Chapter 4
Asymptotic Analysis: An Introduction

The extreme generality of the full NAVIER-STOKES-FOURIER SYSTEM whereby the equations describe the entire spectrum of possible motions—ranging from sound waves, cyclone waves in the atmosphere, to models of gaseous stars in astrophysics—constitutes a serious defect of the equations from the point of view of applications. Eliminating unwanted or unimportant modes of motion, and building in the essential balances between flow fields, allow the investigator to better focus on a particular class of phenomena and to potentially achieve a deeper understanding of the problem. Scaling and asymptotic analysis play an important role in this approach. By scaling the equations, meaning by choosing appropriately the system of the reference units, the parameters determining the behavior of the system become explicit. Asymptotic analysis provides a useful tool in the situations when certain of these parameters called *characteristic numbers* vanish or become infinite.

The main goal of many studies devoted to asymptotic analysis of various physical systems is to derive a simplified set of equations solvable either analytically or at least with less numerical effort. Classical textbooks and research monographs (see Gill [137], Pedlosky [230], Zeytounian [274, 276], among others) focus mainly on the way how the scaling arguments together with other characteristic features of the data may be used in order to obtain, mostly in a very formal way, a simplified system, leaving aside the mathematical aspects of the problem. In particular, the *existence* of classical solutions is always tacitly anticipated, while exact results in this respect are usually in short supply. In fact, not only has the problem not been solved, it is not clear that in general smooth solutions exist. This concerns both the primitive NAVIER-STOKES-FOURIER SYSTEM and the target systems resulting from the asymptotic analysis. Notice that even for the "simple" *incompressible* NAVIER-STOKES SYSTEM, the existence of regular solutions represents an outstanding open problem (see Fefferman [100]). Consequently, given the recent state of art, a rigorous mathematical treatment without any unnecessary restrictions on the size of the observed data as well as the length of the time interval must be based on the concept of *weak solutions* defined in the spirit of Chap. 2. Although suitability

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of this framework might be questionable because of possible loss of information due to its generality, we show that this class of solutions is sufficiently robust to perform various asymptotic limits and to recover a number of standard models in mathematical fluid mechanics (see Sects. 4.2–4.4). Accordingly, the results presented in this book can be viewed as another piece of evidence in support of the mathematical theory based on the concept of *weak solutions*.

In the following chapters, we provide a mathematical justification of several up to now mostly formal procedures, hope to shed some light on the way how the simplified *target problems* can be derived from the *primitive system* under suitable scaling, and, last but not least, contribute to further development of the related numerical methods. We point out that *formal* asymptotic analysis performed with respect to a small (large) parameter tells us only that certain quantities may be negligible in certain regimes because they represent higher order terms in the (formal) asymptotic expansion. However, the specific way *how* they are filtered off is very often more important than the limit problem itself. A typical example are the high frequency *acoustic waves* in meteorological models that may cause the failure of certain numerical schemes. An intuitive argument states that such sizeable elastic perturbations cannot establish permanently in the atmosphere as the fast acoustic waves rapidly redistribute the associated energy and lead to an equilibrium state void of acoustic modes. Such an idea anticipates the existence of an unbounded physical space with a dominating dispersion effect. However any real physical as well as computational domain is necessarily bounded and the interaction of the acoustic waves with its boundary represents a serious problem from both analytical and numerical point of view unless the domain is large enough with respect to the characteristic speed of sound in the fluid. Relevant discussion of these issues is performed formally in Sect. 4.4, and, at a rigorous mathematical level, in Chaps. 7, 8 below. As we shall see, the problem involves an effective interaction of two different time scales, namely the slow time motion of the background incompressible flow interacting with the fast time propagation of the acoustic waves through the convective term in the momentum equation. Another interesting asymptotic regime, studied in detail in Chap. 9, is the situation when the above phenomena are accompanied by vanishing dissipation, here represented by viscosity and heat conductivity of the fluid. In such a case, the resulting target problem is hyperbolic, typically an incompressible Euler system. The lack of compactness characteristic for this singular regime must be compensated by *structural stability* encoded in the relative energy functional introduced in Chap. 9. General issues concerning propagation of acoustic waves are discussed in Chap. 10. It becomes evident that this kind of problem lies beyond the scope of the standard methods based on formal asymptotic expansions.

The key idea pursued in this book is that besides justifying a number of important models, the asymptotic analysis carried out in a rigorous way provides a considerably improved insight into their structure. The purpose of this introductory chapter is to identify some of the basic problems arising in the asymptotic analysis of the complete NAVIER-STOKES-FOURIER SYSTEM along with the relevant limit equations. To begin, we introduce a rescaled system expressed in terms of dimensionless quantities and identify a sample of *characteristic numbers*. The central issue addressed in this study is the passage from *compressible* to *incompressible* fluid models. In particular, we always assume that the speed of sound dominates the characteristic speed of the fluid, the former approaching infinity in the asymptotic limit (see Chap. 5). In addition, we study the effect of strong stratification that is particularly relevant in some models arising in astrophysics (see Chap. 6). Related problems concerning the propagation of acoustic waves in large domains and their interaction with the physical boundary are discussed in Chaps. 7 and 8. In Chap. 9, we consider the situation when the Mach number becomes small but the Reynolds and Péclet number are large. Last but not least, we did not fail to notice a close relation between the asymptotic analysis performed in this book and the method of *acoustic analogies* used in engineering problems (see Chap. 10).

4.1 Scaling and Scaled Equations

For the physical systems studied in this book we recognize four fundamental dimensions: Time, Length, Mass, and Temperature. Each physical quantity that appears in the governing equations can be measured in units expressed as a product of fundamental ones.

The field equations of the NAVIER-STOKES-FOURIER SYSTEM in the form introduced in Chap. 1 do not reveal anything more than the balance laws of certain quantities characterizing the instantaneous state of a fluid. In order to get a somewhat deeper insight into the structure of possible solutions, we can identify *characteristic values* of relevant physical quantities: the *reference time* $T_{\rm ref}$, the *reference length* $L_{\rm ref}$, the *reference density* $\varrho_{\rm ref}$, the *reference temperature* $\vartheta_{\rm ref}$, together with the *reference velocity* $U_{\rm ref}$, and the characteristic values of other composed quantities $p_{\rm ref}$, $e_{\rm ref}$, $\mu_{\rm ref}$, $\eta_{\rm ref}$, $\kappa_{\rm ref}$, and the source terms $f_{\rm ref}$, $Q_{\rm ref}$. Introducing new independent and dependent variables $X' = X/X_{\rm ref}$ and omitting the primes in the resulting equations, we arrive at the following scaled system.

