Limin Jao · Nenad Radakovic *Editors*

Transdisciplinarity in Mathematics Education

Blurring Disciplinary Boundaries

Springer

Transdisciplinarity in Mathematics Education

Limin Jao • Nenad Radakovic
Editors

Transdisciplinarity in Mathematics Education

Blurring Disciplinary Boundaries

Springer

Editors
Limin Jao
Department of Integrated Studies
in Education
McGill University
Montreal, Québec, Canada

Nenad Radakovic
Department of Teacher Education
College of Charleston
Charleston, South Carolina, USA

ISBN 978-3-319-63623-8 ISBN 978-3-319-63624-5 (eBook)
DOI 10.1007/978-3-319-63624-5

Library of Congress Control Number: 2017953870

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Foreword

To a modern reader, the prefix *trans-* can evoke a range of meanings, from the sublime to the mundane, depending on the context of its usage and the particular word that usually follows it. It is a powerful prefix that can create opposing meanings (e.g., transform vs. transgress) and situate meanings depending on the user. For instance, the term "transcendental" colloquially connotes esoteric practices associated with Eastern religions, where as in mathematics it refers to a very specific type of number, one that is not algebraic. It also turns out that although specific transcendental numbers such as π and e are most often referred to and perceived of as exotic, as the term "transcendental" might suggest, there happen to be "as many" of these numbers as there are real numbers, i.e., there are infinitely many! Given the wide range of meanings possible, it is important to know that the prefix *trans-* comes from Latin *transcendere* meaning to climb over or beyond or across. With this meaning in mind, we can better grasp the term transdisciplinary, which is the topic of this book in reference to mathematics education.

The editors of the book, Jao and Radakovic, delve into the etymological aspects of the prefix trans- and clarify how this term is different from prefixes like inter- and multi- to provide clarity to the reader. Simply put, the difference between an interdisciplinary approach and a transdisciplinary approach is that the former typically involves (research) questions that can be approached via more than one discipline and the methods used to answer the question reveal links between the disciplines (e.g., mathematics and physics), whereas the latter suggests (research) questions that lie outside the disciplines with methods and answers (solutions) informing new discipline(s) and offering the possibility of unifying disciplines (e.g., the Human Genome Project). For instance, when we ask simple questions like "What is the cause of gravity?" or "What is the origin of matter?" the answers usually come in the form of theories from different disciplines which have devised their own methods for tackling such questions. These questions are also tackled outside the discipline of science, in the humanities and in religion. Thus, these types of questions cannot be appropriated by any discipline per se! An even simpler question is "What is a human being?" and this as we know is the enterprise not only of entire universities and religious organs but also that of business corporations and the marketing industry.

Other examples are found in issues that deal with sustainable development or with the environment, such as drilling for oil in wilderness areas, which require knowledge from many disciplines to even understand the nature of the questions that arise. Similarly, issues that arise in mathematics education can easily transcend disciplinary boundaries and lead to questions that can only be answered with transdisciplinary thinking. Such questions are as follows:

1. What are the causes of inequity in mathematics classrooms, in schools, and in school systems?
2. Can mathematics as a school subject accommodate indigenous views not subsumed under culture or ethno-labels?

The book takes into account such questions within mathematics education, its genesis and subsequent evolution, into a plurality of perspectives, of theories, and of research design that accommodate diverse groups of stakeholders, and the eventual transformation of these questions into transdisciplinary frameworks that require critical and postmodern modes of thinking. To ease the reader into this progression, the book is organized into five sections based on a cluster model of transdisciplinarity, with chapters in each section that address questions requiring a particular mode of thinking. As a whole, the book makes an excellent case for moving beyond domain-specific thinking, particularly if the goal of mathematics education is to address issues that are important to all its different stakeholders.

Department of Mathematical Sciences
University of Montana-Missoula,
Missoula, MT, USA

Bharath Sriraman

Acknowledgments

Although the idea for an edited book on transdisciplinarity in mathematics education came to us in the summer of 2015, the origins of our interest in transdisciplinarity are much older. The transdisciplinary nature of our personal and professional interests stems from our early experiences as students and classroom teachers struggling to make sense of mathematics and to see it beyond the decontextualized manipulation of symbols. Subsequently, we met as graduate students at the Ontario Institute for Studies in Education of the University of Toronto, where we began our friendship and collaboration. It was there that we were introduced to the mathematics education research community. As emerging scholars, we were inspired by faculty members who infused transdisciplinarity into their research and teaching and conferences that pushed our thinking about the limits and scope of mathematics education. Specifically, we are fortunate and proud to be members of the Canadian Mathematics Education Study Group, a wonderful community of educators, researchers, and practitioners who are advancing mathematics education in Canada and worldwide. As we move forward in our careers, we continue to encounter individuals who enrich, transform, and transcend mathematics education through their explorations across disciplines and cultures.

We are indebted to our previous experiences and the individuals who have informed and shaped our thinking. Without them, this book would not have been possible. We extend our thanks to all of the contributing authors for their dedication and diligence throughout the writing and editing process. Thank you to Peter Trifonas, who gave two recent graduates the confidence to embark on this adventure. For her assistance with editing and proofreading, we thank Madison Fox, a graduate student at the College of Charleston. Bharath Sriraman, thank you for reading our manuscript and writing the Foreword.

We are grateful for the support of the faculty, staff, and students at our respective institutions: the Department of Integrated Studies in Education (Faculty of Education, McGill University) and the Department of Teacher Education (School of Education, Health, and Human Performance, College of Charleston).

As our graduate supervisor, Doug McDougall recognized our potential, and we are grateful for his ongoing mentorship.

To Susan Jagger, our friend and colleague, we are so appreciative of your friendship, encouragement, and sense of humor. We also thank Barb, for her enthusiasm and positive energy.

Finally, we thank our families for their unconditional support and love. We dedicate this book to you.

Contents

Introduction: The Transdisciplinary Nature of Mathematics Education

Mathematics education has evolved from focusing on teaching and learning of disciplinary knowledge to a complex field bridging various disciplines and perspectives. Within the field of mathematics education, there is a focus on multiple ways of knowing and experiencing mathematics including aboriginal and indigenous perspectives, queer theories, and diverse cultural and religious worldviews. Additionally, scholars investigate the relationship between mathematics and different disciplines, for example, mathematics and the arts (visual arts, music, literature) and mathematics and social sciences (including equity and social justice). The field also explores mathematics education in different contexts such as mathematics outside of the classroom, mathematics and social media, mathematics in the media, mathematics and movement, mathematics and gardens, and mathematics in popular culture. Each of the above topics also speaks to the multimodality of mathematics.

The goal of this book is to address the ways in which these various topics, perspectives, and ways of knowing are interconnected and how they inform mathematics education. In order to accomplish our goal, we are using the lens of transdisciplinarity since we believe that it encompasses the complexities within the field of mathematics education.

What Is Transdisciplinarity?

Transdisciplinary research and theories embrace multiple perspectives and approaches. Consistent with its nature, definitions of transdisciplinarity are diverse and in flux (Klein, 2013). A common way to define transdisciplinarity is by outlining a progression from multidisciplinarity, through interdisciplinarity, to transdisciplinarity (Pohl, 2010). This progression often begins with an etymological exercise of analyzing the prefixes multi-, inter-, and trans-. Consistent with this exercise, multidisciplinarity hints at using multiple disciplines in a way that there are clear boundaries between them and without the goal of unifying the disciplines.

Interdisciplinarity is a step beyond multidisciplinarity and brings together different approaches to address a common issue. Specifically, methods, techniques, and skills traditionally used in different disciplines are brought together. Also building upon conceptual, theoretical, and methodological orientations representative of various disciplines, transdisciplinarity shifts its goal and instead focuses on building a common conceptual framework informed by these foundations (Rosenfield, 1992).

For some researchers, the differences between interdisciplinarity and transdisciplinarity concern the participants involved and knowledge contributed (e.g., Klein, 2008; Lawrence, 2004). While interdisciplinarity draws from experts across different disciplines to produce "new and broad knowledge of a particular phenomenon" (Klein, 2008, p.S118), transdisciplinary research involves the contribution of knowledge from stakeholders at different levels (e.g., researchers and the public).

In a review of various definitions of transdisciplinarity, Pohl (2010) identified four recurring features: (1) a focus on socially relevant issues, (2) the transcendence and integration of disciplinary paradigms, (3) engaging in participatory research, and (4) a search for a unity of knowledge beyond disciplines. Pohl found that researchers chose to take up these four features in varying degrees, often only exhibiting a subset rather than all four.

Klein (2013) provides an organizational scheme that goes beyond identifying features of definitions. Instead, she analyzes underlying philosophical foundations (e.g., positivism) that inform these definitions of transdisciplinarity. Klein then produces "clusters" of words that are consistent with each worldview (e.g., unity of knowledge). According to Klein, there are five clusters of keywords. These clusters are shaped by cultural and historic developments, starting with interdisciplinary approaches in the 1960s and 1970s (e.g., Bruner, 1960; Piaget, 1970) followed by an anti-positivist (e.g., Feyerabend, 1975) and postmodern (e.g., Habermas, 1971) turn in research. Current societal problems (e.g., environmental and political) have also informed the clusters. In the next section, we elaborate on Klein's five clusters.

The Five Clusters of Transdisciplinarity

Klein (2013) describes the plurality of its definitions through the introduction of five clusters. Each of these clusters is informed by research programs that share similar philosophical outlooks and work within similar contexts and traditions. Keywords that describe these common perspectives are grouped to form a cluster. The clusters are "not air-tight categories, but they do reveal important differences in how [transdisciplinarity] is constructed" (p.189).

Informed by notions of interdisciplinarity as described earlier, the keywords used to describe Klein's first cluster are integration, synthesis, interaction, holistic thinking, boundary crossing, boundary blurring, and transcendence. As the disciplinary lines blur, the goal of the intellectual activity is to create a unifying science. The unity of science is challenged by many worldviews. These worldviews challenge the "unity" and "unifying" approaches and consider interplay, intersection, and interdependence as defining features of research. These perspectives inform Klein's second cluster characterized by three pillars: complexity, multiple levels of reality, and the logic of included middle.

The third cluster shifts toward ethics by concentrating on collaborative problem-oriented research for the "common good." This cluster is characterized by the focus on "socially relevant issues, transcendence and integration of disciplinary paradigms, conduct of participatory research, and the search for unity of knowledge" (Klein, 2013, p.193). The underlying principle of the third cluster is that research questions and practices should be framed by societal problems rather than by disciplines. The third cluster is characterized by participation, cooperation, collaboration, partnering, networking, and mutual learning.

The fourth cluster frames transdisciplinarity in terms of three forms of knowledge, system, target, and transformation knowledge (Pohl & Hadorn, 2008), which is similar to Habermas' ideas about three domains of human knowledge, namely, technical, practical, and emancipatory (Habermas, 1971). An important feature of this cluster is a shift from technoscientific knowledge to transformational knowledge, as well as the inclusion of perspectives that go beyond Western science and dominant worldviews (Klein, 2013).

Finally, the fifth cluster is characterized by the critical and postmodern turn and involves interrogation, critique, transgression, transformation, reconfiguring, reformulating, and resituating. Consistent with the fifth cluster of meaning, Kellner (1995) discusses transdisciplinarity in the context of pushing boundaries of class, gender, race, ethnicity, and other identities. According to Klein (2013), "one of the transgressive purposes of transdisciplinarity, therefore, is to renounce the logic of instrumental reason by creating new participatory modes of knowledge, discourse, and institutional frameworks across all sectors of academic, private, and public life" (p.197).

Transdisciplinarity and Mathematics Education

There are many themes within contemporary mathematics education research and practice. Some of them focus on developing disciplinary knowledge, for example, research focusing on the US Mathematics Common Core State Standards Initiative (Cobb & Jackson, 2011; Council of Chief State School Offices, 2010; Porter, McMaken, Hwang, & Yang, 2011) and research on proofs and reasoning (Hanna, 2000; Reid, 2002). Other themes follow transdisciplinary perspectives and thus can be organized into all five of Klein's (2013) clusters.

Within the domain of mathematics education, the first cluster focuses on interdisciplinary approaches, namely, the integration of mathematics with other fields. STEM (science, technology, engineering, and mathematics education) and more recently STEAM (where "A" stands for "the arts") can be seen as interdisciplinary. The second cluster can be exemplified in mathematics education research by

Davis and Simmt's (2003) work on integrating complexity science into mathematics teaching practices through reconceptualizing teaching as a complex domain.

Mathematics education research that focuses on issues of sustainability, equity, or social justice belongs within the third cluster of transdisciplinarity. For example, Paige, Lloyd, and Chartres (2008) discuss mathematics and science education for preservice teachers through the sustainability lens. They assert that this teaching approach creates a more well-rounded curriculum and develops students' thinking toward sociopolitical action. Equity, particularly culturally relevant curriculum, is another theme within the third cluster. Na'ilah Suad Nasir focuses on mathematics teaching and learning in the context of African American and other traditionally underrepresented students. In one of her works, Nasir discusses the intersection between goals, identity, and learning to examine the ways in which race, culture, and learning influence minority students' experiences (Nasir, 2002).

The fourth cluster emphasizes indigenous and transformational knowledge. Ethnomathematics, critical mathematics education, and social justice (d'Ambrosio, 1985; Skovsmose, 1994) are consistent with these themes. Finally, the focus on challenging mathematical representations, genderism, and queer perspectives is consistent with the fifth cluster of transdisciplinarity. For example, Esmonde (2011) questions genderism in mathematics education research and practice by moving away from the gender binary (i.e., "males" and "females" or "boys" and "girls"). She reviews literature on differences in achievement based on gender, particularly how these categories have been essentialized. Esmonde then suggests that extending beyond binary categories and providing a complex view of individuals' identities are a more equitable approach to mathematics education research.

Klein's five clusters provide a useful way of capturing different themes and approaches to transdisciplinarity. In order to provide a comprehensive look at transdisciplinarity, we found it useful to use Klein's clusters as an organizing strategy for the chapters of this book, as elaborated on in the next section.

Structure of the Book

In this book, we bring together 13 chapters that reflect themes in transdisciplinarity in mathematics education. The book is organized into five parts mirroring Klein's five clusters.

Part I, "Interdisciplinary Approaches in Mathematics Education," focuses on the integration of mathematics with other fields. In particular, we present chapters that highlight how mathematics concepts can be studied within different disciplinary contexts. In "Euclidean Exploration of Geometry in Islamic Art," Zekeriya Karadag uses the Islamic star as an exemplar of how cultural and religious symbols can be explored in the mathematics classroom. Karadag also outlines a hypothetical instructional sequence of constructing the Islamic star with an aid of a dynamic geometry software (GeoGebra) in order to foster students' geometric thinking. Song An, Daniel Tillman, and Lawrence Lesser describe how music supports mathematics learning by surveying current literature and presenting various interdisciplinary activities highlighting mathematics and music in their chapter, "The Hidden Musicality of Math Class: A Transdisciplinary Approach to Mathematics Education." Similar to music, children's literature (e.g., picture books) serves as an accessible and relatable medium for the learning of mathematics. In "Using Children's Literature to Enhance Math Instruction in K-8 Classrooms," Melissa Luedtke and Karen Sorvaag speak to this by investigating how the use of children's literature in mathematics instruction can help break down barriers to mathematics understanding and increase student success. In summary, chapters in the first part underscore the value and importance of an interdisciplinary approach to mathematics education by exploring the connection between mathematics and other disciplines.

Part II, "Complexity, Difference and/in Mathematics Education," starts with a premise that the goal of mathematics education is not to achieve a unified view of the curriculum but to acknowledge and honor the complexity, diversity, difference, and divergence of pedagogical approaches and mathematical representations. In "Complexity as a Discourse on School Mathematics Reform," Brent Davis concentrates on educational research as a complex, transdisciplinary field. Through the language of complexity theory, Davis describes the nature of mathematics teaching and learning and mathematics education research. Davis then describes how complexity is different from dominant paradigms in mathematics education research particularly evidence-based practice. Continuing to push participatory boundaries, Susan Jagger weaves together the multiple yet inextricably linked texts of philosophy and practice in garden-based learning, entwining poststructuralism and deconstruction with moments of early years and elementary mathematics learning in a school's learning garden. In "Opening a Space of/for Curriculum: The Learning Garden as Context and Content for Difference in Mathematics Education," Jagger traces the children's organic and situated explorations of number and number operations, measurement, geometry, and probability and statistics in the garden, opening up a curricular space and a place for digging into mathematics concepts and processes.

In Part III, "Mathematics for the Common Good," we present chapters positioned within the third cluster of meaning of transdisciplinarity. In the first chapter of this part, "Transdisciplinarity, Critical Mathematics Education, Eco-Justice, and the Politics to Come," Nenad Radakovic, Travis Weiland, and Jesse Bazzul discuss how mathematics education can be situated in the context of the environmental discourse and how this relationship can be used to raise issues such as climate change. In "Using a Mathematics Cultural Resonance Approach for Building Capacity in the Mathematical Sciences for African American Communities," Terrence Richard Blackman and John Belcher describe how to build the mathematics capacity of Black communities by creating opportunities for mathematics research, teaching, and learning that draws upon African American cultural resources. Finally, in "Still Warring After All These Years: Obstacles to a Transdisciplinary Resolution of the Math Wars," Ilona Vashchyshyn and Egan Chernoff propose transdisciplinary thinking as a way of moving beyond the dichotomies of the "math wars" and explore obstacles to a meaningful, transdisciplinary, collaborative, and inclusive dialogue on the future of mathematics education in North America. Each of the authors in Part III views transdisciplinarity through the lens of community engagement, democracy, and active citizenship.

Part IV, “Indigenous and Transformational Mathematical Knowledge,” uses the lens of the fourth cluster to raise equity and social issues in mathematics education. In their chapter, “Echoed Rememberings: Mathematics and Science Education as Reconciliation,” Dawn Wiseman and Lisa Lunney Borden examine how to create opportunities in schools and teacher education to center indigenous knowledges as a place from which learning emerges. Wiseman and Lunney Borden draw from projects using inquiry-based learning with aboriginal communities and a learning garden at a Canadian university to illustrate how valuing indigenous knowledges plays an important role in reconciliation. Gender issues within mathematics education have been a source of discussion for many years. There has been much research regarding issues of achievement, participation, identity, and attitudes. In “Gendered? Gender-Neutral? Views of Gender and Mathematics Held by the Canadian General Public,” Jennifer Hall discusses findings from the Canadian site of a large-scale, international study of the general public’s views of gender and mathematics. The project surveyed adults on the street and in public places (e.g., grocery stores, community centers) about their views of gender and mathematics. Hall asserts that by understanding these views, we can better appreciate the broader cultural milieu in which mathematics teaching and learning take place.

In Part V, “Re-formulating, Re-presenting, and Re-defining Mathematical Knowledge and the Curricula,” authors explore the ways of erasing disciplinary boundaries by redefining what it means to know and do mathematics. The chapters outline two main strategies for achieving this reformulation, namely, embodiment and collaboration. Part V starts with Julie Nurnberger-Haag’s “Borrow, Trade, Regroup, or Unpack? Revealing How Instructional Metaphors Portray Base-Ten Number.” Here, Nurnberger-Haag revisits representations of base-ten arithmetic by introducing embodied metaphors for addition and subtraction and envisions the base-ten problem space as a series of moving pictures rather than the static photographic frames. Continuing on the theme of embodiment, in “Mathematics and Movement,” Susan Gerofsky shows how participation in mathematics can be widened by translating the language of algebra to movement and other embodied experiences. Gerofsky also outlines a process of transdisciplinary collaboration between a mathematics educator and a dancer toward students’ conceptual understanding of mathematics. The theme of transdisciplinary collaboration continues in the final chapter of the book, “Examining the Development of a Transdisciplinary Collaboration,” in which Limin Jao, Melissa Proietti, and Marta Kobiela describe a successful teaching collaboration between a graffiti artist and a mathematics teacher in a grade eight mathematics class in an urban setting in Canada.

The authors presented in this book each share their unique perspectives on transdisciplinarity in mathematics education. By questioning, blurring, and erasing disciplinary boundaries, the book contributes to the inclusion of transdisciplinarity in mathematics education as well as the already rich discussion on transdisciplinarity in general. Although we have tried to offer diverse transdisciplinary perspectives and approaches, we acknowledge these chapters map out only contours of the field of possibilities of interaction between transdisciplinarity and mathematics education. We hope that these possibilities will motivate the mathematics education community to include transdisciplinarity in their own research and practice.

Department of Teacher Education Nenad Radakovic
College of Charleston
Charleston, SC, USA

Department of Integrated Studies in Education Limin Jao
McGill University
Montreal, QC, Canada

References

Bruner, J. S. (1960). *The process of education*. Cambridge, MA: Harvard University Press.

Cobb, P., & Jackson, K. (2011). Assessing the quality of the common core state standards for mathematics. *Educational Researcher*, *40*(4), 183–185.

Council of Chief State School Offices. (2010). *Common core state standards.* Retrieved from http://corestandards.org

d'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*(1), 44–48.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137–167.

Esmonde, I. (2011). Snips and snails and puppy dogs' tails: Genderism and mathematics education. *For the Learning of Mathematics, 31*(2), 27–31.

Feyerabend, P. (1975). *Against method: Outline of an anarchistic theory of knowledge*. London: New Left Books.

Habermas, J. (1971). *Knowledge and human interests* (J. J. Shapiro, Trans.). Boston: Beacon Press.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*(1–2), 5–23.

Kellner, D. (1995). *Media culture: Cultural studies, identity, and politics between the modern and the postmodern*. London: Routledge.

Klein, J. T. (2008). Evaluation of interdisciplinary and transdisciplinary research: A literature review. *American Journal of Preventive Medicine, 35*(2S), S116-S123.

Klein, J. T. (2013). The transdisciplinary moment(um). *Integral Review,* 9(2), 189–199.

Lawrence, R. J. (2004). Housing and health: from interdisciplinary principles to transdisciplinary research and practice. *Futures, 36*(4), 487–502.

Nasir, N. I. S. (2002). Identity, goals, and learning: Mathematics in cultural practice. *Mathematical Thinking and Learning*, *4*(2–3), 213–247.

Paige, K., Lloyd, D., & Chartres, M. (2008). Moving towards transdisciplinarity: An ecological sustainable focus for science and mathematics pre-service education in the primary/middle years. *Asia-Pacific Journal of Teacher Education, 36*(1), 19–33.

Piaget, J. (1970). *Main trends in psychology*. New York: Harper & Row.

Pohl, C. (2010). From transdisciplinarity to transdisciplinary research. *Transdisciplinary Journal of Engineering & Science, 1*(1), 74–83.

Pohl, C., & Hadorn, G. H. (2008). Core terms in transdisciplinary research. In G. H. Hadorn, H. Hoffmann-Riem, S. Biber-Klemm, W. Grossenbacher-Mansuy, D. Joye, C. Pohl, & U. Wiesmann, Elisabeth Zemp (Eds.), *Handbook of transdisciplinary research* (pp. 427–432). Dordrecht: Springer.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards the new US intended curriculum. *Educational Researcher*, *40*(3), 103–116.

Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5–29.

Rosenfield, P. L. (1992). The potential of transdisciplinary research for sustaining and extending linkages between the health and social sciences. *Social Science and Medicine, 35*(11), 1343–1357.

Skovsmose, O. (1994). *Toward a philosophy of critical mathematical education*. Boston: Kluwer Academic Publishers.

Part I
Interdisciplinary Approaches in Mathematics Education

Chapter 1
Euclidean Exploration of Geometry in Islamic Art

Zekeriya Karadag

Abstract This chapter presents Euclidean approach as a learning trajectory to create a common artefact used in Islamic art. The Realistic Mathematics Education sets the theoretical framework for the learning trajectory in two stages, namely, horizontal and vertical mathematization. The chapter provides a brief literature review to explain this framework and exemplify how one could apply the framework in a specific context, Islamic art. Although technology used in creating the artefact provides a number of tools to help learners, the chapter suggests following Euclidean approach rather than taking the activity as technology practice activity. The Euclidean approach fosters learners' geometrical thinking by limiting their tools to an unmarked ruler and a compass. Then, the technology is employed for further explorations as stated in the vertical mathematization stage.

Keywords Islamic art • Geometry in Islamic art • Euclidean approach in geometry • Geometric thinking • Islamic star

Introduction

This chapter explores the geometry of Islamic artefacts as a tool of teaching geometry that is situated in Islamic culture but whose relevance transcends cultural boundaries. Islamic art includes a number of figures that may help learners develop an interest in geometry. We suggest exploring geometry stemming from Islamic art through a process of reinvention and exploration rather than attempting to learn geometry axiomatically—that is, "as a pre-established deductive system" (Freudenthal, 1973, p. 132). Such an approach does not undervalue axiomatization, but rather views axiomatization as a higher level of mathematization, which may be considered part of vertical mathematization (Treffers, 1993).

Z. Karadag, PhD (✉)
Giresun University, Faculty of Education, Giresun, Turkey
e-mail: zekeriya.karadag@giresun.edu.tr

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_1

Traditional Islamic buildings, including masjids,[1] provide opportunities for teachers and students to explore geometry in the ornaments decorating the interiors and exteriors of these buildings, particularly doors and walls. The ornaments are made up of a number of geometric objects, such as stars and polygons, and can also be found in other objects, such as carpets, rugs, and clothes. Scholars interested in the relationships between mathematics, including geometry, and art within the context of Islamic tradition have accumulated quite a number of publications investigating how these two disciplines have been interwoven (Kharazmi, 2016; Lu and Demaine, 2016; Noori and Kiyanmehr, 2016).

The purpose of this chapter is to offer a link between the geometry situated in the ornaments decorating Islamic artefacts and learning geometry via the Euclidean approach. The chapter starts with the outline of the Realistic Mathematics Education (RME) framework (Freudenthal, 1973) created to motivate students learning mathematics. This is followed by providing the real-life context of the artefacts created in the Islamic world. The rest of the chapter presents a hypothetical learning activity of picking up a very specific geometric object, an Islamic star, which is commonly used in Islamic artefacts, to explain two dimensions—horizontal and vertical mathematization—of RME. For the horizontal mathematization process, we suggest creating the Islamic star in a digital environment, GeoGebra, using the Euclidean approach. In terms of vertical mathematization, we discuss how this context could be used to engage students in doing relatively advanced geometry.

Realistic Mathematics Education (RME)

RME is a framework suggested by Freudenthal (1973) to design a learning activity or a set of activities to encourage learners to explore as well as possibly discover and reinvent mathematics situated in real life. Prior to discussing the RME framework, it is important to define some keywords, such as real world, mathematical world, horizontal mathematization, and vertical mathematization. Alongside with Kaput's (1992) perspective in his influential paper, we use real life to denote the physical world where we could pick up a problem scenario and convert it to the mathematical world, which is a mathematically idealized world. For example, we may consider a door in the real world to exemplify rectangles although, in actuality, doors may not really be rectangular as we cannot ensure that the angles in the corners are truly 90° or the opposite sides are equal to each other. On the other hand, we could safely state that a rectangle in a mathematical world has 90-degree angles in the corners and its opposites are equal to each other because this hypothetical world is mathematically idealized and axiomatically constructed. The action of converting a case, a problem, or a scenario from the real world to the mathematical world refers to horizontal mathematization, while working on the scenario to delve into advanced mathematical topics in the mathematical world is known as vertical mathematization.

[1] Masjid is a place to worship in Islamic tradition.

The activities in RME could be problem-solving activities, such as looking for problems or posing problems, solving problems, and ultimately mathematizing any context that can be encountered in real life (Gravemeijer et al., 2000). Unlike teaching mathematics through relations, the framework suggests starting with a real-life problem and translating it into the mathematical world. By converting real-life problems into the mathematical world, mathematization may help students understand the problem and attempt to develop concepts, in contrast to the rote memorization of mathematical facts.

Gravemeijer et al. (2000) conceptualized the characteristics of this mathematization process as a set of the following cognitive activities:

- Generality: generalizing (looking for analogies, classifying, structuring)
- Certainty: reflecting, justifying, and proving (using a systematic approach, elaborating, testing conjectures, etc.)
- Exactness: modeling, symbolizing, and defining (limiting interpretations and validity)
- Brevity: symbolizing and schematizing (developing standard procedures and notations) (Gravemeijer et al., 2000, p. 236)

Consistent with this RME framework, learners should learn to create representations to understand and structure a problem or a case, generalize and conjecture the case in order to gain a better understanding of the mathematics behind it, and test and prove the conjectures to appropriate the case in the world of formal mathematics through formalization.

What really matters in this theory, in terms of the pedagogy of mathematics, is how to start mathematics instruction. Mathematics instruction should start with a problem leading to mathematics to be learned as opposed to starting with a definition, properties, examples, and problem sequence. In other words, rather than starting with formal mathematics and axioms, teachers should "let people get familiar with some mathematical objects, and learn what to expect from them, before you start formalizing everything" (Lockhart, 2002, p. 20).

> Today many would agree that the student should also learn mathematizing unmathematical (or insufficiently mathematical) matters, that is, to learn to organize it into a structure that is accessible to mathematical refinements. Grasping spatial *gestalts* as figures is mathematizing space. Arranging the properties of a parallelogram such that a particular one pops up to base the others on it in order to arrive at a definition of parallelogram, that is mathematizing the conceptual field of the parallelogram. Arranging the geometrical theorems to get all of them from a few, that is mathematizing (or axiomatizing) geometry. Organizing this system by linguistic means is again mathematizing of a subject, now called formalizing. (Freudenthal, 1973, p. 133)

This quote makes two important points: the focus of the activity and the order of the organization of activities. The framework states that the activity should address horizontal mathematization—that is, mathematization of the unmathematical context. For example, one should let elementary school students draw polygons, explore similarities and differences of the polygons, and classify them based on particular features (e.g., grouping quadrilaterals together). They could then be encouraged to explore quadrilaterals to identify further differences among quadrilaterals. Students should not be encouraged to memorize mathematical facts or the formal language of mathematics at this stage. Rather, they should be engaged in activities to understand the context or the problem and to possibly identify the differences between objects or cases within the problem, with the intention of uncovering their mathematical properties. There is no need to make mathematics complicated by starting to teach mathematics by imposing the formal language of mathematics. "Mathematics is not about erecting barriers between ourselves and our intuition, and making simple things complicated. Mathematics is about removing obstacles to our intuition, and keeping simple things simple" (Lockhart, 2002, p. 21). Once students develop an informal understanding of the object, it is easier and more meaningful to define it and set the properties in a formal language through vertical mathematization.

Regarding the order of organization of activities, the RME frameworks suggest starting from mathematization and moving to either axiomatization or formalization depending on the nature of the context. At the horizontal mathematization stage, for example, no formal language use or axiomatic organization is needed. The informal classification of the objects and the use of informal language are both acceptable for the sake of conceptual knowledge development. Referring back to the aforementioned hypothetical example, calling a rectangle a "four-sided figure with right corners" may be totally fine if pupils are able to discern them in a group of objects, such as rectangles, parallelograms, and other quadruples.

This could be a great starting point, and it is rather easy to build mathematical language later. Thus, the goal for teachers should be to organize mathematical activity in a manner that enables learners to mathematize the content and develop meaning, which may not be necessarily supported by formal mathematics language and even with formal definitions and proofs. Lockhart (2002) suggests following the footprints of mathematicians while they learn mathematics by stating that "[n]o mathematician works this way. No mathematician has *ever* worked this way. This is a complete and utter misunderstanding of the mathematical enterprise" (p. 21). Consistent with the RME framework, Lockhart (2002) suggested starting a mathematical topic by stating a problem and then moving mathematics underlying that problem rather than providing definitions, theorems, and proofs, as done in many mathematics courses.

None of the followers of RME have tried to diminish the importance of axiomatization or formalization. Rather, they have suggested changing the place to introduce them such that students follow the path of how mathematicians actually work and learn and that they convert mathematics learning to a business of reinventing mathematics, as has already been done thus far. Let us take Freudenthal's and his successors' view of mathematics as the closing statement of the framework before moving branches of mathematization: "New in Freudenthal's views was not only that he wanted to incorporate everyday reality emphatically in mathematics education, but especially also his fundamental idea to let that rich context of reality serve as a source for learning mathematics" (Treffers, 1993, p. 89).

Treffers (1987) distinguished two mathematization processes, horizontal and vertical mathematization, in the context of the Wiskobas project.[2] First, pupils were encouraged to explore problems collaboratively and to communicate on how to solve them. Following the exploration process, students were engaged in reflecting on the problems and formulating them to find solutions. The exploration process that engaged students in construction or the action is described as the horizontal mathematization, while the reflection process on their construction is accepted as the vertical mathematization.

> In the case of horizontal mathematization, the classroom community develops informal, taken-as-shared ways of speaking, symbolizing, and reasoning as the students attempt to mathematize starting-point problems. By way of contrast, when these ways of describing become the subject of further mathematization, Treffer spoke of vertical mathematization. It is during the interplay of these two processes that symbolizations and symbol use are reinvented. In other words, symbol reinvention emerges as students engage in instructional activities in which they formalize their informal interpretations and solutions. The challenge for the designer (and the teacher) is to anticipate a developmental route for the classroom community that culminates with the powerful use of conventional symbolizations. (Gravemeijer et al., p. 238)

I interpret these mathematization processes somewhat differently than Treffers suggested, although the main approach remains the same. The reason for this difference might be the difference in audience. They apply the framework among lower-grade students, whereas I mostly consider high school and undergraduate students. I take horizontal mathematization as an exploration process to transfer real-life problems into the mathematical world, including the solution processes. These solution processes may include solving a certain type of problem or re-creating an ornament through the Euclidean approach in the digital environment. The vertical mathematization, based on my interpretation, should lead learners to deepen their knowledge on the related topic. Learners could elaborate on the solutions to seek new approaches, pose problems to understand the effects of the variables or constants of the problem, or even alter the problem for new explorations.

Islamic art offers a context for the exploration of geometry through ornaments. The exploration of geometry situated in Islamic art could be taken as the starting point of horizontal mathematization because these ornaments may encourage students to describe the objects, identify the properties of objects, and define the objects if one is different than the other in some aspect. When students are ready to start abstraction, we may lead them to axiomatization and formalization. This second phase of mathematization is called vertical mathematization. However, it is good to remind the readers that the interpretation of horizontal and vertical mathematization could be slightly different; even Freudenthal and Treffers interpreted them differently (Treffers, 1993). In the following context, the horizontal mathematization is taken as a process to create ornaments through the Euclidean approach on a digital platform, GeoGebra, whereas vertical mathematization is considered a generalization

[2] More information about the Wiskobas project may be found in their article.

process, which demands more mathematical thinking features, such as abstraction and formalization.

Islamic Tradition in Art

Islamic art is used to denote artefacts created and influenced by the Islamic tradition. Although these artefacts are generally created in Islamic lands and by Muslim people, it is becoming quite difficult to make such a distinction in today's global village. As demonstrated in the following sections, it is possible to see the Islamic art in non-Muslim countries as well as non-Islamic art in Islamic countries. The common aspect of these artefacts is the spirituality they deliver. Each tradition evolves with its own spirituality, and the artefacts under the influence of this specific tradition may reflect its own religious and ethnic perspective.

Islamic art is usually described as a phenomenon free of time, race, language, and geography, but it is an integration of all of these factors (Burckhardt, 2009). As Burckhardt (2009) explained, it might be quite challenging for an ordinary person—even for a moderate person who has a certain amount of knowledge in art—to discriminate art created in Maghreb (e.g., Morocco) from art created in India or art created in the twelfth century from art created in the eighteenth century. This is because the tradition and belief are integrated while creating art. The reasons for this may include the integrative power of Islam and the way people interpret Islamic rules and the Holy Book, the Qur'an. Indeed, Islam and Qur'an united people in such a way that people from different cultures create similar figures in their artefacts. In order to better understand the spirituality that Islamic art has and conveys, one should review the inspiration and intentions of those who made it. By looking at Islamic art from a broader perspective, Azzam (2013) suggested considering the objects in Islamic art as a language of symbolism and an interpretation of religion through contemplation:

> On the level of metaphysics, traditional art aspires to the highest principles. It acts as a bridge or vehicle to transmit the realm of heaven into our physical world. This metaphysical inspiration cannot be fully explained in rational terms but has to be read and understood through the language of symbolism. Traditional art is a reminder of a higher state of being: it is an essential support for contemplation, and all traditional art forms should be perceived as symbols on earth of the archetypes that are in heaven. (Azzam, 2013, p. 8)

Consistent with the previous quote, the artists who worked on Islamic art reflected on their contemplation in their artefacts in an effort to represent heaven on earth. This approach is not surprising in Islamic tradition, because the Islamic scholars, including the Sufi followers, believe that every word stated in the Qur'an or by the Prophet Muhammed has two meanings: one that is revealed while the other is hidden.

> The physical reality and the metaphysical principle underlying the reality are reflected in Islamic spirituality by the Divine names 'Al-Batin' and 'Al-Zahir' (the Hidden and the Revealed). These two Divine characteristics/ principles maintain the equilibrium of the

> hidden and revealed character of Islamic art, which by extension has an impact on our physical and spiritual levels of being. The language of Islamic art is a contemplative one, and by nature contemplation is a spiritual activity. (Azzam, 2013, p. 9)

Thus, Islamic art is an integration of the physical reality and the metaphysical principles through the artist's own contemplation, such that it becomes a co-existence of both perspectives and even a co-development of them. In other words, the co-existence and integration with the support of contemplation lead to a co-development; thus, artefacts become timeless and free of physical factors such as race, language, and geography. Critchlow's (1976) arguments about the doctrine of unity may help us understand why the integration and coexistence in Islamic art are so important.

> Islamic spirituality could not but develop a sacred art in conformity with its own revealed form as well as with its essence. The doctrine of unity which is central to the Islamic revelation combined with nomadic spirituality which Islam made its own brought into being aniconic art wherein spiritual world was reflected in the sensible world not through various iconic forms but through geometry and rhythm, through arabesques and calligraphy which reflect directly the worlds above and ultimately the supernal sun of Divine Unity. (Critchlow, 1976, p. 6)

Islamic art is an aniconic art because Islamic tradition states that nothing in the physical world could really be a true representation of "the worlds above," such as heaven, as stated in the preceding quote. However, people always tend to interpret Allah's word and implement their interpretations in their work. For example, an eight-cornered, eight-pointed star—a common figure used in Islamic art that sets the context for this chapter—might have emerged from the 17th verse of the Sure Calamity in the Qur'an: "And the angles will be on its sides, and eight will, that Day, bear the Throne of thy Lord above them." (Translated by Yusuf Ali, n.d.). These words may have led people to assume that the heaven is a place with eight doors or eight walls, while the earth is a square—or rectangular—place by comparing Ka'ba's four side walls. This assumption is aligned with Critchlow's (1976) comments, which indicate that Islamic art is usually employed to ennoble the word of God as revealed in the sacred book, the Holy Qur'an, in calligraphic forms supported by geometric and floral ornaments.

