Chapter 2
The Hidden Musicality of Math Class: A Transdisciplinary Approach to Mathematics Education

Song A. An, Daniel A. Tillman, and Lawrence M. Lesser

Abstract This chapter surveys interdisciplinary pedagogy that emphasizes the connections between mathematics and music by contextualizing the mathematics learning process within musical experiences. Both empirical research and international practice have demonstrated a variety of opportunities for music-themed mathematics teaching methods to be developed and implemented across all grade levels, from kindergarten to college. This chapter, which summarizes the current state of research and practice for music-themed interdisciplinary mathematics education, is divided into three main sections: (1) the overview of connection between mathematics and music, (2) theoretical perspectives on music and mathematics learning, and (3) a description of pedagogical approaches appropriate for supporting music-mathematics interdisciplinary lessons. Regarding the overview, the chapter discusses research studies that have investigated the mathematics present within music and the application of mathematics to improving musical composition and musical instrument design. Regarding the theoretical perspectives, the chapter discusses research studies that have investigated passive musical immersion as well as more active musical learning processes and their comparative impacts upon learners' mathematical cognitive processes and capabilities within informal learning settings. Regarding the pedagogical approaches, the chapter presents and evaluates the prevalent mathematics-music-integrated teaching strategies about how student-centered musical activities (i.e., listening and singing, composing and performing, musical notating, and musical instrument design) can be utilized to teach specific mathematics topics.

Keywords Music-mathematics connections • Interdisciplinary curriculum • Interdisciplinary mathematics instruction and mathematics education

S.A. An (✉) • D.A. Tillman • L.M. Lesser
The University of Texas at El Paso,
601 Education Building, 500 W. University Ave., El Paso, TX 79968, USA
e-mail: saan@utep.edu; datillman@utep.edu; lesser@utep.edu

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_2

This chapter will survey interdisciplinary pedagogy that emphasizes the connections between mathematics and music by contextualizing the mathematics learning process within musical experiences. Both empirical research and international practice have demonstrated a variety of opportunities for music-themed mathematics teaching methods to be developed and implemented across all grade levels, from kindergarten to college. This chapter, which summarizes the current state of research and practice for music-themed interdisciplinary mathematics education, is divided into three main sections: (1) the overview of connection between mathematics and music, (2) theoretical perspectives on music and mathematics learning, and (3) a description of pedagogical approaches appropriate for supporting music-mathematics interdisciplinary lessons. Regarding the overview, the chapter will discuss research studies that have investigated the mathematics present within music and the application of mathematics to improving musical composition and musical instrument design. Regarding the theoretical perspectives, the chapter will discuss research studies that have investigated passive musical immersion as well as more active musical learning processes and their comparative impacts upon learners' mathematical cognitive processes and capabilities within informal learning settings. Regarding the pedagogical approaches, the chapter will present and evaluate the prevalent mathematics-music-integrated teaching strategies about how student-centered musical activities (i.e., listening and singing, composing and performing, musical notating, and musical instrument design) can be utilized to teach specific mathematics topics.

Connections Between Mathematics and Music: An Introduction

Mathematics and music have a strong connection and have each contributed to the other's development. In this section, we present examples of how mathematics has played a significant role in acoustics, the mechanics of musical instruments, and music composition.

The Chinese bianzhong (literal meaning: *bell set*) illustrates mathematical sophistication in the construction of musical instruments. The bianzhong, a set of 65 bells, was discovered in the late 1970s inside the ancient tomb of Zeng (430 B.C.E.) in Hubei, China. This instrument was widely recognized as the first musical instrument that can play a range over five octaves (von Falkenhausen, 1993). The bianzhong has a weight of approximately 2.5 tons, and all of the bells are designed to be hung from a large wooden stand 36 feet wide by 9 feet tall. Its construction required an interdisciplinary team consisting of a partnership among musicians, mathematicians, and engineers (Lee & Shen, 1999). The bells increase in size, with the height of the smallest bell being around 8 inches whereas the length of the largest bell is more than 60 inches. Correspondingly, the bells' weights range from 5.3 to 448.9 pounds. This pattern contributes to the acoustics of this musical instrument.

Musicians and mathematicians continue to collaborate to improve musical instruments. One of the major breakthroughs for contemporary music production has been the extensive acceptance of equal temperament, a tuning method based on the computation of logarithms and differential equations in which every note in a chromatic scale has an identical frequency ratio that is artificially divided (Cho, 2003). By distributing inharmonic errors that exist within natural temperament instruments, the tunes across different instruments can be standardized; this technique thus enables musicians to change keys during their performances without changing the instruments. Applications of the equal temperament theory led to the invention of the piano in 1709.

The design of the piano required intensive knowledge of geometry and measurement as well as algebra (Ehrlich, 1990; Gordon, 1996). As an illustration of the application of geometry, cross stringing was a creative invention by Jean-Henri Pape during the 1820s that significantly reduced the size of pianos by transforming string arrangements from a two-dimensional design to a three-dimensional design. In some early versions of pianos, piano strings were arranged parallel to each other to avoid collisions during string vibrations. By utilizing the three-dimensional space, Pape arranged strings into two planes: Bass strings were secured from left to right, under and across the other strings that were secured from right to left. The theory and practice of action (translation of the motion of a piano key into the motion of a hammer that strikes the strings) was developed with the application of algebra. Because action serves as both engine and transmission, certain ratios of the movement distance between the key and the hammer have to be applied in order to create a series of lever systems that magnify the force generated from fingers pressing on keys into hammers striking the strings.

Another link between music and mathematics is the use of mathematical patterns and geometric transformation in the process of musical composition. For example, repeating patterns have been found in almost all works of music, often contained within small sections or a whole movement, and growing patterns such as the Fibonacci sequence as well as geometric transformations such as transpositions and inversions have been massively used by both classical and present-day composers (Beer, 1998; Loy, 2006). The application of mathematics to music includes algorithmic and computational approaches within musical scales (Krantz & Douthett, 2011), algebra within periodic rhythms and scales (Amiot & Sethares, 2011), topology of musical data (Sethares & Budney, 2014), logarithms and differential equations in equal temperament (Cho, 2003), and mathematical patterns within prominent pieces of classical music (Conklin, 2010). However, it should be noted that many of the links between music and mathematics are only deeply understood by music and mathematics specialists. Classroom mathematics teachers who lack familiarity with basic music theory may not have the background knowledge needed to implement music-math beyond the employment of a cover story with entertainment value, incorporating few—if any—pedagogical connections with the mathematics being taught (An, Tillman, Boren, & Wang, 2014; An, Tillman, Shaheen, & Boren, 2014).