■ SCALED NAVIER-STOKES-FOURIER SYSTEM:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p = \frac{1}{\mathrm{Re}}\mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho \mathbf{f}, \tag{4.2}$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}}\mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma + \mathrm{Hr}\varrho\frac{\mathcal{Q}}{\vartheta}, \tag{4.3}$$

together with the associated total energy balance

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\mathrm{d}x = \int\left(\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho \mathbf{f}\cdot\mathbf{u} + \mathrm{Hr}\varrho\mathcal{Q}\right)\mathrm{d}x. \tag{4.4}$$

Here, in accordance with the general principles discussed in Chap. 1, the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ are interrelated through *Gibbs' equation*

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right), \tag{4.5}$$

while

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x{}^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{4.6}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{4.7}$$

and

$$\sigma = \frac{1}{\vartheta}\left(\frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}}\frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{4.8}$$

Note that relation (4.5) requires satisfaction of a natural compatibility condition

$$p_{\mathrm{ref}} = \varrho_{\mathrm{ref}} e_{\mathrm{ref}}. \tag{4.9}$$

The above procedure gives rise to a sample of dimensionless *characteristic numbers* listed below.

■ SYMBOL	■ DEFINITION	■ NAME
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}} U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}} U_{\mathrm{ref}} L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}} f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}} L_{\mathrm{ref}} U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}} \kappa_{\mathrm{ref}})$	Péclet number
Hr	$\varrho_{\mathrm{ref}} Q_{\mathrm{ref}} L_{\mathrm{ref}}/(p_{\mathrm{ref}} U_{\mathrm{ref}})$	Heat release parameter

The set of the chosen characteristic numbers is not unique, however, the maximal number of *independent* ones can be determined by means of Buckingham's Π-theorem (see Curtis et al. [65]).

Much of the subject to be studied in this book is motivated by the situation, where one or more of these parameters approach zero or infinity, and, consequently, the resulting equations contain singular terms. The *Strouhal number* Sr is often set to unity in applications; this implies that the characteristic time scale of flow field evolution equals the convection time scale $L_{\mathrm{ref}}/U_{\mathrm{ref}}$. Large *Reynolds number* characterizes turbulent flows, where the mathematical structure is far less understood than for the "classical" systems. Therefore we concentrate on a sample of interesting and physically relevant cases, with $\mathrm{Sr} = \mathrm{Re} = 1$, the characteristic features of which are shortly described in the rest of this chapter.

4.2 Low Mach Number Limits

In many real world applications, such as atmosphere-ocean flows, fluid flows in engineering devices and astrophysics, velocities are small compared with the speed of sound proportional to $1/\sqrt{\mathrm{Ma}}$ in the *scaled* NAVIER-STOKES-FOURIER SYSTEM. This fact has significant impact on both exact solutions to the governing equations and their numerical approximations. Physically, in the limit of vanishing flow velocity or infinitely fast speed of sound propagation, the elastic features of the fluid become negligible and sound-wave propagation insignificant. The low Mach number regime is particularly interesting when accompanied simultaneously with smallness of other dimensionless parameters such as *Froude*, *Reynolds*, and/or *Péclet numbers*. When the Mach number Ma approaches zero, the pressure is almost constant while the speed of sound tends to infinity. If, simultaneously, the temperature tends to a constant, the fluid is driven to incompressibility. If, in addition, *Froude number* is small, specifically if $\mathrm{Fr} \approx \sqrt{\mathrm{Ma}}$, a formal asymptotic expansion produces a well-known model—the OBERBECK-BOUSSINESQ APPROXIMATION—probably the most widely used simplification in numerous problems in fluid dynamics (see also the introductory part of Chap. 5). An important consequence of the heating process is the appearance of a driving force in the target system, the size of which is proportional to the temperature.

In most applications, we have

$$\mathbf{f} = \nabla_x F,$$

where $F = F(x)$ is a given potential. Taking $\mathrm{Ma} = \varepsilon$, $\mathrm{Fr} = \sqrt{\varepsilon}$, and keeping all other characteristic numbers of order unity, we formally write

$$\begin{aligned} \varrho &= \overline{\varrho} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + \dots, \\ \mathbf{u} &= \mathbf{U} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \dots, \\ \vartheta &= \overline{\vartheta} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + \dots \end{aligned} \tag{4.10}$$

Regrouping the scaled system with respect to powers of ε, we get, again formally comparing terms of the same order,

$$\nabla_x p(\overline{\varrho}, \overline{\vartheta}) = 0. \tag{4.11}$$

Of course, relation (4.11) *does not* imply that both $\overline{\varrho}$ and $\overline{\vartheta}$ must be constant; however, since we are primarily interested in solutions defined on large time intervals, the necessary uniform estimates on the velocity field have to be obtained from the dissipation equation introduced and discussed in Sect. 2.2.3. In particular, the entropy production rate $\sigma = \sigma_\varepsilon$ is to be kept small of order $\varepsilon^2 \approx \mathrm{Ma}^2$. Consequently, as seen from (4.7), (4.8), the quantity $\nabla_x \vartheta$ vanishes in the asymptotic limit $\varepsilon \to 0$. It is therefore natural to assume that $\overline{\vartheta}$ is a positive constant; whence, in agreement with (4.11),

$$\overline{\varrho} = \text{const in } \Omega$$

as soon as the pressure is a strictly monotone function of ϱ. The fact that the density ϱ and the temperature ϑ will be always considered in a vicinity of a *thermodynamic equilibrium* $(\overline{\varrho}, \overline{\vartheta})$ is an inevitable hypothesis in our approach to singular limits based on the concept of *weak solution* and energy estimates "in-the-large".

Neglecting all terms of order ε and higher in (4.1)–(4.4), we arrive at the following system of equations.

