Chapter 12
Bibliographical Remarks

12.1 Fluid Flow Modeling

The material collected in Chap. 1 is standard. We refer to the classical monographs by Batchelor [20] or Lamb [180] for the full account on the mathematical theory of continuum fluid mechanics. A more recent treatment may be found in Truesdell and Noll [259] or Truesdell and Rajagopal [260]. An excellent introduction to the mathematical theory of waves in fluids is contained in Lighthill's book [188].

The constitutive equations introduced in Sect. 1.4, in particular, the mechanical effect of thermal radiation, are motivated by the mathematical models in astrophysics (see Battaner [21]). Relevant material may be also found in the monographs by Bose [31], Mihalas and Weibel-Mihalas [213], Müller and Ruggeri [217], or Oxenius [228]. A general introduction to the theory of equations of state is provided by Eliezer et al. [93].

In the present monograph, we focused on thermodynamics of *viscous* compressible fluids. For the treatment of problems related to inviscid fluids as well as more general systems of *hyperbolic* conservation laws, the literature provides several comprehensive seminal works, for instance, Benzoni-Gavage and Serre [26], Bressan [38], Chen and Wang [58], Dafermos [68], and Serre [247].

The weak solutions in this book are considered on large time intervals. There is a vast literature investigating (strong) solutions with "large" regular external data on short time intervals and/or with "small" regular external data on arbitrary large time intervals for both the Navier-Stokes equations in the barotropic regime and for the Navier-Stokes-Fourier system. These studies were originated by the seminal work of Matsumura and Nishida [206, 207], and further developed by many authors: Beirao da Veiga [23], Cho et al. [59] Danchin [70, 71], Hoff [149–154], Jiang [159], Matsumura and Padula [208, 221], Padula and Pokorný [229], Salvi and Straškraba [241], Valli and Zajaczkovski [264], among others.

As far as the singular limits in the fluid dynamics are concerned, the mathematical literature provides two qualitatively different groups of results. First one

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concerns the investigation of singular limits in the passage from the microscopic description provided by the kinetic models of Boltzmann's type to the macroscopic one represented by the Euler, Navier-Stokes, and Navier-Stokes-Fourier equations and their modifications. The reader may find interesting to compare the methods and techniques used in the present monograph to those developed in the context of kinetic equations and their asymptotic limits by Bardos et al. [17–19], Bardos and Ukai [16], Golse and Saint-Raymond [144], Golse and Levermore [143], Lions and Masmoudi [196, 197], see also the review paper by Villani [266] as well as the references therein. The second group of problems concerns the relations between models at the same conceptual level provided by continuum mechanics studied in this book. We refer to Sect. 12.4 for the corresponding bibliographic remarks.

12.2 Mathematical Theory of Weak Solutions

Variational (weak) solutions represent the most natural framework for a mathematical formulation of the balance laws arising in continuum fluid mechanics, these being originally formulated in the form of integral identities rather than partial differential equations. Since the truly pioneering work of Leray [184], the theory of variational solutions, based on the function spaces of Sobolev type and developed in the work of Ladyzhenskaya [177], Temam [255], Caffarelli et al. [45], Antontsev et al. [11], and, more recently Lions [191], has become an important part of modern mathematical physics.

Although many of the above cited references concern the incompressible fluids, where the weak solutions are expected (but still not proved) to be regular at least for smooth data, the theory of compressible and/or compressible and heat conducting fluids supplemented with arbitrarily large data is more likely to rely on the concept of "genuinely weak" solutions incorporating various types of discontinuities and other irregular phenomena as the case may be (for relevant examples see Desjardins [79], Hoff [153, 154], Hoff and Serre [155], Vaigant [261], among others). Pursuing further this direction some authors developed the theory of *measure valued solutions* in order to handle the rapid oscillations that solutions may develop in a finite time (see DiPerna [83], DiPerna and Majda [86], Málek et al. [201]). The representation of the basic physical principles in terms of conservation laws has been discussed in a recent paper by Chen and Tores [57] devoted to the study of vector fields with divergence measure.