Music and Mathematics Learning: Theoretical Perspectives

Concerning achievement, empirical evidence has shown that learning about math and learning about music are mutually beneficial. For example, in a large-scale study involving a sample of more than 150,000 high school students, researchers found a statistically significant relationship between students' music achievement and their academic success in the core subjects, especially mathematics (Gouzouasis, Guhn, & Kishor, 2007). The finding suggested that students with high mathematics achievement were predicted to have high music achievement and vice versa.

Multiple theoretical perspectives offer rationales for explaining the effectiveness of mathematics education that incorporates music or music-related experiences. In general, the educational theories that encourage music-mathematics instructional connections have two distinct dimensions, which vary in intensity depending upon the particular theory: (1) a focus on the role of music as a catalyst for boosting mathematics learners' cognitive processes by being played as ambient background sounds during mathematics education and/or (2) a focus on the role of music as an educational resource for contextualizing the teaching and learning of mathematics into a meaningful and relatable medium. Along the first dimension, the Mozart effect theory (Rauscher, Shaw, & Ky, 1993) as well as several of its variations (e.g., Hui, 2006; Ivanov & Geake, 2003) have served as a general framework for illuminating the impacts of both active music learning and passive music immersion on mathematics learners' mathematical cognition capacities. Along the second dimension, mathematical motivation theory (e.g., Bursal & Paznokas, 2006; Geist, 2010) has been used to construct an understanding of the effects of placing children in music-contextualized learning environments as well as the impacts of employing a student-centered teaching approach that employs music-themed activities to develop students' conceptual understanding in mathematics and encourage their positive dispositions toward mathematics learning.

Music as a Catalyst for Mathematical Cognition

Among the many studies examining the quantifiable associations between music experiences and their effects on mathematical cognition, the Mozart effect study series (e.g., Rauscher et al., 1993; Rauscher, Shaw, & Ky, 1995; Rauscher et al., 1997) was the most well-known as well as the most controversial research, with about 40 replicated trials involving more than 3,000 participants (Pietschnig, Voracek, & Formann, 2010). In the original research design, Rauscher and his colleagues conducted an experiment comparing three randomized groups: (1) listening to the target music (Mozart's Sonata for Two Pianos in D Major, K.448), (2) listening to comparison music (generic relaxing music), and (3) listening to silence. Results from the study demonstrated that the participants in the Mozart music group significantly outperformed their peers on the spatial reasoning skills sub-tests from the Stanford-Binet Intelligence Scale. Since the publication of the study's results, replications have assessed revisions to the settings, music treatment, and mathematics assessment tasks, accumulating in the process further evidence that listening to Mozart's music—compared to other music or silence—may advance participants' mathematical cognition (Hui, 2006; Ivanov & Geake, 2003; Nantais & Schellenberg, 1999; Rauscher et al., 1995; Rauscher & Zupan, 2000). For example, Nantais and Schellenberg conducted experiments with the random assignment of two groups of participants; treatment group participants listened to both Mozart's and Schubert's music, while the comparison group sat in silence. Two mental visualization assessments, each with 17-item paper folding and cutting tasks, were assigned to the participants after the treatment of music, and the results demonstrated that treatment group students significantly improved their test scores compared to the silence group.

In addition to the laboratory experiments, researchers have investigated music learning experiences (e.g., taking piano lessons, playing an instrument in school band, and practicing vocal music) and their relationship to students' mathematics achievement. In general, learning music and practicing music were positively correlated to students' mathematics achievement, with students who had music-related experiences demonstrating significantly higher mathematics achievement scores than their nonmusical peers. Similar patterns were identified across grade levels, including pre-K (Costa-Giomi, 1999; Rauscher & Zupan, 2000), elementary school (Haley, 2001), middle school (Whitehead, 2001), and high school (Cox & Stephens, 2006). A possible explanation for these consistent findings is that musical experiences can stimulate areas of the brain responsible for mathematical reasoning. On a similar note, Spelke (2008) explained that activating the "musical zone" in the brain may also stimulate the working processes of the "mathematical zone." In other words, within human brains, the areas responsible for processing cognitive functions for musical perception of melody, harmony, and rhythm have overlapping areas with those responsible for processing cognitive functions for mathematical computation, such as geometrical visualization, numerical calculation, and estimation.

Improving Mathematical Proficiency with Music

In an era of standards and accountability, students' academic achievement—especially their scores on high-stakes standardized tests—has become overemphasized (Pinar, 2004; Slattery, 2006). Compared to the importance placed upon developing students' procedural fluency and strategic competence in mathematics, many teachers ignore the development of their students' positive dispositions toward mathematics (Kilpatrick, Swafford, & Findell, 2001). The role of emotion has been recognized as a crucial factor in learning mathematics, and the negative emotions of disengagement and anxiety are now understood as critical obstacles on the road to success with mathematics. Empirical studies have consistently found that students in all grade levels, from kindergarten through college, display negative dispositions toward mathematics—they may believe mathematics is not useful in real life or that learning mathematics is simply too difficult for them (Rameau & Louime, 2007). Compared to peers who display productive dispositions toward mathematics, mathematics learners with negative dispositions not only suffer from significantly higher levels of anxiety during mathematics learning but also exhibit a lack of confidence or motivation to learn and apply mathematics (Ashcraft, 2002; Geist, 2010). Consequently, researchers have found that students with negative mathematics dispositions often have lower mathematics achievement and also generally avoid taking advanced mathematics courses in high school and college, which culminates in their inability to choose any of the STEM-based careers requiring a background in mathematics and/or science (Sullivan, Mousley, & Zevenbergen, 2006).

Unlike other school subjects such as social studies and language arts, which are inherently grounded in meaningful contexts and/or real-life relations, mathematics as a subject is often structured apart from society and culture and is instead based upon a language that employs complex symbols and highly abstract concepts. The distinctive structure and content of mathematics education have resulted in the development of an accompanying pedagogy for this subject, which is vastly different from other school subjects (Kilpatrick et al., 2001). Unfortunately, many school teachers, especially generalists who teach multiple subjects in elementary and middle schools, fail to offer student-appropriate methods when teaching mathematics to the demographics they serve, and this "traditional" instruction model based on the teacher-centered approach has been identified as one of the key factors influencing students' negative dispositions toward learning mathematics (Furner & Berman, 2005).