■ OBERBECK-BOUSSINESQ APPROXIMATION:

$$\mathrm{div}_x \mathbf{U} = 0, \tag{4.12}$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x \Big(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x{}^T \mathbf{U})\Big) + r \nabla_x F, \tag{4.13}$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})\Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa(\overline{\vartheta})\nabla_x \Theta) = 0, \tag{4.14}$$

where

$$G = \overline{\varrho}\, \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) F, \tag{4.15}$$

and

$$r + \overline{\varrho}\alpha(\overline{\varrho}, \overline{\vartheta})\Theta = 0. \tag{4.16}$$

Here r can be identified with $\varrho^{(1)}$ modulo a multiple of F, while $\Theta = \vartheta^{(1)}$. The *specific heat at constant pressure* c_p is evaluated by means of the standard thermodynamic relation

$$c_p(\varrho, \vartheta) = \frac{\partial e}{\partial \vartheta}(\varrho, \vartheta) + \alpha(\varrho, \vartheta)\frac{\vartheta}{\varrho}\frac{\partial p}{\partial \vartheta}(\varrho, \vartheta), \tag{4.17}$$

where the *coefficient of thermal expansion* α reads

$$\alpha(\varrho, \vartheta) = \frac{1}{\varrho} \frac{\partial_\vartheta p}{\partial_\varrho p}(\varrho, \vartheta). \tag{4.18}$$

An interesting issue is a proper choice of the initial data for the limit system. Note that, in order to obtain a non-trivial dynamics, it is necessary to consider general $\varrho^{(1)}$, $\vartheta^{(1)}$, in particular, the initial values $\varrho^{(1)}(0, \cdot)$, $\vartheta^{(1)}(0, \cdot)$ must be allowed to be large. According to the standard terminology, such a stipulation corresponds to the so-called *ill-prepared initial data* in contrast with the *well-prepared data* for which

$$\frac{\varrho(0, \cdot) - \overline{\varrho}}{\varepsilon} \approx \varrho_0^{(1)}, \ \frac{\vartheta(0, \cdot) - \overline{\vartheta}}{\varepsilon} \approx \vartheta_0^{(1)} \text{ provided } \varepsilon \to 0,$$

where $\varrho_0^{(1)}$, $\vartheta_1^{(1)}$ are related to F through

$$\frac{\partial p}{\partial \varrho}(\overline{\varrho}, \overline{\vartheta})\varrho_0^{(1)} + \frac{\partial p}{\partial \vartheta}(\overline{\varrho}, \overline{\vartheta})\vartheta_0^{(1)} = \overline{\varrho} F$$

(cf. Theorem 5.3 in Chap. 5).

Moreover, as we shall see in Chap. 5 below, the initial distribution of the temperature Θ in (4.14) is determined in terms of *both* $\varrho^{(1)}(0, \cdot)$ and $\vartheta^{(1)}(0, \cdot)$. In particular, the knowledge of $\varrho^{(1)}$—a quantity that "disappears" in the target system—is necessary in order to determine $\Theta \approx \vartheta^{(1)}$. The piece of information provided by the initial distribution of the temperature for the full NAVIER-STOKES-FOURIER SYSTEM is not transferred entirely on the target problem because of the initial-time *boundary layer*. This phenomenon is apparently related to the problem termed by physicists the *unsteady data adjustment* (see Zeytounian [275]). For further discussion see Sect. 5.5.

The low Mach number asymptotic limit in the regime of low stratification is studied in Chap. 5.

4.3 Strongly Stratified Flows

Stratified fluids whose densities, sound speed as well as other parameters are functions of a single depth coordinate occur widely in nature. Several so-called mesoscale regimes in the atmospheric modeling involve flows of strong stable stratification but weak rotation. Numerous observations, numerical experiments as well as purely theoretical studies to explain these phenomena have been recently surveyed in the monograph by Majda [200].

From the point of view of the mathematical theory discussed in Sect. 4.1, strong stratification corresponds to the choice

$$\mathrm{Ma} = \mathrm{Fr} = \varepsilon.$$

Similarly to the above, we write

$$\varrho = \tilde{\varrho} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + \dots,$$

$$\mathbf{u} = \mathbf{U} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \dots,$$

$$\vartheta = \overline{\vartheta} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + \dots .$$

Comparing terms of the same order of the small parameter ε in the NAVIER-STOKES-FOURIER SYSTEM (4.1)–(4.4) we deduce

■ HYDROSTATIC BALANCE EQUATION:

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F, \tag{4.19}$$

where we have assumed the driving force in the form $\mathbf{f} = \nabla_x F$, $F = F(x_3)$ depending solely on the depth coordinate x_3. Here the temperature $\overline{\vartheta}$ is still assumed constant, while, in sharp contrast with (4.11), the equilibrium density $\tilde{\varrho}$ depends effectively on the depth (vertical) coordinate x_3.

Accordingly, the standard incompressibility conditions (4.12) has to be replaced by

■ ANELASTIC CONSTRAINT:

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{U}) = 0 \tag{4.20}$$

– a counterpart to the equation of continuity in the asymptotic limit.

In order to identify the asymptotic form of the momentum equation, we assume, for a while, that the pressure p is given by the standard *perfect gas state equation*:

$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta. \tag{4.21}$$

Under these circumstances, the zeroth order terms in (4.2) give rise to

$$\partial_t(\tilde{\varrho}\mathbf{U}) + \mathrm{div}_x(\tilde{\varrho}\mathbf{U} \otimes \mathbf{U}) + \tilde{\varrho}\nabla_x \Pi \tag{4.22}$$

$$= \mu(\overline{\vartheta})\Delta\mathbf{U} + \Big(\frac{1}{3}\mu(\overline{\vartheta}) + \eta(\overline{\vartheta})\Big)\nabla_x \mathrm{div}_x \mathbf{U} - \frac{\vartheta^{(2)}}{\overline{\vartheta}}\tilde{\varrho}\nabla_x F.$$

Note that, similarly to Sect. 4.2, the quantities $\varrho^{(1)}$, $\vartheta^{(1)}$ satisfy the *Boussinesq relation*

$$\tilde{\varrho}\nabla_x\Big(\frac{\varrho^{(1)}}{\tilde{\varrho}}\Big) + \nabla_x\Big(\frac{\tilde{\varrho}}{\overline{\vartheta}}\vartheta^{(1)}\Big) = 0,$$

which, however, does not seem to be of any practical use here. Instead we have to determine $\vartheta^{(2)}$ by means of the entropy balance (4.3).

