Chapter 11
Appendix

For reader's convenience, a number of standard results used in the preceding text is summarized in this chapter. Nowadays classical statements are appended with the relevant reference material, while complete proofs are provided in the cases when a compilation of several different techniques is necessary. A significant part of the theory presented below is related to general problems in mathematical fluid mechanics and may be of independent interest.

In the whole appendix M denotes a positive integer while $N \in \mathbb{N}$ refers to the space dimension. The space dimension is always taken greater or equal than 2, if not stated explicitly otherwise.

11.1 Spectral Theory of Self-Adjoint Operators

Let H be a complex Hilbert space with a scalar produce $\langle \cdot ; \cdot \rangle$. A linear operator $\mathcal{A} : H \to H$ is called self-adjoint, if

- the domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is dense in H;
- $\mathcal{A}$ is symmetric,

$$\langle \mathcal{A}v; w \rangle = \langle v; \mathcal{A}w \rangle$$

  for all $v, w \in \mathcal{D}(\mathcal{A})$;
- if

$$\langle \mathcal{A}x; y \rangle = \langle x; h \rangle \ \text{ for all } x \in \mathcal{D}(\mathcal{A}),$$

  then $y \in \mathcal{D}(\mathcal{A})$ and $h = \mathcal{A}y$.

E. Feireisl, A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, Advances in Mathematical Fluid Mechanics, DOI 10.1007/978-3-319-63781-5_11

The spectrum of a self-adjoint operator $\mathcal{A}$ is a subset of the real axis R, meaning for any complex $\lambda = \lambda_1 + \mathrm{i}\lambda_2$, $\lambda_1, \lambda_2 \in R$, $\lambda_2 \neq 0$, the operator

$$\mathcal{A} + \lambda \mathrm{Id} : \mathcal{D}(\mathcal{A}) \subset H \to H$$

is surjective with bounded inverse.

■ SPECTRAL DECOMPOSITION:

Theorem 11.1 *Let $\mathcal{A}$ be a densely defined self-adjoint operator on a Hilbert space H.*

Then there exists a family of orthogonal projections $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ enjoying the following properties:

- *P_λ, P_μ commute,*

$$P_\lambda \circ P_\mu = P_{\min\{\lambda,\mu\}} \text{ for any } \lambda, \mu \in \mathbb{R};$$

- *P_λ are right continuous,*

$$P_\mu h \to P_\lambda h \text{ in } H \text{ for any } h \in H \text{ whenever } \mu \searrow \lambda;$$

-

$$P_\lambda h \to 0 \text{ in } H \text{ for any } h \in H \text{ if } \lambda \to -\infty,$$

$$P_\lambda h \to h \text{ in } H \text{ for any } h \in H \text{ if } \lambda \to \infty;$$

- *P_λ commutes with $\mathcal{A}$ on $\mathcal{D}(\mathcal{A})$,*

$$u \in \mathcal{D}(\mathcal{A}) \text{ if and only if } \int_{-\infty}^{\infty} \lambda^2 \, \mathrm{d} \langle P_\lambda u; u \rangle < \infty,$$

and

$$\langle \mathcal{A}u; v \rangle = \int_{-\infty}^{\infty} \lambda \, \mathrm{d} \langle P_\lambda u; v \rangle \text{ for any } u \in \mathcal{D}(\mathcal{A}),\ v \in H. \tag{11.1}$$

See Reed and Simon [237], Leinfelder [182] □

The above results is also known as *Spectral Theorem* for self-adjoint operators. The integral in (11.1) is understood in the Lebesgue-Stieltjes sense. We report

■ STONE'S FORMULA:

$$\langle (P_b - P_a)\, u; v\rangle \tag{11.2}$$

$$= \lim_{\delta\to 0+}\left(\lim_{\varepsilon\to 0+}\int_{a+\delta}^{b+\delta}\left\langle\left([\mathcal{A}-(s+\mathrm{i}\varepsilon)\mathbb{I}]^{-1}-[\mathcal{A}-(s-\mathrm{i}\varepsilon)\mathbb{I}]^{-1}\right)u; v\right\rangle \,\mathrm{d}s\right)$$

for any $a < b$ and $u, v \in H$.

See Reed and Simon [238] □

Given the spectral decomposition $\{P_\lambda\}_{\lambda\in\mathbb{R}}$ we may define *functional calculus* associated to $\mathcal{A}$, specifically for any Borel function G defined on $\mathbb{R}$ we define $G(\mathcal{A})$ with a domain

$$u \in \mathcal{D}(G(\mathcal{A})) \text{ if and only if } \int_{-\infty}^{\infty} |G(\lambda)|^2 \,\mathrm{d}\,\langle P_\lambda u; u\rangle < \infty < \infty,$$

and

$$\langle G(\mathcal{A})u; v\rangle = \int_{-\infty}^{\infty} G(\lambda)\,\mathrm{d}\,\langle P_\lambda u; v\rangle\,,\ v \in H,$$

see Reed and Simon [238].

Finally, we introduce the *spectral measure* μ_u associated to $u \in H$ as

$$\langle \mu_u, G\rangle_{\mathcal{M}(R);C_c(R)} = \int_{-\infty}^{\infty} G(\lambda)\,\mathrm{d}\,\langle P_\lambda u; u\rangle\,.$$

We report the following consequence of *Spectral Theorem*.

■ SPECTRAL MEASURE REPRESENTATION:

Theorem 11.2 *Let $\mathcal{A}$ be a densely defined self-adjoint operator on a Hilbert space H, G a Borel function on R. Let $u \in \mathcal{D}(G(\mathcal{A}))$ and let μ_u be the associated spectral measure.*

Then any $\Psi \in H$ admits a representative $\Psi_u \in L^2(\mathbb{R}, \mathrm{d}\mu_u)$,

$$\int_{-\infty}^{\infty} |\Psi_u(\lambda)|^2 \,\mathrm{d}\mu_u \le \|\Psi\|_H^2,$$

such that

$$\langle G(\mathcal{A})u, \Psi \rangle = \int_{-\infty}^{\infty} G(\lambda)\Psi_u(\lambda)\, \mathrm{d}\mu_u.$$

See Reed and Simon [238] □

11.2 Mollifiers

A function $\zeta \in C_c^\infty(\mathbb{R}^M)$ is termed a *regularizing kernel* if

$$\mathrm{supp}[\zeta] \subset (-1,1)^M,\ \zeta(-x) = \zeta(x) \geq 0, \quad \int_{\mathbb{R}^M} \zeta(x)\, dx = 1. \tag{11.3}$$

For a measurable function a defined on $\mathbb{R}^M$ with values in a Banach space X, we denote

$$\begin{gathered} S_\omega[a] = a^\omega(x) = \zeta_\omega * a = \textstyle\int_{\mathbb{R}^M} \zeta_\omega(x-y)a(y)\, dy, \\ \text{where } \zeta_\omega(x) = \frac{1}{\omega^M}\zeta(\frac{x}{\omega}),\ \omega > 0, \end{gathered} \tag{11.4}$$

provided the integral on the right hand-side exists. The operator $S_\omega : a \mapsto a^\omega$ is called a *mollifier*. Note that the above construction easily extends to distributions by setting $a^\omega(x) =< a; \zeta_\omega(x - \cdot) >_{[\mathcal{D}';\mathcal{D}](\mathbb{R}^M)}$.

■ MOLLIFIERS:

Theorem 11.3 *Let X be a Banach space. If $a \in L^1_{loc}(\mathbb{R}^M;X)$, then we have $a^\omega \in C^\infty(\mathbb{R}^M;X)$. In addition, the following holds:*

(i) If $a \in L^p_{\mathrm{loc}}(\mathbb{R}^M;X)$, $1 \leq p < \infty$, then $a^\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^M;X)$, and

$$a^\omega \to a \ \textit{in}\ L^p_{\mathrm{loc}}(\mathbb{R}^M;X) \ \textit{as}\ \omega \to 0.$$

(ii) If $a \in L^p(\mathbb{R}^M;X)$, $1 \leq p < \infty$, then $a^\omega \in L^p(\mathbb{R}^M;X)$,

$$\|a^\omega\|_{L^p(\mathbb{R}^M;X)} \leq \|a\|_{L^p(\mathbb{R}^M;X)},\ \textit{and}\ a^\omega \to a \quad \textit{in} \quad L^p(\mathbb{R}^M;X)\ \textit{as}\ \omega \to 0.$$

(iii) If $a \in L^\infty(\mathbb{R}^M;X)$, then $a^\omega \in L^\infty(\mathbb{R}^M;X)$, and

$$\|a^\omega\|_{L^\infty(\mathbb{R}^M;X)} \leq \|a\|_{L^\infty(\mathbb{R}^M;X)}.$$

iv) If $a \in C^k(U;X)$, *where* $U \subset \mathbb{R}^M$ *is an (open) ball, then* $(\partial^\alpha a)^\omega(x) = \partial^\alpha a^\omega(x)$ *for all* $x \in U$, $\omega \in (0, \mathrm{dist}[x, \partial U])$ *and for any multi-index* α, $|\alpha| \leq k$. *Moreover,*

$$\|a^\omega\|_{C^k(\overline{B};X)} \leq \|a\|_{C^k(\overline{V};X)}$$

for any $\omega \in (0, \mathrm{dist}[\partial B, \partial V])$, *where* B, V *are (open) balls in* $\mathbb{R}^M$ *such that* $\overline{B} \subset V \subset \overline{V} \subset U$. *Finally,*

$$a^\omega \to a \quad \text{in} \quad C^k(\overline{B};X) \text{ as } \omega \to 0.$$

See Amann [8, Chap. III.4], or Brezis [41, Chap. IV.4]. □

11.3 Basic Properties of Some Elliptic Operators

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider a general *elliptic equation* in the divergence form

$$\mathcal{A}(x,u) = -\sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\partial_{x_j} u) + c(x)u = f \text{ for } x \in \Omega, \tag{11.5}$$

supplemented with the boundary condition

$$\delta \mathbf{u} + (\delta - 1)\sum_{j=1}^{N} a_{i,j}\partial_{x_j} u \, n_j|_{\partial\Omega} = g, \tag{11.6}$$

where $\delta = 0, 1$. We suppose that

$$a_{i,j} = a_{j,i} \in C^1(\overline{\Omega}), \ \sum_{i,j} a_{i,j}\xi_i\xi_j \geq \alpha|\xi|^2 \tag{11.7}$$

for a certain $\alpha > 0$ and all $\xi \in \mathbb{R}^N$, $|\xi| = 1$. The case $\delta = 1$ corresponds to the *Dirichlet problem*, $\delta = 0$ is termed the *Neumann problem*.

In several applications discussed in this book, Ω is also taken in the form

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2, \ B_{\text{bottom}}(x_1, x_2) < x_3 < B_{\text{top}}(x_1, x_2)\}, \tag{11.8}$$

where the *horizontal* variable (x_1, x_2) belongs to the *flat torus*

$$\mathcal{T}^2 = \left([-\pi, \pi]|_{\{-\pi,\pi\}}\right)^2.$$

Although all results below are formulated in terms of standard domains, they apply to domains Ω given by (11.8) as well provided we identify

$$\partial\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = B_{\rm bottom}(x_1, x_2)\}$$

$$\cup\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ x_3 = B_{\rm top}(x_1, x_2)\}.$$

This is due to the fact that all theorems concerning regularity of solutions to elliptic equations are of local character.

11.3.1 A Priori Estimates

We start with the classical *Schauder estimates*.

■ HÖLDER REGULARITY:

Theorem 11.4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{k+2,\nu}$, $k = 0, 1, \dots,$ with $\nu > 0$. Suppose, in addition to (11.7), that $a_{i,j} \in C^{k+1,\nu}(\overline{\Omega})$, $i,j = 1, \dots, N$, $c \in C^{k,\nu}(\overline{\Omega})$. Let u be a classical solution of problem (11.5), (11.6), where $f \in C^{k,\nu}(\overline{\Omega})$, $g \in C^{k+\delta+1,\nu}(\partial\Omega)$.*

Then

$$\|u\|_{C^{k+2,\nu}(\overline{\Omega})} \le c\left(\|f\|_{C^{k,\nu}(\overline{\Omega})} + \|g\|_{C^{k+1,\nu}(\partial\Omega)} + \|u\|_{C(\overline{\Omega})}\right).$$

See Ladyzhenskaya and Uralceva [178, Theorems 3.1 and 3.2, Chap. 3], Gilbarg and Trudinger [136, Theorem 6.8]. □

Similar bounds can be also obtained in the L^p-framework. We report the celebrated result by Agmon et al. [2] (see also Lions and Magenes [193]). The hypotheses we use concerning regularity of the boundary and the coefficients $a_{i,j}$, c are not optimal but certainly sufficient in all situations considered in this book.

■ STRONG L^p-REGULARITY:

Theorem 11.5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. In addition to (11.7), assume that $c \in C(\overline{\Omega})$. Let $u \in W^{2,p}(\Omega)$, $1 < p < \infty$, be a (strong) solution of problem (11.5), (11.6), with $f \in L^p(\Omega)$, $g \in W^{\delta+1-1/p,p}(\partial\Omega)$.*

Then

$$\|u\|_{W^{2,p}(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{\delta+1-1/p,p}(\partial\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

See Agmon et al. [2]. □

The above estimates can be extrapolated to "negative" spaces. For the sake of simplicity, we set $g = 0$ in the Dirichlet case $\delta = 1$. In order to formulate the adequate results, let us introduce the Dirichlet form associated to the operator $\mathcal{A}$, namely

$$[\mathcal{A}u, v] := \int_\Omega a_{i,j}(x)\partial_{x_j}u\partial_{x_i}v + c(x)uv \, \mathrm{d}x.$$

In such a way, the operator $\mathcal{A}$ can be regarded as a continuous linear mapping

$$\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega) \text{ for the Dirichlet boundary condition}$$

or

$$\mathcal{A} : W^{1,p}(\Omega) \to [W^{1,p'}(\Omega)]^* \text{ for the Neumann boundary condition,}$$

where

$$1 < p < \infty, \ \frac{1}{p} + \frac{1}{p'} = 1.$$

■ WEAK L^p-REGULARITY:

Theorem 11.6 *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^2, and $1 < p < \infty$. Let $a_{i,j}$ satisfy (11.7), and let $c \in L^\infty(\Omega)$.*

(i) *If $u \in W_0^{1,p}(\Omega)$ satisfies*

$$[\mathcal{A}u, v] = < f, v >_{[W^{-1,p};W_0^{1,p'}](\Omega)} \text{ for all } v \in W_0^{1,p'}(\Omega)$$

for a certain $f \in W^{-1,p}(\Omega)$, then

$$\|u\|_{W_0^{1,p}(\Omega)} \leq c\left(\|f\|_{W^{-1,p}(\Omega)} + \|u\|_{W^{-1,p}(\Omega)}\right).$$

(ii) *If $u \in W^{1,p}(\Omega)$ satisfies*

$$[\mathcal{A}u, v] = < F, v >_{[[W^{1,p'}]^*; W^{1,p'}](\Omega)} \text{ for all } v \in W^{1,p'}(\Omega)$$

for a certain $F \in [W^{1,p'}]^*(\Omega)$, then

$$\|u\|_{W^{1,p}(\Omega)} \leq c\left(\|F\|_{[W^{1,p'}]^*(\Omega)} + \|u\|_{[W^{1,p'}]^*(\Omega)}\right).$$

In particular, if

$$[\mathcal{A}u, v] = \int_\Omega f v \, \mathrm{d}x - \int_{\partial\Omega} g v \, \mathrm{d}S_x \text{ for all } v \in W^{1,p'}(\Omega),$$

then

$$\|u\|_{W^{1,p}(\Omega)} \leq c\left(\|f\|_{[W^{1,p'}]^*(\Omega)} + \|g\|_{W^{-1/p,p}(\partial\Omega)} + \|u\|_{[W^{1,p'}]^*(\Omega)}\right).$$

See Lions [190], Schechter [242]. □

Remark The hypothesis concerning regularity of the boundary can be relaxed to $C^{0,1}$ in the case of the Dirichlet boundary condition, and to $C^{1,1}$ for the Neumann boundary condition.

Remark The norm containing u on the right-hand side of the estimates in Theorems 11.4–11.6 is irrelevant and may be omitted provided that the solution is unique in the given class.

Remark As we have observed, elliptic operators, in general, enjoy the degree of regularity allowed by the data. In particular, the solutions of elliptic problems with constant or (real) *analytic coefficients* are *analytic* on any open subset of their domain of definition. For example, if

$$\Delta u + \mathbf{b} \cdot \nabla_x u + cu = f \text{ in } \Omega \subset \mathbb{R}^N,$$

where $\mathbf{b}$, c are constant, and Ω is a domain, then $\mathbf{u}$ is analytic in Ω provided that f is analytic (see John [162, Chap. VII]). The result can be extended to elliptic systems and even up to the boundary provided the latter is analytic (see Morrey and Nirenberg [216]).

11.3.2 Fredholm Alternative

Now, we focus on the problem of *existence*. Given the scope of applications considered in this book, we consider only the Neumann problem, specifically $\delta = 0$ in system (11.5), (11.6). Similar results hold also for the Dirichlet boundary conditions. A useful tool is the *Fredholm alternative* formulated in the following theorem.

■ FREDHOLM ALTERNATIVE:

Theorem 11.7 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. In addition to (11.7), assume that $c \in C(\overline{\Omega})$, $1 < p < \infty$, $k = 1, 2$, and $\delta = 0$.*

Then either

(i) *Problem (11.5), (11.6) possesses a unique solution $u \in W^{k,p}(\Omega)$ for any f, g belonging to the regularity class*

$$f \in [W^{1,p'}(\Omega)]^*,\ g \in W^{-\frac{1}{p},p}(\partial\Omega) \text{ if } k = 1, \tag{11.9}$$

$$f \in L^p(\Omega),\ g \in W^{1-\frac{1}{p},p}(\partial\Omega) \text{ if } k = 2; \tag{11.10}$$

or

(ii) *the null space*

$$\ker[\mathcal{A}] = \{u \in W^{k,p}(\Omega) \mid u \textit{ solve (11.5), (11.6) with } f = g = 0\}$$

is of finite dimension, and problem (11.5), (11.6) admits a solution for f, g belonging to the class (11.9), (11.10) only if

$$< f; w >_{[[W^{1,p'}]^*;W^{1,p'}](\Omega)} - < g; w >_{[W^{-1/p,p},W^{1/p,p'}](\partial\Omega)} = 0$$

for all $w \in \ker[\mathcal{A}]$.

See Amann [7, Theorem 9.2], Geymonat and Grisvard [135]. □

In the concrete cases, the Fredholm alternative gives existence of a solution u while the estimates of u in $W^{k,p}(\Omega)$ in terms of f and g follow from Theorems 11.5 and 11.6 via a uniqueness contradiction argument.

For example, in the sequel, we shall deal with a simple Neumann problem for generalized Laplacian

$$-\mathrm{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta} \right) \right) = f \text{ in } \Omega, \quad \nabla_x \left(\frac{v}{\eta} \right) \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

where η is a sufficiently smooth and positive function on $\overline{\Omega}$ and $f \in L^p(\Omega)$ with a certain $1 < p < \infty$. In this case the Fredholm alternative guarantees existence of $u \in W^{2,p}(\Omega)$ provided $f \in L^p(\Omega)$, $\int_\Omega f \mathrm{d}x = 0$. The solution is unique in the class $u \in W^{2,p}(\Omega)$, $\int_\Omega \frac{u}{\eta} \mathrm{d}x = 0$ and satisfies estimate

$$\|u\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)}.$$

11.3.3 Spectrum of a Generalized Laplacian

We begin by introducing a densely defined (unbounded) linear operator

$$\Delta_{\eta,\mathcal{N}} = \mathrm{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta} \right) \right), \tag{11.11}$$

with the function η to be specified later, acting from $L^p(\Omega)$ to $L^p(\Omega)$ with domain of definition

$$\mathcal{D}(\Delta_{\eta,\mathcal{N}}) = \{u \in W^{2,p}(\Omega) \mid \nabla_x \left(\frac{v}{\eta} \right) \cdot \mathbf{n}|_{\partial\Omega} = 0\}. \tag{11.12}$$

Further we denote $\Delta_{\mathcal{N}} = \Delta_{1,\mathcal{N}}$ the classical Laplacian with the homogenous Neumann boundary condition.

We shall apply the results of Sects. 11.3.1–11.3.2 to the spectral problem that consists in finding couples (λ, v), $\lambda \in \mathbb{C}$, $v \in \mathcal{D}(\Delta_{\eta,\mathcal{N}})$ that verify

$$-\mathrm{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta} \right) \right) = \lambda v \text{ in } \Omega, \ \nabla_x \left(\frac{v}{\eta} \right) \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

The results announced in the main theorem of this section are based on a general theorem of functional analysis concerning the spectral properties of compact operators.

Let $T : X \to X$ be a linear operator on a Hilbert space X endowed with scalar product $< \cdot ; \cdot >$. We say that a complex number λ belongs to the *spectrum* of T (one writes $\lambda \in \sigma(T)$) if $\ker(T - \lambda\mathbb{I}) \neq \{0\}$ or if $(T - \lambda\mathbb{I})^{-1} : X \to X$ is not a bounded linear operator (here $\mathbb{I}$ denotes the identity operator). We say that λ is an *eigenvalue* of T or belongs to the *discrete (pointwise) spectrum* of T (and write $\lambda \in \sigma_p(T) \subset \sigma(T)$) if $\ker(T - \lambda\mathbb{I}) \neq \{0\}$. In the latter case, the non zero vectors belonging to $\ker(T - \lambda\mathbb{I})$ are called *eigenvectors* and the vector space $\ker(T - \lambda\mathbb{I})$ *eigenspace*.

■ SPECTRUM OF A COMPACT OPERATOR:

Theorem 11.8 *Let H be an infinite dimensional Hilbert space and $T : H \to H$ a compact linear operator. Then*

(i) $0 \in \sigma(T)$;
(ii) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$;
(iii)

$$\begin{cases} \sigma(T) \setminus \{0\} \text{ is finite, or else} \\ \sigma(T) \setminus \{0\} \text{ is a sequence tending to } 0. \end{cases}$$

(iv) *If $\lambda \in \sigma(T) \setminus \{0\}$, then the dimension of the eigenspace* $\ker(T - \lambda \mathbb{I})$ *is finite.*
(v) *If T is a* positive operator, *meaning $< Tv; v > \geq 0$, $v \in H$, then $\sigma(T) \subset [0, +\infty)$.*
(vi) *If T is a* symmetric operator, *meaning $< Tv; w >=< v; Tw >$, $v, w \in H$, then $\sigma(T) \subset \mathbb{R}$. If in addition H is separable, then H admits an orthonormal basis of eigenvectors that consists of eigenvectors of T.*

See Evans [96, Chap. D, Theorems 6,7]

The main theorem of this section reads:

■ SPECTRUM OF THE GENERALIZED LAPLACIAN WITH NEUMANN BOUNDARY CONDITION:

Theorem 11.9 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. Let*

$$\eta \in C^1(\overline{\Omega}), \quad \inf_{x \in \Omega} \eta(x) = \underline{\eta} > 0.$$

Then the spectrum of the operator $-\Delta_{\eta,\mathcal{N}}$, where $\Delta_{\eta,\mathcal{N}}$ is defined in (11.11)–(11.12), coincides with the discrete spectrum and the following holds:

(i) *The spectrum consists of a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of real eigenvalues, where $\lambda_0 = 0$, $0 < \lambda_k < \lambda_{k+1}$, $k = 1, 2, \ldots$, and $\lim_{k\to\infty} \lambda_k = \infty$;*
(ii) $0 < \dim(E_k) < \infty$ *and* $E_0 = \mathrm{span}\{\eta\}$, *where* $E_k = \ker(-\Delta_{\eta,\mathcal{N}} - \lambda_k \mathbb{I})$ *is the eigenspace corresponding to the eigenvalue λ_k;*
(iii) $L^2(\Omega) = \bigoplus_{k=0}^{\infty} E_k$, *where the direct sum is orthogonal with respect to the scalar product*

$$< u; v >_{1/\eta} = \int_\Omega u\overline{v}\, \frac{\mathrm{d}x}{\eta}$$

(here the line over v means the complex conjugate of v).