Some constructions start with four sides at the bottom and continue or end with eight walls on the next floor, illustrating a journey from earth to heaven, which could be interpreted as life (Fig. 1.7). Such an interpretation could lead us to consider Islamic art as a synthesis and co-development of art and Sufism[3] rather than a divine art created by God or allowed by Him for people to create. However, Burckhardt (2009) asserted that "no specific style could be described as being more or less 'Islamic' than any other; this is an example of the phenomenon of diversity in unity, or unity in diversity" (p. 125). This is particularly true because Islamic tradition rejects any type of iconism; no specific styles and no specific figures should be considered Islamic, although some may find connections in the divine inscriptions. Regardless of the reasons why people are encouraged to create this specific type of

[3] Encyclopedia Britannica defines Sufism as a belief and practice in which Muslims seek to find the truth of divine love and knowledge (https://global.britannica.com)

art, we mathematics educators may still benefit from the existence of these artefacts for the sake of helping students engage in learning and doing mathematics.

One might conclude that Islamic art has its roots in the Divine inscriptions, and its development is influenced by interpretations of people from different cultures. The inclusion of various cultures, as well as other factors such as race, language, and geography, does not change the main elements of the art; rather, it helps create a unity in time. Geometry finds a place for itself to connect all these factors, create a balance among them, and avoid using iconic forms.

> The spiritual life has to begin with submission, and the Islamic tradition teaches that without submission there can be no true understanding; without discipline there can be no flowering of the spirit which leads to true and essential knowledge. This is most evident in the relationship between the fundamental aspects of Islamic art which are geometry, islimi and calligraphy. Geometry is an objective manifestation of the principles of creation and forms the underlying framework for the visual expression of the path that leads from unity to multiplicity. (Azzam, 2013, p. 9)

Islamic Star

A number of geometric objects, such as stars and polygons and even composite objects, are presented in Islamic art. In this sense, Islamic art provides a huge number of contexts that geometry teachers can engage their students to explore and work on them. The Euclidean construction of composite objects, such as those presented in Figs. 1.3, 1.5, and 1.6, could be very challenging, even for advanced high school students. In this chapter, the Islamic star will be used as a geometric object to illustrate the hypothetical learning activity because it is a common object used in Islamic art. The following pictures illustrate the context where places and artefacts include the Islamic star and Islamic art in general. The Islamic star is a regular eight-cornered star (Fig. 1.1). One could easily describe the star as the shape obtained by creating two overlapping squares; one square coupled with a second square rotated by 45° around its center of mass. The pictures originated in different places where the Islamic star is used and known.

The picture on the left of Fig. 1.1 is a plate I purchased in Spain, created by an artisan working in the area around Alhambra Palace. It shows not one Islamic star, but two Islamic stars together. Moreover, the octagon surrounding the Islamic star and another version of an eight-cornered star, but not an Islamic star, may inspire teachers and even students to create some other relational geometry problems. The picture on the right is a set of kumis[4] designed in the form of an Islamic star by students at the Kazakhstan Abai University.

The Taj Mahal, an ivory-white marble mausoleum, constructed in Agra, India, in the seventeenth century (Fig. 1.2), was constructed for Mumtaz Mahal, the wife of Shah Jahan, the Mughal emperor of India. The picture in the upper left is a

[4] The set is designed as a gift for Professor Turan Yazgan, the head of the Organization of Research in Turkic World, and presented by A. Sadikov, the rector of Kazakhstan Abai University.

Fig. 1.1 Plate from Spain and set of kumis from Kazakhstan

Fig. 1.2 Taj Mahal, Agra, India

picture of the Taj Mahal and its fabulous gardens. The picture in the upper right illustrates the inner view of the entrance and the gardens. What is important in this picture is the design of the Islamic star together with an altered version of the star on the grass ground of both sides of the pool. The picture in the bottom left was taken inside the main building to demonstrate a combination of star and cross. The picture on the bottom right is the picture of side building, whose towers are constructed with eight walls.

I'timad-ud-Daulah is a tomb constructed for Mirza Ghiyas Beg (Mumtaz Mahal's grandfather) and his wife, Asmat Begum. What is important for this tomb is its ornamentation (Fig. 1.3). A number of geometric designs have been depicted along the exterior. Not only Islamic star designs, but also a number of other designs bringing various geometric figures together provide a wide range of geometry problems for those interested in them. Geometry teachers may encourage their students to look for contexts for geometry problems situated in their own environments, thereby engaging them in thinking about geometry during their daily lives, as opposed to considering geometry as a subject limited to the classroom only.

Figure 1.4 illustrates designs from two different locations in the Greater Toronto Area in Ontario, Canada. The pictures in the top part were taken in a masjid, the Sayeda Khadija Center, in Mississauga. The Islamic star on the left and octagon on the right, among other designs, are related to the context of this chapter.

The bottom pictures were taken at Aga Khan Center in Toronto. The one on the left is a miniature of a monument, decorated with geometric compositions; the one on the right is an Islamic star tile depicting two people facing each other.

Tiles are the most common elements used to decorate interior—and sometimes exterior—walls of the masjids in Turkey (Fig. 1.5). Given the number of masjids in Turkey, it is not surprising that the Turkish tile industry is one of the biggest one in the world. The tile in the upper right is a very common design that can be found in many masjids, whereas the one on the bottom left is rare—not because of its design, but because of its shape—as it decorates a cylindrical surface, which requires a specific skill to construct. Similarly, the one in the bottom right has a sphere whose surface is decorated with geometric figures. The picture in the upper right, representing many Islamic stars, is from Topkapi Palace, Istanbul, whereas the other pictures are from Hopa, Artvin (upper left), Kutahya (bottom right), and Bulancak, Giresun (bottom right).

Fig. 1.3 I'timad-ud-Daulah, Agra, India

Fig. 1.4 Masjid and Aga Khan Center, Toronto, Ontario, Canada

Fig. 1.5 Masjids and Topkapi Palace, Turkey

Sultan Salahuddin Abdul Aziz Mosque, a large mosque located in the city of Shah Alam in Malaysia, has various types of ornament designs decorating its inner walls and other places, such as its minbar and mihrab[5] (Fig. 1.6). It is known as the Blue Mosque because of the colors of its dome, and it has the tallest minarets in the world (upper left). The two bottom-left pictures show the minbar of the mosque.

[5]The minbar and mihrab are where an Imam gives speeches and leads the worshipping ceremony, respectively.

Fig. 1.6 Masjids, Shah Alam, and Johor Bahru, Malaysia

The geometric objects surrounding the Islamic stars in these pictures have a unique look, and each of them could be used as a context for a geometry problem as it may be challenging to create a Euclidian construction of them. The other two pictures, in the upper middle and the left, are the pictures of minaret of the Sultan Abu Bakar Masjid located in Johor Bahru, Malaysia. What is interesting for this minaret is its design. The minaret has a square base at the bottom and an eight-sided prism at the top; it might be constructed to represent a journey (i.e., life) from earth—Ka'ba has four walls—to heaven—the place with eight walls, as previously mentioned.

Figure 1.7 presents a group of pictures from Putrajaya, Malaysia. The picture taken from a distance (upper left) is of the Masjid Putrajaya, which is located next to the artificial Putrajaya Lake. The other pictures are of the mihrab of the masjid (upper right) and ground design of the external part of the masjid. Each includes a representation of an Islamic star surrounded by other geometric objects.

In the following sections, I present two main approaches in learning and using geometry followed by the illustrations of horizontal and vertical mathematization in the context of Islamic star construction.

Geometric Explorations of Islamic Art

Islamic art incorporates a number of ornaments and patterns and provides many examples for explorations in mathematics and geometry. This chapter looks at the artefacts of Islamic art from the perspective of Euclidean geometry. The goal is to re-create the Islamic star in the digital environment of GeoGebra by following the Euclidean approach. The Euclidean approach discerns between *constructing* and

Fig. 1.7 Masjid Putrajaya, Putrajaya, Malaysia

drawing geometric objects. Constructing is the process of using geometric principles to create objects using only an unmarked ruler and a compass, whereas drawing objects involves measuring lengths, angles, and other properties of the geometric object in order to (re-)create it. In order to clarify the differences between these two approaches, I will describe the construction and drawing of the equilateral triangle.

A common approach in drawing an equilateral triangle involves using a ruler to draw the base of the triangle with a specified length of *a* and the end points A and B. One who already knows that each angle of an equilateral triangle should equal 60° could use a protractor to measure and set the angles from each end, in the clockwise and the counter-clockwise directions, respectively. We can find the third point of the triangle, C, by finding the intersection point of the two rays produced by the construction of two angles, or we can measure the length of A and mark it on each ray.

The Euclidean approach, however, constrains students to using a compass and an unmarked ruler. We can draw a line segment with length *a* and end points A and B. We can then draw a circle whose center is A, passing through point B. After repeating the same procedure for a circle with center B, passing through A, we can mark one of the intersection points as the third point, C, of the equilateral triangle.

Constructing the geometric figure, we argue, has at least two advantages to the drawing method. The first one is that the creation of the equilateral triangle through constructions involves the principles of Euclidean geometry, whereas the drawing is based on drawing and measurement, meaning geometric thinking needs not to be evoked. The second advantage is that the follow-up activity (the next step) to the construction of the equilateral triangle can involve more complex construction methods that involve explorations of more general principles—that is, it involves vertical mathematization. Drawing other figures, on the other hand, does not necessarily encourage learners to expand on their mathematical knowledge.

Constructing geometric figures is consistent with Lockhart's (2002) claim that teachers should start geometry lessons with explorations and encourage their students to improve their geometrical knowledge—mathematics in general—in each activity as well as to learn geometry through exploration rather than starting with theorems and axioms. Theorems and axioms should emerge from the justification of students' constructions by asking questions such as: Does the method always work? What is the nature of the relationship between equilateral triangles and circles that makes the construction possible? How can we use the principle between the construction to create other figures such as the square and the regular pentagon?

All these questions may engage students in vertical mathematization. What should the starting point be? Prior to working in an idealized mathematical world, what materials and scenarios could we use to engage learners in geometry? If we try to find an equilateral triangle in the real world, we might be disappointed because no actual triangle is perfectly equilateral, although it is quite possible to find a figure close to the equilateral triangle. Islamic art provides many examples of the real-life approximations of geometric shapes that may trigger students' interest in geometry. The rest of the chapter presents a hypothetical learning activity involving the construction of the Islamic star, as presented in the preceding figures, using the Euclidean approach that could be considered an example of horizontal mathematization. Following this activity, another activity illustrating vertical mathematization is presented.

Creating Islamic Star: Horizontal Mathematization

The RME theoretical framework conceptualizes the pedagogical perspectives of the chapter and serves as the guide to a hypothetical learning trajectory described herein. It has already been pointed out that the RME starts with horizontal mathematization by analyzing a real-world example and converting it into the mathematical world. The real-life example for this chapter is the Islamic star, which can be seen practically worldwide and therefore serves as a common example for many people, including non-Muslims, because it is also used in some other traditions. One could also apply the procedures discussed here in another context by keeping in mind that learning geometry is the ultimate goal. In order to draw the figures, we can use dynamic geometry software, such as GeoGebra.

The first approach, as previously mentioned, is based on a measure-and-apply procedure and could easily be applied to drawing two squares, with one being rotated 45° around the centroid of the other and setting them as to coincide their centroids. Students who want to follow this procedure could draw a square using the ruler and protractor. They could first draw the line segment AB using the ruler and then add perpendicular lines on each end using the protractor. They could find the other two points of the square simply by measuring the length *a* from each end point, A and B.

The big idea behind this approach is setting the second square such that the centroids of both squares coincide. One possible procedure for setting the second square in its place would be to start finding the centroid of the square by intersecting the diagonals (Fig. 1.8).

It might not be a challenge to realize that the perpendicular lines to the sides of the square and passing through the centroid would also pass through the corners of the second square. After drawing the perpendicular lines, students could draw a circle, whose center is the centroid of the square and passing through the corners of the square (Fig. 1.9).

Fig. 1.8 Finding the centroid of the square

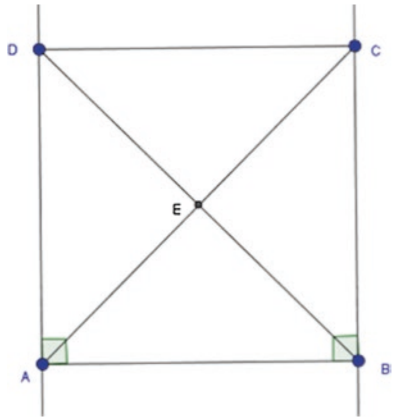
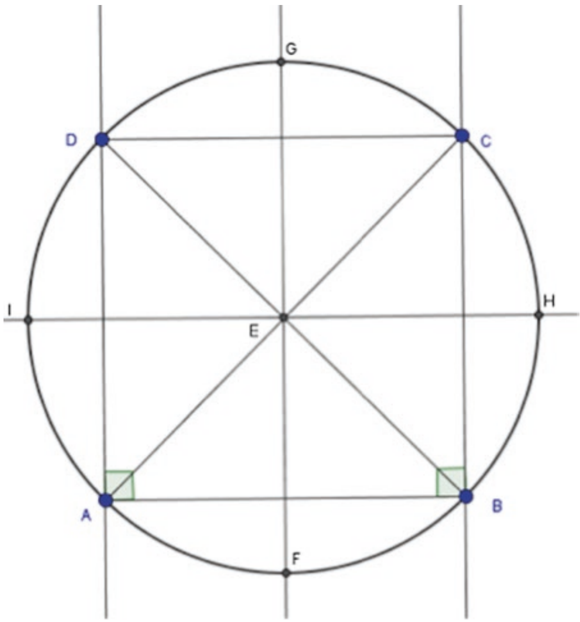


Fig. 1.9 Locating the corners of the second square



The intersecting points of the perpendicular lines and the circle define the corners of the second square (Fig. 1.10).

The final step of this procedure is to remove all the unnecessary elements and leave the Islamic star alone (Fig. 1.11).

The second approach is the Euclidean approach, which requires more geometrical thinking and knowledge in order to construct the figure. The challenge for this approach is to constrain students to using an unmarked ruler and compass only—that is, they are able to draw only line segments and circles whose center and one point are known.

Figure 1.12 illustrates one possible starting point of drawing a line passing through two distinct points (upper left). Given that these two points are going to be the two points of the final construction, we need to construct a perpendicular line passing through either point A or point B. In order to construct the perpendicular line passing through point B, we should draw a circle centered at point B and passing

Fig. 1.10 The squares of an Islamic star

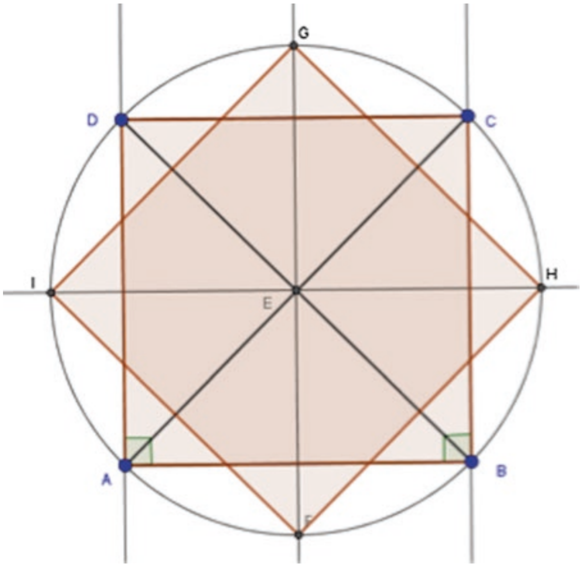
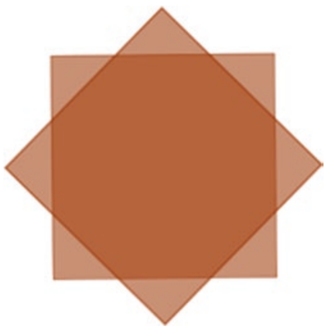


Fig. 1.11 Islamic star



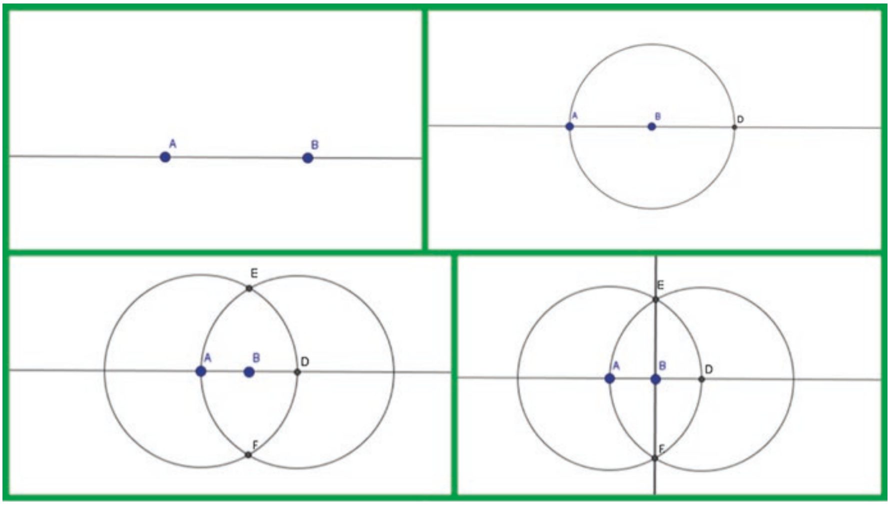


Fig. 1.12 Steps involved in the construction of a square

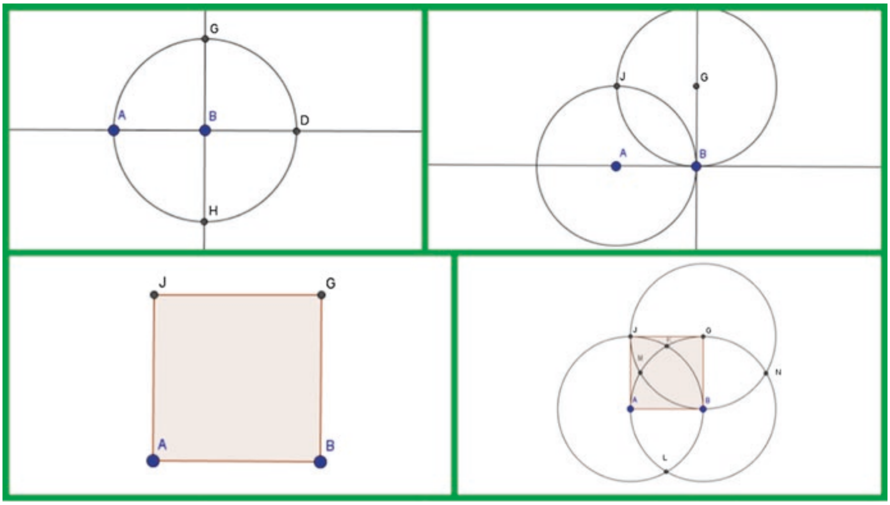


Fig. 1.13 Further steps in constructing a square

through point A such that point B becomes the midpoint of two distinct points, A and D (upper right). Two circles, one centered at A and passing through D and the other just the opposite, are drawn (bottom left). The line passing through points E and F, intersections of the circles, is perpendicular to line AB and passes through point B (bottom right).

It should now be easier to construct a square on the line segment AB (Fig. 1.13). First, the third point of the square should be identified by intersecting the perpendicular line—line EF in Fig. 1.12—and the circle, whose center is point B and whose radius is the distance between A and B (upper left of Fig. 1.13). Following the procedure to find the third point, point G, of the square, students can locate the fourth point by simply constructing the circles whose centers are points A and G and that pass through points G and A, respectively; by intersecting them, we get point J (upper right). Thus, the square ABGJ appears in the bottom left of Fig. 1.13. In order to find the helping points to construct the second and rotated square of the Islamic star, students can use the circles drawn before. The intersection points M and K provide students with the chance to draw perpendicular bisectors of the sides of the square (bottom right).

The steps to construct the second and rotated square are illustrated in Fig. 1.14. The picture in the upper left illustrates the construction of the perpendicular bisectors of the sides of the first square, while the picture in the upper right shows the circle, whose center is O—the point of intersection of the perpendicular bisectors—and that passes through the corners of the first square. The intersection points of the circle and the perpendicular bisectors identify the corners of the second square (bottom left). Finally, students can find the intersection points of two squares to identify the remaining points of the Islamic star (bottom right).

The Islamic star construction steps may help convince readers that this horizontal mathematization of the RME activity could be challenging, even for some advanced secondary students not familiar with constructing a geometric object by applying the Euclidean approach. However, readers might also understand that all of these steps actually stem from Euclidean geometry—that is, there is no need to reinvent the wheel. What students should do is separate the task into pieces, set the goals for each piece, and explore the possibilities to accomplish the task. Other procedures to construct the Islamic star may or may not employ Euclidean approaches.

The following section describes how I used this activity to engage my undergraduate students enrolled in the geometry course in the elementary mathematics teacher education department in the vertical mathematization step. I aimed to help

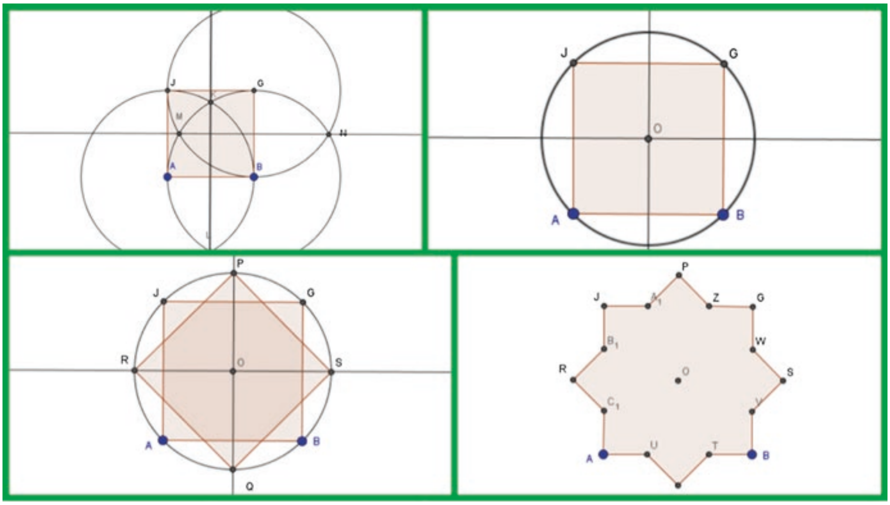


Fig. 1.14 Steps from square to Islamic star

them further their geometry knowledge to make some generalizations—an important dimension of mathematical thinking.

Vertical Mathematization: How Can We Generalize This Construction?

I asked my students, preservice grade 5–8 teachers, to create composite geometry objects found in Islamic art, including but not limited to the Islamic star. The objects they constructed included a number of polygons and stars. It is easier to construct an Islamic star than a 10- or 12-pointed star using the Euclidean approach because creating pentagon is more challenging than creating a square as the basis of the construction. Still, it is a worthwhile task for those interested in engaging their students in vertical mathematizing in this manner. I shifted my method for the second part in my course and encouraged my students to employ transformational geometry rather than following the Euclidean approach for the vertical mathematization. In this section, I describe a generalized version of the construction problem and leave the problem for the readers.

At this stage, the mathematical representation of the Islamic star is still our starting point because we have already transferred it from the real world to the digital world by performing a good amount of geometrical analysis and discussion stemming from the Euclidean approach. Given that the Islamic star is our starting point and we already assume that it is constructed by drawing two squares, one being rotated by 45° around the centroid of the other and superimposed on the first one (upper left in Fig. 1.15), we could start the vertical mathematization step by posing the following question:

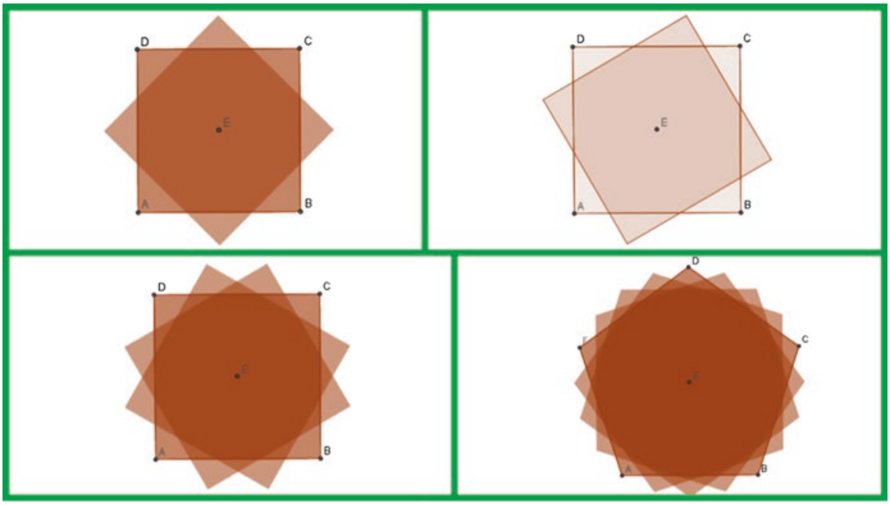


Fig. 1.15 From Islamic star to n-pointed stars −1

What if we rotate the square by 30° rather than by 45° around the centroid of the other and superimposed on the first one (upper right)?

An intuitive answer may come out immediately, as it did in my classroom: "Well, the figure becomes asymmetric!" It is not too difficult to make the figure symmetric. One could rotate the square in increments of 30°, superimposing each figure onto the original square (see the bottom left picture in Fig. 1.15). Further discussion involves explaining why only three new squares appear. We first discovered that, if the square is rotated 90°, then the rotated one completely covers the first one. Second, we all agreed that we divide this 90° by some number. For example, it was 2 while constructing the Islamic star and became 3 in the case of the 12-pointed star. A third discovery followed a discussion of the question I asked: Why do we use a square? What if we use another polygon? What would be the overlapping angle if we use an n-gon rather than a square? For example, we could obtain a 20-pointed star if we take a pentagon, rotate it by 18°, 36°, and 54° around its centroid, and superimpose all four pentagons (bottom right).

We had a great discussion in the classroom exploring the relationships among these numbers. I gave the rest of the vertical mathematization as the assignment. The undergraduates were supposed to generalize the case by assuming that they had an n-sided polygon, the polygon was rotated by α° and its multiples around its centroid, and all polygons were superimposed to get a final p-pointed star. The only variables they had were n, the number of sides of the initial polygon, and k, the number of divisions to get α for each rotation. The parameter p was also to be calculated by using these variables. A case where $n = 5$, $k = 4$, and the maximum value of $\alpha = 72°$ to construct a 20-pointed star is illustrated in Fig. 1.16.

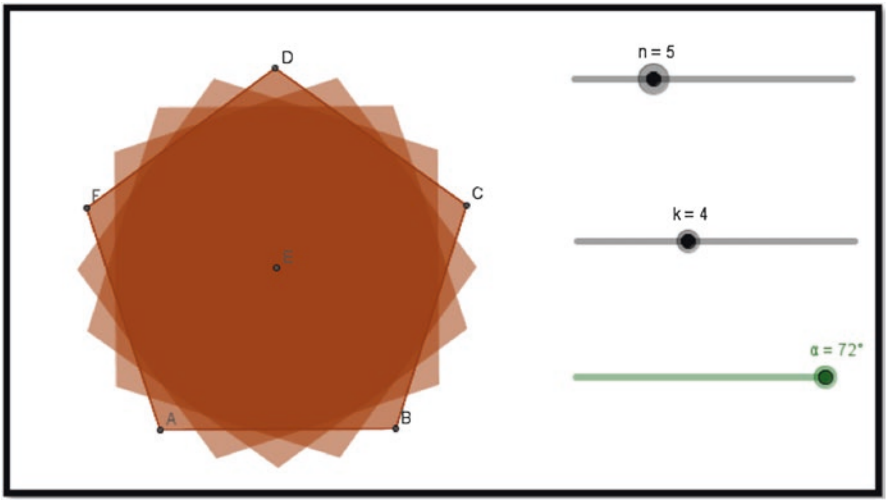


Fig. 1.16 From Islamic star to n-pointed stars −2

Conclusion

Geometry is usually delivered as a set of axioms disconnected from real life or as a course about the application of geometric measurement. However, it is rather challenging for students to develop a deep understanding of geometry if it is delivered as a course bounded by classroom walls and disconnected from real life as it is difficult to engage students' learning that is decontextualized. Fortunately, the world around us is rich in geometric objects. The existence of such a rich environment motivated us to talk about learning geometry stemming from Islamic art using an unmarked ruler and compass only—that is, by following the Euclidean procedure.

The chapter suggested following the Euclidean procedure to encourage students to think geometrically. Regardless of the replication method employed, the process to pick up an ornament from Islamic art and work on it to replicate is referred as the horizontal mathematization of the RME framework. During this process, learners are engaged in the replication activity to become familiar with basic geometric objects and their properties without delving into some axioms and geometrical facts. The chapter presented one possible way as a hypothetical learning activity to follow the Euclidean approach and construct the Islamic star, which is widely seen in Islamic art. It is quite possible to develop some other approaches, which is fine. It is possible to work on a different artefact, which is also fine. As far as the ultimate goal is to encourage students to think geometrically and engage them in learning meaningful geometry, it is quite possible to find a number of artefacts and a number of procedures.

Similarly, the chapter suggested moving into transformational geometry in the vertical mathematization process rather than working on more complicated geometric objects. Although either is fine for the sake of vertical mathematization because learners should be engaged in more advanced topics or processes during this process, introducing transformational geometry and encouraging learners to generalize the construction may open new windows. Learners may realize that mathematics, by assuming that the use of algebra refers to mathematics, and geometry are not distinct topics; rather, they are interconnected across many contexts.

References

Azzam, K. (2013). Introduction. In K. Azzam (Ed.), *Arts and crafts of the Islamic lands: Principles, materials, practice*. London: Thames & Hudson Ltd..

Burckhardt, T. (2009). *Art of Islam: Language and meaning*. Bloomington, IN: World Wisdom.

Critchlow, K. (1976). *Islamic patterns: An analytical and cosmological approach*. London: Thames and Hudson.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: D. Reidel Publishing Company.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modelling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225–273). Mahwah, NJ: Lawrence Erlbaum.

Kharazmi, M. (2016). A study on geometric constructions on brickwork decorations in Iranian architecture. *The Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture* (pp. 301–308).

Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). New York: Macmillan.

Lockhart, P. (2002). A mathematician's lament. MAAOnline. Retrieved from https://www.maa.org/external_archive/devlin/devlin_03_08.html on Apr 16, 2016.

Lu, Y., & Demaine, E. D. (2016). Beautification of Islamic patterns via constraint **s**atisfaction. *The Proceedings of Bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture* (pp. 371–374).

Noori, M., & Kiyanmehr, Q. (2016). The analysis of the geometric decorations of the stone half-columns of friday mosque of Isfahan. *The proceedings of bridges 2016: Mathematics, Music, Art, Architecture, Education, Culture* (pp. 317–324).

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics instruction—The Wiskobas project*. Dordrecht, The Netherlands: D. Reidel Publishing Company.

Treffers, A. (1993). Wiskobas and Freudenthal: Realistic mathematics education. *Educational Studies in Mathematics, 25*, 89–108.

Yusuf Ali, A. (n.d.). *English translation of the Holy Book, Qur'an*. Retrieved from www.kuran.gen.tr on July 17, 2016.

Chapter 2
The Hidden Musicality of Math Class: A Transdisciplinary Approach to Mathematics Education

Song A. An, Daniel A. Tillman, and Lawrence M. Lesser

Abstract This chapter surveys interdisciplinary pedagogy that emphasizes the connections between mathematics and music by contextualizing the mathematics learning process within musical experiences. Both empirical research and international practice have demonstrated a variety of opportunities for music-themed mathematics teaching methods to be developed and implemented across all grade levels, from kindergarten to college. This chapter, which summarizes the current state of research and practice for music-themed interdisciplinary mathematics education, is divided into three main sections: (1) the overview of connection between mathematics and music, (2) theoretical perspectives on music and mathematics learning, and (3) a description of pedagogical approaches appropriate for supporting music-mathematics interdisciplinary lessons. Regarding the overview, the chapter discusses research studies that have investigated the mathematics present within music and the application of mathematics to improving musical composition and musical instrument design. Regarding the theoretical perspectives, the chapter discusses research studies that have investigated passive musical immersion as well as more active musical learning processes and their comparative impacts upon learners' mathematical cognitive processes and capabilities within informal learning settings. Regarding the pedagogical approaches, the chapter presents and evaluates the prevalent mathematics-music-integrated teaching strategies about how student-centered musical activities (i.e., listening and singing, composing and performing, musical notating, and musical instrument design) can be utilized to teach specific mathematics topics.

Keywords Music-mathematics connections • Interdisciplinary curriculum • Interdisciplinary mathematics instruction and mathematics education

---

S.A. An (✉) • D.A. Tillman • L.M. Lesser
The University of Texas at El Paso,
601 Education Building, 500 W. University Ave., El Paso, TX 79968, USA
e-mail: saan@utep.edu; datillman@utep.edu; lesser@utep.edu

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_2

This chapter will survey interdisciplinary pedagogy that emphasizes the connections between mathematics and music by contextualizing the mathematics learning process within musical experiences. Both empirical research and international practice have demonstrated a variety of opportunities for music-themed mathematics teaching methods to be developed and implemented across all grade levels, from kindergarten to college. This chapter, which summarizes the current state of research and practice for music-themed interdisciplinary mathematics education, is divided into three main sections: (1) the overview of connection between mathematics and music, (2) theoretical perspectives on music and mathematics learning, and (3) a description of pedagogical approaches appropriate for supporting music-mathematics interdisciplinary lessons. Regarding the overview, the chapter will discuss research studies that have investigated the mathematics present within music and the application of mathematics to improving musical composition and musical instrument design. Regarding the theoretical perspectives, the chapter will discuss research studies that have investigated passive musical immersion as well as more active musical learning processes and their comparative impacts upon learners' mathematical cognitive processes and capabilities within informal learning settings. Regarding the pedagogical approaches, the chapter will present and evaluate the prevalent mathematics-music-integrated teaching strategies about how student-centered musical activities (i.e., listening and singing, composing and performing, musical notating, and musical instrument design) can be utilized to teach specific mathematics topics.

Connections Between Mathematics and Music: An Introduction

Mathematics and music have a strong connection and have each contributed to the other's development. In this section, we present examples of how mathematics has played a significant role in acoustics, the mechanics of musical instruments, and music composition.

The Chinese bianzhong (literal meaning: *bell set*) illustrates mathematical sophistication in the construction of musical instruments. The bianzhong, a set of 65 bells, was discovered in the late 1970s inside the ancient tomb of Zeng (430 B.C.E.) in Hubei, China. This instrument was widely recognized as the first musical instrument that can play a range over five octaves (von Falkenhausen, 1993). The bianzhong has a weight of approximately 2.5 tons, and all of the bells are designed to be hung from a large wooden stand 36 feet wide by 9 feet tall. Its construction required an interdisciplinary team consisting of a partnership among musicians, mathematicians, and engineers (Lee & Shen, 1999). The bells increase in size, with the height of the smallest bell being around 8 inches whereas the length of the largest bell is more than 60 inches. Correspondingly, the bells' weights range from 5.3 to 448.9 pounds. This pattern contributes to the acoustics of this musical instrument.

Musicians and mathematicians continue to collaborate to improve musical instruments. One of the major breakthroughs for contemporary music production has been the extensive acceptance of equal temperament, a tuning method based on the computation of logarithms and differential equations in which every note in a chromatic scale has an identical frequency ratio that is artificially divided (Cho, 2003). By distributing inharmonic errors that exist within natural temperament instruments, the tunes across different instruments can be standardized; this technique thus enables musicians to change keys during their performances without changing the instruments. Applications of the equal temperament theory led to the invention of the piano in 1709.

The design of the piano required intensive knowledge of geometry and measurement as well as algebra (Ehrlich, 1990; Gordon, 1996). As an illustration of the application of geometry, cross stringing was a creative invention by Jean-Henri Pape during the 1820s that significantly reduced the size of pianos by transforming string arrangements from a two-dimensional design to a three-dimensional design. In some early versions of pianos, piano strings were arranged parallel to each other to avoid collisions during string vibrations. By utilizing the three-dimensional space, Pape arranged strings into two planes: Bass strings were secured from left to right, under and across the other strings that were secured from right to left. The theory and practice of action (translation of the motion of a piano key into the motion of a hammer that strikes the strings) was developed with the application of algebra. Because action serves as both engine and transmission, certain ratios of the movement distance between the key and the hammer have to be applied in order to create a series of lever systems that magnify the force generated from fingers pressing on keys into hammers striking the strings.

Another link between music and mathematics is the use of mathematical patterns and geometric transformation in the process of musical composition. For example, repeating patterns have been found in almost all works of music, often contained within small sections or a whole movement, and growing patterns such as the Fibonacci sequence as well as geometric transformations such as transpositions and inversions have been massively used by both classical and present-day composers (Beer, 1998; Loy, 2006). The application of mathematics to music includes algorithmic and computational approaches within musical scales (Krantz & Douthett, 2011), algebra within periodic rhythms and scales (Amiot & Sethares, 2011), topology of musical data (Sethares & Budney, 2014), logarithms and differential equations in equal temperament (Cho, 2003), and mathematical patterns within prominent pieces of classical music (Conklin, 2010). However, it should be noted that many of the links between music and mathematics are only deeply understood by music and mathematics specialists. Classroom mathematics teachers who lack familiarity with basic music theory may not have the background knowledge needed to implement music-math beyond the employment of a cover story with entertainment value, incorporating few—if any—pedagogical connections with the mathematics being taught (An, Tillman, Boren, & Wang, 2014; An, Tillman, Shaheen, & Boren, 2014).

Music and Mathematics Learning: Theoretical Perspectives

Concerning achievement, empirical evidence has shown that learning about math and learning about music are mutually beneficial. For example, in a large-scale study involving a sample of more than 150,000 high school students, researchers found a statistically significant relationship between students' music achievement and their academic success in the core subjects, especially mathematics (Gouzouasis, Guhn, & Kishor, 2007). The finding suggested that students with high mathematics achievement were predicted to have high music achievement and vice versa.

Multiple theoretical perspectives offer rationales for explaining the effectiveness of mathematics education that incorporates music or music-related experiences. In general, the educational theories that encourage music-mathematics instructional connections have two distinct dimensions, which vary in intensity depending upon the particular theory: (1) a focus on the role of music as a catalyst for boosting mathematics learners' cognitive processes by being played as ambient background sounds during mathematics education and/or (2) a focus on the role of music as an educational resource for contextualizing the teaching and learning of mathematics into a meaningful and relatable medium. Along the first dimension, the Mozart effect theory (Rauscher, Shaw, & Ky, 1993) as well as several of its variations (e.g., Hui, 2006; Ivanov & Geake, 2003) have served as a general framework for illuminating the impacts of both active music learning and passive music immersion on mathematics learners' mathematical cognition capacities. Along the second dimension, mathematical motivation theory (e.g., Bursal & Paznokas, 2006; Geist, 2010) has been used to construct an understanding of the effects of placing children in music-contextualized learning environments as well as the impacts of employing a student-centered teaching approach that employs music-themed activities to develop students' conceptual understanding in mathematics and encourage their positive dispositions toward mathematics learning.

