Chapter 7
Interaction of Acoustic Waves with Boundary

As we have seen in the previous chapters, one of the most delicate issues in the analysis of singular limits for the NAVIER-STOKES-FOURIER SYSTEM in the low Mach number regime is the influence of acoustic waves. If the physical domain is bounded and the complete slip boundary conditions (cf. (5.15)) imposed, the acoustic waves, being reflected by the boundary, inevitably develop high frequency oscillations resulting in the *weak* convergence of the velocity field, in particular, its gradient part converges to zero only in the sense of integral means. This rather unpleasant phenomenon creates additional problems when handling the convective term in the momentum equation (cf. Sects. 5.4.7, 6.6.3 above). In this chapter, we focus on the mechanisms sofar neglected by which the acoustic energy may be dissipated , and the ways how the dissipation may be used in order to show *strong* (pointwise) convergence of the velocities.

The principal mechanism of dissipation in the NAVIER-STOKES-FOURIER SYSTEM is of course *viscosity*, here imposed through Newton's rheological law. At a first glance, the presence of the viscous stress $\mathbb{S}$ in the momentum equation does not seem to play any significant role in the analysis of acoustic waves. In the situation described in Sect. 4.4.1, the acoustic equation can be written in the form

$$\left\{ \begin{array}{c} \varepsilon\partial_t r_\varepsilon + \mathrm{div}_x(\mathbf{V}_\varepsilon) = \text{“small terms''}, \\ \\ \varepsilon\partial_t \mathbf{V}_\varepsilon + \omega\nabla_x r_\varepsilon = \varepsilon \mathrm{div}_x \mathbb{S}_\varepsilon + \text{“small terms''}. \end{array} \right\} \tag{7.1}$$

Replacing for simplicity $\mathrm{div}_x\mathbb{S}_\varepsilon$ by $\Delta\mathbf{V}_\varepsilon$, we examine the associated eigenvalue problem:

$$\begin{aligned} \mathrm{div}_x \mathbf{w} &= \lambda r, \\ \omega\nabla_x r - \varepsilon\Delta_x \mathbf{w} &= \lambda \mathbf{w}. \end{aligned} \tag{7.2}$$

E. Feireisl, A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*,
Advances in Mathematical Fluid Mechanics, DOI 10.1007/978-3-319-63781-5_7

Applying the divergence operator to the second equation and using the first one to express all quantities in terms of r, we arrive at the eigenvalue problem

$$-\Delta_x r = \frac{\lambda^2}{\varepsilon\lambda - \omega} r.$$

Under the periodic boundary conditions, meaning $\Omega = \mathcal{T}^3$, the corresponding eigenvalues are given as

$$\frac{\lambda_n^2}{\varepsilon\lambda_n - \omega} = \Lambda_n,$$

where Λ_n are the (real non-negative) eigenvalues of the Laplace operator supplemented with the periodic boundary conditions. It is easy to check that

$$\lambda_n = \frac{\varepsilon\Lambda_n \pm \mathrm{i}\sqrt{4\omega\Lambda_n - \varepsilon^2\Lambda_n^2}}{2}.$$

Moreover, the corresponding eigenfunctions read

$$\{r_n, \mathbf{w}_n\}, \ \mathbf{w}_n = \frac{\omega - \varepsilon\lambda_n}{\lambda_n}\nabla_x r_n,$$

where r_n are the eigenfunctions of the Laplacian supplemented with the periodic boundary conditions.

The same result is obtained provided the velocity field satisfies the complete slip boundary conditions (1.19), (1.27) leading to the Neumann boundary conditions for r, namely

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x r \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

In particular, the eigenfunctions differ from those of the limit problem with $\varepsilon = 0$ only by a multiplicative constant approaching 1 for $\varepsilon \to 0$.

Physically speaking, the complete slip boundary conditions correspond to the ideal *mechanically smooth* boundary of the physical space. As suggested by the previous arguments, the effect of viscosity in this rather hypothetical situation does not change significantly the asymptotic analysis in the low Mach number limit.

■ CONJECTURE I (NEGATIVE):

The dissipation of the acoustic energy caused by viscosity in domains with mechanically smooth boundaries is irrelevant in the low Mach number regime. The decay of the acoustic waves is exponential with a rate independent of ε.

On the other hand, the decay rate of the acoustic waves may change substantially if the fluid interacts with the boundary, meaning, if some kind of "dissipative" (in terms of the acoustic energy) boundary conditions is imposed on the velocity field. Thus, for instance, the no-slip boundary conditions (1.28) give rise to

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{7.3}$$

Accordingly, system (7.2), supplemented with (7.3), becomes a *singularly perturbed* eigenvalue problem. In particular, if the (overdetermined) limit problem

$$\mathrm{div}_x \mathbf{w} = \lambda r, \ \omega \nabla_x r = \lambda \mathbf{w}, \ \mathbf{w}|_{\partial\Omega} = 0 \tag{7.4}$$

admits only the trivial solution for $\lambda \neq 0$, we can expect that a boundary layer is created in the limit process $\varepsilon \to 0$ resulting in a faster decay of the acoustic waves. This can be seen by means of the following heuristic argument. Suppose that problem (7.2), (7.3) admits a family of eigenfunctions $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ with the associated set of eigenvalues $\{\lambda_\varepsilon\}_{\varepsilon>0}$. Multiplying (7.2) on $\overline{r}_\varepsilon$, $\overline{\mathbf{w}}_\varepsilon$, where the bar stands for the complex conjugate, integrating the resulting expression over Ω, and using (7.3), we obtain

$$\varepsilon \int_\Omega |\nabla_x \mathbf{w}_\varepsilon|^2 \, \mathrm{d}x = (1+\omega)\mathrm{Re}[\lambda_\varepsilon] \int_\Omega \left(|r_\varepsilon|^2 + |\mathbf{w}_\varepsilon|^2 \right) \mathrm{d}x,$$

where Re denotes the real part of a complex number. Normalizing $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3)$ we easily observe that

$$\frac{\mathrm{Re}[\lambda_\varepsilon]}{\varepsilon} \to \infty,$$

since otherwise $\{\mathbf{w}_\varepsilon\}_{\varepsilon>0}$ would be bounded in $W^{1,2}(\Omega; \mathbb{R}^3)$ and any weak accumulation point $(r, \mathbf{w})$ of $\{r_\varepsilon, \mathbf{w}_\varepsilon\}_{\varepsilon>0}$ would represent a nontrivial solution of the overdetermined limit system (7.4).

■ CONJECTURE II (POSITIVE):

Sticky boundaries in combination with the viscous effects may produce a decay rate of the acoustic waves that is considerably faster than their frequency in the low Mach number regime. In particular, the mechanical energy is converted into heat and the acoustic waves are anihilated at a time approaching zero in the low Mach number limit.

Finally, we claim that a similar effect may be produced even if the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ \mathbb{S}\mathbf{n} \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

are imposed, where Ω_ε is a family of domains with "rough" boundaries depending on the scaling parameter ε. More precisely, the boundaries $\partial\Omega_\varepsilon$ differ from a limit shape by a family of small but still smooth asperities approximating the limit boundary in a similar way as the sequence of functions $\varepsilon \sin(x/\varepsilon)$ approaches zero. In particular, as the fluid is viscous, such oscillating boundaries force the fluid velocity to vanish, meaning to satisfy the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0$$

in the asymptotic limit $\varepsilon \to 0$. Thus the scenario predicted by Conjecture II remains valid and we expect to recover strong convergence of the velocity fields.

7.1 Problem Formulation

Motivated by the previous discussion, we examine the low Mach number limit for the NAVIER-STOKES-FOURIER SYSTEM supplemented with either the no-slip boundary condition, or, alternatively, with the complete slip boundary conditions imposed on a family of domains with "oscillating" boundaries. In both cases, the fact that the fluid adheres completely (at least asymptotically in the latter case) to the wall of the physical space imposes additional restrictions on the propagation of acoustic waves. Our goal is to identify the geometrical properties of the domain, for which this implies strong convergence of the velocity field in the asymptotic limit.

7.1.1 Field Equations

We consider the same scaling of the field equations as in Chap. 5. Specifically, we set

$$\mathrm{Ma} = \varepsilon, \ \mathrm{Fr} = \sqrt{\varepsilon}$$

obtaining

■ SCALED NAVIER-STOKES-FOURIER SYSTEM:

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{7.5}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho_\varepsilon, \vartheta_\varepsilon) = \mathrm{div}_x \mathbb{S}_\varepsilon + \frac{1}{\varepsilon}\varrho_\varepsilon \nabla_x F, \tag{7.6}$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}_x\left(\frac{\mathbf{q}_\varepsilon}{\vartheta}\right) = \sigma_\varepsilon, \tag{7.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{\varepsilon^2}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \varepsilon \varrho_\varepsilon F\right) \mathrm{d}x = 0, \tag{7.8}$$

where

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\Big). \tag{7.9}$$

System (7.5)–(7.8) is supplemented, exactly as in Chap. 5, with the constitutive relations:

$$\mathbb{S}_\varepsilon = \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) = \mu(\vartheta_\varepsilon)\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x{}^T \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\Big), \tag{7.10}$$

$$\mathbf{q}_\varepsilon = \mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) = -\kappa(\vartheta_\varepsilon)\nabla_x \vartheta_\varepsilon, \tag{7.11}$$

and

$$p(\varrho_\varepsilon, \vartheta_\varepsilon) = p_M(\varrho_\varepsilon, \vartheta_\varepsilon) + p_R(\vartheta_\varepsilon),\ p_M = \vartheta_\varepsilon{}^{\frac{5}{2}} P\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon{}^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta_\varepsilon{}^4, \tag{7.12}$$

$$e(\varrho_\varepsilon, \vartheta_\varepsilon) = e_M(\varrho_\varepsilon, \vartheta_\varepsilon) + e_R(\varrho_\varepsilon, \vartheta_\varepsilon),\ e_M = \frac{3}{2}\frac{\vartheta_\varepsilon{}^{\frac{5}{2}}}{\varrho_\varepsilon} P\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon{}^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta_\varepsilon{}^4}{\varrho_\varepsilon}, \tag{7.13}$$

$$s(\varrho_\varepsilon, \vartheta_\varepsilon) = s_M(\varrho_\varepsilon, \vartheta_\varepsilon) + s_R(\varrho_\varepsilon, \vartheta_\varepsilon),\ s_M(\varrho_\varepsilon, \vartheta_\varepsilon) = S\Big(\frac{\varrho_\varepsilon}{\vartheta_\varepsilon{}^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta_\varepsilon{}^3}{\varrho_\varepsilon}, \tag{7.14}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{7.15}$$

The reader will have noticed that the bulk viscosity has been neglected in (7.10) for the sake of simplicity.

As always in this book, Eqs. (7.5)–(7.8) are interpreted in the weak sense specified in Chap. 1 (see Sect. 7.2 below). We recall that the technical restrictions imposed on the constitutive functions are dictated by the existence theory developed in Chap. 3 and could be relaxed, to a certain extent, as far as the singular limit passage is concerned.

7.1.2 Physical Domain and Boundary Conditions

As indicated in the introductory part, the geometry of the physical domain plays a crucial role in the study of propagation of the acoustic waves. As already pointed out, the existence of an effective mechanism of dissipation of the acoustic waves is intimately linked to solvability of the (overdetermined) system (7.4) that can be written in a more concise form as

$$-\Delta_x r = \Lambda r \text{ in } \Omega, \ \frac{\lambda^2}{\omega} = -\Lambda, \ \nabla_x r|_{\partial\Omega} = 0. \tag{7.16}$$

The problem of existence of a non-trivial, meaning non-constant, solution to (7.16) is directly related to the so-called *Pompeiu property* of the domain Ω. A remarkable result of Williams [273] asserts that if (7.16) possesses a non-constant solution in a domain in $\mathbb{R}^N$ whose boundary is homeomorphic to the unit sphere, then, necessarily, $\partial\Omega$ must admit a description by a system of charts that are *real analytic*. The celebrated *Schiffer's conjecture* claims that (7.16) admits a non-trivial solution in the aforementioned class of domains only if Ω is a ball.

In order to avoid the unsurmountable difficulties mentioned above, we restrict ourselves to a very simple geometry of the physical space. Similarly to Chap. 6, we assume the motion of the fluid is 2π-periodic in the horizontal variables (x_1, x_2), and the domain Ω is an infinite slab determined by the graphs of two given functions $B_{\rm bottom}$, $B_{\rm top}$,

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2, \ B_{\rm bottom}(x_1, x_2) < x_3 < B_{\rm top}(x_1, x_2)\}, \tag{7.17}$$

where $\mathcal{T}^2$ denotes the flat torus,

$$\mathcal{T}^2 = \left([-\pi, \pi]|_{\{-\pi,\pi\}} \right)^2.$$

Although the specific length of the period is not essential, this convention simplifies considerably the notation used in the remaining part of this chapter.

In the simple geometry described by (7.17), it is easy to see that problem (7.16) admits a non-trivial solution, namely $r = \cos(x_3)$ as soon as the boundary is flat, more precisely, if $B_{\rm bottom} = -\pi$, $B_{\rm top} = 0$. On the other hand, we claim that problem (7.16) possesses only the trivial solution in *domains with variable bottoms* as stated in the following assertion.

Proposition 7.1 *Let Ω be given through (7.17), with*

$$\left\{ \begin{array}{c} B_{\rm bottom} = -\pi - h(x_1, x_2), \ B_{\rm top} = 0, \\ \\ \textit{where} \\ \\ h \in C(\mathcal{T}^2), \ |h| < \pi \textit{ for all } (x_1, x_2) \in \mathcal{T}^2. \end{array} \right\} \tag{7.18}$$

Assume there is a function $r \neq$ const solving the eigenvalue problem (7.16) for a certain Λ.

Then $h \equiv$ constant.

Proof Since r is constant on the top part, specifically $r(x_1, x_2, 0) = r_0$, the function

$$V(x_1, x_2, x_3) = r(x_1, x_2, x_3) - r_0 \cos(\sqrt{\Lambda} x_3)$$

satisfies

$$-\Delta_x V = \Lambda V \text{ in } \Omega, \text{ and, in addition, } \nabla_x V|_{B_{\rm top}} = V|_{B_{\rm top}} = 0.$$

Accordingly, the function V extended to be zero in the upper half plane $\{x_3 > 0\}$ solves the eigenvalue problem (7.16) in $\Omega \cup \{x_3 \geq 0\}$. Consequently, by virtue of the unique continuation property of the elliptic operator $\Delta_x + \Lambda I$ (analyticity of solutions to elliptic problems discussed in Sect. 11.3.1 in Appendix), we get $V \equiv 0$, in other words,

$$r = r_0 \cos(\sqrt{\Lambda} x_3) \text{ in } \Omega.$$

However, as r must be constant on the bottom part $\{x_3 = -\pi - h(x_1, x_2)\}$, we conclude that $h \equiv$ const.

□

Our future considerations will be therefore concerned with fluids confined to domains described through (7.17), with flat "tops" and variables "bottoms" as in (7.18) with $h \not\equiv$ const.

7.2 Main Result: The No-Slip Boundary Conditions

We start by imposing the *no-slip* boundary conditions for the velocity field

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0, \tag{7.19}$$

together with the no-flux boundary condition for the temperature

$$\mathbf{q}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.20}$$

Accordingly, the system is energetically insulated in agreement with (7.8).