Proof We set

$$T : L^2(\Omega) \to L^2(\Omega), \quad Tf = \begin{cases} -\Delta_{\eta,\mathcal{N}}^{-1} f \text{ if } f \in \dot{L}^2(\Omega), \\ \\ 0 \text{ if } f \in \text{span}\{1\}, \end{cases}$$

$$\Delta_{\eta,\mathcal{N}}^{-1} : \dot{L}^2(\Omega) = \{f \in L^2(\Omega) \mid \int_\Omega f \, \mathrm{d}x = 0\} \mapsto \{u \in L^2(\Omega) \mid \int_\Omega \frac{u}{\eta} \, \mathrm{d}x = 0\},$$

$$-\Delta_{\eta,\mathcal{N}}^{-1} f = u \;\Leftrightarrow\; -\Delta_{\eta,\mathcal{N}} u = f.$$

In accordance with the regularity properties of elliptic operators collected in Sects. 11.3.1–11.3.2 (see notably Theorems 11.5 and 11.7), the operator T is a compact operator.

A double integration by parts yields

$$-\int_\Omega \text{div}_x \left(\eta \nabla_x \left(\frac{v}{\eta} \right) \right) u \, \frac{\mathrm{d}x}{\eta} = \int_\Omega \eta \nabla_x \left(\frac{v}{\eta} \right) \cdot \nabla_x \left(\frac{u}{\eta} \right) \mathrm{d}x =$$

$$-\int_\Omega \text{div}_x \left(\eta \nabla_x \left(\frac{u}{\eta} \right) \right) v \, \frac{\mathrm{d}x}{\eta}.$$

Taking in the last formula $u = Tf, f \in L^2(\Omega)$, $v = Tg$, $g \in L^2(\Omega)$ and recalling that functions $\frac{Tf}{\eta}$, $\frac{Tg}{\eta}$ have zero mean, we deduce that

$$\int_\Omega Tf \, g \, \frac{\mathrm{d}x}{\eta} = \int_\Omega f \, Tg \, \frac{\mathrm{d}x}{\eta} \quad \text{and} \quad \int_\Omega Tf \, \overline{f} \, \frac{\mathrm{d}x}{\eta} \geq 0.$$

To resume, we have proved that T is a compact positive linear operator on $L^2(\Omega)$ that is symmetric with respect to the scalar product $< \cdot ; \cdot >_{1/\eta}$. Now, all statements of Theorem 11.9 follow from Theorem 11.8.

□

11.3.4 Neumann Laplacian on Unbounded Domains

In this section, $\Omega \subset R^N$, $N = 2, 3$ is an unbounded *exterior* domain,

$$\Omega = R^N \setminus B,$$

where B is a compact set (the case $B = \emptyset$, $\Omega = R^N$ included). We consider the Neumann Laplacian $\Delta_{\mathcal{N},\Omega}$ defined for sufficiently smooth functions decaying at infinity as

$$\Delta_{\mathcal{N},\Omega}[v] = \Delta v \text{ in } \Omega,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0,\ v(x) \to 0 \text{ as } |x| \to \infty.$$

The standard notation Δ will be used for the same operator if $\Omega = R^N$. Conventionally, the operator $(-\Delta_{\mathcal{N},\Omega})$ is usually considered being self-adjoint and *non-negative*.

In order to apply the abstract spectral theory introduced in Sect. 11.1, we define $(-\Delta_{\mathcal{N},\Omega})$ on the Hilbert space $L^2(\Omega)$ in the following way:

For $u \in D^{1,2}(\Omega), f \in L^2(\Omega)$, *we say that*

$$-\Delta_{\mathcal{N},\Omega}[v] = f \text{ only if} \int_\Omega \nabla_x v \cdot \nabla_x \varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\overline{\Omega}).$$

The domain of $-\Delta_{\mathcal{N},\Omega}$ *in the Hilbert space* $L^2(\Omega)$ *is defined as*

$$\mathcal{D}(-\Delta_{\mathcal{N},\Omega}) = \left\{ v \in L^2(\Omega) \cap D^{1,2}(\Omega) \,\middle|\, -\Delta_{\mathcal{N},\Omega}[v] = f,\ f \in L^2(\Omega) \right\}.$$

If $\partial\Omega$ is at least of class C^2, then $-\Delta_{\mathcal{N},\Omega}$ is a densely defined self-adjoint operator on the Hilbert space $L^2(\Omega)$, with

$$\mathcal{D}(-\Delta_{\mathcal{N},\Omega}) = \left\{ v \in W^{2,2}(\Omega) \,\middle|\, \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the sense of traces} \right\},$$

see e.g. Leis [183].

■ RELLICH'S THEOREM:

Theorem 11.10 *Let* $\Omega \subset R^N$, $N = 2, 3$ *be an exterior domain with* C^2 *boundary. Suppose that*

$$-\Delta u(x) + q(x)u(x) = \lambda u(x) \ \in \Omega,\ \lambda > 0,$$

where q *is Hölder continuous in* $\overline{\Omega}$ *and*

$$|x|q(x) \to 0 \text{ as } |x| \to \infty.$$

Then if $u \in L^2(\{|x| > r_0\})$ *for a certain* $r_0 > 0$, *then*

$$u \equiv 0 \text{ in } \Omega.$$

See Eidus [91, Theorem 2.1] □

As an immediate corollary of Rellich's theorem we deduce that $(-\Delta_{\mathcal{N},\Omega})$ defined on an exterior domain has no eigenvalues—its point spectrum is empty. More specifically, we report the following result.

■ SPECTRUM OF NEUMANN LAPLACIAN ON EXTERIOR DOMAIN:

Theorem 11.11 *Let $\Omega \subset R^3$ be an exterior domain with C^2 boundary.*

Then $-\Delta_{\mathcal{N},\Omega}$ is a non-negative self-adjoint operator with absolutely continuous spectrum $[0,\infty)$—all spectral projection are absolutely continuous with respect to the Lebesgue measure. In addition, $-\Delta_{\mathcal{N},\Omega}$ satisfies the Limiting absorption principle *(LAP):*

$$\left\{ \begin{array}{c} \textit{Operators} \\ \\ \mathcal{V} \circ (-\Delta_{\mathcal{N},\Omega} - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega),\ \mathcal{V}[v] = (1+|x|^2)^{-s/2},\ s > 1 \\ \\ \textit{are bounded uniformly for } \lambda \in [\alpha, \beta],\ 0 < \alpha < \beta,\ \eta > 0, \end{array} \right\}$$

See Leis [183] □

We recall "negative" L^p-estimates for the Neumann Laplacian on exterior domains.

■ NEGATIVE L^p-ESTIMATES FOR THE NEUMANN LAPLACIAN ON EXTERIOR DOMAIN:

Theorem 11.12 *Let $\Omega \subset R^N$ be an exterior domain with C^2 boundary. Then for any $\mathbf{w} \in C_c^\infty(\Omega)$, the problem*

$$\int_\Omega \nabla_x u \cdot \nabla_x \varphi \, \mathrm{d}x = \int_\Omega \mathbf{w} \cdot \nabla_x \varphi \, \mathrm{d}x \textit{ for all } \varphi \in C_c^\infty(\overline{\Omega}) \tag{11.13}$$

admits a unique solution $u \in \mathcal{D}(-\Delta_{\mathcal{N},\Omega})$. Moreover, $u \in D^{1,p}(\Omega)$ and

$$\|\nabla_x u\|_{L^p(\Omega;R^N)} \le c(p)\|\mathbf{w}\|_{L^p(\Omega;R^N)} \textit{ for any } 1 < p < \infty.$$

See e.g. Galdi [131] □

Finally, we consider the operator $U = \exp\left(\pm \mathrm{i}t\sqrt{-\Delta_{\mathcal{N},\Omega}}\right)[h]$ that appears in the variation-of-constants formula associated to the wave equation

$$\partial^2_{t,t} U - \Delta U = 0,\ \nabla_x U \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

■ STRICHARTZ ESTIMATES FOR THE FLAT LAPLACIAN ON R^N:

Theorem 11.13 *Let Δ be the $L^2(R^N)$ self-adjoint realization of the Laplacian defined on the whole space R^N. Suppose that*

$$N \geq 2,\ 2 \leq p < \infty,\ 2 \leq q < \infty,\ \gamma = \frac{N}{2} - \frac{N}{q} - \frac{1}{p},\ \frac{2}{p} \leq \frac{N-1}{2}\left(1 - \frac{2}{q}\right).$$

Then

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm \mathrm{i}t\sqrt{-\Delta}\right)[h] \right\|_{L^q(R^N)}^p \,\mathrm{d}t \leq c(N,p,q,\gamma)\|h\|_{H^{\gamma,2}(R^N)}^p.$$

See Keel and Tao [168] □

Remark Here $H^{\gamma,2}$ denotes the homogeneous Sobolev space of functions having derivatives of order γ square integrable. The norm in $H^{\gamma,2}(R^N)$ can be defined via Fourier transform

$$\|v\|_{H^{\gamma,2}(R^N)}^2 = \int_{\mathbb{R}^N} |\xi|^{2\gamma} \left|\mathcal{F}_{x\to\xi}[v](\xi)\right|^2 \,\mathrm{d}\xi.$$

11.4 Normal Traces

Let Ω be a bounded domain in $\mathbb{R}^N$. For $1 \leq q, p \leq \infty$, we introduce a Banach space

$$E^{q,p}(\Omega) = \{\mathbf{u} \in L^q(\Omega;\mathbb{R}^N)|\ \mathrm{div}\mathbf{u} \in L^p(\Omega)\}. \tag{11.14}$$

endowed with norm

$$\|\mathbf{u}\|_{E^q(\Omega)} := \|\mathbf{u}\|_{E^q(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\mathbf{u}\|_{L^p(\Omega)}. \tag{11.15}$$

We also define

$$E_0^{q,p}(\Omega) = \mathrm{closure}_{E^{q,p}(\Omega)}\left\{C_c^\infty(\Omega;\mathbb{R}^N)\right\}$$

and

$$E^p(\Omega) = E^{p,p}(\Omega), \quad E_0^p(\Omega) = E_0^{p,p}(\Omega).$$

Our goal is to introduce the concept of *normal traces* and to derive a variant of Green's formula for the functions belonging to $E^{q,p}(\Omega)$.

■ NORMAL TRACES:

Theorem 11.14 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and let $1 < p < \infty$. Then there exists a unique linear operator $\gamma_{\mathbf{n}}$ with the following properties:*

(i)

$$\gamma_{\mathbf{n}} : E^p(\Omega) \mapsto [W^{1-\frac{1}{p'},p'}(\partial\Omega)]^* := W^{-\frac{1}{p},p}(\partial\Omega), \tag{11.16}$$

and

$$\gamma_{\mathbf{n}}(\mathbf{u}) = \gamma_0(\mathbf{u}) \cdot \mathbf{n} \text{ a.a. on } \partial\Omega \text{ whenever } \mathbf{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^N). \tag{11.17}$$

(ii) The Stokes formula

$$\int_\Omega v \operatorname{div} \mathbf{u} \, \mathrm{d}x + \int_\Omega \nabla v \cdot \mathbf{u} \, \mathrm{d}x = < \gamma_{\mathbf{n}}(\mathbf{u}) ; \gamma_0(v) >, \tag{11.18}$$

holds for any $\mathbf{u} \in E^p(\Omega)$ and $v \in W^{1,p'}(\Omega)$, where $< \cdot ; \cdot >$ denotes the duality pairing between $W^{1-\frac{1}{p'},p'}(\Omega)$ and $W^{-\frac{1}{p},p}(\Omega)$.

(iii)

$$\ker[\gamma_{\mathbf{n}}] = E_0^p(\Omega). \tag{11.19}$$

(iv) If $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N)$, then $\gamma_{\mathbf{n}}(\mathbf{u})$ in $L^p(\partial\Omega)$, and $\gamma_{\mathbf{n}}(\mathbf{u}) = \gamma_0(\mathbf{u}) \cdot \mathbf{n}$ a.a. on $\partial\Omega$.

Proof of Theorem 11.14 As a matter of fact, Theorem 11.14 is a standard result whose proof can be found in Temam [256, Chap. 1]. We give a concise proof based on the following three lemmas that may be of independent interest.

Step 1 We start with a technical result, the proof of which can be found in Galdi [131, Lemma 3.2]. We recall that a domain $Q \subset \mathbb{R}^N$ is said to be *star-shaped* if there exists $a \in Q$ such that $Q = \{x \in \mathbb{R}^N \mid |x-a| < h(\frac{x-a}{|x-a|})\}$, where h is a positive continuous function on the unit sphere; it is said star-shaped with respect to a ball $B \subset Q$ if it is star-shaped with respect to any of its points.

Lemma 11.1 *Let Ω be a bounded Lipschitz domain.*

Then there exists a finite family of open sets $\{\mathcal{O}_i\}_{i \in I}$ and a family of balls $\{B^{(i)}\}_{i \in I}$ such that each $\Omega_i := \Omega \cap \mathcal{O}_i$ is star-shaped with respect to the ball $B^{(i)}$, and

$$\overline{\Omega} \subset \cup_{i \in I} \mathcal{O}_i.$$

Step 2 The main ingredient of the proof of Theorem 11.14 is the density of smooth functions in the spaces $E^{q,p}(\Omega)$.

Lemma 11.2 *Let Ω be a bounded Lipschitz domain and $1 \le p \le q < \infty$. Then $C^\infty(\overline{\Omega};\mathbb{R}^N) = C_c^\infty(\overline{\Omega})$ is dense in $E^{q,p}(\Omega)$.*

Proof of Lemma 11.2 Hypothesis $q \ge p$ is of technical character and can be relaxed if, for instance, Ω is of class $C^{1,1}$. It ensures that $\mathbf{u}\varphi \in E^{q,p}(\Omega)$ as soon as $\varphi \in C_c^\infty(\Omega)$. Moreover, according to Lemma 11.1, any bounded Lipschitz domain can be decomposed as a finite union of *star-shaped domains* with respect to a ball. Using the corresponding subordinate partition of unity we may assume, without loss of generality, that Ω is a starshaped domain with respect to a ball centered at the origin of the Cartesian coordinate system.

For $\mathbf{u} \in E^{q,p}(\Omega)$ we denote $\mathbf{u}_\tau(x) = \mathbf{u}(\tau x)$, $\tau > 0$, so that if $\tau \in (0,1)$, $\mathbf{u}_\tau \in E^{q,p}(\tau^{-1}\Omega)$ and $\mathrm{div}(\mathbf{u}_\tau) = \tau(\mathrm{div}\mathbf{u})_\tau$ in $\mathcal{D}'(\tau^{-1}\Omega)$, where $\tau^{-1}\Omega = \{x \in \mathbb{R}^N \mid \tau x \in \Omega\}$. We therefore have

$$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_\tau)\|_{L^p(\Omega)} \le (1-\tau)\|\mathrm{div}\mathbf{u}\|_{L^p(\Omega)} + \|\mathrm{div}\mathbf{u} - (\mathrm{div}\mathbf{u})_\tau\|_{L^p(\Omega)}. \tag{11.20}$$

Since the translations $\mathbb{R}^N \ni h \to u(\cdot + h) \in L^s(\mathbb{R}^N)$ are continuous for any fixed $u \in L^s(\mathbb{R}^N)$, $1 \le s < \infty$, the right hand side of formula (11.20) as well as $\|\mathbf{u}-\mathbf{u}_\tau\|_{L^q(\Omega)}$ tend to zero as $\tau \to 1-$. Thus it is enough to prove that $\mathbf{u}_\tau$ can be approximated in $E^{q,p}(\Omega)$ by functions belonging to $C^\infty(\overline{\Omega};\mathbb{R}^N)$.

Since $\overline{\Omega} \subset \tau^{-1}\Omega$, the mollified functions $\zeta_\epsilon * \mathbf{u}_\tau$ belong to $C^\infty(\overline{\Omega};\mathbb{R}^N) \cap E^{q,p}(\Omega)$ provided $0 < \varepsilon < \mathrm{dist}(\Omega, \partial(\tau^{-1}\Omega))$ and tend to $\mathbf{u}_\tau$ in $E^{q,p}(\Omega)$ as $\varepsilon \to 0+$ (see Theorem 11.3). This observation completes the proof of Lemma 11.2.

□

Step 3 We are now in a position to define the operator of normal traces. Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, $v \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$, and $\mathbf{u} \in C^\infty(\overline{\Omega};\mathbb{R}^N)$. According to the trace theorem (see Theorem 6), we have

$$\int_{\partial\Omega} v\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma = \int_\Omega \ell(v)\mathrm{div}\mathbf{u}\,\mathrm{d}x + \int_\Omega \nabla\ell(v)\cdot\mathbf{u}\,\mathrm{d}x,$$

and

$$\left|\int_{\partial\Omega} v\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma\right| \le \|\mathbf{u}\|_{E^p(\Omega)}\,\|\ell(v)\|_{W^{1,p'}(\Omega)} \le c(p,\Omega)\|\mathbf{u}\|_{E^p(\Omega)}\,\|v\|_{W^{1-1/p',p'}(\partial\Omega)},$$

where the first identity is independent of the choice of the lifting operator ℓ. Consequently, the map

$$\gamma_{\mathbf{n}} : \mathbf{u} \to \gamma_0(\mathbf{u}) \cdot \mathbf{n} \tag{11.21}$$

is a linear densely defined (on $C^\infty(\overline{\Omega})$) and continuous operator from $E^p(\Omega)$ to $[W^{1-1/p',p'}(\partial\Omega)]^* = W^{-\frac{1}{p},p}(\partial\Omega)$. Its value at $\mathbf{u}$ is termed the *normal trace* of $\mathbf{u}$ on $\partial\Omega$ and denoted $\gamma_{\mathbf{n}}(\mathbf{u})$ or $(\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega}$.

Step 4 In order to complete the proof of Theorem 11.14, it remains to show that $\ker[\gamma_{\mathbf{n}}] = E^p_0(\Omega)$.

Lemma 11.3 *Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, and let $\gamma_{\mathbf{n}} : E^p(\Omega) \to W^{-\frac{1}{p},p}(\partial\Omega)$ be the operator defined as a continuous extension of the trace operator introduced in (11.21). Then* $\ker[\gamma_{\mathbf{n}}] = E^p_0(\Omega)$.

Proof of Lemma 11.3 Clearly, $C^\infty_c(\Omega) \subset \ker[\gamma_{\mathbf{n}}]$; whence, by continuity of $\gamma_{\mathbf{n}}$, $E^p_0(\Omega) \subset \ker[\gamma_{\mathbf{n}}]$.

Conversely, we set

$$\tilde{\mathbf{u}}(x) = \begin{cases} \mathbf{u}(x) \text{ if } x \in \Omega, \\ \\ 0 \text{ otherwise.} \end{cases}$$

Assumption $\mathbf{u} \in \ker[\gamma_{\mathbf{n}}]$ yields $\int_\Omega v \operatorname{div}\mathbf{u}\,\mathrm{d}x + \int_\Omega \nabla v \cdot \mathbf{u}\,\mathrm{d}x = 0$ for all $v \in C^\infty_c(\mathbb{R}^N)$, meaning that, in the sense the distributions,

$$\operatorname{div}\tilde{\mathbf{u}}(x) = \left\{ \begin{array}{c} \operatorname{div}\mathbf{u}(x) \text{ if } x \in \Omega, \\ \\ 0 \text{ otherwise} \end{array} \right\} \in L^p(\mathbb{R}^N),$$

and, finally, $\tilde{\mathbf{u}} \in E^p(\mathbb{R}^N)$.

In agreement with Lemma 11.2, we suppose, without loss of generality, that Ω is starshaped with respect to the origin of the coordinate system. Similarly to Lemma 11.2, we deduce that $\operatorname{supp}[(\tilde{\mathbf{u}}_{1/\tau})]$ belongs to the set $\overline{\tau\Omega} \subset \Omega$, and, moreover, $\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_{1/\tau}\|_{E^p(\Omega)} \to 0$ as $\tau \to 1-$.

Consequently, it is enough to approximate $\tilde{\mathbf{u}}_{1/\tau}$ by a suitable function belonging to the set $C^\infty_c(\Omega;\mathbb{R}^N)$. However, according Theorem 11.3, functions $\zeta_\varepsilon * \mathbf{u}_{1/\tau}$ belong to $C^\infty_c(\Omega) \cap E^p(\Omega)$ provided $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(\tau\Omega, \partial\Omega)$, and $\zeta_\varepsilon * \tilde{\mathbf{u}}_{1/\tau} \to \tilde{\mathbf{u}}_{1/\tau}$ in $E^p(\Omega)$. This completes the proof of Lemma 11.3 as well as that of Theorem 11.14.

□

11.5 Singular and Weakly Singular Operators

The *weakly singular integral transforms* are defined through formula

$$[T(f)](x) = \int_{\mathbb{R}^N} K(x, x-y) f(y)\, \mathrm{d}y, \tag{11.22}$$

where

$$K(x,z) = \frac{\theta(x,z)}{|z|^\lambda}, \quad 0 < \lambda < N, \quad \theta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N). \tag{11.23}$$

A function K satisfying (11.23) is called *weakly singular kernel*.

The *singular integral transforms* are defined as

$$[T(f)](x) = \lim_{\varepsilon \to 0+} \Big(\int_{|x-y| \geq \varepsilon} K(x, x-y) f(y)\, \mathrm{d}y \Big) := \text{v.p.} \int_{\mathbb{R}^N} K(x, x-y) f(y)\, \mathrm{d}y, \tag{11.24}$$

where

$$\begin{aligned} K(x,z) &= \tfrac{\theta(x, z/|z|)}{|z|^N}, \quad \theta \in L^\infty(\mathbb{R}^N \times S), \\ S &= \{z \in \mathbb{R}^N \mid |z| = 1\}, \quad \textstyle\int_{|z|=1} \theta(x,z)\, \mathrm{d}S_z = 0. \end{aligned} \tag{11.25}$$

The kernels satisfying (11.25) are called *singular kernels of Calderón-Zygmund type*.

The basic result concerning the weakly singular kernels is the *Sobolev theorem*.

■ WEAKLY SINGULAR INTEGRALS:

Theorem 11.15 *The operator T defined in (11.22) with K satisfying (11.23) is a bounded linear operator on $L^q(\mathbb{R}^N)$ with values in $L^r(\mathbb{R}^N)$, where $1 < q < \infty$, $\frac{1}{r} = \frac{\lambda}{N} + \frac{1}{q} - 1$. In particular,*

$$\|T(f)\|_{L^r(\mathbb{R}^N)} \leq c \|f\|_{L^q(\mathbb{R}^N)},$$

where the constant c can be expressed in the form $c_0(q,N)\|\theta\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)}$.

