{N},\Omega_\varepsilon}$ generates a self-adjoint non-negative operator on the space $L^2(\Omega_\varepsilon)$. In particular, as Ω_ε is bounded, the eigenvalue problem

$$-\Delta w_n = \Lambda_n w_n \text{ in } \Omega_\varepsilon,\ \nabla_x w_n \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

admits a countable sequence of eigenvalues $\Lambda_0 = 0 < \Lambda_1 \leq \Lambda_2 \ldots$, where the eigenspace associated to Λ_0 is spanned by constants, cf. (5.146). In particular, we may define the functional calculus and the functions of $-\Delta_{\mathcal{N},\Omega_\varepsilon}$ by means of a simple formula

$$G(-\Delta_{\mathcal{N},\Omega_\varepsilon})[v] = \sum_{j=0}^{\infty} G(\Lambda_n) a_n[v] w_n,\ a_n[v] = \int_{\Omega_\varepsilon} v w_n\ \mathrm{d}x$$

see Sect. 11.1 in Appendix. We may also define a scale of Hilbert spaces

$$\mathcal{D}(-\Delta_{\mathcal{N},\Omega_\varepsilon}{}^{\alpha}) = \left\{ v \in L^2(\Omega_\varepsilon) \ \middle|\ \sum_{j=0}^{\infty} \Lambda_n^{2\alpha} |a_n(v)|^2 < \infty,\ \int_{\Omega_\varepsilon} v\ \mathrm{d}x = 0 \text{ if } \alpha < 0 \right\}$$

Since $\mathcal{D}(-\Delta_{\mathcal{N},\Omega_\varepsilon}) \subset W^{2,2}(\Omega_\varepsilon)$, where $W^{2,2}(\Omega_\varepsilon)$ is compactly embedded in $C(\overline{\Omega}_\varepsilon)$, bounded sets in $\mathcal{M}(\overline{\Omega}_\varepsilon)$ are compact in the dual space $\mathcal{D}((-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1})$. In particular, the linear form

$$\varphi \mapsto \int_{\Omega_\varepsilon} \mathbf{H}_\zeta(t,\cdot)\cdot \nabla_x \varphi\ \mathrm{d}x$$

can be understood as a bounded linear form acting on $\mathcal{D}((-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2})$. Applying the Riesz representation theorem we get

$$\int_{\Omega_\varepsilon} \mathbf{H}_\zeta(t,\cdot)\cdot \nabla_x \varphi\ \mathrm{d}x = \int_{\Omega_\varepsilon} \chi_\zeta^1(t,\cdot)(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}[\varphi]\ \mathrm{d}x, \tag{8.137}$$

with

$$\|\chi^1(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \leq c\|\mathbf{H}_\zeta(t,\cdot)\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)}$$

Next, we take a test function $\nabla_x\varphi$, $\nabla_x\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0$ in (8.130) to obtain

$$\varepsilon\int_0^T\int_{\Omega_\varepsilon}\mathbf{V}_\zeta\cdot\partial_t\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t+\omega\int_0^T\left\langle Z_\zeta(t,\cdot),\Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi]\right\rangle\,\mathrm{d}t$$
$$=-\varepsilon\int_{\Omega_\varepsilon}\mathbf{V}_{0,\zeta}\cdot\nabla_x\varphi(0,\cdot)\,\mathrm{d}x+\varepsilon\int_0^T\left\langle g_\zeta,\Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi]\right\rangle\,\mathrm{d}t$$
$$+\varepsilon\int_0^T\int_{\Omega_\varepsilon}\mathbb{G}_\zeta(t,\cdot):\nabla_x{}^2\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t+\varepsilon\int_0^T\int_{\Omega_\varepsilon}\mathbf{h}_\zeta\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Here, similarly to (8.137), we have

$$\int_{\Omega_\varepsilon}\mathbb{G}_\zeta(t,\cdot):\nabla_x{}^2\varphi\,\mathrm{d}x=\int_{\Omega_\varepsilon}\chi^2(t,\cdot)(-\Delta_{\mathcal{N},\Omega_\varepsilon})^2[\varphi]\,\mathrm{d}x, \tag{8.138}$$

with

$$\|\chi^2(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \leq c\|\mathbb{G}_\zeta(t,\cdot)\|_{L^1(\Omega_\varepsilon;\mathbb{R}^{3\times3})},$$

and, similarly,

$$\int_{\Omega_\varepsilon}\mathbf{h}_\zeta(t,\cdot)\cdot\nabla_x\varphi\,\mathrm{d}x=\int_{\Omega_\varepsilon}\chi^3(t,\cdot)(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}[\varphi]\,\mathrm{d}x \tag{8.139}$$

with

$$\|\chi^3(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \leq c\|\mathbf{h}_\zeta(t,\cdot)\|_{L^1(\Omega_\varepsilon;\mathbb{R}^3)}.$$

Finally, since the embedding $\mathcal{D}((-\Delta_{\mathcal{N},\Omega_\varepsilon})^2)\hookrightarrow C(\overline{\Omega}_\varepsilon)$ is *compact*, we have

$$\left\langle g_\zeta(t,\cdot),\Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi]\right\rangle=\int_{\Omega_\varepsilon}\chi^4(t,\cdot)(-\Delta_{\mathcal{N},\Omega_\varepsilon})^2[\varphi]\,\mathrm{d}x, \tag{8.140}$$

with

$$\|\chi^4(t,\cdot)\|_{L^2(\Omega_\varepsilon)} \leq cD_{\mathcal{M}(\overline{\Omega}_\varepsilon)}[g_\zeta(t,\cdot),0].$$

Writing

$$\mathbf{V}_\zeta=\mathbf{H}[\mathbf{V}_\zeta]+\nabla_x\Psi_\zeta$$

we may reformulate the acoustic system (8.129), (8.130) as

$$\varepsilon \int_0^T < Z_\zeta(t,\cdot), \partial_t \varphi > \ \mathrm{d}t - \int_0^T \int_{\Omega_\varepsilon} \Psi_\zeta \cdot \Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi] \ \mathrm{d}x \ \mathrm{d}t \tag{8.141}$$

$$= -\varepsilon < Z_{0,\zeta}, \varphi(0,\cdot) > +\varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^1 (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}[\varphi] \ \mathrm{d}x \ \mathrm{d}t,$$

for any $\varphi \in C^1([0,T], \mathcal{D}((-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}))$, $\varphi(T,\cdot) = 0$,

$$-\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Psi_\zeta \cdot \partial_t \Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi] \ \mathrm{d}x \ \mathrm{d}t + \omega \int_0^T \left\langle Z_\zeta(t,\cdot), \Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi] \right\rangle \ \mathrm{d}t$$

$$= \varepsilon \int_{\Omega_\varepsilon} \Psi_{0,\zeta} \Delta_{\mathcal{N},\Omega_\varepsilon}[\varphi](0,\cdot) \ \mathrm{d}x + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^4 (-\Delta_{\mathcal{N},\Omega_\varepsilon})^2[\varphi] \mathrm{d}t$$

$$+\varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^2 (-\Delta_{\mathcal{N},\Omega_\varepsilon})^2[\varphi] \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}t + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^3 (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}[\varphi] \ \mathrm{d}x \ \mathrm{d}t,$$

where the latter can be rephrased as

$$-\varepsilon \int_0^T \int_{\Omega_\varepsilon} \Psi_\zeta \cdot \partial_t \varphi \ \mathrm{d}x \ \mathrm{d}t + \omega \int_0^T \left\langle Z_\zeta(t,\cdot), \varphi \right\rangle \ \mathrm{d}t \tag{8.142}$$

$$= \varepsilon \int_{\Omega_\varepsilon} \Psi_{0,\zeta} \varphi(0,\cdot) \ \mathrm{d}x + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^4 (-\Delta_{\mathcal{N},\Omega_\varepsilon})[\varphi] \mathrm{d}t$$

$$+\varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^2 (-\Delta_{\mathcal{N},\Omega_\varepsilon})[\varphi] \ \mathrm{d}x \ \mathrm{d}t \ \mathrm{d}t + \varepsilon \int_0^T \int_{\Omega_\varepsilon} \chi^3 (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}[\varphi] \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C^1([0,T], \mathcal{D}((-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}))$, $\varphi(T,\cdot) = 0$.