This "traditional" teacher-centered model of mathematics instruction is recognized as textbook content lecturing, overreliance on assigning drill problems, single correct answer grading, and consistent use of standardized multiple choice testing. The result is that students' conceptual understanding and strategic competencies are often ignored, while less important but more measurable metrics are pursued (Bursal & Paznokas, 2006; Geist, 2010). As an alternative to the "traditional" teacher-centered model, educational researchers have proposed some common features found among more effective pedagogical methods for instructing mathematics education, including the use of open-ended problem-solving activities in which more than one correct answer is possible, simulations including augmented and virtual reality, game-like challenges that make mathematics learning a friendly competition, and discovery-driven learning where students collect and analyze data to answer real-life contextualized questions that include such themes as finances and urban planning. All of these methods can provide opportunities to facilitate students' communication among peers, connections between and across curriculum, and representation of mathematics in multiple ways (Bursal & Paznokas, 2006; Geist, 2010). Within this sphere, a music-integrated approach to mathematics has been identified as an effective method for teaching student-centered mathematics (Robertson & Lesser, 2013).

Mathematics pedagogy that effectively uses the natural cognitive overlaps between music and mathematics offers students transdisciplinary opportunities to discover, recognize, analyze, and apply mathematics (An, Capraro, & Tillman, 2013). For teachers, music-themed mathematics activities can serve as meaningful and accessible contexts for transforming traditional mathematics pedagogy via entertainment elements (An & Tillman, 2014; Vinson, 2001). Music enables students to represent their mathematical ideas from a different perspective, which supports their learning as they pursue conceptual understanding via multiple cognitive and affective experiences (Gamwell, 2005). Specifically, findings showing positive impacts have accumulated across several studies investigating the effects of music-mathematics-integrated education, including (1) motivating students to undertake more challenging mathematics tasks (Chahine & Montiel, 2015), (2) engaging students in examining relationships among mathematical concepts (An, Tillman, Shaheen, & Boren, 2014), (3) creating a pleasant learning environment for supporting team work (Robertson & Lesser, 2013), (4) providing a teaching environment that minimizes language and culture barriers for English-language learners (Kalinec-Craig, 2015), (5) improving students' academic achievement in mathematics (An & Tillman, 2015; Pinnock, 2015), and (6) developing teachers' self-efficacy for mathematics pedagogy (An, Tillman, & Paez, 2015; An et al., 2016).

Incorporating dynamic auditory approaches in teaching, students could build knowledge cognitively, perceptually, and emotionally (Greene, 2001). However, music is an underused educational resource (An & Tillman, 2014) because teachers are required to pedagogically develop auditory resources into visual and tangible manipulatives for students to make sense of mathematics. In this chapter, we provide an overview of the empirical research studies conducted by An and his colleagues (An et al., 2013; An, Tillman, Shaheen, & Boren, 2014; An, Tillman, & Paez, 2015; An, Zhang, Flores et al., 2015). These studies examined music experiences that occurred during mathematics lessons and were designed to support students' engagement with the topic as they actively manipulated objects, performed creatively, and applied their existing knowledge in the creation of connections among mathematical concepts. During the lessons, music-making and music-sharing experiences enabled students to pursue their original interests along with their curiosity. At the affective domain, the aesthetic appreciation of music encouraged students' mathematics learning behaviors by providing them a meaningful context for completing mathematical tasks. The learning experiences also offered students additional reinforcement of the mathematical concepts by letting them play and share musical works that they had created themselves and increased their efficacy for undertaking further challenging learning tasks in mathematics.

Teaching Mathematics Via Music: Pedagogical Approaches

Curriculum developers and lesson designers have proposed many ways for emphasizing the musical connections available when teaching mathematics. For example, Gelineau (2004) and Cornett (2007), in their books about teaching elementary subjects through the arts, presented several interrelated ideas about the links between mathematics and music. In one of the author's own previous activity books (An & Capraro, 2011), teachers were offered a suite of lessons that put mathematics topics within the context of music composition and musical instrument-designing activities. Subsequently, with a goal of investigating the possibility of pedagogical connections between mathematics and music, An and colleagues conducted an empirical analysis of 78 teacher-generated music-mathematics-integrated lesson plans along with 152 elementary preservice teachers' lesson plans, enabling the identification of 56 different examples of connections between musical (music notating, singing, playing, composing, and instrument designing) and mathematical (number and operation, algebra, geometry, measurement, data analysis, and probability) content areas (An & Tillman, 2015; An, Tillman, & Paez, 2015; An et al., 2016). However, it was also determined that numerous misguided attempts at the contextualization of mathematics pedagogy within music-themed activities have occurred, and the poor implementation of this transdisciplinary concept can hinder the mathematics learning process. In other words, positive learning results in mathematics were only found in scenarios where the music-themed activities were pedagogically relevant to mathematics being taught and not merely serving as a "cover story."

Musical Notation and Fractions: An Example of Weak Pedagogical Integration

When asked "Are there any relationships between music and mathematics?," many mathematics teachers will say "Yes"; however, when asked for examples, the most frequent answer is that "There are quarter notes in music." Moreover, among the lesson plans and activities that can be implemented in classrooms, the music notation system (including musical notes and time signatures) has been used as one of the most popular activity themes for teaching mathematics, especially fractions (An & Tillman, 2014). Unfortunately, fractions are the only mathematics topic that musical notes are often used to address. As a typical instructional design based on teaching fractions connected with the musical notations of note values, one of the participating teachers proposed the following lesson plan:

> The goal of my lesson is to help students understand how musical notes relate to fractions and clap a measure of music by assigning appropriate values to notes. In the lesson I will have [the] following steps: (1) present note names and their values and introduce lesson vocabulary such as whole note [4 beats], half note [2 beats], quarter note [1 beat], eighth note [1/2 beat], sixteenth note [1/4 beat]; (2) present a music note chart to demonstrate the values each note holds and students will be clapping along to gain understanding of the values of the notes in a measure; (3) discuss the value of different notes to help students "hear" the value of those notes, clap a 4-beat measure, a 2-beat measure, and a 1-beat measure and have students join in; (4) ask questions regarding the counting values of music notes (How many beats are in 1 whole note? How many beats are in 2 half notes?) and check and correct the student answers; and (5) on a scratch sheet of paper, in groups, students will solve some math problems such as "two eighth notes equal to ______", and "four quarter notes equal to ______." (An, Tillman, & Paez, 2015, p. 16)

Fig. 2.1 Example of fraction relationships represented within music notation presented by a math teacher participating in the study

In the example presented above, the teacher attempted to integrate music and mathematics by connecting musical notes and beats to the mathematics concept of fractions. This was a weak pedagogical integration, as music symbols—a highly concentrated code developed by composers for facilitating the written communication of music—were not properly connected to fractions. At the visual level, the symbols employed for different note values (Fig. 2.1) failed to demonstrate the proportional relationships for fractions when compared with the traditional approach of conceptualizing fractions based on self-evident pictorial representations, such as area and length (Van de Valle, Karp, & Bay-Williams, 2010). At the auditory level, making non-recorded sounds from hand clapping based on different rhythms is a difficult way for students to comprehend the whole and partial relationships essential for learning fractions.