See Stein [251, Chap. V, Theorem 1] □

The fundamental result concerning the singular kernels is the *Calderón-Zygmund theorem*.

■ SINGULAR INTEGRALS:

Theorem 11.16 *The operator T defined in (11.24) with K satisfying (11.25) is a bounded linear operator on $L^q(\mathbb{R}^N)$ for any $1 < q < \infty$. In particular,*

$$\|T(f)\|_{L^q(\mathbb{R}^N)} \leq c\|f\|_{L^q(\mathbb{R}^N)},$$

where the constant c takes the form $c = c_0(q,N)\|\theta\|_{L^\infty(\mathbb{R}^N\times S)}$.

See Calderón-Zygmund [46, Theorem 2], [47, Sect. 5, Theorem 2]. □

11.6 The Inverse of the div-Operator (Bogovskii Formula)

We consider the problem

$$\mathrm{div}_x\mathbf{u} = f \text{ in } \Omega,\ \mathbf{u}|_{\partial\Omega} = 0 \tag{11.26}$$

for a given function f, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Clearly, problem (11.26) admits many solutions that may be constructed in different manners. Here, we adopt the integral formula proposed by Bogovskii [28] and elaborated by Galdi [131]. In such a way, we resolve (11.26) for any smooth f of zero integral mean. In addition, we deduce uniform estimates that allow us to extend solvability of (11.26) to a significantly larger class of right-hand sides f, similarly to Geissert et al. [134]. The main advantage of our construction is that it requires only Lipschitz regularity of the underlying spatial domain. Extensions to other geometries including unbounded domains are possible. We recommend the interested reader to consult the monograph by Galdi [131] or [224, Chap. III] for both positive and negative results in this direction.

Our result are summarized in the following theorem.

■ THE INVERSE OF THE DIV-OPERATOR:

Theorem 11.17 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

(i) *Then there exists a linear mapping $\mathcal{B}$,*

$$\mathcal{B} : \{f \mid f \in C_c^\infty(\Omega),\ \int_\Omega f \,\mathrm{d}x = 0\} \to C_c^\infty(\Omega;\mathbb{R}^N),$$

such that $\mathrm{div}_x(\mathcal{B}[f]) = f$, meaning, $\mathbf{u} = \mathcal{B}[f]$ solves (11.26).

(ii) *We have*

$$\|\mathcal{B}[f]\|_{W^{k+1,p}(\Omega;\mathbb{R}^N)} \leq c\|f\|_{W^{k,p}(\Omega)} \text{ for any } 1 < p < \infty,\ k = 0,1,\dots, \tag{11.27}$$

in particular, $\mathcal{B}$ can be extended in a unique way as a bounded linear operator

$$\mathcal{B} : \{f \mid f \in L^p(\Omega), \int_\Omega f \, \mathrm{d}x = 0\} \to W_0^{1,p}(\Omega; \mathbb{R}^N).$$

(iii) *If $f \in L^p(\Omega)$, $\int_\Omega f \, \mathrm{d}x = 0$, and, in addition, $f = \mathrm{div}_x \mathbf{g}$, where $\mathbf{g} \in E_0^{q,p}(\Omega)$, $1 < q < \infty$, then*

$$\|\mathcal{B}[f]\|_{L^q(\Omega;\mathbb{R}^3)} \leq c\|\mathbf{g}\|_{L^q(\Omega;\mathbb{R}^3)}. \tag{11.28}$$

(iv) *$\mathcal{B}$ can be uniquely extended as a bounded linear operator*

$$\mathcal{B} : [\dot{W}^{1,p'}(\Omega)]^* = \{f \in [W^{1,p'}(\Omega)]^* \mid \; < f; 1 >= 0\} \to L^p(\Omega; \mathbb{R}^N)$$

in such a way that

$$- \int_\Omega \mathcal{B}[f] \cdot \nabla v \, \mathrm{d}x =< f; v >_{\{[W^{1,p'}]^*; W^{1,p'}\}(\Omega)} \text{ for all } v \in W^{1,p'}(\Omega), \tag{11.29}$$

$$\|\mathcal{B}[f]\|_{L^p(\Omega;R^N)} \leq c\|f\|_{[W^{1,p'}(\Omega)]^*}. \tag{11.30}$$

Here, a function $f \in C_c^\infty(\Omega)$ is identified with a linear form in $[W^{1,p'}(\Omega)]^*$ via the standard Riesz formula

$$< f; v >_{[W^{1,p'}(\Omega)]^*; W^{1,p'}(\Omega)} = \int_\Omega f v \, \mathrm{d}x \text{ for all } v \in W^{1,p'}(\Omega). \tag{11.31}$$

Remark Since $\mathcal{B}$ is linear, it is easy to check that

$$\partial_t \mathcal{B}[f](t,x) = \mathcal{B}[\partial_t f](t,x) \text{ for a.a. } (t,x) \in (0,T) \times \Omega \tag{11.32}$$

provided

$$\partial_t f, \, f \in L^p((0,T) \times \Omega), \; \int_\Omega f(t,\cdot) \, \mathrm{d}x = 0 \text{ for a.a. } t \in (0,T).$$

The proof of Theorem 11.17 is given by means of several steps which may be of independent interest.

Step 1 The first ingredient of the proof is a representation formula for functionals belonging to $[\dot{W}^{1,p'}(\Omega)]^*$.

Lemma 11.4 *Let Ω be a domain in $\mathbb{R}^N$, and let $1 < p \leq \infty$.*
Then any linear form $f \in [\dot{W}^{1,p'}(\Omega)]^*$ admits a representation

$$< f; v >_{[\dot{W}^{1,p'}(\Omega)]^*, \dot{W}^{1,p'}(\Omega)} = \sum_{i=1}^{N} \int_{\Omega} w_i \partial_{x_i} v \, \mathrm{d}x,$$

where

$$\mathbf{w} = [w_1, \dots, w_N] \in L^p(\Omega; \mathbb{R}^N) \text{ and } \|f\|_{[\dot{W}^{1,p'}(\Omega)]^*} = \|\mathbf{w}\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Proof of Lemma 11.4 The operator $I : \dot{W}^{1,p'}(\Omega) \to L^{p'}(\Omega; \mathbb{R}^N)$, $I(u) = \nabla u$ is an isometric isomorphism mapping $\dot{W}^{1,p'}(\Omega)$ onto a (closed) subspace $I(\dot{W}^{1,p'}(\Omega))$ of $L^{p'}(\Omega; \mathbb{R}^N)$. The functional ϕ defined as

$$< \phi; \nabla u > := < f; u >_{[\dot{W}^{1,p'}(\Omega)]^*, \dot{W}^{1,p'}(\Omega)}$$

is a linear functional on $I(\dot{W}^{1,p'}(\Omega))$ satisfying condition

$$\sup \left\{ < \phi; \mathbf{v} > \mid \mathbf{v} \in I(\dot{W}^{1,p'}(\Omega)), \ \|\mathbf{v}\|_{L^{p'}(\Omega;\mathbb{R}^N)} \leq 1 \right\} = \|f\|_{[\dot{W}^{1,p'}(\Omega)]^*}.$$

Therefore by the Hahn-Banach theorem (see e.g. Brezis [41, Theorem I.1]), there exists a linear functional Φ defined on $L^{p'}(\Omega; \mathbb{R}^N)$ satisfying

$$< \Phi; \nabla u > = < \phi; \nabla u >, \quad u \in \dot{W}^{1,p'}(\Omega), \quad \|\Phi\|_{[L^{p'}(\Omega;\mathbb{R}^N)]^*} = \|f\|_{[\dot{W}^{1,p'}(\Omega)]^*}.$$

According to the Riesz representation theorem (cf. Remark following Theorem 2) there exists a unique $\mathbf{w} \in L^p(\Omega; \mathbb{R}^N)$ such that

$$< \Phi; \mathbf{v} > = \int_{\Omega} \mathbf{w} \cdot \mathbf{v}, \quad \mathbf{v} \in L^{p'}(\Omega; \mathbb{R}^N),$$

$$\|\Phi\|_{[L^{p'}(\Omega;\mathbb{R}^N)]^*} = \|\mathbf{w}\|_{L^p(\Omega;\mathbb{R}^N)}.$$

This yields the statement of Lemma 11.4.

□

Step 2 We use Lemma 11.4 to show that $C_c^\infty(\Omega)$ is dense in $[\dot{W}^{1,p'}(\Omega)]^*$.

Lemma 11.5 *Let $\Omega \subset R^N$ be an open set, $1 < p' \leq \infty$.*

Then the set $\{C_c^\infty(\Omega) \,|\, \int_\Omega v \,\mathrm{d}x = 0\}$, identified as a subset of $[\dot{W}^{1,p'}(\Omega)]^*$ via (11.31), is dense in $[\dot{W}^{1,p'}(\Omega)]^*$.

Proof of Lemma 11.5 Let $\mathbf{w} \in L^p(\Omega; R^N)$ be a representant of $f \in [\dot{W}^{1,p'}(\Omega)]^*$ constructed in Lemma 11.4 and let $\mathbf{w}_n \in C_c^\infty(\Omega; R^N)$ be a sequence converging strongly to $\mathbf{w}$ in $L^p(\Omega; \mathbb{R}^N)$. Then a family of functionals $f_n = \mathrm{div}\mathbf{w}_n \in \{v \in C_c^\infty(\Omega) \,|\, \int_\Omega v \,\mathrm{d}x = 0\}$, defined as $< f_n; v >= \int_\Omega \mathbf{w}_n \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \mathrm{div}\mathbf{w}_n v \,\mathrm{d}x$, converges to f in $[\dot{W}^{1,p'}(\Omega)]^*$. This completes the proof.

□

Step 3 Having established the preliminary material, we focus on particular solutions to the problem $\mathrm{div}_x u = f$ with a smooth right hand side f. These solutions have been constructed by Bogovskii [28], and their basic properties are collected in the following lemma.

Lemma 11.6 *Let Ω be a bounded Lipschitz domain.*

Then there exists a linear operator

$$\mathcal{B} : \{f \in C_c^\infty(\Omega) | \int_\Omega f \,\mathrm{d}x = 0\} \mapsto C_c^\infty(\Omega; \mathbb{R}^N) \tag{11.33}$$

such that:

(i)

$$\mathrm{div}_x \mathcal{B}(f) = f, \tag{11.34}$$

and

$$\|\nabla_x \mathcal{B}(f)\|_{W^{k,p}(\Omega;\mathbb{R}^{N\times N})} \leq c\|f\|_{W^{k,p}(\Omega)}, \quad 1 < p < \infty, \ k = 0, 1, \ldots, \tag{11.35}$$

where c is a positive constant depending on k, p, $\mathrm{diam}(\Omega)$ and the Lipschitz constant associated to the local charts covering $\partial\Omega$.

(ii) If $f = \mathrm{div}_x \mathbf{g}$, where $\mathbf{g} \in C_c^\infty(\Omega; \mathbb{R}^N)$, then

$$\|\mathcal{B}(f)\|_{L^q(\Omega;\mathbb{R}^{N\times N})} \leq c\|\mathbf{g}\|_{L^q(\Omega;R^3)}, \ 1 < q < \infty, \tag{11.36}$$

where c is a positive constant depending on q, $\mathrm{diam}(\Omega)$, and the Lipschitz constant associated to $\partial\Omega$.

(iii) If $f, \partial_t f \in \{v \in C_c^\infty(I \times \Omega) \,|\, \int_\Omega v(t,x) \,\mathrm{d}x = 0, \ t \in I\}$, where I is an (open) interval, then

$$\frac{\partial \mathcal{B}(f)}{\partial t}(t,x) = \mathcal{B}\Big(\frac{\partial f}{\partial t}\Big)(t,x) \text{ for all } t \in I, \ x \in \Omega. \tag{11.37}$$

Remark In the case of a domain star-shaped with respect to a ball of radius $\overline{r}$ and for $k = 1$, the estimate of the constants in (11.35), (11.36) are given by formula (11.41) below. In the case of a Lipschitz domain, it may be evaluate by using (11.41) combined with Lemmas 11.1, and 11.7 below.

Step 4 Before starting the proof of Lemma 11.6, we observe that it is enough to consider star-shaped domains.

Lemma 11.7 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and let*

$$f \in C_c^\infty(\Omega), \quad \int_\Omega f \,\mathrm{d}x = 0.$$

Then there exists a family of functions

$$f_i \in C_c^\infty(\Omega_i), \ \int_{\Omega_i} f_i \,\mathrm{d}x = 0, \ \Omega_i = \Omega \cap \mathcal{O}_i \text{ for } i \in I,$$

where $\{\mathcal{O}\}_{i\in I}$ is the covering of Ω constructed in Lemma 11.1, and Ω_i are star-shaped with respect to a ball. Moreover,

$$\|f_i\|_{W^{k,p}(\Omega_i)} \le c\|f\|_{W^{k,p}(\Omega)}, \ 1 \le p \le \infty, \ k = 0, 1, \ldots,$$

where c is a positive constant dependent solely on p, k and $|\mathcal{O}_i|$, $i \in I$.

Proof of Lemma 11.7 Let $\{\varphi_i\}_{i\in I\cup J}$ be a *partition of unity* subordinate to the covering $\{\mathcal{O}_i\}_{i\in I}$ of $\overline{\Omega}$. We set

$$\Omega_1 = \Omega \cap \mathcal{O}_1, \ \Omega^1 = \cup_{i\in I\setminus\{1\}}\Omega_i, \ \text{where } \Omega_i = \mathcal{O}_i \cap \Omega.$$

Next, we introduce

$$f_1 = f\varphi_1 - \kappa_1 \int_{\Omega_1} f\varphi_1 \,\mathrm{d}x, \ g = f\phi - \kappa_1 \int_{\Omega^1} f\phi \,\mathrm{d}x,$$

where

$$\kappa_1 \in C_c^\infty(\Omega_1 \cap \Omega^1), \ \int_\Omega \kappa_1 \,\mathrm{d}x = 1, \ \phi = \sum_{i\in I\setminus\{1\}} \varphi_i.$$

With this choice,

$$f_1 \in C_c^\infty(\Omega_1), \ \int_{\Omega_1} f_1 \,\mathrm{d}x = 0, \ g \in C_c^\infty(\Omega^1), \ \int_{\Omega^1} g \,\mathrm{d}x = 0,$$

and both f_1 and g satisfy $W^{k,p}$-estimates stated in Lemma 11.7. Applying the above procedure to g in place of f and to Ω^1 in place of Ω, we can proceed by induction and complete the proof after a finite number of steps.

□

Step 5: Proof of Lemma 11.6

In view of Lemma 11.7, it is enough to assume that Ω is a star-shaped domain with respect to a ball $B(0;\overline{r})$, where the latter can be taken of radius $\overline{r}$ centered at the origin of the coordinate system.

In such a case, a possible candidate satisfying all properties stated in Lemma 11.6 is the so-called Bogovskii's solution given by the explicit formula:

$$\mathcal{B}[f](x) = \int_\Omega f(y)\Big[\frac{x-y}{|x-y|^N}\int_{|x-y|}^\infty \zeta_{\overline{r}}\Big(y + s\frac{x-y}{|x-y|}\Big)s^{N-1}\,\mathrm{d}s\Big]\,\mathrm{d}y, \tag{11.38}$$

or, equivalently, after the change of variables $z = x - y$, $r = s/|z|$,

$$\mathcal{B}[f](x) = \int_{\mathbb{R}^N}\Big[f(x-z)z\int_1^\infty \zeta_{\overline{r}}(x - z + rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}z, \tag{11.39}$$

where $\zeta_{\overline{r}}$ is a mollifying kernel specified in (11.3)–(11.4). A detailed inspection of these formulas yields all statements of Lemma 11.6.

Thus, for example, we deduce from (11.39) that $\mathcal{B}[f] \in C^\infty(\Omega)$, and that $\mathrm{supp}[\mathcal{B}[f]] \subset M$ where

$$M = \{z \in \Omega \,|\, z = \lambda z_1 + (1-\lambda)z_2, z_1 \in \mathrm{supp}(f), z_2 \in \overline{B(\overline{r};0)}, \lambda \in [0,1]\}.$$

Since M is closed and contained in Ω, (11.33) follows.

Now we explain, how to get (11.34) and estimate (11.35) with $k = 1$. Differentiating (11.39) we obtain

$$\Big(\partial_i\mathcal{B}_j(f)\Big)(x) = \int_{\mathbb{R}^N}\frac{\partial f}{\partial x_i}(x-z)z_j\Big[\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}z$$

$$+ \int_{\mathbb{R}^N} f(x-z)z_j\Big[\int_1^\infty \frac{\partial\zeta_{\overline{r}}}{\partial x_i}\Big(x - z + rz\Big)r^N\,\mathrm{d}r\Big]\,\mathrm{d}z.$$

Next, we split the set $\mathbb{R}^N$ in each integral into a ball $B(0;\varepsilon)$ and its complement realizing that the integrals over $B(0;\varepsilon)$ tend to zero as $\varepsilon \to 0+$. The first of the remaining integrals over $\mathbb{R}^N \setminus B(0;\varepsilon)$ is handled by means of integration by parts.

This direct but rather cumbersome calculation leads to

$$\Big(\partial_i \mathcal{B}_j[f]\Big)(x) = \lim_{\varepsilon\to 0+} \Big\{ \int_{|z|\geq\varepsilon} f(x-z)\times$$

$$\times\Big[\delta_{i,j}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r + z_j\int_1^\infty \frac{\partial\zeta_{\overline{r}}}{\partial_{x_i}}\Big(x-z+rz\Big)r^N\,\mathrm{d}r\Big]\,\mathrm{d}z$$

$$+\int_{|z|=\varepsilon} f(x-z)\Big[z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}\sigma_z\Big\},$$

or, equivalently,

$$\Big(\partial_i \mathcal{B}_j[f]\Big)(x) = \lim_{\varepsilon\to 0+} \Big\{ \int_{|y-x|\geq\varepsilon} f(y)\times$$

$$\times\Big[\frac{\delta_{i,j}}{|x-y|^N}\int_0^\infty \zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1}\,\mathrm{d}r +$$

$$\frac{x_j-y_j}{|x-y|^{N+1}}\int_0^\infty \frac{\partial\zeta_{\overline{r}}}{\partial x_i}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\,\mathrm{d}r\Big]\,\mathrm{d}y\Big\}$$

$$+f(x)\lim_{\varepsilon\to 0+}\Big\{\int_{|z|=\varepsilon}\Big[z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}\sigma_z\Big\},$$

where we have used the fact that

$$\lim_{\varepsilon\to 0+}\Big\{\int_{|z|=\varepsilon}\Big[\Big(f(x-z)-f(x)\Big)z_j\frac{z_i}{|z|}\int_1^\infty \zeta_{\overline{r}}(x-z+rz)r^{N-1}\,\mathrm{d}r\Big]\,\mathrm{d}\sigma_z\Big\} = 0.$$

Developing the expressions $(|x-y|+r)^{N-1}$, $(|x-y|+r)^N$ in the volume integral of the above identity by using the binomial formula, we obtain

$$\begin{aligned}\Big(\partial_i \mathcal{B}_j[f]\Big)(x) &= \mathrm{v.p.}\Big(\textstyle\int_\Omega K_{i,j}(x,x-y)f(y)\,\mathrm{d}y\Big)\\ &+\textstyle\int_\Omega G_{i,j}(x,x-y)f(y)\,\mathrm{d}y + f(x)H_{i,j}(x).\end{aligned} \tag{11.40}$$

The terms on the right hand side have the following properties:

(i) The first kernel reads

$$K_{i,j}(x,z) = \frac{\theta_{i,j}(x,z/|z|)}{|z|^N}$$

with

$$\theta_{i,j}\Big(x, \frac{z}{|z|}\Big) = \delta_{i,j} \int_0^\infty \zeta_{\overline{r}}\Big(x + r\frac{z}{|z|}\Big) r^{N-1}\, \mathrm{d}r + \frac{z_j}{|z|} \int_0^\infty \frac{\partial \zeta_{\overline{r}}}{\partial_{x_i}}\Big(x + r\frac{z}{|z|}\Big) r^N\, \mathrm{d}r.$$

Thus a close inspection shows that

$$\int_{|z|=1} \theta(x, z)\, \mathrm{d}\sigma_z = 0, \ x \in \mathbb{R}^N,$$

$$|\theta(x, z)| \le c(N) \frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N} \Big(1 + \frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big), \ x \in \mathbb{R}^N, \ |z| = 1.$$

We infer that $K_{i,j}$ are singular kernels of Calderón-Zygmund type obeying conditions (11.25) that were investigated in Theorem 11.16.

(ii) The second kernel reads

$$G_{i,j}(x, z) = \frac{\theta_{i,j}(x, z)}{|z|^{N-1}},$$

where

$$|\theta_{i,j}(x, z)| \le c(N) \frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N} \Big(1 + \frac{\mathrm{diam}(\Omega)}{\overline{r}}\Big), \ (x, z) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Thus $G_{i,j}$ are weakly singular kernels obeying conditions (11.23) discussed in Theorem 11.15.

(iii) Finally,

$$H_{i,j}(x) = \int_{\mathbb{R}^N} \frac{z_i z_j}{|z|^2} \zeta_{\overline{r}}(x + z)\, \mathrm{d}z,$$

where

$$|H_{i,j}(x)| \le c(N) \frac{(\mathrm{diam}(\Omega))^N}{\overline{r}^N}, \ x \in \mathbb{R}^N$$

and

$$\sum_{i=1}^N H_{i,i}(x) = 1.$$

Using these facts together with Theorems 11.15, 11.16 we easily verify estimate (11.35) with $k = 1$. We are even able to give an explicit formula for the constant appearing in the estimate, namely

$$c = c_0(p,N)\Big(\frac{\operatorname{diam}(\Omega)}{\overline{r}}\Big)^N\Big(1+\frac{\operatorname{diam}(\Omega)}{\overline{r}}\Big). \tag{11.41}$$

Since

$$\frac{d}{dr}\Big[\zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\Big] =$$

$$\sum_{k=1}^{N}\frac{x_k-y_k}{|x-y|}\frac{\partial\zeta_{\overline{r}}}{\partial x_k}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N$$

$$+N\zeta_{\overline{r}}\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1},$$

we have

$$\sum_{i=1}^{N}\Big(\int_{|x-y|\geq\varepsilon}f(y)(K_{i,i}(x,x-y)+G_{i,i}(x,x-y)\Big)\,\mathrm{d}y = \zeta_{\overline{r}}(x)\int_\Omega f(y)\,\mathrm{d}y = 0.$$

Moreover, evidently,

$$\sum_{i=1}^{N}H_{i,i}(x) = \int_\Omega \zeta_{\overline{r}}(y)\,\mathrm{d}y = 1;$$

whence (11.34) follows directly from (11.40).

In a similar way, the higher order derivatives of $\mathcal{B}[f]$ can be calculated by means of formula (11.39). Moreover, they can be shown to obey a representation formula of type (11.40), where, however, higher derivatives of f do appear; this leads to estimate (11.35) with an arbitrary positive integer k.

Last but not least, formula (11.39) written in terms of $\operatorname{div}_x\mathbf{g}$ yields, after integration by parts, a representation of $\mathcal{B}[\operatorname{div}_x\mathbf{g}]$ of type (11.40), with f replaced by $\mathbf{g}$. Again, the same reasoning as above yields naturally estimate (11.36).

Finally, property (11.37) is a consequence of the standard result concerning integrals dependent on a parameter.

The proof of Lemma 11.6 is thus complete.