As a matter of fact, the approach delineated in this section applies to any bounded and sufficiently smooth spatial domain $\Omega \subset \mathbb{R}^3$, on which the overdetermined problem (7.16) admits only the trivial (constant) solution r. In particular, the arguments in the proof of Proposition 7.1 can be used provided a part of the boundary is flat and the latter is connected.

7.2.1 Preliminaries: Global Existence

Exactly as in Chap. 5, we consider the initial data in the form

$$\left\{ \begin{array}{c} \varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \\ \\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \\ \\ \vartheta_\varepsilon(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \end{array} \right\} \tag{7.21}$$

where

$$\left\{ \begin{array}{c} \int_\Omega \varrho^{(1)}_{0,\varepsilon}\, \mathrm{d}x = 0,\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)} \text{ weakly-(*) in } L^\infty(\Omega), \\ \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly-(*) in } L^\infty(\Omega;\mathbb{R}^3), \\ \\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\, \mathrm{d}x = 0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ weakly in } L^\infty(\Omega), \end{array} \right\} \tag{7.22}$$

with positive constants $\overline{\varrho}$, $\overline{\vartheta}$.

For reader's convenience, we recall the list of hypotheses, under which system (7.5)–(7.15), supplemented with the boundary conditions (7.19), (7.20), and the initial conditions (7.21), possesses a weak solution defined on an arbitrary time interval $(0,T)$. To begin, we need the *hypothesis of thermodynamic stability* (1.44) expressed in terms of the function P as

$$P \in C^1[0,\infty) \cap C^2(0,\infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{7.23}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty, \tag{7.24}$$

together with the coercivity assumption

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{7.25}$$

Similarly to Chap. 5, the transport coefficients μ, η, and κ are assumed to be continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta) \text{ for all } \vartheta \geq 0,\ \mu' \text{ bounded in } [0,\infty), \tag{7.26}$$

and

$$0 < \underline{\kappa}(1 + \vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta \ge 0, \tag{7.27}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

Now, as a direct consequence of the abstract existence result established in Theorem 3.1, we claim that for any $\varepsilon > 0$, the scaled NAVIER-STOKES-FOURIER SYSTEM (7.5)–(7.9), supplemented with the boundary conditions (7.19)–(7.20), and the initial conditions (7.21), possesses a weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ on the set $(0,T) \times \Omega$ such that

$$\varrho_\varepsilon \in L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),\ \mathbf{u}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\ \vartheta_\varepsilon \in L^2(0,T;W^{1,2}(\Omega)).$$

More specifically, we have:

(i) Renormalized equation of continuity:

$$\int_0^T \int_\Omega \varrho_\varepsilon B(\varrho_\varepsilon)\Big(\partial_t\varphi + \mathbf{u}_\varepsilon \cdot \nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.28}$$

$$= \int_0^T \int_\Omega b(\varrho_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon}B(\varrho_{0,\varepsilon})\varphi(0,\cdot)\ \mathrm{d}x$$

for any b as in (2.3) and any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$;

(ii) Momentum equation:

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \partial_t\boldsymbol{\varphi} + \varrho_\varepsilon[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x\boldsymbol{\varphi} + \frac{1}{\varepsilon^2}p(\varrho_\varepsilon,\vartheta_\varepsilon)\mathrm{div}_x\boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x\boldsymbol{\varphi} - \frac{1}{\varepsilon}\varrho_\varepsilon\nabla_x F \cdot \boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x \tag{7.29}$$

for any test function

$$\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \Omega;\mathbb{R}^3);$$

(iii) Total energy balance:

$$\int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon) - \varepsilon\varrho_\varepsilon F\Big)(t)\ \mathrm{d}x \tag{7.30}$$

$$= \int_\Omega \Big(\frac{\varepsilon^2}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon}e(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \varepsilon\varrho_\varepsilon F\Big)\ \mathrm{d}x \text{ for a.a. } t \in (0,T);$$

(iv) Entropy balance:

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\Big(\partial_t \varphi + \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big) \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t \tag{7.31}$$

$$+ < \sigma_\varepsilon; \varphi >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})} = -\int_\Omega \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\varphi(0,\cdot) \,\mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, where $\sigma_\varepsilon \in \mathcal{M}^+([0,T] \times \overline{\Omega})$ satisfies (7.9).

Note that the satisfaction of the no-slip boundary conditions is ensured by the fact that the velocity field $\mathbf{u}_\varepsilon(t,\cdot)$ belongs to the Sobolev space $W_0^{1,2}(\Omega;\mathbb{R}^3)$ defined as a completion of $C_c^\infty(\Omega;\mathbb{R}^3)$ with respect to the $W^{1,2}$-norm. Accordingly, the test functions in the momentum equation (7.29) must be compactly supported in Ω, in particular, the Helmholtz projection $\mathbf{H}[\boldsymbol{\varphi}]$ is no longer an admissible test function in (7.29).

7.2.2 Compactness of the Family of Velocities

In order to avoid confusion, let us point out that the principal result to be shown in this part is *pointwise compactness of the family of velocity fields* $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$. Then following step by step the analysis presented in Chap. 5 we could show that the limit system obtained by letting $\varepsilon \to 0$ is the same as in Theorem 5.2, specifically, the OBERBECK-BOUSSINESQ APPROXIMATION (5.161)–(5.166).

■ COMPACTNESS OF VELOCITIES ON DOMAINS WITH VARIABLE BOTTOMS:

Theorem 7.1 *Let Ω be the infinite slab introduced in (7.17), (7.18), where the "bottom" part of the boundary is given by a function h satisfying*

$$h \in C^3(\mathcal{T}^2), \ |h| < \pi, \ h \not\equiv \text{const.} \tag{7.32}$$

Assume that $\mathbb{S}_\varepsilon$, $\mathbf{q}_\varepsilon$ as well as the thermodynamic functions p, e, and s are given by (7.10)–(7.15), where P meets the structural hypotheses (7.23)–(7.25), while the transport coefficients μ and κ satisfy (7.26), (7.27). Finally, let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to the Navier-Stokes-Fourier system satisfying (7.28)–(7.31), where the initial data are given by (7.21), (7.22).

Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T) \times \Omega; \mathbb{R}^3), \tag{7.33}$$

where $\mathbf{U} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$, $\mathrm{div}_x \mathbf{U} = 0$.

The bulk of the remaining part of this chapter, specifically Sects. 7.3–7.5, is devoted to the proof of Theorem 7.1 which is tedious and rather technical. It is based on careful analysis of the singular eigenvalue problem (7.2), (7.3) in a boundary layer by means of the abstract method proposed by Vishik and Ljusternik [267] and later adapted to the low Mach number limit problems in the context of isentropic fluid flows by Desjardins et al. [81]. In contrast with [81], we "save" one degree of approximation—a fact that simplifies considerably the analysis and makes the proof relatively transparent and easily applicable to other choices of boundary conditions (see [118]).

7.3 Uniform Estimates

We begin the proof of Theorem 7.1 by recalling the uniform estimates that can be obtained exactly as in Chap. 5. Thus we focus only the principal ideas referring to the corresponding parts of Sect. 5.2 for all technical details.

As the initial distribution of the density is a zero mean perturbation of the constant state $\overline{\varrho}$, we have

$$\int_\Omega \varrho_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \varrho_{0,\varepsilon}\,\mathrm{d}x = \overline{\varrho}|\Omega|,$$

in particular,

$$\int_\Omega (\varrho_\varepsilon(t) - \overline{\varrho})\,\mathrm{d}x = 0 \text{ for all } t \in [0,T]. \tag{7.34}$$

To obtain further estimates, we combine (7.30), (7.31) to deduce the dissipation balance equality in the form

$$\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - \varepsilon\varrho_\varepsilon F\Big)\right](\tau)\,\mathrm{d}x + \frac{\overline{\vartheta}}{\varepsilon}\sigma_\varepsilon\Big[[0,\tau]\times\overline{\Omega}\Big] \tag{7.35}$$

$$= \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \varepsilon\varrho_\varepsilon F\Big)\right]\mathrm{d}x \text{ for a.a. } \tau \in [0,T],$$

where $H_{\overline{\vartheta}}$ is the Helmholtz function introduced in (2.48).

As we have observed in (2.49), (2.50), the hypothesis of thermodynamic stability $\partial_\varrho p > 0$, $\partial_\vartheta e > 0$, expressed in terms of (7.23), (7.24), implies that

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) \text{ is a strictly convex function,}$$

while

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho,\vartheta) \text{ attains its strict minimum at } \overline{\vartheta} \text{ for any fixed } \varrho.$$

Consequently, subtracting a suitable affine function of ϱ from both sides of (7.35), and using the coercivity properties of $H_{\overline{\vartheta}}$ stated in Lemma 5.1 we deduce the following list of uniform estimates:

- **Energy estimates:**

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leq c \text{ [cf. (5.49)]}, \tag{7.36}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c \text{ [cf. (5.46)]}, \tag{7.37}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm res} \right\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon^{\frac{1}{5}} c \text{ [cf. (5.45), (5.48)]}, \tag{7.38}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega)} \leq c \text{ [cf. (5.47)]}, \tag{7.39}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| [\vartheta_\varepsilon]_{\rm res} \right\|_{L^4(\Omega)} \leq \varepsilon^{\frac{1}{2}} c \text{ [cf. (5.48)]}, \tag{7.40}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right]_{\rm res} \right\|_{L^1(\Omega)} \leq \varepsilon c \text{ [cf. (5.45), (5.100)]}. \tag{7.41}$$

- **Estimates based on energy dissipation:**

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c \text{ [cf. (5.50)]}, \tag{7.42}$$

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \,{\rm d}t \leq c \text{ [cf. (5.51)]}, \tag{7.43}$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \,{\rm d}t \leq c \text{ [cf. (5.52)]}, \tag{7.44}$$

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \,{\rm d}t \leq c \text{ [cf. (5.53)]}. \tag{7.45}$$

- **Entropy estimates:**

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\right]_{\rm res} \right\|_{L^1(\Omega)} {\rm d}t \leq \varepsilon c \text{ [cf. (5.44)]}, \tag{7.46}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\right]_{\rm res} \right\|^q_{L^q(\Omega)} {\rm d}t \leq c \text{ for a certain } q > 1 \text{ [cf. (5.54)]}, \tag{7.47}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \mathbf{u}_\varepsilon \right]_{\rm res} \right\|^q_{L^q(\Omega;\mathbb{R}^3)} \,{\rm d}t \leq c \text{ for a certain } q > 1 \ \ [\text{ cf. (5.55) }], \tag{7.48}$$

$$\int_0^T \left\| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\|^q_{L^q(\Omega;\mathbb{R}^3)} \,{\rm d}t \to 0 \text{ for a certain } q > 1 \ \ [\text{ cf. (5.56) }]. \tag{7.49}$$

Let us recall that the "essential" component $[h]_{\rm ess}$ of a function h and its "residual" counterpart $[h]_{\rm res}$ have been introduced in (4.44), (4.45).

We conclude with the estimate on the "measure of the residual set" established in (5.46), specifically,

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{\rm res}[t]| \leq \varepsilon^2 c, \tag{7.50}$$

with $\mathcal{M}^\varepsilon_{\rm res}[t] \subset \Omega$ introduced in (4.43).

7.4 Analysis of Acoustic Waves

7.4.1 Acoustic Equation

The acoustic equation governing the time oscillations of the gradient part of the velocity field is essentially the same as in Chap. 5. However, a refined analysis to be performed below requires a more elaborate description of the "small" terms as well as the knowledge of the precise rate of convergence of these quantities toward zero.

We start rewriting the equation of continuity (7.5) in the form

$$\int_0^T \int_\Omega \left(\varepsilon \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \right) {\rm d}x\,{\rm d}t = -\int_\Omega \varepsilon \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} \,{\rm d}x \tag{7.51}$$

for any $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$.

Similarly, the momentum equation (7.29) can be written as

$$\int_0^T \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t \tag{7.52}$$

$$+ \int_0^T \int_\Omega \left(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} - \overline{\varrho} F \right) {\rm div}_x \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t$$

$$- \int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t = -\varepsilon \int_\Omega \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \boldsymbol{\varphi} \,{\rm d}x$$

$$+ \varepsilon \int_0^T \int_\Omega \mathbb{G}^\varepsilon_1 : \nabla_x \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t + \varepsilon \int_0^T \int_\Omega \mathbf{G}^2_\varepsilon \cdot \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t$$

$$+ \int_0^T \int_\Omega \left(G^3_\varepsilon + G^4_\varepsilon \right) {\rm div}_x \boldsymbol{\varphi} \,{\rm d}x\,{\rm d}t,$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$, where we have set

$$\mathbb{G}_\varepsilon^1 = -\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \ \mathbf{G}_\varepsilon^2 = \frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \nabla_x F, \tag{7.53}$$

$$G_\varepsilon^3 = -\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res}}{\varepsilon}, \tag{7.54}$$

and

$$G_\varepsilon^4 = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} - \Big(\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big). \tag{7.55}$$

It is important to notice that validity of (7.52) can be extended to the class of test functions satisfying

$$\boldsymbol{\varphi} \in C_c^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^3), \ \boldsymbol{\varphi}|_{\partial\Omega} = 0 \tag{7.56}$$

by means of a simple density argument. Indeed, in accordance with the integrability properties of the weak solutions established in Theorem 3.2, it is enough to use the density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ for any finite p.

Since $\mathbf{u}_\varepsilon \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3))$, in particular, the trace of $\mathbf{u}_\varepsilon$ vanishes on the boundary, we are allowed to use the Gauss-Green theorem to obtain

$$\int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, {\rm d}x \, {\rm d}t = -\varepsilon \int_0^T \int_\Omega \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \boldsymbol{\varphi}]] \, {\rm d}x \, {\rm d}t \tag{7.57}$$

$$+ \int_0^T \int_\Omega \frac{2\varepsilon\mu(\overline{\vartheta})}{\overline{\varrho}} (\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \boldsymbol{\varphi}]] \, {\rm d}x \, {\rm d}t$$

$$+ \int_0^T \int_\Omega \varepsilon \Big(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}) \Big) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) : \nabla_x \boldsymbol{\varphi} \, {\rm d}x \, {\rm d}t$$

for any $\boldsymbol{\varphi}$ as in (7.56), where we have introduced the notation

$$[[\mathbb{M}]] = \frac{1}{2} \Big[\mathbb{M} + \mathbb{M}^T - \frac{2}{3} {\rm trace}[\mathbb{M}] \, \mathbb{I} \Big].$$

In a similar fashion, the entropy balance (7.31) can be rewritten as

$$\int_0^T \int_\Omega \varepsilon\Big(\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\partial_t\varphi \ \mathrm{d}x\ \mathrm{d}t \tag{7.58}$$

$$= -\int_\Omega \varepsilon\Big(\frac{\varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \varrho_{0,\varepsilon} s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\varphi(0,\cdot)\ \mathrm{d}x - <\sigma_\varepsilon;\varphi>_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}$$

$$+ \int_0^T \int_\Omega \Big(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon\Big)\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$.

Summing up relations (7.51)–(7.58) we obtain, exactly as in Sect. 5.4.3, a linear hyperbolic equation describing the propagation of acoustic waves.