The limited working memory capacity that humans have for processing new knowledge can result in novel situations overloading cognitive capacity. According to cognitive load theory (Sweller, 2016), instructing fractions via associations with musical notation, such as reading and clapping music notes with different values, may increase students' extraneous cognitive load, overwhelming their ability to process the mathematics concepts they are supposed to be learning. In other words, introducing fractions through musical notations by claiming that, for example, the value of a black circle is half of that of a white circle of the same size, instead of helping pedagogically, might be counterproductive as the extraneous cognitive load can result in mathematics learners struggling with the lessons.

Singing and Listening to Music in Mathematics Class: More Than Simply a Cover Story

Many math lessons across all content areas incorporate singing and playing music. The availability of the Internet during the past two decades has greatly increased access to relevant musical resources for supporting such activities, and the popularity of video-sharing sites has especially intensified, with more than 7000 archived educational resources in the format of songs for teaching mathematics and science being prepared by professional musicians, educators, and enterprising students (Crowther, 2012). Numerous mathematics-themed songs—both original music created by educators and popular melodies with new lyrics—are available for teachers and ready to be used. Nevertheless, without the appropriate pedagogical structure, these music resources often only serve as cover stories providing entertainment in a mathematics class (An & Tillman, 2014). A typical instructional design that employs mathematics-themed songs without pedagogical development usually has the following fundamental steps: (1) introduce and play a music video at the beginning of the class, (2) have a mathematics class without music connections, and (3) sing the song together as a summary exercise at the end of class (An & Tillman, 2014). Such a design may set up an environment and facilitate the memorization of a mathematics algorithm or formula, but it fails to present any authentic music connections for students to actively analyze or synthesize mathematical knowledge.

In contrast, a number of mathematics and science education researchers (e.g., Crowther, Davis, Jenkins, & Breckler, 2015; Crowther, McFadden, Fleming, & Davis, 2016; Lesser, 2014; Lesser, 2015) have attempted to go beyond using songs as mere breaks to revive attention or build community during mathematics lessons. Informed by principles of the psychology of learning and some emerging scholarship on the use of songs in the STEM classroom, songs with lyrics based on mathematical concepts, terms, and formulas have been offered as supports for students seeking to understand several topics in mathematics. For example, Lesser (2014, 2015) examined mathematics-themed songs as tools to motivate underrepresented students learning middle school, high school, and college level mathematics. According to his framework, an ideal mathematics-themed song would have six traits that collectively facilitate mathematics learning: "(1) aiding recall (of procedures, properties, definitions, digits of pi, etc.), (2) introducing concepts or terms, (3) reinforcing mathematical thinking processes (e.g., the Pólya's (1945) four-step heuristic for problem solving), (4) connecting to history, (5) connecting to the real world, and (6) humanizing mathematics" (Lesser, 2015, pp. 158–159). Lesser (2015) describes how these six traits are satisfied by his lyrics in "American Pi," which received an award from the National Museum of Mathematics. Its chorus is as follows:

> Find, find the value of π, starts 3.14159
> A good ol' fraction you may hope to define,
> But the decimal never dies, never repeats or dies…. (Lesser, 2015, p. 166)

While a big part of the educational potential of a song is limited by the song itself, another part comes from how the instructor uses it. In other words, one instructor might just let students listen to an online recording of "American Pie," while another instructor might have students sing along and then actively analyze and unpack all of the mathematical references in the lyrics, making connections to their curriculum (e.g., the 16-question sequence of Appendix 2 in Lesser (2015)). A teacher doing the latter would be using the lyrics as a central vehicle to familiarize, contextualize, and conceptualize the meaning of π in multiple memorable ways. A website created by Lesser (http://www.math.utep.edu/Faculty/lesser/Mathemusician.html) provides songs for students to read or listen to as well as links to key articles and websites on the intersection of mathematics and music/songs.

Educators may consider having students consolidate and help recall their knowledge by writing their own songs or lyrics. Lyric writing is a transdisciplinary activity. As Davis (1985, p. xi) noted: "The best lyricists, whether they're aware of it or not, are using elements of phonetics, linguistics, grammar, semantics, metrics, rhyme, rhythm, poetics, phonology, communications, sociology—and even the psychology of verbal behavior." Incorporating mathematical concepts further enriches the transdisciplinary experience.

Music Composition and Playing: Awareness of Mathematical Patterns

Music composition activities offer students opportunities to compose, decompose, and recompose music. These activities can allow students to (1) explore and analyze algebraic patterns and proportional relationships, (2) make geometric transformations and use statistical methods to analyze data, (3) attempt to find multiple solutions during problem solving, and (4) design and conduct experiments that explore probabilities (e.g., explore permutation and the combination of chords and melody development processes) in self-composed or professional music works. As an example of a unique musical notation system used to facilitate novice students in composing and playing their own music, a color-based graphical notation (An & Capraro, 2011; An, Ma, & Capraro, 2011) will be presented that signifies music by using colors, shapes, numbers, and letters to represent the music notes. For example, the colors red, orange, yellow, green, turquoise, blue, and purple were used to represent the musical notes C(Do), D(Re), E(Mi), F(Fa), G(Sol), A(La), and B(Ti) (see Fig. 2.2). Based on this graphical notation system, elementary students composed music by placing a group of color cards on their desks and playing color-matched instruments, such as handbells and boomwhackers (a set of plastic tubes with the same diameter but different lengths). These activities enabled the students to examine mathematical patterns through both visual and auditory approaches while creating solutions to complex music-mathematics challenges. Two sample music composition activities are presented in the following paragraphs.

Composing for Pre-algebra Preparation This activity was designed for upper elementary students as practices for algebra readiness. Students used variables to create their own music. Each music note (a colored square of paper) was assigned a

Fig. 2.2 Color-based notation system and instruments from composition activities

Fig. 2.3 One of the possible music compositions and computation solutions

numerical value (see sample of values in Fig. 2.3), and participants composed a piece of music based on those values. For example, Do has a value of 1, Re has a value of 2, and Mi has a value of 3, so composing a music with five notes (Do, Do, Re, Re, Mi) would have a total value of 9. In one of the activities, participants were asked to compose a piece of music with 24 musical notes. In this music composition work, the sum of the musical value should equal 100 when adding the value of each musical note together (see sample mathematics arrangement and sample music composition in Fig. 2.3). After the participants finished their composition, they played their music using handbells. Unlike traditional drill questions, during which students give answers to rote questions such as 43 + 57 = __, this activity required students to use algebraic thinking while setting variables up as different arrangements in order to obtain a sum of 100. Specifically, students needed to construct an equation with seven variables, such as 1a + 2b + 3c + 4d + 5e + 6f + 7 g = 100 (a, b, c, d, e, f, and g represent values for Do, Re, Mi, Fa, Sol, La, and Ti, respectively) and then figure out the number for each letter to balance this equation. Each student created a different arrangement of colors while solving this problem and then played the different melody that they had composed as a celebration of their success at finding a valid solution. Students who finished fast had an opportunity to play their compositions and rearrange cards to generate a more "pleasing" melody. To vary the activities, teachers sometimes added more restrictions to the computational process or assigned altered values to each color in the activities, such as the use of only 16 musical notes to compose music with a total value of 120.