□

Step 6: End of the Proof of Theorem 11.17 . For

$$< f; v >_{[\dot{W}^{1,p'}(\Omega)]^*,\dot{W}^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}\cdot\nabla v\,\mathrm{d}x, \text{ with } \mathbf{w}\in L^p(\Omega;\mathbb{R}^N),$$

we can take

$$< f_\varepsilon; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}_\varepsilon \cdot \nabla v \, \mathrm{d}x,$$

where $\mathbf{w}_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^N)$ have been constructed in Lemma 11.5.

Furthermore, let $\mathbf{h}_\varepsilon \in L^p(\Omega; \mathbb{R}^N)$,

$$\int_\Omega f_\varepsilon v \, \mathrm{d}x = -\int_\Omega \mathbf{h}_\varepsilon \cdot \nabla v \, \mathrm{d}x \text{ for all } v \in C^\infty(\overline{\Omega}),$$

$$\|f_\varepsilon\|_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \|\mathbf{h}_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)},$$

be a sequence of representants of f_ε introduced in Lemma 11.4. The last formula yields

$$f_\varepsilon = \mathrm{div}\mathbf{h}_\varepsilon, \quad \int_\Omega \Big(v \mathrm{div}\mathbf{h}_\varepsilon + \mathbf{h}_\varepsilon \cdot \nabla v \Big) \, \mathrm{d}x = 0,$$

meaning, in particular,

$$\gamma_{\mathbf{n}}(\mathbf{h}_\varepsilon) = 0 \text{ and, equivalently, } \mathbf{h}_\varepsilon \in E_0^p(\Omega), \ 1 < p < \infty$$

(see (11.19) in Theorem 11.14).

In view of the basic properties of the spaces $E_0^p(\Omega)$, we can replace $\mathbf{h}_\varepsilon$ by $\mathbf{g}_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^N)$ so that

$$\|\mathbf{h}_\varepsilon - \mathbf{g}_\varepsilon\|_{E^p(\Omega)} \to 0.$$

In particular, the sequence $\tilde f_\varepsilon$, $< \tilde f_\varepsilon; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{g}_\varepsilon \cdot \nabla v \, \mathrm{d}x$, converges to f, $< f; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w} \cdot \nabla v \, \mathrm{d}x$, strongly in $[\dot W^{1,p'}(\Omega)]^*$.

Due to estimate (11.36), the operator $\mathcal{B}$ is densely defined and continuous from $[\dot W^{1,p'}(\Omega)]^*$ to $L^p(\Omega; \mathbb{R}^N)$, therefore it can be extended by continuity to the whole space $[\dot W^{1,p'}(\Omega)]^*$.

If $< f; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega w v \, \mathrm{d}x$, with $w = W_0^{k,p}(\Omega) \cap \dot L^p(\Omega)$, we take f_ε such that $< f_\varepsilon; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega w_\varepsilon v \, \mathrm{d}x$, $w_\varepsilon = \zeta_\varepsilon * w - \kappa \int_\Omega (\zeta_\varepsilon * w) \, \mathrm{d}x$, where $\kappa \in C_c^\infty(\Omega)$, $\int_\Omega \kappa \, \mathrm{d}x = 0$ so that

$$C_c^\infty(\Omega) \ni f_\varepsilon = w_\varepsilon \to f = w \text{ in } W^{k,p}(\Omega).$$

If $< f; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w} \cdot \nabla v \, \mathrm{d}x$ with $\mathbf{w} \in E_0^{q,p}(\Omega)$, we take a sequence f_ε such that $< f_\varepsilon; v >_{[\dot W^{1,p'}(\Omega)]^*, \dot W^{1,p'}(\Omega)} = \int_\Omega \mathbf{w}_\varepsilon \cdot \nabla v \, \mathrm{d}x$, with $\mathbf{w} \in L^p(\Omega; \mathbb{R}^N) = \int_\Omega \mathrm{div}\mathbf{w}_\varepsilon v \, \mathrm{d}x$, where $\mathbf{w}_\varepsilon \in C_c^\infty(\Omega; \mathbb{R}^N)$, $\mathbf{w}_\varepsilon \to \mathbf{w}$ in $E_0^{q,p}(\Omega)$.

By virtue of estimates (11.35), (11.36), the operator $\mathcal{B}$ is in both cases a densely defined bounded linear operator on $W_0^{k,p}(\Omega)$ ($\hookrightarrow [\dot W^{1,p'}(\Omega)]^*$) ranging in

$W_0^{k+1,p}(\Omega)$, and on $E_0^{q,p}(\Omega)$ ($\hookrightarrow [\dot{W}^{1,p'}(\Omega)]^*$) with values in $L^q(\Omega) \cap W_0^{1,p}(\Omega)$; in particular, it can be continuously extended to $W_0^{k,p}(\Omega)$, and $E_0^{q,p}(\Omega)$, respectively.

This completes the proof of Theorem 11.17.

□

11.7 Helmholtz Decomposition

Let Ω be a domain in $\mathbb{R}^N$. Set

$$L^p_\sigma(\Omega; \mathbb{R}^N) = \{\mathbf{v} \in L^p(\Omega; \mathbb{R}^N) \mid \mathrm{div}_x \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$L^p_{g,\eta}(\Omega; \mathbb{R}^N) = \{\mathbf{v} \in L^p(\Omega; \mathbb{R}^N) \mid \mathbf{v} = \eta \nabla_x \Psi,\ \Psi \in W^{1,p}_{\mathrm{loc}}(\Omega)\},$$

where $\eta \in C(\overline{\Omega})$. The definition and the basic properties of the *Helmholtz decomposition* are collected in the following theorem.

■ HELMHOLTZ DECOMPOSITION:

Theorem 11.18 *Let Ω be a bounded domain of class $C^{1,1}$, and let*

$$\eta \in C^1(\overline{\Omega}),\ \inf_{x \in \Omega} \eta(x) = \underline{\eta} > 0.$$

Then the Lebesgue space $L^p(\Omega; \mathbb{R}^N)$ admits a decomposition

$$L^p(\Omega; \mathbb{R}^N) = L^p_\sigma(\Omega; \mathbb{R}^N) \oplus L^p_{g,\eta}(\Omega; \mathbb{R}^N),\ 1 < p < \infty,$$

more precisely,

$$\mathbf{v} = \mathbf{H}_\eta[\mathbf{v}] + \mathbf{H}^\perp_\eta[\mathbf{v}] \text{ for any } \mathbf{v} \in L^p(\Omega; \mathbb{R}^N),$$

with $\mathbf{H}^\perp_\eta[\mathbf{v}] = \eta \nabla_x \Psi$, where $\Psi \in W^{1,p}(\Omega)$ is the unique (weak) solution of the Neumann problem

$$\int_\Omega \eta \nabla_x \Psi \cdot \nabla_x \varphi \, \mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \, \mathrm{d}x \text{ for all } \varphi \in C^\infty(\overline{\Omega}),\ \int_\Omega \Psi \, \mathrm{d}x = 0.$$

In the particular case $p = 2$*, the decomposition is orthogonal with respect to the weighted scalar product*

$$< \mathbf{v}; \mathbf{w} >_{1/\eta} = \int_\Omega \mathbf{v} \cdot \mathbf{w} \frac{\mathrm{d}x}{\eta}.$$

Proof We start the proof with a lemma which is of independent interest.

Lemma 11.8 *Let* Ω *be a bounded domain of class* $C^{0,1}$ *and* $1 < p < \infty$*. Then*

$$L^p_\sigma(\Omega; \mathbb{R}^N) = \text{closure}_{L^p(\Omega;\mathbb{R}^N)} C^\infty_{c,\sigma}(\Omega; \mathbb{R}^N),$$

where

$$C^\infty_{c,\sigma}(\Omega; \mathbb{R}^N) = \{\mathbf{v} \in C^\infty_c(\Omega; \mathbb{R}^N) \mid \mathrm{div}_x \mathbf{v} = 0\}.$$

Proof of Lemma 11.8 Let $\mathbf{u} \in L^p_\sigma(\Omega; \mathbb{R}^3)$. Due to Lemma 11.3, there exists a sequence $\mathbf{w}_\varepsilon \in C^\infty_c(\Omega, \mathbb{R}^N)$, such that $\mathbf{w}_\varepsilon \to \mathbf{u}$ in $L^p(\Omega; \mathbb{R}^3)$ and $\mathrm{div}_x \mathbf{w}_\varepsilon \to 0$ in $L^p(\Omega)$ as $\varepsilon \to 0+$. Next we take the sequence $\mathbf{u}_\varepsilon = \mathbf{w}_\varepsilon - \mathcal{B}[\mathrm{div}_x \mathbf{w}_\varepsilon]$, where $\mathcal{B}$ is the Bogovskii operator introduced in Sect. 11.6. According to Theorem 11.17, the functions $\mathbf{u}_\varepsilon$ belong to $C^\infty_{c,\sigma}(\Omega; \mathbb{R}^N)$ and the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ converges to $\mathbf{u}$ in $L^p(\Omega; \mathbb{R}^N)$. This completes the proof of Lemma 11.8.

□

Existence and uniqueness of Ψ follow from Theorems 11.6, 11.7. Evidently, according to the definition, $\mathbf{H}_\eta[\mathbf{v}] = \mathbf{v} - \eta \nabla_x \Psi \in L^p_\sigma(\Omega; \mathbb{R}^N)$. Finally, we may use density of $C^\infty_{c,\sigma}(\Omega; \mathbb{R}^N)$ in $L^p_\sigma(\Omega; R^N)$ and integration by parts to show that the spaces $L^2_\sigma(\Omega; R^N)$ and $L^2_{g,\eta}(\Omega; \mathbb{R}^N)$ are orthogonal with respect to the scalar product $< \cdot; \cdot >_{1/\eta}$. This completes the proof of Theorem 11.18.

□

Remark In accordance with the regularity properties of the elliptic operators reviewed in Sect. 11.3.1, both $\mathbf{H}_\eta$ and $\mathbf{H}^\perp_\eta$ are continuous linear operators on $L^p(\Omega; \mathbb{R}^N)$ and $W^{1,p}(\Omega; \mathbb{R}^N)$ for any $1 < p < \infty$ provided Ω is of class $C^{1,1}$.

If $\eta = 1$, we recover the classical Helmholtz decomposition denoted as $\mathbf{H}$, $\mathbf{H}^\perp$ (see, for instance, Galdi [131, Chap. 3]). The result can be extended to a considerably larger class of domains, in particular, it holds for *any* domain $\Omega \subset \mathbb{R}^3$ if $p = 2$. For more details about this issue in the case of arbitrary $1 < p < \infty$ see Farwig et al. [99] or Simader, Sohr [248], and references quoted therein.

If $\Omega = \mathbb{R}^N$, the operator $\mathbf{H}^\perp$ can be defined by means of the Fourier multiplier

$$\mathbf{H}^\perp[\mathbf{v}](x) = \mathcal{F}^{-1}_{\xi \to x}\left[\frac{\xi \otimes \xi}{|\xi|^2} \mathcal{F}_{x \to \xi}[\mathbf{v}]\right].$$

11.8 Function Spaces of Hydrodynamics

Let Ω be a domain in $\mathbb{R}^N$. We introduce the following closed subspaces of the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$, $1 \le p \le \infty$:

$$W^{1,p}_{0,\sigma}(\Omega) = \{\mathbf{v} \in W^{1,p}_0(\Omega;\mathbb{R}^N) \,|\, \mathrm{div}_x\, \mathbf{v} = 0\},$$

$$W^{1,p}_{\mathbf{n}}(\Omega) = \{\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N) \,|\, \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$

$$W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N) = \{\mathbf{v} \in W^{1,p}_{\mathbf{n}}(\Omega) \,|\, \mathrm{div}_x\mathbf{v} = 0\}.$$

We also consider the vector spaces

$$C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N) = \{\mathbf{v} \in C^\infty_c(\Omega;\mathbb{R}^N) \,|\, \mathrm{div}\mathbf{v} = 0\},$$

$$C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) = \{\mathbf{v} \in C^{k,\nu}_c(\overline{\Omega};\mathbb{R}^N) \,|\, \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$

$$C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega},\mathbb{R}^N) = \{\mathbf{v} \in C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) \,|\, \mathrm{div}_x\mathbf{v} = 0\},$$

$$C^\infty_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N) = \cap_{k=1}^\infty C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N), \quad C^\infty_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N) = \cap_{k=1}^\infty C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N).$$

Under certain regularity assumptions on the boundary $\partial\Omega$, these spaces are dense in the afore-mentioned Sobolev spaces, as stated in the following theorem.

■ DENSITY OF SMOOTH FUNCTIONS:

Theorem 11.19 *Suppose that Ω is a bounded domain in $\mathbb{R}^N$, and $1 < p < \infty$. Then we have:*

(i) If the domain Ω is of class $C^{0,1}$, then the vector space $C^\infty_{c,\sigma}(\Omega;\mathbb{R}^N)$ is dense in $W^{1,p}_{0,\sigma}(\Omega;\mathbb{R}^N)$.
(ii) Suppose that Ω is of class $C^{k,\nu}$, $\nu \in (0,1)$, $k = 2,3,\ldots$, then the vector space $C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ is dense in $W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N)$.
(iii) Finally, if Ω is of class $C^{k,\nu}$, $\nu \in (0,1)$, $k = 2,3,\ldots$, then the vector space $C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ is dense in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$.

Proof **Step 1** In order to show statement (i), we reproduce the proof of Galdi [131, Sect. II.4.1]. Let $\mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega) \hookrightarrow W^{1,p}_0(\Omega;\mathbb{R}^N)$. There exists a sequence of smooth functions $\mathbf{w}_\varepsilon \in C^\infty_c(\Omega;\mathbb{R}^N)$ such that $\mathbf{w}_\varepsilon \to \mathbf{v}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, and, obviously, $\mathrm{div}\mathbf{w}_\varepsilon \to 0$ in $L^p(\Omega)$. Let $\mathbf{u}_\varepsilon = \mathcal{B}[\mathrm{div}_x\mathbf{w}_\varepsilon]$, where $\mathcal{B} \approx \mathrm{div}_x^{-1}$ is the operator constructed in Theorem 11.17. In accordance with Theorem 11.17, $\mathbf{u}_\varepsilon \in C^\infty_c(\Omega;\mathbb{R}^N)$, $\mathrm{div}\mathbf{u}_\varepsilon = \mathrm{div}\mathbf{w}_\varepsilon$, and $\|\mathbf{u}_\varepsilon\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \to 0$.

In view of these observations, we have

$$\mathbf{v}_\varepsilon = \mathbf{w}_\varepsilon - \mathbf{u}_\varepsilon \in C_c^\infty(\Omega;\mathbb{R}^N), \quad \mathrm{div}_x \mathbf{v}_\varepsilon = 0,$$

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)$$

yielding part (i) of Theorem 11.19.

Step 2 Let $\mathbf{v} \in W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N) \hookrightarrow W^{1,p}(\Omega;\mathbb{R}^N)$. Take $\mathbf{w}_\varepsilon \in C_c^\infty(\overline{\Omega};\mathbb{R}^N)$ such that $\mathbf{w}_\varepsilon \to \mathbf{v}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$. Obviously, we have

$$\mathrm{div}\mathbf{w}_\varepsilon \to 0 \text{ in } L^p(\Omega),\ \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} \to 0 \text{ in } W^{1-\frac{1}{p},p}(\partial\Omega).$$

Let $\varphi_\varepsilon \in C_c^{k,\nu}(\overline{\Omega})$, $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x = 0$ be an auxiliary function satisfying

$$\Delta\varphi_\varepsilon = \mathrm{div}\mathbf{w}_\varepsilon, \quad \nabla\varphi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega}.$$

Then, in accordance with Theorem 11.4,

$$C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N) \ni \mathbf{w}_\varepsilon - \nabla\varphi_\varepsilon \to \mathbf{v} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

This finishes the proof of part (ii).

Step 3 Let $\mathbf{v} \in W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$. We take $\mathbf{u} = \mathcal{B}(\mathrm{div}_x\mathbf{v})$, where $\mathcal{B}$ is the Bogovskii operator constructed in Theorem 11.17, and set $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Clearly $\mathbf{w} \in W^{1,p}_{\mathbf{n},\sigma}(\Omega;\mathbb{R}^N)$.

In view of statement (ii), there exists a sequence $\mathbf{w}_\varepsilon \in C^{k,\nu}_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ such that

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

On the other hand $\mathbf{u}$ belonging to $W^{1,p}_0(\Omega;\mathbb{R}^N)$, there exists a sequence $\mathbf{u}_\varepsilon \in C_c^\infty(\Omega;\mathbb{R}^N)$ such that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } W^{1,p}(\Omega;\mathbb{R}^N).$$

The sequence $\mathbf{v}_\varepsilon = \mathbf{w}_\varepsilon + \mathbf{u}_\varepsilon$ belongs to $C^{k,\nu}_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ and converges in $W^{1,p}(\Omega;\mathbb{R}^N)$ to $\mathbf{v}$.

This completes the proof of Theorem 11.19

□

The hypotheses concerning regularity of the boundary in statements (ii), (iii) are not optimal but sufficient in all applications for all treated in this book.

If the domain Ω is of class C^∞, the density of the space $C^\infty_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^N)$ in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$ and of $C^\infty_{\mathbf{n},\sigma}(\overline{\Omega};\mathbb{R}^N)$ in $W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^N)$ is a consequence of the theorem.

11.9 Poincaré Type Inequalities

The Poincaré type inequalities allow to estimate the L^p-norm of a function by the L^p-norms of its derivatives. The basic result in this direction is stated in the following lemma.

■ POINCARÉ INEQUALITY:

Lemma 11.9 *Let $1 \leq p < \infty$, and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then the following holds:*

(i) For any $A \subset \partial\Omega$ with the non zero surface measure there exists a positive constant $c = c(p, N, A, \Omega)$ such that

$$\|v\|_{L^p(\Omega)} \leq c \left(\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} + \int_{\partial\Omega} |v| \, \mathrm{d}S_x \right) \text{ for any } v \in W^{1,p}(\Omega).$$

(ii) There exists a positive constant $c = c(p, \Omega)$ such that

$$\|v - \frac{1}{|\Omega|}\int_{\Omega} v \, \mathrm{d}x\|_{L^p(\Omega)} \leq c\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} \text{ for any } v \in W^{1,p}(\Omega).$$

The above lemma can be viewed as a particular case of more general results, for which we refer to Ziemer [277, Chap. 4, Theorem 4.5.1].

Applications in fluid mechanics often require refined versions of Poincaré inequality that are not directly covered by the standard theory. Let us quote Babovski, Padula [13] or [87] as examples of results going in this direction. The following version of the refined Poincaré inequality is sufficiently general to cover all situations treated in this book.

■ GENERALIZED POINCARÉ INEQUALITY:

Theorem 11.20 *Let $1 \leq p \leq \infty$, $0 < \Gamma < \infty$, $V_0 > 0$, and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then there exists a positive constant $c = c(p, \Gamma, V_0)$ such that

$$\| v \|_{W^{1,p}(\Omega)} \leq c\Big[\|\nabla_x v\|_{L^p(\Omega;\mathbb{R}^N)} + \Big(\int_V |v|^\Gamma \mathrm{d}x \Big)^{\frac{1}{\Gamma}} \Big]$$

for any measurable $V \subset \Omega$, $|V| \geq V_0$ and any $v \in W^{1,p}(\Omega)$.

Proof Fixing the parameters p, Γ, V_0 and arguing by contradiction, we construct sequences $w_n \in W^{1,p}(\Omega)$, $V_n \subset \Omega$ such that

$$\|w_n\|_{L^p(\Omega)} = 1, \quad \|\nabla w_n\|_{W^{1,p}(\Omega;\mathbb{R}^N)} + \left(\int_{V_n} |w_n|^\Gamma \,\mathrm{dx}\right)^{\frac{1}{\Gamma}} < \frac{1}{n}, \tag{11.42}$$

$$|V_n| \geq V_0. \tag{11.43}$$

By virtue of (11.42), we have at least for a chosen subsequence

$$w_n \to \overline{w} \text{ in } W^{1,p}(\Omega) \text{ where } \overline{w} = |\Omega|^{-\frac{1}{p}}.$$

Consequently, in particular,

$$\left|\{w_n \leq \frac{\overline{w}}{2}\}\right| \to 0. \tag{11.44}$$

On the other hand, by virtue of (11.42)

$$\left|\{w_n \geq \frac{\overline{w}}{2}\} \cap V_n\right| \leq \left(2/\overline{w}\right)^\Gamma \int_{V_n} w_n^\Gamma \,\mathrm{d}x \to 0,$$

in contrast to

$$\left|\{w_n \geq \frac{\overline{w}}{2}\} \cap V_n\right| = \left|V_n \setminus \{w_n < \frac{\overline{w}}{2}\}\right| \geq \left|V_n\right| - \left|\{w_n < \frac{\overline{w}}{2}\}\right| \geq V_0,$$

where the last statement follows from (11.43), (11.44).

□

Another type of Poincaré inequality concerns norms in the negative Sobolev spaces in the spirit of Nečas [219].

■ Poincaré inequality in negative spaces:

Lemma 11.10 *Let Ω be a bounded Lipschitz domain, $1 < p < \infty$, and $k = 0, 1, \ldots$. Let $\kappa \in W_0^{k,p'}(\Omega)$, $\int_\Omega \kappa \,\mathrm{d}x = 1$ be a given function.*

(i) Then we have

$$\|f\|_{W^{-k,p}(\Omega)} \leq c\Big(\|\nabla_x f\|_{W^{-k-1,p}(\Omega;\mathbb{R}^N)} + \left|\sum_{|\alpha|\leq k}(-1)^{|\alpha|}\int_\Omega w_\alpha \partial^\alpha \kappa \, dx\right|\Big) \text{ for any } f \in W^{-k,p}(\Omega), \tag{11.45}$$

where $\{w_\alpha\}_{|\alpha|\leq k}$, $w_\alpha \in L^p(\Omega)$ is an arbitrary representative of f constructed in Theorem 3, and c is a positive constant depending on p, N, Ω.

(ii) In particular, if $k = 0$, inequality (11.45) reads

$$\|f\|_{L^p(\Omega)} \le c\Big(\|\nabla f\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega f\kappa\,\mathrm{d}x\Big|\Big).$$

Proof Since $C_c^\infty(\Omega)$ is dense in $W^{-k,p}(\Omega)$, it is enough to suppose that f is smooth. By direct calculation, we get

$$\|f\|_{W^{-k,p}(\Omega)} = \sup_{g\in W_0^{k,p'}(\Omega)} \frac{\int_\Omega fg\,\mathrm{d}x}{\|g\|_{W^{k,p'}(\Omega)}} \le$$

$$\sup_{g\in W_0^{k,p'}(\Omega)} \Big(\frac{\int_\Omega f\,[g-\kappa\int_\Omega g\,\mathrm{d}x]\,\mathrm{d}x}{\|g-\kappa\int_\Omega g\,\mathrm{d}x\|_{W^{k,p'}(\Omega)}} \times \frac{\|g-\kappa\int_\Omega g\,\mathrm{d}x\|_{W^{k,p'}(\Omega)}}{\|g\|_{W^{k,p'}(\Omega)}}\Big)$$

$$+ \sup_{g\in W_0^{k,p'}(\Omega)} \frac{(\int_\Omega g\,\mathrm{d}x)(\int_\Omega f\kappa\,\mathrm{d}x)}{\|g\|_{W^{k,p'}(\Omega)}} \le$$

$$c(p,\Omega)\Big(\sup_{\mathbf{v}\in W_0^{k+1,p'}(\Omega;\mathbb{R}^N)} \frac{\int_\Omega f\mathrm{div}_x\mathbf{v}\,\mathrm{d}x}{\|\mathbf{v}\|_{W^{k+1,p'}(\Omega;\mathbb{R}^N)}} + \Big|\sum_{|\alpha|\le k}(-1)^\alpha\int_\Omega w_\alpha\partial^\alpha\kappa\,\mathrm{d}x\Big|\Big),$$

where $\{w_\alpha\}_{\alpha\le k}$ is any representative of f (see formula (3) in Theorem 3), and where the quantity $W_0^{k+1,p'}(\Omega) \ni \mathbf{v} = \mathcal{B}(g-\kappa\int g\,\mathrm{d}x)$ appearing on the last line is a solution of problem

$$\mathrm{div}_x\mathbf{v} = g-\kappa\int_\Omega g\,\mathrm{d}x, \quad \|\mathbf{v}\|_{W^{k+1,p'}\Omega)} \le c(p,\Omega)\Big\|g-\kappa\int_\Omega g\,\mathrm{d}x\Big\|_{W^{k,p'}(\Omega)}$$

constructed in Theorem 11.17.