Music as a Catalyst for Mathematical Cognition

Among the many studies examining the quantifiable associations between music experiences and their effects on mathematical cognition, the Mozart effect study series (e.g., Rauscher et al., 1993; Rauscher, Shaw, & Ky, 1995; Rauscher et al., 1997) was the most well-known as well as the most controversial research, with about 40 replicated trials involving more than 3,000 participants (Pietschnig, Voracek, & Formann, 2010). In the original research design, Rauscher and his colleagues conducted an experiment comparing three randomized groups: (1) listening to the target music (Mozart's Sonata for Two Pianos in D Major, K.448), (2) listening to comparison music (generic relaxing music), and (3) listening to silence. Results from the study demonstrated that the participants in the Mozart music group significantly outperformed their peers on the spatial reasoning skills sub-tests from the Stanford-Binet Intelligence Scale. Since the publication of the study's results, replications have assessed revisions to the settings, music treatment, and mathematics assessment tasks, accumulating in the process further evidence that listening to Mozart's music—compared to other music or silence—may advance participants' mathematical cognition (Hui, 2006; Ivanov & Geake, 2003; Nantais & Schellenberg, 1999; Rauscher et al., 1995; Rauscher & Zupan, 2000). For example, Nantais and Schellenberg conducted experiments with the random assignment of two groups of participants; treatment group participants listened to both Mozart's and Schubert's music, while the comparison group sat in silence. Two mental visualization assessments, each with 17-item paper folding and cutting tasks, were assigned to the participants after the treatment of music, and the results demonstrated that treatment group students significantly improved their test scores compared to the silence group.

In addition to the laboratory experiments, researchers have investigated music learning experiences (e.g., taking piano lessons, playing an instrument in school band, and practicing vocal music) and their relationship to students' mathematics achievement. In general, learning music and practicing music were positively correlated to students' mathematics achievement, with students who had music-related experiences demonstrating significantly higher mathematics achievement scores than their nonmusical peers. Similar patterns were identified across grade levels, including pre-K (Costa-Giomi, 1999; Rauscher & Zupan, 2000), elementary school (Haley, 2001), middle school (Whitehead, 2001), and high school (Cox & Stephens, 2006). A possible explanation for these consistent findings is that musical experiences can stimulate areas of the brain responsible for mathematical reasoning. On a similar note, Spelke (2008) explained that activating the "musical zone" in the brain may also stimulate the working processes of the "mathematical zone." In other words, within human brains, the areas responsible for processing cognitive functions for musical perception of melody, harmony, and rhythm have overlapping areas with those responsible for processing cognitive functions for mathematical computation, such as geometrical visualization, numerical calculation, and estimation.

Improving Mathematical Proficiency with Music

In an era of standards and accountability, students' academic achievement—especially their scores on high-stakes standardized tests—has become overemphasized (Pinar, 2004; Slattery, 2006). Compared to the importance placed upon developing students' procedural fluency and strategic competence in mathematics, many teachers ignore the development of their students' positive dispositions toward mathematics (Kilpatrick, Swafford, & Findell, 2001). The role of emotion has been recognized as a crucial factor in learning mathematics, and the negative emotions of disengagement and anxiety are now understood as critical obstacles on the road to success with mathematics. Empirical studies have consistently found that students in all grade levels, from kindergarten through college, display negative dispositions toward mathematics—they may believe mathematics is not useful in real life or that learning mathematics is simply too difficult for them (Rameau & Louime, 2007). Compared to peers who display productive dispositions toward mathematics, mathematics learners with negative dispositions not only suffer from significantly higher levels of anxiety during mathematics learning but also exhibit a lack of confidence or motivation to learn and apply mathematics (Ashcraft, 2002; Geist, 2010). Consequently, researchers have found that students with negative mathematics dispositions often have lower mathematics achievement and also generally avoid taking advanced mathematics courses in high school and college, which culminates in their inability to choose any of the STEM-based careers requiring a background in mathematics and/or science (Sullivan, Mousley, & Zevenbergen, 2006).

Unlike other school subjects such as social studies and language arts, which are inherently grounded in meaningful contexts and/or real-life relations, mathematics as a subject is often structured apart from society and culture and is instead based upon a language that employs complex symbols and highly abstract concepts. The distinctive structure and content of mathematics education have resulted in the development of an accompanying pedagogy for this subject, which is vastly different from other school subjects (Kilpatrick et al., 2001). Unfortunately, many school teachers, especially generalists who teach multiple subjects in elementary and middle schools, fail to offer student-appropriate methods when teaching mathematics to the demographics they serve, and this "traditional" instruction model based on the teacher-centered approach has been identified as one of the key factors influencing students' negative dispositions toward learning mathematics (Furner & Berman, 2005).

This "traditional" teacher-centered model of mathematics instruction is recognized as textbook content lecturing, overreliance on assigning drill problems, single correct answer grading, and consistent use of standardized multiple choice testing. The result is that students' conceptual understanding and strategic competencies are often ignored, while less important but more measurable metrics are pursued (Bursal & Paznokas, 2006; Geist, 2010). As an alternative to the "traditional" teacher-centered model, educational researchers have proposed some common features found among more effective pedagogical methods for instructing mathematics education, including the use of open-ended problem-solving activities in which more than one correct answer is possible, simulations including augmented and virtual reality, game-like challenges that make mathematics learning a friendly competition, and discovery-driven learning where students collect and analyze data to answer real-life contextualized questions that include such themes as finances and urban planning. All of these methods can provide opportunities to facilitate students' communication among peers, connections between and across curriculum, and representation of mathematics in multiple ways (Bursal & Paznokas, 2006; Geist, 2010). Within this sphere, a music-integrated approach to mathematics has been identified as an effective method for teaching student-centered mathematics (Robertson & Lesser, 2013).

Mathematics pedagogy that effectively uses the natural cognitive overlaps between music and mathematics offers students transdisciplinary opportunities to discover, recognize, analyze, and apply mathematics (An, Capraro, & Tillman, 2013). For teachers, music-themed mathematics activities can serve as meaningful and accessible contexts for transforming traditional mathematics pedagogy via entertainment elements (An & Tillman, 2014; Vinson, 2001). Music enables students to represent their mathematical ideas from a different perspective, which supports their learning as they pursue conceptual understanding via multiple cognitive and affective experiences (Gamwell, 2005). Specifically, findings showing positive impacts have accumulated across several studies investigating the effects of music-mathematics-integrated education, including (1) motivating students to undertake more challenging mathematics tasks (Chahine & Montiel, 2015), (2) engaging students in examining relationships among mathematical concepts (An, Tillman, Shaheen, & Boren, 2014), (3) creating a pleasant learning environment for supporting team work (Robertson & Lesser, 2013), (4) providing a teaching environment that minimizes language and culture barriers for English-language learners (Kalinec-Craig, 2015), (5) improving students' academic achievement in mathematics (An & Tillman, 2015; Pinnock, 2015), and (6) developing teachers' self-efficacy for mathematics pedagogy (An, Tillman, & Paez, 2015; An et al., 2016).

Incorporating dynamic auditory approaches in teaching, students could build knowledge cognitively, perceptually, and emotionally (Greene, 2001). However, music is an underused educational resource (An & Tillman, 2014) because teachers are required to pedagogically develop auditory resources into visual and tangible manipulatives for students to make sense of mathematics. In this chapter, we provide an overview of the empirical research studies conducted by An and his colleagues (An et al., 2013; An, Tillman, Shaheen, & Boren, 2014; An, Tillman, & Paez, 2015; An, Zhang, Flores et al., 2015). These studies examined music experiences that occurred during mathematics lessons and were designed to support students' engagement with the topic as they actively manipulated objects, performed creatively, and applied their existing knowledge in the creation of connections among mathematical concepts. During the lessons, music-making and music-sharing experiences enabled students to pursue their original interests along with their curiosity. At the affective domain, the aesthetic appreciation of music encouraged students' mathematics learning behaviors by providing them a meaningful context for completing mathematical tasks. The learning experiences also offered students additional reinforcement of the mathematical concepts by letting them play and share musical works that they had created themselves and increased their efficacy for undertaking further challenging learning tasks in mathematics.

Teaching Mathematics Via Music: Pedagogical Approaches

Curriculum developers and lesson designers have proposed many ways for emphasizing the musical connections available when teaching mathematics. For example, Gelineau (2004) and Cornett (2007), in their books about teaching elementary subjects through the arts, presented several interrelated ideas about the links between mathematics and music. In one of the author's own previous activity books (An & Capraro, 2011), teachers were offered a suite of lessons that put mathematics topics within the context of music composition and musical instrument-designing activities. Subsequently, with a goal of investigating the possibility of pedagogical connections between mathematics and music, An and colleagues conducted an empirical analysis of 78 teacher-generated music-mathematics-integrated lesson plans along with 152 elementary preservice teachers' lesson plans, enabling the identification of 56 different examples of connections between musical (music notating, singing, playing, composing, and instrument designing) and mathematical (number and operation, algebra, geometry, measurement, data analysis, and probability) content areas (An & Tillman, 2015; An, Tillman, & Paez, 2015; An et al., 2016). However, it was also determined that numerous misguided attempts at the contextualization of mathematics pedagogy within music-themed activities have occurred, and the poor implementation of this transdisciplinary concept can hinder the mathematics learning process. In other words, positive learning results in mathematics were only found in scenarios where the music-themed activities were pedagogically relevant to mathematics being taught and not merely serving as a "cover story."

Musical Notation and Fractions: An Example of Weak Pedagogical Integration

When asked "Are there any relationships between music and mathematics?," many mathematics teachers will say "Yes"; however, when asked for examples, the most frequent answer is that "There are quarter notes in music." Moreover, among the lesson plans and activities that can be implemented in classrooms, the music notation system (including musical notes and time signatures) has been used as one of the most popular activity themes for teaching mathematics, especially fractions (An & Tillman, 2014). Unfortunately, fractions are the only mathematics topic that musical notes are often used to address. As a typical instructional design based on teaching fractions connected with the musical notations of note values, one of the participating teachers proposed the following lesson plan:

> The goal of my lesson is to help students understand how musical notes relate to fractions and clap a measure of music by assigning appropriate values to notes. In the lesson I will have [the] following steps: (1) present note names and their values and introduce lesson vocabulary such as whole note [4 beats], half note [2 beats], quarter note [1 beat], eighth note [1/2 beat], sixteenth note [1/4 beat]; (2) present a music note chart to demonstrate the values each note holds and students will be clapping along to gain understanding of the values of the notes in a measure; (3) discuss the value of different notes to help students "hear" the value of those notes, clap a 4-beat measure, a 2-beat measure, and a 1-beat measure and have students join in; (4) ask questions regarding the counting values of music notes (How many beats are in 1 whole note? How many beats are in 2 half notes?) and check and correct the student answers; and (5) on a scratch sheet of paper, in groups, students will solve some math problems such as "two eighth notes equal to ______", and "four quarter notes equal to ______." (An, Tillman, & Paez, 2015, p. 16)

Fig. 2.1 Example of fraction relationships represented within music notation presented by a math teacher participating in the study

In the example presented above, the teacher attempted to integrate music and mathematics by connecting musical notes and beats to the mathematics concept of fractions. This was a weak pedagogical integration, as music symbols—a highly concentrated code developed by composers for facilitating the written communication of music—were not properly connected to fractions. At the visual level, the symbols employed for different note values (Fig. 2.1) failed to demonstrate the proportional relationships for fractions when compared with the traditional approach of conceptualizing fractions based on self-evident pictorial representations, such as area and length (Van de Valle, Karp, & Bay-Williams, 2010). At the auditory level, making non-recorded sounds from hand clapping based on different rhythms is a difficult way for students to comprehend the whole and partial relationships essential for learning fractions.

The limited working memory capacity that humans have for processing new knowledge can result in novel situations overloading cognitive capacity. According to cognitive load theory (Sweller, 2016), instructing fractions via associations with musical notation, such as reading and clapping music notes with different values, may increase students' extraneous cognitive load, overwhelming their ability to process the mathematics concepts they are supposed to be learning. In other words, introducing fractions through musical notations by claiming that, for example, the value of a black circle is half of that of a white circle of the same size, instead of helping pedagogically, might be counterproductive as the extraneous cognitive load can result in mathematics learners struggling with the lessons.

Singing and Listening to Music in Mathematics Class: More Than Simply a Cover Story

Many math lessons across all content areas incorporate singing and playing music. The availability of the Internet during the past two decades has greatly increased access to relevant musical resources for supporting such activities, and the popularity of video-sharing sites has especially intensified, with more than 7000 archived educational resources in the format of songs for teaching mathematics and science being prepared by professional musicians, educators, and enterprising students (Crowther, 2012). Numerous mathematics-themed songs—both original music created by educators and popular melodies with new lyrics—are available for teachers and ready to be used. Nevertheless, without the appropriate pedagogical structure, these music resources often only serve as cover stories providing entertainment in a mathematics class (An & Tillman, 2014). A typical instructional design that employs mathematics-themed songs without pedagogical development usually has the following fundamental steps: (1) introduce and play a music video at the beginning of the class, (2) have a mathematics class without music connections, and (3) sing the song together as a summary exercise at the end of class (An & Tillman, 2014). Such a design may set up an environment and facilitate the memorization of a mathematics algorithm or formula, but it fails to present any authentic music connections for students to actively analyze or synthesize mathematical knowledge.

In contrast, a number of mathematics and science education researchers (e.g., Crowther, Davis, Jenkins, & Breckler, 2015; Crowther, McFadden, Fleming, & Davis, 2016; Lesser, 2014; Lesser, 2015) have attempted to go beyond using songs as mere breaks to revive attention or build community during mathematics lessons. Informed by principles of the psychology of learning and some emerging scholarship on the use of songs in the STEM classroom, songs with lyrics based on mathematical concepts, terms, and formulas have been offered as supports for students seeking to understand several topics in mathematics. For example, Lesser (2014, 2015) examined mathematics-themed songs as tools to motivate underrepresented students learning middle school, high school, and college level mathematics. According to his framework, an ideal mathematics-themed song would have six traits that collectively facilitate mathematics learning: "(1) aiding recall (of procedures, properties, definitions, digits of pi, etc.), (2) introducing concepts or terms, (3) reinforcing mathematical thinking processes (e.g., the Pólya's (1945) four-step heuristic for problem solving), (4) connecting to history, (5) connecting to the real world, and (6) humanizing mathematics" (Lesser, 2015, pp. 158–159). Lesser (2015) describes how these six traits are satisfied by his lyrics in "American Pi," which received an award from the National Museum of Mathematics. Its chorus is as follows:

> Find, find the value of π, starts 3.14159
> A good ol' fraction you may hope to define,
> But the decimal never dies, never repeats or dies…. (Lesser, 2015, p. 166)

While a big part of the educational potential of a song is limited by the song itself, another part comes from how the instructor uses it. In other words, one instructor might just let students listen to an online recording of "American Pie," while another instructor might have students sing along and then actively analyze and unpack all of the mathematical references in the lyrics, making connections to their curriculum (e.g., the 16-question sequence of Appendix 2 in Lesser (2015)). A teacher doing the latter would be using the lyrics as a central vehicle to familiarize, contextualize, and conceptualize the meaning of π in multiple memorable ways. A website created by Lesser (http://www.math.utep.edu/Faculty/lesser/Mathemusician.html) provides songs for students to read or listen to as well as links to key articles and websites on the intersection of mathematics and music/songs.

Educators may consider having students consolidate and help recall their knowledge by writing their own songs or lyrics. Lyric writing is a transdisciplinary activity. As Davis (1985, p. xi) noted: "The best lyricists, whether they're aware of it or not, are using elements of phonetics, linguistics, grammar, semantics, metrics, rhyme, rhythm, poetics, phonology, communications, sociology—and even the psychology of verbal behavior." Incorporating mathematical concepts further enriches the transdisciplinary experience.

Music Composition and Playing: Awareness of Mathematical Patterns

Music composition activities offer students opportunities to compose, decompose, and recompose music. These activities can allow students to (1) explore and analyze algebraic patterns and proportional relationships, (2) make geometric transformations and use statistical methods to analyze data, (3) attempt to find multiple solutions during problem solving, and (4) design and conduct experiments that explore probabilities (e.g., explore permutation and the combination of chords and melody development processes) in self-composed or professional music works. As an example of a unique musical notation system used to facilitate novice students in composing and playing their own music, a color-based graphical notation (An & Capraro, 2011; An, Ma, & Capraro, 2011) will be presented that signifies music by using colors, shapes, numbers, and letters to represent the music notes. For example, the colors red, orange, yellow, green, turquoise, blue, and purple were used to represent the musical notes C(Do), D(Re), E(Mi), F(Fa), G(Sol), A(La), and B(Ti) (see Fig. 2.2). Based on this graphical notation system, elementary students composed music by placing a group of color cards on their desks and playing color-matched instruments, such as handbells and boomwhackers (a set of plastic tubes with the same diameter but different lengths). These activities enabled the students to examine mathematical patterns through both visual and auditory approaches while creating solutions to complex music-mathematics challenges. Two sample music composition activities are presented in the following paragraphs.

Composing for Pre-algebra Preparation This activity was designed for upper elementary students as practices for algebra readiness. Students used variables to create their own music. Each music note (a colored square of paper) was assigned a

Fig. 2.2 Color-based notation system and instruments from composition activities

Fig. 2.3 One of the possible music compositions and computation solutions

numerical value (see sample of values in Fig. 2.3), and participants composed a piece of music based on those values. For example, Do has a value of 1, Re has a value of 2, and Mi has a value of 3, so composing a music with five notes (Do, Do, Re, Re, Mi) would have a total value of 9. In one of the activities, participants were asked to compose a piece of music with 24 musical notes. In this music composition work, the sum of the musical value should equal 100 when adding the value of each musical note together (see sample mathematics arrangement and sample music composition in Fig. 2.3). After the participants finished their composition, they played their music using handbells. Unlike traditional drill questions, during which students give answers to rote questions such as 43 + 57 = __, this activity required students to use algebraic thinking while setting variables up as different arrangements in order to obtain a sum of 100. Specifically, students needed to construct an equation with seven variables, such as 1a + 2b + 3c + 4d + 5e + 6f + 7 g = 100 (a, b, c, d, e, f, and g represent values for Do, Re, Mi, Fa, Sol, La, and Ti, respectively) and then figure out the number for each letter to balance this equation. Each student created a different arrangement of colors while solving this problem and then played the different melody that they had composed as a celebration of their success at finding a valid solution. Students who finished fast had an opportunity to play their compositions and rearrange cards to generate a more "pleasing" melody. To vary the activities, teachers sometimes added more restrictions to the computational process or assigned altered values to each color in the activities, such as the use of only 16 musical notes to compose music with a total value of 120.

Total value of the music: (1 × 4) + (2 × 1) + (3 × 3) + (4 × 4) + (5 × 7) + (6 × 1) + (7 × 4) = 100.

Composing with Ordered Pairs This activity was proposed and taught by one of the participating teachers in our previous study (An et al., 2016) for a group of fifth- and sixth-grade students. The designed lessons illustrated how upper elementary students (fourth and fifth grades) could perceive ordered pairs through visual representation during music composition and auditory representation during music playing by composing music within a Cartesian coordinate system in which the *x*-axis and the *y*-axis represent two simultaneous melodies. In this activity, Cartesian coordinates were used for students to represent harmonic intervals (i.e., a pair of notes with the same or different sounds) (see Fig. 2.4).

Several recent studies (An & Tillman, 2014; An, Tillman, & Paez, 2015) have indicated that many inquiry-based strategies for teaching mathematics can be implemented within music composition activities. For example, when teaching numbers and operations, teachers can help students (1) conceptualize a base-eight numeration system through music scales, (2) explore rational numbers through an analysis

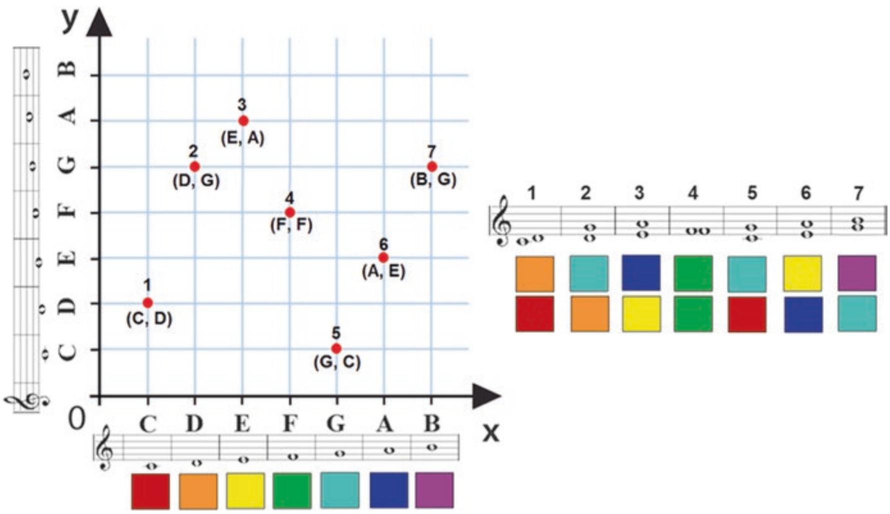


Fig. 2.4 A sample composition displaying harmonic intervals within a coordinate system

of existing or self-created music works, and (3) understand operation rules through the demonstration of chord variations. When teaching algebra, teachers can help students (1) identify ratios and proportions through an analysis of musical works, (2) find unknowns and missing values in music works, and (3) explore functions, sequence, and factors through music composition. When teaching geometry and measurement, teachers can help students (1) compose music through geometric transformations, including reflection and rotation, and (2) explore concepts about time through music composing and playing. When teaching probability and data analysis, teachers can help students (1) collect and analyze data based on music works, (2) develop statistical graphs based on music works, (3) conduct an analysis of events within musical compositions, and (4) explore combinations and permutations through chord and melody composition.

Musical Instrument Design and Construction: Mathematics-Embedded Tasks

By designing musical instruments on paper and then constructing the instrument with different materials, students can learn to understand principles of scientific inquiry and investigation as they formulate hypotheses about how changing the properties of an instrument will affect its sound and then test the hypotheses. Students have been offered opportunities to explore one-, two-, and three-dimensional geometric concepts and relationships within different types of musical instruments (e.g., idiophones, membranophones, chordophones, and aerophones) and explored acoustical physics to understand how the patterns of shapes, dimensions, and materials affect instrument sounds and tones. Specifically, empirical studies (An et al., 2013; An, Tillman, Boren, & Wang, 2014; An, Tillman, & Paez, 2015) demonstrated that musical instrument designing activities have provided students with opportunities to (1) use geometry and measurement concepts to construct different types of instruments; (2) fabricate musical instruments by using 3D printers with a variety of plastic, metal, and hybrid materials; (3) apply knowledge of sound production for basic acoustic instrument types to develop combinations of vibrating strings, pipes, bells, membranes, and reeds that allow the manipulation of variables (e.g., length, size, volume, shape, material, and tension); (4) recognize the iterative process by which a set of simple musical instruments were designed to produce a palette of music "colors" (i.e., color-coordinated musical notes); (5) determine the impact of variable manipulation on the sound properties of pitch, tone timbre, loudness, and resonance time; and (6) test how the combinations of sound waves with patterns of regular or irregular pitch intervals can cause different feelings or emotions.

Algebra in Musical Instrument Design Algebra is widely used during the musical instrument-making process, an example being that musical scales were developed based on proportional relationships. During the instrument-making process, string instruments such as guitars required that instrument designers calculate the position of frets on the finger board; likewise, wind instruments such as saxophones required designers to calculate the positions of finger keys, and percussive instruments such as glockenspiels required designers to calculate the size of component pieces. One of the many activities wherein students can apply geometric sequencing is to design a glockenspiel by cutting and pasting tape or paper strips (see Fig. 2.5).

Using Geometric Sequences to Create a Glockenspiel As a percussion instrument that was constructed with 24 tuned pieces of steel bars, the construction of a glockenspiel requires making a series of rectangles with the same width, although of different lengths, having a common ratio of 0.94. Teachers have directed students to design a glockenspiel by cutting and pasting paper strips based on this geometric progression. For example, An and Capraro (2011) introduced the following instructional steps to create a glockenspiel for fifth graders. Students can cut paper strips

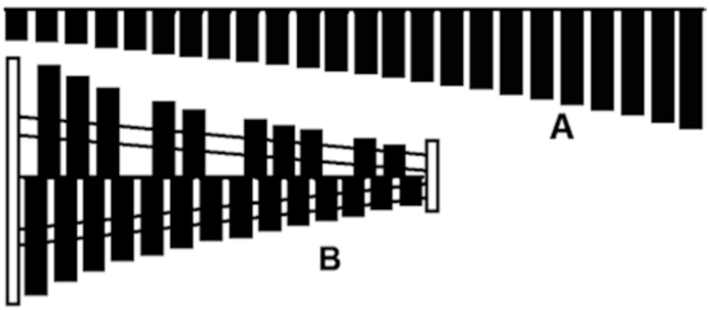


Fig. 2.5 The process of designing a glockenspiel through geometric progression

by using a geometric sequence to compute the accurate length of each tape or paper piece, with the common ratio of approximately 0.94. Based on the paper glockenspiel that their students constructed, teachers can provide additional open-ended mathematics questions. For example, students can examine the total area of the glockenspiel pieces that they used in designing their instrument, and they also can explore the geometric sequence formula describing the length of each piece.

Linear Equations to Investigate Trumpet Tube Length In an activity investigating the tube length changes in a trumpet, students needed to set up equations in order to figure out the volume of the vibrating air column inside the trumpet when a player is changing the pitch. As a wind instrument, the sound from a trumpet is produced by the players' lip vibrations as well as the follow-up pressing of the piston valve, changing the length of the tube within the trumpet. A typical trumpet has three valves, and the length of the tube can be increased when players are pressing one or more of these valves with different combinations. Specifically, the instrument's pitch will be lowered by (1) a major second interval when the first valve is pressed (9/8 longer than the original tube), (2) a minor second interval when the second valve is pressed (16/15 longer than the original tube), or (3) a minor third interval when the third valve is pressed (6/5 longer than the original tube) (see Fig. 2.6). Students set up equations based on this given information to calculate the length of tubes in different conditions when a specific valve was pressed. As An and Capraro (2011) proposed in their lesson designed for sixth-grade students:

Let's suppose the tube length when no valve was pressed is 100 cm. For example, the press of the first valve will lower the instrument's pitch by a major second interval. Let's represent the increased length of tube as x, yielding the following equation:

$100 + x = 100 \times (9/8)$

$x = 12.5$ (cm)

So, the press of the first valve will increase the length of the tube by 12.5 cm.

Use the same method to create equations for the second valve (lowering a minor second interval), and the third valve (lowing a minor third interval). What are the equations for the second and the third valve? Discuss your equations with your classmates. (p. 67)

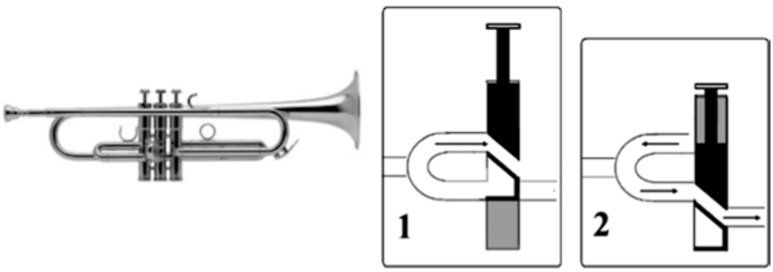


Fig. 2.6 Illustration of the changing tube length in a trumpet

Geometry in Musical Instrument Design Musical instrument design activities have provided links for associating geometry with music; many types of shapes have been used in the history of musical instrument design, and this process often involved combining different simple shapes to create irregular curves. In addition to investigating properties of shapes as well as their unique properties, by designing musical instruments, students have additional chances to apply geometric transformations and improve visualization skills.

Two guitar-themed activities (one for lower elementary grades and another for upper elementary grades) serve to illustrate how geometry has been contextualized in music instrument design. In the Flying V guitar design activity (see Fig. 2.7), first-grade students played with triangles by making transformations such as translating, rotating, and resizing; they also worked with geometric concepts, such as obtuse triangles, isosceles triangles, and congruence. Specifically, students were directed to create two congruent triangles and then rotate one of the triangles and put the two sides (legs) together to create a V-shape figure as the guitar body. The same V shape was then resized at 1/3 scale to make the guitar head the smaller V shape, which was rotated for placement at the other end of the guitar neck opposite the guitar body. In the other activity, circles were employed as a geometrical shape for students to use as the basis for creating the outline of a classical guitar (see Fig. 2.8). An and Capraro (2011) presented this activity with the following instructional steps for fifth graders: (1) make four congruent circles tangent to each other and outline the edges of the two middle circles and the spaces; (2) find the symmetrical line and cut the line off; (3) regroup the two pieces by leaving an uneven, nonparallel space; and (4) redraw the outline and then all the sound holes with a smaller circle. In this classical guitar activity, students investigated geometric concepts such as tangent circles, parallelism, and symmetry. In both guitar design activities, students explored the area and perimeter in the figures that they designed, and students were ready to conduct additional measurements based on real instruments.

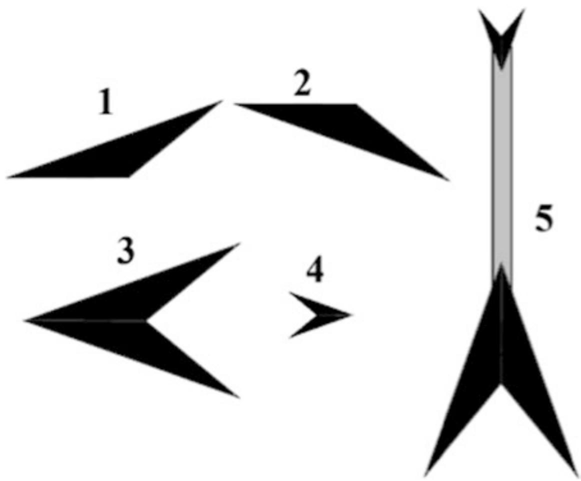


Fig. 2.7 The basic steps for designing a Flying V guitar

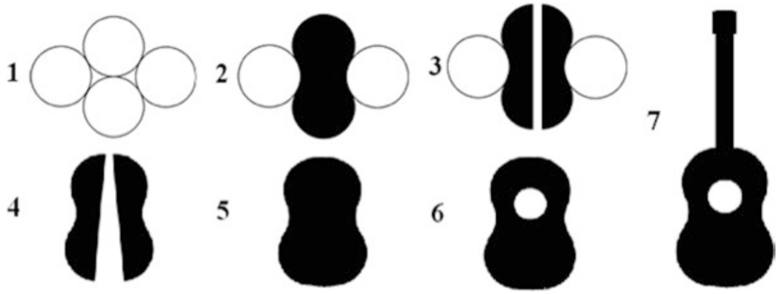


Fig. 2.8 The basic steps for designing a classical guitar

Concluding Remarks

This chapter has provided a review and summary of research studies and activities identifying music-themed mathematics as a valid and worthwhile transdisciplinary pedagogy, which may, when well-implemented, promote both mathematics teaching and learning. However, music is not a panacea for the ailments that plague direct mathematics instruction. Many of the existing resources and popular strategies for using music in mathematics teaching are primarily entertainment oriented, simply providing a cover story for mathematics word problems or playing background music instead of connecting with the mathematics content. Teachers can help reveal the inherent music-themed pedagogy of a math class. They can focus on musical resources that can truly assist in supporting instruction, which may develop students' understanding of the mathematical concepts with connections to music.

Teaching mathematics through music composition and musical instrument design is an application of constructivist learning because teachers need to direct students to engage in complex tasks and then facilitate students' learning by transforming difficult tasks into accessible, manageable tasks within students' zone of proximal development (Vygotsky, 1978). Only when aspects of student-centered pedagogy are thoroughly implemented, such as proposing open-ended tasks for students to provide diverse answers or facilitating group discussion for students to exchange and evaluate their ideas, can students learn mathematics effectively (Schoenfeld, 2004). In our previous studies analyzing more than 200 teachers' instructional designs (An & Tillman, 2014; An, Tillman, & Paez, 2015) and more than implementations of 80 lessons to students (An et al., 2013; An & Tillman, 2015; An et al., 2016), we identified that effective mathematics learning only happened when students were cognitively engaged in participating with the mathematics tasks by manipulating objects, performing activities, and applying the skills in generalized mathematical structures within arithmetic situations. Based on our collective research findings, the common feature among effective mathematics-music-integrated lessons is a mathematical process (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000) orientation in which teachers allowed students to (a) explore algebraic patterns and geometric transformations as composing methods in planning rhythm, investigating intervals, and transferring chords; (b) apply statistical knowledge such as measurement and data analysis as mathematical tools for supporting music analysis and creation processes; and (c) represent mathematical ideas through multiple representations, including singing, playing, composing, decomposing, and recomposing music works.

Music and mathematics are two intelligence domains of recognized importance in human learning, and using music to enhance students' enjoyment and understanding of mathematics has been shown to help learners develop improved logical/mathematical intelligence (Gardner, 1993). There is much potential in the integration of mathematics education and the arts; however, this potential has yet to be fully realized. Unfortunately, the pressure exerted by high-stakes standardized assessment has forced many creative teachers to involuntarily marginalize and ignore the arts, especially when teaching mathematics. Quantifiable standardization in education threatens to seriously harm the fundamental principles of liberal pedagogy and the ongoing quest for nurturing the next generation of innovative young minds (Pinar, 2004). As Slattery (2006) argued, curriculum should be a "kaleidoscope" that opens the eyes, with the ultimate goal of teaching students how to generate critical and original thought. Without the arts, many colors and patterns are eliminated from the kaleidoscope's viewscape, and the "complicated conversation" between students and teachers becomes limited. Teaching mathematics through music offers an opportunity to restore the kaleidoscopic nature of the curriculum.

References

Amiot, E., & Sethares, W. A. (2011). An algebra for periodic rhythms and scales. *Journal of Mathematics and Music, 5*(3), 149–169.

An, S. A., & Capraro, M. M. (2011). *Music-math integrated activities for elementary and middle school students*. Irvine, CA: Education for All.

An, S. A., Capraro, M. M., & Tillman, D. (2013). Elementary teachers integrate music activities into regular mathematics lessons: Effects on students' mathematical abilities. *Journal for Learning through the Arts, 9*(1), 1–20.

An, S. A., Ma, T., & Capraro, M. M. (2011). Preservice teachers' beliefs and attitude about teaching and learning mathematics through music: An intervention study. *School Science and Mathematics, 111*(5), 235–247.

An, S. A., & Tillman, D. (2014). Elementary teachers' design of arts based teaching: Investigating the possibility of developing mathematics-music integrated curriculum. *Journal of Curriculum Theorizing, 30*(2), 20–38.

An, S. A., & Tillman, D. (2015). Music activities as a meaningful context for teaching elementary students mathematics: A quasi-experiment time series design with random assigned control group. *European Journal of Science and Mathematics Education, 3*(1), 45–60.

An, S. A., Tillman, D., Boren, R., & Wang, J. (2014). Fostering elementary students' mathematics disposition through music-mathematics integrated lessons. *International Journal for Mathematics Teaching and Learning, 15*(3), 1–18.

An, S. A., Tillman, D., & Paez, C. (2015). Music-themed mathematics education as a strategy for improving elementary preservice teachers' mathematics pedagogy and teaching self-efficacy. *Journal of Mathematics Education at Teachers College, 6*(1), 9–24.

An, S. A., Tillman, D., Shaheen, A., & Boren, R. (2014). Preservice teachers' perception about teaching mathematics through music. *Journal of Interdisciplinary Teaching and Learning, 4*(3), 150–171.

An, S. A., Zhang, M., Flores, M., Chapman, J. R., Tillman, D. A., & Serna, L. (2015). Music activities as an impetus for Hispanic elementary students' mathematical disposition. *Journal of Mathematics Education, 8*(2), 39–55.

An, S. A., Zhang, M., Tillman, D., Lesser, L., Siemssen, A., & Tinajero, J. (2016). Learning to teach music-themed mathematics: An examination of preservice teachers' beliefs about developing and implementing interdisciplinary mathematics pedagogy. *Mathematics Teacher Education and Development, 18*(1), 20–36.

Ashcraft, M. H. (2002). Math anxiety: Personal, educational, and cognitive consequences. *Current Directions in Psychological Science, 11*(5), 181–185.

Beer, M. (1998). *How do mathematics and music relate to each other?* Brisbane, Queensland, Australia: East Coast College of English.

Bursal, M., & Paznokas, L. (2006). Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science. *School Science and Mathematics, 106*(4), 173–180.

Chahine, I., & Montiel, M. (2015). Teaching modeling in algebra and geometry using musical rhythms: Teachers' perceptions on effectiveness. *Journal of Mathematics Education, 8*(2), 126–138.

Cho, G. (2003). *The discovery of musical equal temperament in China and Europe in the sixteenth century*. Lewiston, NY: Edwin Mellen Press.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved Oct. 22, 2016, from http://www.corestandards.org/Math/

Conklin, D. (2010). Distinctive patterns in the first movement of Brahms' string quartet in C minor. *Journal of Mathematics and Music, 4*(2), 85–92.

Cornett, C. (2007). *Creating meaning through literature and the arts*. Upper Saddle River, NJ: Merrill.

Costa-Giomi, E. (1999). The effects of three years of piano instruction on children's cognitive development. *Journal of Research in Music Education, 47*(3), 198–212.

Cox, H. A., & Stephens, L. J. (2006). The effect of music participation on mathematical achievement and overall academic achievement of high school students. *International Journal of Mathematical Education in Science and Technology, 37*(7), 757–763.

Crowther, G. (2012). Using science songs to enhance learning: An interdisciplinary approach. *CBE Life Sciences Education, 11*(1), 26–30.

Crowther, G. J., Davis, K., Jenkins, L. D., & Breckler, J. L. (2015). Integration of math jingles into physiology courses. *Journal of Mathematics Education, 8*(2), 56–73.

Crowther, G. J., McFadden, T., Fleming, J. S., & Davis, K. (2016). Leveraging the power of music to improve science education. *International Journal of Science Education, 38*(1), 73–95.

Davis, S. (1985). *The craft of lyric writing*. Cincinnati, OH: Writer's Digest Books.

Ehrlich, C. (1990). *The piano: A history*. Oxford: Oxford University Press.

Furner, J., & Berman, B. (2005). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high-tech globally competitive world. *Dimensions in Mathematics, 18*(1), 28–31.

Gamwell, P. (2005). Intermediate students' experiences with an arts-based unit: An action research. *Canadian Journal of Education, 28*(3), 359–383.

Gardner, H. (1993). *Multiple intelligences: The theory in practice*. New York: Basic Books.

Geist, E. (2010). The anti-anxiety curriculum: Combating math anxiety in the classroom. *Journal of Instructional Psychology, 37*(1), 24–31.

Gelineau, R. P. (2004). *Integrating arts across the curriculum*. Belmont, CA: Thompson Wadsworth.

Gordon, S. (1996). *A history of keyboard literature: Music for the piano and its forerunners*. New York: Schirmer Books.

Gouzouasis, P., Guhn, M., & Kishor, N. (2007). The predictive relationship between achievement and participation in music and achievement in core grade 12 academic subjects. *Music Education Research, 9*(1), 81–92.