In the absence of the source $\mathcal{Q}$, comparing the zeroth order terms in (4.3) yields

$$\mathrm{div}_x(\tilde{\varrho}s(\tilde{\varrho}, \overline{\vartheta})\mathbf{U}) = 0.$$

However, this relation is compatible with (4.20) only if

$$U_3 \equiv 0. \tag{4.23}$$

In such a case, the system of equations (4.20)–(4.22) coincides with the so-called *layered two-dimensional incompressible flow equations in the limit of strong stratification* studied by Majda [200, Sect. 6.1]. The flow is layered horizontally, whereas the motion in each layer is governed by the incompressible NAVIER-STOKES EQUATIONS, the vertical stacking of the layers is determined through the hydrostatic balance, and the viscosity induces transfer of horizontal momentum through vertical variations of the horizontal velocity.

Even more complex problem arises when, simultaneously, the *Péclet number* Pe is supposed to be small, specifically, $\mathrm{Pe} = \varepsilon^2$. A direct inspection of the entropy balance equation (4.3) yields, to begin with,

$$\vartheta^{(1)} \equiv 0,$$

and, by comparison of the terms of zeroth order,

$$\tilde{\varrho}\nabla_x F \cdot \mathbf{U} + \kappa(\overline{\vartheta})\Delta\vartheta^{(2)} = 0. \tag{4.24}$$

Equations (4.20)–(4.22), together with (4.24), form a closed system introduced by Chandrasekhar [53] as a simple alternative to the OBERBECK-BOUSSINESQ APPROXIMATION when both *Froude* and *Péclet numbers* are small. More recently, Ligniéres [188] identified a similar system as a suitable model of flow dynamics in stellar radiative zones. Indeed, under these circumstances, the fluid behaves as a plasma characterized by the following features: (1) a strong radiative transport predominates the molecular one; therefore the *Péclet number* is expected to be vanishingly small; (2) strong stratification effect due to the enormous gravitational potential of gaseous celestial bodies determines many of the properties of the fluid in the large; (3) the convective motions are much slower than the speed rendering the *Mach number* small.

On the point of conclusion, it is worth-noting that system (4.20)–(4.22) represents the so-called ANELASTIC APPROXIMATION introduced by Ogura and Phillipps [225], and Lipps and Hemler [198]. The low Mach number limits for strongly stratified fluids are examined in Chap. 6.

4.4 Acoustic Waves

Acoustic waves, as their proper name suggests, are intimately related to *compressibility* of the fluid and as such should definitely disappear in the incompressible limit regime. Accordingly, the impact of the acoustic waves on the fluid motion is neglected in a considerable amount of practical applications. On the other hand, a fundamental issue is to understand the way how the acoustic waves disappear and to which extent they may influence the motion of the fluid in the course of the asymptotic limit.

4.4.1 Low Stratification

The so-called *acoustic equation* provides a useful link between the first order terms $\varrho^{(1)}$, $\vartheta^{(1)}$, and the zeroth order velocity field $\mathbf{U}$ introduced in the formal asymptotic expansion (4.10). Introducing the fast time variable $\tau = t/\varepsilon$ and neglecting terms of order ε and higher in (4.1)–(4.3), we deduce

$$\left.\begin{aligned} \partial_\tau \varrho^{(1)} + \mathrm{div}_x(\overline{\varrho}\mathbf{U}) = 0 \\ \partial_\tau(\overline{\varrho}\mathbf{U}) + \nabla_x\Big[\partial_\varrho p(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)} - \overline{\varrho}F\Big] = 0 \\ \partial_\tau\Big[\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\Big] = 0. \end{aligned}\right\} \tag{4.25}$$

Thus after a simple manipulation we easily obtain

■ ACOUSTIC EQUATION:

$$\begin{aligned} \partial_\tau r + \mathrm{div}_x \mathbf{V} = 0, \\ \partial_\tau \mathbf{V} + \omega\nabla_x r = 0, \end{aligned} \tag{4.26}$$

where we have set

$$r = \frac{1}{\omega}\Big(\partial_\varrho p(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)} - \overline{\varrho}F\Big),\ \mathbf{V} = \overline{\varrho}\mathbf{U},$$

$$\omega = \partial_\varrho p(\overline{\varrho},\overline{\vartheta}) + \frac{|\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})|^2}{\overline{\varrho}^2\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})}.$$

System (4.26) can be viewed as a wave equation, where the wave speed $\sqrt{\omega}$ is a real number as soon as hypothesis of thermodynamic stability (1.44) holds.

Moreover, the kernel $\mathcal{N}$ of the generator of the associated evolutionary system reads

$$\mathcal{N} = \{(r, \mathbf{V}) \mid r = \text{const},\ \mathrm{div}_x \mathbf{V} = 0\}. \tag{4.27}$$

Consequently, decomposing the vector field $\mathbf{V}$ in the form

$$\mathbf{V} = \underbrace{\mathbf{H}[\mathbf{V}]}_{\text{solenoidal part}} + \underbrace{\mathbf{H}^{\perp}[\mathbf{V}]}_{\text{gradient part}},\ \text{where } \mathrm{div}_x \mathbf{H}[\mathbf{V}] = 0,\ \mathbf{H}^{\perp}[\mathbf{V}] = \nabla_x \Psi$$

(cf. Sect. 11.7 and Theorem 11.18 in Appendix), system (4.26) can be recast as

$$\begin{aligned} \partial_\tau r + \Delta \Psi &= 0, \\ \partial_\tau (\nabla_x \Psi) + \omega \nabla_x r &= 0. \end{aligned} \tag{4.28}$$

Returning to the original time variable $t = \varepsilon\tau$ we infer that the rapidly oscillating acoustic waves are supported by the gradient part of the fluid velocity, while the time evolution of the solenoidal component of the velocity field remains essentially constant in time, being determined by its initial distribution. In terms of stability of the original system with respect to the parameter $\varepsilon \to 0$, this implies *strong* convergence of the solenoidal part $\mathbf{H}[\mathbf{u}_\varepsilon]$, while the gradient component $\mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$ converges, in general, only *weakly* with respect to time. Here and in what follows, the subscript ε refers to quantities satisfying the scaled primitive system (4.1)–(4.3). The hypothetical oscillations of the gradient part of the velocity field reveal one of the fundamental difficulties in the analysis of asymptotic limits in the present study, namely the problem of "weak compactness" of the convective term $\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$.