The proof of Lemma 11.10 is complete.

□

11.10 Korn Type Inequalities

Korn's inequality has played a central role not only in the development of linear elasticity but also in the analysis of viscous incompressible fluid flows. The reader interested in this topic can consult the review paper of Horgan [157], the recent article of Dain [69], and the relevant references cited therein. While these results rely mostly on the Hilbertian L^2-setting, various applications in the theory of compressible fluid flows require a general L^p-setting and even more.

We start with the standard formulation of Korn's inequality providing a bound of the L^p-norm of the gradient of a vector field in terms of the L^p-norm of its symmetric part.

■ KORN'S INEQUALITY IN L^p:

Theorem 11.21 *Assume that* $1 < p < \infty$.

(i) There exists a positive constant $c = c(p, N)$ *such that*

$$\|\nabla \mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq c \|\nabla \mathbf{v} + \nabla^T \mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^N; \mathbb{R}^N)$.

(ii) Let $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Then there exists a positive constant* $c = c(p, N, \Omega) > 0$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq c\Big(\|\nabla \mathbf{v} + \nabla^T \mathbf{v}\|_{L^p(\Omega,\mathbb{R}^{N\times N})} + \int_\Omega |\mathbf{v}| \, \mathrm{d}x \Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega; \mathbb{R}^N)$.

Proof **Step 1** Since $C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N; \mathbb{R}^N)$, we may suppose that $\mathbf{v}$ is smooth with compact support. We start with the identity

$$\partial_{x_k} \partial_{x_j} v_s = \partial_{x_j} D_{s,k} + \partial_{x_k} D_{s,j} - \partial_{x_s} D_{j,k}, \tag{11.46}$$

where

$$\mathbb{D} = (D_{i,j})_{i,j=1}^N, \ D_{i,j} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

Relation (11.46), rewritten in terms of the Fourier transform, reads

$$\xi_k \xi_j \mathcal{F}_{x\to\xi}(v_s) = -\mathrm{i}\Big(\xi_j \mathcal{F}_{x\to\xi}(D_{s,k}) + \xi_k \mathcal{F}_{x\to\xi}(D_{s,j}) - \xi_s \mathcal{F}_{x\to\xi}(D_{j,k}) \Big).$$

Consequently,

$$\mathcal{F}_{x\to\xi}(\partial_{x_k} v_s) = \mathcal{F}_{x\to\xi}(D_{s,k}) + \frac{\xi_j \xi_k}{|\xi|^2} \mathcal{F}_{x\to\xi}(D_{s,j}) - \frac{\xi_j \xi_s}{|\xi|^2} \mathcal{F}_{x\to\xi}(D_{j,k}).$$

Thus estimate (i) follows directly from the Hörmander-Mikhlin theorem (Theorem 9).

Step 2 Similarly to the previous part, it is enough to consider smooth functions $\mathbf{v}$. Lemma 11.10 applied to formula (11.46) yields

$$\|\nabla \mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \leq c\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \Big|\int_\Omega \nabla \mathbf{v}\kappa \,\mathrm{d}x\Big|\Big),$$

where $\kappa \in C_c^\infty(\Omega)$, $\int_\Omega \kappa \,\mathrm{d}x = 1$. Consequently, estimate (ii) follows.

□

In applications to models of *compressible* fluids, it is useful to replace the symmetric gradient in the previous theorem by its *traceless* part. The adequate result is stated in the following theorem.

■ GENERALIZED KORN'S INEQUALITY:

Theorem 11.22 *Let* $1 < p < \infty$, *and* $N > 2$.

(i) There exists a positive constant $c = c(p,N)$ *such that*

$$\|\nabla \mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq c\|\nabla \mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\mathbb{I}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$, *where* $\mathbb{I} = (\delta_{i,j})_{i,j=1}^N$ *is the identity matrix.*

(ii) Let $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Then there exists a positive constant* $c = c(p,N,\Omega) > 0$ *such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq c\Big(\|\nabla \mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\mathbb{I}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} + \int_\Omega |\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$.

Remark As a matter of fact, part (i) of Theorem 11.22 holds for any $N \geq 1$. On the other hand, statement (ii) may fail for $N = 2$ as shown by Dain [69].

Proof **Step 1** In order to show (i), we suppose, without loss of generality, that $\mathbf{v}$ is smooth and has a compact support in $\mathbb{R}^N$. A straightforward algebra yields

$$\partial_{x_k}\partial_{x_j}v_s = \partial_{x_j}D_{s,k} + \partial_{x_k}D_{s,j} - \partial_{x_s}D_{j,k} + \frac{1}{N}\Big(\delta_{s,k}\partial_{x_j}\mathrm{div}_x\mathbf{v} + \delta_{s,j}\partial_{x_k}\mathrm{div}_x\mathbf{v} - \delta_{j,k}\partial_{x_s}\mathrm{div}_x\mathbf{v}\Big), \tag{11.47}$$

$$(N-2)\partial_{x_s}\mathrm{div}_x\mathbf{v} = 2N\partial_{x_k}D_{s,k} - N\Delta v_s, \tag{11.48}$$

$$\partial_{x_j}(\Delta v_s) = \partial_{x_j}\partial_{x_k}D_{s,k} + \Delta D_{j,s} - \partial_{x_s}\partial_{x_k}D_{j,k} + \frac{1}{N-1}\delta_{j,s}\partial_{x_k}\partial_{x_n}D_{k,n}, \tag{11.49}$$

where $\mathbb{D} = (D_{i,j})_{i,j=1}^N$ denotes the tensor

$$\mathbb{D} = \frac{1}{2}(\nabla_x \mathbf{v} + \nabla_x{}^T \mathbf{v}) - \frac{1}{N}\mathrm{div}_x \mathbf{v}\mathbb{I}.$$

Moreover, we deduce from (11.47) that

$$\begin{aligned} \mathcal{F}_{x\to\xi}(\partial_{x_k} v_s) = \mathcal{F}_{x\to\xi}(D_{s,k}) + \tfrac{\xi_k\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{s,j}) \\ -\tfrac{\xi_s\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{j,k}) + \tfrac{1}{N}\delta_{s,k}\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{v}), \end{aligned} \tag{11.50}$$

where, according to (11.48), (11.49),

$$\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{v}) = \frac{N}{N-2}\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}\big(\partial_s(\Delta v_s)\big) + \frac{2N}{N-2}\frac{\xi_s\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}(D_{s,j}),$$

with

$$\frac{1}{|\xi|^2}\mathcal{F}_{x\to\xi}\big(\partial_s(\Delta v_s)\big) = -\Big(\mathcal{F}_{x\to\xi}(D_{s,s}) + \frac{N}{N-1}\frac{\xi_k\xi_n}{|\xi|^2}\mathcal{F}(D_{k,n})\Big).$$

Thus, estimate (i) follows from (11.50) via the Hörmander–Mikhlin multiplier theorem.

Step 2 Similarly to the previous step, it is enough to show (ii) for a smooth $\mathbf{v}$. By virtue of Lemma 11.10, we have

$$\|\partial_{x_k} v_j\|_{L^p(\Omega)} \le c(p,\Omega)\Big(\|\nabla_x \partial_{x_k} v_j\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \partial_{x_k} v_j \kappa \,\mathrm{d}x\Big|\Big), \tag{11.51}$$

and

$$\|\Delta v_s\|_{W^{-1,p}(\Omega)} \le c(p,\Omega)\Big(\|\nabla_x \Delta v_s\|_{W^{-2,p}(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \Delta v_s \tilde{\kappa}\,\mathrm{d}x|\Big) \tag{11.52}$$

for any $\kappa \in L^{p'}(\Omega)$, $\int_\Omega \kappa \,\mathrm{d}x = 1$, $\tilde{\kappa} \in W_0^{1,p'}(\Omega)$, $\int_\Omega \tilde{\kappa}\,\mathrm{d}x = 1$.

Using the basic properties of the $W^{-1,p}$-norm we deduce from identities (11.47)–(11.48) that

$$\|\nabla_x \partial_{x_k} v_j\|_{W^{-1,p}(\Omega;\mathbb{R}^N)} \le c\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^N)} + \|\Delta \mathbf{v}\|_{W^{-1,p}(\Omega;\mathbb{R}^N)}\Big),$$

where the second term at the right-hand side is estimated by help of identity (11.49) and inequality (11.52). Coming back to (11.51) we get

$$\|\partial_{x_k} v_j\|_{L^p(\Omega)} \le c(p,\Omega)\Big(\|\mathbb{D}\|_{L^p(\Omega;\mathbb{R}^N)} + \Big|\int_\Omega \partial_{x_k} v_j\kappa\,\mathrm{d}x\Big| + \Big|\int_\Omega \Delta v_j \tilde{\kappa}\,\mathrm{d}x\Big|\Big),$$

which, after by parts integration and with a particular choice $\kappa \in C_c^1(\Omega), \tilde{\kappa} \in C_c^2(\Omega)$, yields estimate (ii).

□

We conclude this part with another generalization of the previous results.

■ GENERALIZED KORN-POINCARÉ INEQUALITY:

Theorem 11.23 *Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded Lipschitz domain, and let $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 1$.*

Then there exists a positive constant $c = c(p, M_0, K, \gamma)$ such that the inequality

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \tag{11.53}$$

$$\leq c\left(\left\|\nabla_x \mathbf{v} + \nabla_x{}^T \mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\,\mathbb{I}\right\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r|\mathbf{v}|\,\mathrm{d}x\right)$$

holds for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and any non negative function r such that

$$0 < M_0 \leq \int_\Omega r\,\mathrm{d}x,\ \int_\Omega r^\gamma\,\mathrm{d}x \leq K. \tag{11.54}$$

.

Proof Without loss of generality, we may assume that $\gamma > \max\{1, \frac{Np}{(N+1)p-N}\}$. Indeed replacing r by $T_k(r)$, where $T_k(z) = \max\{z, k\}$, we can take $k = k(M_0, \gamma)$ large enough. Moreover, it is enough to consider smooth functions $\mathbf{v}$.

Fixing the parameters K, M_0, γ we argue by contradiction. Specifically, we construct a sequence $\mathbf{w}_n \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\|\mathbf{w}_n\|_{W^{1,p}(\Omega;\mathbb{R}^N)} = 1, \quad \mathbf{w}_n \to \mathbf{w} \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N) \tag{11.55}$$

and

$$\left\|\nabla_x \mathbf{w}_n + \nabla_x{}^T \mathbf{w}_n - \frac{2}{N}\mathrm{div}_x \mathbf{w}_n\,\mathbb{I}\right\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r_n|\mathbf{w}_n|\,\mathrm{d}x < \frac{1}{n} \tag{11.56}$$

for certain

$$r_n \to r \text{ weakly in } L^\gamma(\Omega),\ \int_\Omega r\,\mathrm{d}x \geq M_0 > 0. \tag{11.57}$$

Consequently, due to the compact embedding $W^{1,p}(\Omega)$ into $L^p(\Omega)$, and by virtue of Theorem 11.22,

$$\mathbf{w}_n \to \mathbf{w} \text{ strongly in } W^{1,p}(\Omega;\mathbb{R}^N). \tag{11.58}$$

Moreover, in agreement with (11.55)–(11.58), the limit $\mathbf{w}$ satisfies the identities

$$\|\mathbf{w}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} = 1, \tag{11.59}$$

$$\nabla\mathbf{w} + \nabla^T\mathbf{w} - \frac{2}{N}\mathrm{div}\mathbf{w}\mathbb{I} = 0, \tag{11.60}$$

$$\int_\Omega r|\mathbf{w}|\,\mathrm{d}x = 0. \tag{11.61}$$

Equation (11.60) which is valid provided $N > 2$, implies that $\Delta\mathrm{div}\mathbf{w} = 0$ and $\Delta\mathbf{w} = \frac{2-N}{N}\mathrm{div}\mathbf{w}$, see (11.48), (11.49). In particular, in agreement with remarks after Theorem 11.4 in Appendix, $\mathbf{w}$ is analytic in Ω. On the other hand, according to (11.61), $\mathbf{w}$ vanishes on the set $\{x \in \Omega \mid r(x) > 0\}$ of a nonzero measure; whence $\mathbf{w} \equiv 0$ in Ω in contrast with (11.61).

Theorem 11.23 has been proved.

□

Finally, we address the question how the constant in Theorem 11.23 depends on the geometry of the spatial domain Ω. To this end, we assume that $\partial\Omega$ can be described by a finite number of charts based on balls of radius r and Lipschitz constant L. Then it turns out that c depends only on these two parameters.

■ GENERALIZED KORN-POINCARÉ INEQUALITY—DOMAIN DEPENDENCE:

Theorem 11.24 *Under the hypotheses of Theorem 11.23, suppose that there exists a radius r and a constant L such that $\partial\Omega$ can be covered by a finite number of balls $B(x,r)$, on each of which $\partial\Omega$ is expressed as a graph of a Lipschitz function with the Lipschitz constant L.*

Then the generalized Korn inequality (11.53) holds with a constant depending only on r and L.

Proof See [42]. □

11.11 Estimating $\nabla\mathbf{u}$ by Means of $\mathrm{div}_x\mathbf{u}$ and $\mathbf{curl}_x\mathbf{u}$

■ ESTIMATING $\nabla\mathbf{u}$ IN TERMS OF $\mathrm{div}_x\mathbf{u}$ AND $\mathbf{curl}_x\mathbf{u}$:

Theorem 11.25 *Assume that* $1 < p < \infty$.

(i) Then

$$\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq c(p,N)\Big(\|\mathrm{div}_x \mathbf{u}\|_{L^p(\mathbb{R}^N)} + \|\mathbf{curl}_x \mathbf{u}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}\Big),$$
$$\text{for any } \mathbf{u} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N). \tag{11.62}$$

(ii) If $\Omega \subset \mathbb{R}^N$ *is a bounded domain, then*

$$\|\nabla \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \leq c\Big(\|\mathrm{div}_x \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{curl}_x \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{N\times N})}\Big),$$
$$\text{for any } \mathbf{u} \in W^{1,p}_0(\Omega;\mathbb{R}^N). \tag{11.63}$$

Proof To begin, observe that it is enough to show the estimate for $\mathbf{u} \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$. To this end, we write

$$\mathrm{i}\sum_{k=1}^N \xi_k \mathcal{F}_{x\to\xi}(u_k) = \mathcal{F}_{x\to\xi}(\mathrm{div}_x \mathbf{u}),$$

$$\mathrm{i}\Big(\xi_k \mathcal{F}_{x\to\xi}(u_j) - \xi_j \mathcal{F}_{x\to\xi}(u_k)\Big) = \mathcal{F}_{x\to\xi}([\mathrm{curl}]_{j,k}\mathbf{u}), \quad j \neq k.$$

Solving the above system we obtain

$$\mathrm{i}|\xi|^2 \mathcal{F}_{x\to\xi}(u_k) = \xi_k \mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{u}) + \sum_{j\neq k} \xi_j \mathcal{F}_{x\to\xi}([\mathrm{curl}]_{k,j}\mathbf{u}),$$

for $k = 1, \ldots, N$. Consequently, we deduce

$$\mathcal{F}_{x\to\xi}(\partial_{x_r} u_k) = \frac{\xi_k \xi_r}{|\xi|^2}\mathcal{F}_{x\to\xi}(\mathrm{div}\mathbf{u}) + \sum_{j\neq k} \frac{\xi_j \xi_r}{|\xi|^2}\mathcal{F}_{x\to\xi}([\mathrm{curl}]_{k,j}\mathbf{u}).$$

Thus estimate (11.62) is obtained as a direct consequence of Hörmander-Mikhlin theorem on multipliers (Theorem 9).

□

If the trace of $\mathbf{u}$ does not vanish on $\partial\Omega$, the estimates of type (11.62) depend strongly on the geometrical properties of the domain Ω, namely on the values of its first and second *Betti numbers*.

For example, the estimate

$$\|\nabla \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} \leq c(p, N, \Omega)\Big(\|\mathrm{div}_x \mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{curl}_x \mathbf{u}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})}\Big)$$

holds

(i) for any $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$, $\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0$, provided Ω is a bounded domain with the boundary of class $C^{1,1}$ and the set $\mathbb{R}^3 \setminus \overline{\Omega}$ is (arcwise) connected (meaning $\mathbb{R}^3 \setminus \Omega$ does not contain a bounded (arcwise) connected component);

(ii) for any $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3)$, $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$, if Ω is a bounded domain with the boundary of class $C^{1,1}$ whose boundary $\partial\Omega$ is a connected and compact two-dimensional manifold.

The interested reader should consult the papers of von Wahl [270] and Bolik and von Wahl [29] for a detailed treatment of these questions including more general results in the case of non-vanishing tangential and/or normal components of the vector field $\mathbf{u}$.

11.12 Weak Convergence and Monotone Functions

We start with a straightforward consequence of the De la Vallée Poussin criterion of the L^1-weak compactness formulated in Theorem 10.

Corollary 11.1 *Let $Q \subset \mathbb{R}^N$ be a domain and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^1(Q)$ satisfying*

$$\sup_{n>0} \int_Q \Phi(|f_n|)\,\mathrm{d}x < \infty, \tag{11.64}$$

where Φ is a non negative function continuous on $[0,\infty)$ such that $\lim_{z\to\infty} \Phi(z)/z = \infty$.

Then

$$\sup_{n>0} \left\{ \int_{\{|f_n|\geq k\}} |f_n(x)|\,\mathrm{d}x \right\} \to 0 \quad \text{as } k \to \infty, \tag{11.65}$$

in particular,

$$k \sup_{n>0} \{|\{|f_n| \geq k\}|\} \to 0 \text{ as } k \to \infty.$$

Typically, $\Phi(z) = z^p$, $p > 1$, in which case we have

$$|\{|f_n| \geq k\}| \leq \frac{1}{k} \int_{\{|f_n| \geq k\}} |f_n(x)| \mathrm{d}x \leq \frac{1}{k} \left(\int_Q |f_n|^p \mathrm{d}x \right)^{1/p} |\{|f_n| \geq k\}|^{1/p'}.$$

Consequently, we report the following result.

Corollary 11.2 *Let $Q \subset \mathbb{R}^N$ be a domain and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions bounded in $L^p(Q)$, where $p \in [1, \infty)$.*

Then

$$\int_{\{|f_n| \geq k\}} |f_n|^s \mathrm{d}x \leq \frac{1}{k^{p-s}} \sup_{n>0} \left\{ \|f_n\|^p_{L^p(Q)} \right\}, \ s \in [0, p]. \tag{11.66}$$

In particular

$$|\{|f_n| \geq k\}| \leq \frac{1}{k^p} \sup_{n>0} \left\{ \|f_n\|^p_{L^p(Q)} \right\}. \tag{11.67}$$

In the remaining part of this section, we review a mostly standard material based on monotonicity arguments. There are several variants of these results scattered in the literature, in particular, these arguments have been extensively used in the monographs of Lions [192], or [102, 224]. Our aim is to formulate these results at such a level of generality so that they may be directly applicable to all relevant situations investigated in this book.

■ WEAK CONVERGENCE AND MONOTONICITY:

Theorem 11.26 *Let $I \subset \mathbb{R}$ be an interval, $Q \subset \mathbb{R}^N$ a domain, and*

$$(P, G) \in C(I) \times C(I) \quad \text{a couple of non-decreasing functions.} \tag{11.68}$$

Assume that $\varrho_n \in L^1(Q; I)$ is a sequence such that

$$\left. \begin{array}{c} P(\varrho_n) \to \overline{P(\varrho)}, \\ \\ G(\varrho_n) \to \overline{G(\varrho)}, \\ \\ P(\varrho_n)G(\varrho_n) \to \overline{P(\varrho)G(\varrho)} \end{array} \right\} \quad \text{weakly in } L^1(Q). \tag{11.69}$$

(i) Then

$$\overline{P(\varrho)}\ \overline{G(\varrho)} \leq \overline{P(\varrho)G(\varrho)}. \tag{11.70}$$

(ii) If, in addition,

$$G \in C(\mathbb{R}), \quad G(\mathbb{R}) = \mathbb{R}, \quad G \text{ is strictly increasing},$$
$$P \in C(\mathbb{R}), \quad P \text{ is non-decreasing}, \tag{11.71}$$

and

$$\overline{P(\varrho)G(\varrho)} = \overline{P(\varrho)}\ \overline{G(\varrho)}, \tag{11.72}$$

then

$$\overline{P(\varrho)} = P \circ G^{-1}(\overline{G(\varrho)}). \tag{11.73}$$

(iii) In particular, if $G(z) = z$, then

$$\overline{P(\varrho)} = P(\varrho). \tag{11.74}$$

Proof We shall limit ourselves to the case $I = (0, \infty)$ already involving all difficulties encountered in other cases.

Step 1 If P is bounded and G strictly increasing, the proof is straightforward. Indeed, in this case,

$$0 \le \lim_{n\to\infty} \int_B \Big[P(\varrho_n) - (P \circ G^{-1})\Big(\overline{G(\varrho)}\Big)\Big]\Big(G(\varrho_n) - \overline{G(\varrho)}\Big) \mathrm{d}\,x =$$
$$\int_B \Big(\overline{P(\varrho)G(\varrho)} - \overline{P(\varrho)}\ \overline{G(\varrho)}\Big) \mathrm{d}\,x \tag{11.75}$$
$$- \lim_{n\to\infty} \int_B P \circ G^{-1}(\overline{G(\varrho)})\Big(G(\varrho_n) - \overline{G(\varrho)}\Big) \mathrm{d}\,x,$$

where B is a ball in Q and $P \circ G^{-1}(\overline{G(\varrho)}) = \lim_{s\to\overline{G(\varrho)}} P \circ G^{-1}(s)$. By virtue of assumption (11.69), the second term at the right hand side of the last formula tends to 0; whence the desired inequality (11.70) follows immediately from the standard result on the Lebesgue points.

Step 2 If P is bounded and G non-decreasing, we replace G by a strictly increasing function, say,

$$G_k(z) = G(z) + \frac{1}{k} \arctan(z), \quad k > 0.$$

In accordance with **Step 1** we obtain

$$\overline{P(\varrho)G(\varrho)} + \frac{1}{k}\overline{P(\varrho)\arctan(\varrho)} \geq \overline{P(\varrho)}\ \overline{G(\varrho)} + \frac{1}{k}\overline{P(\varrho)}\ \overline{\arctan(\varrho)},$$

where we have used the De la Vallé Poussin criterion (Theorem 10) to guarantee the existence of the weak limits. Letting $k \to \infty$ in the last formula yields (11.70).