■ ACOUSTIC EQUATION:

$$\int_0^T \int_\Omega \Big(\varepsilon r_\varepsilon \partial_t\varphi + \mathbf{V}_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.59}$$

$$= -\int_\Omega \varepsilon r_{0,\varepsilon}\varphi(0,\cdot)\ \mathrm{d}x + \frac{A}{\omega}\Big(\int_0^T\int_\Omega \mathbf{G}_5^\varepsilon\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - <\sigma_\varepsilon,\varphi>\Big)$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \Big(\varepsilon\mathbf{V}_\varepsilon\cdot\partial_t\boldsymbol{\varphi} + \omega r_\varepsilon \mathrm{div}_x\boldsymbol{\varphi} + \varepsilon D\mathbf{V}_\varepsilon\cdot\mathrm{div}_x[[\nabla_x\boldsymbol{\varphi}]]\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.60}$$

$$= -\int_\Omega \varepsilon\mathbf{V}_{0,\varepsilon}\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x$$

$$+ \int_0^T\int_\Omega \Big(\mathbf{G}_6^\varepsilon\cdot\mathrm{div}_x[[\nabla_x\boldsymbol{\varphi}]] + \mathbb{G}_7^\varepsilon : \nabla_x\boldsymbol{\varphi} + G_8^\varepsilon\mathrm{div}_x\boldsymbol{\varphi} + \mathbf{G}_9^\varepsilon\cdot\boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^3)$, $\varphi|_{\partial\Omega} = 0$,

---

where we have set

$$r_\varepsilon = \frac{1}{\omega}\Big(\omega\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + A\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F\Big),\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon, \tag{7.61}$$

$$r_{0,\varepsilon} = \frac{1}{\omega}\Big(\omega\frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} + A\varrho_{0,\varepsilon}\frac{s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \overline{\varrho}F\Big),\ \mathbf{V}_{0,\varepsilon} = \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}, \tag{7.62}$$

with

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2\, \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})},\ A = \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\overline{\varrho}\, \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})},\ D = \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}}. \tag{7.63}$$

Note that the integral identities (7.59), (7.60) represent a weak formulation of Eq. (7.1), where the "small" terms read as follows:

$$\mathbf{G}_5^\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon, \tag{7.64}$$

$$\mathbf{G}_6^\varepsilon = \varepsilon D(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon, \tag{7.65}$$

$$\mathbb{G}_7^\varepsilon = 2\varepsilon(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}))[[\nabla_x \mathbf{u}_\varepsilon]] - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \tag{7.66}$$

$$\begin{aligned} G_8^\varepsilon = A\varrho_\varepsilon \Big[\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big]_{\rm res} - \Big[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon}\Big]_{\rm res} \\ + A\Big\{\Big[\varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\Big]_{\rm ess} \\ - \overline{\varrho}\Big(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm ess} + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big)\Big\} \\ - \Big\{\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big)\Big\} \\ + \omega\Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big]_{\rm res}, \end{aligned} \tag{7.67}$$

and

$$\mathbf{G}_9^\varepsilon = (\overline{\varrho} - \varrho_\varepsilon)\nabla_x F. \tag{7.68}$$

7.4.2 Spectral Analysis of the Acoustic Operator

In this part, we are concerned with the spectral analysis of the linear operator associated to problem (7.59), (7.60), namely we examine the differential operator

$$\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} \mapsto \mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} + \varepsilon \mathcal{B}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix}, \tag{7.69}$$

with

$$\mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \omega \mathrm{div}_x \mathbf{w} \\ \nabla_x v \end{bmatrix}, \ \mathcal{B}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ D\mathrm{div}_x[[\nabla_x \mathbf{w}]] \end{bmatrix}$$

that can be viewed as the formal adjoint of the generator in (7.59), (7.60). In accordance with (7.19), we impose the homogeneous Dirichlet boundary condition for **w**,

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{7.70}$$

Let us start with the limit eigenvalue problem

$$\mathcal{A}\begin{bmatrix} v \\ \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} v \\ \mathbf{w} \end{bmatrix}, \text{ meaning } \left\{ \begin{array}{c} \omega \mathrm{div}_x \mathbf{w} = \lambda v \\ \\ \nabla_x v = \lambda \mathbf{w} \end{array} \right\} \tag{7.71}$$

which can be *equivalently* reformulated as

$$-\Delta_x v = \Lambda v, \ \Lambda = -\frac{\lambda^2}{\omega}, \tag{7.72}$$

where the boundary condition (7.70) transforms to $\nabla_x v|_{\partial\Omega} = 0$, in particular,

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.73}$$

Note that the null space (kernel) of $\mathcal{A}$ is

$$\mathrm{Ker}[\mathcal{A}] = \begin{bmatrix} \mathrm{span}\{1\} \\ L^2_\sigma(\Omega; \mathbb{R}^3) \end{bmatrix} \tag{7.74}$$

$$= \{(v, \mathbf{w}) \mid v = \mathrm{const}, \ \mathbf{w} \in L^2(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \mathbf{w} = 0, \ \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

As is well-known, the Neumann problem (7.72), (7.73) admits a countable set of real eigenvalues $\{\Lambda_n\}_{n=0}^\infty$,

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 \dots,$$

where the associated family of real eigenfunctions $\{v_{n,m}\}_{n=0,m=1}^{\infty,m_n}$ forms an orthonormal basis of the Hilbert space $L^2(\Omega)$. Moreover, we denote

$$E_n = \mathrm{span}\{v_{n,m}\}_{m=1}^{m_n}, \ n = 0, 1, \dots$$

the eigenspace corresponding to the eigenvalue Λ_n of multiplicity m_n. In particular, $m_0 = 1$, $E_0 = \mathrm{span}\{1\}$ (see Theorem 11.9 in Appendix).

Under hypothesis (7.32), Proposition 7.1 implies that $v_0 = 1/\sqrt{|\Omega|}$ is the only eigenfunction that satisfies the supplementary boundary condition $\nabla_x v_0|_{\partial\Omega} = 0$. Thus the term $\varepsilon\mathcal{B}$, together with (7.70), may be viewed as a *singular perturbation* of the operator $\mathcal{A}$.

Accordingly, the eigenvalue problem (7.71), (7.73) admits a system of eigenvalues

$$\lambda_{\pm n} = \pm \mathrm{i}\sqrt{\omega\Lambda_n},\ n = 0, 1, \dots$$

lying on the imaginary axis. The associated eigenspaces are

$$\left\{ \begin{array}{c} \mathrm{span}\{1\} \times L^2_\sigma(\Omega;\mathbb{R}^3) \text{ for } n = 0, \\ \\ \mathrm{span}\left\{(v_{n,m}, \mathbf{w}_{\pm n,m}) = \frac{1}{\lambda_{\pm n}}\nabla_x v_{n,m}\right\}_{m=1}^{m_n} \text{ for } n = 1, 2, \dots \end{array} \right\}$$

Here and hereafter, we fix $n > 0$ and set

$$\lambda = \lambda_n = \mathrm{i}\sqrt{\omega\Lambda_n},\ v = v_{n,1},\ \mathbf{w} = \mathbf{w}_{n,1} = \frac{1}{\lambda_n}\nabla_x v_{n,1}, \tag{7.75}$$

together with

$$E = E_n = \mathrm{span}\{v_{(1)}, \dots, v_{(m)}\},\ v_{(j)} = v_{n,j},\ m = m_n. \tag{7.76}$$

In order to match the incompatibility of the boundary conditions (7.70), (7.73), we look for "approximate" eigenfunctions of the perturbed problem (7.80), (7.82) in the form

$$v_\varepsilon = (v^{\mathrm{int},0} + v^{\mathrm{bl},0}) + \sqrt{\varepsilon}(v^{\mathrm{int},1} + v^{\mathrm{bl},1}), \tag{7.77}$$

$$\mathbf{w}_\varepsilon = (\mathbf{w}^{\mathrm{int},0} + \mathbf{w}^{\mathrm{bl},0}) + \sqrt{\varepsilon}(\mathbf{w}^{\mathrm{int},1} + \mathbf{w}^{\mathrm{bl},1}), \tag{7.78}$$

where we set

$$v^{\mathrm{int},0} = v,\ \mathbf{w}^{\mathrm{int},0} = \mathbf{w}. \tag{7.79}$$

The functions v_ε, $\mathbf{w}_\varepsilon$ are determined as solutions to the following approximate problem.

■ APPROXIMATE EIGENVALUE PROBLEM:

$$\mathcal{A}\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \varepsilon\mathcal{B}\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} = \lambda_\varepsilon\begin{bmatrix} v_\varepsilon \\ \mathbf{w}_\varepsilon \end{bmatrix} + \sqrt{\varepsilon}\begin{bmatrix} s^1_\varepsilon \\ \mathbf{s}^2_\varepsilon \end{bmatrix},$$

meaning,

$$\left\{ \begin{array}{c} \omega \mathrm{div}_x \mathbf{w}_\varepsilon = \lambda_\varepsilon v_\varepsilon + \sqrt{\varepsilon} s^1_\varepsilon, \\ \\ \nabla_x v_\varepsilon + \varepsilon D \mathrm{div}_x [[\nabla_x \mathbf{w}_\varepsilon]] = \lambda_\varepsilon \mathbf{w}_\varepsilon + \sqrt{\varepsilon} \mathbf{s}^2_\varepsilon, \end{array} \right\} \tag{7.80}$$

where

$$\lambda_\varepsilon = \lambda^0 + \sqrt{\varepsilon} \lambda^1, \text{ with } \lambda^0 = \lambda, \tag{7.81}$$

supplemented with the homogeneous Dirichlet boundary condition

$$\mathbf{w}_\varepsilon|_{\partial\Omega} = 0. \tag{7.82}$$

There is a vast amount of literature, in particular in applied mathematics, devoted to formal asymptotic analysis of singularly perturbed problems based on the so-called WKB (Wentzel-Kramers-Brilbuin) expansions for boundary layers similar to (7.77), (7.78). An excellent introduction to the mathematical aspects of the theory is the book by Métivier [211]. The "interior" functions $v^{\mathrm{int},k} = v^{\mathrm{int},k}(x)$, $\mathbf{w}^{\mathrm{int},k} = \mathbf{w}^{\mathrm{int},k}(x)$ depend only on $x \in \Omega$, while the "boundary layer" functions $v^{\mathrm{bl},k}(x,Z) = v^{\mathrm{bl},k}(x,Z)$, $\mathbf{w}^{\mathrm{bl},k} = \mathbf{w}^{\mathrm{bl},k}(x,Z)$ depend on x and the fast variable $Z = d(x)/\sqrt{\varepsilon}$, where d is a generalized distance function to $\partial\Omega$,

$$d \in C^3(\overline{\Omega}), \ d(x) = \left\{ \begin{array}{l} \mathrm{dist}[x, \partial\Omega] \text{ for all } x \in \overline{\Omega} \text{ such that } \mathrm{dist}[x, \partial\Omega] \leq \delta, \\ \\ d(x) \geq \delta \text{ otherwise.} \end{array} \right. \tag{7.83}$$

Note that the distance function enjoys the same regularity as the boundary $\partial\Omega$, namely as the function h appearing in hypothesis (7.32).

The rest of this section is devoted to identifying all terms in the asymptotic expansions (7.77), (7.78), the remainders s^1_ε, $\mathbf{s}^2_\varepsilon$, and the value of λ^1. In accordance with the heuristic arguments in the introductory part of this chapter, we expect to recover $\lambda^1 \neq 0$, specifically, $\mathrm{Re}[\lambda^1] < 0$ yielding the desired exponential decay rate of order $\sqrt{\varepsilon}$ (no contradiction with the sign of $\mathrm{Re}[\lambda_\varepsilon]$ in the introductory section as the elliptic part of problem (7.80)–(7.82) has negative spectrum!). This rather tedious task is accomplished in several steps.

Differential Operators Applied to the Boundary Layer Correction Functions To avoid confusion, we shall write $\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})$ for the gradient of the composed function $x \mapsto \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})$, while $\nabla_x \mathbf{w}^{\mathrm{bl},k}(x,Z)$, $\partial_Z \mathbf{w}^{\mathrm{bl},k}(x,Z)$ stand for the differential operators applied to a function of two variables x and Z. It is a routine matter to compute:

$$[[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})]] = [[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, Z)]] +$$
$$\frac{1}{2\sqrt{\varepsilon}}\left[\partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z) \otimes \nabla_x d + \nabla_x d \otimes \partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z) - \frac{2}{3}\partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z) \cdot \nabla_x d \, \mathbb{I}\right].$$

Similarly, we get

$$\mathrm{div}_x[\mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})] = \mathrm{div}_x \mathbf{w}^{\mathrm{bl},k}(x, Z) + \frac{1}{\sqrt{\varepsilon}}\partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z) \cdot \nabla_x d(x),$$

$$\nabla_x[v^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})] = \nabla_x v^{\mathrm{bl},k}(x, Z) + \frac{1}{\sqrt{\varepsilon}}\partial_Z v^{\mathrm{bl},k}(x, Z)\nabla_x d(x),$$

and

$$\mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, d(x)/\sqrt{\varepsilon})]] = \mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},k}(x, Z)]]$$
$$+\frac{1}{\sqrt{\varepsilon}}\Big\{[\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},k}(x, Z)]\nabla_x d(x) + \frac{1}{6}(\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},k}(x, Z))\nabla_x d(x)$$
$$+\frac{1}{6}[\partial_Z \nabla_x{}^T \mathbf{w}^{\mathrm{bl},k}(x, Z)]\nabla_x d(x)$$
$$+\frac{1}{2}\partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z)\Delta_x d(x) + \frac{1}{6}[\nabla_x^2 d(x)]\partial_Z \mathbf{w}^{\mathrm{bl},k}(x, Z)\Big\}$$
$$+\frac{1}{2\varepsilon}\Big\{\partial_Z^2 \mathbf{w}^{\mathrm{bl},k}(x, Z)|\nabla_x d(x)|^2 + \frac{1}{3}\partial_Z^2 \mathbf{w}^{\mathrm{bl},k}(x, Z) \cdot \nabla_x d(x)\nabla_x d(x)\Big\}$$

for $k = 0, 1$, where Z stands for $d(x)/\sqrt{\varepsilon}$.