Total value of the music: $(1 \times 4) + (2 \times 1) + (3 \times 3) + (4 \times 4) + (5 \times 7) + (6 \times 1) + (7 \times 4) = 100$.

Composing with Ordered Pairs This activity was proposed and taught by one of the participating teachers in our previous study (An et al., 2016) for a group of fifth- and sixth-grade students. The designed lessons illustrated how upper elementary students (fourth and fifth grades) could perceive ordered pairs through visual representation during music composition and auditory representation during music playing by composing music within a Cartesian coordinate system in which the *x*-axis and the *y*-axis represent two simultaneous melodies. In this activity, Cartesian coordinates were used for students to represent harmonic intervals (i.e., a pair of notes with the same or different sounds) (see Fig. 2.4).

Several recent studies (An & Tillman, 2014; An, Tillman, & Paez, 2015) have indicated that many inquiry-based strategies for teaching mathematics can be implemented within music composition activities. For example, when teaching numbers and operations, teachers can help students (1) conceptualize a base-eight numeration system through music scales, (2) explore rational numbers through an analysis

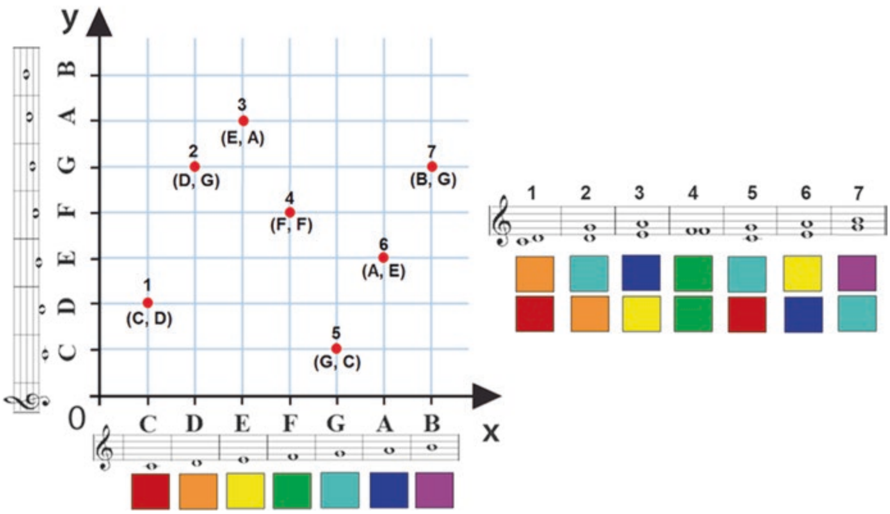

Fig. 2.4 A sample composition displaying harmonic intervals within a coordinate system

of existing or self-created music works, and (3) understand operation rules through the demonstration of chord variations. When teaching algebra, teachers can help students (1) identify ratios and proportions through an analysis of musical works, (2) find unknowns and missing values in music works, and (3) explore functions, sequence, and factors through music composition. When teaching geometry and measurement, teachers can help students (1) compose music through geometric transformations, including reflection and rotation, and (2) explore concepts about time through music composing and playing. When teaching probability and data analysis, teachers can help students (1) collect and analyze data based on music works, (2) develop statistical graphs based on music works, (3) conduct an analysis of events within musical compositions, and (4) explore combinations and permutations through chord and melody composition.

Musical Instrument Design and Construction: Mathematics-Embedded Tasks

By designing musical instruments on paper and then constructing the instrument with different materials, students can learn to understand principles of scientific inquiry and investigation as they formulate hypotheses about how changing the properties of an instrument will affect its sound and then test the hypotheses. Students have been offered opportunities to explore one-, two-, and three-dimensional geometric concepts and relationships within different types of musical instruments (e.g., idiophones, membranophones, chordophones, and aerophones) and explored acoustical physics to understand how the patterns of shapes, dimensions, and materials affect instrument sounds and tones. Specifically, empirical studies (An et al., 2013; An, Tillman, Boren, & Wang, 2014; An, Tillman, & Paez, 2015) demonstrated that musical instrument designing activities have provided students with opportunities to (1) use geometry and measurement concepts to construct different types of instruments; (2) fabricate musical instruments by using 3D printers with a variety of plastic, metal, and hybrid materials; (3) apply knowledge of sound production for basic acoustic instrument types to develop combinations of vibrating strings, pipes, bells, membranes, and reeds that allow the manipulation of variables (e.g., length, size, volume, shape, material, and tension); (4) recognize the iterative process by which a set of simple musical instruments were designed to produce a palette of music "colors" (i.e., color-coordinated musical notes); (5) determine the impact of variable manipulation on the sound properties of pitch, tone timbre, loudness, and resonance time; and (6) test how the combinations of sound waves with patterns of regular or irregular pitch intervals can cause different feelings or emotions.

Algebra in Musical Instrument Design Algebra is widely used during the musical instrument-making process, an example being that musical scales were developed based on proportional relationships. During the instrument-making process, string instruments such as guitars required that instrument designers calculate the position of frets on the finger board; likewise, wind instruments such as saxophones required designers to calculate the positions of finger keys, and percussive instruments such as glockenspiels required designers to calculate the size of component pieces. One of the many activities wherein students can apply geometric sequencing is to design a glockenspiel by cutting and pasting tape or paper strips (see Fig. 2.5).