Step 3 If $\lim_{z\to 0+} P(z) \in \mathbb{R}$ and if P is unbounded, we may approximate P by a family of bounded non-decreasing functions,

$$P \circ \mathcal{T}_k, \quad k > 0,$$

where

$$\mathcal{T}_k(z) = k\mathcal{T}(\frac{z}{k}), \quad C^1(\mathbb{R}) \ni \mathcal{T}(z) = \left\{ \begin{array}{c} z \text{ if } z \in [0,1] \\ \\ \text{concave in } (0,\infty) \\ \\ 2 \text{ if } z \geq 3 \\ \\ -\mathcal{T}(-z) \text{ if } z \in (-\infty, 0) \end{array} \right\}. \tag{11.76}$$

Reasoning as in the previous step, we obtain

$$\overline{(P \circ \mathcal{T}_k)(\varrho)G(\varrho)} \geq \overline{(P \circ \mathcal{T}_k)(\varrho)}\ \overline{G(\varrho)}. \tag{11.77}$$

In order to let $k \to \infty$, we observe first that

$$\|\overline{(P \circ \mathcal{T}_k)(\varrho)} - \overline{P(\varrho)}\|_{L^1(Q)} \leq$$

$$\liminf_{n\to\infty} \|(P \circ \mathcal{T}_k)(\varrho_n) - P(\varrho_n)\|_{L^1(Q)} \leq 2 \sup_{n\in N} \left\{ \int_{\{\varrho_n \geq k\}} |P(\varrho_n)| \mathrm{d}\, x \right\},$$

where the last integral is arbitrarily small provided k is sufficiently large (see Theorem 10). Consequently,

$$\overline{(P \circ \mathcal{T}_k)(\varrho)} \to \overline{P(\varrho)} \quad \text{a.e. in } Q.$$

Similarly,

$$\overline{P \circ \mathcal{T}_k(\varrho)G(\varrho)} \to \overline{P(\varrho)G(\varrho)} \quad \text{a.e. in } Q.$$

Thus, letting $k \to \infty$ in (11.77) we obtain again (11.70).

Step 4 Finally, if $\lim_{z\to 0+} P(z) = -\infty$, we approximate P by

$$P_h(z) = \left\{ \begin{array}{ll} P(h) & \text{if } z \in (-\infty, h) \\ & \\ P(z) & \text{if } z \geq h \end{array} \right\}, \quad h > 0, \tag{11.78}$$

so that, according to **Step 3**,

$$\overline{P_h(\varrho)G(\varrho)} \geq \overline{P_h(\varrho)}\ \overline{G(\varrho)}, \tag{11.79}$$

As in the previous step, in accordance with Theorem 10,

$$\begin{aligned} &\|\overline{P_h(\varrho)} - \overline{P(\varrho)}\|_{L^1(Q)} \leq \liminf_{n\to\infty} \|P_h(\varrho_n) - P(\varrho_n)\|_{L^1(Q)} \\ &\leq 2\sup_{n\in N} \left\{ \int_{\{|P(\varrho_n)|\geq |P(h)|\}} |P(\varrho_n)| \mathrm{d}\,x \right\} \to 0 \quad \text{as } h \to 0+, \end{aligned} \tag{11.80}$$

and

$$\begin{aligned} &\|\overline{P_h(\varrho)G(\varrho)} - \overline{P(\varrho)G(\varrho)}\|_{L^1(Q)} \\ &\leq 2\sup_{n\in N} \left\{ \int_{\{|P(\varrho_n)|\geq |P(h)|\}} |P(\varrho_n)G(\varrho_n)| \mathrm{d}\,x \right\} \to 0 \quad \text{as } h \to 0+. \end{aligned} \tag{11.81}$$

Thus we conclude the proof of part **(i)** of Theorem 11.26 by letting $h \to 0+$ in (11.79).

Step 5 Now we are in a position to prove part (ii). We set

$$M_k = \left\{ x \in B \mid \sup_{s\in[-1,1]} G^{-1}\left(\overline{G(\varrho)} + s\right)(x) \leq k \right\},$$

where B is a ball in Q, and $k > 0$. Thanks to monotonicity of P and G, we can write

$$\begin{aligned} 0 \leq \int_B 1_{M_k} \left[P(\varrho_n) - (P \circ G^{-1})\left(\overline{G(\varrho)} \pm \epsilon\varphi\right) \right] \times \\ \left(G(\varrho_n) - \overline{G(\varrho)} \mp \epsilon\varphi \right) \mathrm{d}\,x = \\ \int_B 1_{M_k} \left(P(\varrho_n)G(\varrho_n) - P(\varrho_n)\ \overline{G(\varrho)} \right) \mathrm{d}\,x \\ - \int_B 1_{M_k} (P \circ G^{-1})\left(\overline{G(\varrho)} \pm \epsilon\varphi\right)\left(G(\varrho_n) - \overline{G(\varrho)} \right) \mathrm{d}\,x \\ \mp\varepsilon \int_B 1_{M_k} \left[P(\varrho_n) - (P \circ G^{-1})\left(\overline{G(\varrho)} \pm \epsilon\varphi\right) \right] \varphi \mathrm{d}\,x, \end{aligned} \tag{11.82}$$

where $\epsilon > 0$, $\varphi \in C_c^\infty(B)$ and 1_{M_k} is the characteristic function of the set M_k.

For $n \to \infty$ in (11.82), the first integral on the right-hand side tends to zero by virtue of (11.69), (11.72). Recall that $1_{M_k}\overline{G(\varrho)}$ is bounded. On the other hand, the second integral approaches zero by virtue of (11.69). Recall that $1_{M_k}(P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)$ is bounded.

Thus we are left with

$$\int_B 1_{M_k}\Big[\overline{P(\varrho)} - (P \circ G^{-1})\Big(\overline{G(\varrho)} \pm \epsilon\varphi\Big)\Big]\varphi \mathrm{d}x = 0, \quad \varphi \in C_c^\infty(B); \tag{11.83}$$

whence (11.73) follows by sending $\epsilon \to 0+$ and realizing that $\cup_{k>0} M_k = B$. This completes the proof of statement **(ii)**.

□

11.13 Weak Convergence and Convex Functions

The idea of monotonicity can be further developed in the framework of *convex functions*. Similarly to the preceding section, the material collected here is standard and may be found in the classical books on convex analysis as, for example, Ekeland and Temam [92], or Azé [12].

Consider a functional

$$F : \mathbb{R}^M \to (-\infty, \infty], \ M \geq 1. \tag{11.84}$$

We say that F is *convex* on a convex set $O \subset \mathbb{R}^M$ if

$$F(tv + (1-t)w) \leq tF(v) + (1-t)F(w) \text{ for all } v, w \in O, \ t \in [0, 1]; \tag{11.85}$$

F is *strictly convex* on O if the above inequality is strict whenever $v \neq w$.

Compositions of convex functions with weakly converging sequences have a remarkable property of being lower semi-continuous with respect to the weak L^1-topology as shown in the following assertion (cf. similar results in Visintin [268], Balder [15]).

■ WEAK LOWER SEMI-CONTINUITY OF CONVEX FUNCTIONS:

Theorem 11.27 *Let $O \subset \mathbb{R}^N$ be a measurable set and $\{\mathbf{v}_n\}_{n=1}^\infty$ a sequence of functions in $L^1(O; \mathbb{R}^M)$ such that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(O; \mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ *be a lower semi-continuous convex function. Then*

$$\int_O \Phi(\mathbf{v})\mathrm{d}\,x \le \liminf_{n\to\infty} \int_O \Phi(\mathbf{v}_n)\mathrm{d}\,x.$$

Moreover if

$$\Phi(\mathbf{v}_n) \to \overline{\Phi(\mathbf{v})} \text{ weakly in } L^1(O),$$

then

$$\Phi(\mathbf{v}) \le \overline{\Phi(\mathbf{v})} \text{ a.a. on } O. \tag{11.86}$$

If, in addition, Φ *is strictly convex on an open convex set* $U \subset \mathbb{R}^M$*, and*

$$\Phi(\mathbf{v}) = \overline{\Phi(\mathbf{v})} \text{ a.a. on } O,$$

then

$$\mathbf{v}_n(\mathbf{y}) \to \mathbf{v}(\mathbf{y}) \text{ for a.a. } \mathbf{y} \in \{\mathbf{y} \in O \mid \mathbf{v}(\mathbf{y}) \in U\} \tag{11.87}$$

extracting a subsequence as the case may be.

Proof **Step 1** Any convex lower semi-continuous function with values in $(-\infty, \infty]$ can be written as a supremum of its affine minorants:

$$\Phi(\mathbf{z}) = \sup\{a(\mathbf{z}) \mid a \text{ an affine function on } \mathbb{R}^M, a \le \Phi \text{ on } \mathbb{R}^M\} \tag{11.88}$$

(see Theorem 3.1 of Chap. 1 in [92]). Recall that a function is called *affine* if it can be written as a sum of a linear and a constant function.

On the other hand, if $B \subset O$ is a measurable set, we have

$$\int_B \overline{\Phi(\mathbf{v})}\,\mathrm{d}y = \lim_{n\to\infty} \int_B \Phi(\mathbf{v}_n)\,\mathrm{d}y \ge \lim_{n\to\infty} \int_B a(\mathbf{v}_n)\,\mathrm{d}y = \int_B a(\mathbf{v})\,\mathrm{d}y$$

for any affine function $a \le \Phi$. Consequently,

$$\overline{\Phi(\mathbf{v})}(\mathbf{y}) \ge a(\mathbf{v})(\mathbf{y})$$

for any $\mathbf{y} \in O$ which is a Lebesgue point of both $\overline{\Phi(\mathbf{v})}$ and $\mathbf{v}$.

Thus formula (11.88) yields (11.86).

Step 2 As any open set $U \subset \mathbb{R}^M$ can be expressed as a countable union of compacts, it is enough to show (11.87) for

$$\mathbf{y} \in M_K \equiv \{\mathbf{y} \in O \mid \mathbf{v}(\mathbf{y}) \in K\},$$

where $K \subset U$ is compact.

Since Φ is strictly convex on U, there exists an open set V such that

$$K \subset V \subset \overline{V} \subset U,$$

and $\Phi : \overline{V} \to R$ is a Lipschitz function (see Corollary 2.4 of Chap. I in [92]). In particular, the subdifferential $\partial\Phi(\mathbf{v})$ is non-empty for each $\mathbf{v} \in K$, and we have

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) \text{ for any } \mathbf{w} \in \mathbb{R}^M,\ \mathbf{v} \in K,$$

where $\underline{\partial}\Phi(\mathbf{v})$ denotes the linear form in the subdifferential $\partial\Phi(\mathbf{v}) \subset (\mathbb{R}^M)^*$ with the smallest norm (see Corollary 2.4 of Chap. 1 in [92]).

Next, we shall show the existence of a function ω,

$$\begin{gathered} \omega \in C[0,\infty),\ \omega(0) = 0, \\ \omega \text{ non-decreasing on } [0,\infty) \text{ and strictly positive on } (0,\infty), \end{gathered} \tag{11.89}$$

such that

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) + \omega(|\mathbf{w} - \mathbf{v}|) \text{ for all } \mathbf{w} \in \overline{V},\ \mathbf{v} \in K. \tag{11.90}$$

Were (11.90) not true, we would be able to find two sequences $\mathbf{w}_n \in \overline{V}$, $\mathbf{z}_n \in K$ such that

$$\Phi(\mathbf{w}_n) - \Phi(\mathbf{z}_n) - \underline{\partial}\Phi(\mathbf{z}_n) \cdot (\mathbf{w}_n - \mathbf{z}_n) \ \to 0 \text{ for } n \to \infty$$

while

$$|\mathbf{w}_n - \mathbf{z}_n| \geq \delta > 0 \text{ for all } n = 1, 2, \ldots$$

Moreover, as K is compact, one can assume

$$\mathbf{z}_n \to \mathbf{z} \in K,\ \Phi(\mathbf{z}_n) \to \Phi(\mathbf{z}),\ \mathbf{w}_n \to \mathbf{w} \text{ in } \overline{V},\ \underline{\partial}\Phi(\mathbf{z}_n) \to L \in \ \mathbb{R}^M,$$

and, consequently,

$$\Phi(\mathbf{y}) - \Phi(\mathbf{z}) \geq L \cdot (\mathbf{y} - \mathbf{z}) \text{ for all } \mathbf{y} \in \mathbb{R}^M,$$

that is $L \in \partial\Phi(\mathbf{z})$.

Now, the function

$$\Psi(\mathbf{y}) \equiv \Phi(\mathbf{y}) - \Phi(\mathbf{z}) - L \cdot (\mathbf{y} - \mathbf{z})$$

is non-negative, convex, and

$$\Psi(\mathbf{z}) = \Psi(\mathbf{w}) = 0, \ |\mathbf{w} - \mathbf{z}| \geq \delta.$$

Consequently, Ψ vanishes on the whole segment $[\mathbf{z}, \mathbf{w}]$, which is impossible as Φ is strictly convex on U.

Seeing that the function

$$a \mapsto \Phi(\mathbf{z} + a\mathbf{y}) - \Phi(\mathbf{z}) - a\underline{\partial}\Phi(\mathbf{z}) \cdot \mathbf{y}$$

is non-negative, convex and non-decreasing for $a \in [0, \infty)$ we infer that the estimate (11.90) holds without the restriction $\mathbf{w} \in \overline{V}$. More precisely, there exists ω as in (11.89) such that

$$\Phi(\mathbf{w}) - \Phi(\mathbf{v}) \geq \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}) + \omega(|\mathbf{w} - \mathbf{v}|) \text{ for all } \mathbf{w} \in \mathbb{R}^M, \ \mathbf{v} \in K. \tag{11.91}$$

Taking $\mathbf{w} = \mathbf{v}_n(\mathbf{y})$, $\mathbf{v} = \mathbf{v}(\mathbf{y})$ in (11.91) and integrating over the set M_K we get

$$\int_{M_K} \omega(|\mathbf{v}_n - \mathbf{v}|) \, \mathrm{d}y \leq \int_{M_K} \Phi(\mathbf{v}_n) - \Phi(\mathbf{v}) - \underline{\partial}\Phi(\mathbf{v}) \cdot (\mathbf{v}_n - \mathbf{v}) \, \mathrm{d}y,$$

where the right-hand side tends to zero for $n \to \infty$. Note that the function $\underline{\partial}\Phi(\mathbf{v})$ is bounded measurable on M_k as Φ is Lipschitz on $\overline{V}$, and

$$\underline{\partial}\Phi(\mathbf{v}) = \lim_{\varepsilon \to 0} \nabla\Phi_\varepsilon(\mathbf{v}) \text{ for any } \mathbf{v} \in V,$$

where

$$\Phi_\varepsilon(\mathbf{v}) \equiv \min_{\mathbf{z} \in \mathbb{R}^M} \left\{ \frac{1}{\varepsilon} |\mathbf{z} - \mathbf{v}| + \Phi(\mathbf{z}) \right\} \tag{11.92}$$

is a convex, continuously differentiable function on $\mathbb{R}^M$ (see Propositions 2.6, 2.11 of Chap. 2 in [40]).

Thus

$$\int_{M_K} \omega(|\mathbf{v}_n - \mathbf{v}|) \, \mathrm{d}y \to 0 \text{ for } n \to \infty$$

which yields pointwise convergence (for a subsequence) of $\{\mathbf{v}_n\}_{n=1}^\infty$ to $\mathbf{v}$ a.a. on M_K. □

11.14 Div-Curl Lemma

The celebrated Div-Curl Lemma of Tartar [254] (see also Murat [218]) is a cornerstone of the theory of compensated compactness and became one of the most efficient tools in the analysis of problems with lack of compactness. Here, we recall its L^p-version.

Lemma 11.11 *Let $Q \subset \mathbb{R}^N$ be an open set, and $1 < p < \infty$. Assume*

$$\begin{aligned} \mathbf{U}_n &\to \mathbf{U} \text{ weakly in } L^p(Q;\mathbb{R}^N), \\ \mathbf{V}_n &\to \mathbf{V} \text{ weakly in } L^{p'}(Q;\mathbb{R}^N). \end{aligned} \tag{11.93}$$

In addition, let

$$\left.\begin{aligned} \operatorname{div} \mathbf{U}_n &\equiv \nabla\cdot\mathbf{U}_n, \\ \operatorname{curl} \mathbf{V}_n &\equiv (\nabla\mathbf{V}_n - \nabla^T\mathbf{V}_n) \end{aligned}\right\} \text{be precompact in} \left\{\begin{aligned} &W^{-1,p}(Q), \\ &W^{-1,p'}(Q;\mathbb{R}^{N\times N}). \end{aligned}\right. \tag{11.94}$$

Then

$$\mathbf{U}_n\cdot\mathbf{V}_n \to \mathbf{U}\cdot\mathbf{V} \text{ in } \mathcal{D}'(Q).$$

Proof Since the result is local, we can assume that $Q = \mathbb{R}^N$. We have to show that

$$\int_{\mathbb{R}^N} \big(\mathbf{H}[\mathbf{U}_n] + \mathbf{H}^\perp[\mathbf{U}_n]\big)\cdot\big(\mathbf{H}[\mathbf{V}_n] + \mathbf{H}^\perp[\mathbf{V}_n]\big)\,\varphi\,\mathrm{d}x \to$$

$$\int_{\mathbb{R}^N} \big(\mathbf{H}[\mathbf{U}] + \mathbf{H}^\perp[\mathbf{U}]\big)\cdot\big(\mathbf{H}[\mathbf{V}] + \mathbf{H}^\perp[\mathbf{V}]\big)\,\varphi\,\mathrm{d}x$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $\mathbf{H}$, $\mathbf{H}^\perp$ are the Helmholtz projections introduced in Sect. 11.7. We have

$$\mathbf{H}^\perp[\mathbf{U}_n] = \nabla\Psi_n^U,\ \mathbf{H}^\perp[\mathbf{V}_n] = \nabla\Psi_n^V,$$

where, in accordance with hypothesis (11.94) and the standard elliptic estimates discussed in Sects. 11.3.1 and 11.11,

$$\nabla\Psi_n^U \to \nabla\Psi^U = \mathbf{H}^\perp[\mathbf{U}] \text{ in } L^p(B;\mathbb{R}^N),$$

$$\mathbf{H}[\mathbf{V}_n] \to \mathbf{H}[\mathbf{V}] \text{ in } L^{p'}(B;\mathbb{R}^N),$$

and

$$\mathbf{H}[\mathbf{U}_n] \to \mathbf{H}[\mathbf{U}] \text{ weakly in } L^p(B;\mathbb{R}^N),$$

$$\nabla\Psi_n^V \to \nabla\Psi^V = \mathbf{H}^{\perp}[\mathbf{V}] \text{ weakly in } L^{p'}(B;\mathbb{R}^N),$$

where $B \subset \mathbb{R}^N$ is a ball containing the support of φ.

Consequently, it is enough to handle the term $\mathbf{H}[\mathbf{U}_n] \cdot \nabla_x \Psi_n^V \varphi$. However,

$$\int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}_n] \cdot \nabla_x \Psi_n^V \varphi \, \mathrm{d}x = -\int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}_n] \cdot \nabla\varphi \Psi_n^V \, \mathrm{d}x \to$$

$$-\int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}] \cdot \nabla\varphi \Psi^V \, \mathrm{d}x = \int_{\mathbb{R}^N} \mathbf{H}[\mathbf{U}] \cdot \nabla_x \Psi^V \varphi \, \mathrm{d}x.$$

□

The following variant of Div-Curl Lemma seems more convenient from the perspective of possible applications.

■ DIV-CURL LEMMA:

Theorem 11.28 *Let $Q \subset \mathbb{R}^N$ be an open set. Assume*

$$\begin{aligned} \mathbf{U}_n &\to \mathbf{U} \textit{ weakly in } L^p(Q;\mathbb{R}^N),\\ \mathbf{V}_n &\to \mathbf{V} \textit{ weakly in } L^q(Q;\mathbb{R}^N), \end{aligned} \tag{11.95}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

In addition, let

$$\left.\begin{aligned} \operatorname{div} \mathbf{U}_n &\equiv \nabla \cdot \mathbf{U}_n,\\ \operatorname{curl} \mathbf{V}_n &\equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{aligned}\right\} \textit{ be precompact in } \left\{\begin{aligned} &W^{-1,s}(Q),\\ &W^{-1,s}(Q;\mathbb{R}^{N\times N}), \end{aligned}\right. \tag{11.96}$$

for a certain $s > 1$. Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \textit{ weakly in } L^r(Q).$$

The proof follows easily from Lemma 11.11 as soon as we observe that precompact sets in $W^{-1,s}$ that are bounded in $W^{-1,p}$ are precompact in $W^{-1,m}$ for any $s < m < p$.

11.15 Maximal Regularity for Parabolic Equations

We consider a parabolic problem:

$$\left\{ \begin{array}{c} \partial_t u - \Delta u = f \text{ in } (0,T) \times \Omega, \\ \\ u(0,x) = u_0(x), \ x \in \Omega, \\ \\ \nabla_x u \cdot \mathbf{n} = 0 \text{ in } (0,T) \times \partial\Omega, \end{array} \right\} \tag{11.97}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In the context of the so-called *strong solutions*, the first equation is satisfied a.e. in $(0,T) \times \Omega$, the initial condition holds a.e. in Ω, and the homogenous Neumann boundary condition is satisfied in the sense of traces.

The following statement holds.

■ MAXIMAL $L^p - L^q$ REGULARITY:

Theorem 11.29 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2, $1 < p, q < \infty$. Suppose that*

$$f \in L^p(0,T;L^q(\Omega)), \ u_0 \in X_{p,q}, \ X_{p,q} = \{L^q(\Omega); \mathcal{D}(\Delta_{\mathcal{N}})\}_{1-1/p,p},$$

$$\mathcal{D}(\Delta_{\mathcal{N}}) = \{v \in W^{2,q}(\Omega) \mid \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$

where $\{\cdot;\cdot\}_{\cdot,\cdot}$ denotes the real interpolation space.

Then problem (11.97) admits a solution u, unique in the class

$$u \in L^p(0,T;W^{2,q}(\Omega)), \ \partial_t u \in L^p(0,T;L^q(\Omega)),$$

$$u \in C([0,T];X_{p,q}).$$

Moreover, there exists a positive constant $c = c(p,q,\Omega,T)$ such that

$$\|u(t)\|_{X_{p,q}} + \|\partial_t u\|_{L^p(0,T;L^q(\Omega))} + \|\Delta u\|_{L^p(0,T;L^q(\Omega))} \leq \tag{11.98}$$

$$c\left(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{X_{p,q}}\right)$$

for any $t \in [0,T]$.

See Amann [7, 8]. □

For the definition of real interpolation spaces see e.g. Bergh, Löfström [27, Chap. 3]. It is well known that

$$X_{p,q} = \begin{cases} B^{2-\frac{2}{p}}_{q,p}(\Omega) \text{ if } 1-\frac{2}{p}-\frac{1}{q} < 0, \\ \{u \in B^{2-\frac{2}{p}}_{q,p}(\Omega) \mid \nabla_x u \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \text{ if } 1-\frac{2}{p}-\frac{1}{q} > 0, \end{cases}$$

see Amann [7]. In the above formula, the symbol $B^s_{q,p}(\Omega)$ refers to the Besov space.