A rigorous mathematical theory of compressible barotropic fluids with large data was presented only recently in the pioneering work by Lions [192] (see also a very interesting related result by Vaigant and Kazhikhov [262]). The fundamental idea discussed already by Hoff [152] and Serre [246] is based on a "weak continuity" property of a physical quantity termed effective viscous pressure, together with a clever use of the renormalized equation of continuity in order to describe possible density oscillations. A survey of the relevant recent results in this direction can be found in the monograph [224].

12.3 Existence Theory

The seminal work of Lions [192] on the existence for compressible viscous barotropic fluids requires certain growth restrictions on the pressure, specifically, the adiabatic exponent $\gamma \geq \frac{9}{5}$ in the nonsteady case, and $\gamma > \frac{5}{3}$ in the steady case. These result have been improved by means of a more precise description of the density oscillations in [101, 117] up to the adiabatic exponents $\gamma > \frac{3}{2}$. Finally, Frehse, Goj, Steinhauer in [128] and Plotnikov, Sokolowski in [232] derived, independently, new estimates, which has been quite recently used, at least in the steady case, to extend the existence theory to smaller adiabatic exponents, see [39, 223, 233] and [160], among others. Recently, Plotnikov and Weigant [234] succeeded in applying these ideas also to the evolutionary *isothermal* case in two space dimensions. Time periodic solutions have been investigated in [116, 121]. All above mentioned results deal with no inflow/outflow boundary conditions. A few existence results with large inflow/outflow boundary data are available at least in the barotropic case in [139] and recently in [54, 115]. A progress has been made also in another direction, namely relaxation of certain hypotheses concerning the structure of the viscous stress tensor as well as pressure required by the theory based on Lions' ideas, see Bresch and Jabin [36].

The existence theory presented in this book can be viewed as a part of the program originated in the monograph [102]. In comparison with [102], the present study contains some new material, notably, the constitutive equations are much more realistic, with structural restrictions based on purely physical principles, and the transport coefficients are allowed to depend on the temperature. These new ingredients of the existence theory have been introduced in a series of papers [103–105, 107] and recently revisited in [126].

Recently, several works appeared constructing weak solutions from the convergent numerical schemes, see for example Eymard et al. [98] for the compressible Stokes problem, Karper [163] for the equations in barotropic regime, [124] for the full system or monograph [125] and references quoted there.

Several new ideas related to the existence problem for the full Navier-Stokes-Fourier system with density dependent shear and bulk viscosities satisfying a particular differential relation have been developed recently in a series of papers by Bresch and Desjardins [34, 35]. Making a clever use of the structure of the equations, the authors discovered a new integral identity which allows to obtain uniform estimates on the density gradient and which may be used to prove existence of global-in-time solutions in some particular situations, see Vasseur, Yu [265], Li, Xin [185].

12.4 Analysis of Singular Limits

Many recent papers and research monographs explain the role of formal scaling arguments in the physical and numerical analysis of complex models arising in mathematical fluid dynamics. This approach has become of particular relevance in meteorology, where the huge scale differences in atmospheric flows give rise to a large variety of qualitatively different models, see the survey papers by Klein et al. [173], Klein [171, 172], the lecture notes of Majda [200], and the monographs by Chemin et al. [56], Zeytounian [274–276]. The same is true for applications in astrophysics, see the classical book of Chandrasekhar [53], or the more recent treatment by Gilman, Glatzmeier [138, 140], Lignières [189], among others.

The "incompressible limit" Ma $\to 0$ for various systems arising in mathematical fluid dynamics was rigorously studied in the seminal work by Klainerman and Majda [170] (see also Ebin [89]). One may distinguish two kinds of qualitatively different results based on different techniques. The first approach applies to strong solutions defined on possibly short time intervals, the length of which, however, is independent of the value of the parameter Ma $\to$ 0. In this framework, the most recent achievements for the full Navier-Stokes-Fourier system can be found in the recent papers by Alazard [3, 4] (for earlier results see the survey papers by Danchin [72], Métivier and Schochet [212], Schochet [244], and the references cited therein).