Remark Formally, the system of equations (8.141), (8.142) can be written as

$$\varepsilon \partial_t Z_\zeta + \Delta_{\mathcal{N},\Omega_\varepsilon} \Psi_\zeta = -\varepsilon (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{3/2}[\chi^1],$$

$$\varepsilon \partial_t \Phi_\zeta + \omega Z_\zeta = \varepsilon (-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^4] + \varepsilon (-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^2] + \varepsilon (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}[\chi^3].$$

Such a formulation can be rigorously justified at the level of individual projections onto the eigenfunctions of the operator $\Delta_{\mathcal{N},\Omega_\varepsilon}$, which corresponds to taking the test functions in (8.141), (8.142) in the form

$$\varphi = G(-\Delta_{\mathcal{N},\Omega_\varepsilon})[w], \ G \in C_c^\infty(0,\infty).$$

Note that such a procedure has already been performed in Sect. 5.4.6.

Solutions, or rather their spectral projections, of the linear system (8.141), (8.142) can be conveniently expressed by means of the *variation-of-constants formula*, namely

$$\Psi_\zeta(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}\right)\left[\Psi_{0,\zeta} + \mathrm{i}\omega\,(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1/2}\,[Z_{0,\zeta}]\right] \tag{8.143}$$
$$+\frac{1}{2}\exp\left(-\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}\right)\left[\Psi_{0,\zeta} - \mathrm{i}\omega\,(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1/2}\,[Z_{0,\zeta}]\right]$$
$$+\frac{1}{2}\int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}\right)\big[(-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^4]$$
$$+(-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^2] + (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}[\chi^3] - \mathrm{i}\omega\,(-\Delta_{\mathcal{N},\Omega_\varepsilon})\,[\chi^1]\big]\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}\right)\big[(-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^4] + (-\Delta_{\mathcal{N},\Omega_\varepsilon})[\chi^2]$$
$$+(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{1/2}[\chi^3] + \mathrm{i}\omega\,(-\Delta_{\mathcal{N},\Omega_\varepsilon})\,[\chi^1]\big]\,\mathrm{d}s,$$

where we have set

$$\nabla_x \Psi_{0,\zeta} = \mathbf{H}^\perp[\mathbf{V}_{0,\zeta}].$$

In accordance with (8.132), we have

$$\|\nabla_x \Psi_{0,\zeta}\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \le \zeta;\ \text{whence}\ \Psi_\zeta = (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1/2}[\psi_\zeta],\ \|\psi_\zeta\|_{L^2(\Omega_\varepsilon)} \le c\zeta.$$

Remark A similar formula holds for Z_ζ, however, we do not need it here.

The identity between Ψ_ζ and the expression on the right-hand side of (8.143) is to be understood in the sense of the Fourier coefficients

$$a_n = \int_{\Omega_\varepsilon} \Psi_\zeta w_n\,\mathrm{d}x,\ n = 1, 2, \ldots$$

w_n being the eigenfunctions of $(-\Delta_{\mathcal{N},\Omega_\varepsilon})$. In view of the uniform bounds established in (8.131)–(8.135), in combination with (8.137)–(8.140), it is easy to deduce from formula (8.143) that

$$\Psi_\zeta(t,\cdot) = (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1}[\psi_\zeta(t,\cdot)],$$

where

$$\sup_{t\in[0,T]} \|\psi_\zeta(t,\cdot)\|_{L^2(\Omega_\varepsilon)} = o(\zeta),\ o(\zeta)\to 0 \text{ as } \zeta\to 0.$$

Going back to (8.136) we easily observe that

$$\int_{\Omega_\varepsilon} \mathbf{V}_\zeta(t,\cdot)\cdot \mathbf{H}^\perp[\varphi]\,\mathrm{d}x = -\int_{\Omega_\varepsilon} \Psi_\zeta \mathrm{div}_x\varphi\,\mathrm{d}x$$

$$= -\int_{\Omega_\varepsilon} (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1}[\psi_\zeta(t,\cdot)]\mathrm{div}_x\varphi\,\mathrm{d}x$$

$$= -\int_{\Omega_\varepsilon} (-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1}\psi_\zeta(t,\cdot)(-\Delta_{\mathcal{N},\Omega_\varepsilon})^{-1}[\mathrm{div}_x\varphi]\,\mathrm{d}x;$$

whence (8.136) follows as $\varphi\in C_c^\infty(\Omega_\varepsilon)$.

Step 3: Extension to Ω As shown in the previous two steps, the desired property (8.120) can be verified replacing the original problem (with irregular data) by the problem with regularized and compactly supported data specified in Sect. 8.6.1. Moreover, extending the data to be zero in $\Omega_\varepsilon\setminus\Omega$ we may use the finite speed of propagation property established in (8.124), together with Property (L), to observe that we may consider the problem defined on the target domain Ω. Thus our task reduces to the following problem

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■ PROBLEM (D):

For a given $\varphi\in C_c^\infty(\Omega)$ show that

$$\left\{t\mapsto \int_\Omega \mathbf{V}_\varepsilon\cdot\varphi\,\mathrm{d}x\right\} \text{ is precompact in } L^2(0,T) \text{ as } \varepsilon\to 0, \tag{8.144}$$

where $[Z_\varepsilon,\mathbf{V}_\varepsilon]$ is a family of (regular) solutions of the acoustic system

$$\varepsilon\partial_t Z_\varepsilon + \mathrm{div}_x\mathbf{V}_\varepsilon = \varepsilon\mathrm{div}_x\mathbf{H}_\varepsilon \tag{8.145}$$

$$\varepsilon\partial_t\mathbf{V}_\varepsilon + \omega\nabla_x Z_\varepsilon = \varepsilon\left(\mathrm{div}_x\mathbb{G}_\varepsilon + \mathbf{g}\right) \tag{8.146}$$

with the Neumann boundary conditions

$$\mathbf{V}_\varepsilon\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{8.147}$$

and the far field conditions

$$\mathbf{V}_\varepsilon,\ Z_\varepsilon\to 0 \text{ as } |x|\to\infty, \tag{8.148}$$

and the initial data

$$Z_\varepsilon(0,\cdot) = Z_{0,\varepsilon}, \mathbf{V}_{\varepsilon,0}(0,\cdot) = \mathbf{V}_{0,\varepsilon}. \tag{8.149}$$

The data enjoy the following regularity properties:

$$\left\{ \begin{array}{c} Z_{0,\varepsilon} \in C_c^\infty(\Omega), \ \mathbf{V}_{0,\varepsilon} \in C_c^\infty(\Omega;\mathbb{R}^3), \\ \\ \|Z_{0,\varepsilon}\|_{(L^1+L^2+D^{1,2})(\Omega)} \leq c, \ \ \|\mathbf{V}_\varepsilon\|_{(L^2\cap L^\infty)(\Omega;\mathbb{R}^3)} \leq c, \end{array} \right\} \tag{8.150}$$

$$\mathbf{H} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3), \int_0^T \|\mathbf{H}\|^2_{(L^1+L^2)(\Omega;\mathbb{R}^3)} \,\mathrm{d}t \leq c, \tag{8.151}$$

$$\mathbb{G} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^{3\times 3}), \int_0^T \|\mathbf{H}\|^2_{(L^1+L^2)(\Omega;\mathbb{R}^{3\times 3})} \,\mathrm{d}t \leq c, \tag{8.152}$$

and

$$\mathbf{g} \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3), \ \sup_{t\in[0,T]} \|\mathbf{g}(t,\cdot)\|_{L^{5/3}(\Omega;\mathbb{R}^3)} \leq c, \tag{8.153}$$

where all constants are independent of ε.