Consequently, substituting ansatz (7.77), (7.78) in (7.80), (7.81), we arrive at the following system of equations:

$$\omega \mathrm{div}_x \mathbf{w}^{\mathrm{int},1}(x) = \lambda^0 v^{\mathrm{int},1}(x) + \lambda^1 v^{\mathrm{int},0}(x), \tag{7.84}$$

$$\nabla_x v^{\mathrm{int},1}(x) = \lambda^0 \mathbf{w}^{\mathrm{int},1}(x) + \lambda^1 \mathbf{w}^{\mathrm{int},0}(x), \tag{7.85}$$

$$\partial_Z \mathbf{w}^{\mathrm{bl},0}(x, Z) \cdot \nabla_x d(x) = 0, \tag{7.86}$$

$$\omega\left(\mathrm{div}_x \mathbf{w}^{\mathrm{bl},0}(x, Z) + \partial_Z \mathbf{w}^{\mathrm{bl},1}(x, Z) \cdot \nabla_x d(x)\right) = \lambda^0 v^{\mathrm{bl},0}(x, Z), \tag{7.87}$$

$$\partial_Z v^{\mathrm{bl},0}(x, Z)\nabla_x d(x) = 0, \tag{7.88}$$

and

$$\Big(\nabla_x v^{\mathrm{bl},0}(x,Z) + \partial_Z v^{\mathrm{bl},1}(x,Z)\nabla_x d(x)\Big) \tag{7.89}$$

$$+\frac{D}{2}\Big(\partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x,Z)|\nabla_x d(x)|^2 + \frac{1}{3}\partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x,Z)\cdot\nabla_x d(x)\nabla_x d(x)\Big) = \lambda^0 \mathbf{w}^{\mathrm{bl},0}(x,Z).$$

Moreover, the remainders s_ε^1, $\mathbf{s}_\varepsilon^2$ are determined by means of (7.80) as

$$\begin{aligned} s_\varepsilon^1 &= \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},1}(x,Z)) - \lambda^0 v^{\mathrm{bl},1}(x,Z) \\ &-\lambda^1 v^{\mathrm{bl},0}(x,Z) - \sqrt{\varepsilon}\lambda^1\Big(v^{\mathrm{int},1}(x) + v^{\mathrm{bl},1}(x,Z)\Big), \end{aligned} \tag{7.90}$$

$$\begin{aligned} \mathbf{s}_\varepsilon^2 = D\Big\{&[\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x) + \frac{1}{6}[\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x) \\ &+\frac{1}{6}[\partial_Z \nabla_x{}^T \mathbf{w}^{\mathrm{bl},0}(x,Z)]\nabla_x d(x) + \frac{1}{2}\partial_Z \mathbf{w}^{\mathrm{bl},0}(x,Z)\Delta_x d(x) \\ &+\frac{1}{6}[\nabla_x^2 d(x)]\partial_Z \mathbf{w}^{\mathrm{bl},0}(x,Z) + \frac{1}{2}\partial_Z^2 \mathbf{w}^{\mathrm{bl},1}(x,Z)|\nabla_x d(x)|^2 \\ &+\frac{1}{6}\partial_Z^2 \mathbf{w}^{\mathrm{bl},1}(x,Z)\cdot\nabla_x d(x)\nabla_x d(x)\Big\} \\ &+\nabla_x v^{\mathrm{bl},1}(x,Z) - \lambda^0 \mathbf{w}^{\mathrm{bl},1}(x,Z) - \lambda^1 \mathbf{w}^{\mathrm{bl},0}(x,Z) \\ &+\sqrt{\varepsilon}\Big\{D\Big(\mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{int},0}(x)]] + \mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},0}(x,Z)]] + \\ &[\partial_Z \nabla_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x) + \frac{1}{6}[\partial_Z \mathrm{div}_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x) \\ &+\frac{1}{6}[\partial_Z \nabla_x{}^T \mathbf{w}^{\mathrm{bl},1}(x,Z)]\nabla_x d(x) \\ &+\frac{1}{2}\partial_Z \mathbf{w}^{\mathrm{bl},1}(x,Z)\Delta_x d(x) + \frac{1}{6}[\nabla_x^2 d(x)]\partial_Z \mathbf{w}^{\mathrm{bl},1}(x,Z)\Big) \\ &-\lambda^1 \mathbf{w}^{\mathrm{int},1}(x) - \lambda^1 \mathbf{w}^{\mathrm{bl},1}(x,Z)\Big\} \\ &+\varepsilon\Big\{\mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{int},1}(x)]] + \mathrm{div}_x[[\nabla_x \mathbf{w}^{\mathrm{bl},1}(x,Z)]]\Big\}, \end{aligned} \tag{7.91}$$

where $Z = d(x)/\sqrt{\varepsilon}$.

Finally, in agreement with (7.82), we require

$$\mathbf{w}^{\mathrm{bl},k}(x,0) + \mathbf{w}^{\mathrm{int},k}(x,0) = 0 \text{ for all } x \in \partial\Omega,\ k = 0,1. \tag{7.92}$$

Determining the Zeroth Order Terms System (7.84)–(7.89) consists of six equations for the unknowns $v^{\mathrm{bl},0}$, $\mathbf{w}^{\mathrm{bl},0}$, $v^{\mathrm{int},1}$, $\mathbf{w}^{\mathrm{int},1}$, and $v^{\mathrm{bl},1}$, $\mathbf{w}^{\mathrm{bl},1}$. Note that, in agreement with (7.79),

$$\omega \mathrm{div}_x \mathbf{w}^{\mathrm{int},0} = \lambda^0 v^{\mathrm{int},0},\ \lambda^0 \mathbf{w}^{\mathrm{int},0} = \nabla v^{\mathrm{int},0},$$
$$\mathbf{w}^{\mathrm{int},0} \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x v^{\mathrm{int},0} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{7.93}$$

Moreover, since the matrix $\{\int_{\partial\Omega} \nabla_x v_{(i)} \cdot \nabla_x \mathbf{v}_{(j)}\ \mathrm{d}S_x\}_{i,j=1}^m$ is diagonalizable, the basis $\{v_{(1)}, \ldots, v_{(m)}\}$ of the eigenspace E introduced in (7.75), (7.76) may be chosen in such a way that

$$\int_\Omega v_{(i)} v_{(j)}\ \mathrm{d}x = \delta_{i,j},\ \int_{\partial\Omega} \nabla_x v_{(i)} \cdot \nabla_x v_{(j)}\ \mathrm{d}S_x = 0 \text{ for } i \neq j, \tag{7.94}$$

where $v^{\mathrm{int},0} = v_{(1)}$.

Since there are no boundary conditions imposed on the component v, we can take

$$v^{\mathrm{bl},0}(x, Z) \equiv v^{\mathrm{bl},1}(x, Z) \equiv 0, \tag{7.95}$$

in particular, Eq. (7.88) holds.

Furthermore, Eq. (7.86) requires the quantity $\mathbf{w}^{\mathrm{bl},0}(x, Z) \cdot \nabla_x d(x)$ to be independent of Z. On the other hand, by virtue of (7.73), (7.92), the function $x \mapsto \mathbf{w}^{\mathrm{bl},0}(x, d(x)/\sqrt{\varepsilon})$ must have zero normal trace on $\partial\Omega$. Since $d(x) = 0$, $\nabla_x d(x) = -\mathbf{n}(x)$ for any $x \in \partial\Omega$, we have to take

$$\mathbf{w}^{\mathrm{bl},0}(x, Z) \cdot \nabla d(x) = 0 \text{ for all } x \in \overline{\Omega}, Z \geq 0. \tag{7.96}$$

Consequently, Eq. (7.89) reduces to

$$\frac{D}{2} \partial_Z^2 \mathbf{w}^{\mathrm{bl},0}(x, Z) |\nabla_x d(x)|^2 = \lambda^0 \mathbf{w}^{\mathrm{bl},0}(x, Z) \text{ to be satisfied for } Z > 0. \tag{7.97}$$

For a fixed $x \in \overline{\Omega}$, relation (7.97) represents an ordinary differential equation of second order in Z, for which the initial conditions $\mathbf{w}^{\mathrm{bl},0}(x, 0)$ are uniquely determined by (7.92), namely

$$\mathbf{w}^{\mathrm{bl},0}(x, 0) = -\mathbf{w}^{\mathrm{int},0}(x) \text{ for all } x \in \partial\Omega. \tag{7.98}$$

It is easy to check that problem (7.97), (7.98) admits a unique solution that *decays to zero* for $Z \to \infty$, specifically,

$$\mathbf{w}^{\mathrm{bl},0}(x, Z) = -\chi(d(x)) \mathbf{w}^{\mathrm{int},0} (x - d(x) \nabla_x d(x)) \exp(-\Gamma Z), \tag{7.99}$$

where $\chi \in C^{\infty}[0, \infty)$,

$$\chi(d) = \begin{cases} 1 \text{ for } d \in [0, \delta/2] \\ \\ 0 \text{ if } d > \delta, \end{cases} \tag{7.100}$$

and

$$\Gamma^2 = \frac{2\lambda^0}{D}, \ \mathrm{Re}[\Gamma] > 0. \tag{7.101}$$

It seems worth-noting that formula (7.99) is compatible with (7.96) as for $x \in \Omega$ the point $x - \nabla_x d(x)/d(x)$ is the nearest to x on $\partial\Omega$ as soon as $d(x)$ coincides with $\mathrm{dist}[x, \partial\Omega]$.

First Order Terms Equation (7.87), together with the ansatz made in (7.95), give rise to

$$\partial_Z\Big(\mathbf{w}^{\mathrm{bl},1}(x, Z) \cdot \nabla_x d(x)\Big) = -\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z)). \tag{7.102}$$

In view of (7.99), Eq. (7.102) admits a unique solution with *exponential decay* for $Z \to \infty$ for any fixed $x \in \overline{\Omega}$, namely

$$\mathbf{w}^{\mathrm{bl},1}(x, Z) \cdot \nabla_x d(x) = \frac{1}{\Gamma}\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z)).$$

Thus we can set

$$\mathbf{w}^{\mathrm{bl},1}(x, Z) = \frac{1}{\Gamma}\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, Z))\nabla_x d(x) + \mathbf{H}(x)\exp(-\Gamma Z), \tag{7.103}$$

for a function $\mathbf{H}$ such that

$$\mathbf{H}(x) \cdot \nabla_x d(x) = 0 \tag{7.104}$$

to be determined below. Note that, in accordance with formula (7.99), $|\nabla_x d(x)| = |\nabla_x \mathrm{dist}[x, \partial\Omega]| = 1$ on the set where $\mathbf{w}^{\mathrm{bl},0} \neq 0$.

Determining $\boldsymbol{\lambda}^1$ Our ultimate goal is to identify $v^{\mathrm{int},1}$, $\mathbf{w}^{\mathrm{int},1}$, and, in particular λ^1, by help of equations of (7.84), (7.85). In accordance with (7.92), the normal trace of the quantity $\mathbf{w}^{\mathrm{int},1}(x) + \mathbf{w}^{\mathrm{bl},1}(x, 0)$ must vanish for $x \in \partial\Omega$; whence, by virtue of (7.103),

$$0 = \mathbf{w}^{\mathrm{int},1}(x) \cdot \mathbf{n}(x) + \mathbf{w}^{\mathrm{bl},1}(x, 0) \cdot \mathbf{n}(x) = \mathbf{w}^{\mathrm{int},1}(x) \cdot \mathbf{n}(x) - \frac{1}{\Gamma}\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x, 0)) \tag{7.105}$$

for any $x \in \partial\Omega$.

As a consequence of (7.93), system (7.84), (7.85) can be rewritten as a second order elliptic equation

$$\Delta_x v^{\mathrm{int},1} + \Lambda v^{\mathrm{int},1} = 2\frac{\lambda^1\lambda^0}{\omega} v^{\mathrm{int},0} \text{ in } \Omega, \tag{7.106}$$

where $\Lambda = -(\lambda^0)^2/\omega$. Problem (7.106) is supplemented with the *non-homogeneous* Neumann boundary condition determined by means of (7.93), (7.85), and (7.105), namely

$$\nabla_x v^{\mathrm{int},1} \cdot \mathbf{n}(x) = \frac{\lambda^0}{\Gamma} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) \text{ for all } x \in \partial\Omega. \tag{7.107}$$

According to the standard Fredholm alternative for elliptic problems (see Sect. 11.3.2 in Appendix), system (7.106), (7.107) is solvable as long as

$$\frac{\omega}{\Gamma}\int_{\partial\Omega} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x = 2\lambda^1 \int_\Omega v^{\mathrm{int},0} v_{(k)} \, \mathrm{d}x \text{ for } k = 1, \ldots, m,$$

where $\{v_{(1)}, \ldots, v_{(m)}\}$ is the system of eigenvectors introduced in (7.94). In accordance with our agreement $v_{(1)} = v^{\mathrm{int},0}$, therefore we set

$$\lambda^1 = \frac{\omega}{2\Gamma}\int_{\partial\Omega} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) v^{\mathrm{int},0} \, \mathrm{d}S_x \tag{7.108}$$

and verify that

$$\int_{\partial\Omega} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x = 0 \text{ for } k = 2, \ldots, m. \tag{7.109}$$

To this end, use (7.93), (7.99) to compute

$$\begin{aligned}
\mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) &= -\mathrm{div}_x\left(\mathbf{w}^{\mathrm{int},0}(x - d(x)\nabla_x d(x))\right) \\
&= -\frac{1}{\lambda^0}\mathrm{div}_x\left(\nabla_x v^{\mathrm{int},0}(x - d(x)\nabla_x d(x))\right) \\
&= -\frac{1}{\lambda^0}\nabla_x^2 v^{\mathrm{int},0}\,(x - d(x)\nabla_x d(x)) : \left(\mathbb{I} - \nabla_x d(x) \otimes \nabla_x d(x) - d(x)\nabla^2 d(x)\right)
\end{aligned}$$

whenever $\mathrm{dist}[x, \partial\Omega] < \delta/2$. Consequently,

$$\begin{aligned}
&\int_{\partial\Omega} \mathrm{div}_x(\mathbf{w}^{\mathrm{bl},0}(x,0)) v_{(k)} \, \mathrm{d}S_x \\
&= -\frac{1}{\lambda^0}\int_{\partial\Omega} \nabla_x^2 v^{\mathrm{int},0} : (\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) v_{(k)} \, \mathrm{d}S_x \\
&= \frac{1}{\lambda^0}\int_{\partial\Omega} \Delta_S v^{\mathrm{int},0} v_{(k)} \, \mathrm{d}S_x = \frac{1}{\lambda^0}\int_{\partial\Omega} \nabla_x v^{\mathrm{int},0} \cdot \nabla_x v_{(k)} \, \mathrm{d}S_x,
\end{aligned}$$

where the symbol Δ_S denotes the Laplace-Beltrami operator on the (compact) Riemannian manifold $\partial\Omega$. Indeed expression $\left[\nabla_x^2 v^{\mathrm{int},0} : (\mathbf{n} \otimes \mathbf{n} - \mathbb{I})\right]$ represents the standard "flat" Laplacian of the function $v^{\mathrm{int},0}$ with respect to the tangent plane at each point of $\partial\Omega$ that *coincides* (up to a sign) with the associated Laplace-Beltrami operator on the manifold $\partial\Omega$ applied to the restriction of $v^{\mathrm{int},0}|_{\partial\Omega}$ provided $\nabla_x v^{\mathrm{int},0} \cdot \mathbf{n} = 0$ on $\partial\Omega$ (see Gilbarg and Trudinger [136, Chap. 16]).

In accordance with (7.94), we infer that

$$\int_{\partial\Omega} \nabla_x v^{\mathrm{int},0} \cdot \nabla_x v_{(k)} \, \mathrm{d}S_x = \begin{cases} \int_{\partial\Omega} |\nabla_x v^{\mathrm{int},0}|^2 \, \mathrm{d}S_x \text{ if } k = 1, \\ \\ 0 \text{ for } k = 2, \dots, m. \end{cases}$$

In particular, we get (7.109), and, using (7.72), (7.101),

$$\lambda^1 = -\Gamma \frac{D}{4\Lambda} \int_{\partial\Omega} |\nabla_x v^{\mathrm{int},0}|^2 \, \mathrm{d}S_x.$$

Seeing that $\Lambda > 0$, and, by virtue of (7.101), $\mathrm{Re}[\Gamma] > 0$, we utilize hypothesis (7.32) together with Proposition 7.1 to deduce the desired conclusion

$$\mathrm{Re}[\lambda_1] < 0. \tag{7.110}$$

This is the crucial point of the proof of Theorem 7.1.