Writing

$$\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \approx \mathrm{div}_x(\overline{\varrho} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$$

$$= \overline{\varrho}\ \mathrm{div}_x(\mathbf{u}_\varepsilon \otimes \mathbf{H}[\mathbf{u}_\varepsilon]) + \overline{\varrho}\ \mathrm{div}_x(\mathbf{H}[\mathbf{u}_\varepsilon] \otimes \nabla_x \Psi_\varepsilon) + \frac{1}{2}\overline{\varrho} \nabla_x |\nabla_x \Psi_\varepsilon|^2 + \overline{\varrho}\ \Delta \Psi_\varepsilon \nabla_x \Psi_\varepsilon,$$

where we have set $\Psi_\varepsilon = \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$, we realize that the only problematic term is $\Delta \Psi_\varepsilon \nabla_x \Psi_\varepsilon$ as the remaining quantities are either weakly pre-compact or can be written as a gradient of a scalar function therefore irrelevant in the target system (4.12), (4.13), where they can be incorporated in the pressure.

A bit naive approach to solving this problem would be to rewrite the material derivative in (4.13) by means of (4.12) in the form

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) = \varrho_\varepsilon \partial_t \mathbf{u}_\varepsilon + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon \approx \overline{\varrho} \partial_t \mathbf{u}_\varepsilon + \overline{\varrho} \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon$$

$$\approx \overline{\varrho} \partial_t \mathbf{u}_\varepsilon + \overline{\varrho} \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{H}[\mathbf{u}_\varepsilon] + \overline{\varrho} \mathbf{H}[\mathbf{u}_\varepsilon] \cdot \nabla_x \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon] + \overline{\varrho} \frac{1}{2} \nabla_x |\nabla_x \Psi_\varepsilon|^2.$$

Unfortunately, in the framework of the weak solutions, such a step is not allowed at least in a direct fashion.

Alternatively, we can use the acoustic equation (4.28) in order to see that

$$\Delta\Psi_\varepsilon \nabla_x \Psi_\varepsilon = -\partial_\tau (r_\varepsilon \nabla_x \Psi_\varepsilon) - \frac{\omega}{2}\nabla_x r_\varepsilon^2 = -\varepsilon\partial_t (r_\varepsilon \nabla_x \Psi_\varepsilon) - \frac{\omega}{2}\nabla_x r_\varepsilon^2,$$

where the former term on the right-hand side is pre-compact (in the sense of distributions) while the latter is a gradient. This is the heart of the so-called local method developed in the context of isentropic fluid flows by Lions and Masmoudi [194].

4.4.2 *Strong Stratification*

Propagation of the acoustic waves becomes more complex in the case of a strongly stratified fluid discussed in Sect. 4.3. Similarly to Sect. 4.4.1, introducing the fast time variable $\tau = t/\varepsilon$ and supposing the pressure in the form $p = \varrho\vartheta$, we deduce the *acoustic equation* in the form

$$\begin{aligned} \partial_\tau r + \tfrac{1}{\tilde{\varrho}}\mathrm{div}_x \mathbf{V} &= 0, \\ \partial_\tau \mathbf{V} + \overline{\vartheta}\, \tilde{\varrho}\nabla_x r + \nabla_x(\tilde{\varrho}\vartheta^{(1)}) &= 0, \end{aligned} \tag{4.29}$$

where we have set $r = \varrho^{(1)}/\tilde{\varrho}$, $\mathbf{V} = \tilde{\varrho}\mathbf{U}$.

Assuming, in addition, that $\mathrm{Pe} = \varepsilon^2$ we deduce from (4.3) that $\vartheta^{(1)} \equiv 0$; whence Eq. (4.29) reduces to

■ STRATIFIED ACOUSTIC EQUATION:

$$\left.\begin{aligned} \partial_\tau r + \tfrac{1}{\tilde{\varrho}}\mathrm{div}_x \mathbf{V} &= 0, \\ \partial_\tau \mathbf{V} + \overline{\vartheta}\, \tilde{\varrho}\nabla_x r &= 0. \end{aligned}\right\} \tag{4.30}$$

Apparently, in sharp contrast with (4.26), the wave speed in (4.30) depends effectively on the vertical coordinate x_3.

4.4.3 *Attenuation of Acoustic Waves*

There are essentially three rather different explanations why the effect of the acoustic waves should be negligible.

Well-Prepared vs. Ill-Prepared Initial Data For the sake of simplicity, assume that $F = 0$ in (4.25). A proper choice of the initial data for the primitive system can eliminate the effect of acoustic waves as the acoustic equation preserves the norm in the associated energy space. More specifically, taking

$$\varrho^{(1)}(0,\cdot) \approx \vartheta^{(1)}(0,\cdot) \approx 0,\ \mathbf{U}(0,\cdot) \approx \mathbf{V},$$

where $\mathbf{V}$ is a solenoidal function, we easily observe that the amplitude of the acoustic waves remains small uniformly in time. As a matter of fact, the problem is more complex, as the "real" acoustic equation obtained in the course of asymptotic analysis contains forcing terms of order ε therefore not negligible in the "slow" time of the limit system. These issues are discussed in detail in Chap. 10.

Moreover, we point out that, in order to obtain an interesting limit problem, we need

$$\vartheta^{(1)} \approx \Theta$$

to be large (see Sect. 4.2). Consequently, the initial data for the primitive system considered in this book are always *ill-prepared*, meaning compatible with the presence of large amplitude acoustic waves.

The Effect of the Kinematic Boundary Although it is sometimes convenient to investigate a fluid confined to an unbounded spatial domain, any *realistic* physical space is necessarily bounded. Accordingly, the interaction of the acoustic waves with the boundary of the domain occupied by the fluid represents an intrinsic feature of any incompressible limit problem.

Viscous fluids adhere completely to the boundary. That means, if the latter is at rest, the associated velocity field $\mathbf{u}$ satisfies the *no-slip boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0.$$

The no-slip boundary condition, however, is not compatible with free propagation of acoustic waves, unless the boundary is flat or satisfies very particular geometrical constraints. Consequently, a part of the acoustic energy is dissipated through a boundary layer causing a uniform time decay of the amplitude of acoustic waves. Such a situation is discussed in Chap. 7.