Remark Note that system (8.145), (8.146) is formally the same as (8.92), (8.93). However, there are two essential features that make the present setting definitely more convenient for future discussion: system (8.145), (8.146) is defined on the (ε independent) target domain Ω and admits unique classical solutions compactly supported in $[0,T]\times\overline{\Omega}$.

8.7 Dispersive Estimates and Time Decay of Acoustic Waves

Our goal in this section is to give a positive answer to Problem (D) and thus complete the proof of the strong (a.a. pointwise) convergence of the velocity fields claimed in (8.61). To this end, we use the dispersive decay estimates for solutions of the acoustic system (8.145), (8.146) on the *unbounded* domain Ω. The method, formally similar to that used in the previous section, is based on the spectral properties of the Neumann Laplacian $-\Delta_{\mathcal{N},\Omega}$,

$$\Delta_{\mathcal{N},\Omega} v = \Delta v \text{ in } \Omega, \ \nabla_x v\cdot\mathbf{n}|_{\partial\Omega} = 0, \ v \in C_c^\infty(\overline{\Omega})$$

and its extension to a self-adjoint non-negative operator on that Hilbert space $L^2(\Omega)$, see Sect. 11.3.4 in Appendix. As a consequence of Rellich's theorem (Theorem 11.10 in Appendix), the point spectrum of $-\Delta_{\mathcal{N},\Omega}$ is empty in sharp contrast with its bounded domain counterpart $-\Delta_{\mathcal{N},\Omega_\varepsilon}$. Moreover, the spectrum of $-\Delta_{\mathcal{N},\Omega}$ is absolutely continuous and coincides with the half-line $[0,\infty)$, see Sect. 11.3.4 in Appendix. In particular, we may develop the spectral theory, define functions $G(-\Delta_{\mathcal{N},\Omega})$ for $G \in C(0,\infty)$, and the associated Hilbert spaces $\mathcal{D}((-\Delta_{\mathcal{N},\Omega})^\alpha)$, $\alpha \in \mathbb{R}$, see Sect. 11.1 in Appendix.

8.7.1 Compactness of the Solenoidal Components

Similarly to the preceding part, we observe that (8.144) holds true for solenoidal functions, in particular

$$\left\{ t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \mathbf{H}[\varphi] \, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \text{ as } \varepsilon \to 0.$$

Writing $\mathbf{V}_\varepsilon$ in terms of its Helmholtz decomposition

$$\mathbf{V}_\varepsilon = \mathbf{H}[\mathbf{V}_\varepsilon] + \nabla_x \Psi_\varepsilon,$$

we therefore conclude that it is enough to show

$$\left\{ t \mapsto \int_\Omega \nabla_x \Psi_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \text{ as } \varepsilon \to 0.$$

Moreover, as the gradient part $\nabla_x \Psi_\varepsilon$ is expected to disappear in the asymptotic limit (cf. (8.60)), we may anticipate a stronger statement

$$\left\{ t \mapsto \int_\Omega \nabla_x \Psi_\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right\} \to 0 \text{ (strongly) in } L^2(0,T) \text{ as } \varepsilon \to 0. \tag{8.154}$$

for any fixed $\boldsymbol{\varphi} \in C_c^\infty(\Omega;\mathbb{R}^3)$.

Remark Note that (8.154) *cannot* hold on any domain, where $-\Delta_{\mathcal{N},\Omega}$ admits positive eigenvalues, in particular if Ω was a bounded domain, as can be observed from the variation-of-constants formula (8.143). On the other hand, we will see that the absence of eigenvalues is basically sufficient to produce (8.154).

8.7.2 Analysis of Acoustic Waves

Similarly to the preceding section, system (8.145), (8.146) can be written in the form of

■ LINEAR WAVE EQUATION:

$$\varepsilon\partial_t Z_\varepsilon + \Delta_{\mathcal{N},\Omega}\Psi_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{H}_\varepsilon, \tag{8.155}$$

$$\varepsilon\partial_t \Psi_\varepsilon + \omega Z_\varepsilon = \varepsilon(\Delta_{\mathcal{N},\Omega})^{-1}\mathrm{div}_x\,(\mathrm{div}_x\mathbb{G}_\varepsilon + \mathbf{g}) \tag{8.156}$$

with the Neumann boundary conditions

$$\nabla_x\Psi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{8.157}$$

the far field conditions

$$\Psi_\varepsilon,\ Z_\varepsilon \to 0 \text{ as } |x| \to \infty, \tag{8.158}$$

and the initial data

$$Z_\varepsilon(0,\cdot) = Z_{0,\varepsilon},\, \Psi_{\varepsilon,0}(0,\cdot) = \Delta_{\mathcal{N},\Omega}{}^{-1}\mathrm{div}_x\mathbf{V}_{0,\varepsilon}. \tag{8.159}$$

Our aim is to rewrite the linear operators on the right-hand sides of (8.155), (8.156) in the form

$$G(-\Delta_{\mathcal{N},\Omega})[h] \text{ where } h \in L^2(0,T;L^2(\Omega)),$$

cf. **Step 2** in Sect. 8.6.2.

- As $\mathbf{H}$ admits the bound (8.151) and is compactly supported in Ω, the linear form

$$\varphi \mapsto \int_\Omega \mathrm{div}_x\mathbf{H}(t,\cdot)\varphi\ \mathrm{d}x = -\int_\Omega \mathbf{H}(t,\cdot)\cdot\nabla_x\varphi\ \mathrm{d}x$$

 is continuous on the space of functions φ having their gradient $\nabla_x\varphi$ bounded in $L^2 \cap L^\infty$, in particular, it is continuous on the Hilbert space

$$\mathcal{D}((-\Delta_{\mathcal{N},\Omega})^{1/2}) \cap \mathcal{D}((-\Delta_{\mathcal{N},\Omega})^{3/2}).$$

 Indeed, by virtue of the standard elliptic regularity estimates (see Theorem 11.12 in Appendix), the gradients of functions in $\mathcal{D}((-\Delta_{\mathcal{N},\Omega})^{1/2}) \cap \mathcal{D}((-\Delta_{\mathcal{N},\Omega})^{3/2})$ belong to $L^2(\Omega)$, with their second derivatives bounded in $L^2(\Omega)$; whence bounded in $W^{2,2}(\Omega) \subset (L^2 \cap L^\infty)(\Omega)$. Thus we can write

$$\mathrm{div}_x\mathbf{H} = ((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{1/2})[\chi^1],\ \|\chi^1\|_{L^2(0,T;L^2(\Omega))} \leq c. \tag{8.160}$$

- Similarly,

$$\mathrm{div}_x \mathbf{g} = ((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{1/2})[\chi^2]$$

therefore, by virtue of (8.153),

$$\Delta_{\mathcal{N},\Omega}{}^{-1}\mathrm{div}_x \mathbf{g} = ((-\Delta_{\mathcal{N},\Omega})^{1/2} + (-\Delta_{\mathcal{N},\Omega})^{-1/2})[\chi^2], \ \sup_{t\in[0,T]} \|\chi^2(t,\cdot)\|_{L^2(\Omega)} \le c. \tag{8.161}$$