Having identified $v^{\mathrm{int},1}$ by means of (7.106), (7.107) we use (7.85) to compute

$$\mathbf{w}^{\mathrm{int},1} = \frac{1}{\lambda^0} \left(\nabla_x v^{\mathrm{int},1} - \lambda^1 \mathbf{w}^{\mathrm{int},0} \right).$$

Finally, in order to meet the boundary conditions (7.92), we set

$$\mathbf{H}(x) = -\chi(d(x)) \Big(\mathbf{w}^{\mathrm{int},1}(x) - (\mathbf{w}^{\mathrm{int},1} \cdot \nabla_x d(x)) \nabla_x d(x) \Big) \text{ for } x \in \overline{\Omega}$$

in (7.103), with χ given by (7.100).

Conclusion By a direct inspection of (7.90), (7.91), where all quantities are evaluated by means (7.95), (7.99), (7.103), we infer that

$$|s_\varepsilon^1| + |\mathbf{s}_\varepsilon^2| \le c \left(\sqrt{\varepsilon} + \exp\left(-\mathrm{Re}[\Gamma] \frac{d(x)}{\sqrt{\varepsilon}} \right) \right),$$

in particular s_ε^1, $\mathbf{s}_\varepsilon^2$ are uniformly bounded in $\overline{\Omega}$ and tend to zero uniformly on any compact $K \subset \Omega$.

The results obtained in this section are summarized in the following assertion.

Proposition 7.2 *Let Ω be given through (7.17), with*

$$B_{\rm top} = 0,\ B_{\rm bottom} = -\pi - h,$$
$$h \in C^3(\mathcal{T}^2),\ |h| < \pi,\ h \not\equiv {\rm const}.$$

Assume that $v^{{\rm int},0}$, $\mathbf{w}^{{\rm int},0}$, and $\lambda^0 \neq 0$ solve the eigenvalue problem (7.71), (7.73).

Then the approximate eigenvalue problem (7.80)–(7.82) admits a solution in the form (7.77), (7.78), where

- *the functions $v^{{\rm int},1} = v^{{\rm int},1}(x)$, $\mathbf{w}^{{\rm int},1} = \mathbf{w}^{{\rm int},1}(x)$ belong to the class $C^2(\overline{\Omega})$;*
- *the boundary layer corrector functions $v^{{\rm bl},0} = v^{{\rm bl},1} = 0$, $\mathbf{w}^{{\rm bl},0} = \mathbf{w}^{{\rm bl},0}(x,Z)$, $\mathbf{w}^{{\rm bl},1} = \mathbf{w}^{{\rm bl},1}(x,Z)$ are all of the form $\mathbf{h}(x)\exp(-\Gamma Z)$, where $\mathbf{h} \in C^2(\overline{\Omega};\mathbb{R}^3)$, and ${\rm Re}[\Gamma] > 0$;*
- *the approximate eigenvalue λ_ε is given by (7.81), where*

$${\rm Re}[\lambda^1] < 0; \tag{7.111}$$

- *the remainders s_ε^1, $\mathbf{s}_\varepsilon^2$ satisfy*

$$s_\varepsilon^1 \to 0 \text{ in } L^q(\Omega),\ \mathbf{s}_\varepsilon^2 \to 0 \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ as } \varepsilon \to 0 \text{ for any } 1 \le q < \infty. \tag{7.112}$$

7.5 Strong Convergence of the Velocity Field

We are now in a position to establish the main result of this chapter stated in Theorem 7.1, namely

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ strongly in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{7.113}$$

We recall that, in accordance with (7.43),

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)); \tag{7.114}$$

at least for a suitable subsequence. Moreover, exactly as in Sect. 5.3.1, we have

$${\rm div}_x\,\mathbf{U} = 0.$$

Consequently, it remains to control possible oscillations of the velocity field in time. To this end, similarly to Chap. 5, the problem is reduced to a finite number of acoustic modes that can be treated by means of the spectral theory developed in the preceding section.

7.5.1 Compactness of the Solenoidal Component

It follows from the uniform estimates (7.36)–(7.38) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho}\mathbf{U} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{7.115}$$

Using quantities

$$\boldsymbol{\varphi}(t,x) = \psi(t)\boldsymbol{\phi}(x),\ \psi \in C_c^\infty(0,T),\ \boldsymbol{\phi} \in C_c^\infty(\Omega),\ \mathrm{div}_x\boldsymbol{\phi} = 0$$

as test functions in the momentum equation (7.29) we deduce, by means of the standard Arzelà-Ascoli theorem, that the scalar functions

$$t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x \text{ are precompact in } C[0,T].$$

Note that

$$\int_\Omega \frac{1}{\varepsilon}\varrho_\varepsilon \nabla_x F \cdot \boldsymbol{\phi}\ \mathrm{d}x = \int_\Omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\nabla_x F\boldsymbol{\phi}\ \mathrm{d}x$$

as $\boldsymbol{\phi}$ is a divergenceless vector field.

Consequently, by help of (7.115) and a simple density argument, we infer that the family

$$t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{H}[\boldsymbol{\phi}]\ \mathrm{d}x \text{ is precompact in } C[0,T]$$

for any $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^3)$, where $\mathbf{H}$ denotes the Helmholtz projection introduced in Sect. 5.4.1. In other words,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho}\mathbf{H}[\mathbf{U}] = \overline{\varrho}\mathbf{U} \text{ in } C_{\mathrm{weak}}([0,T];L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{7.116}$$

Let us point out that $\mathbf{H}[\phi]$ *is not* an admissible test function in (7.29), however, it can be approximated in $L^p(\Omega;\mathbb{R}^3)$ by smooth solenoidal functions with compact support for finite p (see Sect. 11.7 in Appendix).

Thus, combining relations (7.114), (7.116), we infer

$$\int_0^T \int_\Omega \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{H}[\mathbf{u}_\varepsilon]\ \mathrm{d}x\ \mathrm{d}t \to \overline{\varrho}\int_0^T \int_\Omega |\mathbf{H}[\mathbf{U}]|^2\ \mathrm{d}x\ \mathrm{d}t,$$

which, together with estimates (7.37), (7.38), gives rise to

$$\int_0^T \int_\Omega |\mathbf{H}[\mathbf{u}_\varepsilon]|^2\ \mathrm{d}x\ \mathrm{d}t \to \int_0^T \int_\Omega |\mathbf{U}|^2\ \mathrm{d}x\ \mathrm{d}t$$

yielding, finally, the desired conclusion

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{U} \text{ (strongly) in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{7.117}$$

7.5.2 Reduction to a Finite Number of Modes

Exactly as in (5.146), we decompose the space L^2 as a sum of the subspace of solenoidal vector fields L^2_σ and gradients L^2_g:

$$L^2(\Omega;\mathbb{R}^3) = L^2_\sigma(\Omega;\mathbb{R}^3) \oplus L^2_g(\Omega;\mathbb{R}^3).$$

Since we already know that the solenoidal components of the velocity field $\mathbf{u}_\varepsilon$ are precompact in L^2, the proof of (7.113) reduces to showing

$$\mathbf{H}^\perp[\mathbf{u}_\varepsilon] \to \mathbf{H}^\perp[\mathbf{U}] = 0 \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3).$$

Moreover, since the embedding $W^{1,2}_0(\Omega;\mathbb{R}^3) \hookrightarrow L^2(\Omega;\mathbb{R}^3)$ is compact, it is enough to show

$$\left[t \mapsto \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{w}\, \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T), \tag{7.118}$$

for any fixed $\mathbf{w} = \frac{1}{\lambda}\nabla_x v$, where v, $\mathbf{w}$, $\lambda \neq 0$ solve the eigenvalue problem (7.71), (7.73) (cf. Sect. 5.4.6).

In view of (7.37), (7.38), relation (7.118) follows as soon as we show

$$\left[t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}\, \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T),$$

where the latter quantity can be expressed by means of the acoustic equation (7.59), (7.60). In addition, since the solutions of the eigenvalue problem (7.71), (7.73) come in pairs $[v, \mathbf{w}, \lambda]$, $[v, -\mathbf{w}, -\lambda]$, it is enough to show

$$\left[t \mapsto \int_\Omega \left(r_\varepsilon v + \mathbf{V}_\varepsilon \cdot \mathbf{w}\right) \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T) \tag{7.119}$$

for any solution v, $\mathbf{w}$ of (7.71), (7.73) associated to an eigenvalue $\lambda \neq 0$, where r_ε, $\mathbf{V}_\varepsilon$ are given by (7.61).

Finally, in order to exploit the information on the spectrum of the perturbed acoustic operator, we claim that (7.119) can be replaced by

$$\left[t \mapsto \int_\Omega \left(r_\varepsilon v_\varepsilon + \mathbf{V}_\varepsilon \cdot \mathbf{w}_\varepsilon\right) \mathrm{d}x\right] \to 0 \text{ in } L^2(0,T), \tag{7.120}$$

where v_ε, $\mathbf{w}_\varepsilon$ are the solutions of the approximate eigenvalue problem (7.80), (7.82) constructed in the previous section. Indeed, by virtue of Proposition 7.2, we have

$$v_\varepsilon \to v \text{ in } C(\overline{\Omega}),\ \mathbf{w}_\varepsilon \to \mathbf{w} \text{ in } L^q(\Omega;\mathbb{R}^3) \text{ for any } 1 \le q < \infty.$$

Accordingly, the proof of Theorem 7.1 reduces to showing (7.120). This will be done in the following section.

7.5.3 *Strong Convergence*

In order to complete the proof of Theorem 7.1, our ultimate goal consists in showing (7.120). To this end, we make use of the specific form of the acoustic equation (7.59), (7.60), together with the associated spectral problem (7.80), (7.82). Taking the quantities $\psi(t)v_\varepsilon(x)$, $\psi(t)\mathbf{w}_\varepsilon(x)$, $\psi \in C_c^\infty(0,T)$, as test functions in (7.59), (7.60), respectively, we obtain

$$\int_0^T \Big(\varepsilon\chi_\varepsilon\partial_t\psi + \lambda_\varepsilon\chi_\varepsilon\psi\Big)\,\mathrm{d}t + \sqrt{\varepsilon}\int_0^T \psi\int_\Omega \Big(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon\cdot\mathbf{s}_\varepsilon^2\Big)\,\mathrm{d}x\,\mathrm{d}t = \sum_{m=1}^7 I_m^\varepsilon, \tag{7.121}$$

where we have set

$$\chi_\varepsilon(t) = \int_\Omega \Big(r_\varepsilon(t,\cdot)v_\varepsilon + \mathbf{V}_\varepsilon(t,\cdot)\cdot\mathbf{w}_\varepsilon\Big)\,\mathrm{d}x,$$

and the symbols I_m^ε stand for the "small" terms:

$$I_1^\varepsilon = \frac{A}{\omega}\int_0^T \psi\int_\Omega \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon\Big]\cdot\nabla_x v_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_2^\varepsilon = -\frac{A}{\omega} < \sigma_\varepsilon; \psi v_\varepsilon >_{[\mathcal{M};C]([0,T]\times\overline{\Omega})},$$

$$I_3^\varepsilon = D\int_0^T \psi\int_\Omega \varepsilon^2\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\mathbf{u}_\varepsilon\cdot\mathrm{div}_x[[\nabla_x\mathbf{w}_\varepsilon]]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_4^\varepsilon = \int_0^T \psi\int_\Omega \varepsilon^2\Big(\frac{\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta})}{\varepsilon}\Big)[[\nabla_x\mathbf{u}_\varepsilon]] : \nabla_x\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_5^\varepsilon = -\int_0^T \psi\int_\Omega \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_6^\varepsilon = \int_0^T \psi\int_\Omega \varepsilon\Big(\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon}\Big)\nabla_x F\cdot\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_7^\varepsilon = \int_0^T \psi \int_\Omega G_8^\varepsilon \, {\rm div}_x \mathbf{w}_\varepsilon \, {\rm d}x \, {\rm d}t,$$

where G_8^ε is given by (7.67).

Our aim is to show that each of the integrals can be written in the form

$$I^\varepsilon \approx \int_0^T \psi(t)\Big(\varepsilon \gamma^\varepsilon(t) + \varepsilon^{1+\beta} \Gamma^\varepsilon(t)\Big) \, {\rm d}t,$$

where

$$\left\{ \begin{array}{c} \{\gamma_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q>1, \\ \\ \{\Gamma_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T), \text{ and } \beta > 0. \end{array} \right\}$$

This rather tedious task, to be achieved by means of Proposition 7.2 combined with the uniform estimates listed in Sect. 7.3, consists in several steps as follows:

(i) By virtue of Hölder's inequality, we have

$$\left| \int_\Omega \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x v_\varepsilon \, {\rm d}x \right| \tag{7.122}$$

$$\leq \varepsilon \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \left| \int_\Omega \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm ess} \left| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right| {\rm d}x \right| + \left| \int_\Omega \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm res} \left| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right| {\rm d}x \right|$$

$$= \varepsilon \gamma_{1,1}^\varepsilon, \text{ with } \{\gamma_1^\varepsilon\}_{\varepsilon>1} \text{ bounded in } L^q(0,T) \text{ for a certain } q>1,$$

where we have used estimates (7.44) and (7.49). Note that, in accordance with Proposition 7.2, both correction terms $v^{\rm bl,0}$, $v^{\rm bl,1}$ vanish identically, in particular,

$$\|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \leq c \text{ uniformly in } \varepsilon. \tag{7.123}$$

In a similar way,

$$\left| \int_\Omega \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathbf{u}_\varepsilon \cdot \nabla_x v_\varepsilon \, {\rm d}x \right| \tag{7.124}$$

$$\leq \varepsilon \|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \left[\int_\Omega \left| \left[\frac{\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\rm ess} \right| |\mathbf{u}_\varepsilon| \, {\rm d}x \right.$$

$$\left. + \int_\Omega \left| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\rm res} \mathbf{u}_\varepsilon \right| {\rm d}x \, {\rm d}t + |s(\overline{\varrho}, \overline{\vartheta})| \int_\Omega \left[\frac{\varrho_\varepsilon}{\varepsilon} \right]_{\rm res} |\mathbf{u}_\varepsilon| \, {\rm d}x \right] = \varepsilon \gamma_{1,2}^\varepsilon.$$

Thus we can use Proposition 5.2, together with estimates (7.37)–(7.39), (7.43), (7.48), (7.50), in order to conclude that

$$\{\gamma_{1,2}^{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q > 1.$$

Summing up (7.122), (7.124) we infer that

$$I_1^{\varepsilon} = \varepsilon \int_0^T \psi(t)\gamma_1^{\varepsilon}(t)\,\mathrm{d}t, \text{ with } \{\gamma_1^{\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^q(0,T) \text{ for a certain } q > 1. \tag{7.125}$$

(ii) As a straightforward consequence of estimate (7.42) we obtain

$$I_2^{\varepsilon} = \varepsilon^2 < \Gamma_2^{\varepsilon}; \psi >_{[\mathcal{M};C][0,T]}, \text{ where } \{\Gamma_2^{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } \mathcal{M}^+[0,T]. \tag{7.126}$$