Greene, M. (2001). *Variations on a blue guitar: The Lincoln lectures in aesthetic education*. New York: Teachers College Press.
Haley, J. A. (2001). The relationship between instrumental music instruction and academic achievement in fourth grade students. (Doctoral dissertation, Pace University, New York). Dissertation Abstracts International, 62 (09), 2969A.
Hui, K. (2006). Mozart effect in preschool children? *Early Child Development and Care, 176*(3–4), 411–419.
Ivanov, V. K., & Geake, J. G. (2003). The Mozart effect and primary school children. *Psychology of Music, 31*(4), 405–413.
Kalinec-Craig, C. A. (2015). Uncovering the mathematical challenges and connections when using mariachi music as a representation for teaching equivalent fractions. *Journal of Mathematics Education, 8*(2), 3–20.
Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up*. Washington, DC: National Academy Press.
Krantz, R., & Douthett, J. (2011). Algorithmic and computational approaches to pure-tone approximations of equal-tempered musical scales. *Journal of Mathematics and Music, 5*(3), 171–194.
Lee, Y. Y., & Shen, S. Y. (1999). *Chinese musical instruments*. Chicago: Chinese Music Society of North America.
Lesser, L. M. (2014). Mathematical lyrics: Noteworthy endeavours in education. *Journal of Mathematics and the Arts, 8*(1–2), 46–53.
Lesser, L. M. (2015). "American Pi": The story of a song about pi. *Journal of Mathematics Education, 8*(2), 158–168.
Loy, R. (2006). *Musimathics: The mathematical foundations of music*. Cambridge, MA: MIT Press.
Nantais, K. M., & Schellenberg, E. G. (1999). The Mozart effect: An artifact of preference. *Psychological Science, 10*(4), 370–373.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Pietschnig, J., Voracek, M., & Formann, A. K. (2010). Mozart effect–Shmozart effect: A meta-analysis. *Intelligence, 38*(3), 314–323.
Pinar, W. F. (2004). *What is curriculum theory?* Mahwah, NJ: Lawrence Erlbaum Associates.
Pinnock, G. (2015). Using live reggae instrumental acoustic music to influence students' mathematics test scores. *Journal of Mathematics Education, 8*(2), 115–125.
Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.
Rameau, P., & Louime, C. (2007). Mathematics phobia: Are the mathematical sciences a pothole in the road of life? *Current Science (Bangalore), 93*(11), 1481–1482.
Rauscher, F. H., Shaw, G. L., & Ky, K. N. (1993). Music and spatial task performance. *Nature, 365*(6447), 611.
Rauscher, F. H., Shaw, G. L., & Ky, K. N. (1995). Listening to Mozart enhances spatial-temporal reasoning: Towards a neurophysiological basis. *Neuroscience Letters, 185*(1), 44–47.
Rauscher, F. H., Shaw, G. L., Levine, L. J., Wright, E. L., Dennis, W. R., & Newcomb, R. L. (1997). Music training causes long-term enhancement of preschool children's spatial-temporal reasoning. *Neurological Research, 19*(1), 2–8.
Rauscher, F. H., & Zupan, M. A. (2000). Classroom keyboard instruction improves kindergarten children's spatial-temporal performance: A field experiment. *Early Child Research Quarterly, 15*(2), 215–228.
Robertson, W., & Lesser, L. M. (2013). Scientific skateboarding and mathematical music: Edutainment that actively engages middle school students. *European Journal of Science and Mathematics Education, 1*(2), 60–68.
Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.
Sethares, W. A., & Budney, R. (2014). Topology of musical data. *Journal of Mathematics and Music, 8*(1), 73–92.
Slattery, P. (2006). *Curriculum development in the postmodern era* (2nd ed.). New York: Routledge.

Spelke, E. (2008). Effects of music instruction on developing cognitive systems at the foundations of mathematics and science. In C. Asbury & B. Rich (Eds.), *Learning, arts, and the brain* (pp. 17–49). New York/Washington, DC: The Dana Foundation.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education, 4*(1), 117–143.

Sweller, J. (2016). Cognitive load theory, evolutionary educational psychology, and instructional design. In *Evolutionary perspectives on child development and education* (pp. 291–306). New York, NY: Springer.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston: Allyn & Bacon.

Vinson, B. M. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal, 29*(2), 89–94.

von Falkenhausen, L. (1993). *Suspended music: Chime-bells in the culture of Bronze Age China*. Berkeley, CA: University of California.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Whitehead, B. J. (2001). The effect of music-intensive intervention on mathematics scores of middle and high school students (Doctoral dissertation, Capella University, Minneapolis, Minnesota).

Chapter 3
Using Children's Literature to Enhance Math Instruction in K-8 Classrooms

Melissa Luedtke and Karen Sorvaag

Abstract The integration of math and literacy enriches student experience and increases the chances of successful math learning for students. Elementary-age students typically spend hours each day building reading skills in literacy classes. Literacy skills that are emphasized in reading programs, including effective listening, comprehension, predicting, and questioning, are a logical fit with mathematics and help teachers and students dispel the myth that mathematics is a separate subject unrelated to other disciplines. Using children's books to connect literacy skills to mathematics, a subject that often produces great anxiety in students, is a reasonable and positive approach. Educational structures such as standardized testing and instruction that utilizes repetitive drill and practice as evidence of standards mastery contributes to this anxiety. Integration of math and literacy frees students to use knowledge they already possess in new applications. The use of children's literature, therefore, can free students from boundaries that limit their confidence, motivating them to see that success in mathematics is possible. Relevant children's books also form connections between math and its application in the world. This helps students to understand that math goes beyond what they see in a math textbook. How can the use of children's literature in math instruction help break down barriers to math understanding and increase student success? Foundational studies, empirical research, and specific classroom examples are combined in this chapter to answer this question.

Keywords Math and children's books • Children's literature in math • Story and math • Literacy and math • Math anxiety • Children's book and motivation in math

M. Luedtke (✉) • K. Sorvaag
Saint Mary's University of Minnesota, 700 Terrace Heights, #23, Winona, MN 55987, USA
e-mail: mluedtke@smumn.edu; ksorvaag@smumn.edu

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_3

Introduction

With the continued focus on academic achievement in math and science in schools, partially aimed at preparing learners to work in STEM fields, teachers must consider a variety of alternative instructional approaches to meet the various needs of learners. Children's literature can be a valuable tool in helping students understand math by placing math concepts in contextual settings. The alignment of meaning and context is essential when learning any new content, and most educators agree that learning should be based on meaningful understanding of concepts placed in a context of reality. In fact, the ideas of respected educational theorists in history set the foundation for this alignment (Barton & Heidema, 2002).

Vygotsky taught educators that cognitive, language, and social development require interactive dialogue where learners share experiences (Woolfolk, 2013). These experiences are often sensory and include concrete learning activities that are developmentally appropriate, as supported by Piaget's work in stages of development (Woolfolk, 2013). Bruner's conclusions that learning should be active and engaging rather than passive (Barton & Heidema, 2002; Bruner, 1996) adds to this solid foundation that supports social, developmental, and active participation in the teaching and learning process. This social, developmental, and active approach helps students make meaning, and the use of narrative can be a tool that plays a part in that process (O'Neill, Pearson, & Pick, 2004). While instructional approaches that align with these ideas have been routinely applied in teaching literacy skills such as reading, speaking, and writing, they can also be effective in the instruction of math, where learners must build new knowledge based on previous understandings in contextual settings (Columba, Kim, & Moe, 2005).

Before addressing how children's literature can effectively enhance learning in a math classroom, it is important to identify how social, developmental, and active learning do or do not occur in literacy and math classrooms. Literacy and language are, historically, disciplines where social and active learning occur in educational settings. Students read and discuss, write and share, collaborate to make predictions, and solve problems. They dialogue, debate, observe, and hypothesize (Columba et al., 2005; Schiro, 1997). Yet in math classrooms, such activities often do not occur. Instead, students may passively observe, while the teacher explains content and models practice and then work on problems individually. Some research criticizes such an approach (Burns, 2015; National Council of Teachers of Mathematics [NCTM], 2014), yet active and constructivist learning in math instruction, even at primary and intermediate grade levels where active learning may be present in other disciplines, exists only in pockets and isolated classrooms. It seems logical to consider that the positive benefits of using social, active learning in literary areas could be also applied to the teaching of math.

In elementary schools, students spend significant time engaged with literacy skills that build from prior knowledge. When reading and math meet, past learning in reading can lead to new learning in math, giving students the freedom to use knowledge they already possess in one area to help make sense of new ideas in another. For example, children who learn to use prediction skills to aid in comprehending a story can also use those skills to predict a reasonable answer in a math word problem. The integration of math and literacy, then, enriches student experience and increases the chances of successful math learning (Columba et al., 2005). Because understanding math is a constructive process, using skills already developed through literacy instruction can provide the foundation upon which math learning can be built (Barton & Heidema, 2002).

Children's literature is one of many resources that can bridge the gap between a passive tradition in math learning and active participation and construction of knowledge for students (Columba et al., 2005), and studies support the idea that the integration of children's literature with math content enhances a constructivist pedagogy (Capraro & Capraro, 2006; Elia, Van den Heuvel-Panhuizen, & Georgiou, 2010). In current practice, primary teachers frequently include children's literature in math instruction, often as an introduction to the lesson, but picture books and other literature can be used as more than merely an interesting opening to a lesson. It is also important to remember that children's literature can be a valuable instructional tool for older, as well as younger, students. The use of children's literature to help students see meaning in math can be strengthened through intentional planning and instruction and can benefit children of all ages. This natural teaming of literature and math begs the question, "How can the use of children's literature in math instruction help break down barriers to math understanding and increase student success?"

The Power of Story

To answer this question, one must first recognize the power that story has for human beings. The use of story can be a powerful learning and reflection tool, and this can be especially true in a classroom setting. Imagine an elementary classroom as students come in shortly after recess and transition to the next learning block. Students are sitting on the floor around the teacher with bright eyes. Many cannot wait for the book the teacher holds to come alive in the classroom. As soon as the teacher lifts the book to introduce the title and show the cover, hands shoot up. Students want to share what they think the book will be about or what it reminds them of in their own lives. Some are so excited to share that they stretch their hand higher and higher. The energy and excitement builds as the new adventure unfolds.

Story has this kind of power. It invites the reader into a world where personal understanding, perspective, and experience matter. It draws on emotions and feelings to capture the reader and brings a new sense of knowing that relates to other stories, experiences, or knowledge. A teacher who understands this kind of power can tap into it to create scenarios like the one described above in a variety of disciplinary settings. What if the start of a math class looked and felt like this? Story can take students there.

According to Zazkis and Liljedahl (2009), storytelling supports memory, increases motivation, encourages engagement, and improves analytic skills.

> One result of the development of language was the discovery that words can be used to evoke images in the minds of their hearers, and that these images can have as powerful emotional effects as reality might, and in some cases even more. (p. 15)

Capturing the attention and imagination of listeners is crucial, and in some situations, images can be used to promote direct math understanding. Connecting students' interests with story is a beginning step, and it can be the first step in motivating them to think about a math problem in a certain way. By piquing students' interests, the material presented may become more accessible. Students may also feel more relaxed because they are comfortable with story as a non-threatening means of delivering ideas. In a math classroom, the use of story also serves as a break in routine, further motivating students to ready their minds for mathematical thinking and learning (Zazkis & Lilijedahl, 2009). As students progress to practice and master a math concept, images in story can be a reflective base to which they can return as needed.

Stories can be "creative, suspenseful, imaginative, meaningful, poetic, humorous, adventurous, vivid and colorful, mysterious, engaging, gripping, relaxing, touching, emotional, inspirational, conversational, easy to understand, and beautifully written" (Columba et al., 2005, p. 2). When well chosen and well presented, literature moves the reader inside the story, making it easier to consider difficult and complex issues. In the process, the reader is allowed to recall positive memories of literary experiences that connect to life. They connect intellectually through familiar memories that reach back to past emotions. Such experiences may include laughter and fun, further relaxing students to be open to learning (Flanagan, 2014). Because story is often a community process, sharing responses to stories can also contribute to a feeling of community and group cohesion, building on the benefits of social and active learning for students.

The use of story is a fundamental way for people to gain control of the world around them. Responding to story is not a conscious and deliberate activity; it is the way in which the mind naturally works (Wells, 1986, as cited in Columba et al., 2005, p. 197). Narratives mirror life. They allow people to see a view of self that is presented in familiar as well as unfamiliar ways. Stories are not merely vessels for delivery of knowledge; they are intentionally chosen to achieve a specific purpose that goes beyond the acquisition of basic information (Columba et al., 2005). They do not present opportunities for finding the one, right answer. Instead, they greet students like an old friend who engages them in conversation. As a result, children's responses to narratives are natural, not forced, and feelings are comfortable, not anxious. It is clear that the use of story is positive for students. But it can be an ignored resource when the content to be taught is not related specifically to literacy. Teachers must understand that the power of story is a natural tool that can be used in many subjects, including developing mathematical understanding and ways of thinking (Columba et al., 2005).

The Math and Literacy Link

To use the power of story effectively in math instruction, teachers must intentionally link math and literacy skills. Literacy skills emphasized in reading programs include effective listening, comprehension, predicting, and questioning (Gear, 2006). These skills are a logical fit with math and help teachers and students break down the myth that math is a separate subject unrelated to other disciplines (Bay-Williams & Martinie, 2004). The lessons learned from literacy instruction that apply to math are many. Students communicate orally and in writing. Teachers expect a variety of responses, even when the topic is the same. Vocabulary instruction is essential. Books provide students with common experiences that allow them to learn at a variety of levels. There can be a blend of whole class, small group, and individual work (Burns, 2015). These skills serve as a bridge as teachers move from reading to math. For example, in math, teachers often ask students to make estimates before problem-solving (predicting). Students write and discuss math ideas that can develop understanding (discourse). A variety of methods for solving one problem is encouraged (critical thinking). Students explain and present solutions to problems (problem-solving). Teachers use math vocabulary and explain thinking aloud and ask students to do the same (academic language). These are only some of the similarities between approaches in literacy and math, and by using children's books to stimulate math thinking and problem-solving, strategies that align between the two content areas can be used appropriately and effectively in both (Burns, 2015).

A key similarity in literacy and math is the importance of vocabulary development. The purpose of developing vocabulary in literacy is to enhance comprehension and communication. This is important when meanings of words in story and math are the same, but even when the mathematical meaning of a word is different and specific to math, intentional vocabulary instruction is important (Burns, 2015). Consider the many words that have meanings specific to math. When first introduced to these words, students will rely on their previous knowledge of the meaning of the word. For this reason, it is important for teachers to recognize the importance of prior knowledge in learning vocabulary and to be intentional about teaching the meaning of math words (Burns, 2015). For example, students who hear words such as *difference*, *product*, *factor*, *power*, *face*, *remainder*, *dividend*, *times*, *compass*, *expression*, *positive*, *negative*, *improper*, *rational*, *irrational*, and *real* may have existing contexts for these words. A student who views a *product* as something bought in a store may struggle to relate that word to the answer in a multiplication problem. It is the task of the teacher to convey that the purpose in learning the language of math is to be able to communicate about mathematical ideas, just like words in reading are used to communicate ideas about life.

The connections between literacy and math in vocabulary development alone are worth making intentional instructional decisions (Burns, 2015). In literacy, students learn to break words apart into roots, prefixes, and suffixes. In math, the same should be done. The word *circumference* begins with *circ*. Asking students to name other words that begin in the same way builds a bridge. Students reflect on the meaning of *circle, circulation*, *circuit*, *circumnavigation*, *circumstances*, and *circumspect.* When students learn that *circ* relates to something round, they have built connections among words that refer to math, social studies, science, and life. Children's books can present the opportunity that leads to this learning. By asking students why the main character's name in *Sir Cumference and the Dragon of Pi* (Neuschwander, 1999) is "Sir Cumference," the door is opened to connecting the character with the feat he must accomplish to break the spell that changed his father to a dragon. The meaning of *circumference* as the distance around a circle is part of the story, is visual, and is the solution to the problem. Add in a little humor as to why *circumference* is "Sir Cumference" and *pi* is "pie" as the family has pie to celebrate, and students have two ways, story and math, to lock this content into memory and enjoy the process of learning.

Vocabulary instruction also applies in connecting children's literature to the study of geometric shapes. A study completed by Skoumpourdi and Mpakopoulo (2011) showed that there are misconceptions as children learn about the attributes of two- and three-dimensional objects. When tangrams and geoboards are used by teachers to represent plane figures, students may think that the names of plane figures, such as *square* or *rectangle*, can be used to name solid shapes such as a cube or rectangular prism. They may understand that an object is a ball, a head, a pizza, or a table but are unable to name the three-dimensional mathematical term that aligns with each. In the study, *The Prints*, a book developed by the researchers that includes both story and math, was used to help alleviate this misconception (Skoumpourdi & Mpakopoulou, 2011). In this book, tangible objects such as a piece of wood or a tent are used to make prints on the ground that create a circular, triangular, or rectangular shape. The conclusions of the study showed that using the picture book for instruction shifted all students' understanding so that each could correctly identify the print that a tangible object would make. Students were better able to understand that plane shapes are the result of a print made by a solid object but are not the object itself. By engaging with the book, students helped the main character make discoveries while also learning math ideas.

In another study involving geometry, one group of middle school students was exposed to literature in addition to the textbook and regular instruction, while another group of students was given unstructured seatwork as the added factor in place of the literature addition (Capraro & Capraro, 2006). In this study, the children's book, *Sir Cumference and the Dragon of Pi* (Neuschwander, 1999), was used as a follow-up to instruction with one group of students. Students were asked to connect characters and events in the story to the content they had learned in the classroom. They developed understanding through investigation and discourse of geometry vocabulary as it related to the story. This occurred after, and in addition to, a hands-on experience and memorization experienced by both groups (Capraro & Capraro, 2006). Conclusions showed that students exposed to both the instruction and connection to literature performed significantly better on the posttest. Many students also reported that the book was a favorite part of the lesson.

The overlap between math vocabulary and everyday English vocabulary stresses the relationship between words and disciplines (Barton & Heidema, 2002). Direct and intentional instruction about words that are integral to content has a dramatic effect on learning. As follow-up activities to reading children's literature, students can be asked to list, group, and label words, especially in a second reading of a story that includes vocabulary. For example, students may do a geometry word and characteristic sort after reading *The Greedy Triangle* (Burns, 2008a). Students may complete a *question/answer/relationship* activity that asks them to make conceptual connections that lead to critical thinking. They may express math learning through the retelling of a story using a RAFT activity (role, audience, format, topic) where they consider their *role* as the writer, the *audience*, the *format*, and the *topic*. This strategy serves as a post-reading reflective exercise. Many other strategies designed as literacy skills that aid in comprehension, such as *Write to Learn* (using short, informal writing responses such journals or logs that help students think through key concepts or ideas) are also effective in making meaning in math (Barton & Heidema, 2002).

Considering the link between math and literacy is important for the reasons previously mentioned, but it also becomes important when considering the confidence of a teacher. Using literacy resources to teach ways of learning and knowing is often comfortable for elementary teachers who have experience teaching reading concepts and skills. Teachers' eyes light up when they find the perfect book to use as the basis for a literacy lesson, but many do not feel the same when teaching math (Burns, 2015). Along with other benefits, the pairing of a subject in which a teacher has confidence with one where there may be more doubt is a reasonable way to allow for teacher improvement in math instruction.

Seeing and Experiencing Math Content

To set the stage for the natural union of story and math, it is important to form a positive vision of what effective math instruction looks like. That is, how do teachers want students to experience math content? Traditional approaches to teaching math often involve symbols that can have no meaning to students when used in isolated mathematical problems that do not connect to student lives (Barton & Heidema, 2002).

To make meaning from math, students must grasp the big picture of the mathematical concept or of the problem being presented. They must see the problem as a whole, a main idea, as they would see a plot in a story (Barton & Heidema, 2002). To provide this context, students can be offered opportunities to process what they are learning in a variety of ways, as well as a chance to reflect on what they have learned so that learning can be extended to new areas. An example would be allowing students to sketch what they hear and see or to view pictures of the situation. Research shows that when using sensory systems such as colorful visuals, sounds, objects, smell, or taste, student attention is more focused (Gregory & Kaufeldt, 2015). The use of a cartoon, a photograph, music, or a video can help students see the meaning in what is being learned. The brain is naturally curious, so providing puzzles, mysteries, or other discrepant events promotes a desire for students to seek answers. The use of picture books to teach math does this by connecting text with illustration.

When deciding how students will experience math, teacher must determine how math concepts, both simple and complex, can be presented to readers of all levels in ways that allow them to make sense of what they are seeing and doing based on individual understanding (Columba et al., 2005; O'Neill, Pearson, & Pick 2004). Teachers must consider purpose, audience, structure, and standards when planning math instruction that is meaningful. It is also important to consider how math might be integrated with other subjects, as recommended by math experts (Columba et al., 2005). Too often students learn a mathematical operation but fail to see where and how it should be used, and when math concepts are taught passively, rather than actively, students may not understand what the math means. Active learning in math is essential. Instruction should be child-centered, based on opportunities that allow children to learn at his or her personal level and to construct knowledge through engagement.

Providing students with opportunities to encounter math ideas in children's literature can accomplish these important characteristics of effective math learning. In some cases, the book chosen focuses on the meaning, the intellectual point, of the math (Columba et al., 2005). The math is at the center of the book, and the story is the avenue for delivery. In these types of books, the math to be learned is explicit. An example of a book that is specifically written to teach math skills is *Spaghetti and Meatballs for All* (Burns, 2008b), a story presenting the problem of arranging tables to seat all the dinner guests. Students learn that a set of tables arranged in different ways will have the same area but different perimeters. Another example is *The Greedy Triangle* (Burns, 2008a), a story whose characters have names and characteristics of geometric shapes. Both books were written by Marilyn Burns, a well-known expert in math education in the United States. The stories present direct mathematical concepts and instruction and include pages titled "For Parents, Teachers, and Other Adults" that provide ideas for how the books can be used to teach the embedded math skills.

In some children's books, the development of the story is at the forefront, and math instruction is secondary, like a passenger going along for the ride. In this case, content may be explicit or implicit in the story. For example, *Sir Cumference and the Dragon of Pi: A Math Adventure* (Neuschwander, 1999) combines humor, a story whose characters' names are based on math vocabulary words, and a spell that must be broken, to teach children that the distance across the middle of a circle through the center fits around the outside of a circle three times and a little bit more. Characters' names are Sir Cumference, Lady Di-ameter, Radius, Geo of Metry, and Sym Metry. Lines such as *It's also the dose, so be clever, or a dragon he will stay... forever* and a celebration with pie when the spell is broken appeal to the imagination of students. Another story, *One Grain of Rice* (Demi, 1997), introduces the doubling function in the tale of a village girl named Rani who outsmarts a cruel Raja by asking for a reward of one grain of rice to be doubled each day for 30 days. How much rice can Rani gain to help feed the people in her village when she is granted this reward by the unknowing Raja? A pull-out visual on the final pages of the book depicts camels carrying the bushels of rice that result, a powerful illustration of the mathematical concept.

Occasionally, the math in a children's book is invisible, with the teacher finding effective ways to make a connection between an excellent piece of literature and math. For example, a teacher might create an investigation that asks students to answer the question, "How big is Hagrid?" as they read *Harry Potter and the Sorcerer's Stone* by J.K. Rowling (Bay-Williams & Martinie, 2004). In the novel, Hagrid is described as twice as tall and five times as wide as a usual person. This investigation asks students to think critically to first identify what a "usual person" looks like. The purpose for using the literature in cases like these is to connect a story that students are already familiar with to math, hopefully motivating them to become engaged with the math content. Another example is *Snowflake Bentley* (Martin, 1998), the story of Willie Bentley and his interest in the world around him. The story tells of his love for all forms of moisture—rain, ice crystals, and snowflakes. Willie finds that snowflakes are "masterpieces of design," an idea that could lead a teacher to discuss weather or geometric shapes. An example of this type of story for younger children is *A Little Bit of Winter* (Stewart, 1998.) The story about the coming of winter as shared by Rabbit and Hedgehog includes quantity words such as *a little bit*, *more*, *less*, *small*, and *big*. Teachers of very young children might use it to teach words that show an amount or comparison words such as *greater than* or *less than.*

Choices made by a teacher to enhance math learning in any of these three ways are supported by research showing that pictures with representational function, rather than purely informational function, resulted in increased math statements during story discussions by participants (Elia et al., 2010). In other words, learning about the math while seeing it portrayed in the story provides students with a big picture of a math concept. Research also supports the idea that it is the teacher's responsibility to help students connect informal math understanding, which can be presented through picture books and resulting discussions, with formal math content (Van den Huevel-Panhuizen, Van den Boogard, & Doig, 2009).

For example, a study conducted with fourth grade students integrating children's literature into the instruction of long division showed an increased understanding of this math operation when children's literature along with other methods that allowed students to see math was used in extended teaching or reteaching (Thomas & Feng, 2015). Researchers evaluated student understanding without the use of children's literature for a 2-week long division unit and then again during a 2-week unit using children's literature. Only instruction and assessment connected to the algorithm were used to lead student learning to the first posttest, while in the second round of assessment, literature, and also manipulatives and graphic organizers related to the children's literature, was added.

Whatever the approach, stories provide contexts for students that allow them to think critically, solve problems, and make connections to the world as they see and know it (Columba et al., 2005). Books can lead students to discover patterns and relationships, to reason, and to confront authentic problems through an inquiry approach. Children's literature provides opportunities for students to listen actively as they see math concepts humanized and as they confront abstract ideas brought to life through story.

When considering children's books that contribute to understanding the meaning of math, teachers must be careful to choose literature that is engaging to the reader, meets the purpose of the lesson, is age appropriate, and includes meaningful and relevant math concepts (Burns, 2015; Columba et al., 2005). Resources must be viewed as high quality from a literary perspective, and content presented must be mathematically sound in that it helps students learn to think and reason mathematically. Books chosen may introduce the connection to math through an investigation, a problem, or an exploration (Burns, 2015). Connections to national standards may be considered, or books may be chosen that align with reading comprehension strategies taught in literacy classes (Columba et al., 2005). When chosen and used effectively, children's literature provides "vehicles for math lessons in unique ways" that help students see the meaning in math and experience math content (Burns, 2015, p. 129).

The Importance of Prior Knowledge

One of the unique ways that children's literature can function in the classroom is by activating prior knowledge. Inadequate prior knowledge activation or prior knowledge that is not organized and accessible to long-term memory can be a major obstacle in math learning (Barton & Heidema, 2002). In math, students must develop organized constructs that help them understand and explain how concepts and procedures are related to one another. How information is organized and integrated in the mind (schema theory) promotes the activation of prior knowledge, an essential piece in meaning construction (Columba et al., 2005). By spiraling back to previous learning, concepts, words, and symbols can be developed and practiced. If students develop this structure related to the content, they will be able to recall and use prior knowledge quickly and effectively. Learning is goal oriented and links to old information, so asking students to pre-activate thought prepares them for new learning (Barton & Heidema, 2002; Gregory & Kaufeldt, 2015).

The use of story as pre-learning can serve this purpose, because understanding narrative is based on past experience. When story is heard, existing knowledge is awakened, and new learning can be connected to that knowledge. If, on the other hand, connections do not occur, questions can be formed that can help bridge the gap between old and new learning as instruction continues (Barton & Heidema, 2002). In this case, math is learned through the interaction between what students already know and what they can learn by reading children's books that are carefully chosen. The role of the teacher in this process is facilitator, one who selects resources and prepares opportunities for learning (Columba et al., 2005). Teachers may choose to supplement the prior knowledge connection through literature by questioning, brainstorming, or previewing, as a story is introduced and read. This connection can also be enhanced through interactive and reflective activities related to the literature being used (Barton & Heidema, 2002). Activities may include questions, sharing responses with peers, or more formal measures such as graphic organizers (Barton & Heidema, 2002; Gregory & Kaufeldt, 2015). Children's literature can also be used after instruction to help students develop a strong understanding of math vocabulary and academic language (Capraro & Capraro, 2006). In these cases, student references to the characters and plot in a story help form the long-term memories that will serve as prior knowledge in the future when math concepts are reviewed, sustained, and evaluated for mastery.

Students learn best when they connect new learning to existing knowledge and skills, and this is true also for teachers (Burns, 2015). By carrying what teachers already know about activating prior knowledge in reading instruction into the planning instruction in math, they, like students, also build learning in a new application. During reading instruction, teachers want students to read fluently, love reading, develop good word attack skills, and comprehend what they read. They hope students will make predictions about what might come next in a story, retell a story in their own words, identify what is important and what is not as important in what they read, and experience shared reading through guided reading, independent reading, and read alouds (Burns, 2015). Prior knowledge is the foundation for all of these literacy skills, for both students and teachers, and they can also be effectively applied to math.

Reading of picture books can stimulate mathematical thinking, and the intentional choice to use literature recognizes that children possess a great deal of informal understanding of math even before math instruction occurs. Children's books can serve as a springboard from this prior math knowledge to more formal levels of mathematical understanding. A study showing this connection used the picture book *The Surprise* (Van Ommen, 2003). The main character in the book is a thick wooly sheep who measures the thickness of his fleece, cuts his wool, and delivers it to another character, a poodle, for spinning. The sheep knits a jumper from the spun wool and gives it to a giraffe as a present. Throughout the story, the sheep graphs the data in this process. Children responding to the story discussed what the sheep is doing as the story progresses. Without any instruction, students understood that the sheep was measuring and keeping track of something and marking the results on a line plot, shown in the pictures in the book. The learning process in this case begins with prior knowledge about math that students already possess. Children had a general notion of what the chart in the story showed and understood that the upward line represented an increase (Van den Huevel-Panhuizen et al., 2009).

While many books used for the purpose of activating prior knowledge are fiction, nonfiction books are also valuable instructional tools. *The History of Counting* (Schmandt-Besserat, 1999) introduces students to 20 years of research completed by an archeologist on the history of numbers and counting. The information integrates math with social studies as it discusses the rise of cities, past cultures and how they adapted, and the modern decimal system. Questions are posed that students will hopefully find intriguing: *What did people do when there were no numbers? How did our current number system evolve from the system used in ancient* *times in the Middle East?* The combination of history and math, combined with the beautiful illustrations of Michael Hays, results in an appealing book that promotes critical thinking. As math and social studies are intertwined in *the History of Counting*, the primary purpose of nonfiction literature can be to intentionally integrate subjects, helping students see that math is not isolated in the world. The book, *MATH-terpieces: The Art of Problem-Solving* (Tang, 2003), can be used to teach addition to younger students and problem-solving to older students, making it an appropriate choice for teaching math skills. But primarily, the book may be used as an introduction to art history. The novel, *Shipwreck at the Bottom of the World: The Extraordinary True Story of Shackleton and the Endurance* (Armstrong, 2000), presents the story of 27 men who set out to become the first team to cross Antarctica in 1921. While the opportunities for math instruction abound (estimation, measurement, mapping, timelines), the focus of the activity might be helping students understand how math is used in exploration and connected to science and geography.

These resources, when read aloud, require children to listen in a different way. Students listen to the facts presented and assimilate them into what they already know about the subject addressed (Bay-Williams & Martinie, 2004). Prior knowledge is, again, key to building new learning on existing knowledge. Authors of nonfiction books are sensitive to this and carefully consider how to present new information to make it accessible to children (Barton & Heidema, 2002). The use of literature to activate prior knowledge, then, can be effective for both primary and intermediate students and can include both fiction and nonfiction genres (Gregory & Kaufeldt, 2015).

Student Motivation

While the use of children's literature is an effective way to activate prior knowledge, it may be even more important when considering student interest and motivation. Traditional math teaching that relies on teacher modeling followed by individual student work is often not motivating to students (NCTM, 2014). Multiple problems practiced in drill fashion may not pique the interest of students or hold their attention. The use of children's literature, however, can eliminate some of the obstacles to motivation and heighten student interest in a variety of ways (Barton & Heidema, 2002).

For example, it is possible that the storyline in a children's book can provide the foundation for an entire day's lesson, focusing student attention on one story, one problem, and finding the solution to that problem. Students move from the often-overwhelming task of completing many problems, a task that can seem to have no purpose in their minds except to complete an assignment, to solving one complex problem that is presented in the context of a story. This can be highly motivating. For example, after reading *Spaghetti and Meatballs for All* (Burns, 2008b), students may draw and calculate the perimeter and area of a set number of tables organized in different configurations to find the arrangement that can seat the most people. As students complete this task, they compare their findings with other groups of students, and patterns that emerge are shared. Through activity and discourse, students discover that no matter how the tables are arranged, the area is the same, but the perimeter, the relevant measure when considering where people will sit to solve the problem in the story, is different. In this type of mathematical instruction, students go through a process of drawing and calculating while considering a real-life situation they already understand, tables and people. Past knowledge is combined with knowledge gained in a socially interactive setting, making it much more likely that students will be motivated to both learn and remember the concept addressed.

Math problems paired with children's literature are also often interdisciplinary, connecting to science, reading, social studies, or the arts. This can increase the chance that the story will appeal to student interests, which may in turn, increase student interest in the math. For example, *The 39 Apartments of Ludwig van Beethoven* (Winter, 2006) illustrates the use of simple machines to move pianos, the importance of neighborhood as community, and the artistic genius of Beethoven. Mathematically, the story focuses on the five legless pianos owned by Beethoven and the task of moving them by posing the question, *How hard is it to move 5 legless pianos 39 times?* As the story unfolds, the reader visualizes the pianos and the pages of Beethoven's great works spread out over the floor of his apartment in the Vienna theater district. But when Beethoven forgets to pay the rent, he has to move. Because his second apartment is in a dangerous part of town, he moves again, and the pianos follow on a series of pulleys. A third apartment with a view of the Danube is abandoned because of neighbors' complaints about noise, and a fourth apartment is in an attic. Through all of the moves, pianos are bought, left behind, and moved on pulleys and slides to make it possible for Beethoven to compose his great musical works for the world. Math, science, and the arts are integrated in telling the story of this great master, capturing a variety of student interests through one resource.

Such integration of disciplines helps teachers reach the whole child and eliminates the separate compartments that can be created by teaching subjects in isolation (Clements & Sarama, 2004). For example, plots in children's books may align math with the study of faith and love in an immigrant family (*The Keeping Quilt* – Polacco, 2001), environmental issues and the conservation of natural resources (*The Great Kapok Tree* – Cherry, 1990), or Bernoulli's principle and the flight of Charles Lindbergh (*Flight: The Journey of Charles Lindbergh* – Burleigh, 1997) (Columba et al., 2005). In these examples of literature, math may be the main focus of the book or math may be in an even partnership with an exceptionally strong piece of literature.

Integration of math with other topics also allows teachers to consider student interests that may be connected to gender, age, reading ability, ethnicity, religion, and experience with literature (Clements & Sarama, 2004; Columba et al., 2005). Teachers can intentionally seek to discover student interests and can have them in mind as choices in literature are made (Gregory & Kaufeldt, 2015; Willis, 2010). The teacher may be aware of books that students have already read and may choose resources that connect to those, or a teacher may reference materials previously or currently used in another discipline and connect them to math. The selection and inclusion of quality literature that aligns with student interest is one way to create a motivating learning environment in a math classroom (Columba et al., 2005).

Using literature to present math content is also motivating in another way, in that through this process, students are allowed to determine responses and find meaning at an individual level (Columba et al., 2005). The triangle formed when the reader, the text, and the context for the content are all considered is important and unique to each student. Because children come from different backgrounds, they have different interests and abilities, and they have different levels of motivation. By carefully choosing appropriate literature, a teacher can consider these characteristics and differentiate for students. Teachers can also consider the aesthetic response to reading that students may have, whether they enjoy reading or being read to or whether reading is a struggle. Considering student interests also helps teachers make math relevant to student lives. Books involving math concepts can be chosen intentionally to make personal connections with the reader, providing opportunities for students to see themselves in the characters (Columba et al., 2005).

While closely related, recognizing student interests and understanding student motivation are two different things. Appealing to student interest is one thing, but considering how to get students to pay attention and engage in rigorous mathematical tasks is another (Gregory & Kaufeldt, 2015). The desire to want to learn is the energy that leads to engagement with both peers and content, which must be sustained for learning to occur. When motivation is high, teachers and students together are able to create a learning environment that is built on habits of care, such as support, encouragement, listening, respect, and the ability to positively negotiate differences. These conditions, however, must be intentionally created and supported by teachers to be maintained. Teachers must understand that students often arrive at school with a well-developed self-image that is perceived by the student as competent, incompetent, or somewhere in between. A student's feelings about the ability to succeed in school, either in general or in a particular subject, are well ingrained. Students who view themselves as incompetent math students are likely to have low motivation in math; therefore, it is crucial that these students see that success is possible for motivation to increase.

Planning instructional frameworks is also important when considering what will motivate students to learn. How a lesson is introduced or the format in which the math content is delivered is key to engaging students (Barton & Heidema, 2002). Literature can serve as the hook that grabs student attention or presents content in ways that vary from traditional formats. If students view themselves as unsuccessful in math, the use of story can build confidence by changing the focus and how they view the content. Children's books can be effective in motivating children to think and reason mathematically. Books can stimulate imagination while also teaching important concepts and skills, thus motivating students to learn math, sometimes without them even being aware this is happening.

Studies support the idea that picture books and children's literature can increase motivation in students. One study showed that picture books elicit mathematical utterances from the reader without prompting (Van den Huevel-Pahnuizen & Van den Boogard, 2008). When books with limited text were used with kindergarten students in this study, students responded with mathematical observations that expressed how many toys were present or used the language *all* or *everyone* or *none* or *nobody* to describe the characters. Other student responses were spatially related and expressed with words such as *here* or *there* or *one out* or *one in.* This study showed that picture books can motivate students to respond with mathematical thinking even without the direct involvement of the teacher through instruction. The book alone provoked responses related to math.

At other times, however, the direct involvement of the teacher is important. In another study involving kindergarten students, children were invited to tell what was happening in a story by viewing illustrations before any text was read (Rathé Torbeyns, Hannula-Sormunen, & Vershchaffel, 2016). While student utterances did involve math, students were unable to see the larger picture of the mathematical concepts when asked to make predictions at quarter points throughout the book. This study shows that while it is not always possible for students to gain the knowledge intended from a picture book alone (without intentional instruction and involvement from the teacher), the picture book did encourage student responses, demonstrating that students were motivated to interact with the text.

Beyond considering how lessons are introduced and what resources are used to present content in a math lesson, the method of instruction must also be considered. Active learning strategies, those that ask students to read, talk, and explore while making sense of their world, are naturally motivating to most students (Columba et al., 2005; Willis, 2010). Books stimulate and support conversations among students, and conversations can build excitement for math (Burns, 2015). Gregory and Kaufeldt (2015) express this idea clearly when they state, "Brains don't like to be bored" (p. 148). Boredom is a clear path to disengagement with content, and disengagement negatively affects learning. According to Gregory and Kaufeldt, school, in some situations, is "the least responsive institution in today's society" (p. 146), clinging to traditional methods of teaching that originated from factory model ideas instead of moving to a thinking model. This certainly can be true in math classrooms. Teachers must remember that children are thinkers. They are naturally curious and want to figure out their environments (Columba et al., 2005; Willis, 2010). A picture book used effectively can pique curiosity and focus student interest, providing an avenue for students to find meaning from content and allowing them to engage socially at various stages of development (Columba et al., 2005).

Since all information enters the brain as sensory input, the use of picture books in instruction provides several ways to encourage curiosity (Willis, 2010). Children listening to a well-read story hear animated reading that includes voices of characters, suspenseful pauses, creative word order used by authors, and color and art provided by illustrators. Well-constructed and presented stories that include math content can elicit positive responses in students and can help alleviate negative feelings about the content. Students feel safe with story, and safety is a highly motivating condition.