- The expression $\mathrm{div}_x\mathrm{div}_x\mathbb{G}$ can be identified with

$$\mathrm{div}_x\mathrm{div}_x\mathbb{G} = ((-\Delta_{\mathcal{N},\Omega})^2 + (-\Delta_{\mathcal{N},\Omega})^{1/2})[\chi^3];$$

whence, by virtue of (8.151),

$$\Delta_{\mathcal{N},\Omega}{}^{-1}\mathrm{div}_x\mathrm{div}_x\mathbb{G} = ((-\Delta_{\mathcal{N},\Omega}) + (-\Delta_{\mathcal{N},\Omega})^{-1/2})[\chi^3], \ \|\chi^3\|_{L^2(0,T;L^2(\Omega))} \le c. \tag{8.162}$$

- Finally, in accordance with (8.150), the initial data can be written as

$$\left\{ \begin{aligned} Z_{0,\varepsilon} &= \left((-\Delta_{\mathcal{N},\Omega})^2 + (-\Delta_{\mathcal{N},\Omega})^{-1/2}\right)[\chi^4], \\ \Psi_{0,\varepsilon} &= (-\Delta_{\mathcal{N},\Omega})^{-1/2}[\chi^5], \ \|\chi^j\|_{L^2(\Omega)} \le c. \end{aligned} \right\}$$

Consequently, system (8.155), (8.156) takes the form

$$\varepsilon\partial_t Z_\varepsilon + \Delta_{\mathcal{N},\Omega}\Psi_\varepsilon = \varepsilon((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{1/2})[f^1_\varepsilon] \tag{8.163}$$

$$\varepsilon\partial_t \Psi_\varepsilon + \omega Z_\varepsilon = \varepsilon((-\Delta_{\mathcal{N},\Omega}) + (-\Delta_{\mathcal{N},\Omega})^{-1/2})[f^2_\varepsilon] \tag{8.164}$$

where

$$\|f^1_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|f^2_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le c, \tag{8.165}$$

$$Z_{0,\varepsilon} = \left((-\Delta_{\mathcal{N},\Omega})^2 + (-\Delta_{\mathcal{N},\Omega})^{-1/2}\right)[z_{0,\varepsilon}], \ \Psi_{0,\varepsilon} = (-\Delta_{\mathcal{N},\Omega})^{-1/2}[\psi_{0,\varepsilon}]$$

$$\|z_{0,\varepsilon}\|_{L^2(\Omega)} + \|\psi_{0,\varepsilon}\|_{L^2(\Omega)} \le c. \tag{8.166}$$

Remark We have used a simple observation that

$$F(-\Delta_{\mathcal{N},\Omega})[a] + G(-\Delta_{\mathcal{N},\Omega})[b] = (F(-\Delta_{\mathcal{N},\Omega}) + G(-\Delta_{\mathcal{N},\Omega}))[d],$$

$$d = \frac{F(-\Delta_{\mathcal{N},\Omega})}{(F(-\Delta_{\mathcal{N},\Omega}) + G(-\Delta_{\mathcal{N},\Omega}))}[a] + \frac{F(-\Delta_{\mathcal{N},\Omega})}{(F(-\Delta_{\mathcal{N},\Omega}) + G(-\Delta_{\mathcal{N},\Omega}))}[b] \in L^2(\Omega)$$

whenever $F, G \ge 0$, $a, b \in L^2(\Omega)$.

At this stage, we evoke the variation-of-constants formula introduced in (8.143) to compute Ψ_ε:

$$\Psi_\varepsilon(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)\Big[(-\Delta_{\mathcal{N},\Omega})^{-1/2}[\psi_{0,\varepsilon}] \tag{8.167}$$

$$+\mathrm{i}\omega\left((-\Delta_{\mathcal{N},\Omega})^{3/2}+(-\Delta_{\mathcal{N},\Omega})^{-1}\right)[z_{0,\varepsilon}]\Big]$$
$$+\frac{1}{2}\exp\left(-\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)\Big[(-\Delta_{\mathcal{N},\Omega})^{-1/2}[\psi_{0,\varepsilon}]$$
$$-\mathrm{i}\omega\left((-\Delta_{\mathcal{N},\Omega})^{3/2}+(-\Delta_{\mathcal{N},\Omega})^{-1}\right)[z_{0,\varepsilon}]\Big]$$
$$+\frac{1}{2}\int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)\Big[((-\Delta_{\mathcal{N},\Omega})+(-\Delta_{\mathcal{N},\Omega})^{-1/2})[f_\varepsilon^2]$$
$$+\mathrm{i}\omega((-\Delta_{\mathcal{N},\Omega})+\mathrm{Id})[f_\varepsilon^1]\Big]\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)\Big[((-\Delta_{\mathcal{N},\Omega})+(-\Delta_{\mathcal{N},\Omega})^{-1/2})[f_\varepsilon^2]$$
$$-\mathrm{i}\omega((-\Delta_{\mathcal{N},\Omega})+\mathrm{Id})[f_\varepsilon^1]\Big]\,\mathrm{d}s.$$

Now, take $G_\zeta \in C_c^\infty(0,\infty)$ such that

$$0 \le G_\zeta \le 1,\ G_\zeta(z) = 1 \text{ for } z \in [\zeta, \frac{1}{\zeta}],\ \zeta > 0.$$

Going back to (8.154), we write

$$\int_\Omega \nabla_x \Psi_\varepsilon \cdot \boldsymbol{\varphi}\,\mathrm{d}x = -\int_\Omega \Psi_\varepsilon \mathrm{div}_x \boldsymbol{\varphi}\,\mathrm{d}x = \tag{8.168}$$

$$-\int_\Omega G_\zeta^2(-\Delta_{\mathcal{N},\Omega})[\Psi_\varepsilon]\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega \Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega})-\mathrm{Id}\Big)[\Psi_\varepsilon]\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x,$$

where

$$\int_\Omega \Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega})-\mathrm{Id}\Big)[\Psi_\varepsilon]\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \Psi_\varepsilon\Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega})-\mathrm{Id}\Big)[\mathrm{div}_x\boldsymbol{\varphi}]\,\mathrm{d}x.$$

In accordance with the explicit formula (8.167) and the bounds (8.165), (8.166), we have

$$\Psi_\varepsilon = \left((-\Delta_{\mathcal{N},\Omega})^{3/2}+(-\Delta_{\mathcal{N},\Omega})^{-1}\right)[\psi_\varepsilon],$$

where

$$\sup_{t\in[0,T]} \|\psi_\varepsilon(t,\cdot)\|_{L^2(\Omega)} \le c.$$

Consequently, writing

$$\int_\Omega \Psi_\varepsilon\Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega}) - \mathrm{Id}\Big)[\mathrm{div}_x \boldsymbol{\varphi}]\ \mathrm{d}x$$
$$= \int_\Omega \left((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{-1}\right)[\psi_\varepsilon]\Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega}) - \mathrm{Id}\Big)[\mathrm{div}_x \boldsymbol{\varphi}]\ \mathrm{d}x$$
$$= \int_\Omega \psi_\varepsilon\Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega}) - \mathrm{Id}\Big)\left[\left((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{-1}\right)\mathrm{div}_x \boldsymbol{\varphi}\right]\mathrm{d}x$$

we get

$$\left|\int_\Omega \Big(G_\zeta^2(-\Delta_{\mathcal{N},\Omega}) - \mathrm{Id}\Big)[\Psi_\varepsilon]\mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\right| < o(\zeta),\ o(\zeta) \to 0 \text{ as } \zeta \to 0$$

uniformly in ε as soon as we observe that

$$\left((-\Delta_{\mathcal{N},\Omega})^{3/2} + (-\Delta_{\mathcal{N},\Omega})^{-1}\right)[\mathrm{div}_x \boldsymbol{\varphi}] \in L^2(\Omega).$$