(iii) Taking advantage of the form of $\mathbf{w}^{\mathrm{bl},0}$, $\mathbf{w}^{\mathrm{bl},1}$ specified in Proposition 7.2, we obtain

$$\|\varepsilon \mathrm{div}_x[[\nabla_x \mathbf{w}_{\varepsilon}]]\,\|_{L^{\infty}(\Omega;\mathbb{R}^3)} \leq c$$

uniformly for $\varepsilon \to 0$. This fact, combined with the uniform bounds established in (7.37), (7.38), (7.43), and the standard embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, gives rise to

$$I_3^{\varepsilon} = \varepsilon \int_0^T \psi(t)\gamma_3^{\varepsilon}(t)\,\mathrm{d}t, \tag{7.127}$$

where

$$\{\gamma_3^{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

(iv) Similarly to the preceding step, we deduce

$$\|\sqrt{\varepsilon}\mathbf{w}_{\varepsilon}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)} \leq c; \tag{7.128}$$

whence, by virtue of (7.40), (7.43), and (7.44),

$$I_4^{\varepsilon} = \varepsilon^{3/2} \int_0^T \psi(t)\Gamma_4^{\varepsilon}(t)\,\mathrm{d}t, \tag{7.129}$$

where

$$\{\Gamma_4^{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T).$$

(v) Probably the most delicate issue is to handle the integrals in I_5^ε. To this end, we first write

$$\int_0^T \psi \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \overline{\varrho} \int_0^T \psi \int_\Omega \varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t,$$

where, by virtue of (7.37), (7.38), (7.43), and the gradient estimate established in (7.128),

$$\int_0^T \psi \int_\Omega \varepsilon^2 \Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big) \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = \varepsilon^{3/2} \int_0^T \psi(t) \Gamma_{5,1}^\varepsilon(t) \, \mathrm{d}t, \tag{7.130}$$

with

$$\{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^1(0,T).$$

On the other hand, a direct computation yields

$$\int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w}_\varepsilon \, \mathrm{d}x = -\int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x - \int_\Omega (\nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x. \tag{7.131}$$

Now, we have

$$\int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x = \int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{ess}} \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x + \int_\Omega \mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{res}} \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x,$$

where, in accordance with estimates (7.36), (7.43),

$$\{\mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{ess}}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^1(\Omega;\mathbb{R}^3)),$$

while

$$\|\mathrm{div}_x \mathbf{u}_\varepsilon [\mathbf{u}_\varepsilon]_{\mathrm{res}}\|_{L^1(0,T;L^1(\Omega;\mathbb{R}^3))}$$
$$\leq c\varepsilon^{2/3} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))},$$

where we have used (7.43), the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, and the bound on the measure of the "residual set" established in (7.50).

Applying the same treatment to the latter integral on the right-hand side of (7.131) and adding the result to (7.130) we conclude that

$$I_5^\varepsilon = \varepsilon^{3/2} \int_0^T \psi(t)\Gamma_{5,1}^\varepsilon \, dt + \varepsilon \int_0^T \psi(t)\gamma_5^\varepsilon(t) \, dt + \varepsilon^{5/3} \int_0^T \psi(t)\Gamma_{5,2}^\varepsilon \, dt, \tag{7.132}$$

where

$$\{\gamma_5^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T),$$

and

$$\{\Gamma_{5,1}^\varepsilon\}_{\varepsilon>0}, \; \{\Gamma_{5,2}^\varepsilon\}_{\varepsilon>0} \text{ are bounded in } L^1(0,T).$$

(vi) In view of estimates (7.37), (7.38), it is easy to check that

$$I_6^\varepsilon = \varepsilon \int_0^T \psi(t)\gamma_6^\varepsilon(t) \, dt, \tag{7.133}$$

with

$$\{\gamma_6^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T).$$

(vii) Finally, in accordance with the first equation in (7.80) and Proposition 7.2,

$$\|\mathrm{div}_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega)} \le c;$$

therefore relations (7.38)–(7.41), (7.46), together with Proposition 5.2, can be used in order to conclude that

$$I_7^\varepsilon = \varepsilon \int_0^T \psi(t)\gamma_7^\varepsilon(t) \, dt, \tag{7.134}$$

where

$$\{\gamma_7^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

We are now in a position to use relation (7.121) in order to show (7.120). To begin, we focus on the integral

$$\sqrt{\varepsilon} \int_0^T \psi \int_\Omega \left(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \right) dx$$

appearing on the left-hand side of (7.121), with r_ε, $\mathbf{V}_\varepsilon$ specified in (7.61). Writing

$$\sqrt{\varepsilon}\int_0^T \psi \int_\Omega \left(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \right) \, dx$$
$$= \sqrt{\varepsilon}\int_0^T \psi \int_\Omega \left([r_\varepsilon]_{\rm ess} s_\varepsilon^1 + [r_\varepsilon]_{\rm res} s_\varepsilon^1 + (\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot \mathbf{s}_\varepsilon^2 \right) \, dx$$

we can use the uniform estimates (7.36)–(7.41), together with pointwise convergence of the remainders established in (7.112), in order to deduce that

$$\sqrt{\varepsilon}\int_0^T \psi \int_\Omega \left(r_\varepsilon s_\varepsilon^1 + \mathbf{V}_\varepsilon \cdot \mathbf{s}_\varepsilon^2 \right) \, dx = \sqrt{\varepsilon}\int_0^T \psi(t)\beta_\varepsilon(t) \, dt, \tag{7.135}$$

where

$$\beta_\varepsilon \to 0 \text{ in } L^\infty(0,T). \tag{7.136}$$

Next, we use a family of standard regularizing kernels

$$\zeta_\delta(t) = \frac{1}{\delta}\zeta\left(\frac{t}{\delta}\right), \ \delta \to 0,$$

$$\zeta \in C_c^\infty(-1,1), \ \zeta \geq 0, \ \int_{-1}^1 \zeta(t) \, dt = 1$$

in order to handle the "measure-valued" term in (7.121). To this end, we take ζ_δ as a test function in (7.121) to obtain

$$\frac{d}{dt}\chi_{\varepsilon,\delta} - \frac{\lambda_\varepsilon}{\varepsilon}\chi_{\varepsilon,\delta} = \sqrt{\varepsilon} h^1_{\varepsilon,\delta} + h^2_{\varepsilon,\delta} + \frac{1}{\sqrt{\varepsilon}} h^3_{\varepsilon,\delta}, \tag{7.137}$$

where we have set

$$\chi_{\varepsilon,\delta}(t) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)\psi_\delta(s) \, ds$$

for $t \in (\delta, T-\delta)$.

In accordance with the uniform estimates (7.122)–(7.134), we have

$$\{h^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^1(0,T), \ \{h^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^p(0,T) \text{ for a certain } p>1, \tag{7.138}$$

uniformly for $\delta \to 0$, where we have used the standard properties of mollifiers recorded in Theorem 11.3 in Appendix. Similarly, by virtue of (7.135), (7.136),

$$\sup_{\delta>0} \|h^3_{\varepsilon,\delta}\|_{L^\infty(0,T)} \leq \nu(\varepsilon), \ \nu(\varepsilon) \to 0 \text{ for } \varepsilon \to 0. \tag{7.139}$$

Here all functions in (7.138), (7.139) have been extended to be zero outside $(\delta, T - \delta)$.

The standard variation-of-constants formula yields

$$|\chi_{\varepsilon,\delta}(t)| \leq \exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-\delta)\Big)\mathrm{ess}\sup_{s\in(0,T)} |\chi_{\varepsilon,\delta}(s)| + \sqrt{\varepsilon}\int_0^T |h^1_{\varepsilon,\delta}(s)|\,\mathrm{d}s$$

$$+\int_\delta^t \exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)|h^2_{\varepsilon,\delta}(s)|\,\mathrm{d}s + \int_\delta^t \frac{1}{\sqrt{\varepsilon}}\exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)|h^3_{\varepsilon,\delta}(s)|\,\mathrm{d}s;$$

therefore letting first $\delta \to 0$ and then $\varepsilon \to 0$ yields the desired conclusion (7.120). Note that, in accordance with (7.111),

$$\mathrm{Re}[\lambda_\varepsilon/\varepsilon] \leq -\frac{c}{\sqrt{\varepsilon}} \text{ for a certain } c > 0,$$

in particular

$$\int_0^t \frac{1}{\sqrt{\varepsilon}}\exp\Big(\mathrm{Re}[\lambda_\varepsilon/\varepsilon](t-s)\Big)\,\mathrm{d}s < c$$

uniformly for $\varepsilon \to 0$. The proof of Theorem 7.1 is now complete.

7.6 Asymptotic Limit on Domains with Oscillatory Boundaries and Complete Slip Boundary Conditions

Although the no-slip boundary condition (7.19) is probably the most widely accepted for viscous fluids in contact with an impermeable boundary, it is sometimes more convenient to approximate a complicated topography of the real physical boundary by a smooth one endowed with a suitable wall law similar to the slip boundary condition (5.15) rather than (7.19) (see Jaeger and Mikelic [158], Mohammadi et al. [215], among others).

Similarly to the preceding part, we consider the infinite slab (7.17), (7.18), with flat top and variable bottom determined through a function

$$B_{\mathrm{bottom}} = -\pi - h(x_1,x_2) - \omega_\varepsilon(x_1,x_2),\ \omega_\varepsilon(x_1,x_2) = \frac{1}{k(\varepsilon)}\omega\left(k(\varepsilon)x_1, k(\varepsilon)x_2\right),\ \omega \geq 0 \tag{7.140}$$

where $h \in C^2(\mathcal{T}^2)$ is the same as in (7.18), $\omega \in C^2(\mathcal{T}^2)$, and $k(\varepsilon)$ is a sequence of positive integers, $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Thus the functions ω_ε are $2\pi/k(\varepsilon)$-periodic, with amplitude proportional to $1/k(\varepsilon)$.

We set

$$\Omega_\varepsilon = \left\{(x_1, x_2, x_3) \,\middle|\, (x_1, x_2) \in \mathcal{T}^2,\ -\pi - h(x_1, x_2) - \omega_\varepsilon(x_1, x_2) < x_3 < 0\right\} \tag{7.141}$$

and impose the following boundary conditions for the velocity:

$$\left\{ \begin{array}{c} \mathbf{u}|_{\{x_3=0\}} = 0, \\ \mathbf{u}\cdot\mathbf{n}|_{\{x_3=-\pi-h(x_1,x_2)-\omega_\varepsilon(x_1,x_2)\}} = 0,\ \mathbb{S}\mathbf{n}\times\mathbf{n}|_{\{x_3=-\pi-h(x_1,x_2)-\omega_\varepsilon(x_1,x_2)\}} = 0. \end{array} \right\} \tag{7.142}$$

The no-slip boundary condition is therefore prescribed only on the top part while complete slip boundary conditions, used in the preceding Chaps. 5 and 6, are required at the bottom part of Ω_ε. Our goal is to show that (7.142) provides the same effect as the no-slip boundary conditions provided the "oscillatory" part of the boundary here represented by ω_ε is non-degenerate, meaning not constant in any direction. In particular, the velocities $\mathbf{u}_\varepsilon$ in the asymptotic low Mach number limit will approach the limit profile $\mathbf{u}$ strongly with respect to the L^1-topology.

We claim the following variant of Theorem 7.1.

■ COMPACTNESS OF VELOCITIES ON DOMAINS WITH VARIABLE BOTTOMS: THE COMPLETE SLIP BOUNDARY CONDITIONS

Theorem 7.2 *Let Ω_ε be a family of domains determined through (7.140), (7.141), where the "bottom" part of the boundary is given by functions h, ω satisfying*

$$h,\ \omega \in C^3(\mathcal{T}^2),\ |h| < \pi,\ h \not\equiv \text{const},\ \omega \geq 0, \tag{7.143}$$

and ω is non-degenerate, specifically, for any $\mathbf{w} = [w_1, w_2] \neq 0$ there is $(x_1, x_2) \in \mathcal{T}^2$ such that

$$\nabla\omega(x_1, x_2)\cdot\mathbf{w} \neq 0. \tag{7.144}$$

Let

$$k(\varepsilon) \geq \varepsilon^{-m} \text{ for a certain } m > 1. \tag{7.145}$$

Let $F \in W^{1,\infty}(\mathbb{R}^3)$ be given such that

$$\int_\Omega F \,\mathrm{d}x = 0,$$

where

$$\Omega = \left\{(x_1, x_2, x_3) \,\middle|\, (x_1, x_2) \in \mathcal{T}^2,\ -\pi - h(x_1, x_2) < x_3 < 0\right\}. \tag{7.146}$$

Assume that $\mathbb{S}$, $\mathbf{q}$ *as well as the thermodynamic functions* p, e, *and* s *are given by (7.10)–(7.15), where* P *meets the structural hypotheses (7.23)–(7.25), while the transport coefficients* μ *and* κ *satisfy (7.26), (7.27).*

Finally, let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the Navier-Stokes-Fourier system satisfying (7.5)–(7.9) in* $(0,T) \times \Omega_\varepsilon$, *with the boundary conditions (7.20), (7.142), with the initial data*

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}.\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

where

$$\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \int_{\Omega_\varepsilon} \varrho^{(1)}_{0,\varepsilon}\, \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon}\, \mathrm{d}x = 0 \textit{ for all } \varepsilon > 0,$$

and

$$\left\{ \begin{array}{c} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \textit{ weakly-(*) in } L^\infty(\mathbb{R}^3), \\ \\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \textit{ weakly-(*) in } L^\infty(\mathbb{R}^3;\mathbb{R}^3), \\ \\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \textit{ weakly-(*) in } L^\infty(\mathbb{R}^3). \end{array} \right\}$$

Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \textit{ in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{7.147}$$

where $\mathbf{U} \in L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$, $\mathrm{div}_x \mathbf{U} = 0$.

Remark It is worth noting that the limit velocity profile $\mathbf{U}$ satisfies the *no-slip* boundary condition on both the top and the bottom part of the boundary of the limit domain Ω. Similarly to the preceding part, we leave to the reader to show that the limit quantities satisfy the Oberbeck-Boussinesq system introduced in Sect. 5.

Remark The weak solutions are defined exactly as in Sect. 7.2.1, with the obvious modifications

$$\mathbf{u}_\varepsilon \in L^2(0,T;W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)),\ \mathbf{u}_\varepsilon|_{\{x_3=0\}} = 0,\ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\{x_3=-\pi-h(x_1,x_2)-\omega_\varepsilon(x_1,x_2)\}} = 0,$$

whereas the test functions φ in the momentum equation (7.29) are taken from the space

$$\varphi \in C^2([0,T)\times\Omega_\varepsilon;\mathbb{R}^3),\ \ \varphi|_{\{x_3=0\}} = 0,\ \varphi\cdot\mathbf{n}|_{\{x_3=-\pi-h(x_1,x_2)-\omega_\varepsilon(x_1,x_2)\}} = 0.$$

Remark For the sake of simplicity, we have assumed that the initial data are defined on the whole physical space $\mathbb{R}^3$. As $\Omega \subset \Omega_\varepsilon$, the statement (7.147) makes sense.

The rest of this chapter is devoted to the proof of Theorem 7.2. The idea is that the rapidly oscillating boundary along with the effect of viscosity will force the fluid to be at rest on the boundary of the limit domain; whence the methods developed in Sects. 7.4, 7.5 can be applied.