The second group of results is based on a global-in-time existence theory for the weak solutions of the underlying primitive system of equations, asserting convergence towards solutions of the target system on an arbitrary time interval. Results of this type for the isentropic Navier-Stokes system have been obtained by Lions and Masmoudi [194, 195], and later extended by Desjardins et al. [81], Bresch et al. [37]. For a survey of these as well as of many other related results, see the review paper by Masmoudi [204].

The investigation of singular limits for the full Navier–Stokes–Fourier system in the framework of weak variational solutions has been originated in [108] and [109]. The spectral analysis of acoustic waves in the presence of strong stratification exposed in Chap. 6 follows the book of Wilcox [272], while the weighted Helmholtz decomposition used throughout the chapter has been inspired by Novotný and Pileckas [222]. Related results based on the so-called local method were obtained only recently by Masmoudi [205]. The refined analysis of the acoustic waves presented in Chap. 7 is based on the asymptotic expansion technique developed by Vishik and Ljusternik [267] to handle singular perturbations of elliptic operators, later adopted in the pioneering paper of Desjardins et al. [81] to the wave operator framework. Related techniques are presented in the monograph of Métivier [211].

12.5 Propagation of Acoustic Waves

There is a vast literature concerning acoustics in fluids, in general, and acoustic analogies and equations, in particular. In the study of the low Mach number limits, we profited from the theoretical work by Schochet [243–245]. Besides, the truly pioneering work in the context of weak solutions is the paper by Desjardins and Grenier [80], where the Strichartz estimates are used. A nice introduction to the linear theory of wave propagation is the classical monograph by Lighthill [188].

The nonlinear acoustic phenomena together with the relevant mathematical theory are exposed in the book by Enflo and Hedberg [94].

Lighthill's acoustic analogy in the spirit of Chap. 10 has been used by many authors, let us mention the numerical results obtained by Golanski et al. [141, 142].

Clearly, this topic is closely related to the theory of wave equation both in linear and nonlinear setting. Any comprehensive list of the literature in this area goes beyond the scope of the present monograph, and we give only a representative sample of results: Bahouri and Chemin [14], Burq [43], Christodoulou and Klainerman [61], Smith and Tataru [250], or, more recently, Metcalfe and Tataru [210].

12.6 Relative Energy, Inviscid Limits

The method of relative entropies (energies) proposed in the truly pioneering work of Dafermos [67] has been used by many authors in rather different context, see Masmoudi [203, 204], Saint–Raymond [240], Sueur [253] to name only a few. Applications to the Navier-Stokes-Fourier system have been established only recently in [110] (see also [106, 119, 120]). This tool proved to be very efficient in investigating singular limits with lack of compactness for the velocity field occurring typically in the high Reynolds number regime, see [111, 112, 253] and also Chap. 9 of this monograph. It appeared also to be efficient in investigation of multiply scaled distinguished singular limits, see [113, 114] or in the dimension reduction of the compressible models, see [24, 25, 202]. Note that similar ideas in the context of purely barotropic fluids were exploited by Masmoudi [203], Wang and Jiang [271], or Jiang et al. [161], among others. Applications of the relative energy method to the numerical schemes with goal to establish error estimates has been started in Gallouet et al. [133].

The problem of inviscid (zero dissipation) limits was considered in Chap. 9 in its "mild" form, in particular, the effect of the boundary layer was eliminated by a proper choice of the boundary conditions. In general, the velocity component of the primitive system on domains with boundaries is expected to take the form

$$\mathbf{u}_\varepsilon = \mathbf{U} + \mathbf{U}_{BL} \tag{12.1}$$

where $\mathbf{U}$ is the solution of the limit inviscid problem and $\mathbf{U}_{BL}$ is small except at a small neighborhood of the boundary. The behavior of $\mathbf{U}_{BL}$ is determined by Prandtl's equation, however, rigorous results concerning validity of (12.1) are in a short supply, see the survey papers by W. E [88], Grenier et al. [146], or Masmoudi [204].