Indeed

$$(-\Delta_{\mathcal{N},\Omega})^{3/2}[\mathrm{div}_x \boldsymbol{\varphi}] \in L^2(\Omega)$$

as $\boldsymbol{\varphi}$ is smooth and compactly supported, while, by the same token,

$$\mathrm{div}_x \boldsymbol{\varphi} \in L^p(\Omega) \text{ for any } 1 \le p \le \infty,$$

therefore, by the L^p-elliptic estimates (see Theorem 11.12 in Appendix),

$$(-\Delta_{\mathcal{N},\Omega})^{-1}[\mathrm{div}_x \boldsymbol{\varphi}] \in D^{1,p}(\Omega) \text{ for any } 1 < p < \infty,$$

and the desired conclusion

$$(-\Delta_{\mathcal{N},\Omega})^{-1}[\mathrm{div}_x \boldsymbol{\varphi}] \in L^2(\Omega)$$

follows from Sobolev inequality.

Consequently, in view of (8.168), verifying validity of (8.154) amounts to showing

$$\left\{t \mapsto \int_\Omega G_\zeta^2(-\Delta_{\mathcal{N},\Omega})[\Psi_\varepsilon] \cdot \mathrm{div}_x \boldsymbol{\varphi}\ \mathrm{d}x\right\} \to 0 \text{ (strongly) in } L^2(0,T) \text{ as } \varepsilon \to 0 \tag{8.169}$$

for any fixed $\varphi \in C_c^\infty(\Omega;\mathbb{R}^3)$ and any fixed $\zeta > 0$. As Ψ_ε is given (8.167), the problem reduces to suitable time decay properties of

$$\chi G_\zeta(-\Delta_{\mathcal{N},\Omega}) \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}(\omega\Delta_{\mathcal{N},\Omega})^{1,2}\right)[h],\ \chi \in C_c^\infty(\Omega), \tag{8.170}$$

with h belonging to a bounded set in $L^2(\Omega)$, and

$$\chi G_\zeta(-\Delta_{\mathcal{N},\Omega}) \int_0^t \exp\left(\pm \mathrm{i}\frac{t-s}{\varepsilon}(\omega\Delta_{\mathcal{N},\Omega})^{1,2}\right)[h(s)]\,\mathrm{d}s,\ \chi \in C_c^\infty(\Omega), \tag{8.171}$$

with h belonging to a bounded set in $L^2(0,T;L^2(\Omega))$.

8.7.3 Decay Estimates via RAGE Theorem

In order to establish (8.170), (8.171) we use the celebrated *RAGE Theorem*, see Reed and Simon [237, Theorem XI.115], Cycon et al. [66]. The reader may consult Sect. 11.1 in Appendix for the relevant part of the spectral theory for self-adjoint operators used in the text below.

■ RAGE THEOREM

Theorem 8.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto H_c, where*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{w \in H \mid w \text{ an eigenvector of } A\}\Big\}.$$

Then

$$\left\| \frac{1}{\tau}\int_0^\tau \exp(-\mathrm{i}tA)CP_c\exp(\mathrm{i}tA)\,\mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \text{ for } \tau \to \infty. \tag{8.172}$$

We apply Theorem 8.1 to

$$H = L^2(\Omega),\ A = (-\omega\Delta_{\mathcal{N},\Omega})^{1/2},\ C = \chi^2 G(-\Delta_{\mathcal{N},\Omega}),\ P_c = \mathrm{Id},$$

with

$$\chi \in C_c^\infty(\Omega),\ \chi \ge 0,\ G \in C_c^\infty(0,\infty),\ 0 \le G \le 1.$$

Remark The operator $C = \chi^2 G(-\Delta_{\mathcal{N},\Omega})$ represents a cut-off both in the physical space $\mathbb{R}^3$ represented by the compactly supported function χ and in the "frequency" space represented by picking up a compact part of the spectrum of $-\Delta_{\mathcal{N},\Omega}$ belonging to the support of G. It is easy to see that

$$G(-\Delta_{\mathcal{N},\Omega}) \subset \mathcal{D}((\Delta_{\mathcal{N},\Omega})^{\alpha}) \text{ for any } \alpha \in \mathbb{R},$$

in particular

$$\|\nabla_x{}^k G(-\Delta_{\mathcal{N},\Omega})[v]\|_{L^2(\Omega)} \leq c(k)\|v\|_{L^2(\Omega)} \text{ for any } k \geq 0$$

ensuring local compactness in L^2.

Taking $\tau = 1/\varepsilon$ in (8.172) we obtain

$$\int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{N,\Omega})^{1/2}\right)\chi^2 G(-\Delta_{N,\Omega})\exp\left(\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{N,\Omega})^{1/2}\right)X; Y\right\rangle_{L^2(\Omega)} \mathrm{d}t$$
$$\leq o(\varepsilon)\|X\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)},\ o(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Thus for $Y = G(-\Delta_{\mathcal{N},\Omega})[X]$ we deduce that

$$\int_0^T \left\|\chi G(-\Delta_{\mathcal{N},\Omega})\exp\left(\mathrm{i}\frac{t}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)[X]\right\|^2_{L^2(\Omega)} \mathrm{d}t \tag{8.173}$$
$$\leq o(\varepsilon)\|X\|^2_{L^2(\Omega)} \text{ for any } X \in L^2(\Omega),\ o(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

yielding (8.169) for the component of Ψ_ε given by (8.170).

Similarly, we have

$$\left\|\chi\int_0^T G(-\Delta_{\mathcal{N},\Omega})\exp\left(\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{N,\Omega})^{1/2}\right)[Y(s)]\,\mathrm{d}s\right\|^2_{L^2((0,T)\times\Omega)} \tag{8.174}$$

$$\leq \int_0^T \left(\left\|\int_0^T \chi G(-\Delta_{\mathcal{N},\Omega})\exp\left(\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)[Y(s)]\,\mathrm{d}s\right\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$
$$\leq \int_0^T\int_0^T \left\|\chi G(-\Delta_{\mathcal{N},\Omega})\exp\left(\mathrm{i}\frac{t-s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)[Y(s)]\right\|^2_{L^2(\Omega)} \mathrm{d}t\,\mathrm{d}s$$
$$\leq o(\varepsilon)\int_0^T \left\|\exp\left(-\mathrm{i}\frac{s}{\varepsilon}(-\omega\Delta_{\mathcal{N},\Omega})^{1/2}\right)[Y(s)]\right\|^2_{L^2(\Omega)} \mathrm{d}s$$
$$= o(\varepsilon)\int_0^T \|Y(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s,\ o(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

which implies (8.169) for the component of Ψ_ε given by (8.171).

Having completed the proof of (8.144) we have shown the strong convergence of the velocities claimed in (8.61).