Dispersion of the Acoustic Waves on Large Domains As already pointed out, *realistic* physical domains are always bounded. However, it is still reasonable to consider the situation when the diameter of the domain is sufficiently large with respect to the characteristic speed of sound in the fluid. The acoustic waves redistribute quickly the energy and, leaving a fixed bounded subset of the physical space, they will be reflected by the boundary but never come back in a finite lapse of time as the boundary is far away. In practice, such a problem can be equivalently posed on the whole space $\mathbb{R}^3$. Accordingly, the gradient component of the velocity field decays to zero locally in space with growing time. This problem is analyzed in detail in Chap. 8.

Vanishing Dissipation Limit Even more interesting situation arises when the diffusion terms like viscosity or heat conductivity become negligible in the asymptotic regime. In such a case the limit system is inviscid and the lack of compactness must be compensated by structural stability properties of the system, see Chap. 9.

4.5 Acoustic Analogies

The mathematical simulation of aeroacoustic sound presents in many cases numerous technical problems related to modeling of its generation and propagation. Its importance for diverse industrial applications is nowadays without any doubt in view of various demands in relation to the user comfort or environmental regulations. A few examples where aeroacoustics enters into the game are the sounds produced by jet engines of an airliner, the noise produced in high speed trains and cars, wind noise around buildings, ventilator noise in various household appliances, etc.

The departure point of many methods of acoustic simulations (at least those called hybrid methods) is *Lighthill's theory* [186, 187]. The starting point in Lighthill's approach is the system of NAVIER-STOKES EQUATIONS describing the motion of a viscous compressible gas in *isentropic regime*, with unknown functions density ϱ and velocity $\mathbf{u}$. The system of equations reads:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}_x \varrho \mathbf{u} &= 0, \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}, \end{aligned} \tag{4.31}$$

where $p = p(\varrho)$, and

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x{}^T \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0.$$

We can rewrite this system in the form

$$\begin{aligned} \partial_t R + \mathrm{div}_x \mathbf{Q} &= 0, \\ \partial_t \mathbf{Q} + \omega \nabla_x R &= \mathbf{F} - \mathrm{div}_x \mathbb{T}, \end{aligned} \tag{4.32}$$

where

$$\mathbf{Q} = \varrho \mathbf{u}, \quad R = \varrho - \overline{\varrho} \tag{4.33}$$

are the momentum and the density fluctuations from the basic constant density distribution $\overline{\varrho}$ of the background flow. Moreover, we have set

$$\omega = \frac{\partial p}{\partial \varrho}(\overline{\varrho}) > 0, \quad \mathbf{F} = \varrho \mathbf{f},$$
$$\mathbb{T} = \varrho \mathbf{u} \otimes \mathbf{u} + \Big(p - \omega(\varrho - \overline{\varrho})\Big)\mathbb{I} - \mathbb{S}. \tag{4.34}$$

The reader will have noticed apparent similarity of system (4.32) to the *acoustic equation* discussed in the previous part. Condition $\omega > 0$ is an analogue of the hypothesis of thermodynamics stability (3.10) expressing positive compressibility property of the fluid, typically a gas.

Taking the time derivative of the first equation in (4.32) and the divergence of the second one, we convert the system to a "genuine" wave equation

$$\partial_t^2 R - \omega \Delta_x R = -\mathrm{div}_x \mathbf{F} + \mathrm{div}_x(\mathrm{div}_x\, \mathbb{T}), \tag{4.35}$$

with wave speed $\sqrt{\omega}$. The viscous component is often neglected in $\mathbb{T}$ because of the considerable high Reynolds number of typical fluid regimes.

The main idea behind the method of *acoustic analogies* is to view system (4.32), or, equivalently (4.35), as a linear wave equation supplemented with a source term represented by the quantity on the right-hand side. In contrast with the original problem, the source term is assumed to be known or at least it can be determined by solving a kind of simplified problem. Lighthill himself completed system (4.32) adding extra terms corresponding to acoustic sources of different types. The resulting problem in the simplest possible form captures the basic acoustic phenomena in fluids and may be written in the following form.

■ LIGHTHILL'S EQUATION:

$$\partial_t R + \mathrm{div}_x \mathbf{Q} = \Sigma,$$
$$\partial_t \mathbf{Q} + \omega \nabla_x R = \mathbf{F} - \mathrm{div}_x \mathbb{T}. \tag{4.36}$$

According to Lighthill's interpretation, system (4.36) is a non-homogenous wave equation describing the acoustic waves (fluctuations of the density), where the terms on the right-hand side correspond to the *mononopolar* (Σ), *bipolar* ($-\mathbf{F}$), and *quadrupolar* ($\mathrm{div}_x \mathbb{T}$) acoustic sources, respectively. These source terms are considered as known and calculable from the background fluid flow field. The physical meaning of the source terms can be interpreted as follows:

- The first term Σ represents the acoustic sources created by the changes of control volumes due to changes of pressure or displacements of a rigid surface: this source can be schematically described via a particle whose diameter changes (pulsates) creating acoustic waves (density perturbations). It may be interpreted as a non stationary injection of a fluid mass rate $\partial_t \Sigma$ per unit volume. The acoustic noise of a gun shot is a typical example.

- The second term $\mathbf{F}$ describes the acoustic sources due to external forces (usually resulting from the action of a solid surface on the fluid). This sources are responsible for the most of the acoustic noise in the machines and ventilators.
- The third term $\mathrm{div}_x(\mathbb{T})$ is the acoustic source due to the turbulence and viscous effects in the background fluid flow which supports the density oscillations (acoustic waves). The noise of steady or non steady jets in aero-acoustics is a typical example.
- The tensor $\mathbb{T}$ is called the *Lighthill tensor*. It is composed of three tensors whose physical interpretation is the following: the first term is the Reynolds tensor with components $\varrho u_i u_j$ describing the (nonlinear) turbulence effects, the term $(p - \omega(\varrho - \overline{\varrho}))\mathbb{I}$ expresses the entropy fluctuations and the third one is the viscous stress tensor $\mathbb{S}$.