Using Geometric Sequences to Create a Glockenspiel As a percussion instrument that was constructed with 24 tuned pieces of steel bars, the construction of a glockenspiel requires making a series of rectangles with the same width, although of different lengths, having a common ratio of 0.94. Teachers have directed students to design a glockenspiel by cutting and pasting paper strips based on this geometric progression. For example, An and Capraro (2011) introduced the following instructional steps to create a glockenspiel for fifth graders. Students can cut paper strips

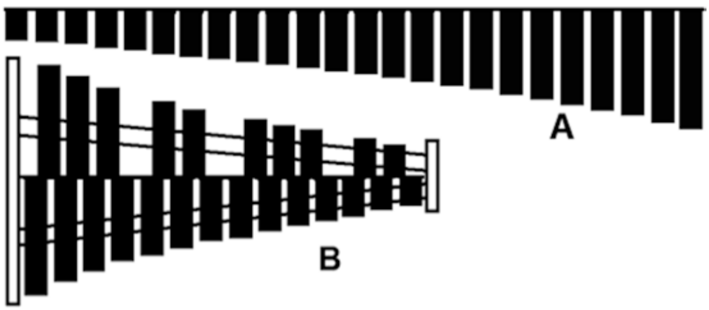

Fig. 2.5 The process of designing a glockenspiel through geometric progression

by using a geometric sequence to compute the accurate length of each tape or paper piece, with the common ratio of approximately 0.94. Based on the paper glockenspiel that their students constructed, teachers can provide additional open-ended mathematics questions. For example, students can examine the total area of the glockenspiel pieces that they used in designing their instrument, and they also can explore the geometric sequence formula describing the length of each piece.

Linear Equations to Investigate Trumpet Tube Length In an activity investigating the tube length changes in a trumpet, students needed to set up equations in order to figure out the volume of the vibrating air column inside the trumpet when a player is changing the pitch. As a wind instrument, the sound from a trumpet is produced by the players' lip vibrations as well as the follow-up pressing of the piston valve, changing the length of the tube within the trumpet. A typical trumpet has three valves, and the length of the tube can be increased when players are pressing one or more of these valves with different combinations. Specifically, the instrument's pitch will be lowered by (1) a major second interval when the first valve is pressed (9/8 longer than the original tube), (2) a minor second interval when the second valve is pressed (16/15 longer than the original tube), or (3) a minor third interval when the third valve is pressed (6/5 longer than the original tube) (see Fig. 2.6). Students set up equations based on this given information to calculate the length of tubes in different conditions when a specific valve was pressed. As An and Capraro (2011) proposed in their lesson designed for sixth-grade students:

Let's suppose the tube length when no valve was pressed is 100 cm. For example, the press of the first valve will lower the instrument's pitch by a major second interval. Let's represent the increased length of tube as x, yielding the following equation:

$100 + x = 100 \times (9/8)$

$x = 12.5$ (cm)

So, the press of the first valve will increase the length of the tube by 12.5 cm.

Use the same method to create equations for the second valve (lowering a minor second interval), and the third valve (lowing a minor third interval). What are the equations for the second and the third valve? Discuss your equations with your classmates. (p. 67)

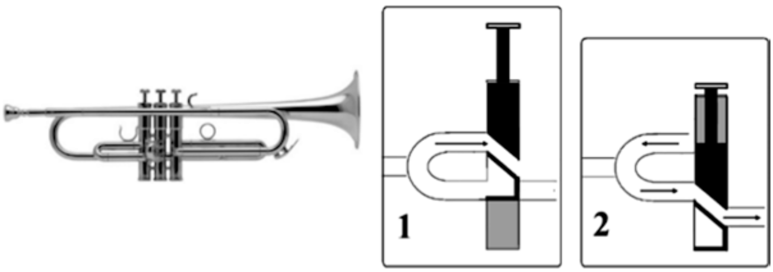

Fig. 2.6 Illustration of the changing tube length in a trumpet

Geometry in Musical Instrument Design Musical instrument design activities have provided links for associating geometry with music; many types of shapes have been used in the history of musical instrument design, and this process often involved combining different simple shapes to create irregular curves. In addition to investigating properties of shapes as well as their unique properties, by designing musical instruments, students have additional chances to apply geometric transformations and improve visualization skills.

Two guitar-themed activities (one for lower elementary grades and another for upper elementary grades) serve to illustrate how geometry has been contextualized in music instrument design. In the Flying V guitar design activity (see Fig. 2.7), first-grade students played with triangles by making transformations such as translating, rotating, and resizing; they also worked with geometric concepts, such as obtuse triangles, isosceles triangles, and congruence. Specifically, students were directed to create two congruent triangles and then rotate one of the triangles and put the two sides (legs) together to create a V-shape figure as the guitar body. The same V shape was then resized at 1/3 scale to make the guitar head the smaller V shape, which was rotated for placement at the other end of the guitar neck opposite the guitar body. In the other activity, circles were employed as a geometrical shape for students to use as the basis for creating the outline of a classical guitar (see Fig. 2.8). An and Capraro (2011) presented this activity with the following instructional steps for fifth graders: (1) make four congruent circles tangent to each other and outline the edges of the two middle circles and the spaces; (2) find the symmetrical line and cut the line off; (3) regroup the two pieces by leaving an uneven, nonparallel space; and (4) redraw the outline and then all the sound holes with a smaller circle. In this classical guitar activity, students investigated geometric concepts such as tangent circles, parallelism, and symmetry. In both guitar design activities, students explored the area and perimeter in the figures that they designed, and students were ready to conduct additional measurements based on real instruments.

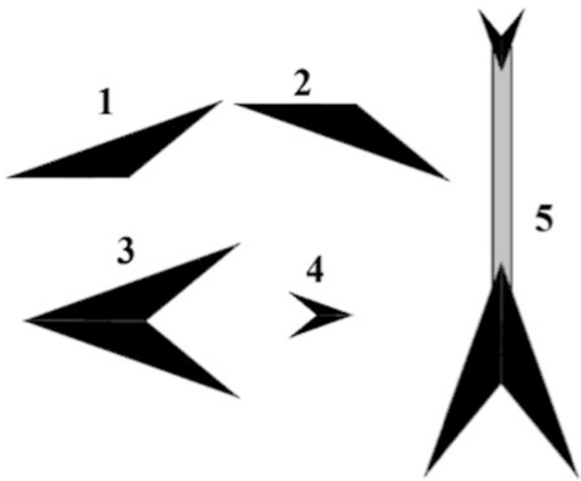

Fig. 2.7 The basic steps for designing a Flying V guitar

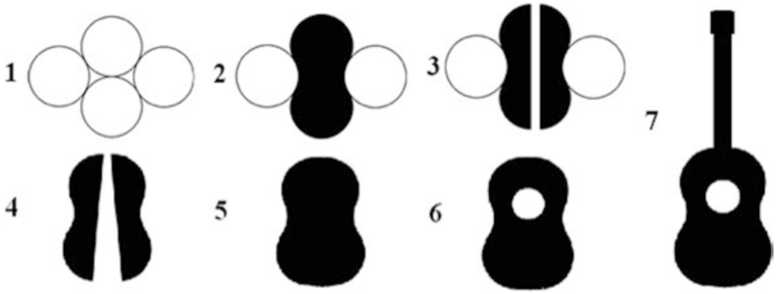

Fig. 2.8 The basic steps for designing a classical guitar

Concluding Remarks

This chapter has provided a review and summary of research studies and activities identifying music-themed mathematics as a valid and worthwhile transdisciplinary pedagogy, which may, when well-implemented, promote both mathematics teaching and learning. However, music is not a panacea for the ailments that plague direct mathematics instruction. Many of the existing resources and popular strategies for using music in mathematics teaching are primarily entertainment oriented, simply providing a cover story for mathematics word problems or playing background music instead of connecting with the mathematics content. Teachers can help reveal the inherent music-themed pedagogy of a math class. They can focus on musical resources that can truly assist in supporting instruction, which may develop students' understanding of the mathematical concepts with connections to music.