For the definition and properties of the scale of Besov spaces $B^s_{q,p}(\mathbb{R}^N)$ and $B^s_{q,p}(\Omega)$, $s \in \mathbb{R}$, $1 \le q, p \le \infty$ see Bergh and Löfström [27, Sect. 6.2], Triebel [257, 258]. A nice overview can be found in Amann [7, Sect. 5]. Many of the classical spaces are contained as special cases in the Besov scales. It is of interest for the purpose of this book that

$$B^s_{p,p}(\Omega) = W^{s,p}(\Omega), \; s \in (0,\infty) \setminus \mathbb{N}, \; 1 \le p < \infty,$$

where $W^{s,p}(\Omega)$ is the Sobolev-Slobodeckii space.

Extension of Theorem 11.29 to general classes of parabolic equations and systems as well as to different type of boundary conditions are available. For more information concerning the $L^p - L^q$ maximal regularity for parabolic systems with general boundary conditions, we refer to the book of Amann [8] or to the papers by Denk et al. [77, 78, 148].

Maximal regularity in the classes of smooth functions relies on classical argument. A result in this direction reads as follows.

■ MAXIMAL HÖLDER REGULARITY:

Theorem 11.30 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$. Suppose that*

$$f \in C([0,T]; C^{0,\nu}(\overline{\Omega})), \; u_0 \in C^{2,\nu}(\overline{\Omega}), \; \nabla_x u_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then problem (11.97) admits a unique solution

$$u \in C([0,T]; C^{2,\nu}(\overline{\Omega})), \; \partial_t u \in C([0,T]; C^{0,\nu}(\overline{\Omega})).$$

Moreover, there exists a positive constant $c = c(p,q,\Omega,T)$ such that

$$\|\partial_t u\|_{C([0,T];C^{0,\nu}(\overline{\Omega}))} + \|u\|_{C([0,T];C^{2,\nu}(\Omega))} \le c\Big(\|u_0\|_{C^{2,\nu}(\overline{\Omega})} + \|f\|_{C([0,T];C^{0,\nu}(\overline{\Omega}))}\Big). \tag{11.99}$$

See Lunardi [199, Theorem 5.1.2] □

Unlike most of the classical existence theorems that can be found in various monographs on parabolic equation (see e.g. Ladyzhenskaya et al. [179]), the above results requires merely the continuity in time of the right hand side. This aspect is very convenient for the applications in this book.

11.16 Quasilinear Parabolic Equations

In this section we review a well known result solvability of the quasilinear parabolic problem:

$$\left\{\begin{array}{c} \partial_t u - \sum_{i,j=1}^N a_{ij}(t,x,u)\partial_{x_i}\partial_{x_j}u + b(t,x,u,\nabla_x u) = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \\ \sum_{i,j=1}^N n_i\, a_{ij}\partial_{x_j}u + \psi = 0 \quad \text{on } S_T, \\ \\ u(0,\cdot) = u_0, \end{array}\right\} \tag{11.100}$$

where

$$a_{ij} = a_{ij}(t,x,u),\ i,j = 1,\dots,N, \quad \psi = \psi(t,x),\ b(t,x,u,\mathbf{z}) \text{ and } u_0 = u_0(x)$$

are continuous functions of their arguments $(t,x) \in [0,T]\times\overline{\Omega}$, $u \in \mathbb{R}$, $\mathbf{z} \in \mathbb{R}^N$, $S_T = [0,T]\times\partial\Omega$ and $\mathbf{n} = (n_1,\dots,n_N)$ is the outer normal to the boundary $\partial\Omega$.

The results stated below are taken over from the classical book by Ladyzhenskaya et al. [179]. We refer the reader to this work for all details, and also for the further properties of quasilinear parabolic equations and systems.

■ EXISTENCE AND UNIQUENESS
FOR THE QUASILINEAR PARABOLIC NEUMANN PROBLEM:

Theorem 11.31 *Let $\nu \in (0,1)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\nu}$. Suppose that*

(i)

$$u_0 \in C^{2,\nu}(\overline{\Omega}), \quad \psi \in C^1([0,T]\times\overline{\Omega}), \quad \nabla_x\psi \text{ is Hölder continuous}$$

$$\text{in the variables } t \text{ and } x \text{ with exponents } \nu/2 \text{ and } \nu, \text{ respectively,}$$

$$\sum_{i,j=1}^N n_i(x)\, a_{ij}\partial_{x_j}(0,x,u_0(x)) + \psi(0,x) = 0,\ x \in \partial\Omega;$$

(ii)

$$a_{ij} \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R}),$$

$$\nabla_x a_{ij}, \partial_u a_{ij} \text{ are } \nu - \text{ Hölder continuous in the variable } x;$$

(iii)

$$b \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N),$$

$$\nabla_x b, \partial_u b, \nabla_{\mathbf{z}} b \text{ are } \nu - \text{ Hölder continuous in the variablex};$$

(iv) there exist positive constants $\underline{c}$, $\overline{c}$, c_1, c_2 *such that*

$$0 \le a_{ij}(t,x,u)\xi_i\xi_j \le \overline{c}|\xi|^2, \quad (t,x,u,\xi) \in (0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N,$$

$$a_{ij}(t,x,u)\xi_i\xi_j \ge \underline{c}|\xi|^2, \quad (t,x,u,\xi) \in S_T \times \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

$$-ub(t,x,u,z) \le c_0|z|^2 + c_1u^2 + c_2, \quad (t,x,u,\xi) \in [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N;$$

(v) for any $L > 0$ *there are positive constants* $\underline{C}$ *and* $\overline{C}$ *such that*

$$\underline{C}(L)|\xi|^2 \le a_{ij}(t,x,u)\xi_i\xi_j, \quad (t,x,u,\xi) \in [0,T] \times \overline{\Omega} \times [-L,L] \times \mathbb{R}^N,$$

$$\left| b, \partial_t b, \partial_u b, (1+z)\nabla_z b \right| (t,x,u,z)$$

$$\le \overline{C}(L)(1+|z|^2), \quad (t,x,u,z) \in [0,T] \times \overline{\Omega} \times [-L,L] \times \mathbb{R}^N.$$

Then problem (11.100) admits a unique classical solution u belonging to the Hölder space $C^{1,\nu/2;2,\nu}([0,T] \times \overline{\Omega})$, *where the symbol* $C^{1,\nu/2;2,\nu}([0,T] \times \overline{\Omega})$ *stands for the Banach space with norm*

$$\|u\|_{C^1([0,T]\times\overline{\Omega})} + \sup_{(t,\tau,x)\in[0,T]^2\times\overline{\Omega}} \frac{|\partial_t u(t,x) - \partial_t u(\tau,x)|}{|t-\tau|^{\nu/2}}$$

$$+ \sum_{i,j=1}^{3} \|\partial_{x_i}\partial_{x_j} u\|_{C([0,T]\times\overline{\Omega})}$$

$$+ \sum_{i,j=1}^{3} \sup_{(t,x,y)\in[0,T]\times\overline{\Omega}^2} \frac{|\partial_{x_i}\partial_{x_j} u(t,x) - \partial_{x_i}\partial_{x_j} u(t,y)|}{|x-y|^\nu}.$$

See Ladyzhenskaya et al. [179, Theorems 7.2, 7.3, 7.4]. □

11.17 Basic Properties of the Riesz Transform and Related Operators

Various *(pseudo) differential operators* used in the book are identified through their Fourier symbols:

- the Riesz transform:

$$\mathcal{R}_j \approx \frac{i\xi_j}{|\xi|},\ j = 1, \dots, N,$$

  meaning that

$$\mathcal{R}_j[v] = \mathcal{F}^{-1}_{\xi \to x}\left[\frac{i\xi_j}{|\xi|} \mathcal{F}_{x \to \xi}[v] \right];$$

- the “double” Riesz transform:

$$\mathcal{R} = \{\mathcal{R}_{k,j}\}_{k,j=1}^N,\ \mathcal{R} = \Delta_x^{-1} \nabla_x \otimes \nabla_x,\ \mathcal{R}_{i,j} \approx \frac{\xi_i \xi_j}{|\xi|^2},\ i,j = 1, \dots, N,$$

  meaning that

$$\mathcal{R}_{k,j}[v] = \mathcal{F}^{-1}_{\xi \to x}\left[\frac{\xi_k \xi_j}{|\xi|^2} \mathcal{F}_{x \to \xi}[v] \right];$$

- the inverse divergence:

$$\mathcal{A} = \{\mathcal{A}_j\}_{j=1}^N,\ \mathcal{A}_j = \partial_{x_j} \Delta_x^{-1} \approx -\frac{\mathrm{i}\xi_j}{|\xi|^2},\ j = 1, \dots, N,$$

  meaning that

$$\mathcal{A}_j[v] = -\mathcal{F}^{-1}_{\xi \to x}\left[\frac{i\xi_j}{|\xi|^2} \mathcal{F}_{x \to \xi}[v] \right];$$

- the inverse Laplacian:

$$(-\Delta)^{-1} \approx \frac{1}{|\xi|^2},$$

  meaning that

$$(-\Delta)^{-1}[v] = \mathcal{F}^{-1}_{\xi \to x}\left[\frac{1}{|\xi|^2} \mathcal{F}_{x \to \xi}[v] \right].$$

In the sequel, we shall investigate boundedness of these pseudo- differential operators in various function spaces. The following theorem is an immediate consequence of the Hörmander-Mikhlin theorem (Theorem 9).

■ CONTINUITY OF THE RIESZ OPERATOR:

Theorem 11.32 *The operators $\mathcal{R}_k$, $\mathcal{R}_{k,j}$ are continuous linear operators mapping $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$. In particular, the following estimate holds true:*

$$\|\mathcal{R}[v]\|_{L^p(\mathbb{R}^N)} \le c(N,p)\|v\|_{L^p(\mathbb{R}^N)} \text{ for all } v \in L^p(\mathbb{R}^N), \tag{11.101}$$

where $\mathcal{R}$ stands for $\mathcal{R}_k$ or $\mathcal{R}_{k,j}$.

As a next step, we examine the continuity properties of the inverse divergence operator. To begin, we recall that for Banach spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, the sum $X + Y = \{w = u + v \,|\, u \in X, v \in Y\}$ and the intersection $X \cap Y$ can be viewed as Banach spaces endowed with norms $\|w\|_{X+Y} = \inf\left\{\max\{\|u\|_X, \|v\|_Y\}, \,\middle|\, w = u + v\right\}$ and $\|w\|_{X\cap Y} = \|w\|_X + \|w\|_Y$, respectively.

■ CONTINUITY PROPERTIES OF THE INVERSE DIVERGENCE:

Theorem 11.33 *Assume that $N > 1$.*

(i) The operator $\mathcal{A}_k$ is a continuous linear operator mapping $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, and $L^p(\mathbb{R}^N)$ into $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$ for any $1 < p < N$.

(ii) In particular,

$$\|\mathcal{A}_k[v]\|_{L^\infty(\mathbb{R}^N)+L^2(\mathbb{R}^N)} \le c(N)\|v\|_{L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)} \quad \text{for all } v \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \tag{11.102}$$

and

$$\|\mathcal{A}_k[v]\|_{L^{\frac{Np}{N-p}}(\mathbb{R}^N)} \le c(N,p)\|v\|_{L^p(\mathbb{R}^N)} \text{ for all } v \in L^p(\mathbb{R}^N),\ 1 < p < N. \tag{11.103}$$

(iii) If $v, \frac{\partial v}{\partial t} \in L^p(I \times \mathbb{R}^N)$, where I is an (open) interval, then

$$\frac{\partial \mathcal{A}_k(f)}{\partial t}(t,x) = \mathcal{A}_k\left(\frac{\partial f}{\partial t}\right)(t,x) \text{ for a. a. } (t,x) \in I \times \mathbb{R}^N. \tag{11.104}$$

Proof **Step 1** We write

$$-\mathcal{A}_k[v] = \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{i\xi_k}{|\xi|^2}1_{\{|\xi|\le 1\}}\mathcal{F}_{x\to\xi}[v]\Big] + \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{i\xi_k}{|\xi|^2}1_{\{|\xi|>1\}}\mathcal{F}_{x\to\xi}[v]\Big].$$

Since v belongs to $L^1(\mathbb{R}^N)$, the function $\mathcal{F}_{x\to\xi}[v]$ is uniformly bounded; whence the quantity $\frac{i\xi_k}{|\xi|^2}1_{\{|\xi|\le 1\}}\mathcal{F}_{x\to\xi}[v]$ is integrable. Similarly, v being square integrable, $\mathcal{F}_{x\to\xi}[v]$ enjoys the same property so that $\frac{i\xi_k}{|\xi|^2}1_{\{|\xi|>1\}}\mathcal{F}_{x\to\xi}[v]$ is square integrable as well. After these observations, estimate (11.102) follows immediately from the basic properties of the Fourier transform, see Sect. 5.

Step 2 We introduce $\mathcal{E}(x)$—the fundamental solution of the Laplace operator, specifically,

$$\Delta_x\mathcal{E} = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{11.105}$$

where δ denotes the Dirac distribution. If $N \ge 2$, $\partial_{x_k}\mathcal{E}$ takes the form

$$\partial_{x_k}\mathcal{E}(x) = \frac{1}{a_N}\frac{1}{|x|^{N-1}}\frac{x_k}{|x|}, \text{ where } a_N = \left\{ \begin{array}{c} 2\pi \text{ if } N=2 \\ \\ (N-2)\sigma_N \text{ if } N>2 \end{array}\right\} \tag{11.106}$$

with σ_N being the area of the unit sphere. From (11.105) we easily deduce that

$$\mathcal{F}_{x\to\xi}[\partial_{x_k}\mathcal{E}] = \frac{1}{(2\pi)^{N/2}}\frac{i\xi_k}{|\xi|^2}.$$

Consequently,

$$\partial_{x_k}\mathcal{E} * v = \mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{F}_{x\to\xi}[\partial_{x_k}\mathcal{E} * v]\Big] = \frac{1}{(2\pi)^{N/2}}\mathcal{F}^{-1}_{\xi\to x}\Big[\frac{i\xi_k}{|\xi|^2}\mathcal{F}_{x\to\xi}[v]\Big]$$

where the weakly singular operator $v \to \partial_{x_k}\mathcal{E} * v$ is continuous from $L^p(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$, $\frac{1}{r} = \frac{N-1}{N} + \frac{1}{p} - 1$, provided $1 < p < N$ as a consequence of the classical results of harmonic analysis stated in Theorem 11.15. This completes the proof of parts (i), (ii).

Step 3 If $v \in C^\infty_c(\bar{I} \times \mathbb{R}^3)$, statement (iii) follows directly from the theorem on differentiation of integrals with respect to a parameter. Its L^p-version can be proved via the density arguments.

□

In order to conclude this section, we recall several elementary formulas that can be verified by means of direct computation.

$$\mathcal{R}_{j,k}[f] = \partial_j \mathcal{A}_k[f] = -\mathcal{R}_j\Big[\mathcal{R}_k[f]\Big],$$

$$\mathcal{R}_j\Big[\mathcal{R}_k[f]\Big] = \mathcal{R}_k\Big[\mathcal{R}_j[f]\Big],$$

$$\textstyle\sum_{k=1}^N \mathcal{R}_k\Big[\mathcal{R}_k[f]\Big] = f \qquad (11.107)$$

$$\textstyle\int_\Omega \mathcal{A}_k[f]\overline{g}\,\mathrm{d}x = -\int_\Omega f\overline{\mathcal{A}_k[g])}\,\mathrm{d}x,$$

$$\textstyle\int_\Omega \mathcal{R}_j\Big[\mathcal{R}_k[f]\Big]\overline{g}\,\mathrm{d}x = \int_\Omega f\overline{\mathcal{R}_j\Big[\mathcal{R}_k[g]\Big]}\,\mathrm{d}x.$$

These formulas hold for all $f, g \in \mathcal{S}(\mathbb{R}^N)$ and can be extended by density in accordance with Theorems 11.33, 11.32 to $f \in L^p(\mathbb{R}^N)$, $g \in L^{p'}(\mathbb{R}^N)$, $1 < p < \infty$, whenever the left and right hand sides make sense. We also notice that functions $\mathcal{A}_k(f)$, $\mathcal{R}_{j,k}(f)$ are real valued functions provided f is real valued.

11.18 Commutators Involving Riesz Operators

This section presents two important results involving Riesz operators. The first one represents a keystone in the proof of the weak continuity property of the effective pressure. Its formulation and proof are taken from [101, 117].

■ COMMUTATORS INVOLVING RIESZ OPERATORS, WEAK CONVERGENCE:

Theorem 11.34 *Let*

$$\mathbf{V}_\varepsilon \to \mathbf{V} \text{ weakly in } L^p(\mathbb{R}^N; \mathbb{R}^N),$$

$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } L^q(\mathbb{R}^N; \mathbb{R}^N),$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1$. *Then*

$$\mathbf{U}_\varepsilon \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \mathcal{R}[\mathbf{U}_\varepsilon] \cdot \mathbf{V}_\varepsilon \to \mathbf{U} \cdot \mathcal{R}[\mathbf{V}] - \mathcal{R}[\mathbf{U}] \cdot \mathbf{V} \text{ weakly in } L^s(\mathbb{R}^N).$$

Proof Writing

$$\mathbf{U}_\varepsilon \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] = \Big(\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]\Big) \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \Big(\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]\Big) \cdot \mathcal{R}[\mathbf{U}_\varepsilon]$$

we easily check that

$$\mathrm{div}_x\Big(\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]\Big) = \mathrm{div}_x\Big(\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]\Big) = 0,$$

while $\mathcal{R}[\mathbf{U}_\varepsilon]$, $\mathcal{R}[\mathbf{V}_\varepsilon]$ are gradients, in particular

$$\mathbf{curl}_x \mathcal{R}[\mathbf{U}_\varepsilon] = \mathbf{curl}_x \mathcal{R}[\mathbf{V}_\varepsilon] = 0.$$

Thus the desired conclusion follows from Div-Curl Lemma (Theorem 11.28). □

The following result is in the spirit of Coifman, Meyer [62]. The main ideas of the proof are taken over from [87].

■ COMMUTATORS INVOLVING RIESZ OPERATORS, BOUNDEDNESS IN SOBOLEV-SLOBODECKII SPACES:

Theorem 11.35 *Let $w \in W^{1,r}(\mathbb{R}^N)$ and $\mathbf{V} \in L^p(\mathbb{R}^N;\mathbb{R}^N)$ be given, where*

$$1 < r < N, \quad 1 < p < \infty, \quad \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1.$$

Then for any s satisfying

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1,$$

there exists

$$\beta = \beta(s,p,r) \in (0,1), \quad \frac{\beta}{N} = \frac{1}{s} + \frac{1}{N} - \frac{1}{p} - \frac{1}{r}$$

such that

$$\Big\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \Big\|_{W^{\beta,s}(\mathbb{R}^N;\mathbb{R}^N)} \le c\|w\|_{W^{1,r}(\mathbb{R}^N)} \|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)},$$

where $c = c(s,p,r)$ is a positive constant.

Proof We may suppose without loss of generality that $w \in C_c^\infty(\mathbb{R}^N)$, $\mathbf{V} \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$. First we notice that the norms

$$\|\mathbf{a}\|_{W^{1,m}(\mathbb{R}^N;\mathbb{R}^N)} \text{ and } \|\mathbf{a}\|_{L^m(\mathbb{R}^N;\mathbb{R}^N)} + \|\mathbf{curl}_x\mathbf{a}\|_{L^m(\mathbb{R}^N;\mathbb{R}^N)} + \|\mathrm{div}_x\mathbf{a}\|_{L^m(\mathbb{R}^N)} \qquad (11.108)$$

are equivalent for $1 < m < \infty$, see Theorem 11.25. We also verify by a direct calculation that

$$[(\mathrm{curl}_x(\mathcal{R}[w\mathbf{V}])]_{j,k} = 0, \; [\mathrm{curl}_x(w\mathcal{R}[\mathbf{V}])]_{j,k} = \partial_{x_k} w\, \mathcal{R}_{j,s}[V_s] - \partial_{x_j} w\, \mathcal{R}_{k,s}[V_s], \qquad (11.109)$$

and

$$\mathrm{div}_x(\mathcal{R}[w\mathbf{V}]) - \mathrm{div}_x\Big(w\mathcal{R}[\mathbf{V}]\Big) = \sum_{j=1}^N \partial_{x_j} w\, V_j - \sum_{i,j=1}^N \partial_{x_i} w\, \mathcal{R}_{i,j}[V_j]. \qquad (11.110)$$

Next we observe that for any s, $\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$ there exist $1 \leq r_1 = r_1(s,p) < r < r_2 = r_2(s,p) < \infty$ such that

$$\frac{1}{r_1} + \frac{1}{p} - \frac{1}{N} = \frac{1}{s} = \frac{1}{r_2} + \frac{1}{p}.$$

Taking advantage of (11.108)–(11.110) and using Theorem 11.32 together with the Hölder inequality, we may infer that

$$\Big\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\Big\|_{W^{1,s}(\mathbb{R}^N;\mathbb{R}^N)} \leq c\|w\|_{W^{1,r_2}(\mathbb{R}^N)}\|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}. \qquad (11.111)$$

On the other hand, Theorem 11.32 combined with the continuous embedding $W^{1,r_1}(\mathbb{R}^N) \hookrightarrow L^{\frac{Nr_1}{N-r_1}}(\mathbb{R}^N)$, and the Hölder inequality yield

$$\Big\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\Big\|_{L^s(\mathbb{R}^N;\mathbb{R}^N)} \leq c\|w\|_{W^{1,r_1}(\mathbb{R}^N)}\|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}. \qquad (11.112)$$

We thus deduce that, for any fixed $\mathbf{V} \in L^p(\Omega;\mathbb{R}^N)$, the linear operator $w \to \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]$ is a continuous linear operator from $W^{1,r_2}(\Omega)$ to $W^{1,s}(\Omega,\mathbb{R}^N)$ and from $W^{1,r_1}(\Omega)$ to $L^s(\Omega;\mathbb{R}^N)$. Now we conclude by the Riesz-Thorin interpolation theorem (see [257]) that this operator is as well continuous from $W^{1,r}(\Omega)$ to $W^{\beta,s}(\Omega)$, where $\beta \in (0,1)$ verifies the formula $\frac{\beta}{r_1} + \frac{1-\beta}{r_2} = \frac{1}{r}$.

This finishes the proof.

□

11.19 Renormalized Solutions to the Equation of Continuity

In this section we explain the main ideas of the regularization technique developed by DiPerna and Lions [85] and discuss the basic properties of the renormalized solutions to the equation of continuity. To begin, we introduce a variant of the classical Friedrichs commutator lemma.