7.7 Uniform Bounds

The uniform bounds on the sequence of solutions $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ are essentially the same as in Sect. 7.3. However, we should keep in mind that the underlying spatial domains Ω_ε depend on the scaling parameter ε. Accordingly, the constants appearing in Korn's and Poicaré's inequality used in Sect. 7.3 may depend on ε. Fortunately, by virtue of hypotheses (7.140), (7.141), the family Ω_ε is uniformly Lipschitz, and, consequently, the corresponding constants are the same for all Ω_ε, see Theorem 11.24 in Appendix. With this observation in mind, we report the following list of estimates.

- **Energy estimates:**

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \leq c, \tag{7.148}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{7.149}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\mathrm{res}} \right\|_{L^{\frac{5}{3}}(\Omega_\varepsilon)} \leq \varepsilon^{\frac{1}{5}} c, \tag{7.150}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\mathrm{ess}} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{7.151}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| [\vartheta_\varepsilon]_{\mathrm{res}} \right\|_{L^4(\Omega_\varepsilon)} \leq \varepsilon^{\frac{1}{2}} c, \tag{7.152}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon}\right]_{\mathrm{res}} \right\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon c. \tag{7.153}$$

- **Estimates based on energy dissipation:**

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega}_\varepsilon)} \leq \varepsilon^2 c, \tag{7.154}$$

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)}\,\mathrm{d}t \leq c, \tag{7.155}$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} \mathrm{d}t \leq c, \tag{7.156}$$

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} \mathrm{d}t \leq c. \tag{7.157}$$

- **Entropy estimates:**

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\mathrm{res}} \right\|_{L^1(\Omega_\varepsilon)} \mathrm{d}t \leq \varepsilon c, \tag{7.158}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \right]_{\mathrm{res}} \right\|^q_{L^q(\Omega_\varepsilon)} \mathrm{d}t \leq c \text{ for a certain } q > 1, \tag{7.159}$$

$$\int_0^T \left\| \left[\frac{\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \mathbf{u}_\varepsilon \right]_{\mathrm{res}} \right\|^q_{L^q(\Omega_\varepsilon;\mathbb{R}^3)} \mathrm{d}t \leq c \text{ for a certain } q > 1, \tag{7.160}$$

$$\int_0^T \left\| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\mathrm{res}} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\|^q_{L^q(\Omega_\varepsilon;\mathbb{R}^3)} \mathrm{d}t \to 0 \text{ for a certain } q > 1. \tag{7.161}$$

7.8 Convergence of the Velocity Trace on Oscillatory Boundary

Our goal is to show that the traces of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ vanish on the boundary of the limit domain Ω in the asymptotic limit $\varepsilon \to 0$.

Proposition 7.3 *Let Ω_ε be a family of domains satisfying the hypotheses of Theorem 7.2.*

Then

$$\int_{\{x_3 = -\pi - h(x_1,x_2)\}} |\mathbf{v}|^2 \, \mathrm{dS}_x \leq c \frac{1}{k(\varepsilon)} \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \leq c\varepsilon^m \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \tag{7.162}$$

for any $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$ satisfying

$$\mathbf{v} \cdot \mathbf{n}|_{\{x_3 = -\pi - h(x_1,x_2) - \omega_\varepsilon(x_1,x_2)\}} = 0,$$

where the constant is independent of $\varepsilon \to 0$.

Proof Obviously, we can restrict ourselves to the strip

$$S_\varepsilon = \Big\{(x_1, x_2, x_3) \ \Big| \ (x_1, x_2) \in \mathcal{T}^2, \\ -\pi - h(x_1, x_2) - \omega_\varepsilon(x_1, x_2) < x_3 < -\pi - h(x_1, x_2) + \frac{1}{k(\varepsilon)}\Big\}$$

containing the bottom part B of the boundary $\partial\Omega$,

$$B = \Big\{x_3 = -\pi - h(x_1, x_2) \ \Big| \ (x_1, x_2) \in \mathcal{T}^2\Big\}.$$

Next, writing

$$S_\varepsilon = \cup_{n=0,m=0}^{n=k(\varepsilon)-1,m=k(\varepsilon)-1} S_\varepsilon^{n,m},$$

$$S_\varepsilon^{n,m} = \Big\{(x_1, x_2, x_3) \ \Big| \ \frac{n}{k(\varepsilon)} < x_1 < \frac{n+1}{k(\varepsilon)}, \frac{m}{k(\varepsilon)} < x_2 < \frac{m+1}{k(\varepsilon)}, \\ -\pi - h(x_1, x_2) - \omega_\varepsilon(x_1, x_2) < x_3 < -\pi - h(x_1, x_2) + \frac{1}{k(\varepsilon)}\Big\}$$

we observe that it is enough to show (7.162) on each $S_\varepsilon^{n,m}$.

Finally, after the scaling $x \approx k(\varepsilon)x$ and an obvious space shift, the problem reduces to proving

$$\int_{(x_1,x_2)\in(0,1)^2} \int_{\{x_3=\chi_\varepsilon(x_1,x_2)\}} |\mathbf{v}|^2 \, \mathrm{dS}_x \\ \leq c \int_{(x_1,x_2)\in(0,1)^2} \int_{\{\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)<x_3<\chi_\varepsilon(x_1,x_2)+1\}} |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \tag{7.163}$$

for any $\mathbf{v} \in W^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$,

$$\mathbf{v} \cdot \mathbf{n}|_{x_3=\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)} = 0 \text{ in the sense of traces,}$$

where $\chi_\varepsilon \to \chi$ in $C^1[0,1]^2$ as $\varepsilon \to 0$, and where χ is an affine function.

Arguing by contradiction, we obtain a sequence $\mathbf{v}_\varepsilon \in W^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$,

$$\mathbf{v}_\varepsilon \cdot \mathbf{n}|_{x_3=\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)} = 0 \text{ in the sense of traces,}$$

$$g(\varepsilon) \int_{(x_1,x_2)\in(0,1)^2} \int_{\{x_3=\chi_\varepsilon(x_1,x_2)\}} |\mathbf{v}_\varepsilon|^2 \, \mathrm{dS}_x \\ \geq \int_{(x_1,x_2)\in(0,1)^2} \int_{\{\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)<x_3<\chi_\varepsilon(x_1,x_2)+1\}} |\nabla_x \mathbf{v}_\varepsilon|^2 \, \mathrm{d}x,$$

where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. In addition, we may assume

$$\int_{(x_1,x_2)\in(0,1)^2} \int_{\{\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)<x_3<\chi_\varepsilon(x_1,x_2)+1\}} |\mathbf{v}_\varepsilon|^2 \, \mathrm{d}x = 1,$$

and

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } W^{1,2}(\mathbb{R}^3, \mathbb{R}^3).$$

Consequently, in view of the compact embedding $W^{1,2}(K;\mathbb{R}^3) \hookrightarrow L^2(K;\mathbb{R}^3)$, $K \subset \mathbb{R}^3$ compact, the limit function $\mathbf{v}$ satisfies

$$\int_Q |\mathbf{v}|^2 \, \mathrm{d}x = 1, \ \nabla_x \mathbf{v}|_Q = 0, \tag{7.164}$$

where

$$Q = \left\{ (x_1, x_2, x_3) \;\middle|\; (x_1, x_2) \in (0,1)^2, \ \chi(x_1,x_2) - \omega(x_1,x_2) < x_3 < \chi(x_1,x_2) + 1 \right\}.$$

Finally, we claim that

$$\mathbf{v} \cdot \mathbf{n}|_{\{x_3=\chi(x_1,x_2)-\omega(x_1,x_2)\}} = 0. \tag{7.165}$$

Indeed seeing that

$$\int_{(x_1,x_2)\in(0,1)^2} \int_{\{\chi_\varepsilon(x_1,x_2)-\omega(x_1,x_2)<x_3<\chi_\varepsilon(x_1,x_2)+1\}} \left(\mathrm{div}_x \mathbf{v}_\varepsilon \varphi - \mathbf{v}_\varepsilon \cdot \nabla_x \varphi\right) \, \mathrm{d}x = 0$$

for any

$$\varphi \in C^1(R^3), \ \varphi|_{x_3=\chi_\varepsilon(x_1,x_2)+1}$$

we infer that

$$\int_{(x_1,x_2)\in(0,1)^2} \int_Q \left(\mathrm{div}_x \mathbf{v}_\varepsilon \varphi - \mathbf{v}_\varepsilon \cdot \nabla_x \varphi\right) \, \mathrm{d}x = 0$$

for any

$$\varphi \in C^1(R^3), \ \varphi|_{x_3=\chi(x_1,x_2)+1},$$

which implies (7.165).

In view of (7.164), the limit $\mathbf{v}$ is constant in Q, which is incompatible with (7.165) as long as ω satisfies the non-degeneracy condition (7.144).

□

Thus Proposition 7.3, together with the uniform bound (7.155) yield

$$\int_0^T \int_{\partial\Omega} |\mathbf{u}_\varepsilon|^2 \, \mathrm{dS}_x \leq c\varepsilon^m. \tag{7.166}$$

The remaining part of the proof is obvious although technically involved. We restrict ourselves to the limit domain Ω, where the boundary terms arising in by parts integration will be controlled by (7.166).

7.9 Strong Convergence of the Velocity Field Revisited

Our final goal is to establish the strong convergence of the velocities claimed in (7.147). As a consequence of (7.155), we may assume that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\mathcal{T}^2 \times (-\infty,0), \mathbb{R}^3) \tag{7.167}$$

provided $\mathbf{u}_\varepsilon$ were extended to the set where $x_3 \leq -\pi - h(x_1,x_2) - \omega_\varepsilon(x_1,x_2)$. Moreover, as a consequence of (7.166),

$$\mathrm{div}_x \mathbf{U} = 0, \ \mathbf{U}|_{\partial\Omega} = 0. \tag{7.168}$$

7.9.1 Solenoidal Component

The velocity fields, restricted to the *target* domain Ω, decompose as

$$\mathbf{u}_\varepsilon = \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\mathbf{u}_\varepsilon],$$

where $\mathbf{H}$ denotes the Helmholtz projection defined on Ω. Using the uniform bounds obtained in Sect. 7.7 and repeating the arguments of Sect. 7.5.1 we deduce that the family of scalar functions

$$t \mapsto \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot) \cdot \phi \, \mathrm{d}x \text{ is precompact in } C[0,T]$$

for any $\phi \in C_c^\infty(\Omega;\mathbb{R}^3)$, $\mathrm{div}_x \phi = 0$; whence, in accordance with (7.168),

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho}\mathbf{H}[\mathbf{U}] = \overline{\varrho}\mathbf{U} \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/4}(\Omega;\mathbb{R}^3)).$$

Consequently, by virtue of (7.167) and compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^5(\Omega)$, we may infer that

$$\int_0^T \int_\Omega \mathbf{H}[\varrho_\varepsilon, \mathbf{u}_\varepsilon] \cdot \mathbf{H}[\mathbf{u}_\varepsilon] \, \mathrm{d}x \, \mathrm{d}t \to \overline{\varrho} \int_0^T \int_\Omega |\mathbf{H}(\mathbf{U})|^2 \, \mathrm{d}x \, \mathrm{d}t = \overline{\varrho} \int_0^T \int_\Omega |\mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t$$

yielding

$$\mathbf{H}[\mathbf{u}_\varepsilon] \to \mathbf{U} \text{ (strongly) in } L^2((0,T) \times \Omega; \mathbb{R}^3). \tag{7.169}$$

7.9.2 Acoustic Waves

In view of (7.169), it remains to show

$$\mathbf{H}^\perp[\mathbf{u}_\varepsilon] \to 0 \text{ (strongly) in } L^2((0,T) \times \Omega; \mathbb{R}^3). \tag{7.170}$$

We use arguments similar to those in Sect. 7.4 starting with the acoustic equation (7.59), (7.60) for the unknowns

$$r_\varepsilon = \frac{1}{\omega}\Big(\omega \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + A \varrho_\varepsilon \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F\Big), \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon,$$

$$r_{0,\varepsilon} = \frac{1}{\omega}\Big(\omega \frac{\varrho_{0,\varepsilon} - \overline{\varrho}}{\varepsilon} + A \varrho_{0,\varepsilon} \frac{s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} - \overline{\varrho} F\Big), \mathbf{V}_{0,\varepsilon} = \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon}.$$

The equation of continuity together with the entropy balance give rise to

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon r_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi\Big) \, \mathrm{d}x \, \mathrm{d}t \tag{7.171}$$

$$= -\int_{\Omega_\varepsilon} \varepsilon r_{0,\varepsilon} \varphi(0,\cdot) \, \mathrm{d}x + \frac{A}{\omega}\Big(\int_0^T \int_\Omega \mathbf{G}_5^\varepsilon \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t - < \sigma_\varepsilon, \varphi > \Big)$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}_\varepsilon)$, with

$$\omega = \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})|^2}{\overline{\varrho}^2 \, \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})}, \ A = \frac{\partial_\vartheta p(\overline{\varrho})}{\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta})},$$

and

$$\mathbf{G}_5^\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho}, \overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\Big) \mathbf{u}_\varepsilon,$$

cf. (7.59).

Next, as $\Omega \subset \Omega_\varepsilon$, we may consider

$$\boldsymbol{\varphi} \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3),$$

as a test function in (7.29) obtaining

$$\int_0^T \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \tag{7.172}$$

$$+ \int_0^T \int_\Omega \Big(\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} - \overline{\varrho} F \Big) {\rm div}_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = -\varepsilon \int_\Omega \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$+ \varepsilon \int_0^T \int_\Omega \mathbb{G}_1^\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \varepsilon \int_0^T \int_\Omega \mathbf{G}_\varepsilon^2 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \Big(G_\varepsilon^3 + G_\varepsilon^4 \Big) {\rm div}_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t,$$

with

$$\mathbb{G}_\varepsilon^1 = -\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \ \mathbf{G}_\varepsilon^2 = \frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon} \nabla_x F,$$

$$G_\varepsilon^3 = - \frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res}}{\varepsilon},$$

and

$$G_\varepsilon^4 = \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\rm ess} + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\rm ess} - \Big(\frac{[p(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm ess} - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big),$$

cf. (7.52). It is easy to observe that validity of (7.172) can be extended to

$$\boldsymbol{\varphi} \in W^{1,\infty}([0,T) \times \overline{\Omega}; \mathbb{R}^3), \ \boldsymbol{\varphi}|_{\partial\Omega} = 0, \ \boldsymbol{\varphi}(T, \cdot) = 0.$$

Next, we may compute

$$\int_0^T \int_\Omega \varepsilon \mathbb{S}_\varepsilon : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = -\varepsilon \int_0^T \int_\Omega \frac{2\mu(\overline{\vartheta})}{\overline{\varrho}} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \boldsymbol{\varphi}]] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ 2\varepsilon \int_0^T \int_{\partial\Omega} \mu(\overline{\vartheta}) ([[\nabla_x \boldsymbol{\varphi}]] \mathbf{u}_\varepsilon) \cdot \mathbf{n} \, \mathrm{d}\sigma + \int_0^T \int_\Omega \frac{2\varepsilon \mu(\overline{\vartheta})}{\overline{\varrho}} (\varrho_\varepsilon - \overline{\varrho}) \mathbf{u}_\varepsilon \cdot {\rm div}_x [[\nabla_x \boldsymbol{\varphi}]] \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \varepsilon \Big(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}) \Big) \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\perp \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t,$$

for all

$$\boldsymbol{\varphi} \in W^{2,\infty}((0,T)\times\Omega;\mathbb{R}^3),\ \boldsymbol{\varphi}|_{\partial\Omega} = 0,\ \boldsymbol{\varphi}(T,\cdot) = 0,$$

recalling that

$$[[\mathbb{M}]] = \frac{1}{2}\Big[\mathbb{M} + \mathbb{M}^t - \frac{2}{3}\mathrm{trace}[\mathbb{M}]\ \mathbb{I}\Big].$$