The social interaction provided through discussion of literature is, by itself, motivating to students (Columba et al., 2005). The social aspects of learning: play, dialogue, small group tasks, and whole class work, provide opportunities for multiple interactions for students (Columba et al., 2005; Young-Loveridge, 2004). While traditionally math is often taught in a line that runs only between teacher and student, when taught using methods that encourage discussion and social interaction, a web of connections is created, increasing the chances that something will make sense to a student and improve math understanding. Math instructional choices that create such a network can benefit students by embracing the natural inclination students have to be social with their peers.

Not only does social interaction motivate students to learn, the use of children's literature can also humanize math for children. The inclusion of story in a math lesson challenges the stereotype of math as a noncreative, unimaginative, or cold subject. It builds on the positive reaction many children have to hearing or reading stories, providing the teacher with yet another way to communicate about math. Literature becomes the connecting agent among math content, student interest, and student background. Because story is personal to the student, the connection between story and math can motivate students to learn, building confidence, interest, and enjoyment (Columba et al., 2005).

Math Anxiety

There is no question that a well-constructed lesson relying on the use of children's literature can be motivating to students, but it is important to recognize that the use of children's literature can also reduce math anxiety, further increasing a student's ability to learn. High levels of anxiety negatively affect student success in math on a regular basis. Math anxiety can be exacerbated by standardized testing and instruction based on repetitive drill and practice used to evidence mastery of standards. It can exist because of previous real or perceived failures in math learning. Working in isolation can reinforce the fear that a student cannot be successful, and this fear can inhibit math learning. Students enter math classrooms with a preformed mental disposition, some even perceiving that they are under threat (Gregory & Kaufeldt, 2015). There is often a fixed mindset about one's ability in math, and students may not believe that they have the potential to grow or be successful while learning math. Negative attitudes about learning math can decrease motivation, which, in turn, can seriously affect achievement. Even if students get off to a positive start in math, math anxiety can cause motivation to wane over time, either as students get into new content that confuses them or as they move from primary grades to intermediate grades when math content becomes more complex. Students who become discouraged may fight instructional choices that emphasize math understanding over rules that can be easily presented by the teacher, memorized, and applied. (Barton & Heidema, 2002). Students may seek the easiest way to conquer assignments, which often is the memorization of rules, out of fear that they are incapable of understanding math on their own.

A way to reduce anxiety created by the fear of isolation is to remove the focus on the individual and instead, place the individual in a community where students construct knowledge together. In a constructivist classroom that uses children's literature as one means of instruction, the teacher is intentionally placing the students in the center of the picture, making them a prominent part of the learning experience (Willis, 2010). In such a positive collaborative environment, attentive listening and mutual respect abound, and building community is part of the daily routine. When students believe that they are part of a team of learners in a positive and supportive environment, anxiety is reduced. The use of children's literature can contribute to such an environment. When reading together or responding to a book being read by a teacher, students are asked to collaborate, discussing the story and the math involved. Because the reader is the interpreter of story, there is room for multiple views and opportunities to recognize differences and build respect (Willis, 2010).

Working intentionally to reduce math anxiety is important to help reduce emotions that can negatively affect math learning. There are few subjects that push "emotional buttons" the way math does (Willis, 2010). To be successful in math, competence is required in reasoning and abstract thinking, pattern and relationship recognition, and conceptual understanding. Students are asked to learn, use, and apply knowledge in new ways. Too often, these rigorous expectations are threatening to students. For teachers, the first step in reversing math negativity is to recognize it. Some statistics report that one third or more of all students hate math, more than twice any other reported subject (Willis, 2010).

A key to helping teachers recognize that math anxiety exists and changing negative attitudes is to create an achievable challenge for students by differentiating for each student's level of intellectual and emotional capacity (Willis, 2010). Connecting children's literacy skills and math is just one of many ways to provide this achievable challenge and, thus, help reduce anxiety in students. Integrating skills students have mastered in literacy to new applications in math eases insecurities. Students are able to use knowledge they already possess in new applications. Even a single strategy, such as using children's literature to teach math, can help free students from boundaries that limit their confidence, motivating them to see that success in math is possible.

A study using literature to help reduce math anxiety brings all these points together. In the study conducted with eighth grade math students, researchers examined student learning that resulted from instruction with and without the inclusion of children's literature (Green, 2013). In the study, both math achievement scores and math anxiety scores were analyzed. Results showed that there was a significant gain in math weekly scores and a significant decrease in math anxiety scores when pre- and posttests were compared.

There are many factors that likely contributed to these results. When relating math content to children's literature, learning happens through integration of skills and subjects (Columba et al., 2005). There is less pressure created by timelines that expect students to complete large numbers of problems. Instead, lessons focus on one or two big ideas rather than trivializing content into 25 problems done in the same way, often seen by students as busy work. Through the context of children's literature, math makes more sense to the child, and the content can be adjusted to individual differences and interests, providing a rich context for the development of a wide range of skills. When literature includes active learning strategies, students learn in the safety of community. Repetition can help students master concepts, and while repeating problems in math may seem like drudgery to students, rereading a favorite book may not. Teacher questions during the reading of a book, a natural process when reading, can help students clarify understanding and build on the ideas of others.

Before children can love math, they have to be comfortable with it (Willis, 2010). Helping students reach that comfort level involves reducing or eliminating math anxiety by creating a learning environment where students see that successful learning in math is possible.

Math Applications in the World

There are other advantages that emerge when moving students from known skills in reading to new applications in math. Children's books provide opportunities for students to make connections, integrate ideas, and synthesize concepts. This kind of thinking enables them to connect mathematical concepts to life experiences (Van den Heuvel-Panhuizen, Van den Boogard, & Doig, 2009). It is difficult to expect students to be excited about learning anything that is not relevant in their lives and in the world. Connections between story and life can be natural for students and can help them see that something they are learning connects to someone's life, so possibly it also connects to their own. Books can help students generalize novel, causal information from stories to the world and encourage them to look for similarities and non-similarities between the story and their own life. If a story is realistic, children relate the story to what they know of life, and they are more likely to generalize problems and solutions to the problem presented (Walker, Gopnik, & Ganea, 2015).

Students often report that math is not relevant to their lives, so it is important for teachers to help students see that math is more than what they see in a math textbook (Bay-Williams & Martinie, 2004; Gregory & Kaufeldt, 2015). By connecting math to real-world experiences, students learn that they are consumers of math and that math is all around them and has value in their lives (Barton & Heidema, 2002; Gregory & Kaufeldt, 2015). When math is not seen as relevant and practical in life, dangerous results can follow. A 2005 Associated Press-America Online poll of 1,000 adults in the United States revealed that 37% of respondents "hated" math in school (Willis, 2010, p. 5). This was more than twice the number who reported hating any other subject. In a random sampling of adults in an evaluation of math literacy, 71% could not calculate miles per gallon or determine a 10% tip for a lunch bill (Phillips, 2007, as cited in Willis, 2010). Myths about math ability still abound, perpetuating the idea that a person must be very intelligent to be good at math, that it is OK to be bad at math because most people are, or that math is not often used outside a math classroom (Willis, 2010). Some parents are not bothered when their children struggle in math because they do not perceive that they are good at math either. An important role for a teacher is showing students that math is everywhere, that math counts, and that every child can be a successful math learner.

Bringing the real world into a math classroom is a teacher's responsibility (Willis, 2010). One way to accomplish this is by aligning children's literature with practical math concepts that can be applied in life. A plethora of literary resources exist that can help teachers accomplish this goal. Books about topics including geometry, measurement, quantities, fractions, integers, number systems, mathematicians are everywhere. Whatever math a teacher is teaching, there are undoubtedly a variety of children's books that can be used to introduce, instruct, apply, or review that concept. Finding these materials is easy; however, when the decision is made to incorporate children's literature into a math lesson, it is important for teachers to evaluate the learning that results from this choice.

Evaluating Math Instruction that Includes Children's Literature

The research discussed in this chapter supports the use of children's literature in math instruction, and numerous math lesson plans include children's literature as a strategy for teaching math. But how can a teacher be confident that literature enhances math learning in the classroom?

To answer this question, it is important to consider research that has been conducted on the evaluation of children's books. Generally, this research states that using low-quality books, or using any book ineffectively, can negatively affect student learning (Flevares & Schiff, 2014), resulting in instruction that does not engage or motivate students. To avoid such negative results, teachers can consider using tested criteria for the selection of children's literature that is supported through years of research. The progression of such criteria has developed from the work of Shiro (1997), who identified eleven specific evaluative criteria, to the work of Hellwig, Monroe, and Jacobs (2000), who narrowed the criteria to five. Hunsader (2004) believed important points from Shiro's work were lost in this simplification and increased the list to six, adding specific detail to each criterion. Nesmith and Cooper (2010) extended Hunsader's work, considering the possibility that multiple interpretations can affect how literature is scored against these criteria, even when a detailed rubric is provided.

Today, the five criteria established by Hellwig, Monroe, and Jacobs (*accuracy*, *visual and verbal appeal*, *connections*, *audience*, and *the "wow" factor*) are commonly accepted as a good starting point. The first criterion of accuracy is ensuring that the book is accurate in its representation of the mathematical concepts. Another important component to consider is whether there is visual and verbal appeal. This includes how inviting the illustrations and cover art are, as well as whether or not the story keeps the reader engaged. Stories that are too repetitive or do not have that element of surprise may not hold the readers' interests. Connections from the math to the real world or from the math to a reader's experience are another way to evaluate the book. Teachers need to look for books that have authentic connections and draw on the interest and experience of the readers in order to keep them engaged.

This also becomes important when considering the audience for the book. There should be layers of complexity in the book so that multiple readings will expose the reader to deeper levels of understanding (Hellwig et al., 2000). It is also necessary to consider the gender and culture of the audience to ensure it has broad appeal. Finally, the "wow" factor is a quality that cannot be ignored. Books with this criterion will so strongly capture the attention of the reader that the reader becomes excited about investigating the ideas further. Books with the "wow" factor represent the mathematical concept in a unique way that most audiences have never considered. Most books will not address all five criteria, but using them can assist a teacher in determining the most effective book for the concept to be taught. Research continues on this topic, but it is clear that while past research can help teachers effectively adopt children's literature in a classroom setting, it is up to the professional practices of a teacher to determine that learning in math is truly enhanced by the addition of children's literature.

A classroom teacher whose main concern is the learning of the students in his or her care must take responsibility for instructional choices made. Those choices should be informed by research but should also include a teacher's professional knowledge of best practices and how those practices directly affect students. As with any instructional choice, a teacher must be prepared to analyze results that verify learning has been enhanced through the alignment of math and literacy. In this process, it is essential for a teacher to support decisions with evidence of improved student learning. We have created a set of guiding question for self-reflection (see Table 3.1) that can help a teacher to intentionally work toward this goal.

As shown in Table 3.1, the starting point in the effective use of children's literature is the identification of purpose, and this may very well be the most critical step. Table 3.1 is a tool that can support teachers with the intentional development of lessons using children's literature. The table provides a structure for lesson development that helps a teacher consider the possible benefits and challenges that might come along with the integration of children's literature and math. It provides questions to consider before planning a lesson and also identifies key statements that may help a teacher choose a book that will support mathematical learning and development. Using such a tool guides the teacher through a reflective process, which can result in highly effective learning opportunities for students. This tool offers an opportunity to pause and consider the potential benefits and/or challenges that using a new resource may provide. By choosing children's literature that includes math content, a teacher is deciding that the book chosen will help students feel more comfortable in learning math, understand academic language, comprehend math concepts, see how math exists in the world, or serve another purpose that the teacher has identified. The book becomes a teaching tool that helps a teacher meet an intended goal.

Table 3.1 Guiding questions for teacher self-reflection on the use of children's books in math instruction

Teacher actions	Guiding questions	Possible responses
Step 1: Initial reflection	Why am I considering adding children's literature to math instruction? What is my purpose is adding a children's book to this specific math lesson?	I believe the book will help develop vocabulary and academic language. I believe the book will motivate students to learn this math concept. I believe the book will help reduce math anxiety. I believe the book will help students master the math content or skill. I believe the book will help students connect this math skill to a real-world application.
Step 2: Defining end goals	What do I expect to see that will tell me that the use of the book has caused learning to occur? How will I know I have met the goals I set when including this book in this lesson?	Students are able to correctly use the academic language as it relates to the book. Students are engaged with the book, participate in dialogue with peers, ask questions, and offer responses to teacher questions about the book. Students are relaxed as the story is read and respond to teacher prompts. Students are able to demonstrate mastery of the math concept and explain the connection between the concept and the story. Students respond to one or two survey questions on an assignment that ask them to rate the effectiveness of the book in helping them understand the math concept. Students can give an example that connects the math content in the story to how it is used in the world.
Step 3: Choosing assessment tools	How will I measure the learning I expect to see? How will I know that what I expected to see has, indeed, occurred?	Take notes on the use of academic language in teacher-observed peer conversations and/or written responses to journal prompts. Record observations of student body language, eye contact, questions being asked, and on- or off-task behavior. Observe student interactions, and create journal prompts that address students' comfort level in the class and with the content. Create assignments asking students to draw diagrams and pictures and write explanations of math concepts on exit cards, to provide correct responses to practice problems, and to participate in oral conferences that explain the math concept as it connects to the story. Include survey questions that relate to the use of the book. Ask students to write a book reflection, retell the story (including the math), respond to teacher questions about the math in the story, and define a situation that connects the math understanding in the story to a new situation.

(continued)

Table 3.1 (continued)

Teacher actions	Guiding questions	Possible responses
Step 4: Documenting results to meet student needs and inform future choices	How will I document the data that results from my assessment process? What conclusions can I draw from this process to inform future instructional choices?	To record results from all assessments used, create a whole class chart listing individual student names on the left and tools used to gather information on the top. Use key words and symbols to record notes from each assessment tool that show mastery of identified vocabulary and academic language and math skills. Include teacher anecdotal notes from observations and conferences, student scores on assignments, and key words from survey responses for future reference. Such a chart provides a snapshot of math understanding for each individual student as well as the whole class. From this information, a teacher can form flexible groups for the future instruction based on students' levels of success with the math concept or skill. From the compiled data resulting from the use of a variety of assessment tools, a teacher can determine both the math understanding of students and the effectiveness of children's literature used to enhance that understanding.

But it is not enough for teachers to believe that using literature will improve math instruction. Evidence must show that this is true or not true. Whatever the initial purpose for choosing a children's book, the results must be measured to verify that learning has been enhanced by the use of literature as an instructional tool. This is done through assessment best practices, identifying specific criteria and then creating assessments that measure both individual and whole class mastery of the identified criteria. Data must be gathered, analyzed, interpreted, and evaluated to inform future instructional decisions. The assessment process defined by these steps is important for teachers in all subjects. Good teaching does not just happen on its own; it is intentionally planned and concretely measured. This is not different when considering the place children's literature holds in the instruction of math.

Conclusion

When teachers are intentional about choosing children's literature, using it effectively in math instruction, and analyzing the results, student learning can be positively affected. Yet, it is important to remember that while the effective use of children's literature can enhance math instruction, it is only one strategy in helping students succeed in math. The inclusion of literature does not take the place of manipulatives and hands-on activities that help students see math, of teacher modeling that guides math processes, of word walls and other literacy tools that aid in understanding the academic language of math, or of the practice required to help students reinforce new understanding. A complete math program uses every effective tool available to help students succeed in math, and children's literature can be one of these, a tool that, when used in conjunction with other best practices, can greatly improve the chances that a student will succeed in learning math.

Children's literature provides a powerful opportunity to share knowledge with students and foster unique learning experiences (Columba et al., 2005). Topics in this chapter have emphasized the power of story, the intentional alignment of literacy and math skills, and the importance of helping students "see" math. Considering prior knowledge, reducing math anxiety, and appealing to student interests have been presented as ways to increase student motivation, thus creating a positive learning environment for students. Helping students see that math is not a separate or isolated subject, but one that can be seen everywhere in the world, can greatly improve the chances that students will view math as a relevant subject worthy of their time and efforts. All of these objectives can be partially achieved through the use of children's literature in a math classroom. By understanding the math concepts to be taught, the books used to teach them, and the children who will be experiencing the instruction, teachers can greatly improve student learning in math (Clements & Sarama, 2004; Columba et al., 2005).

References

Armstrong, J. (2000). *Shipwreck at the bottom of the world: The extraordinary true story of Shackleton and the endurance*. New York: Crown Publishers.

Barton, M., & Heidema, C. (2002). *Teaching reading in mathematics* (2nd ed.). Aurora, CO: McRel.

Bay-Williams, J., & Martinie, S. (2004). *Math and literature: Grades 6-8*. Sausalito, CA: Math Solutions Publications.

Bruner, J. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.

Burleigh, R. (1997). *Flight: The journey of Charles Lindbergh*. New York: The Putnam & Grosset Group.

Burns, M. (2008a). *The greedy triangle*. New York: Scholastic Press.

Burns, M. (2008b). *Spaghetti and meatballs for all*. New York: Scholastic Press.

Burns, M. (2015). *About teaching mathematics, 4th edition: A K-8 resource*. Sausalito, CA: Math Solutions Publications.

Capraro, R. M., & Capraro, M. M. (2006). Are you really going to read us a story? Learning geometry through children's mathematics literature. *Reading Psychology, 27*, 21–36. doi:10.1080/02702710500468716.

Cherry, L. (1990). *The great kapok tree*. Boston: Hougton Mifflin Harcourt.

Clements, D. H., & Sarama, J. (2004). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Retrieved from http://samples.sainsburysebooks.co.uk/9781135635343_sample_823160.pdf

Columba, L., Kim, C., & Moe, A. (2005). *The power of picture books in teaching math and science: Grades preK-8*. Scottsdale, AZ: Holcomb Hathaway, Publishers, Inc..

Demi. (1997). *One grain of rice*. New York: Scholastic Press.

Elia, I., Van den Heuvel-Panhuizen, M., & Georgiou, A. (2010). The role of pictures in picture books on children's cognitive engagement with mathematics. *European Early Childhood Education Research Journal, 18*(3), 125–147. doi:10.1080/1350293X.2010.500054.

Flanagan, S. (2014). How does storytelling within higher education contribute to the learning experience of early years students? *Journal of Practice Teaching & Learning, 13*(2-3), 146–168.

Flevares, L., & Schiff, J. (2014). Learning mathematics in two dimensions: A review and look ahead at teaching and learning early childhood mathematics with children's literature. *Frontiers in Psychology, 5*, 459. doi:10.3389/fpsyg.2014.00459.

Gear, A. (2006). *Reading powers*. Ontario, Canada: Pembroke Publishers.

Green, S. (2013). Improving comprehension in middle school math by incorporating children's literature in the instruction of mathematics. (Doctoral dissertation). Retrieved from ProQuest LLC. (ED554063).

Gregory, G., & Kaufeldt, M. (2015). *The motivated brain: Improving student attention, engagement, and perseverance*. Alexandria, VA: ASCD.

Hellwig, S., Monroe, E. E., & Jacobs, J. S. (2000). Making informed choices: Selecting children's trade books for mathematics instruction. *Teaching Children Mathematics, 7*(3), 138–143.

Hunsader, P. D. (2004). Trade books: Establishing their value and assessing their quality. *Reading Teacher, 57*, 618–629.

Martin, J. B. (1998). *Snowflake Bentley*. New York: Scholastic.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Nesmith, S., & Cooper, S. (2010). Trade books in the mathematics classroom: The impact of many, varied perspectives on determination of quality. *Journal of Research in Childhood Education, 24*(4), 279–297. doi:10.1080/02568543.2010.510086.

Neuschwander, C. (1999). *Sir Cumference and the dragon of Pi: A math adventure*. Watertown, MA: Charlesbridge.

O'Neill, D. K., Pearce, M. J., & Pick, J. L. (2004). Preschool children's narrative and performance on the Peabody individualized achievement test - revised: Evidence of a relation between early narrative and later mathematical ability. *First Language, 24*, 149–183. doi:10.1177/0142723704043529.

Polacco, P. (2001). *The keeping quilt*. New York: Aladdin Paperbacks.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., & Verschaffel, L. (2016). Kindergarteners' spontaneous focusing on numerosity in relation to their number related utterances during numerical picture book reading, *Mathematical Thinking and Learning, 18*(2), 125–141, doi: 10.1080.10986065.1148531

Schiro, M. (1997). *Integrating children's literature and mathematics in the classroom: Children as meaning makers, problem solvers, and literary critics*. New York: Teachers College Press.

Schmandt-Besserat, D. (1999). *The history of counting*. New York: William Morrow and Company, Inc.

Skoumpourdi, C., & Mpakopoulou, I. (2011). The prints: A picture book for pre-formal geometry. *Early Childhood Education Journal, 39*, 197–206. doi:10.1007/s10643-011-0454-0.

Stewart, P. (1998). *A little bit of winter*. New York: Scholastic Inc..

Tang, G. (2003). *MATH-terpieces: The art of problem-solving*. New York: Scholastic Press.

Thomas, L., & Feng, J. (2015, October). *Integrating children's literature in elementary mathematics.* Paper presented at Georgia Educational Research Association Annual Conference, Savannah, Georgia.

Van den Heuvel-Pahnuizen, M., & Van den Boogard, S. (2008). Picture books as an impetus for kindergarteners' mathematical thinking. *Mathematical Thinking and Learning, 10*(4), 341–373. doi:10.1080/10986060802425539.

Van den Heuvel-Panhuizen, M., Van den Boogard, S., & Doig, S. (2009). Picture books stimulate the learning of mathematics. *Australasian Journal of Early Childhood, 34*(2), 30–39.

Van Ommen, S. (2003). *The surprise*. Homesdale, PA: Front Street.

Walker, C. M., Gopnik, A., & Ganea, P. A. (2015). Learning to learn from stories: Children's developing sensitivity to the causal structure of fictional worlds. *Child Development, 86*(1), 310–318. doi:10.1111/cdev.12287.

Willis, J. (2010). *Learning to love math: Teaching strategies that change student attitudes and get results*. Alexandria, VA: ASCD.

Winter, J. (2006). *The 39 apartments of Ludwig van Beethoven*. New York: Random House.

Woolfolk, A. (2013). *Educational psychology* (12th ed.). New York: Pearson.

Young-Loveridge, J. M. (2004). Effects on early numeracy of program using number books and games. *Early Child Research Quarterly, 19*(1), 82–98. doi:10.1016/j.ecresq.2004.01.001.

Zazkis, R., & Liljedahl, P. (2009). *Teaching mathematics as storytelling*. Rotterdam, Netherlands: Sense Publishers.

Part II
Complexity, Difference and/in Mathematics Education

Chapter 4
Complexity as a Discourse on School Mathematics Reform

Brent Davis

Abstract This writing begins with a brief introduction of complexity thinking, coupled to a survey of some of the disparate ways that it has been taken up within mathematics education. That review is embedded in a report on a teaching experiment that was developed around the topic of exponentiation, and that report is in turn used to highlight three elements that may be critical to school mathematics reform. Firstly, complexity is viewed in curricular terms for how it might affect the content of school mathematics. Secondly, complexity is presented as a discourse on learning, which might influence how topics and experiences are formatted for students. Thirdly, complexity is interpreted as a source of pragmatic advice for those tasked with working in the complex space of teaching mathematics.

Keywords Complexity thinking • School mathematics • Mathematics curriculum

One of the most common criticisms of contemporary school mathematics is that its contents are out of step with the times. The curriculum, it is argued, comprises many facts and skills that have become all but useless, while it ignores a host of concepts and competencies that have emerged as indispensible. Often the problem is attributed to a system that is prone to accumulation and that cannot jettison its history. Programs of study have thus become not-always-coherent mixes of topics drawn from ancient traditions, skills imagined necessary for a citizen of the modern (read: industry-based, consumption-driven) world, necessary preparations for postsecondary study, and ragtag collections of other topics that were seen to add some pragmatic value at one time or another over the past few centuries – all carried along by a momentum of habit and familiarity. Somewhat ironically, a domain that has not been particularly influential in these evolutions is mathematics itself. As a result, few current curricula have any substantial content that is reflective of developments in mathematics over the past few centuries.

B. Davis (✉)
University of Calgary, Werklund School of Education,
2500 University Drive NW, Calgary, AB, Canada, T2N 1N4
e-mail: brent.davis@ucalgary.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_4

Oriented by a deep concern for this situation, I am currently involved in a longitudinal investigation of “changing the culture of mathematics teaching at the school level.” Through this design-based inquiry, a group of university researchers has teamed with the staff of a school in a 7-year commitment to work together in transforming how mathematics is seen and engaged.

The project has three foci, distilled from preliminary discussions with the teacher-participants:

- Mathematics curriculum – e.g., what mathematics is important to teach? Is that the same as what is in the curriculum? Where did that curriculum come from?
- Individual understanding – e.g., how does understanding of a concept develop? Is there a “best” way to structure/sequence teaching to support robust conceptual development? Are individuals’ understandings necessarily unique, or is there a way of nudging learners to “true” interpretations of concepts?
- Social process – e.g., how do groups support/frustrate the development of individual understanding? How does individual understanding support/frustrate the work of groups?

At first blush, the range of topics represented in these clusters of questions may seem to be so broad as to disable inquiry. In truth, even as one of the principal researchers, I was at first taken aback with the full range of concerns raised by the research partners. However, while these three clusters of questions might seem on the surface to be focused on disparate matters, “inside” them there is a uniting theme: complexity.

More precisely, each of these clusters of issues concerns a category of emergent phenomena. That is, each points to a form or agent that obeys an evolutionary dynamic and that arises in and transforms through the interactions of other forms and agents. That realization helped to shift the principal focus from the three clusters of questions above to a single unifying theme. In the process, as is reported below, a space was opened both to move toward productive and pragmatic responses to the questions posed and to make meaningful strides toward the grander intention of the project.

What Is “Complexity” within Mathematics Education?

Before getting into some of the specifics of those developments, it is important to situate the intended meaning of *complexity*. Unfortunately, there is no unified or straightforward definition of the word. Indeed, most commentaries on complexity research begin with the observation that there is no singular meaning of complexity, principally because researchers tend to define it in terms of their particular research foci. One thus finds quite focused-and-technical definitions in such fields as mathematics and software engineering, more-indistinct-but-operational meanings in chemistry and biology, and quite flexible interpretations in the social sciences (cf. Mitchell, 2009).

Within mathematics education, the range of interpretations of complexity is almost as divergent as it is across all academic discourses. This variety can in part be attributed to the way that mathematics education straddles two very different domains. On one side, mathematics offers precise definitions and strategies. On the other side, education cannot afford such precision, as it sits at the nexus of disciplinary knowledge, social engineering, and other cultural enterprises. Conceptions of complexity among mathematics education researchers thus vary from the precise to the vague, depending on how and where the notion is taken up.

However, diverse interpretations do collect around a few key qualities. In particular, *complex* systems adapt and are thus distinguishable from *complicated* (i.e., mechanical) systems that may be composed of many interacting components and which can be described and predicted using laws of classical physics. A complex system comprises many interacting agents – and those agents, in turn, may comprise many interacting subagents – presenting the possibility of global behaviors that are rooted in but not reducible to the actions or qualities of the constituting agents. In other words, a complex system is better described by using Darwinian dynamics than Newtonian mechanics.

Complexity research only cohered as a discernible movement in the physical and information sciences in the middle of that twentieth century, with the social sciences and humanities joining in its development in more recent decades. To a much lesser (but noticeably accelerating) extent, complex systems research has been embraced by educationists whose interests extend across such levels of phenomena as genomics, neurological process, subjective understanding, interpersonal dynamics, mathematical modeling, cultural evolution, and global ecology. As discussed elsewhere (Davis & Simmt, 2014, 2016), these topics can be seen across three strands of interest among mathematics education researchers – namely:

- Regarding the contents of curriculum, complexity as a disciplinary discourse – i.e., as a digitally enabled, modeling-based branch of mathematics
- Regarding beliefs on learning, complexity as a theoretical discourse – i.e., as the study of learning systems, affording insight into the structures of knowledge domains, the social dynamics of knowledge production, and the intricacies of individual sense-making
- Regarding pedagogical strategies, complexity as a pragmatic discourse – i.e., as a means to nurture emergent possibility, with advice on how to design tasks, structure interactions, etc.

For the most part, to my reading, researchers in mathematics education have tended to treat these issues singularly. That is perhaps not surprising, since each represents a significant departure from entrenched, commonsense beliefs. However, as I attempt to illustrate in the example I turn to presently, there may be great transformative potential in treating these considerations as necessary simultaneities.

Importantly, the resonance between these three strands of interest among mathematics education researchers and the three foci of the project (mentioned earlier) are not accidental. Engaging with teachers about such matters is, I believe, integral to bringing possibilities afforded by complexity thinking to the realities experienced by teachers. This thought has oriented much of my own research efforts over the past several years, particularly around efforts to co-design and co-teach units of study with teachers in our design-based research study. To that end, in the following account, I endeavor to highlight how complexity can serve, simultaneously, as a theory of curriculum, learning, and pedagogy.

A Teaching Experiment on Exponentiation

As already noted, for centuries, the basics of school mathematics tend to be construed as addition, subtraction, multiplication, and division. Notably, these operations are "basic" not because they are foundational to mathematics knowledge, but because they were vital to a newly industrialized and market-driven economy a few hundred years ago. It is easy to see why computational competence would be useful to a citizen of that era and to ours as well. If anything, the need has been amplified in our number-dense world. However, it is not clear that these four operations are a sufficient set of basics today, given that some of the most pressing issues – such as population growth, the rise of greenhouse gases, ocean acidification, decline in species diversity, cultural change, increases in debt, and so on – have strongly exponential characters. More descriptively, these sorts of pressing issues are instances of complexity, evidenced in part by their potentials for rapid change and unpredictability.

Understandings and appreciations of the volatility of prediction have become rather commonplace, evidenced in the way the "butterfly effect" has captured the collective imagination. However, while awareness of this popular trope might suggest that complexivist sensibilities have gained traction, it might also indicate limited understanding of the actual mechanisms at work inside complex dynamical systems. The butterfly effect is most often stated in terms of a system's sensitivity to initial conditions, but what really matters is the power of iteration to amplify or dampen. That is, the butterfly effect – like any complex dynamic – only makes sense within a frame of exponentiation.

I mentioned that thought in a social conversation with an eighth-grade teacher in Calgary, and she promptly challenged me to design and teach a brief unit in which exponentiation was treated as a useful interpretive tool rather than a site for symbolic manipulations. The major impetus for the work was thus professional curiosity rather than a predefined research intention. (Appropriate ethical clearances and permissions were secured.) She generously offered a week of lessons, and a few weeks later, I found myself in her regular-stream class of 32 students. Not wanting to interrupt established routines much, I mimicked the teacher's structures of frequent full-group discussion, modulated with small-group work. No individual seatwork and no deliberate homework were assigned during the week. That decision was made for several reasons. Firstly, the brevity of the project made it difficult for me to get to know the students and communicate expectations in ways that made

Table 4.1 An overview of a weeklong unit on exponentiation

Day	Focus	Activities
Monday	Images of exponentiation	Drawing pictures of exponential change Web searches ("exponentiation," "exponential growth," "powers of two," and related terms)
Tuesday	Exponentiation lattice	Collectively assembling a lattice Looking for patterns Contrasts to addition and multiplication lattices
Wednesday	Analogies to other binary operations	Symbolism and vocabulary Noting similarities between addition and multiplication, and extending these to exponentiation
Thursday	Exploring the validity of those analogies	Justifying and questioning Thinking about the structure of mathematics and mathematical ideas
Friday	Consolidation and examples	Other illustrations of exponentiation Instances of exponentiation in the world we inhabit

me confident such emphases would be effective. Secondly, and closely related, a driving intention of the unit was to trouble the conflation of "mathematics" and "computation" – and, to my mind, individual seatwork and homework presented risks of pressing those two constructs together. Thirdly, as a champion of collective sense-making, I am personally much more comfortable in settings where learners have ample opportunity to express their thinking, to challenge one another, and to openly speculate.

The outline of lesson topics for that week is presented in Table 4.1. A more detailed, general overview of the classroom activities has been presented elsewhere (Davis, 2015), and so only summary descriptions are offered here.

The unit's opening task was an invitation to create images of exponential change. Students were instructed on drawing grid-based images of sequential doubling – starting by outlining a single square, then doubling the figure to enclose two squares, and so on, to the limits of their sheets of paper. T-tables were incorporated into the activity to record quantities and make number patterns more apparent, and students were then tasked with creating similar images and tables for bases of 3–9. They were encouraged to do Web searches and together generated a rich range of associated figures that included images of exponential growth/decay and exponential curves.

On the second day, students were asked to compare exponentiation to addition and multiplication. Earlier in the school year, the class had created poster-sized lattices for addition, subtraction, multiplication, and division on *xy*-coordinate grids. On these charts, values on the *x*-axis served, respectively, as augend, subtrahend, multiplier, and dividend; values on the on the *y*-axis as addend, minuend, multiplicand, and divisor; and corresponding positions on the grid as locations for sums, differences, products, and quotients. Figure 4.1 presents small portions of these lattices.

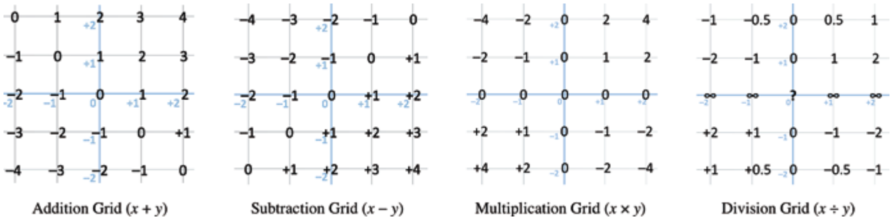


Fig. 4.1 Core portions of the addition, subtraction, multiplication, and division lattices generated earlier in the school year

In the earlier unit, these devices proved to be powerful tools for noticing patterns and, in the process, interpreting identity elements, commutativity, and other concepts and properties. We imagined a chart for exponentiation might serve similar purposes and began the second class with the construction of an exponentiation lattice spanning values of −10 to +10 on both axes – that is, covering the range of -10^{-10} to 10^{10}. A core portion of the exponentiation lattice is presented in Fig. 4.2.

The collective analysis of the result began by examining the first quadrant. Students compared its patterns to those in the addition and multiplication lattices, posted nearby. Three observations were immediately noted. First, students remarked on the "steeper and crazy-steeper" increases in values as one moves away from the origin, contrasted with the "flattening" feel of the addition lattice and the "gentler rising" of the multiplication lattice. Second, it was noted that the exponentiation chart "doesn't fold over like adding and multiplying" – that is, whereas the addition and multiplication lattices are symmetric about the line $y = x$, the exponentiation lattice is not. Third, "the diagonal of one table is the 2-row of the next." That is, just as the values along the $y = x$ diagonal of the addition lattice correspond to those of the $y = 2$ row of the multiplication lattice, so the values along the $y = x$ diagonal of the multiplication lattice correspond to those of the $y = 2$ row of the exponentiation lattice. Discussions touched on such topics as commutativity and other symmetries, the mathematics of rapid change, logarithms, imaginary numbers, and mathematical notations (see Davis, 2015, for a more complete account on how discussions of these observations unfolded).

The third session dealt with analogies between exponentiation and the operations of addition and multiplication. Prompted by the problems encountered with x^x the previous day, we began by noting that the symbolism for exponentiation might obscure the relationship to other operations. To highlight similarities to "2 + 3" and "2 × 3," we proposed "2 ↑ 3," which is one of several accepted notations (Cajori, 2007). The resulting set of pairs

$x + x = 2x$.

$x \times x = x^2$.

$x \uparrow x = x^x$.

seemed to satisfy the desire for parallel representations that had emerged the day before.

We set up the day's task with a version of Table 4.2 (below), which was an extension of a chart they had done earlier in the year comparing properties of addition to properties of multiplication. We reminded them of that detail to get things started and then invited suggestions for completing the row labeled "commutative property."

The main point of this activity was to deepen understandings of exponentiation. A second purpose was to support understandings of the relationship among concepts, based on a vital difference between topics studied at elementary and secondary levels. Whereas almost all the concepts encountered at the elementary level can be interpreted in terms of (i.e., are analogical to) objects and actions in the physical world, the analogies for concepts at the secondary level are mostly mathematical objects (see Hofstadter & Sander, 2013). Making analogies, then, is both a mechanism for extending mathematical insight and a window into the structure of mathematics knowledge.

Before setting the students to work on their own, we indicated that they should not worry about the last column, as we had already planned that for the focus of the fourth session. The rest of the class was devoted to filling in blank cells, an effort that began in small groups and that ended in whole-group negotiations of acceptable, parallel phrasings for each entry (see the second row in Table 4.3). Notably, the final three rows of the chart were additions proposed by the students themselves.

The fourth session was devoted to exploring the truth or falsity of the conjectures from the day before. Students worked in small groups and focused on speculations of their choosing. They also made free use of the Internet to help them in their deliberations. Topics in the follow-up discussion included a problem with the speculation on inverse values (i.e., that for every a there is a $\downarrow a$ such that $a \uparrow (\downarrow a) = 1$), because the exponentiation grid suggested $a \uparrow 0 = 1$ (for all $a \neq 0$). If the speculation were true, it would mean that the exponentiative inverse of every number would be 0, which most felt to be nonsensical – in addition to rendering the speculation on "operating on the opposite" similarly troublesome. We elected to leave these discussions unsettled, suggesting that our simple analogies might be misleading. We also suggested that further studies in high school would shed some light on a few of the details – a point that was supported by topics that came up in students' Web searches, including logarithms, imaginary and complex number systems, and tetration.

The final session was devoted to review and consolidation. We framed the session by developing the table presented in Table 4.4, through which we suggested that the geometric image best fitted to addition is the line, to multiplication is a rectangle, and to exponentiation is a fractal. That thought was tied in to a "fractal card" activity (Simmt & Davis, 1998) that the students had undertaken earlier in the school year.

The balance of the lesson was given to conducting searches and looking across instances of exponential growth and decay (e.g., creating fractal cards, population growth, species decline, greenhouse gas increase, technology evolution), framed by Charles and Ray Eames' (1977) film, *Powers of Ten* and Cary and Michael Huang's (2012) interactive Prezi, *The Scale of the Universe*. Exponential growth curves emerged to be a uniting image across these explorations and also proved useful as a

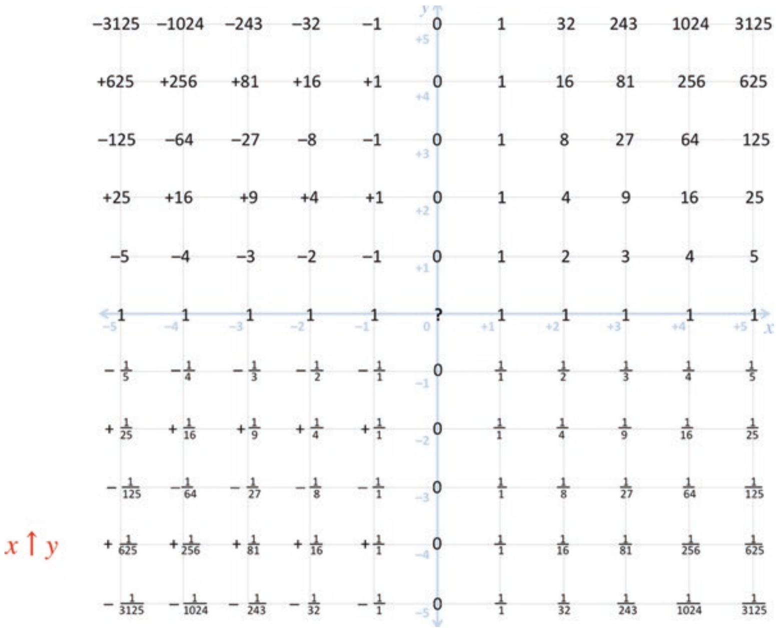


Fig. 4.2 A core portion of the exponentiation lattice

recap on the week as they linked back to the images and grid developed on Monday and Tuesday.