■ FRIEDRICHS' COMMUTATOR LEMMA IN SPACE:

Lemma 11.12 *Let* $N \geq 2$, $\beta \in [1, \infty)$, $q \in [1, \infty]$, *where* $\frac{1}{q} + \frac{1}{\beta} = \frac{1}{r} \in (0, 1]$. *Suppose that*

$$\varrho \in L^{\beta}_{\mathrm{loc}}(\mathbb{R}^N), \ \mathbf{u} \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N).$$

Then

$$\mathrm{div}_x\Big(S_\epsilon[\varrho\mathbf{u}]\Big) - \mathrm{div}_x\Big(S_\epsilon[\varrho]\mathbf{u}\Big) \to 0 \textit{ in } L^r_{\mathrm{loc}}(\mathbb{R}^N), \tag{11.113}$$

where S_ε *is the mollifying operator introduced in (11.3)–(11.4).*

Proof We have

$$\mathrm{div}_x\Big(S_\epsilon[\varrho\mathbf{u}]\Big) - \mathrm{div}_x\Big(S_\epsilon[\varrho]\mathbf{u}\Big) = I_\varepsilon - S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u},$$

where

$$I_\varepsilon(x) = \int_{\mathbb{R}^N} \varrho(y)[\mathbf{u}(y) - \mathbf{u}(x)] \cdot \nabla_x \zeta_\varepsilon(x - y)\mathrm{d}y. \tag{11.114}$$

According to Theorem 11.3,

$$S_\varepsilon(\varrho)\mathrm{div}_x\mathbf{u} \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^N);$$

whence it is enough to show that

$$I_\varepsilon \to \varrho\mathrm{div}_x\mathbf{u} \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^N). \tag{11.115}$$

After a change of variables $y = x + \varepsilon z$, formula (11.114) reads

$$\begin{aligned} I_\varepsilon(x) &= \textstyle\int_{|z|\leq 1} \varrho(x + \varepsilon z)\frac{\mathbf{u}(x+\varepsilon z) - \mathbf{u}(x)}{\varepsilon} \cdot \nabla_x \zeta(z)\mathrm{d}z \\ &= \textstyle\int_0^1 \int_{|z|\leq 1} \varrho(x + \varepsilon z)\, z \cdot \nabla_x \mathbf{u}(x + \varepsilon t z) \cdot \nabla_x \zeta(z)\mathrm{d}z\,\mathrm{d}t, \end{aligned} \tag{11.116}$$

where we have used the *Lagrange formula*

$$\mathbf{u}(\xi + \varepsilon z) - \mathbf{u}(\xi) = \varepsilon \int_0^1 z \cdot \nabla_x \mathbf{u}(\xi + \varepsilon t z) \mathrm{d}t.$$

From (11.116) we deduce a general estimate

$$\|I_\varepsilon\|_{L^s(B_R)} \leq c(\overline{r}, s, p, q) \, \|\varrho\|_{L^p(B_{\overline{r}+1})} \| \, \|\nabla_x \vec{u}\|_{L^q(B_{\overline{r}+1})}, \tag{11.117}$$

where $B_{\overline{r}}$ is a ball of radius $\overline{r}$ in $\mathbb{R}^N$, and where

$$\left\{ \begin{array}{c} s \text{ is arbitrary in } [1, \infty) \text{ if } p = q = \infty, \\ \\ \frac{1}{s} = \frac{1}{q} + \frac{1}{p} \text{ if } \frac{1}{q} + \frac{1}{p} \in (0, 1] \end{array} \right\}.$$

Formula (11.117) can be used with $\varrho_n - \varrho$ and $p = \beta$, q and $s = r$, where $\varrho_n \in C_c(\mathbb{R}^N)$, $\varrho_n \to \varrho$ strongly in $L^\beta_{\text{loc}}(\mathbb{R}^N)$, in order to justify that it is enough to show (11.115), with ϱ belonging to $C_c(\mathbb{R}^N)$. For such a ϱ, we evidently have

$$I_\varepsilon(x) \to [\varrho \mathrm{div}_x \mathbf{u}](x) \text{ a. a. in } \mathbb{R}^N$$

as is easily seen from (11.116). Moreover, formula (11.117) now with $p = \infty$, yields I_ε bounded in $L^s(B_{\overline{r}})$ with $s > r$. This observation allows us obtain the desired conclusion by means of Vitali's convergence theorem.

□

In the case of a time dependent scalar field ϱ and a vector field $\mathbf{u}$, Lemma 11.117 gives rise to the following corollary.

■ FRIEDRICHS COMMUTATOR LEMMA IN TIME-SPACE:

Corollary 11.3 *Let* $N \geq 2$, $\beta \in [1, \infty)$, $q \in [1, \infty]$, $\frac{1}{q} + \frac{1}{\beta} = \frac{1}{r} \in (0, 1]$. *Suppose that*

$$\varrho \in L^\beta_{\text{loc}}((0, T) \times \mathbb{R}^N), \ \mathbf{u} \in L^q_{\text{loc}}(0, T; W^{1,q}_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)).$$

Then

$$\mathrm{div}_x \Big(S_\epsilon[\varrho \mathbf{u}] \Big) - \mathrm{div}_x \Big(S_\epsilon[\varrho] \mathbf{u} \Big) \to 0 \textit{ in } L^r_{\text{loc}}((0, T) \times \mathbb{R}^N), \tag{11.118}$$

where S_ε is the mollifying operator introduced in (11.3)–(11.4) acting solely on the space variables.

With Lemma 11.12 and Corollary 11.3 at hand, we can start to investigate the renormalized solutions to the continuity equation.

■ RENORMALIZED SOLUTIONS OF THE CONTINUITY EQUATION I:

Theorem 11.36 *Let* $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} \in (0,1]$. *Suppose that the functions* $(\varrho, \mathbf{u}) \in L^{\beta}_{\rm loc}((0,T)\times\mathbb{R}^N) \times L^q_{\rm loc}(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, *where* $\varrho \geq 0$ *a. e. in* $(0,T)\times\mathbb{R}^N$, *satisfy the transport equation*

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = f \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N), \tag{11.119}$$

where $f \in L^1_{\rm loc}((0,T)\times\mathbb{R}^N)$.

Then

$$\partial_t b(\varrho) + {\rm div}_x\Big((b(\varrho)\mathbf{u}\Big) + \Big(\varrho b'(\varrho) - b(\varrho)\Big){\rm div}_x\mathbf{u} = f b'(\varrho) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N) \tag{11.120}$$

for any

$$b \in C^1([0,\infty)), \quad b' \in C_c([0,\infty)). \tag{11.121}$$

Proof Taking convolution of (3.198) with ζ_ε (see (11.3)–(11.4)), that is to say using $\zeta_\varepsilon(x-\cdot)$ as a test function, we obtain

$$\partial_t\Big(S_\varepsilon[\varrho]\Big) + {\rm div}_x\Big(S_\varepsilon[\varrho]\mathbf{u}\Big) = \wp_\varepsilon(\varrho,\mathbf{u}), \tag{11.122}$$

where

$$\wp_\varepsilon(\varrho,\mathbf{u}) = {\rm div}_x\Big(S_\varepsilon[\varrho]\mathbf{u}\Big) - {\rm div}_x S_\varepsilon[\varrho\mathbf{u}] \text{ a.e. in } (0,T)\times\mathbb{R}^N.$$

Equation (11.122) can be multiplied on $b'(S_\varepsilon[\varrho])$, where b is a globally Lipschitz function on $[0,\infty)$; one obtains

$$\partial_t b\left(S_\varepsilon[\varrho]\right) + {\rm div}_x\left[b\left(S_\varepsilon[\varrho]\right)\mathbf{u}\right] \tag{11.123}$$

$$+\Big[S_\varepsilon[\varrho]b'\left(S_\varepsilon[\varrho]\right) - b\left(S_\varepsilon[\varrho]\right)\Big] = \wp_\varepsilon(\varrho,\mathbf{u})\, b'\left(S_\varepsilon[\varrho]\right).$$

It is easy to check that for $\varepsilon \to 0+$ the left hand side of (11.123) tends to the desired expression appearing in the renormalized formulation of the continuity equation (11.120). Moreover, the right hand side tends to zero as a direct consequence of Corollary 11.3.

□

Once the renormalized continuity equation is established for any b belonging to (11.121), it is satisfied for any "renormalizing" function b belonging a larger class. This is clarified in the following lemma.

■ RENORMALIZED SOLUTIONS OF THE CONTINUITY EQUATION II:

Lemma 11.13 *Let* $N \geq 2$, $\beta \in [1,\infty)$, $q \in [1,\infty]$, $\frac{1}{q} + \frac{1}{\beta} \in (0,1]$. *Suppose that the functions* $(\varrho, \mathbf{u}) \in L^{\beta}_{\rm loc}((0,T) \times \mathbb{R}^N) \times L^q_{\rm loc}(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, *where* $\varrho \geq 0$ *a. e. in* $(0,T)\times\mathbb{R}^N$, *satisfy the renormalized continuity equation (11.120) for any b belonging to the class (11.121).*

Then we have:

(i) *If* $f \in L^p_{\rm loc}((0,T)\times\mathbb{R}^N)$ *for some* $p > 1$, $p'(\frac{\beta}{q'} - 1) \leq \beta$, *then Eq. (11.120) holds for any*

$$b \in C^1([0,\infty)),\ |b'(s)| \leq cs^{\lambda},\ \textit{for } s > 1,\ \textit{where } \lambda \leq \frac{\beta}{q'} - 1. \tag{11.124}$$

(ii) *If* $f = 0$, *then Eq. (11.120) holds for any*

$$b \in C([0,\infty)) \cap C^1((0,\infty)),$$

$$\lim_{s\to 0+}\Big(sb'(s) - b(s)\Big) \in \mathbb{R}, \tag{11.125}$$

$$|b'(s)| \leq cs^{\lambda} \textit{ if } s \in (1,\infty) \textit{ for a certain } \lambda \leq \frac{\beta}{q'} - 1$$

(iii) *The function* $z \to b(z)$ *in any of the above statements* **(i)–(ii)** *can be replaced by* $z \to cz + b(z)$, $c \in \mathbb{R}$, *where b satisfies (11.124) or (11.125) as the case may be.*

(iv) *If* $f = 0$, *then*

$$\partial_t\Big(\varrho B(\varrho)\Big) + {\rm div}_x\Big(\varrho B(\varrho)\mathbf{u}\Big) + b(\varrho){\rm div}_x\mathbf{u} = 0 \textit{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N) \tag{11.126}$$

for any

$$b \in C([0,\infty)) \cap L^{\infty}(0,\infty),\ B(\varrho) = B(1) + \int_1^{\varrho} \frac{b(z)}{z^2}{\rm d}z \tag{11.127}$$

Proof Statement (**i**) can be deduced from (11.120) by approximating conveniently the functions b satisfying relation (11.124) by functions belonging to the class $C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ and using consequently the Lebesgue dominated or Vitali's and the Beppo-Levi monotone convergence theorems. We can take a sequence $S_{\frac{1}{n}}(b\circ\mathcal{T}_n)$, $n\to\infty$, where $\mathcal{T}_n$ is defined by (11.76), and with the mollifying operator $S_{\frac{1}{n}}$ introduced in (11.3)–(11.4).

Statement **(ii)** follows from **(i)**: The renormalized continuity equation (11.121) certainly holds for $b_h(\cdot) := b(h+\cdot)$. Thus we can pass to the limit $h\to 0+$, take advantage of condition $\lim_{s\to 0+}(sb'(s)-b(s))\in R$, and apply the Lebesgue dominated convergence.

Statement **(iii)** results from summing the continuity equation with the renormalized continuity equation.

The function $z\to zB(z)$ satisfies assumptions (11.125). Statement **(iv)** thus follows immediately from **(ii)**.

□

Next, we shall investigate the pointwise behavior of renormalized solutions with respect to time.

■ TIME CONTINUITY OF RENORMALIZED SOLUTIONS

Lemma 11.14 *Let* $N\geq 2$, $\beta,q\in(1,\infty)$, $\frac{1}{q}+\frac{1}{\beta}\in(0,1]$. *Suppose that the functions* $(\varrho,\mathbf{u})\in L^\infty(0,T;L^\beta_{\rm loc}(\mathbb{R}^N))\times L^q(0,T;W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, $\varrho\geq 0$ *a.a. in* $(0,T)\times\mathbb{R}^N$, *satisfy continuity equation (11.119) with* $f\in L^s_{\rm loc}((0,T)\times\Omega)$, $s>1$, *and renormalized continuity equation (11.120) for any b belonging to class (11.121).*

Then

$$\varrho\in C_{\rm weak}([0,T];L^\beta(O))\cap C([0,T],L^p(O))$$

with any $1\leq p<\beta$ *and* O *any bounded domain in* $\mathbb{R}^N$.

Proof According to Lemma 11.13,

$$\partial_t\sigma+{\rm div}_x(\sigma\mathbf{u})=\frac{1}{2}\sigma{\rm div}_x\mathbf{u}\ \text{in}\ \mathcal{D}'((0,T)\times\mathbb{R}^N),$$

where we have set $\sigma=\sqrt{\varrho}$; we may therefore assume that

$$\sigma\in C_{\rm weak}([0,T];L^{2\beta}(O))\quad\text{for any bounded domain } O\subset\mathbb{R}^N. \tag{11.128}$$

Regularizing the latter equation over the space variables, we obtain

$$\partial_t\left(S_\varepsilon[\sigma]\right)+{\rm div}_x\left(S_\varepsilon[\sigma]\mathbf{u}\right)=\frac{1}{2}S_\varepsilon\left[\sigma{\rm div}_x\mathbf{u}\right]+\wp_\varepsilon(\sigma,\mathbf{u})\quad\text{a.a. in } (0,T)\times\mathbb{R}^N,$$

where S_ε and $\wp_\varepsilon$ are the same as in the proof of Theorem 11.36. Now, applying to the last equation Theorem 11.36 and Lemma 11.13, we get

$$\partial_t (S_\varepsilon[\sigma])^2 + \mathrm{div}_x\Big((S_\varepsilon[\sigma])^2 \mathbf{u}\Big) = S_\varepsilon[\sigma] S_\varepsilon (\sigma \mathrm{div}_x \mathbf{u})$$

$$+ 2S_\varepsilon[\sigma]\wp_\varepsilon(\sigma, \mathbf{u}) - (S_\varepsilon[\sigma])^2 \mathrm{div}_x \mathbf{u} \quad \text{a.a. in } (0,T) \times \mathbb{R}^N. \tag{11.129}$$

We employ Eq. (11.129) together with Theorem 11.3 and Corollary 11.3 to verify that the sequence $\{\int_\Omega (S_\varepsilon[\sigma])^2 \eta \,\mathrm{d}x\}_{\varepsilon>0}$, $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfies assumptions of Arzelà-Ascoli theorem on $C([0,T])$. Combining this information with separability of $L^{\beta'}(O)$ and the density argument, we may infer that

$$\int_O (S_\varepsilon[\sigma])^2 \eta \,\mathrm{d}x \to \int_O \overline{\sigma^2}(t) \eta \,\mathrm{d}x \quad \text{in } C([0,T]).$$

for any $\eta \in L^{\beta'}(O)$.

On the other hand, Theorem 11.3 yields

$$(S_\varepsilon[\sigma])^2 (t) \to \sigma^2(t) \quad \text{in } L^\beta(O) \text{ for all } t \in [0,T];$$

therefore $\int_O \overline{\sigma^2} \eta \,\mathrm{d}x = \int_O \sigma^2 \eta \,\mathrm{d}x$ on $[0,T]$ and

$$\sigma^2 \in C_{\mathrm{weak}}([0,T]; L^\beta(O)). \tag{11.130}$$

Relations (11.128) and (11.130) yield $\sigma \in C([0,T]; L^2(O))$, whence we complete the proof by a simple interpolation argument.

□

We conclude this section with a compactness result involving the renormalized continuity equation.

Theorem 11.37 *Let* $N \geq 2$, $\beta > \frac{2N}{N+2}$, Ω *be a bounded Lipschitz domain in* $\mathbb{R}^N$, $T > 0$, *and*

$$B \in C([0,T] \times \overline{\Omega} \times [0,\infty)), \quad \sup_{(t,x)\in(0,T)\times\Omega} |B(t,x,s)| \leq c(1+s^p), \tag{11.131}$$

where c *is a positive constant, and* $0 < p < \frac{N+2}{2N}\beta$ *is a fixed number.*

Suppose that $\{\varrho_n \geq 0, \mathbf{u}_n\}_{n=1}^\infty$ *is a sequence with the following properties:*

(i)

$$\begin{aligned} &\varrho_n \to \varrho \ \ \textit{weakly} \ -(*) \ \textit{in} \ L^\infty(0,T; L^\beta(\Omega)), \\ &\mathbf{u}_n \to \mathbf{u} \ \textit{weakly in} \ L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^N)); \end{aligned} \tag{11.132}$$

(ii)

$$\int_0^T \int_\Omega \Big(a(\varrho_n)\partial_t\varphi + a(\varrho_n)\mathbf{u}_n \cdot \nabla_x\varphi - (\varrho_n a'(\varrho_n) - a(\varrho_n))\mathrm{div}_x\mathbf{u}_n \Big)\,\mathrm{d}x\mathrm{d}t = 0 \tag{11.133}$$

for all $a \in C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$, *and for all* $\varphi \in C_c^\infty((0,T)\times\overline{\Omega})$.

Then the sequence $\{B(\cdot,\cdot,\varrho_n)\}_{n=1}^\infty$ *is precompact in the space* $L^s(0,T;W^{-1,2}(\Omega))$ *for any* $s \in [1,\infty)$.

Proof **Step 1** Due to Corollary 11.2 and in accordance with assumptions (11.131)–(11.133),

$$\sup_{n\in\mathbb{N}} \|B(\cdot,\cdot,\mathcal{T}_k(\varrho_n)) - B(\cdot,\cdot,\varrho_n)\|_{L^{\frac{2N}{N+2}}(\Omega)} \to 0 \text{ as } k \to \infty,$$

where $\mathcal{T}_k$ is the truncation function introduced in (11.76). Since $L^\beta(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ whenever $\beta > \frac{2N}{N+2}$, it is enough to show precompactness of the sequence of composed functions $B(\cdot,\cdot,\mathcal{T}_k(\varrho_n))$.

Step 2 According to the Weierstrass approximation theorem, there exists a polynomial A_ε on $\mathbb{R}^{N+2}$ such that

$$\|A_\varepsilon - B\|_{C([0,T]\times\overline{\Omega}\times[0,2k])} < \varepsilon,$$

where $\varepsilon > 0$. Therefore,

$$\sup_{n\in\mathbb{N}} \|A_\varepsilon(\cdot,\cdot,\mathcal{T}_k(\varrho_n) - B(\cdot,\cdot,\mathcal{T}_k(\varrho_n)\|_{L^\infty((0,T)\times\Omega)} < \varepsilon.$$

Consequently, it is merely enough to show precompactness of any sequence of type $a_1(t)a_2(x)a(\varrho_n)$, where $a_1 \in C^1([0,T])$, $a_2 \in C^1(\overline{\Omega})$, and where a belongs to $C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$. However, this is equivalent to proving precompactness of the sequence $a(\varrho_n)$, $a \in C^1([0,\infty))$.

Step 3 Since ϱ_n, $\mathbf{u}_n$ solve Eq. (11.133), we easily check that the functions $t \to [\int_\Omega a(\varrho_n)\varphi\,\mathrm{d}x](t)$ form a bounded and equi-continuous sequence in $C([0,T])$ for all $\varphi \in C_c^\infty(\Omega)$. Consequently, the standard Arzelà-Ascoli theorem combined with the separability of $L^{\beta'}(\Omega)$ yields, via density argument and a diagonalization procedure, the existence of a function $\overline{a(\varrho)} \in C_{\mathrm{weak}}([0,T];L^\beta(\Omega))$ satisfying

$$\int_\Omega a(\varrho_n)\varphi\,\mathrm{d}x \to \int_\Omega \overline{a(\varrho)}\varphi\,\mathrm{d}x \text{ in } C([0,T]) \text{ for all } \varphi \in L^{\beta'}(\Omega)$$

at least for a chosen subsequence. Since $L^\beta(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$, we deduce that

$$a(\varrho_n)(t,\cdot) \to \overline{a(\varrho)}(t,\cdot) \text{ strongly in } W^{-1,2}(\Omega) \text{ for all } t \in [0,T].$$

Thus applying Vitali's theorem to the sequence $\{\|a(\varrho_n)\|_{W^{-1,2}(\Omega)}\}_{n=1}^{\infty}$, which is bounded in $L^{\infty}(0,T)$ completes the proof.

□

11.20 Transport Equation and the Euler System

For a given vector field $\mathbf{w} = \mathbf{w}(t,x)$, consider the transport equation

$$\partial_t U + \mathbf{w} \cdot \nabla_x U = 0,\ U(0,x) = U_0(x). \tag{11.134}$$

We also define a *weak solution* to the transport equation in $(0,T) \times R^N$ via a family of integral identities

$$\int_0^T \int_{\mathbb{R}^N} (U \partial_t \varphi + U \mathbf{w} \cdot \nabla_x \varphi + U \mathrm{div}_x \mathbf{w} \varphi)\ \mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^N} U_0 \varphi(0,\cdot)\,\mathrm{d}x \tag{11.135}$$

for any $\varphi \in C_c^{\infty}([0,T) \times R^N)$.

Solutions of (11.134) can be computed by the method of characteristics. Specifically, supposing we can solve the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,x) = \mathbf{w}\left(t, \mathbf{X}(t,x)\right),\ \mathbf{X}(0,x) = x,$$

we may take

$$U\left(t, \mathbf{X}(t,x)\right) = U_0(x),\ t \geq 0,\ x \in \mathbb{R}^N.$$

More specifically, the following holds.

■ CHARACTERISTICS AND TRANSPORT EQUATION:

Theorem 11.38 *Let the vector field* $\mathbf{w}$ *belongs to the class*

$$\mathbf{w} \in L^{\infty}((0,T) \times \mathbb{R}^N; \mathbb{R}^N),\ \nabla_x \mathbf{w} \in L^1(0,T; L^{\infty}(\mathbb{R}^N; \mathbb{R}^{N\times N})).$$

Then for any $U_0 \in L^{\infty}(R^N)$ *the problem (11.134) admits a solution* U *determined by the method of characteristics. Moreover, the solution is unique in the class of weak solutions satisfying (11.135).*

See DiPerna and Lions [85] □

Finally, we consider the incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0,\ \mathrm{div}_x \mathbf{v} = 0,\ \mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{11.136}$$

■ Classical solutions to the Euler system:

Theorem 11.39 *Let* $\mathbf{v}_0 \in W^{m,2}(\mathbb{R}^N)$ *be given such that*

$$m > \left[\frac{N}{2}\right] + 1,\ N = 2, 3,\ \mathrm{div}_x \mathbf{v}_0 = 0.$$

Then the initial-value problem (11.136) admits a classical solution $\mathbf{v}$*, unique in the class*

$$\mathbf{v} \in C([0, T_{\max}); W^{m,2}(\mathbb{R}^N; \mathbb{R}^N)),\ \Pi \in C([0, T_{\max}); W^{m,2}(\mathbb{R}^N)),$$

$$\partial_t \mathbf{v} \in C([0, T_{\max}); W^{m-1,2}(\mathbb{R}^N; \mathbb{R}^N))$$

defined on some maximal time interval $[0, T_{\max})$*, where* $T_{\max} > 0$ *if* $N = 3$ *and* $T_{\max} = \infty$ *if* $N = 2$.

See Kato and Lai [167] □

Finally, we remark that vorticity $\mathbf{w} = \mathbf{curl}_x \mathbf{v}$ satisfies the pure transport equation

$$\partial_t \mathbf{w} + \mathbf{v} \cdot \nabla_x \mathbf{w} = 0 \text{ if } N = 2,$$

and

$$\partial_t \mathbf{w} + \mathbf{v} \cdot \nabla_x \mathbf{w} = \mathbf{w} \cdot \nabla_x \mathbf{v} \text{ if } N = 3.$$

Therefore the theory of the transport equation (11.134) may be applied as long as the velocity field $\mathbf{v}$ is smooth.