Teaching mathematics through music composition and musical instrument design is an application of constructivist learning because teachers need to direct students to engage in complex tasks and then facilitate students' learning by transforming difficult tasks into accessible, manageable tasks within students' zone of proximal development (Vygotsky, 1978). Only when aspects of student-centered pedagogy are thoroughly implemented, such as proposing open-ended tasks for students to provide diverse answers or facilitating group discussion for students to exchange and evaluate their ideas, can students learn mathematics effectively (Schoenfeld, 2004). In our previous studies analyzing more than 200 teachers' instructional designs (An & Tillman, 2014; An, Tillman, & Paez, 2015) and more than implementations of 80 lessons to students (An et al., 2013; An & Tillman, 2015; An et al., 2016), we identified that effective mathematics learning only happened when students were cognitively engaged in participating with the mathematics tasks by manipulating objects, performing activities, and applying the skills in generalized mathematical structures within arithmetic situations. Based on our collective research findings, the common feature among effective mathematics-music-integrated lessons is a mathematical process (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000) orientation in which teachers allowed students to (a) explore algebraic patterns and geometric transformations as composing methods in planning rhythm, investigating intervals, and transferring chords; (b) apply statistical knowledge such as measurement and data analysis as mathematical tools for supporting music analysis and creation processes; and (c) represent mathematical ideas through multiple representations, including singing, playing, composing, decomposing, and recomposing music works.

Music and mathematics are two intelligence domains of recognized importance in human learning, and using music to enhance students' enjoyment and understanding of mathematics has been shown to help learners develop improved logical/mathematical intelligence (Gardner, 1993). There is much potential in the integration of mathematics education and the arts; however, this potential has yet to be fully realized. Unfortunately, the pressure exerted by high-stakes standardized assessment has forced many creative teachers to involuntarily marginalize and ignore the arts, especially when teaching mathematics. Quantifiable standardization in education threatens to seriously harm the fundamental principles of liberal pedagogy and the ongoing quest for nurturing the next generation of innovative young minds (Pinar, 2004). As Slattery (2006) argued, curriculum should be a "kaleidoscope" that opens the eyes, with the ultimate goal of teaching students how to generate critical and original thought. Without the arts, many colors and patterns are eliminated from the kaleidoscope's viewscape, and the "complicated conversation" between students and teachers becomes limited. Teaching mathematics through music offers an opportunity to restore the kaleidoscopic nature of the curriculum.

References

Amiot, E., & Sethares, W. A. (2011). An algebra for periodic rhythms and scales. *Journal of Mathematics and Music, 5*(3), 149–169.

An, S. A., & Capraro, M. M. (2011). *Music-math integrated activities for elementary and middle school students*. Irvine, CA: Education for All.

An, S. A., Capraro, M. M., & Tillman, D. (2013). Elementary teachers integrate music activities into regular mathematics lessons: Effects on students' mathematical abilities. *Journal for Learning through the Arts, 9*(1), 1–20.

An, S. A., Ma, T., & Capraro, M. M. (2011). Preservice teachers' beliefs and attitude about teaching and learning mathematics through music: An intervention study. *School Science and Mathematics, 111*(5), 235–247.

An, S. A., & Tillman, D. (2014). Elementary teachers' design of arts based teaching: Investigating the possibility of developing mathematics-music integrated curriculum. *Journal of Curriculum Theorizing, 30*(2), 20–38.

An, S. A., & Tillman, D. (2015). Music activities as a meaningful context for teaching elementary students mathematics: A quasi-experiment time series design with random assigned control group. *European Journal of Science and Mathematics Education, 3*(1), 45–60.

An, S. A., Tillman, D., Boren, R., & Wang, J. (2014). Fostering elementary students' mathematics disposition through music-mathematics integrated lessons. *International Journal for Mathematics Teaching and Learning, 15*(3), 1–18.

An, S. A., Tillman, D., & Paez, C. (2015). Music-themed mathematics education as a strategy for improving elementary preservice teachers' mathematics pedagogy and teaching self-efficacy. *Journal of Mathematics Education at Teachers College, 6*(1), 9–24.

An, S. A., Tillman, D., Shaheen, A., & Boren, R. (2014). Preservice teachers' perception about teaching mathematics through music. *Journal of Interdisciplinary Teaching and Learning, 4*(3), 150–171.

An, S. A., Zhang, M., Flores, M., Chapman, J. R., Tillman, D. A., & Serna, L. (2015). Music activities as an impetus for Hispanic elementary students' mathematical disposition. *Journal of Mathematics Education, 8*(2), 39–55.

An, S. A., Zhang, M., Tillman, D., Lesser, L., Siemssen, A., & Tinajero, J. (2016). Learning to teach music-themed mathematics: An examination of preservice teachers' beliefs about developing and implementing interdisciplinary mathematics pedagogy. *Mathematics Teacher Education and Development, 18*(1), 20–36.

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science, 11*(5), 181–185.

Beer, M. (1998). *How do mathematics and music relate to each other?* Brisbane, Queensland, Australia: East Coast College of English.

Bursal, M., & Paznokas, L. (2006). Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science. *School Science and Mathematics, 106*(4), 173–180.

Chahine, I., & Montiel, M. (2015). Teaching modeling in algebra and geometry using musical rhythms: Teachers' perceptions on effectiveness. *Journal of Mathematics Education, 8*(2), 126–138.

Cho, G. (2003). *The discovery of musical equal temperament in China and Europe in the sixteenth century*. Lewiston, NY: Edwin Mellen Press.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved Oct. 22, 2016, from http://www.corestandards.org/Math/

Conklin, D. (2010). Distinctive patterns in the first movement of Brahms' string quartet in C minor. *Journal of Mathematics and Music, 4*(2), 85–92.

Cornett, C. (2007). *Creating meaning through literature and the arts*. Upper Saddle River, NJ: Merrill.