Complexity as a Disciplinary Discourse: Moving from Computation to Modeling

Revisiting the three ways that complexity has been taken up by mathematics education researchers, I would assert that the above teaching episode is an instantiation of those diverse but complementary perspectives on the discourse:

- Complexity as a theory of curriculum – specifically, in this case, an examination of the mathematics of rapid change, which is vital for appreciating the dynamics involved in complex modeling; more generally, approaching mathematics as a means to model experiences and phenomena
- Complexity as a theory of learning – using principles of complexity to interpret individual sense-making, collective knowledge production, and mathematics itself as responsive, adaptive systems that require disequilibration, interactivity, and other conditions of emergence (see Davis & Sumara, 2006)
- Complexity as a theory of pedagogy – used, for example, to inform the distribution of tasks across the collective, to balance redundancy and specialization of agents, and to blend emergent possibilities with preconceived intentions (Davis & Simmt, 2003)

Table 4.2 The blank speculation table

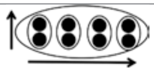
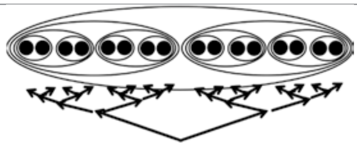
Topic/property	How it looks for addition ($x + y$)	How it looks for multiplication ($x \times y$)	Speculation for exponentiation ($x \uparrow y$)	T/F
Commutative property				
Reverse operation				
Identity element				
Inverse values				

Table 4.3 Conjectures for exponentiation based on analogies to addition and multiplication

Topic/property	How it looks for addition ($x + y$)	How it looks for multiplication ($x \times y$)	Speculation for exponentiation ($x \uparrow y$)	T/F
Commutative property	$a + b = b + a$	$a \times b = b \times a$	$a \uparrow b = b \uparrow a$	False: $2\uparrow 3 \neq 3\uparrow 2$
Reverse operation	Subtraction (−)	Division (÷)	De-exponentiation (↓)	
Identity element	0 … as in $a + 0 = 0 + a = a$	1 … as in $a \times 1 = 1 \times a = a$	1? … since $a \uparrow 1 = a$ … although $1 \uparrow a = 1$	
Inverse values	Additive inverse of a is $0 - a$, or $-a$; $a + (-a) = 0$	Multiplicative inverse of a is $1 \div a$, or $\frac{1}{a}$; $a \times \frac{1}{a} = 1$	Exponentiative inverse of a is $1 \downarrow a$, or $\downarrow a$; $a \uparrow (\downarrow a) = 1$	
Operating on the opposite	Subtraction can be done by adding the [additive] inverse: $a - b = a + (-b)$	Division can be done by multiplying the [multiplicative] inverse: $a \div b = a \times \frac{1}{b}$	De-exponentiation must be doable by exponentiating the [exponentiative] inverse: $a \downarrow b = a \uparrow (\downarrow b)$	
"Next" operation	A repeated addition is a multiplication	A repeated multiplication is an exponentiation	A repeated exponentiation must be a … something	
"Next" set of numbers	When you allow subtraction, you need signed numbers	When you allow division, you need rational numbers	When you allow de-exponentiation, you need another set of numbers	

Each of these points merits considerable elaboration. However, given constraints on space, I focus on the first, with the suggestion that school mathematics might be reconstrued in terms of modeling rather than the currently dominant computation-heavy emphasis. Repeating an assertion made earlier, as I hope is illustrated with the account of the teaching experiment, all three elements must occur simultaneously – and so I acknowledge the artificiality of focusing on the first point. (The citations included in the second and third points provide detailed discussions of those elements.)

Table 4.4 Some geometric analogies to arithmetic operations

Operation	Principal visual metaphors	Common applications/interpretations (using whole number values)
2 + 4		Combining of sets or lengths along 1 dimension Can be consistently represented in linear form
2 × 4		Sets of sets or array/area generated by crossing dimensions Can often be represented as a rectangle
2 ↑ 4		Sets of sets of sets (etc.) or multidimensional form Representable in a fractalesque, recursively generated and/or branching image

This suggestion is, of course, anchored to a conviction that being mathematically competent is about being able to interpret and simulate real-life situations with mathematical constructs. It was in this spirit that exponentiation was studied in the reported classroom episode. While some calculations were involved, computation was always a means to an end. It was a tool within the modeling activity.

To elaborate, a "model" is a representation – a description, an image, a copy – which is intended to highlight vital, defining attributes of some phenomenon. Most often, a model is a simplification, one that is useful as a tool for understanding. A "mathematical model" is thus a description of a phenomenon using mathematical constructs. Examples abound and range from the mundane to the enormously complex. On the more familiar end of the spectrum, every act of counting or measuring is an act of mathematical modeling – that is, of representing a situation in terms of an appropriate number system. At the more complex end of the spectrum, mathematical models are used in the natural sciences (e.g., physics, chemistry, biology, geology, meteorology, astronomy), engineering, and the social sciences (e.g., economics, psychology, political science, sociology) to interpret, explain, and predict phenomena that arise in the interactions of many, many interacting agents.

In this sense, the discipline of mathematics has always been about modeling – although this core emphasis has often been obscured by the computational demands of some models. In particular, prior to rapid and inexpensive computing, the modeling of systems was largely focused on those dynamics that could be studied through differential linear equations. Poincaré was notable among those who examined nonlinear dynamical systems, doing so from a theoretical perspective (Bell, 1937). The computational power of digital technologies in the second half of the twentieth century was necessary for the investigation of dynamical systems began to flourish. Computing power brought about possibility of doing "experimental mathematics" (Borwein & Devlin, 2008) and numerical analysis, triggering a rebirth of the modeling of nonlinear dynamical systems. Importantly, digital computing provided not only a means of computing extremely large data sets and iterating functions through hundreds of thousands of repetitions, it also provided means for converting numerical data to visual representations, enabling the generation of new insights and, consequently, new forms of mathematics (Mitchell, 2009).

It might be tempting to characterize the ever-growing gap between the research mathematics and school mathematics in terms of the contrast between the emphasis on modeling in the former and the emphasis on computation in the latter. That distinction would be unfair, however. Every topic in school mathematics was originally selected for its power to model, and this detail helps to explain the traditional pedagogical emphasis has been on rote application. In the first public schools, learners were being trained not to model, but to apply established mathematical models, and to do so efficiently and effectively. Routinized, repetitive instruction that does not allow for much divergent thinking is arguably the best way to do that.

In other words, schooling's emphasis on computation was a once-fitting educational emphasis, aimed at exploiting mathematics' capacities to model critical elements of one's world. However, circumstances and sensibilities have changed, along with the needs of a mathematically literate citizen. But so too have the affordances of the world in which we live, such as access to data, computational speed, and spatio-visual interfaces. Such evolutions were behind Lesh's (2010) assertion that complexity has emerged as "an important topic to be included in any mathematics curriculum that claims to be preparing students for full participation in a technology-based age of information" (p. 563).

To be clear on the point of this writing, the suggestion is *not* that study of complex systems is new, but that the mathematics of complexity could represent a significant shift from traditional emphases on computation to a new emphasis on (complex) modeling – and, in that shift, possibly nudge school mathematics closer to its parent discipline. As Stewart (1989) has reported, mathematicians have long seen their work in terms of modeling. Just as significantly, they were perfectly aware when they were using linear approximations and other reductions in order to avoid computational intractability. Lecturers and texts followed suit in omitting nonlinear accounts; hence generations of students were exposed to over-simplified, linearized versions of natural phenomena. In other words, non-complex mathematics prevailed in public schools not because it was ideal but because it lent itself to calculations that could be done by hand. The power of digital technologies has not just opened up new vistas of calculation, they have triggered epistemic shifts as they contribute to redefinitions of what counts as possible and what is expressible, and this insight has been engaged by many mathematics education researchers (e.g., English, 2011; Hoyles & Noss, 2008; Moreno-Armella, Hegedus, & Kaput, 2008).

Notable in this the movement toward recasting school mathematics in terms of modeling is the seminal work of Papert (e.g., 1980), particularly his development of the Logo programming language in the late 1970s. The language was designed to be usable by young novices and advanced experts alike. It enabled users to solve problems using a mobile robot, the "Logo turtle," and eventually a simulated turtle on the computer screen. While not intended explicitly for the study of complexity, Logo lent itself to recursive programming and was thus easily used to generate fractal-like images and to explore applications dynamically – opening the door to more complexity-specific topics. To that end, different developers have since offered Logo-based platforms that are explicitly intended to explore complex systems (and other) applications. For example, StarLogo (lead designer, M. Resnick; http://education.mit.edu/starlogo/) and NetLogo (lead designer, U. Wilensky; http://ccl.northwestern.edu/netlogo/). Both platforms were developed in the 1990s and extended Papert's original Logo program by presenting the possibility of multiple, interacting agents (turtles). This feature renders the applications useful for simulating ranges of complex phenomena. Both StarLogo and NetLogo include extensive online libraries of already-programmed simulations of familiar phenomena (e.g., flocking birds, traffic jams, disease spread, and population dynamics) and less-familiar applications in a variety of domains such as economics, biology, physics, chemistry, neurology, and psychology. At the same time, the platforms preserve the simplicity of programming that distinguished the original Logo (e.g., utilizing switches, sliders, choosers, inputs, and other interface elements), making them accessible for even young learners. Other visual programming languages have been developed that are particularly appropriate to students (e.g., Scratch, http://scratch.mit.edu, and ToonTalk, http://www.toontalk.com).

Over the past few decades, hundreds of speculative essays and research reports (see, e.g., http://ccl.northwestern.edu/netlogo/references.shtml) have been published on these and other multi-turtle programs. Regarding matters of potential innovations for school mathematics, in addition to well-developed resources, there have been extensive discussions, and there exists a substantial empirical basis for moving forward on the selection and development of curriculum content that is fitted to themes of complexity. Not surprisingly, then, with the ready access to computational and imaging technologies in most school classrooms, some (e.g., Jacobson & Wilensky 2006) have advocated for the inclusion of such topics as computer-based modeling and simulation languages, including networked collaborative simulations (see Kaput Center for Research and Innovation in STEM Education, http://www.kaputcenter.umassd.edu). In this vein, complexity is understood as a digitally enabled, modeling-based branch of mathematics that opens spaces (particularly in secondary and tertiary education) for new themes such as recursive functions, fractal geometry, and modeling of complex phenomena with mathematical tools such as iteration, cobwebbing, and phase diagrams.

The shift in sensibility from linearity to complexity is more important than the development of the computational competencies necessary for modeling. The very role of mathematics in one's life is transformed through this shift in curriculum emphasis. As Lesh (2010) described, "whereas the entire traditional K–14 mathematics curriculum can be characterized as a step-by-step line of march toward the study of single, solvable, differentiable functions, the world beyond schools contains scarcely a few situations of single actor–single outcome variety" (p. 564). Extending this thought, Lesh highlighted that questions and topics in complexity and data management are not only made more accessible in K-14 settings through digital technologies, but current tools have also made it possible to render some key principles comprehensible to young learners in manners that complement traditional curriculum emphases.

Despite the growing research base and the compelling arguments, however, few contemporary programs of study in school mathematics have heeded such admonitions for change. It is perhaps for this reason that many mathematics education researchers have focused on familiar topic areas (such as those just mentioned; see Davis & Simmt, 2016, for other examples) as means to incorporate studies of complexity into school mathematics. Discussions of and research into possible sites of integration have spanned all grade levels and several content areas, and proponents have tended to advocate for complexity content but in a less calculation-dependent format.

Closing Remarks

For many mathematics educators, complexity thinking might seem like a Pandora's box. If the field were to open it and take up the topic seriously, an array to world-changing possibilities would impose themselves. Complexity thinking challenges many of the deeply engrained, commonsensical assumptions on how humans think and learn. It interrupts much of the orthodoxy on group process and collective knowledge. And, in particular, as a curriculum topic, there is no straightforward way to fit complex modeling into the mold of contemporary school mathematics. It transcends procedures with its invitation to experiment; it demands precision, but in the service of playful possibility; it is rooted in computation but off-loads most of that work onto digital technologies; it requires facility with symbol manipulation, but that manipulation is more for description than deriving solutions. In other words, merely considering complex modeling as a possible topic for today's classrooms forces a rethinking of not just *what* is being taught, but *why* some topics maintain such prominence and *how* topics might be formatted to engage learners meaningfully and effectively.

Indeed, as the example of exponentiation might be used to illustrate, if complexity were to be seriously considered as a curriculum topic, it would compel reexamination of the very foundations of school mathematics. Not only must the "basics" be available for interrogation and revision, emphases of computation-heavy and symbol-based processes would have to be complemented with modeling-rich and spatial-based possibilities. Importantly, this is not an either-or situation. Taking up modeling as a focus of school mathematics does not negate computation and symbolic manipulation, but such a shift does reposition them as means rather than ends.

It will be interesting to see if and when the culture of school mathematics is able to move in the direction of complexity thinking. The discourse itself suggests that, while a sudden and dramatic shift could happen at any time, it is more likely that the grander system will find ways to maintain its current emphases for some time longer. Caught in a tangle of popular expectation, deep-rooted practice, entrenched curricula, uninterrogated beliefs, and lucrative publishing and testing industries, school mathematics is an exemplar of a complex unity. This insight, more than any other, is the one that sustains my interest. Sooner or later, a well-situated wing flapping will trigger that moment of exponential change through a cascade of transformations that pull school mathematics into a new era.

References

Alberta Education. (2007). *Mathematics – Kindergarten to Grade 9*. Edmonton, AB: Alberta Education.

Bell, E. T. (1937). *Men of mathematics: The lives and achievements of the great mathematicians from Zeno to Poincaré*. New York: Simon and Schuster.

Borwein, J., & Devlin, K. (2008). *The computer as crucible: An introduction to experimental mathematics*. Natick, MA: A.K. Peters.

Cajori, F. (2007). *A history of mathematical notations – Volume 1*. New York: Cosimo.

Davis, B. (2015). Exponentiation: A new basic? *Mathematics Teaching in the Middle School, 21*(1), 34–41.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics teaching and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137–167.

Davis, B., & Simmt, E. (2014). Complexity in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 87–91). Berlin, DE: Springer.

Davis, B., & Simmt, E. (2016). Perspectives on complexity in mathematics learning. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed.). London: Taylor & Francis.

Davis, B., & Sumara, D. (2006). *Complexity and education: Inquiries into learning, teaching, and research*. Mahwah, NJ: Erlbaum.

Eames, C., & Eames, R. (1977). *Powers of ten* (movie). Available at www.powers of10.coms

English, L. D. (2011). Complex learning through cognitively demanding tasks. *Mathematics Enthusiast, 8*(3), 483–506.

Hofstadter, D., & Sander, E. (2013). *Surfaces and essences: Analogy as the fuel and fire of thinking*. New York: Basic Books.

Hoyles, C., & Noss, R. (2008). Next steps in implementing Kaput's research programme. *Educational Studies in Mathematics, 68*(2), 85–94.

Huang, C., & Huang, M. (2012). *The scale of the universe* (interactive Prezi). Available at htwins.net/scale2/

Jacobson, M., & Wilensky, U. (2006). Complex systems in education: Scientific and educational importance and research challenges for the learning sciences. *The Journal of the Learning Sciences, 15*(1), 11–34.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*. New York: Basic Books.

Lesh, R. (2010). The importance of complex systems in K–12 mathematics education. In B. Sriraman & L. D. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 563–566). Berlin: Springer.

Mitchell, M. (2009). *Complexity: A guided tour*. Oxford, UK: Oxford University Press.

Moreno-Armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics: Historical and representational perspectives. *Educational Studies in Mathematics, 68*(2), 99–111.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.

Stewart, I. (1989). *Does God play dice?* Cambridge, MA: Blackwell.

Simmt, E., & Davis, B. (1998). Fractal cards: A space for exploration in geometry and discrete mathematics. *Mathematics Teacher,* 91 (February): 102–108.

Chapter 5
Opening a Space of/for Curriculum: The Learning Garden as Context and Content for Difference in Mathematics Education

Susan Jagger

Abstract The late Ted Aoki playfully identified curriculum as a "weasel word," one that eludes definition, its slipperiness not allowing for it to be pinned down to any one universal meaning (Aoki, 1993). Instead, curriculum is inclusive of all learning contents and contexts; it extends and interacts rhizomatically and without boundaries. The curricular space, that rich space for learning and of learning, is similarly unbounded and endlessly open and interactive. This openness pushes beyond the four walls of the classroom, disrupting and dismantling the very structure of modern understandings of a curriculum that is framed by disciplines and disciplinary spaces. The learning garden grows a space, a space beyond, for such multidisciplinary curricular possibilities. As both content and context, the garden allows for difference to be recognized and realized in the planned and lived curriculum of learners of all ages, and this is particularly true for early years and elementary school students. This paper weaves together the multiple yet inextricably linked garden-based curricular moments of early years and elementary mathematics learning in the garden. Drawing on a participatory research study with elementary school students on their experience of their urban school garden and through shared curricular vignettes, it traces children's organic and situated explorations of number sense and numeration, measurement, geometry and spatial sense, patterns and algebra, and data management and probability in the garden, opening up a space and a place for digging into mathematics concepts and processes and into curriculum itself.

Keywords Curriculum • Elementary education • Mathematics • School garden • Transdisciplinarity

> *An information- and technology-based society requires individuals who are able to think critically about complex issues, analyze and adapt to new situations, solve problems of various kinds, and communicate their thinking effectively. The study of mathematics equips students with knowledge, skills, and habits of mind that are essential for successful and*

S. Jagger (✉)
School of Early Childhood Studies, Ryerson University,
350 Victoria Street, Toronto, ON M5B 2K3, Canada
e-mail: susan.jagger@ryerson.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_5

rewarding participation in such a society. To learn mathematics in a way that will serve them well throughout their lives, students need classroom experiences that help them to develop mathematical understanding; learn important facts, skills, and procedures; develop the ability to apply these procedures; and acquire a positive attitude towards mathematics. (Ontario Ministry of Education, 2005, p. 3)

. . .

The above statement from the Ontario Ministry of Education puts forth a purpose for mathematics, and for mathematics education, that is focused on a society that is founded on and values information and technology. Mathematics learners need to be able to apply what they have learned in the classroom to participate successfully in such a society. While this is certainly important in purpose for both students and educators of mathematics, we must also open up this guiding statement beyond information and technology, and beyond our social community. Indeed many of the *complex issues* and *new situations* that we are now, and will be, challenged by extend beyond our social networks and permeate deeply within our much broader and inherently interconnected environmental communities and, along with informational and technological concerns, present us with ecological *problems of various kinds*. In order to recognize the rhizomatic expanse of our communities, the entangled network of organic threads that quietly connect and support us, and realize our membership within those relations, we must release from structural rigidity our curriculum and instruction and invite into our teaching situated knowledge and understandings, multidisciplinary contents, and organic contexts. We need to unravel the web of assemblages that coalesce around the mathematics curriculum and its teaching and learning in order to understand the materiality of its pedagogical ecologies and formations.

Taking Learning Outside

Being in the garden is always fun because
Being in the classroom is so boring.

—John Cena[1]

The long history of outdoor learning has seen a revival in recent decades as educators are realizing and responding to children's increasingly limited time spent in the natural world and subsequent lack of relationship that children have with the environment characterized by nature-deficit disorder (Louv, 2005). Outdoor and environmental learning's roots can be traced back hundreds of years to the calls from Rousseau, Froebel, Pestalozzi, and Montessori, to name a few, to situate teaching and learning in children's natural everyday and authentic experiences in the world beyond the classroom. To Rousseau, children's lived experiences and interactions with their environments are central in their learning: "we begin to

[1] To ensure confidentiality, all names of participants and places were changed to pseudonyms and participants chose their own pseudonyms.

instruct ourselves when we begin to live" (1911/1966, p. 42). Wilbur Jackman's *Nature Study for Common Schools* (1891) formalized these calls and encouraged taking school children outside to learn across disciplines through first-hand explorations of the natural world. It was the beginning of a holistic and place-based educational movement bringing the learner into intimate and emotional contact with nature. At the same time, Dewey (1902/1966) espoused the centrality of experience in learning and the importance of recognizing those experiences within the curriculum. Nature study was prevalent in early childhood and elementary education well into the 1920s until it was eventually overshadowed by the rigorous rise of a structured and "modernist" education in the 1940s and 1950s. With exceptions in due course, this culture of progress and accountability has since dominated education and in turn directed teaching and learning into standardized models of curriculum and instruction with quantifiable and easily ranked outcomes. This has particularly been the case across the disciplines of mathematics and mathematics education.

Mathematics education can be released from the bounding structure of schooling through the active embrace of the natural environment and the actualization of organic learning experiences in its curriculum and instruction. The context of the outdoor environment can support the development of mathematical content knowledge by supplementing theory with experience. Learning outdoors is a sensual experience; children are invited to touch, see, hear, smell, and sometimes taste elements of the environment, and these stimulations can enhance and enrich knowledge and understandings of mathematics processes and products. By engaging mathematically in everyday situations and settings, learners can become more aware of mathematics in the world and disposed to viewing their environment through a mathematical lens. Learning about mathematics in the outdoors allows children to find a variability of solutions to problems that would otherwise be more challenging and use strategies that would less efficient in another setting. The everyday and known materials and experiences of the outdoors can be the starting off point for questioning, wondering, and exploring mathematical ideas that can then be transferred into other contexts and abstracted (Pratt, 2011).

Mathematics education can benefit from the principles of outdoor and environmental learning that is becoming an increasingly common approach in K–12 education, and one setting that is being embraced by many elementary and secondary schools, particularly in urban areas, is the school garden (Cutter-Mackenzie, 2009; Ozer, 2007; Williams & Brown, 2012). The school garden presents to the learner the living metaphors of garden as environment, garden as community, and garden as transformation (Gaylie, 2009). Together, these organically infuse theory with practice in garden-based ecologies of teaching and learning whose assemblages are found in the experience of nature. Garden as environment, community, and transformation at once supports the development of environmental understandings and the fostering of respectful interactions with place and with each other as well as the deep changes in students through garden-based learning (Gaylie, 2009).

School gardening projects and pedagogies have provided a wide range of benefits to the learner, the school, and the community. Participation in school gardening can be a way for students to learn about their place on many levels and across many systems in a way that can add curricular value and redefine their relationships with the environment. School gardens can provide students and other members of the school community with a pleasing place for learning within a natural setting. Through urban development, transformation of rural areas into suburban sprawl, and fears of physical and personal dangers, many children have lost access to traditional free play environments such as wild spaces, forts, and fields (Malone & Tranter, 2003). School gardens open a much needed space for children to enjoy, appreciate, and contemplate nature and reconnect with and find comfort in the natural environment (Bradley, 1995; Carrier, 2009; Rahm, 2002). Children's learning related to school gardens has spanned disciplinary boundaries and uprooted curriculum organizations and orientations. Teachers at schools with gardens have been able to include outdoor learning opportunities in their planning across the curriculum. Garden-based learning has helped to develop children's science and environmental knowledge, skills, and attitudes as well as language, art, geography, gardening, and cooking skills (Alexander, North, & Hendren, 1995; Carrier, 2009; Cronin-Jones, 2000; Mayer-Smith, Bartosh, & Peterat, 2007; Miller, 2007; Morgan, Hamilton, Bentley, & Myrie, 2009; Skelly & Bradley, 2007). In addition to gains in traditional academic understandings and skills, children can develop positive social and affective learning and development. School gardens can open up unique learning opportunities for collaborative meaning making with peers, community members, and the environment that would not have been realized within the four walls of the traditional classroom (Alexander et al., 1995; Evergreen, 2000; Mayer-Smith et al., 2007; Morgan et al., 2009; Rahm, 2002). Through school gardening and related environmental understandings and attitudes, children can develop an environmental ethic and responsibility and be motivated to respect and take care of others in their environmental community (Carrier, 2009; Morgan et al., 2007; Skelly & Bradley, 2007). This paper extends existing research on school gardening and mathematics learning (see, e.g., Civil, 2007; Clarkson, 2010) as it shares the botanical experience of mathematics of urban elementary school students.

The following four stories share some of the "curricular moments" realized by teaching and learning *about* and *in* the school garden, of the content and context of the garden, the organic spaces grown for mathematical thinking, acting, and being outside of the disciplinary walls of the elementary school classroom. The stories stem from a year-long participatory research project done at City Public School, an urban elementary school in Ontario with well-established school learning gardens and several dedicated teachers regularly bringing garden-based curriculum and pedagogy into their teaching. The stories are created *composite narratives*[2] that weave together direct quotes from participating children and events documented in observational field notes and activity artifacts with fictional text to present the essence of the experience and are invitations to explore and perhaps expand perceptions to reconceptions of possibilities for mathematics education and for curriculum.

[2] Composite narratives bring together the themes, experiences, and context of the research space(s) into a story that invites the reader into the place of inquiry. Fictional elements (e.g., characters, settings, events) are woven together with actual individuals and occurrences into a narrative reflective of the research (see Dawson, 2007; McRobbie & Tobin, 1995; Tippins, Tobin, & Nichols, 1995).

Planting Seeds

I made the flower bed on the other side of the garden,
we made the flower beds.

—Awesome Blue

Mathematical processes can be seen as the processes through which students acquire and apply mathematical knowledge and skills. These processes are interconnected …. The mathematical processes cannot be separated from the knowledge and skills that students acquire throughout the year. Students must problem solve, communicate, reason, reflect, and so on, as they develop the knowledge, the understanding of concepts, and skills required in all the strands in every grade. (Ontario Ministry of Education, 2005, p. 11)

. . .

The grade six students had some work to do in the garden. Following the winter weather and the spring thaw, the raised garden beds were a bit worse for wear and needed to be repaired. Some of the wooden slats framing the beds had separated from each other and become damaged and needed replacing. A small group of students joined April, the garden coordinator, outside to do the repair work.

—"Thank you so much for volunteering to help me out with fixing up the raised beds," said April to the grade six students. "After winter and playground wear and tear, we need to rejoin some slats together. Some we'll need to replace because they have begun to rot. Some are just broken."

—"Cool. We can do that," said The Cool Guy.

—"Yeah, we know how to build raised beds. I remember when we made those beds. And the beds on the other side of the garden. We even made the bricks to make that bed with," recalled Awesome Blue.

—"That's right! That was an awesome field trip to the brick works. OK, let's fix these borders. I want to use the drill," said B4.

B4 and The Cool Guy set to work on fixing the weakened frames by reattaching the wooden slats. The Cool Guy held together the slats while using the long bolts and the drill B4 carefully joined them together.

Awesome Blue and Lara worked with April to measure and cut replacement slats.

—"Well, the pieces of wood that we have are 4 x 4 by 10 feet. Four inches by four inches," said April.

—"OK. Is the bed the length of one piece of wood?" asked Lara.

—"It looks like it," said Awesome Blue. "Can you please hold the end of the measuring tape for me?" Awesome Blue walked the length of the bed and measured it.

—"It's 300 cm. How many feet is that?"

—"A ruler is about a foot long, and it has 30 cm on it. So 10 rulers would be 300 cm. 10 x 30, right?" said Lara. "We don't need to cut anything then."

The girls removed the broken slat and put the new slat in its place. Awesome Blue then walked the width of the bed and measured it to be 150 cm.

—"So we need one piece of wood for the length and then half of a piece of wood for the width," said Lara.

—"Let's double check before cutting, and look as well at how the slats fit together to make the frame," reminded April.

—"OK, the bed measures 300 cm by 150 cm," thought Awesome Blue aloud. "The old slats don't all go to the edge though. Each side is made up of a slat lengthwise and then the end of the other slat. Can you measure the end of the piece of wood?"

—"It is 4 inches, that is what April told us, and…" says Lara, measuring the end of the piece of wood, "that is about 10 cm."

—"So, for the long side of the bed, we'll need a piece of wood that is 300 cm minus 10 cm for the end of the other piece. 290 cm. And for the shorter side, we'll need 150 cm minus 10 cm, so 140 cm," calculated Awesome Blue. "Right?"

Lara and Awesome Blue looked at April.

—"That sounds right to me. Let's measure those lengths and get the wood cut," said April.

—"And good luck getting the drill away from B4 and The Cool Guy!" laughed Lara.

. . .

In this curricular story, the garden afforded a context for practical problem solving. The grade six students needed to fix the garden bed and to do that they had to determine the length of wood to cut. This seemingly simple action required students to actively apply their understandings and skills from across strands of mathematics. Awesome Blue and Lara drew on their measurement skills as they worked with and between different nonstandard (i.e., ruler lengths) and standard units of length (i.e., inches and feet) and systems of measurement (i.e., metric centimeters and imperial inches). To convert units, they applied their number sense and knowledge of number operations as they multiplied to get all measurements in centimeters and thus allow for measurements to be related. Furthermore, Lara applied her own way of converting between feet and centimeters by beginning with the nonstandard unit of a ruler length and then applying her understanding that a ruler length was 1 foot and had about 30 centimeters to calculate that a 10 foot length was equal to 300 centimeters. This garden-based problem solving has also been realized by elementary school students using measurement and geometry content knowledge and related process skills to plan for the watering of their garden during a time of drought (Clarkson, 2010). Students have also used proportion and data management to set up a healthy worm bin to make an organic fertilizer for their garden (Clarkson, 2010). The use of mathematics to complete an authentic and situated task, rather than solve a textbook question, can allow for students to make connections to place and reconnections between abstract and concrete applied mathematics and view and be in the world mathematically.

These grade six students had a unique relationship with this part of their school garden: they helped to actually build the foundation and the beds themselves when they were in the fourth grade. This long-standing connection, as well as their regular class visits to the garden, made a well-known and personally relevant context for learning and applying mathematics concepts and skills. The care and upkeep of the garden opened curricular moments for learning in mathematics and across subject areas that were intrinsically motivated—the students cared for their garden—rather than simply because they were assigned by their teacher. These moments in the familiar space of the garden parallel, and it can even be said that they are, those same situations where students, and then adults, will be bringing together and applying knowledge and skills from across disciplinary boundaries to work with and hopefully solve real and situated problems, issues, and concerns in the environment.

Growing Food for Mathematical Thought

There are these plants that taste really, really good.
—Vintage Beef

Measurement concepts and skills are directly applicable to the world in which students live. Many of these concepts are also developed in other subject areas, such as science, social studies, and physical education. (Ontario Ministry of Education, 2005, p. 9)

. . .

The garden was full of life and ready for a harvest of herbs and vegetables. Thea's grade three class had been learning about healthy living and healthy eating, and Thea decided that it was a good time for her students to have a farm-to-table, garden-to-classroom, meal. She took a group of students down to the garden in front of the school.

—"There are a lot of plants that are ready to harvest. In fact, except for a few ingredients, we will be able to pick our snack today. We'll be making tabbouleh. How many of you have had tabbouleh before?" asked Thea.

Several students raised their hands.

—"My dad makes it," Vintage Beef said, "It is soooooo good!"

Bautista looked a little hesitant. He had never heard of tabbouleh before and wasn't too sure about vegetables in general.

—"Bautista, can you work with Vintage Beef and Alice to pick lots of sour leaf[3] and parsley? We'll need about three cups of leaves," asked Thea. "Mighty Robot and Marinda, would you please pick a nice bunch of mint? Brett, you and I will pick some cherry tomatoes. There are lots of bright red ripe ones that will be perfect for our tabbouleh."

[3] *Sorrel* was commonly, and interchangeably, identified as *sour leaf* by students and teachers.

Bautista followed Vintage Beef and Alice to the patches of sour leaf and parsley. Vintage Beef immediately started to pick and eat sour leaves. Together, the three students filled a bowl with sour leaf and curly and flat leaf parsley and then rejoined Ms. D and the other students to head back into class to make the tabbouleh.

—“Before we do anything, we all need to wash our hands. Then I will get Alice and Brett to fill the sink to wash our tomatoes and greens. Marinda, will you please give the mint and lemon balm a good cleaning?” asked Thea.

Vintage Beef and Bautista joined Thea at the big work table where she was unpacking some other ingredients and a set of measuring cups and spoons.

—“Here is my recipe, but we are going to double it so there is enough for everyone to have some for snack.”

1 cup cooked quinoa, 1 ½ cups parsley + sorrel mix, 1 cup tomatoes, 5 green onions, 2 tablespoons mint, ¼ cup olive oil, ¼ cup lemon juice, ¾ teaspoon salt, ¼ teaspoon pepper

—“I can do that. My dad cooks with me at home,” said Vintage Beef. “The quinoa and tomatoes are easy. Doubling one cup makes two cups. And 5 green onions plus 5 green onions is 10 green onions.”

—“And 2 tablespoons of mint doubled is 4 tablespoons,” figured Bautista.

Vintage Beef next drew some circles and divided them each into four parts.

—“OK, so if each of these parts is ¼, the two quarters is the same as half of the circle. So we need ½ cup of olive oil and ½ cup of lemon juice and ½ teaspoon of pepper.”

—“If we double the parsley and sorrel, we need 2 cups and 2 half cups. And two halves make a whole. So that is 3 cups altogether,” added Bautista. “But what about the ¾ teaspoons of salt?”

—“We’ll just double the ¾ like everything else. So 3 parts plus 3 parts is 6 parts. Four parts make one whole, and then there are two parts left. And 2 quarters is one half. So we need one whole and one half teaspoon of salt.”

Thea got out a big bowl for the tabbouleh and a small bowl to mix together the dressing.

—“Thank you, Vintage Beef and Bautista, for doubling our recipe. Mighty Robot and Brett, can you help me with the dressing?” asked Thea. “The rest of you can chop the green onions and tear the greens and mint into pieces. Then you can go ahead and start measuring out those ingredients into the big bowl. Here is some cooked quinoa to add.”

Working together, the grade threes chopped and carefully measured the ingredients of the tabbouleh. Mighty Robot poured the dressing over the salad and Marinda gave it a careful stir and scooped spoonfuls onto everyone’s plate.

—“Let’s eat!” said Vintage Beef.

Bautista was a little hesitant. He did like the sour leaf on its own but all of those tomatoes, he wasn’t sure how much he liked tomatoes. To his surprise, he really liked the tabbouleh, so much so that he asked for another spoon of the salad.

And he got a piece of paper to write down the recipe. He wanted to teach his mom how to make it.

. . .

Food is a shared way of including the garden and garden-based pedagogy across the curriculum while also meeting provincial curricular requirements. Thea's students' study of food included cooking and, inherent to this, mathematics content (e.g., number sense, measurement) and process skills (e.g., counting, weighing). Along with engaging students in tasks rich in mathematics concepts and skills applications, food explorations are directly related to grade three curriculum expectations in science [e.g., "assess ways in which plants have an impact on society and the environment, and ways in which human activity has an impact on plants and plant habitats" (Ontario Ministry of Education, 2007, p. 70)] and social studies [e.g., "compare ways of life among specific groups in Canada around the beginning of the nineteenth century, and describe some changes between that era and the present day" (Ontario Ministry of Education, 2013, p. 86)]. Cooking also opened up learning opportunities related to healthy living and character education.

Many have highlighted the need for issues of food, food security and insecurity, and health to be infused throughout the curriculum (see, e.g., Gaylie, 2011; Williams & Brown, 2012) to support sustainability and well-being on many levels—personal, social, cultural, economic, and ecological. As content and context for the study of food, the school garden provides important opportunities for students and teachers to reconnect with what they eat through sensory experiences of food; students smelled, touched, and tasted the fruits, vegetables, and herbs grown in the garden. Studies have suggested that active participation in gardening can increase the likelihood that children willingly eat more vegetables (see, e.g., Parmer, Salisbury-Glennon, Shannon, & Struempler, 2009). Students were also able to realize and reconnect to the path that their food took, from seed to plant to meal; they were a part of the physical foodway of the produce produced in the garden as they planted seeds, transplanted seedlings, tended to the growing plants, and prepared and ate food from the garden.

The garden, and harvesting and cooking with its produce, opens up a bridge between academic mathematics and everyday mathematics (Civil, 2007). School gardening is a unique pedagogical approach that grows in the space in-between out-of-school learning and classroom curriculum and instruction. Much garden learning comes forth through apprenticeship and contextualized exploration. Teaching and learning in the school garden often (and perhaps should more often) take root in students' experiences, interests, and wonderings; teachable moments pop up and can bloom in the garden. Mathematics in out-of-school learning tends to be hidden despite its ever-presence in everyday situations (Civil, 2007). Here, opportunities for math exploration can be overlooked and those teachable moments in mathematics missed. The garden's curricular and pedagogical richness allows for students and teachers to dig into those moments and uproot the hidden math of the garden, making explicit the implicit.

Nurturing Mathematical Inquiry

I let it sit in the sunshine…
I watered it and I always kept an eye on it
and I let it have free air.

—Ruby

One of the central themes in mathematics is the study of patterns and relationships. This study requires students to recognize, describe, and generalize patterns and to build mathematical models to simulate the behaviour of real-world phenomena that exhibit observable patterns. (Ontario Ministry of Education, 2005, p. 9)

. . .

Alex's kindergarten class had been learning about spring and the changes that happen to plants and animals when seasons change. The children had been growing beans from a seed in only a moist paper towel in a plastic freezer bag. The beans had sprouted and Ruby asked Alex how big the beans would get if they were taken out of the plastic bags and planted.

—"Let's find out," said Alex. "What do you think these little beans need to grow tall and strong?"

—"Milk! My mum says I have to drink milk to be strong and have strong bones," called out Clint.

—"But you're not a plant, Clint," said Ruby. "Plants need sunlight and water and love."

—"Maybe milk will help to grow Clint's bean plant tall and strong. Why don't you try that, Clint? And Ruby, you make sure that your plant has sunlight and water and love."

Determined to grow the tallest bean plant, Ruby set to work. She filled a small pot with soil and carefully planted her bean plant into a hole she made with her finger. She gently pressed the soil down around the bean—she had seen her grandfather do this when he planted seeds in his garden in the springtime—making sure to not break the delicate young sprout. Ruby then watered the plant and placed the pot on the windowsill of the classroom. Each day she checked on her growing bean plant, and when she felt that the soil was getting dry, she watered it. Using the colored cubes, Ruby measured how tall her bean plant was by counting up the cubes and then recording its height on her clipboard.

When a few weeks had passed, the children were excited to find out whose bean plant was the tallest. Ruby's plant was second tallest in the class. Only Omar's bean plant was taller.

—"How did you get the bean plant to be so tall, Ruby? What sorts of things did you do to help it grow?" asked Alex.

—"I let it sit in the sunshine in the pot. I watered it and I always let it have free air."

—“OK, so sunshine and water and free air. Is this fresh air?”

—“Yes, fresh clean air.”

—“Yes! How about you, Omar? What did you do?” asked Alex.

—“Water and sunshine, too,” said Omar.

—“Clint, tell me about your bean plant. How did your plant do?”