■ LOCAL DECAY OF ACOUSTIC WAVES:

Theorem 8.2 *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of bounded domains in $\mathbb{R}^3$, with $C^{2+\nu}$ boundaries*

$$\partial\Omega_\varepsilon = \Gamma \cup \Gamma_\varepsilon$$

enjoying PROPERTY (L). *Let F be determined through (8.18), where $m \geq 0$ is a bounded measurable function,*

$$\mathrm{supp}[m] \subset \mathbb{R}^3 \setminus \Omega\,,$$

Ω being the exterior domain, $\partial\Omega = \Gamma$. Assume that the thermodynamic functions p, e, s as well as the transport coefficients μ, κ satisfy the structural hypotheses (8.41)–(8.48). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a weak solution of the NAVIER-STOKES-FOURIER SYSTEM *(8.1)–(8.6) in $(0,T)\times\Omega_\varepsilon$ with the complete slip boundary conditions (8.11) in the sense specified in Sect. 5.1.2. Finally, let the initial data satisfy (8.28)–(8.31).*

Then, at least for a suitable subsequence, we have

$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ in } L^2((0,T)\times K;\mathbb{R}^3) \textit{ for any compact } K \subset \Omega\,,$$

with

$$\mathbf{U} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),\ \mathrm{div}_x\mathbf{U} = 0.$$

Remark Smoothness of the boundaries $\partial\Omega_\varepsilon$ is necessary as we have repeatedly used the regularity theory for the Neumann Laplacian. Recall that RAGE Theorem is applicable under the mere assumption of the absence of eigenvalues of $\Delta_{\mathcal{N},\Omega}$. On the other hand, we have no information on the rate of decay. In Sect. 8.9 below, we shall discuss other possibilities to deduce dispersive estimates with an explicit decay rate in terms the parameter $\varepsilon > 0$.

8.8 Convergence to the Target System

Since we have shown strong pointwise (a.a.) convergence of the family of the velocity fields $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ we may let $\varepsilon \to 0$ in the weak formulation of the NAVIER-STOKES-FOURIER SYSTEM to deduce as in Sect. 5.3 that

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(K)) \text{ for any compact } K \subset \Omega,$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

cf. (8.59), and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \begin{cases} \text{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \\ \\ \text{and (strongly) in } L^2((0,T)\times K) \ \text{ for any compact } K \subset \Omega, \end{cases}$$

cf. (8.60), where $[r,\Theta,\mathbf{U}]$ solves the OBERBECK–BOUSSINESQ APPROXIMATION (8.14)–(8.17) in $(0,T)\times\Omega$. Specifically, we have

$$\mathrm{div}_x \mathbf{U} = 0 \text{ a.a. on } (0,T)\times\Omega,$$

$$\int_0^T \int_\Omega \left(\overline{\varrho}(\mathbf{U}\cdot\partial_t\varphi + (\mathbf{U}\otimes\mathbf{U}) : \nabla_x\varphi)\right) \,\mathrm{d}x\,\mathrm{d}t \tag{8.175}$$

$$= -\int_\Omega \overline{\varrho}\mathbf{U}_0\cdot\varphi \,\mathrm{d}x + \int_0^T\int_\Omega \mathbb{S} : \nabla_x\varphi - r\nabla_x F \,\mathrm{d}x\,\mathrm{d}t$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$, $\mathrm{div}_x\varphi = 0$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, where

$$\mathbb{S} = \mu(\overline{\vartheta})(\nabla_x\mathbf{U} + \nabla_x{}^t\mathbf{U}).$$

Furthermore,

$$\overline{\varrho}c_p(\overline{\varrho},\overline{\vartheta})\Big[\partial_t\Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big] - \kappa\Delta\Theta - \overline{\varrho}\overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})\mathrm{div}_x(F\mathbf{U}) = 0 \text{ a.a. in } (0,T)\times\Omega,$$

$$\nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \Theta(0,\cdot) = \Theta_0,$$

and

$$r + \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta = 0 \text{ a.a. in } (0,T)\times\Omega.$$

Similarly to the primitive system, the limit velocity field $\mathbf{U}$ satisfies the complete slip boundary conditions condition

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ and } [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0,$$

where the latter holds implicitly through the choice of test functions in the momentum equation (8.175).

Exactly as in Sect. 5.5.3 the adjustment of the initial temperature distribution experiences some difficulties related to the initial time boundary layer. While the initial conditions for the limit velocity are determined through

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega; \mathbb{R}^3),$$

the initial value of the temperature deviation Θ_0 reads

$$\Theta_0 = \frac{\overline{\vartheta}}{c_p(\overline{\varrho}, \overline{\vartheta})} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} \right), \tag{8.176}$$

where

$$\tilde{\varrho}_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)}, \ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ weakly in } L^2(\Omega).$$

Thus if $\varrho_0^{(1)}$, $\vartheta_0^{(1)}$ satisfy

$$\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho_0^{(1)} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} = 0,$$

which is nothing other than linearization of the pressure at the constant state $(\overline{\varrho}, \overline{\vartheta})$ applied to the vector $[\varrho_0^{(1)}, \vartheta_0^{(1)}]$, relation (8.176) reduces to

$$\Theta_0 = \vartheta_0^{(1)}.$$

We have shown the following result.

■ LOW MACH NUMBER LIMIT: LARGE DOMAINS

Theorem 8.3 *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of bounded domains in $\mathbb{R}^3$, with $C^{2+\nu}$ boundaries*

$$\partial\Omega_\varepsilon = \Gamma \cup \Gamma_\varepsilon$$

enjoying PROPERTY (L). *Let F be determined through (8.18), where $m \geq 0$ is a bounded measurable function,*

$$\text{supp}[m] \subset \mathbb{R}^3 \setminus \Omega,$$

Ω being the exterior domain, $\partial\Omega = \Gamma$. Assume that the thermodynamic functions p, e, s as well as the transport coefficients μ, κ satisfy the structural hypotheses (8.41)–(8.48). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a weak solution of the NAVIER-STOKES-FOURIER SYSTEM *(8.1)–(8.6) in $(0,T)\times\Omega_\varepsilon$ with the complete slip boundary conditions (8.11) in the sense specified in Sect. 5.1.2. Finally, let the initial data satisfy*

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon\tilde{\varrho}^{(1)}_{0,\varepsilon},\ \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon},$$

where

$$\|\tilde{\varrho}^{(1)}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon)} \le c,\ \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon)} \le c,$$

$$\int_{\Omega_\varepsilon} \tilde{\varrho}^{(1)}_{0,\varepsilon}\,\mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon}\,\mathrm{d}x = 0;$$

$$\tilde{\varrho}^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \textit{ weakly in } L^2(\Omega),\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \textit{ weakly in } L^2(\Omega),$$

and

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon;\mathbb{R}^3)} \le c,\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \textit{ weakly in } L^2(\Omega;\mathbb{R}^3).$$

Then, at least for a suitable subsequence, we have

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \textit{ weakly-(*) in } L^\infty(0,T;L^{5/3}(K)) \textit{ for any compact } K \subset \Omega,$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \begin{cases} \textit{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \\ \\ \textit{and (strongly) in } L^2((0,T)\times K) \textit{ for any compact } K \subset \Omega, \end{cases}$$

where $[r,\Theta,\mathbf{U}]$ is a weak solution OBERBECK–BOUSSINESQ APPROXIMATION *(8.14)–(8.17) in $(0,T)\times\Omega$, with the initial data*

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{u}_0],\ \Theta(0,\cdot) = \frac{\overline{\vartheta}}{c_p(\overline{\varrho},\overline{\vartheta})}\left(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\varrho^{(1)}_0 + \frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}_0\right).$$

Remark We have tacitly assumed that the initial data were suitable extended outside Ω_ε to the whole space $\mathbb{R}^3$.