Costa-Giomi, E. (1999). The effects of three years of piano instruction on children's cognitive development. *Journal of Research in Music Education, 47*(3), 198–212.

Cox, H. A., & Stephens, L. J. (2006). The effect of music participation on mathematical achievement and overall academic achievement of high school students. *International Journal of Mathematical Education in Science and Technology, 37*(7), 757–763.

Crowther, G. (2012). Using science songs to enhance learning: An interdisciplinary approach. *CBE Life Sciences Education, 11*(1), 26–30.

Crowther, G. J., Davis, K., Jenkins, L. D., & Breckler, J. L. (2015). Integration of math jingles into physiology courses. *Journal of Mathematics Education, 8*(2), 56–73.

Crowther, G. J., McFadden, T., Fleming, J. S., & Davis, K. (2016). Leveraging the power of music to improve science education. *International Journal of Science Education, 38*(1), 73–95.

Davis, S. (1985). *The craft of lyric writing*. Cincinnati, OH: Writer's Digest Books.

Ehrlich, C. (1990). *The piano: A history*. Oxford: Oxford University Press.

Furner, J., & Berman, B. (2005). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high-tech globally competitive world. *Dimensions in Mathematics, 18*(1), 28–31.

Gamwell, P. (2005). Intermediate students' experiences with an arts-based unit: An action research. *Canadian Journal of Education, 28*(3), 359–383.

Gardner, H. (1993). *Multiple intelligences: The theory in practice*. New York: Basic Books.

Geist, E. (2010). The anti-anxiety curriculum: Combating math anxiety in the classroom. *Journal of Instructional Psychology, 37*(1), 24–31.

Gelineau, R. P. (2004). *Integrating arts across the curriculum*. Belmont, CA: Thompson Wadsworth.

Gordon, S. (1996). *A history of keyboard literature: Music for the piano and its forerunners*. New York: Schirmer Books.

Gouzouasis, P., Guhn, M., & Kishor, N. (2007). The predictive relationship between achievement and participation in music and achievement in core grade 12 academic subjects. *Music Education Research, 9*(1), 81–92.

Greene, M. (2001). *Variations on a blue guitar: The Lincoln lectures in aesthetic education*. New York: Teachers College Press.

Haley, J. A. (2001). The relationship between instrumental music instruction and academic achievement in fourth grade students. (Doctoral dissertation, Pace University, New York). Dissertation Abstracts International, 62 (09), 2969A.

Hui, K. (2006). Mozart effect in preschool children? *Early Child Development and Care, 176*(3–4), 411–419.

Ivanov, V. K., & Geake, J. G. (2003). The Mozart effect and primary school children. *Psychology of Music, 31*(4), 405–413.

Kalinec-Craig, C. A. (2015). Uncovering the mathematical challenges and connections when using mariachi music as a representation for teaching equivalent fractions. *Journal of Mathematics Education, 8*(2), 3–20.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up*. Washington, DC: National Academy Press.

Krantz, R., & Douthett, J. (2011). Algorithmic and computational approaches to pure-tone approximations of equal-tempered musical scales. *Journal of Mathematics and Music, 5*(3), 171–194.

Lee, Y. Y., & Shen, S. Y. (1999). *Chinese musical instruments*. Chicago: Chinese Music Society of North America.

Lesser, L. M. (2014). Mathematical lyrics: Noteworthy endeavours in education. *Journal of Mathematics and the Arts, 8*(1–2), 46–53.

Lesser, L. M. (2015). "American Pi": The story of a song about pi. *Journal of Mathematics Education, 8*(2), 158–168.

Loy, R. (2006). *Musimathics: The mathematical foundations of music*. Cambridge, MA: MIT Press.

Nantais, K. M., & Schellenberg, E. G. (1999). The Mozart effect: An artifact of preference. *Psychological Science, 10*(4), 370–373.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pietschnig, J., Voracek, M., & Formann, A. K. (2010). Mozart effect–Shmozart effect: A meta-analysis. *Intelligence, 38*(3), 314–323.

Pinar, W. F. (2004). *What is curriculum theory?* Mahwah, NJ: Lawrence Erlbaum Associates.

Pinnock, G. (2015). Using live reggae instrumental acoustic music to influence students' mathematics test scores. *Journal of Mathematics Education, 8*(2), 115–125.

Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Rameau, P., & Louime, C. (2007). Mathematics phobia: Are the mathematical sciences a pothole in the road of life? *Current Science (Bangalore), 93*(11), 1481–1482.

Rauscher, F. H., Shaw, G. L., & Ky, K. N. (1993). Music and spatial task performance. *Nature, 365*(6447), 611.

Rauscher, F. H., Shaw, G. L., & Ky, K. N. (1995). Listening to Mozart enhances spatial-temporal reasoning: Towards a neurophysiological basis. *Neuroscience Letters, 185*(1), 44–47.

Rauscher, F. H., Shaw, G. L., Levine, L. J., Wright, E. L., Dennis, W. R., & Newcomb, R. L. (1997). Music training causes long-term enhancement of preschool children's spatial-temporal reasoning. *Neurological Research, 19*(1), 2–8.

Rauscher, F. H., & Zupan, M. A. (2000). Classroom keyboard instruction improves kindergarten children's spatial-temporal performance: A field experiment. *Early Child Research Quarterly, 15*(2), 215–228.

Robertson, W., & Lesser, L. M. (2013). Scientific skateboarding and mathematical music: Edutainment that actively engages middle school students. *European Journal of Science and Mathematics Education, 1*(2), 60–68.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.

Sethares, W. A., & Budney, R. (2014). Topology of musical data. *Journal of Mathematics and Music, 8*(1), 73–92.

Slattery, P. (2006). *Curriculum development in the postmodern era* (2nd ed.). New York: Routledge.

Spelke, E. (2008). Effects of music instruction on developing cognitive systems at the foundations of mathematics and science. In C. Asbury & B. Rich (Eds.), *Learning, arts, and the brain* (pp. 17–49). New York/Washington, DC: The Dana Foundation.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education, 4*(1), 117–143.

Sweller, J. (2016). Cognitive load theory, evolutionary educational psychology, and instructional design. In *Evolutionary perspectives on child development and education* (pp. 291–306). New York, NY: Springer.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston: Allyn & Bacon.

Vinson, B. M. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal, 29*(2), 89–94.

von Falkenhausen, L. (1993). *Suspended music: Chime-bells in the culture of Bronze Age China*. Berkeley, CA: University of California.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Whitehead, B. J. (2001). The effect of music-intensive intervention on mathematics scores of middle and high school students (Doctoral dissertation, Capella University, Minneapolis, Minnesota).