—“It didn’t grow very tall. I kept it in my cubby and put milk on it.”

—“Why do you think it didn’t grow very tall?” questioned Alex.

—“Well, my cubby is kind of small. There isn’t much room to grow. And it is not by the window so there is no sunshine. I don’t think the milk helped it to grow like it helps me to grow,” said Clint.

—“Maybe the more sunshine and water a plant gets, the taller it grows,” predicted Ruby.

—“That sounds like a good conclusion, Ruby,” said Alex.

—“I’ll bet if I used soy milk my bean plant would grow,” Clint thought aloud.

. . .

The kindergartners’ exploration of plant growth and growth rates and plants’ needs was an organic bringing together of the curricular disciplines of science and mathematics through an inquiry-based approach. Inquiry-based teaching and learning opens up the learning experience and process to the learner as they are invited to work with ideas, concepts, and understandings in a way similar to the thinking and working of scientists and mathematicians (Artigue & Blomhøj, 2013). Entangled with and in the rhizomatic roots from which nature study is grounded, inquiry-based learning can be formally traced, as can nature study, back to Dewey and his consideration of the interaction and change of elements of an environment in relation to each other. He described inquiry as “the controlled or directed transformation of an indeterminate situation into one that is as determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole” (Dewey, 1938, p. 108). With a foundational rooting in the inherent relations and connectedness of contextual elements, inquiry brings the learner into this interplay in which their reflective and recursive exploration of interacting elements allows for their own reflective and recursive learning. Inquiry can open up the curriculum to a multiplicity of engagements and explorations.

Alex opened up his kindergarten students’ learning experience and allowed Ruby’s initial wondering about how big their bean seedlings would grow when they were planted to guide their inquiry. His questioning and openness to possibility in the activity’s unfolding invited his students to draw from their prior knowledge from school and from home. Here, we see Ruby and Clint both making connections to what they already know about requirements for healthy growth as Ruby shares that plants need sunlight, water, and care, and Clint recalls that milk helps people to grow strong. Ruby further follows her grandfather's careful gardening as she gently plants her bean seedling like she has seen him do in his garden in the spring. The students' academic understandings together with their everyday knowledge helped to inform their inquiry.

Like the grade threes and their cooking, the mathematics that the kindergarten students explored and applied as they grew their bean plants—measurement, data collection and management, number sense and numeration, and patterns and relations—was "situated, dilemma-driven" (Lave, 1992) and personally motivated. Students wanted to grow the tallest plants and, to do so, had to determine how to ensure the needs of the growing plant were best met. Mathematics was inherent in this component of the students' inquiry process. Students were actively applying the number sense and measurement understandings and skills as they measured and compared heights of their bean plants ["demonstrate an understanding of numbers, using concrete materials to explore and investigate counting, quantity, and number relationships," "measure, using non-standard units of the same size, and compare objects, materials, and spaces in terms of their lengths…" (Ontario Ministry of Education, 2016, p. 181)]. Also in their exploration of plant growth and factors to supporting growth, students used data displays and interpretation in their inquiry to determine the optimal growing conditions for the bean plants ["collect, organize, display, and interpret data to solve problems and to communicate information…" (Ontario Ministry of Education, 2016, p. 182). The mathematics that was inherent to the students' exploration allowed them to find answers to their wonderings and nurture and grow their mathematical knowledge and skills while enriching their understanding of plants.

Exploring Curricular Diversity

If you, like, learn about it,
you know what it needs,
what stuff helps it grow.

– John Cena

The related topics of data management and probability are highly relevant to everyday life. Graphs and statistics bombard the public in advertising, opinion polls, population trends, reliability estimates, descriptions of discoveries by scientists, and estimates of health risks, to name just a few. (Ontario Ministry of Education, 2005, p. 9)

. . .

The raised beds in front of City Public School needed to be thinned as they were becoming overgrown with sweet grass (*Hierochloe odorata*). Sweet grass is a plant native to North America and has many traditional uses by Aboriginal peoples. The grass is one of the four sacred medicines used ceremonially and spiritually; the others are cedar, sage, and tobacco. Sweet grass represents the hair of Mother

Earth, and when it is braided, its three strands symbolize the love, honesty, and kindness of the earth. The sweet grass at City was going to be thinned out and shared with a neighboring school community that wished to add sweet grass to their own school garden.

Sidney used this gardening opportunity to bring together mathematics and science with hands on gardening as her grade six students looked at the biodiversity of species in the raised beds using a study of quadrats. The students gathered by the native plant garden in front of the school.

—"We've been studying biodiversity in science and today we'll start to look at the biodiversity in our native plant garden," introduced Sidney. "Let's work together in groups of four and use quadrats that we learned about last class to divide up the garden. With your groups, I would like you to plan and record how you will collect data in your biodiversity study and then do your study. Then you'll need to analyze your data. What does it tell you about the biodiversity of this garden? Remember that biodiversity includes the number of different species and the number of individuals of those species. When we put out data together, we'll have a good idea of the diversity of species in the garden. I have brought out a lot of materials and some identification guides for you to use. OK, let's see how diverse our native garden is."

Dynamite and his group had a look through the materials and chose a magnifying glass, 4 m loop of rope, 4 tagged pegs, a measuring tape, sheets of graph paper, pencils, and a camera. Using the pegs and loop of rope, they marked a 1 meter square of the garden to explore.

—"We used the graph paper last class when we learned about quadrats to figure out percentages. We can imagine that our square is a 10 x 10 grid and mark the squares on graph paper with different plants," suggested Dynamite.

—"OK," said Adam. "How will we collect data? Let me write it down. We've already measured and marked our quadrat with the rope. Maybe we should first figure out how many different plants there are."

—"Don't forget animals and mushrooms. The fungus among us," laughed Dynamite.

The students looked at their garden square and found that about one third of the square was covered with sweet grass.

—"Almost all of the rest is just soil. Except for a dandelion and maybe a strawberry, I think that is what this is," said Adam.

—"So, we need one square on the graph for dandelion," said Dynamite, "two for the strawberry, and 33 and one-third for sweet grass. And the rest blank. Let's take a picture of our square so we remember what it looks like and we can ask Sidney if this is a strawberry plant."

—"Class, it is getting close to lunch," noted Sidney. "I think everyone has figured out how many different types of species are in their quadrats. Purple Roses, could you share with us what plants you found and the percentages that each covered?"

—"Sure. Our group had lots of sweet grass, about 95%. And we had about 2% dandelion and 3% another one that we didn't know," said Purple Roses, pointed to the unknown plant.

—"OK, so mostly sweet grass. Laura, what about your group? Did you have mostly sweet grass as well?" asked Sidney.

—"No, we had just under half sweet grass, 47%, 7% clover, 2% of the same unknown plant that Purple Roses had, 4% of another unknown, 8% twigs, and 32% soil. It was more diverse than Purple Roses' square," shared Laura.

—"Ours was one third sweet grass, 1% dandelion, and 2% strawberry, we think," said Dynamite. "And the rest was just bare soil. We had the same number of species as Purple Roses' group. But way less sweet grass."

—"I agree," said Sidney. "Let's leave our quadrats marked through lunch and we will come out this afternoon to do our counts of the individual species."

—"Our square might not have as many species as Laura's group but at least we don't have as much sweet grass as Purple Roses' group. That will take forever to count!" laughed Adam.

. . .

The biodiversity study of one of the school garden beds presented a realization of the weaving together of mathematical strands. The students' application of mathematical understandings and process skills related to number sense and numeration and data management were essential to their examination of garden biodiversity. Students initially organized their data and recorded their observations in a 10x10 grid that was a scale representation of their one square metre garden quadrat ["collect and organize discrete or continuous primary data and secondary data and display the data using charts and graphs..." (Ontario Ministry of Education, 2005, p. 88)]. Recognizing that the 10x10 grid presented 100 squares, they were then easily able to interpret their data in terms of percent coverage of each species in the garden, which ["demonstrate an understanding of relationships between percent, ratio, and unit rate" (Ontario Ministry of Education, 2005, p. 88) and "read, describe, and interpret data, and explain relationships between sets of data" (Ontario Ministry of Education, 2005, p. 95)]. Content areas in mathematics always and inherently inform each other in understandings and applications.

Stemming from the practical need to thin and desire to share the sweet grass that was dominating the native plant garden, curricular boundaries between mathematics, science, and social studies were blended as the diversity of species was explored. The grade six students were able to apply their knowledge of the prescribed science curriculum, ["demonstrate an understanding of biodiversity as the variety of life on earth…," "describe ways in which biodiversity within species is important for maintaining the resilience of those species," and "describe ways in which biodiversity within and among communities is important for maintaining the resilience of these communities" (Ontario Ministry of Education, 2007, p. 114)] through their engagement with and in the learning garden context. The study also drew from understandings of indigenous cultures and knowledge as students restored balance in the garden and gifted plants (here, the culturally significant sweet grass) to other school communities and their learning gardens ["assess contributions to Canadian identity made by various groups and by various features of Canadian communities and features" (Ontario Ministry of Education, 2013, p. 118)]. In the grade six students' biodiversity study, the garden provided a curricular context for realizing the rhizomatic relation of mathematics, science, and social studies knowledge and skills.

Mathematics in the learning garden supported students' care for the garden and the garden's health. By viewing the garden through a mathematical lens, students were able to recognize the dominance of one species, the broadly spreading sweet grass, and how this dominance impacted the diversity of the garden. They saw how the spread of the sweet grass made it difficult for other plants to obtain food, water, sunlight, and space, and that this influenced the health of the garden. The students took positive action by removing and sharing some of the sweet grass plants with another school and their learning garden. The application of mathematics knowledge and skills, along with science and social studies understandings, allowed the grade six students to engage with an authentic and situated exploration of biodiversity. This garden experience is a starting point for further thinking and learning about and taking action to plant, grow, and nurture diversity and ecological health and well-being in other contexts.

The Garden as an Opening of Curriculum

I think it's in our curriculum or something
to go outside and garden or stuff.

– Cherry

*Although no overall or specific expectations explicitly address environmental education, in each of the strands the learning context could be used to foster in students the development of environmental understanding (e.g., problems relating to climate or waste management could be the focus of problem solving). In addition, the mathematical processes (e.g., problem solving, connecting) address skills that can be used to support the development of environmental literacy. (*Ontario Ministry of Education, 2011*, p. 18)*

. . .

The time spent learning in the school garden was time spent in a space both outside of and within the overlapping space of difference of many structural contexts. Learning about the garden, in the garden, and with the garden grows curriculum well beyond the binding of prescribed learning outcomes, the siloing of disciplines, and the confining structures of school and schooling. It was outside of the structure of the classroom: the four walls, the desks and chairs, and the rules and routines. The movement from the classroom made room for the class to open another space for experiencing the curriculum. And here, the curriculum that was experienced was outside, outside of the framing boundaries of the structured and prescribed curriculum. Mathematics, along with science, social studies, and literacy, blurred into a decentered curriculum that recentered on the centrality of the sensuous experience of place within the context of the learning garden. As such, "the garden [became] part of the curriculum and a vehicle through which academic content [was] elaborated" (Richardson, 2011, p. 117). Sights, smells, and sounds of the garden framed the experience rather than the exclusive direction of the prescribed and disciplinary objectives. Students and teachers were able to work with and in the structure of the prescribed curriculum while at the same time extending beyond, disrupting the disciplinary curricular framework. Their learning and teaching met curricular requirements in mathematics, science, and social studies, but did so through organic and inherently multidisciplinary ideas. This was a different way of knowing the garden, of knowing place, and of knowing home and one that opened up the impossibility of a curricular metanarrative, a single and unquestioned structure for teaching and learning, and the possibility of a multiplicity of narratives and experiences of and in place of lived meanings. Aoki (1993) reflected on this curricular multiplicity, noting that "it is time not to reject but to decenter the modernist-laden curricular landscape and to replace it with the C&C landscape that accommodates lived meanings, thereby legitimating thoughtful everyday narratives" (p. 263). It is a call for the recognition of both the lived curriculum and the curriculum-as-plan, for an upsetting of the dominance of curriculum-as-plan to allow for the acknowledgment of the lived curriculum that quietly persists, and that is unspoken, always and already there.

Curriculum is a "weasel word" (Aoki, 1993). It eludes definition and its slipperiness does not allow it to be pinned down. It instead plays in and out of content and contexts. It climbs above, burrows below, and navigates around structural borders. It tangles threads so carefully and meticulously separated. So rather than struggle to restrain the weasel's movement, let us instead embrace the ease and fluidity with which it moves. Let us marvel in how it twists and turns and ties together those seemingly separate disciplines. Let us allow the weasel to guide us in play with and in the open and interactive rhizomatic network of organic transdisciplinary possibilities of difference in teaching and learning.

. . .

> *Mathematics is a powerful learning tool. As students identify relationships between mathematical concepts and everyday situations and make connections between mathematics and other subjects, they develop the ability to use mathematics to extend and apply their knowledge in other curriculum areas, including science, music, and language.* (Ontario Ministry of Education, 2005, p. 3)

References

Alexander, J., North, M., & Hendren, D. (1995). Master Gardener Classroom Garden Project: An evaluation of the benefits to children. *Children's Environments, 12*(2), 123–133.

Aoki, T. (1993). Legitimating lived curriculum: Towards a curricular landscape of multiplicity. *Journal of Curriculum and Supervision, 8*(3), 255–268.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM, 45*(6), 797–810.

Bradley, L. K. (1995). Terra Buena: The creation of an urban wildlife habitat in an elementary school in the inner city. *Children's Environments, 12*(2), 102–110.

Carrier, S. (2009). Environmental learning in the schoolyard: Learning styles and gender. *Journal of Environmental Education, 40*(3), 2–12.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. S. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 105–117). New York: Teachers College Press.

Clarkson, P. (2010). Mathematics and water in the garden: Weaving mathematics into the students' lived environment. *Australian Primary Mathematics Classroom, 15*(1), 11–14.

Cronin-Jones, L. L. (2000). The effectiveness of schoolyards as sites for elementary science instruction. *School Science and Mathematics, 100*(4), 203–211.

Cutter-Mackenzie, A. (2009). Multicultural school gardens: Creating engaging garden spaces in learning about language, culture, and environment. *Canadian Journal of Environmental Education, 14*, 122–135.

Dawson, V. (2007). Exploring students' learning experiences through narrative tales. In P. C. Taylor & J. Wallace (Eds.), *Contemporary qualitative research: Exemplars for science and mathematics education* (pp. 83–91). Dordrecht: Springer.

Dewey, J. (1902/1966). *The child and the curriculum*. University of Chicago Press.

Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Holt.

Evergreen. (2000). *Stewards and storytellers: The greening of British Columbia school grounds*. Toronto, ON: Author.

Gaylie, V. (2009). *The learning garden: Ecology, teaching, and transformation*. New York: Peter Lang.

Gaylie, V. (2011). *Roots and research in urban school gardens*. New York: Peter Lang.

Jackman, W. (1891). *Nature study for common schools*. New York: Henry Holt and Company.

Lave, J. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (Eds.), *Context and cognition: Ways of learning and knowing* (pp. 74–92). New York: Harvester Wheatsheaf.

Louv, R. (2005). *Last child in the woods: Saving our children from nature-deficit disorder*. Chapel Hill, NC: Algonquin Books.

Malone, K., & Tranter, P. J. (2003). School grounds as sites for learning: Making the most of environmental opportunities. *Environmental Education Research, 9*(3), 283–303.

Mayer-Smith, J., Bartosh, O., & Peterat, L. (2007). Teaming children with elders to grow food and environmental consciousness. *Applied Environmental Education and Communication, 6*, 77–85.

McRobbie, C., & Tobin, K. (1995). Restraints to reform: The congruence of teacher and student actions in a chemistry classroom. *Journal of Research in Science Teaching, 32*(4), 373–385.

Miller, D. (2007). The seeds of learning: Young children develop important skills through their gardening activities at a midwestern early education program. *Applied Environmental Education and Communication, 6*, 49–66.

Morgan, S. C., Hamilton, S. L., Bentley, M. L., & Myrie, S. (2009). Environmental education in botanic gardens: Exploring Brooklyn Botanic Garden's Project Green Reach. *Journal of Environmental Education, 40*(4), 35–52.

Ontario Ministry of Education. (2005). *The Ontario curriculum grades 1–8 mathematics.* Toronto, ON: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf

Ontario Ministry of Education. (2007). *The Ontario curriculum grades 1–8 science and technology*. Toronto, ON: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/scientec18currb.pdf

Ontario Ministry of Education. (2011). *Environmental education scope and sequence of expectations*. Toronto, ON: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/environ18curr.pdf

Ontario Ministry of Education. (2013). *The Ontario curriculum social studies grades 1 to 6; history and geography grades 7 and 8*. Toronto, ON: Queen's Printer. Retrieved from http://www.edu.gov.on.ca/eng/curriculum/elementary/sshg18curr2013.pdf

Ontario Ministry of Education. (2016). *The kindergarten program*. Toronto, ON: Queen's Printer. Retrieved from https://files.ontario.ca/books/kindergarten-program-en.pdf?_ga=1.205142672.1699203898.1474124060

Ozer, E. J. (2007). The effect of school gardens on students and schools: Conceptualization and consideration for maximizing healthy development. *Health Education and Behaviour, 34*(6), 846–863.

Parmer, S., Salisbury-Glennon, J., Shannon, D., & Struempler, B. (2009). School gardens: An experiential approach for a nutrition education program to increase fruit and vegetable knowledge, preference, and consumption among second-grade students. *Journal of Nutrition Education and Behaviour, 41*(3), 212–217.

Pratt, N. (2011). Mathematics outside the classroom. In S. Waite (Ed.), *Children learning outside the classroom: From birth to eleven* (pp. 80–93). Los Angeles: Sage.

Rahm, J. (2002). Emergent learning opportunities in an inner-city youth gardening program. *Journal of Research in Science Teaching, 39*(2), 164–184.

Richardson, T. (2011). At the garden gate: Community building through food: Revisiting the critique of "food, folk, and fun" in multicultural education. *Urban Review, 43*, 107–123.

Rousseau, J.-J. (1911/1966). *Émile: Or, on education* (B. Foxley, Trans.). London: J. M. Dent and Sons.

Skelly, S. M., & Bradley, J. C. (2007). The growing phenomenon of school gardens: Measuring their variation and their effect on students' sense of responsibility and attitudes toward science and the environment. *Applied Environmental Education and Communication, 6*, 97–104.

Tippins, D., Tobin, K., & Nichols, S. (1995). A constructivist approach to change in elementary science teaching and learning. *Research in Science Education, 25*(2), 135–149.

Williams, D., & Brown, J. (2012). *Learning gardens and sustainability education: Bringing life to schools and schools to life*. New York: Routledge.

Part III
Mathematics for the Common Good

Chapter 6
Transdisciplinarity, Critical Mathematics Education, Eco-justice, and the Politics to Come

Nenad Radakovic, Travis Weiland, and Jesse Bazzul

Abstract This chapter proposes a vision of transdisciplinary mathematics education that takes into account the sociopolitical nature of mathematics education and approaches to sustainability that go beyond the savior status of mathematics. In the three sections of the chapter, we discuss transdisciplinarity by positioning mathematics as equal partner with other disciplines and worldviews, argue that mathematics and mathematics education should be also viewed as perpetrators in the sustainability and social justice discourse, and explore transdisciplinary mathematics education for sustainability in pedagogical settings. Finally, we offer a list of possible discussion questions for educators considering the topic of food waste.

Keywords Transdisciplinarity • Mathematics education for sustainability • Environmental education • Eco-justice • Sociopolitical turn in mathematics education, critical math education

The goal of this chapter is to put forth a vision of mathematics education for sustainability that acknowledges the sociopolitical turn in mathematics education, underscores the importance of resisting the neoliberal tendency of mainstream interdisciplinary approaches such as STEM, and goes beyond the savior status of mathematics education in addressing contemporary issues and crises. The authors

N. Radakovic (✉)
Department of Teacher Education, College of Charleston,
66 George Street, Charleston, SC 29424, USA
e-mail: radakovicn@cofc.edu

T. Weiland
Department of Mathematical Sciences, Appalachian State University,
342 Walker Hall, 121 Bodenheimer Dr, Boone, NC 28608, USA

J. Bazzul
Faculty of Education, University of Regina,
3737 Waskana Pkwy, Regina, SK S4S 0A2, Canada

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_6

(Jesse, Nenad, and Travis) come from different backgrounds: Jesse is a science and environmental educator, Nenad is a mathematics educator whose primary research goal evolved from the advancement of students' disciplinary knowledge to activism and social justice, and Travis is interested in the intersection of critical and statistical literacies in the K–12 setting.

This chapter includes three sections. In the first section, we stress the importance of transdisciplinarity as well as dethroning mathematics and mathematics education, making them equal partners with other disciplines in the discussion of sustainability. We also underscore the importance of taking a sociopolitical turn in mathematics education (Gutiérrez, 2013). In the second section, we talk about the importance of moving away from the savior status of mathematics in relation to environmental problems, instead shifting toward an understanding of the ways that mathematics has played the role of perpetrator in environmental crimes. Finally, we move toward pedagogical considerations by envisioning what mathematics education for sustainability could look like in practice.

Transdisciplinarity, Sustainability, and Mathematics Education

A transdisciplinary mathematics perspective is a necessary requirement if any kind of robust, widespread environmental education agenda is to take shape in schools across the globe, quite simply because all aspects of living in the world involve interconnectedness on a scale educators are only beginning to understand. Achieving any kind of transdisciplinarity related to sustainability, eco-justice, and environmental education means entangling politics, eco-justice, ethics, and mathematics education together in a way such that educators would have to be irresponsible and/or very uninformed to try and untangle them. Sadly, STEM education has unequivocally placed short-term economic gain over the needs of the vast majority of human and nonhuman communities (Pierce, 2012, 2015); therefore, it comes as no surprise that it is difficult, both practically and politically, to begin with mathematics education and then move toward transdisciplinary (although we try!). The route to integration, interaction, and recombination is altogether different depending on where an educational community chooses to begin, its philosophical/cultural outlook, and its ethical and political values—in other words, sociopolitical context and subjectivity matter (Bazzul, 2016). If mathematics is colloquially considered by many as the "queen of the sciences," its flattening-out may provoke "a loud fall from epistemic grace." More importantly, however, making mathematics and mathematics education one equal partner among many ways of doing and knowing will be highly productive as the discipline assumes a primary place in the scope of human activity that works to make the world a more ecologically and socially just place. Like other disciplines, mathematics education is undergoing a sociopolitical turn (Gutiérrez, 2013). According to Gutiérrez, the sociopolitical turn in mathematics education entails positioning social struggles (e.g., against white supremacy and global economic hegemony) and systemic configurations of power—what Hardt and Negri (2009) call *Empire*—at the center of pedagogy and research. Gutiérrez (2013) captured the revolutionary spirit of this turn when she declared that "any resistance to the sociopolitical turn is a form of hegemony" (p. 22).

It may be easier for an outsider to the mathematics education community to see this. For example, the life sciences, and consequently science education, are embodied in the body politic through institutions such as schooling and public health as well as the integration of biology into daily life through popular media (Rabinow & Rose, 2006). For the sciences, a turn toward embracing multiple ways of viewing the material world may be easier simply because both politics (the institutions that give science its legitimacy) and materiality (the substrate for all scientific knowledge) are a little bit closer and their relations slightly more visible. Since mathematics and mathematics education have traditionally been tied to abstraction, refocus of math education toward addressing unjust material conditions in the form of environmental education—and away from career preparation and depoliticized knowledge—will likely be more noticeable. The turning of mathematics curricula and pedagogy toward politics, such as the politicization of finance, risk, and big data, will be a strategic site of engagement, where the power of quantification will be turned from the service of hegemonic power toward meeting the needs of communities.

To this end, "the queen" (mathematics) will have to put itself level with other disciplines and, in many cases, defer to other disciplines. In mathematics education research, environmental education seems not to be taken too seriously, save the work of a few scholars (e.g., Barwell, 2013; Renert, 2011, Renert & Davis 2012; Sriraman & Knott, 2009). Educators can begin a criticism of mathematics education like Lynn White (1996) did in the essay "The Historical Roots of Our Ecological Crisis" simply by admitting that certain assumptions (e.g., that humans are separate and superior to nature) and institutions of power have allowed educational institutions to be complicit with the destruction of our shared world. Yet one of White's not-so-subtle points is that the plethora of already existing myths that hold meaning for us can be combined and retold to suit life better, rather than blindly following ideologies and belief systems only known to exacerbate the ecological problems faced by communities exposed to environmental destruction and inequality. The same is true for educational life, especially for the indigenous theories, philosophies, and narratives that have existed in many places for millennia (McGregor, 2005). Indigenous knowledges and eco-justice have much in common, as they both have at their center the notions of relationality, dependence, and intersectionality. They also stress the relationship between the natural world, values, meaning, and the politics of existence. Thus, given the way knowledge systems are constructed, their resulting social and ecological consequences matter (Deloria & Wildcat, 2001).

Humans have irreversibly altered Earth's geology and ecosystems, which is manifesting itself in the sixth mass extinction (species extinction rate being 100 to 1000 times higher than normal). This has prompted a proposal to call this new epoch the

Anthropocene (Lewis & Maslin, 2015). What makes the declaration of the Anthropocene by scientists most relevant to the dissolution of disciplinary boundaries in mathematics education is the blurring of the nature/culture divide; notably, this blurring is coming from the side of science. Events once thought to be historical, cultural, and political, such as the eradication of indigenous peoples by Europeans in places like North America or the refinement of postwar production, can now also be seen as events that permanently mark Earth's geological and biodiversity record or, more provocatively, as geological and biological events. Transdisciplinarity is needed to meet the demands of the Anthropocene (Lloro-Bidart, 2015). As Latour (2004) noted, for too long have fields like science been kept from politics and politics and the social world kept from the study of science.

Like many others, we feel that mathematics and mathematics education should also not be kept from engaging politics and the social world (Frankenstein, 1994; Gutiérrez, 2013; Gutstein & Peterson, 2013; Wager & Stinson, 2012), ecology, or the study of nature. In this way, we endeavor to think about what a transdisciplinary way of relating entails and then provide some examples of how environmental and mathematics education can be brought together in the hopes of fostering deeper connection and care for our shared world. To think ecologically is also to think in a transdisciplinary way that involves relationships molded by strong historical forces that must constantly be negotiated.

As many environmental educators have pointed out, social inequality and radical change cannot be delivered through statements like "more should be done" or "I need to change my habits" (Henderson & Hursh, 2014; McKenzie, 2012). Mathematics education that covers the curriculum to meet the needs of unjust labor and commodity markets of a globalized world is more than short-sighted; it is, collectively speaking, morally and ethically reprehensible. We are wary of seeing transdisciplinary environmental education initiatives within fields like mathematics education leave traditional curriculum and pedagogy untransformed. Traditional curricula and pedagogy prepare students to exist uncritically in a world governed by racial, sex/gender, and economic hierarchies rather than care for and produce a sustainable world in common.

Our intention is not to argue for a form of sustainability or environmental education that is "correct," as there are many currents in environmental education to consider (naturalist, feminist, bioregionalist, holistic, etc.) (Sauvé, 2005). However, we recognize that an eco-literate mathematics education requires different knowledge systems—and different social activist movements—speaking with each other for the betterment of the planet. Fostering transdisciplinarity and entangling eco-literacy in mathematics education will likely involve a revolution in values that will require educators to question authority and boundaries. Environmental educator Marcia McKenzie (2009) emphasized that transgressing boundaries or limits is not about exploding past them (if only it were that easy!), but mostly about illuminating limits. For McKenzie, borderland pedagogies are vital for transdisciplinary and critical work in education. Of course, environmental and eco-mathematics education will take place under the constraints of strong corporate and governmental political forces that want to push/force the marketization of schooling. Therefore, it is important to exercise a certain degree of caution concerning what sustainability or environmental education entails (Jickling & Wals, 2008). In other words, does our use of terms like sustainability allow us to enter into disagreement or implement radical new practices of what this could entail? If not, are educators engaging in sustainability education for "Big Brother" (Jickling, 2001; Jickling & Wals, 2008)? To engage in sustainability education, as Orr (2004) asserted, communities will need to define what exactly they want to sustain.

Mathematics and Environmental Sustainability: Understanding and Accepting the Perpetrator Status

There is an understandable impulse for mathematics educators (particularly those whose work is centered around the notion of social justice) to embed sustainability into mathematics education research and practice. After all, mathematics is seen as a lens through which the extent of the ecological damage can be observed and potentially addressed. As Barwell (2013) pointed out, mathematics helps us understand climate science in the form of description, prediction, and communication. Renert's (2011) ideas complement Barwell's by further noting that the communication element of climate mathematics requires a sophisticated understanding of rational numbers as well as large quantities.

However, mathematics (together with science and technology) is not just the judge of the state of our climate and potential environmental hazards; it is also the perpetrator of the hazards (Beck 2009). Furthermore, Barwell (2013), d'Ambrosio (2010), and Skovsmose (1994) pointed out that mathematics as a discipline is also responsible for climate change, as mathematics has enabled the economic–industrial–scientific–political system that led to anthropogenic climate change. Industrial revolution(s), fossil fuel technology, financial risk management, and stock market (hedge funds) systems would not be possible without the mathematical apparatus. As d'Ambrosio (2010) wrote, we have a responsibility, as mathematics educators, "to question the role of mathematics and mathematics education in arriving at the present global predicaments of mankind [sic]" (p. 51).

The idea that the action of mathematicians can be ethically questionable has not escaped the mathematical mainstream. For example, the American Mathematical Society (AMS), the leading body for professional mathematicians in the United States, has recognized in its own ethical guidelines that:

> [w]hen mathematical work may affect the public health, safety or general welfare, it is the responsibility of mathematicians to disclose the implications of their work to their employers and to the public, if necessary. Should this bring retaliation, the Society will examine the ways in which it may want to help the "whistle-blower", particularly when the disclosure has been made to the Society. (AMS, 2001, para. 13)

So it appears that even the mathematical establishment recognizes that mathematics does not happen in the Platonic vacuum and that there may be some unethical consequences of mathematicians' work. It is important to note that the AMS's ethical guidelines do not imply that mathematicians should not be involved in such work; rather, they simply state that any implications should be disclosed. It is questionable whether, based on the reading of the guidelines, most members of the mathematics research community would feel responsible for contributing to the current environmental crisis.

Can we actually understand to what extent mathematics as a discipline (or potentially mathematics as a community of practice) can be held responsible for the current state of the planet? Van de Poel, Nihlen Fahlquist, Doorn, Zwart, and Royakkers (2012) outlined conditions under which an agent may be held morally responsible by focusing specifically on climate change. One of the conditions is causality: We can hold mathematics responsible for the environmental woes if we can establish the causal link between the content and the methods of the mathematical practice and the current state of our planet. A possible way to establish causality is to argue that mathematics as practiced has created unsustainable conditions that enabled the emergence of the Anthropocene through the mathematical models that brought us everything from coal power plants to the internal combustion engine. Another condition is knowledge. It is possible to claim that mathematicians working on the models may not be aware of how they may be used by, for example, the fossil fuel industry. However, to paraphrase Van de Poel et al. (2012), mathematicians have a duty to know and research how the models may be applied.

As educators and researchers proceed to take a closer look into the existing literature on mathematics education and sustainability, they have to be careful of the narratives that describe mathematics as part of the solution to the problem of sustainability and climate change as well as accept mathematics as one of the culprits of climate change. As educators, we realize that this is not an easy task because the focus of the mathematics education reforms has been on making students develop a productive disposition toward mathematics, which includes that mathematics is simply a worthwhile enterprise.

Sustainability in Mathematics Education

Sustainability is not a major topic in mathematics education research and practice. However, researchers who have written about sustainability (e.g., Barwell, 2013; Renert, 2011; Sriraman & Knott, 2009) recognize that sustainability should be more than just the context for enriching mathematical instruction. One possible approach to sustainability education is to identify the mathematical concepts necessary for understanding sustainability and use it to educate others about the planet. For example, Renert (2011) called for an understanding of large numbers and ratios, and Sriraman and Knott (2009) offered an empirical study of preservice teachers as they make sense of the amount of waste generated in the United States through the mathematical concepts of rate, proportion, and ratio. In addition to their calculations, preservice teachers in the study were also urged to think critically about their estimates. The authors concluded that, through critical thinking about estimates, "[pre-service teachers] begin thinking about solutions to problems brought about by the new found awareness of the threat to our planet because of over-population and overuse of natural resources" (p. 220).

The idea that students will gain environmental awareness through the logical and mathematical analysis of an environmental problems has been challenged by the work of Kahan et al. (2012), in which the statistical analysis of a representative sample of American adults ($n = 1540$) suggested that a person's perceptions about climate change do not depend on the individual's level of scientific literacy and numeracy, but on their cultural worldviews. For example, among the individuals who could be described as hierarchical individualists, many perceived climate change risk was actually significantly negatively correlated with the individuals' scientific literacy and numeracy. The study, widely cited, casts a doubt on the notion that ecological awareness can be increased simply through mathematical instruction. The study is also consistent with Levinson, Kent, Pratt, Kapadia, and Yogui's (2012) claim that individual risk estimates are informed by values, experiences, personal and social commitments, as well as representations.

Transformation in the School Curriculum

When discussing the idea of bringing issues from environmental studies and sustainability into the mathematics curriculum, we are not simply talking about creating a bridge or collaborating between mathematics and science. There also needs to be an element of activism and criticality. Criticality in the context of sustainability refers to interrogating discourses, institutions, as well as social and economic structures to identify and problematize those creating unsustainable conditions and, in turn, envision how to transform these structures for a more just and sustainable world. A key element of criticality is not only to use mathematics to read and write the world (Gutstein, 2006) but also to see and interrogate the formatting power of mathematics (Skovsmose, 1994) and its responsibility in creating issues of injustice, such as sustainability issues like climate change (Barwell, 2013) and food waste.

An example of the formatting power of mathematical models that directly affects the lived realities of students are those used to determine what annual income will represent the poverty level for families in the U.S. (for details, see U.S. Census Bureau, 2015). Such models are also used to determine cutoffs based on the percent of the poverty level that families' annual income needs to be below for their children to qualify for free or reduced lunch at school. These models become realized abstractions (Skovsmose, 1994) that construct our reality, determining who does or does not have to pay for lunch at school. In this way, mathematical models have very real power over which children receive or do not receive lunch at schools. Mathematically determined poverty thresholds have much further reaching importance than merely determining who gets free or reduced cost lunch. These thresholds are also often used to create groups as an operationalization of socioeconomic status in complex statistical analyses meant to shape policy and funding that have far-reaching implications for “who gets what” in their various communities and settings.

Presenting such models as the poverty threshold, and the power it has in shaping reality, would be important for both students and teachers to experience in their own mathematics education. This also relates to sustainability in a number of ways. For one, it relates to the sustaining of human life by providing nutrition for children in need. Also other issues of ecological sustainability emerge as the production, transportation, refrigeration, and disposal of food waste involved in providing free and reduced lunch to children in need also require the use of fossil fuels, deforestation of arable land for agriculture, water pollution from farmland runoff, and loss of land to waste disposal, to name just a few considerations. These issues also connect directly to climate change. For example, the use of fossil fuels to produce, transport, and refrigerate food creates greenhouse gases, as does the decomposition of food waste in landfills. Advanced mathematical models are also used heavily in climate change science, which Barwell (2013) discussed in detail related to mathematics education. These interconnections also point to another aspect of sustainability: When thinking of ecosystems and ecology, everything is interconnected. This interconnectedness brings a complexity to considering such issues as no clear boundaries exist to delineate them from one another.

Complexity is in fact an important consideration in creating transdisciplinary mathematics education and sustainable ecological practices in communities. It also provides some promise for change because, with everything being interconnected in complex webs of relationships, a change in a single node or relationship can have a rippling effect through other nodes and relationships. In this way, small changes can have widespread effects in ways that are not easily predictable (Hamilton & Pfaff, 2014; Renert, 2011; Renert & Davis, 2012). It is crucial that students and teachers are aware of the complex nature of sustainability and our ecosystems. This also makes the task of investigating issues of sustainability in the mathematics classroom inherently difficult, messy, and time-consuming. However, at the same time, if framed the right way, such interconnectedness can also be used to show the agency students and teachers have to effect change for a more just and sustainable future.

Such complexity is also another reason why we need to break down disciplinary boundaries and blur the lines between subject areas, which can be strong in the K–12 curriculum, to tackle the complex issues many communities are facing. Very dangerous events are unfolding that impact the health of our planet and the survival of human and other-than-human life, such as climate change, environmental pollution, water contamination, food shortages and waste, ozone depletion, and ocean acidification. These issues are further compounded by misinformation campaigns and the misdirection of the public by corporations, special interest groups, and politicians more interested in personal gains than social goods. This can be seen front and center in today’s American context, with the dramatic shift in political direction after the last presidential election; indeed, the agency tasked with protecting the environment, the United States Environmental Protection Agency (US EPA), is now run by someone who has attempted to dismantle it in the past and has historically been an advocate for companies and special interest groups opposed to the EPA’s very mission (Lipton & Davenport, 2017). Dealing with such large-scale environmental sustainability issues requires not only awareness but also activism in order to work to alleviate such issues as well as critique their causes.

According to the US EPA's (n.d.) webpage, "sustainability isn't *part* of our work—it's a guiding influence for *all* of our work." In a similar direction, sustainability should not be just an inserted unit or a project in a unit or topic in mathematics education; it should be a guiding influence for all our work in mathematics education. As a note, when we originally wrote this chapter, it was during the waning days of the Obama administration, and despite changes in the leadership to the US EPA, we stand in support of those faithful employees of that agency that still fight for the mission of environmental sustainability. An important question to again come back to here is what are we fighting to sustain? As we shift to discussing possibilities for practice for a transdisciplinary approach to mathematics and environmental/sustainability education, this should be a constant open question for dialogue as no single right answer exists. Instead, it needs to be an open question constantly considered and negotiated by the local and global communities, all of which are interrelated and affected by environmental issues.

It is important to point out that making a shift toward a transdisciplinary approach is not like flipping a light switch. It may not be possible to go directly from a traditional mathematics perspective to transformation where sustainability is intertwined and inseparable from mathematics education, making it not just part of our work, but also the guiding influence. It requires a transition. And this transitioning involves recognizing that mathematics is *not* neutral, nor should it be seen or taught as an isolated discipline. As discussed earlier, such a shift is beginning to occur with the sociopolitical turn (Gutiérrez, 2013). However, much more work is needed for such a transition. To try and relate what we have been discussing in terms of mathematics education and sustainability to the classroom, some examples would be helpful.

Possibilities in Classroom Practice

The goal of this section is to provide some examples of how sustainability could be integrated as a guiding influence into the mathematics curriculum threaded around a common issue—namely, food waste. We are choosing to use food waste as an issue for several reasons. First, food is something that students can relate to because everyone needs it to survive and, generally, students have some interest in foods they like. Food is also tied to different contextual considerations, such as culture, geographical contexts, flows of global capital, and socioeconomic status, all of which are important issues to consider in critical mathematics education (Frankenstein, 2009; Skovsmose, 1994; Wager & Stinson, 2012). Furthermore, food is something that is visible and relevant in schools. A frequently heard lament of school teachers in the United States is on the waste of food they have seen in their school cafeterias, with students throwing away entire uneaten lunches. At the same time, programs are in place to provide students with free or reduced lunches at

American schools because so many students come from homes that cannot provide the basic nutrition children need to be healthy.