8.9 Dispersive Estimates Revisited

The crucial arguments used to derive the dispersion estimates in Sect. 8.7.3 were all based on the decay rate $d = d(\varepsilon, \varphi, G)$ of the integral

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2} \right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \leq d(\varepsilon, \varphi, G) \|\Psi\|^2_{L^2(\Omega)}. \tag{8.177}$$

In particular, we have shown, by means of RAGE Theorem, that $d(\varepsilon, \varphi, G) \to 0$ as $\varepsilon \to 0$ for any fixed $\varphi \in C_c^\infty(\Omega)$ and $G \in C_c^\infty(0, \infty)$ as long as $-\Delta_{\mathcal{N},\Omega}$ does not possesses any proper eigenvalues in its spectrum. In this section, we examine (8.177) in more detail and show that certain piece of qualitative information concerning d may be available at least on a special class of domains including the exterior domains considered sofar in this chapter. To this end, refined tools of the spectral theory will be used, in particular the properties of the spectral measure associated to the function φ. The reader may consult Sect. 11.1 in Appendix for the relevant results used in the text below.

8.9.1 RAGE Theorem via Spectral Measures

We start by rewriting the integral

$$\left\langle \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2} \right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)}$$

to a more tractable form. Following the language of quantum mechanics, notably the work by Last [181], we use the *spectral measure* μ_φ associated to the function φ. Given μ_φ, any function Ψ possesses its representative Ψ_φ such that

$$\Psi_\varphi \in L^2([0,\infty), \mu_\varphi), \ \|\Psi_\varphi\|_{L^2([0,\infty),\mu_\varphi)} \leq \|\Psi\|_{L^2(\Omega)}$$

and

$$\langle H(-\Delta_{\mathcal{N},\Omega})[\Psi], \varphi \rangle_{L^2(\Omega)} = \int_{[0,\infty)} H(\lambda) \Psi_\varphi(\lambda) \, \mathrm{d}\mu_\varphi,$$

in particular

$$\left\langle \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2} \right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \tag{8.178}$$

$$= \int_{[0,\infty)} \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} \lambda^{1/2} \right) G(\lambda) \Psi_\varphi(\lambda) \, \mathrm{d}\mu_\varphi.$$

Accordingly, we write

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\,(-\Delta_{\mathcal{N},\Omega})^{1/2}\right)[\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \tag{8.179}$$

$$= \int_0^T \left| \int_{[0,\infty)} \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\lambda^{1/2}\right) G(\lambda)\Psi_\varphi(\lambda)\ \mathrm{d}\mu_\varphi \right|^2 \mathrm{d}t$$

$$\leq e \int_{-\infty}^{\infty} \exp(-(t/T)^2) \left| \int_{[0,\infty)} \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\lambda^{1/2}\right) G(\lambda)\Psi_\varphi(\lambda)\ \mathrm{d}\mu_\varphi \right|^2\ \mathrm{d}t$$

$$= e \int_{[0,\infty)} \int_{[0,\infty)} \left[\int_{-\infty}^{\infty} \exp(-(t/T)^2) \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\left(x^{1/2} - y^{1/2}\right)\right) \mathrm{d}t \right] \times$$

$$\times G(x)G(y)\Psi_\varphi(x)\overline{\Psi}_\varphi(y)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)$$

$$= eT\sqrt{\pi} \int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) G(x)G(y)\Psi_\varphi(x)\overline{\Psi}_\varphi(y)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y).$$

Remark We have used the explicit formula

$$\int_{-\infty}^{\infty} \exp(-t^2)\exp(\pm \mathrm{i}\Lambda t) = \sqrt{\pi}\exp\left(\frac{-\Lambda^2}{4}\right).$$

Thus, finally, by means of Hölder's inequality,

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\,(-\Delta_{\mathcal{N},\Omega})^{1/2}\right)[\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \tag{8.180}$$

$$\leq eT\sqrt{\pi} \int_{[0,\infty)} \left[\int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) |G(x)||\Psi_\varphi(x)|\mathrm{d}\mu_\varphi(x) \right] \times$$

$$\times |G(y)||\Psi_\varphi(y)|\ \mathrm{d}\mu_\varphi(y)$$

$$\leq eT\sqrt{\pi} \sup_{z\in[0,\infty)} |G(z)|^2 \times$$

$$\times \left(\int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y) \right)^{1/2} \|\Psi_\varphi\|^2_{L^2[0,\infty)}$$

$$\leq eT\sqrt{\pi} \sup_{z\in[0,\infty)} |G(z)|^2 \times$$

$$\times \left(\int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y) \right)^{1/2} \|\Psi\|^2_{L^2(\Omega)}.$$

We infer that (8.177) holds with

$$d(\varepsilon, \varphi, G)$$

$$= eT\sqrt{\pi} \sup_{z\in[0,\infty)} |G(z)|^2 \left(\int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2} \right) \mathrm{d}\mu_\varphi(x) \mathrm{d}\mu_\varphi(y) \right)^{1/2},$$

where

$$\left(\int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2} \right) \mathrm{d}\mu_\varphi(x) \mathrm{d}\mu_\varphi(y) \right)^{1/2} \to 0 \text{ for } \varepsilon \to 0$$

as long as the spectral measure μ_φ does not charge points in $[0, \infty)$, meaning as long as the point spectrum of the operator $\Delta_{\mathcal{N},\Omega}$ is empty (cf. Sect. 11.1 in Appendix). We have recovered the statement shown in the previous section by means of RAGE Theorem.

8.9.2 Decay Estimates via Kato's Theorem

An alternative approach to study the local decay of acoustic waves is based on an abstract result of Tosio Kato [166] (see also Burq et al. [44], Reed and Simon [237, Theorem XIII.25 and Corollary]).

■ KATO'S THEOREM

Theorem 8.4 *Let C be a closed densely defined linear operator and A a self-adjoint densely defined linear operator in a Hilbert space H. For $\lambda \notin \mathbb{R}$, let $R_A[\lambda] = (A - \lambda \mathrm{Id})^{-1}$ denote the resolvent of A. Suppose that*

$$\Gamma = \sup_{\lambda\notin\mathbb{R},\ v\in\mathcal{D}(C^*),\ \|v\|_H=1} \|C \circ R_A[\lambda] \circ C^*[v]\|_H < \infty. \tag{8.181}$$

Then

$$\sup_{w\in X,\ \|w\|_H=1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|C \exp(-\mathrm{i}tA)[w]\|_X^2 \,\mathrm{d}t \le \Gamma^2.$$

Anticipating, for a while, that $A = (-\Delta_{\mathcal{N},\Omega})^{1/2}$, C—the projection onto the $1D$-space spanned by φ, satisfy the hypotheses of Kato's theorem, we get

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i}(-\Delta_{\mathcal{N},\Omega})^{1/2} \frac{t}{\varepsilon} \right) [\Psi], \varphi \right\rangle_{L^2(\Omega)} \right|^2 \, \mathrm{d}t \tag{8.182}$$

$$= \varepsilon \int_0^{T/\varepsilon} \left| \left\langle \exp\left(\pm \mathrm{i}(-\Delta_{\mathcal{N},\Omega})^{1/2} \tau \right) [\Psi], \varphi \right\rangle_{L^2(\Omega)} \right|^2 \, \mathrm{d}\tau \le \varepsilon \Gamma^2(\varphi) \|\Psi\|^2_{L^2(\Omega)},$$

meaning (8.177) holds with an explicit decay of d of order ε. This is because the piece of information hidden in hypothesis (8.181) is definitely stronger than the mere absence of eigenvalues required by RAGE Theorem. In fact, as we shall se bellow, relation (8.181) is basically equivalent to the so-called limiting absorption principle for the operator $\Delta_{\mathcal{N},\Omega}$, cf. Vaĭnberg [263]. Our plan for the remaining part of this section is to use a direct argument, based on the spectral measure representation introduced above, to show explicit decay rate for d in (8.177), among which (8.182) as a special case. To this end, we adopt an extra assumptions on the cut-off function G, namely