The method for predicting noise based on *Lighthill's equation* is usually referred to as a hybrid method since noise generation and propagation are treated separately. The first step consists in using data provided by numerical simulations to identify the sound sources. The second step then consists in solving the wave equation (4.36) driven by these source terms to determine the sound radiation. The main advantage of this approach is that most of the conventional flow simulations can be used in the first step.

In practical numerical simulations, the *Lighthill tensor* is calculated from the velocity and density fields obtained by using various direct numerical methods and solvers for compressible NAVIER-STOKES EQUATIONS. Then the acoustic effects are evaluated from *Lighthill's equation* by means of diverse direct numerical methods for solving the non-homogenous wave equations (see e.g. Colonius [63], Mitchell et al. [214], Freud et al. [129], among others). For flows in the low Mach number regimes the direct simulations are costly, unstable, inefficient and non-reliable, essentially due to the presence of rapidly oscillating acoustic waves (with periods proportional to the Mach number) in the equations themselves (see the discussion in the previous part). Thus in the low Mach number regimes the acoustic analogies as *Lighthill's equation*, in combination with the incompressible flow solvers, give more reliable results, see [129].

Acoustic analogies, in particular Lighthill's approach in the low Mach number regime, will be discussed in Chap. 10.

4.6 Initial Data

Motivated by the formal asymptotic expansion discussed in the previous sections, we consider the initial data for the scaled NAVIER-STOKES-FOURIER SYSTEM in the form

$$\varrho(0,\cdot) = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\varepsilon) = \mathbf{u}_{0,\varepsilon},\ \vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

where $\varepsilon = \mathrm{Ma}$, $\varrho^{(1)}_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ are given functions, and $\tilde{\varrho}$, $\overline{\vartheta}$ represent an equilibrium state. Note that the apparent inconsistency in the form of the initial data is a consequence of the fact that smallness of the velocity with respect to the speed of sound is already incorporated in the system by scaling.

The initial data are termed *ill-prepared* if

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^p(\Omega),\ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^p(\Omega;\mathbb{R}^3)$$

for a certain $p \geq 1$, typically $p = 2$ or even $p = \infty$. If, in addition,

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0,\ \mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] \to 0 \text{ a.a. in } \Omega,$$

where $\varrho^{(1)}_0$, $\vartheta^{(1)}_0$ satisfy certain *compatibility conditions*, we say that the data are *well-prepared*. For instance, in the situation described in Sect. 4.2, we require

$$\frac{\partial p}{\partial \varrho}(\overline{\varrho},\overline{\vartheta})\varrho^{(1)}_0 + \frac{\partial p}{\partial \vartheta}(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}_0 = \overline{\varrho}F.$$

In particular, the common definition of the well-prepared data, namely $\varrho^{(1)}_0 = \vartheta^{(1)}_0 = 0$, is recovered as a special case provided $F = 0$.

As observed in Sect. 4.4, the ill-prepared data are expected to generate large amplitude rapidly oscillating acoustic waves, while the well-prepared data are not. Alternatively, following Lighthill [188], we may say that the well-prepared data may be successfully handled by the linear theory, while the ill-prepared ones require the use of a nonlinear model.

4.7 A General Approach to Singular Limits for the Full Navier-Stokes-Fourier System

The overall strategy adopted in this book leans on the concept of *weak solutions* for both the primitive system and the associated asymptotic limit. The starting point is always the complete NAVIER-STOKES-FOURIER SYSTEM introduced in Chap. 1 and discussed in Chaps. 2, 3, where one or several characteristic numbers listed in Sect. 4.1 are proportional to a small parameter. We focus on the passage to incompressible fluid models, therefore the *Mach number* Ma is always of order $\varepsilon \to 0$. On the contrary, the *Strouhal number* Sr as well as the *Reynolds number* Re are assumed to be of order 1 with exception of Chap. 9, where the Reynolds number approaches infinity. Consequently, the velocity of the fluid in the target system will satisfy a variant of incompressible (viscous) NAVIER-STOKES EQUATIONS coupled with a balance of the internal energy identified through a convenient choice of the characteristic numbers Fr and Pe.

Our theory applies to *dissipative fluid systems* that may be characterized through the following properties.

■ DISSIPATIVE FLUID SYSTEM:

(P1) The total mass of the fluid contained in the physical space Ω is constant at any time.

(P2) In the absence of external sources, the total energy of the fluid is constant or non-increasing in time.

(P3) The system produces entropy, in particular, the total entropy is a non-decreasing function of time. In addition, the system is thermodynamically stable, that means, the maximization of the total entropy over the set of all allowable states with the same total mass and energy delivers a unique equilibrium state provided the system is thermally and mechanically insulated.

The key tool for studying singular limits of dissipative fluid systems is the dissipation balance, or rather inequality, analogous to the corresponding equality introduced in (2.52). Neglecting, for simplicity, the source terms in the scaled system (4.1)–(4.3), we deduce

■ SCALED DISSIPATION INEQUALITY:

$$\int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) (\tau, \cdot)\, \mathrm{d}x \tag{4.37}$$

$$+\sigma\Big[[0, \tau] \times \overline{\Omega} \Big]$$

$$\leq \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} + H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right) \mathrm{d}x$$

$$\text{for a.a. } \tau \in (0, T),$$

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{4.38}$$

where $H_{\overline{\vartheta}} = \varrho e - \overline{\vartheta} \varrho s$ is the Helmholtz function introduced in (2.48). Note that, in accordance with **(P2)**, there is an inequality sign in (4.37) because we admit systems that dissipate energy.

The quantities $\overline{\varrho}$ and $\overline{\vartheta}$ are positive constants characterizing the static distribution of the density and the absolute temperature, respectively. In accordance with **(P1)**, we have

$$\int_{\Omega} (\varrho(t,\cdot) - \overline{\varrho}) \, \mathrm{d}x = 0 \text{ for any } t \in [0,T],$$

while $\overline{\vartheta}$ is determined by the asymptotic value of the total energy for $t \to \infty$. In accordance with **(P3)**, the static state $(\overline{\varrho}, \overline{\vartheta})$ maximizes the entropy among all states with the same total mass and energy and solves the NAVIER-STOKES-FOURIER SYSTEM with the velocity field $\mathbf{u} = 0$, in other words, $(\overline{\varrho}, \overline{\vartheta})$ is an equilibrium state. In Chap. 6, the constant density equilibrium state $\overline{\varrho}$ is replaced by $\tilde{\varrho} = \tilde{\varrho}(x_3)$.