Thus (7.172) finally reads as

$$\int_0^T \int_\Omega \Big(\varepsilon\mathbf{V}_\varepsilon \cdot \partial_t\boldsymbol{\varphi} + \omega r_\varepsilon \mathrm{div}_x\boldsymbol{\varphi} + \varepsilon D\mathbf{V}_\varepsilon \cdot \mathrm{div}_x[[\nabla_x\boldsymbol{\varphi}]]\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{7.173}$$

$$= -\int_\Omega \varepsilon\mathbf{V}_{0,\varepsilon}\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x$$

$$+\int_0^T\int_\Omega \Big(\mathbf{G}_6^\varepsilon \cdot \mathrm{div}_x[[\nabla_x\boldsymbol{\varphi}]] + \mathbb{G}_7^\varepsilon : \nabla_x\boldsymbol{\varphi} + G_8^\varepsilon \mathrm{div}_x\boldsymbol{\varphi} + \mathbf{G}_9^\varepsilon\cdot\boldsymbol{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+2\varepsilon\int_0^T\int_{\partial\Omega} \mu(\overline{\vartheta})\ ([[\nabla_x\boldsymbol{\varphi}]]\mathbf{u}_\varepsilon)\cdot\mathbf{n}\ \mathrm{dS}_x$$

for any

$$\boldsymbol{\varphi} \in W^{2,\infty}((0,T)\times\Omega;\mathbb{R}^3),\ \boldsymbol{\varphi}|_{\partial\Omega} = 0,\ \boldsymbol{\varphi}(T,\cdot) = 0,$$

where

$$\mathbf{G}_5^\varepsilon = \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon,$$

$$\mathbf{G}_6^\varepsilon = \varepsilon D(\varrho_\varepsilon - \overline{\varrho})\mathbf{u}_\varepsilon,$$

$$\mathbb{G}_7^\varepsilon = 2\varepsilon(\mu(\vartheta_\varepsilon) - \mu(\overline{\vartheta}))[[\nabla_x\mathbf{u}_\varepsilon]] - \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon,$$

$$G_8^\varepsilon = A\varrho_\varepsilon\Big[\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big]_{\mathrm{res}} - \Big[\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big]_{\mathrm{res}}$$

$$+A\Big\{\Big[\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big]_{\mathrm{ess}}$$

$$-\overline{\varrho}\Big(\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\rm ess}+\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big)\Big\}$$

$$-\Big\{\frac{[p(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm ess}-p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}-\Big(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\rm ess}+\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big]_{\rm ess}\Big)\Big\}$$

$$+\omega\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\rm res},$$

$$\mathbf{G}^\varepsilon_9=(\overline{\varrho}-\varrho_\varepsilon)\nabla_x F \text{ and } D=\frac{2\mu(\overline{\vartheta})}{\overline{\varrho}},$$

cf. (7.60). Unlike its counterpart (7.59), (7.60), however, Eqs. (7.171), (7.173) are considered on *different* spatial domains Ω_ε, Ω, respectively.

7.9.3 Strong Convergence of the Gradient Component

In view of exactly the same arguments as in Sect. 7.5.2, the proof of (7.170) can be reduced to showing

$$\Big[t\mapsto\int_\Omega\Big(r_\varepsilon v_\varepsilon+\mathbf{V}_\varepsilon\cdot\mathbf{w}_\varepsilon\Big)\,{\rm d}x\Big]\to 0 \text{ in } L^2(0,T), \tag{7.174}$$

where r_ε, $\mathbf{V}_\varepsilon$ satisfy the acoustic system (7.171), (7.173), and v_ε, $\mathbf{w}_\varepsilon$ are the solutions of the approximate eigenvalue problem (7.80), (7.82).

A natural idea is to use v_ε, $\mathbf{w}_\varepsilon$ as test functions in (7.171), (7.173), respectively. Unfortunately, however, v_ε is defined only on the set Ω and therefore must be extended as $\tilde{v}_\varepsilon$ to Ω_ε in such a way that

$$\tilde{v}_\varepsilon\in W^{1,\infty}(\mathbb{R}^3),\ \tilde{v}_\varepsilon|_\Omega=v,\ \|\tilde{v}_\varepsilon\|_{W^{1,\infty}(\mathbb{R}^3)}\le c\|v_\varepsilon\|_{W^{1,\infty}(\Omega)},$$

where c is independent of ε. As the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is equi-Lipschitz, such an extension exists (see Theorem 8). Accordingly, we will show

$$\Big[t\mapsto\Big(\int_{\Omega_\varepsilon}r_\varepsilon\tilde{v}_\varepsilon\,{\rm d}x+\int_\Omega\mathbf{V}_\varepsilon\cdot\mathbf{w}_\varepsilon\,{\rm d}x\Big)\Big]\to 0 \text{ in } L^2(0,T) \tag{7.175}$$

instead of (7.174).

Following the line of arguments in Sect. 7.5.3, we take the quantities $\psi(t)\tilde{v}_\varepsilon(x)$, $\psi(t)\mathbf{w}_\varepsilon(x)$, $\psi\in C^\infty_c(0,T)$, as test functions in (7.171), (7.173) to obtain

$$\int_0^T\Big(\varepsilon\chi_\varepsilon\partial_t\psi+\lambda_\varepsilon\chi_\varepsilon\psi\Big)\,{\rm d}t+\sqrt{\varepsilon}\int_0^T\psi\int_\Omega\Big(r_\varepsilon s^1_\varepsilon+\mathbf{V}_\varepsilon\cdot\mathbf{s}^2_\varepsilon\Big)\,{\rm d}x\,{\rm d}t=\sum_{m=1}^{10}I^\varepsilon_m, \tag{7.176}$$

where

$$\chi_\varepsilon(t) = \int_{\Omega_\varepsilon} r_\varepsilon(t,\cdot)\tilde{v}_\varepsilon \, \mathrm{d}x + \int_\Omega \mathbf{V}_\varepsilon(t,\cdot)\cdot \mathbf{w}_\varepsilon \, \mathrm{d}x,$$

and

$$I_1^\varepsilon = \frac{A}{\omega}\int_0^T \psi \int_{\Omega_\varepsilon} \Big[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\vartheta_\varepsilon + \Big(\varrho_\varepsilon s(\overline{\varrho},\overline{\vartheta}) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathbf{u}_\varepsilon\Big]\cdot\nabla_x\tilde{v}_\varepsilon \, \mathrm{d}x\,\mathrm{d}t,$$

$$I_2^\varepsilon = -\frac{A}{\omega} < \sigma_\varepsilon; \psi\tilde{v}_\varepsilon >_{[\mathcal{M};C]([0,T]\times\overline{\Omega_\varepsilon})},$$

$$I_3^\varepsilon = D\int_0^T \psi\int_\Omega \varepsilon^2\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\mathbf{u}_\varepsilon\cdot \mathrm{div}_x[[\nabla_x\mathbf{w}_\varepsilon]]\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_4^\varepsilon = \int_0^T \psi\int_\Omega \varepsilon^2\Big(\frac{\mu(\vartheta_\varepsilon)-\mu(\overline{\vartheta})}{\varepsilon}\Big)[[\nabla_x\mathbf{u}_\varepsilon]] : \nabla_x\mathbf{w}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t,$$

$$I_5^\varepsilon = -\int_0^T \psi\int_\Omega \varepsilon\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_6^\varepsilon = \int_0^T \psi\int_\Omega \varepsilon\Big(\frac{\overline{\varrho}-\varrho_\varepsilon}{\varepsilon}\Big)\nabla_x F\cdot\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_7^\varepsilon = \int_0^T \psi\int_\Omega G_8^\varepsilon\,\mathrm{div}_x\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_8^\varepsilon = 2\varepsilon\int_{\partial\Omega}\mu(\overline{\vartheta})\,([[\nabla_x\mathbf{w}_\varepsilon]]\mathbf{u}_\varepsilon)\cdot\mathbf{n}\,\mathrm{dS}_x,$$

$$I_9^\varepsilon = -\int_0^T \psi\int_{\Omega_\varepsilon\setminus\Omega}\mathbf{V}_\varepsilon\cdot\nabla_x\tilde{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

$$I_{10}^\varepsilon\lambda_\varepsilon = \int_0^T\psi\int_{\Omega_\varepsilon\setminus\Omega} r_\varepsilon\tilde{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Note that the integrals I_1^ε–I_7^ε are the same as their counterparts in (7.121). In particular, the same arguments as in Sect. 7.5.3 can be used to show that each of them can be written in the form

$$I_m^\varepsilon = \int_0^T \psi(t)\Big(\varepsilon\gamma_m^\varepsilon(t) + \varepsilon^{1+\beta}\Gamma_m^\varepsilon(t)\Big)\,\mathrm{d}t,\ m=1,\dots,7$$

where

$$\left\{\begin{array}{c}\{\gamma_m^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^q(0,T) \text{ for a certain } q>1,\\[1em] \{\Gamma_m^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(0,T),\ \text{and } \beta>0.\end{array}\right\}$$

Now, we come to the crucial point of the proof using the bound (7.166) on the trace of the solution $\mathbf{u}_\varepsilon$ on $\partial\Omega$ to obtain

$$\int_{\partial\Omega} ([[\nabla_x \mathbf{w}_\varepsilon]]\ \mathbf{u}_\varepsilon) \cdot \mathbf{n}\ \mathrm{dS}_x \le \varepsilon^{\frac{m-1}{2}} \|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \|\sqrt{\varepsilon}\nabla_x \mathbf{w}_\varepsilon\|_{L^\infty(\Omega)}.$$

With help of (7.155) and Proposition 7.2 yielding the necessary bound for $\sqrt{\varepsilon}\nabla_x \mathbf{w}_\varepsilon$ we conclude that

$$I_8^\varepsilon = \varepsilon^{1+\frac{m-1}{2}} \int_0^T \gamma_8^\varepsilon\ \mathrm{d}t,\ \{\gamma_8^\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T).$$

Next, as $|\Omega_\varepsilon \setminus \Omega| \approx \varepsilon$, we get

$$\Big| \int_{\Omega_\varepsilon\setminus\Omega} \mathbf{V}_\varepsilon \cdot \nabla_x \tilde{v}_\varepsilon\ \mathrm{d}x \Big|$$
$$\le \|\nabla_x \tilde{v}_\varepsilon\|_{L^\infty(R^3)} \Big(\varepsilon \int_{\Omega_\varepsilon} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big| |\mathbf{u}_\varepsilon|\ \mathrm{d}x + \overline{\varrho} \int_{\Omega_\varepsilon\setminus\Omega} |\mathbf{u}_\varepsilon|\ \mathrm{d}x \Big).$$

Furthermore, by means of Hölder's inequality,

$$\int_{\Omega_\varepsilon\setminus\Omega} |\mathbf{u}_\varepsilon|\ \mathrm{d}x \le c\varepsilon^{5m/6} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega_\varepsilon;\mathbb{R}^3)}$$

therefore, by virtue of (7.155) and Proposition 7.2,

$$I_9^\varepsilon = \varepsilon^{\min\{\frac{5m/6}{,}1\}} \int_0^T \psi(t)\gamma_9^\varepsilon\ \mathrm{d}t,$$

where

$$\{\gamma_9^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T).$$

Finally, to control the integral I_{10}^ε, we write

$$\Big| \int_{\Omega_\varepsilon\setminus\Omega} r_\varepsilon \tilde{v}_\varepsilon\ \mathrm{d}x \Big|$$
$$\le c\|\tilde{v}_\varepsilon\|_{L^\infty(R^3)} \Big(\int_{\Omega_\varepsilon\setminus\Omega} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|\ \mathrm{d}x + \int_{\Omega_\varepsilon\setminus\Omega} \varrho_\varepsilon \Big| \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big|\ \mathrm{d}x + \int_{\Omega_\varepsilon\setminus\Omega} \overline{\varrho}|F|\ \mathrm{d}x \Big),$$

where,

$$\int_{\Omega_\varepsilon\setminus\Omega} \Big| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big|\ \mathrm{d}x$$
$$\le \int_{\Omega_\varepsilon\setminus\Omega} \Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} \Big|\ \mathrm{d}x + \int_{\Omega_\varepsilon} \Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{res}} \Big|\ \mathrm{d}x,$$

and, similarly,

$$\int_{\Omega_\varepsilon \setminus \Omega} \varrho_\varepsilon \Big| \frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big| \, \mathrm{d}x$$

$$\leq c\Big(\int_{\Omega_\varepsilon \setminus \Omega} \Big(\Big| \Big[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \Big]_{\mathrm{ess}} \Big| + \Big| \Big[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \Big]_{\mathrm{ess}} \Big| \, \mathrm{d}x + \int_{\Omega_\varepsilon} \varrho_\varepsilon \Big| \Big[\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \Big]_{\mathrm{res}} \Big| \, \mathrm{d}x \Big).$$

Thus the uniform bounds (7.149)–(7.153) yield the desired conclusion

$$I_{10}^{\varepsilon} = \varepsilon^{\min\{\frac{m}{2},1\}} \int_0^T \psi(t) \gamma_{10}^{\varepsilon} \, \mathrm{d}t,$$

where

$$\{\gamma_{10}^{\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T).$$

Having controlled all the integrals in (7.176) and seeing that $m > 1$, we are in the situation described in the last part of Sect. 7.5.3. Consequently, repeating step by step the arguments used therein, we can show (7.174) and therefore complete the proof of Theorem 7.2.

7.10 Concluding Remarks

We have shown that the no-slip boundary conditions and the complete slip boundary conditions considered on "oscillatory" boundary produce the same effect in the low Mach number limit, specifically, the acoustic waves are effectively damped as long as the boundary of the target domain is non-degenerate (non-flat). As a matter of fact, a proper choice of the boundary conditions for the velocity of a viscous fluid confined to a bounded physical space has been discussed by many prominent physicists and mathematicians over the last two centuries (see the survey paper by Priezjev and Troian [235]).

For a long time, the no-slip boundary conditions have been the most widely accepted for their tremendous success in reproducing the observed velocity profiles for macroscopic flows. Still the no-slip boundary condition is not intuitively obvious. Recently developed technologies of micro and nano-fluidics have shown the slip of the fluid on the boundary to be relevant when the system size approaches the nanoscale. The same argument applies in the case when the shear rate is sufficiently strong in comparison with the characteristic length scale as in some meteorological models (see Priezjev and Troian [235]). As a matter of fact, an alternative microscopic explanation of the no-slip condition argues that because most real surfaces are *rough*, the viscous dissipation as the fluid passes the surface irregularities brings it to rest regardless the character of the intermolecular forces acting between the fluid and the solid wall. A rigorous mathematical evidence of this hypothesis has been provided in a series of papers by Amirat et al. [9, 10], Casado-Díaz et al. [51] or, more recently, [52]. Thus the roughness argument, used also in this chapter, reconciles convincingly the ubiquitous success of the no-slip condition with the boundary behaviour of real fluids predicted by molecular dynamics (cf. Qian et al. [236]).