$$\mathrm{supp}[G] \subset [a, b], \ 0 < a < b < \infty. \tag{8.183}$$

Exactly as in (8.179), we have

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2} \right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \tag{8.184}$$

$$\le eT\sqrt{\pi} \int_{[0,\infty)} \int_{[0,\infty)} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2} \right) G(x)G(y)\Psi_\varphi(x)\overline{\Psi}_\varphi(y) \, \mathrm{d}\mu_\varphi(x) \, \mathrm{d}\mu_\varphi(y)$$

$$= eT\sqrt{\pi} \int_{[a,b]} \int_{[a,b]} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2} \right) G(x)G(y)\Psi_\varphi(x)\overline{\Psi}_\varphi(y) \, \mathrm{d}\mu_\varphi(x) \, \mathrm{d}\mu_\varphi(y)$$

$$\le \int_{[a,b]} G^2(x)|\Psi_\varphi(x)|^2 \left(\int_{[a,b]} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2} \right) \mathrm{d}\mu_\varphi(y) \right) \mathrm{d}\mu_\varphi(x),$$

where we have used the Cauchy-Schwartz inequality and Fubini's theorem in the following way:

$$\int \int K(x,y)f(x)f(y) \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int \left(\int K(x,y)f(y)\mathrm{d}\mu(y) \right) f(x)\mathrm{d}\mu(x)$$

$$\le \int |f(x)| \left(\int K(x,y)f^2(y)\mathrm{d}\mu(y) \right)^{1/2} \left(\int K(x,y)\mathrm{d}\mu(y) \right)^{1/2} \mathrm{d}\mu(x)$$

$$\left(\int \left(\int K(x,y)\mathrm{d}\mu(x) \right) f^2(y)\mathrm{d}\mu(y) \right)^{1/2} \left(\int \left(\int K(x,y)\mathrm{d}\mu(y) \right) f^2(x)\mathrm{d}\mu(x) \right)^{1/2}$$

yielding the desired conclusion for the symmetric kernel

$$K(x,y) = \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) = K(y,x).$$

Now, the kernel in the last integral in (8.184) can be written as

$$\int_{[a,b]} \exp\left(-T^2 \frac{|x^{1/2} - y^{1/2}|^2}{4\varepsilon^2}\right) \, \mathrm{d}\mu_\varphi(y)$$

$$= \sum_{n=0}^{\infty} \int_{\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1),\ y \in [a,b]} \exp\left(-\frac{|x^{1/2} - y^{1/2}|^2}{\varepsilon^2} \frac{T^2}{4}\right) \mathrm{d}\mu_\varphi(y)$$

$$\leq \sup_{n \geq 0} \left\{ \int_{\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1)} 1_{[a,b]} \mathrm{d}\mu_\varphi(y) \right\} \sum_{n=0}^{\infty} \exp\left(-\frac{n^2 T^2}{4}\right).$$

As only the points $x \in [a,b]$ are relevant in evaluating

$$\sup_{n \geq 0} \left\{ \int_{\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1)} 1_{[a,b]} \mathrm{d}\mu_\varphi(y) \right\}$$

relation (8.184) gives rise to

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2}\right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \tag{8.185}$$

$$\leq c(T) \sup_{z \in [a,b]} |G^2(z)| \; \|\Psi\|^2_{L^2(\Omega)} \times$$

$$\times \sup_{n \geq 0, x \in [a,b]} \left\{ \int_{\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1)} 1_{[a,b]}(y) \mathrm{d}\mu_\varphi(y) \right\}.$$

For each fixed $\sqrt{x}$, n, the length of the interval of $y's$ satisfying

$$\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1), \ a \leq y \leq b$$

does not exceed $\varepsilon \left(a^{1/2} + b^{1/2}\right)$ therefore

$$\int_{\varepsilon n \leq |y^{1/2} - x^{1/2}| < \varepsilon(n+1)} 1_{[a,b]}(y) \mathrm{d}\mu_\varphi(y) < \varepsilon c(a,b,\varphi) \left(a^{1/2} + b^{1/2}\right) \tag{8.186}$$

provided μ_φ is absolutely continuous with respect to the Lebesgue measure on $[a, b]$ and

$$\mu_\varphi[I] \leq c(a, b, \varphi)|I| \text{ for any closed interval } I \subset [a, b]. \tag{8.187}$$

Relations (8.186), (8.187) give rise to (8.177) with

$$d(\varepsilon, \varphi, G) = \varepsilon c(\varphi, G);$$

it remains to show sufficient conditions for (8.187) to hold. The value of $\mu_\varphi[\alpha, \beta]$ can be evaluated by means of Stone's formula (formula (11.1) in Appendix)

$$\mu_\varphi[\alpha, \beta] \tag{8.188}$$

$$= \lim_{\delta \to 0+} \lim_{\eta \to 0+} \int_{\alpha+\delta}^{\beta+\delta} \left\langle \left(\frac{1}{-\Delta_{\mathcal{N},\Omega} - \lambda - \mathrm{i}\eta} - \frac{1}{-\Delta_{\mathcal{N},\Omega} - \lambda + \mathrm{i}\eta} \right) \varphi, \varphi \right\rangle_{L^2(\Omega)} \mathrm{d}\lambda,$$

consequently, (8.187) holds as soon as the operator $-\Delta_{\mathcal{N},\Omega}$ satisfies the so-called *limiting absorption principle* (LAP).

■ LIMITING ABSORPTION PRINCIPLE:

We say that $-\Delta_{\mathcal{N},\Omega}$ satisfies limiting absorption principle (LAP) if

$$\left\{ \begin{array}{c} \textit{Operators} \\ \mathcal{V} \circ (-\Delta_{\mathcal{N},\Omega} - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega),\ \mathcal{V}[v] = (1 + |x|^2)^{-s/2},\ s > 1 \\ \textit{are bounded uniformly for } \lambda \in [\alpha, \beta],\ 0 < \alpha < \beta,\ \eta > 0, \end{array} \right\}$$

It is known that $-\Delta_{\mathcal{N},\Omega}$ satisfies (LAP) if Ω is an exterior domain with a smooth boundary considered in this chapter, see Theorem 11.11 in Appendix. Accordingly, we have

$$\int_0^T \left| \left\langle \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} (-\Delta_{\mathcal{N},\Omega})^{1/2} \right) [\Psi], G(-\Delta_{\mathcal{N},\Omega})[\varphi] \right\rangle_{L^2(\Omega)} \right|^2 \mathrm{d}t \leq \varepsilon c(\varphi, a, b) \|\Psi\|^2_{L^2(\Omega)} \tag{8.189}$$

provided

$$G \in C_c^\infty(0, \infty),\ \mathrm{supp}[G] \subset [a, b],\ 0 \leq G \leq 1.$$

8.10 Conclusion

Apart form the exterior domains considered in this chapter, there is a vast class of domains on which the operator $-\Delta_{\mathcal{N},\Omega}$ has empty point spectrum or even satisfies the limiting absorption principle. Obviously our method can be extended to the situation when these domains are approximated by a suitable family of bounded domains. A relevant example is the perturbed half-space studied in [123].

Another possibility how to exploit the stronger decay rate stated in (8.189) is the situation, where the boundary of Ω_ε varies with ε, in particular, it may contains one or several "holes" vanishing in the asymptotic limit $\varepsilon \to 0$, see [122].

There are intermediate decay rates of $d(\varepsilon, G, \varphi)$ for spectral measures that are α-Hölder continuous with respect to the Lebesgue measure, see Strichartz [252]. Other interesting extensions were obtained by Last [181].