Basically all available bounds on the family of solutions to the scaled system are provided by (4.37), (4.38). If the Mach number Ma is proportional to a small parameter ε, and, simultaneously $\mathrm{Re} = \mathrm{Pe} \approx 1$, relations (4.37), (4.38) yield a bound on the gradient of the velocity field provided the integral on the right-hand side of (4.37) *divided on* ε^2 remains bounded.

On the other hand, seeing that

$$H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) - (\varrho_0 - \overline{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \approx c \left(|\varrho_0 - \overline{\varrho}|^2 + |\vartheta_0 - \overline{\vartheta}|^2 \right)$$

at least in a neighborhood of the static state $(\overline{\varrho}, \overline{\vartheta})$, we conclude, in agreement with the formal asymptotic expansion discussed in Sect. 4.2, that the quantities

$$\varrho^{(1)}_{0,\varepsilon} = \frac{\varrho(0,\cdot) - \overline{\varrho}}{\varepsilon} \text{ and } \vartheta^{(1)}_{0,\varepsilon} = \frac{\vartheta(0,\cdot) - \overline{\vartheta}}{\varepsilon}, \text{ and } \mathbf{u}_{0,\varepsilon} = \mathbf{u}(0,\cdot)$$

have to be bounded uniformly for $\varepsilon \to 0$, or, in the terminology introduced in Sect. 4.6, the initial data must be at least ill-prepared.

As a direct consequence of the structural properties of $H_{\overline{\vartheta}}$ established in Sect. 2.2.3, it is not difficult to deduce from (4.37) that

$$\varrho^{(1)}(t,\cdot) = \frac{\varrho(t,\cdot) - \overline{\varrho}}{\varepsilon} \text{ and } \vartheta^{(1)} = \frac{\vartheta(t,\cdot) - \overline{\vartheta}}{\varepsilon}$$

remain bounded, at least in $L^1(\Omega)$, uniformly for $t \in [0,T]$ and $\varepsilon \to 0$.

Now, we introduce the set of essential values $\mathcal{O}_{\mathrm{ess}} \subset (0,\infty)^2$,

$$\mathcal{O}_{\mathrm{ess}} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \,\middle|\, \overline{\varrho}/2 < \varrho < 2\overline{\varrho},\ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta} \right\}, \tag{4.39}$$

together with its residual counterpart

$$\mathcal{O}_{\mathrm{res}} = (0,\infty)^2 \setminus \mathcal{O}_{\mathrm{ess}}. \tag{4.40}$$

Let $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of solutions to a scaled NAVIER-STOKES-FOURIER SYSTEM. In agreement with (4.39), (4.40), we define the *essential set* and *residual set* of points $(t,x) \in (0,T) \times \Omega$ as follows.

■ ESSENTIAL AND RESIDUAL SETS:

$$\mathcal{M}^\varepsilon_{\rm ess} \subset (0,T) \times \Omega,$$

$$\mathcal{M}^\varepsilon_{\rm ess} = \{(t,x) \in (0,T) \times \Omega \mid (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x)) \in \mathcal{O}_{\rm ess}\}, \tag{4.41}$$

$$\mathcal{M}^\varepsilon_{\rm res} = ((0,T) \times \Omega) \setminus \mathcal{M}^\varepsilon_{\rm ess} \tag{4.42}$$

We point out that $\mathcal{O}_{\rm ess}$, $\mathcal{O}_{\rm res}$ are *fixed* subsets of $(0,\infty)^2$, while $\mathcal{M}^\varepsilon_{\rm ess}$, $\mathcal{M}^\varepsilon_{\rm res}$ are measurable subsets of the time-space cylinder $(0,T) \times \Omega$ *depending* on ϱ_ε, ϑ_ε.

It is also convenient to introduce the "projection" of the set $\mathcal{M}^\varepsilon_{\rm ess}$ for a fixed time $t \in [0,T]$,

$$\mathcal{M}^\varepsilon_{\rm ess}[t] = \{x \in \Omega \mid (t,x) \in \mathcal{M}^\varepsilon_{\rm ess}\}$$

and

$$\mathcal{M}^\varepsilon_{\rm res}[t] = \Omega \setminus \mathcal{M}^\varepsilon_{\rm ess}[t], \tag{4.43}$$

where both are measurable subsets of Ω for a.a. $t \in (0,T)$.

Finally, each measurable function h can be decomposed as

$$h = [h]_{\rm ess} + [h]_{\rm res}, \tag{4.44}$$

where we set

$$[h]_{\rm ess} = h\, 1_{\mathcal{M}^\varepsilon_{\rm ess}},\ [h]_{\rm res} = h\, 1_{\mathcal{M}^\varepsilon_{\rm res}} = h - [h]_{\rm ess}. \tag{4.45}$$

Of course, we should always keep in mind that such a decomposition depends on the actual values of ϱ_ε, ϑ_ε.

The specific choice of $\mathcal{O}_{\rm ess}$ is not important. We can take $\mathcal{O}_{\rm ess} = \mathcal{U}$, where $\mathcal{U} \subset \overline{\mathcal{U}} \subset (0,\infty)^2$ is a bounded open neighborhood of the equilibrium state $(\overline{\varrho}, \overline{\vartheta})$. A general idea exploited in this book asserts that the "essential" component $[h]_{\rm ess}$ carries all information necessary in the limit process, while its "residual" counterpart $[h]_{\rm res}$ vanishes in the asymptotic limit for $\varepsilon \to 0$. In particular, the Lebesgue measure of the residual sets $|\mathcal{M}_{\rm res}[t]|$ becomes small uniformly in $t \in (0,T)$ for small values of ε.

Another characteristic feature of our approach is that the entropy production rate σ is small, specifically of order ε^2, in the low Mach number limit. Accordingly, in contrast with the primitive NAVIER-STOKES-FOURIER SYSTEM, the target problem can be expressed in terms of equations rather than inequalities. The ill-prepared data, for which the perturbation of the equilibrium state is proportional to the Mach number, represent a sufficiently rich scaling leading to non-trivial target problems.