Food is also a complex system that includes issues of production, transportation, cost, ethics, and environmental impact in both its production and disposal. All of these issues are deeply intertwined in economic, social, historical, and cultural concerns as well (Stuart, 2009). Such complex ecological and sustainability issues are ostensibly considered as part of science in the American K–12 educational setting (NGSS Lead States, 2013). For example, the NGSS includes human impacts on Earth systems and global climate change as part of the disciplinary core ideas to be taught in the science curriculum. Sustainability issues stretch well beyond science and the strict, inadequate disciplinary boundaries in K–12 schools and are important to consider in the context of mathematics education and other fields, such as visual and language arts (Renert, 2011; Renert & Davis, 2012).

For our example about food waste, we draw on the work of Frankenstein (2009, 2011), who often talks about how she attempts to present her students with *real* real-life examples of mathematics that are also outrageously horrible, yet represent the lived reality for millions of people. An example is presenting students with figures on the proportion of the federal budget spent on national defense and past, present, and future wars (Frankenstein, 1994). According to Frankenstein (2009, p. 114), "real real-life mathematical problems occur in broad contexts, integrated with other knowledge of the world." They are not just calculation problems dressed up with words and a meaningless context. They are problems that allow students to make sense of the social world around them and the political context in which they are situated. Part of her rationale for presenting such examples is because "part of struggling to change our world in the direction of more justice is knowing how to clearly and powerfully communicate the outrageousness" (Frankenstein, 2011, p. 10).

We consider the issue of food waste in school cafeterias as a real real-world problem situated in students' daily reality and drawing upon other knowledge of the world outside of mathematics. Drawing from this notion, an initial step to spark awareness and draw students' interest into exploring, investigating, and acting upon issues of sustainability could begin by presenting them with some outrageous figures and statistics around food waste. The issue of food waste is also intimately connected to the issue of climate change, which we focused on earlier, for a number of reasons, including fossil fuel costs from producing, distributing, refrigerating, and disposing of food; greenhouse gas production by decomposing food waste; methane production by cattle due to demand for beef; and deforestation for clearing arable land for farming.

To begin our food waste example, consider the following: The US EPA (2014) estimated that the amount of municipal solid waste in 2010—that is, food waste from both retail and consumer sources—was 98,940,000,000 pounds. To put that into perspective for students, if we assume the average school bus without students weighs around 17,000 pounds (this value varies greatly, which is why we are using a conservative assumed measure here), Americans throw away approximately 5,820,000 school busses worth of trash a year. To help students develop a sense of the quantity, if we also assume that a school bus is about 30 feet long, the total equates to an end-to-end train of busses 174,600,000 feet long (5,820,000 school busses × 30 feet per bus), or 33,068 miles long (174600,000 feet long or 33,068 miles), or long enough to go around the Earth 1.33 times (33,068 miles long compared to the approximate circumference of the Earth of 24,901 miles). There is a significant amount of quantitative reasoning involved in such measurement conversions that help to also provide a clearer picture of the immensity of the figure.

Another way to put 98,940,000,000 pounds of municipal solid waste into perspective for students would be to consider the amount of waste per person in the United States per year: 319.9 pounds (U.S. EPA, 2014). To hit the point home, a teacher could physically present students with that amount of trash, perhaps from their own school dumpster, so they can visually see and experience that amount of waste, rather than just considering it as an abstract number. This is only considering solid food waste; if all postharvest food losses are considered, the US Department of Agriculture puts that figure at 133 billion pounds or 31% of all harvested food was lost in 2010 (Buzby, Wells, Axtman, & Mickey, 2009). That is approximately 7,823,529 school busses worth (133,000,000,000 pounds of harvested food waste compared to the average school bus weight of 17,000 pounds) or 431 pounds per person in the United States per year (133,000,000,000 pounds of harvested food waste per 308,745,538 people, the population of the United States in 2010 according to the US Census Bureau [2010]).

We have shown the calculations involved in each of the changes in the perspective we have presented to emphasize the significant amount of unit conversion and proportional reasoning used in just this small example, which could be incorporated into the mathematics classroom targeting key topics in mathematics (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010). Furthermore, such experiences could make such a quantity of waste seem outrageous to students, which acts to raise awareness and hopefully students' interest, while simultaneously presenting students with experiences seeing the power of mathematics to view the world.

Part of awareness is to destabilize and problematize the "sustainability" of society's current path, or the status quo, to begin to work toward transforming society and mathematics education toward a more just and sustainable future. Awareness is but one component though. Even if we are aware of the outrageously unsustainable practices around us, it does not mean we act to do anything about it. As Kahan et al. (2012) found in their survey, numeracy can actually be negatively correlated with individuals' perceived risk of climate issues. Furthermore, in taking an approach to foster awareness by presenting outrageous real real-life examples, a fine line must also be tread carefully so as not to stray into fear and despair. Some scientists take the shock and awe route to awareness, in turn sending out messages of fear and despair, as if no one individual's work or change could ever turn around such large, complex issues to prevent ecological disaster (Renert, 2011; Renert & Davis, 2012). It is important to avoid such apocalyptic and defeatist rhetoric, which can lead students to take up positions that there is no hope, so why bother, or that problems are too large for anyone to handle or resolve.

One way to balance such a line is to also present ways that individuals have the agency to transform their own context in an effort to mediate current ecological trends. Spurning outrage in students will hopefully result in motivating or precipitating students' desire to investigate such issues themselves, beginning in a local setting, such as students' own school or local community. A teacher could initially begin by starting a discussion with students to open a space for them to discuss their reactions to being presented with such outrageous figures of the amount of food waste in the United States. These questions could also be used to connect this issue more to the lives of students by having them reflect upon what such figures mean for their community. In the Appendix, we have provided some possible initial discussion questions to help teachers introduce the issues of food waste.

After having a discussion with students, a next step would be to begin to investigate ways to transform their own communities to be more sustainable. Initially, students could come up with their own issues they want to investigate in their school cafeteria, such as just how much food is thrown away each day? What type of food is being thrown away? Are certain types of food more likely to be thrown away than others? How could the amount of food waste be reduced? What seem to be the major causes of food being thrown away at school? Where does food waste from our school go? What are the influences of such food waste on the community and environment? These are all questions that may spark students' interest in investigating and critiquing food waste in their own school. Such questions could then drive investigations in which students consider how they would collect data to investigate or answer those questions. These investigations could then be followed by analyzing the data statistically or using mathematical modeling, followed by interpreting such models and analyses relative to the questions posed. Such mathematical processes are important components of the mathematics curriculum (NCTM, 2000; NGA Center & CCSSO, 2010) and, as shown with our example, can be taught in *real* real-world problems and contexts rather than in terms of abstract algorithms and calculations or contrived or fictitious contexts (Frankenstein, 2009).

Still, merely investigating and critiquing issues do not necessarily lead to change. Change requires action and transformation. Fostering awareness and investigating issues related to food waste only create the potential for change. We would argue, though, that awareness increases the potential for change more than if students are not aware or given opportunities to explore and critique such critical sustainability issues. Providing students with an open space in which to investigate sustainability issues important to them and related to their daily lives in the mathematics classroom is a first step in fostering students' agency to investigate such issues on their own.

Students need to feel empowered to effect change or transform current conditions for a better and more sustainable future (Hamilton & Pfaff, 2014). For example, Donnay (2013) reported on how post-secondary level mathematics students were interested in exploring the idea of their cafeterias going tray-free. By comparing the weight of food waste on a day the cafeteria used trays versus a day they did not, students estimated the cafeteria could reduce food waste by 4600 pounds per year by going tray-free. The students also created surveys to survey the student body in order to help determine how to gain students' buy-in for such a move. As a result of the action of students in their mathematics class, within 2 years the college's cafeterias went tray-free. Though this example played out at the college level, Donnay (2013) reported that much of the mathematics content (e.g., linear functions, proportions, unit conversion) was applicable to the K–12 mathematics curriculum (NCTM, 2000; NGA Center & CCSSO, 2010). This example also highlights how the post-secondary level could also be a powerful setting for the transformation of mathematics education because it is where future classroom teachers are educated. There is, after all, a bit of a chicken-or-egg paradox in where to start a transformation in education: with teachers or with students? Teachers play a large role in shaping the mathematics curriculum that students experience (Remillard & Heck, 2014). Realistically, to effect change, educators must work at all levels of the system: students, teachers, teacher educators, administrators, and policymakers. After all, education is a complex system, just like our planet's ecosystem.

Conclusion

There are no certainties, guarantees, cure-alls, or quick fixes in doing transdisciplinary work for justice and sustainability. What we have tried to do in this chapter is question and problematize the predominant direction mathematics education has been taking in terms of its isolation from other disciplines. If educators continue to treat mathematics as an isolated, politically neutral subject, how can they expect students to see the power of mathematics to not only investigate important and meaningful issues but also change such issues for a more just and sustainable world? We hope that this work provokes further discussion, which is desperately needed to transform the educational reality we see unfolding before our eyes. We have also tried to move beyond critique and problematization by providing some possibilities for moving forward in hopes that others will take up this charge. We hope other researchers, teacher educators, and teachers may find such possibilities as a starting point to consider how they can move toward a transdisciplinary mathematics education for sustainability in their own settings and contexts. One guarantee that can be made is that if those who exercise power do not change, this planet will no longer be able to sustain us. Change needs to happen, and transdisciplinary mathematics education can and should be part of that process.

Appendix: Possible Discussion Questions for Teachers

1. How important is food in your daily routine?
2. How much food would you estimate you throw away on a weekly basis?
3. What factors contribute to you throwing food away?
4. How much food would you estimate is being thrown away in the cafeteria in your school each day?

5. Does food being wasted at your school bother you?
6. What factors do you think contribute to food being wasted at your school?
7. How do you think mathematization might play a role in contributing to the food wasted at your school?
8. How might mathematization be used to help reduce the food wasted at your school?
9. What might you do to better understand the issues of food waste at your school?
10. What mathematics do you think would be involved in investigating the issue of food waste at your school?
11. How might you engage your students in investigating the issue of food waste at your school connected to your mathematics curriculum?
12. What are the ethical issues of the food waste/consumption in your community?

References

American Mathematical Society. (2001, January). *Policy statement on ethical guidelines*. Retrieved from http://www.ams.org/about-us/governance/policy-statements/sec-ethics

Barwell, R. (2013). The mathematical formatting of climate change: Critical mathematics education and post-normal science. *Research in Mathematics Education, 15*(1), 1–16.

Bazzul, J. (2016). *Ethics and science education: How subjectivity matters*. Dordrecht, Netherlands: Springer. doi:10.1007/978-3-319-39132-8

Beck, U. (2009). *World at risk*. Cambridge, MA: Polity.

Buzby, J. C., Wells, H. F., Axtman, B., & Mickey, J. (2009). *Supermarket loss estimates for fresh fruit, vegetables, meat, poultry, and seafood and their use in the ERS loss-adjusted food availability data*. United States Department of Agriculture, Economic Research Service. Retrieved from https://ideas.repec.org/p/ags/uersib/58313.html

d'Ambrosio, U. (2010). Mathematics education and survival with dignity. In H. Alrø, O. Ravn, & P. Valero (Eds.), *Critical mathematics education: Past, present and future* (pp. 51–63). Rotterdam, Netherlands: Sense Publishers.

Deloria, V., Jr., & Wildcat, D. (2001). *Power and place: Indian education in America*. Golden, CO: Fulcrum Publishing.

Donnay, V. J. (2013). Using sustainability to incorporate service-learning into a mathematics course: A case study. *Primus, 23*(6), 519–537.

Frankenstein, M. (1994). Understanding the politics of mathematical knowledge as an integral part of becoming critically numerate. *Radical Statistics, 56*, 22–40.

Frankenstein, M. (2009). Developing a critical mathematical numeracy through real real-life word problems. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.), *Words and worlds: Modelling verbal descriptions of situations* (pp. 111–130). Rotterdam, Netherlands: Sense Publishers.

Frankenstein, M. (2011). Quantitative form in arguments. In S. Mukhopadhyay & W.-M. Roth (Eds.), *Alternative forms of knowing (in) mathematics* (pp. 283–297). Rotterdam, Netherlands: Sense Publishers.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. New York, NY: Routledge.

Gutstein, E., & Peterson, B. (2013). *Rethinking mathematics: Teaching social justice by the numbers* (2nd ed.). Milwaukee, WI: Rethinking Schools, LDT.
Hamilton, J., & Pfaff, T. J. (2014). Sustainability education: The what and how for mathematics. *Primus, 24*(1), 61–80.
Hardt, M., & Negri, A. (2009). *Commonwealth*. Cambridge, MA: Harvard University Press.
Henderson, J. A., & Hursh, D. (2014). Economics and education for human flourishing: Wendell Berry and the oikonomic alternative to neoliberalism. *Educational Studies, 50*(2), 167–186.
Jickling, B. (2001). Environmental thought, the language of sustainability, and digital watches. *Environmental Education Research, 7*(2), 167–180.
Jickling, B., & Wals, A. E. (2008). Globalization and environmental education: Looking beyond sustainable development. *Journal of Curriculum Studies, 40*(1), 1–21.
Kahan, D. M., Peters, E., Wittlin, M., Slovic, P., Ouellette, L. L., Braman, D., & Mandel, G. (2012). The polarizing impact of science literacy and numeracy on perceived climate change risks. *Nature Climate Change, 2*, 732–735.
Latour, B. (2004). *Politics of nature*. Cambridge, MA: Harvard University Press.
Levinson, R., Kent, P., Pratt, D., Kapadia, R., & Yogui, C. (2012). Risk-based decision making in a scientific issue: A study of teachers discussing a dilemma through a microworld. *Science Education, 96*(2), 212–233.
Lewis, S. L., & Maslin, M. A. (2015). Defining the anthropocene. *Nature, 519*(7542), 171–180.
Lipton, E., & Davenport, C. (2017, January 15). Scott Pruit, Trump's E.P.A. pick , backed industry donors over regulators. *The New York Times*, p. 1A.
Lloro-Bidart, T. (2015). A political ecology of education in/for the anthropocene. *Environment and Society: Advances in Research, 6*(1), 128–148.
McGregor, D. (2005). Coming full circle: Indigenous knowledge, environment and our future. *The American Indian Quarterly, 28*(3), 385–410.
McKenzie, M. (2009). Pedagogical transgression: Toward intersubjective agency and action. In M. McKenzie, P. Hart, H. Bai, & B. Jickling (Eds.), *Fields of green: Restorying culture, environment, and education* (pp. 211–226). New York, NY: Hampton Press.
McKenzie, M. (2012). Education for y'all: Global neoliberalism and the case for a politics of scale in sustainability education policy. *Policy Futures in Education, 10*(2), 165–177.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
National Governors Association Center for Best Practices (NGA Center) & Council of Chief State School Officers (CCSSO). (2010). *Common core state standards for mathematics*. Washington, DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
NGSS Lead States. (2013). *Next generation science standards: For states, by states*. Retrieved from http://www.nextgenscience.org/
Orr, D. W. (2004). *Earth in mind: On education, environment, and the human prospect*. Washington, DC: Island Press.
Pierce, C. (2012). *Education in the age of biocapitalism: Optimizing educational life for a flat world*. New York, NY: Palgrave Macmillan.
Pierce, C. (2015). Learning about a fish from an ANT: Actor network theory and science education in the postgenomic era. *Cultural Studies of Science Education, 10*(1), 83–107.
Rabinow, P., & Rose, N. (2006). Biopower today. *BioSocieties, 1*(2), 195–217.
Remillard, J. T., & Heck, D. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM: The International Journal on Mathematics Education, 46*(5), 705–718.
Renert, M. (2011). Mathematics for life: Sustainable mathematics education. *For the Learning of Mathematics, 31*(1), 20–26.
Renert, M., & Davis, B. (2012). Ecological sustainability and mathematics education: Integrally connected. *Journal of Integral Theory and Practice, 7*(1), 94–104.
Sauvé, L. (2005). Currents in environmental education: Mapping a complex and evolving pedagogical field. *Canadian Journal of Environmental Education, 10*(1), 11–37.

Skovsmose, O. (1994). *Toward a philosophy of critical mathematical education*. Boston, MA: Kluwer Academic Publishers.

Sriraman, B., & Knott, L. (2009). The mathematics of estimation: Possibilities for interdisciplinary pedagogy and social consciousness. *Interchange, 40*(2), 205–223.

Stuart, T. (2009). *Waste: Uncovering the global food scandal*. New York, NY: W. W. Norton & Company.

United States Census Bureau. (2010). *Interactive population map*. Retrieved from https://www.census.gov/2010census/popmap/

United States Census Bureau. (2015). *How the Census Bureau measures poverty.* Retrieved from https://www.census.gov/topics/income-poverty/poverty/guidance/poverty-measures.html

United States Environmental Protection Agency. (n.d.). *Sustainability*. Retrieved from https://www.epa.gov/sustainability

United States Environmental Protection Agency. (2014). *Food waste scoping analysis*. Washington, DC: Author. Retrieved from https://www.epa.gov/sites/production/files/2016-01/documents/msw_task11-1_foodwastescopinganalysis_508_fnl.pdf

Van de Poel, I., Nihlen Fahlquist, J., Doorn, N., Zwart, S., & Royakkers, L. (2012). The problem of many hands: Climate change as an example. *Science and Engineering Ethics, 18*, 49–67.

Wager, A. A., & Stinson, D. W. (Eds.). (2012). *Teaching mathematics for social justice: Conversations with educators*. Reston, VA: National Council of Teachers of Mathematics.

White, L. (1996). The historical roots of our ecological crisis. In R. S. Gottlieb (Ed.), *This sacred earth: Religion, nature, environment* (pp. 184–193). London, UK: Routledge.

Chapter 7
Using a Mathematics Cultural Resonance Approach for Building Capacity in the Mathematical Sciences for African American Communities

Terrence Richard Blackman and John Belcher

Abstract The underrepresentation of African Americans in the mathematical sciences in post-secondary education and in professional settings has been well documented. This state of affairs has persisted despite multiple and varied efforts over the years to address the concern. We assert that defining efforts around closing achievement gaps and/or through making moral arguments, such as has often been the case, is insufficient for compelling the levels of commitment and action needed to address meaningfully issues that contribute to the seeming intractability of Black underrepresentation in the mathematical sciences. The equity and access issues at play are embedded in the histories of oppression and devaluation faced by Black people in this nation.

In this chapter, we introduce a mathematics cultural resonance framework (MCRF) to inform mathematics teaching, learning, and knowledge production in ways that affirm and draw upon African American cultural resources. We argue for strategies that link mathematics pedagogy with active mathematics research and with the mathematical sciences knowledge, practices, and dispositions embedded within African, African American, and/or other African Diasporic cultural traditions. We posit that culturally resonant approaches facilitate African Americans developing robust mathematics identities and maintain that these approaches provide ripe opportunities for producing new, groundbreaking mathematics knowledge, thereby benefiting the mathematics community (and society) as a whole. In considering implications for using an MCRF to build the mathematics capacity of

We dedicate this chapter to the memory of Brother Jimmi Griffin, our friend and colleague, who played a major role in our efforts to define and begin to implement a mathematics cultural resonance framework.

T.R. Blackman (✉)
Medgar Evers College, 1650 Bedford Avenue, Brooklyn, NY 11225, USA
e-mail: tblackman@mec.cuny.edu

J. Belcher
Mathematics Cultural Resonance Project, School of Science, Health and Technology, Medgar Evers College, 1650 Bedford Avenue, Brooklyn, NY 11225, USA
e-mail: jbelcher57@gmail.com

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_7

African American communities, we describe in relative detail our work in the West Ocala community of Ocala, Florida.

Keywords Cultural resonance • Black communities • Mathematics practices • Knowledge production • Cultural traditions • Black mathematicians

Preliminaries

We write this chapter from the perspective of Black (African American) cultural-racial membership. As a writing convention, we use Black and African American interchangeably, knowing that these two labels are not technically synonymous. When we refer to "the" Black community or "the" African American community or Black/African American mathematicians, we are speaking about Black people from Africa and/or the African Diaspora who reside in the United States. We recognize that there is no monolithic Black community or monolithic African American culture, though we write, generally, about African American culture and African American/Black communities rather than making attempts to qualify these characterizations throughout this chapter. Further, as Black mathematics professionals ourselves and as an expression of an alignment of our racial, cultural, and mathematics identities, where appropriate, we write in terms of "our" communities and "our" histories, traditions, and culture, rather than making third person, object references.

Though we recognize that the primary audience of this publication is, more than likely, the mathematics education academic community, we write with members of our African American communities in mind, who might not necessarily be in academia but who, nonetheless, have a stake in and a claim to issues, such as under-representation in the mathematical sciences, which have serious implications for the vitality of these communities. We hope for, aspire towards and intend a broadened conversation and to engaged participation, including through efforts defined and led by members of African American communities to build the mathematics capacity of these communities.

Beyond Closing Gaps

Our motivations for articulating the vision for building mathematics capacity of our communities extend beyond aims to close gaps of one sort or another between African Americans and our White American and/or Asian American counterparts. We are guided by questions of what it takes to repair and to prepare our communities. We are guided by considerations of what it will take to compel action toward revitalizing our communities, action toward thriving, not surviving.

We think of our forebears who built institutions, communities, and even towns in this country, despite having been born to enslavement and having endured brutal oppression. We relate to Murrell's contextual framing for White teachers teaching Black children to consider:

> where you might be now if your ancestors had been denied access to education, or barred from entering a commonwealth, and denied citizenship and basic rights by the government? Where might you be if your ancestors were publicly flogged every thirty days until they left the state (as they did in Oregon in 1857) or denied citizenship by the United States Supreme Court, stating that they had no rights that either persons or governments were bound to respect? Or you might ask where you would be if your grandparents attended schools in a region of the country where state governments preferred to have no quality public schools rather than have quality public schools that served you, your family, and your kin? (Murrell, 2002, p. 30)

Clearly any approach toward effectively addressing educational and professional inequities must be informed by considerations of African American history and cultural traditions. We think about the requisite infrastructures for building the mathematics capacity of Black communities and how knowing our histories and traditions reminds us that we've done this type of building before and that the more important gap to close is the gap that separates us from our cultural richness.

More of the Same (Kind of) Data

The underrepresentation of African Americans in the mathematical sciences in post-secondary education as well as in professional settings has been well documented. Though African Americans constitute approximately 12% of the United States population, at the college and university level, 5.7% bachelor's degrees in mathematics and statistics are awarded to Blacks. In graduate school, African Americans are approximately 2.9% of master's degree recipients and 2.0% of PhD recipients.

There are an estimated 300 living African American mathematicians in the United States (Walker, 2014). For context, according to the American Mathematical Society (AMS) website, "there are over 35,800 individual members of the four leading professional mathematical sciences societies in the U.S." (AMS, n.d.) a number which, of course, excludes those mathematicians who don't belong to any of these societies. As far as we have been able to determine, it is still the case that "You Can Count on One Hand all the Black Mathematics Professors at the Highest-Ranking American Universities," which was the title of an article that appeared in the *Journal of Blacks in Higher Education* (Winter, 1998–99).

These data tell a familiar and persisting, year-to-year tale of an unacceptable state of affairs regarding diversity in the mathematical sciences. This begs the question, however, of unacceptable for whom? Hopefully, this persisting state of affairs becomes unacceptable to more and more of us, at levels visceral enough to feel urgent enough to compel needed responses.

Unexcavated Stereotypes About Black People and Mathematics

After Benjamin Banneker showed his almanacs containing complicated trigonometric calculations to then Secretary of State Thomas Jefferson, Jefferson communicated his conclusion to colleagues that Banneker must have received assistance from Whites, going so far as to say that "I have not yet found one of them [Blacks] that could solve the geometric problems of Euclid" (as cited in JBHE, 1998). In 1916, Lewis Terman, writing of Black people and other non-White racial-ethnic groups in his book *The Measurement of Intelligence*, in which he introduced the Stanford-Binet IQ test he created, stated that, "They cannot master abstractions but they can often be made into efficient workers."

The editors of the book *Black Mathematicians and Their Works* took an approach of countering stereotypes about Black people's lack of capacity for abstract thought by showing the works of Black people demonstrating this ability at the highest levels. Included in this volume is a reprint of an April 1952 Negro History Bulletin editorial, "Science and Mathematics":

> …It has been assumed that Negroes have special talents along the lines of the arts, but that they are inherently weak in science and mathematics. Upon this assumption are based the watered-down programs of science and mathematics that frequently are offered to Negro students on all levels of instruction…The idea of Negroes having minds that would become 'confused by figures' is similar to an idea in earlier generations that Negroes would not make good factory workers. The 'hum of the machinery would put them to sleep,' it was said. Thousands of Negroes employed at machines in war-expanded industry enjoyed the inflated salaries, delivered the goods and did their sleeping at home. The whole idea proved to be a myth and not a matter of a racial trait at all. (Newell, Gipson, Rich, & Stubblefield, 1980, p. 305)

Black people have had long histories of "making a way out of no way," as the African American expression goes, in confronting and overcoming stereotypes about our capacities, whether these stereotypes be about our abilities to work in factories, or serve as officers in the military, or be quarterbacks (Shanahan, 2014, p. 3), or play baseball, or to be elected officials (without chaos such as portrayed in D. W. Griffith's racially demeaning film *The Birth of a Nation*). As the African American authors of this chapter, informed by our lived experience, we view dismantling and excavating stereotypes about our capacity for abstract thought, as a final frontier of sorts for Black people. We continue to witness, without surprise, manifestations of the survival of these stereotypes. We experience their durability in the self-doubts we ourselves have internalized, triggered when we find ourselves in mathematics spaces that feel unwelcoming. We know that stereotypical perceptions and portrayals that have persisted and been perpetuated, oftentimes by design, for hundreds of years don't magically disappear, even with the best of intentions and pledges to do the right thing because it's the right, moral, socially just thing to do. Changing curricula, changing pedagogy, or making rallying calls to close achievement gaps or opportunity gaps is insufficient for compelling the education community, the mathematics community, the society at large, and Black people ourselves to examine and take action at the depth of levels needed to address meaningfully issues that contribute to the seeming intractability of Black underrepresentation in the mathematical sciences. This requires intentional, sustained effort, not at the branch, twig, or flower petals levels but, rather, at the levels of the trunk and the roots.

Introducing a Mathematics *Cultural Resonance Framework*

We introduce a mathematics cultural resonance framework (MCRF) to inform mathematics knowledge production and pedagogy in ways that affirm and draw upon African American cultural resources. We are intentional in speaking specifically about a "mathematics" cultural resonance framework, instead of discussing a culturally resonant framework more generally, even though the same principles apply, we believe. The status of the mathematical sciences as an epitomization of abstract thought has bearing on how we shape and view our overall intellectual identities as Black people.

Along with considerations about developing culturally resonant intellectual identities, there are additional factors that speak to the value for Black communities of a mathematics cultural resonance framework informed through lenses of African American culture(s). The mathematical sciences are integral to our everyday lives. Further, given the prestige attached to having expertise in using the tools of mathematics, concentrating this prestige in the hands of a subset of the population limits the types of mathematics that are valued, pursued, and developed (Herzig, 2004). In today's world, access to mathematics learning at the highest levels and to active participation in setting and pursuing mathematical sciences research agendas are inextricably linked to the economic and political empowerment and overall vitality and quality of life of African American communities. We can't afford to operate on the fringe and/or in the primary capacity as consumers. Nor can the contributions of those from the community of Black mathematicians remain invisible, under-recognized, and unheralded. Without active, visible participation in the practices of mathematics, including its knowledge production, not only will achievement and opportunity gaps persist, but our communities will not be assured that the practices and research agendas of the mathematics community at large meaningfully align with the needs of African American communities or with the assets of our communities, including the cultural capital of our histories and varied knowledge traditions.

The Mathematics and Physics of Resonance

Our combined research interests and work over the years align with the importance we place upon maintaining strong links between mathematics teaching, learning, and knowledge production dimensions while being mindful of cultural and socioeconomic considerations. Terrence is a mathematician specializing in number theory. Specifically, he studies the discrete spectrum and the eigenfunctions of the Laplacian for a special class of arithmetic surfaces of fundamental interest in mathematics and physics. Concomitantly, he has pursued a mathematics education research agenda grounded in identifying and addressing issues impeding the meaningful participation of Black people in the mathematical sciences. Based upon his backgrounds in drumming and mathematics, John has for many years investigated relationships between music and mathematics. Among his areas of focus are the various ways that pulses can be grouped and subdivided and that rhythmic patterns can be sequenced and layered and organized. He has made significant use of mathematical "tools" and concepts in these investigations, which have included composing music based upon mathematics structures (Belcher, 2006). As a drummer/composer/"rhythmologist," he has worked in a variety of educational, therapeutic, and performance settings.

In 2012, we decided to develop and pursue a collaborative research agenda (Belcher & Blackman, 2013) organized around the question "Can one hear the shape of a drum?," posed by mathematician Mark Kac (Kac, 1966). Essentially, Kac asks in an intuitive manner if one can identify the specific shape of a vibrating membrane (such as a drumhead) given the knowledge of all the frequencies at which the membrane vibrates. We have found that various points of entry to the question and various twists to the question suggest many possibilities for interesting collaborations, such as our own, and provide multiple opportunities for engaging mathematics learners of all levels around a deep and very active research area (Belcher & Blackman, 2013).

The phenomenon of resonance is at the heart of this research. A resonant frequency is the frequency at which an object tends to vibrate "naturally," in the absence of some external force being applied to the object. When an external force is applied at a frequency equal to the resonant frequency, the effect is magnified. A familiar example is pushing a person in a playground swing in time with the resonant frequency making the swing go higher and higher with minimal effort. Indeed, if the pushes occur at a faster of slower rate than the rate at which the swing makes its arcs, then the motion of the swing is disrupted.

The resonance phenomenon can also be observed when you vibrate a taut string at different frequencies. At most of the frequencies applied, nothing remarkable occurs. However, at resonant frequencies, the amplitude of the string's response increases dramatically and different patterns occur, according to the particular resonant frequency.

The resonance that occurs in drums (two-dimensional membranes) is an extrapolation of what happens with strings, which are considered to be one-dimensional in an abstract mathematical sense. An impressive display of resonance in two-dimensional objects can be seen in so-called Chladni patterns. These emerge when you sprinkle sand (or powder) on metal plates of various shapes and sizes and vibrate them at their resonant frequencies.[1] Figure 7.1 shows images of Chladni patterns produced on equipment in our community-based office/lab space.

[1] Chladni patterns are named in honor of late eighteenth-/early nineteenth-century musician-phys-

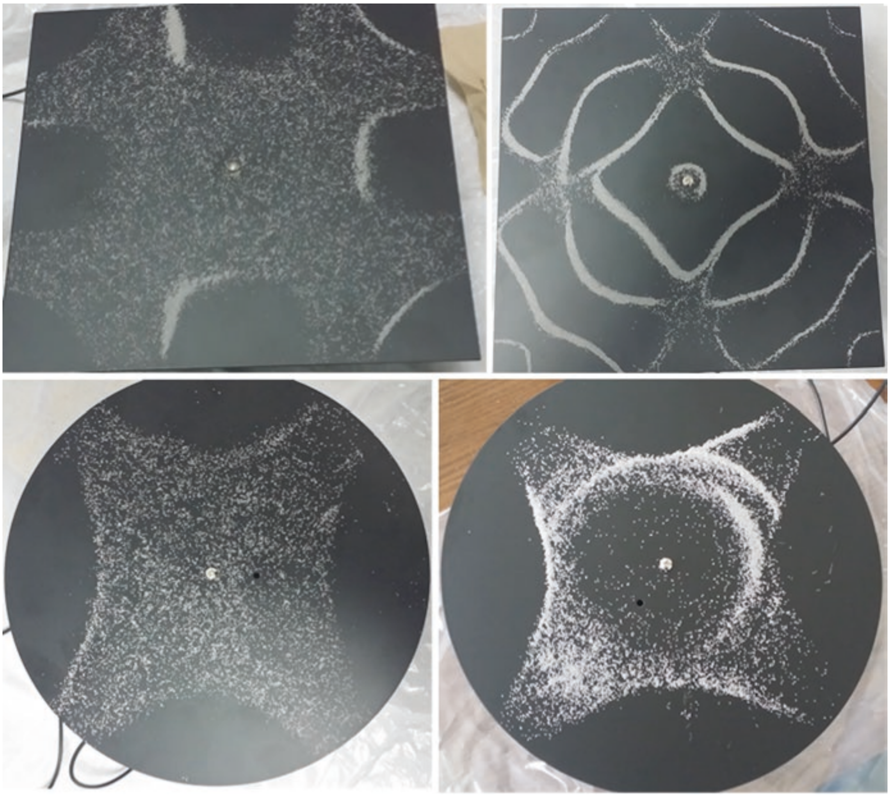


Fig. 7.1 Chladni patterns with the frequencies that generated them on a square plate (24 cm sides) and a circular plate (24 cm diameter)

It is an exhilarating, awe-inspiring experience to witness particles on a Chladni plate become excited and move into beautiful, largely unpredictable patterns when you find a resonant frequency – non-resonant frequencies leave the particles unaffected. Any object that vibrates, be it a string, a drumhead, or an atom, has resonant frequencies. Information about an object can be gleaned from knowledge of these frequencies.

It was actually during one of our conversations about our investigations into the mathematics and physics of resonance that we recognized the aptness of resonance as an illuminating frame for engaging individuals and communities (broadly defined) around mathematics teaching, learning, and knowledge production. We are all familiar with the palpable feeling that accompanies an encounter with an idea or experience that strikes us in a particular way. We feel excited. We feel moved (e-motion). In the noted conversation, as we, the authors, considered ways of more

icist Ernst Chladni who made drawings of the figures created when he used his violin bow to excite a metal plate sprinkled with sand.

effectively activating the untapped potential that exists in our Black communities, we recognized the critical importance of finding words and actions that resonate with the members of these communities, with the culture(s) and histories of these communities. We recognized the critical importance of approaches, of words, and of actions that resonate with the social, cultural, racial identities of Black people. We began to envision and imagine the potential for transformative impact both with respect to how African Americans have positive engagement with mathematics and with respect to how the discipline of mathematics – its practice and its knowledge base in today's world – might be affected by contributions of Black people who access more fully our cultural assets and resources.

Envisioning and Imagining the Contours of an MCRF

We articulate the contours of a mathematics cultural resonance framework by describing how we envision its expression through the lenses of African American culture. In describing our framework through these lenses, we do not intend to convey a message that this MCRF is designed exclusively for Black culture. We posit that mathematics as it is practiced in academic settings has already been operating within an MCRF, experienced predominately through the lenses of European cultures. Our position aligns with arguments made against long-held claims that mathematics is a neutral, acultural discipline (D'Ambrosio, 2001; Nasir, Hand, & Taylor, 2008; Powell & Frankenstein, 1997). We are intentional about exploring an MCRF at the level of granularity and specificity we have chosen, i.e., what it might look like operating through African American cultural lenses. Extracting and distilling generalizable MCRF core principles, particularly principles generalizable to diverse communities that have experienced underrepresentation and marginalization, require having meaningful participation of individuals from these communities.

The MCRF shares core principles with various other frameworks that consider racial, social, and cultural factors in seeking to address issues of equity and access in the mathematical sciences. We inform our MCRF with culturally relevant/responsive pedagogy[2] (Ladson-Billings, 1995) and African-centered pedagogy[3] (Murrell, 2002). We draw upon what we characterize as an expanded success/brilliance of Black children in mathematics framework, which aims to create counter-narratives to the deficit-based depictions of African American students, their

[2] According to Ladson-Billings (1995), culturally relevant pedagogy is a pedagogical practice "that not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (p. 469).

[3] Murrell (2002) describes *African-Centered Pedagogy* as a synthesis of five relevant frameworks: learning communities or communities of learning, culturally responsive or culturally relevant teaching, teaching for understanding, situated learning, and cultural and racial identity development.

families, and communities. The expanded success/brilliance framework calls for "embrac[ing] the cultural backgrounds and knowledge that Black learners bring to their school and classroom contexts" (Leonard & Martin, 2013, p. xvi). Walker's research and framing approaches used to investigate the cultivation of African American mathematics excellence and identity[4] (Walker, 2014) provide further key influence to our MCRF.

In our work to date shaping an MCRF, we have identified the following set of organizing/design principles:

- A different take on mathematics pedagogy: intentional teaching and learning of mathematics practices
- Using African American cultural resources to inform mathematics teaching, learning, and knowledge production
- Diversity as a means and not an end
- Expanded definitions of mathematics success
- Creating intentional culturally resonant mathematics spaces

A Different Take on Mathematics Pedagogy: Intentional Teaching and Learning of Mathematics Practices

Typically, discussions on mathematics education focus on mathematics content. Pedagogical considerations and efforts are directed toward how to more effectively teach and learn the subject matter. Thinking about mathematics as the primary ideas that lend themselves to presentation in conventional mathematics texts makes invisible the cultural dimensions of mathematics practices.

In our MCRF approach, we highlight the importance of distinguishing between the content of mathematics as a discipline and the practice of mathematics as a profession. Included among the skills and activities required of practicing mathematicians are developing and pursuing research agendas, writing mathematics papers, presenting mathematics talks, acquiring mathematics knowledge (staying current within one's area of focus), and, in general, participation in the local and broader mathematical communities. These skills and activities, among others, contribute to the development of robust mathematics identities (Herzig, 2004).

This skill-activity set stands in glaring contrast to what students of mathematics typically experience at the graduate school level (Herzig, 2004), much less at K-12 and undergraduate levels. As it turns out, a focus on the practices of mathematicians, on mathematics as a "community of practice" (Lave & Wenger, 1998; Wenger,

[4] "What has emerged as a key factor in the success of high achievers and mathematicians alike is the important role that out-of-school experiences and relationships, many rooted in specific cultural and social contexts, have played in their mathematics knowledge development and socialization" (Walker, 2012).

2009), reveals the cultural dimensions of mathematics (Burton, 1998; Nasir et al., 2008). In summarizing her interview-based study of 70 research mathematicians focused on how they "come to know" mathematics, Burton (1998) observed that the mathematics practices of the study participants were "highly varied, from one to another, and personally and culturally dominated" (p. 140). We believe that prominent consideration of cultural features of mathematics practices will clarify the transformational opportunities arising from meaningful African American participation in the discipline.

One example of an African American community of mathematics practice is the Conference for African American Researchers in the Mathematical Sciences (CAARMS). The organization, co-founded in 1995 by Bill Massey and other colleagues, was formed "to highlight current research by African-American researchers and graduate students in mathematics, strengthen the mathematical sciences by encouraging increased participation of African Americans and members of other underrepresented groups, facilitate working relations among them, and provide assistance to them in cultivating their careers" (CAARMS website). Massey has characterized his CAARMS' and other mentoring efforts as producing new colleagues "rather than waiting around to see" (Kenschaft, 2005, p. 194) if they would emerge. As attendees of CAARMS conferences over the years, we contend that what distinguishes the CAARMS community is something much more than the melanin content of the participants. Greater significance comes from participants having shared histories of navigating (attempting to navigate) the challenging trajectories through the stages of becoming a mathematician and having a fruitful mathematics career while Black.

Using African American Cultural Resources to Inform Mathematics Teaching, Learning, and Knowledge Production

What are the implications for mathematics teaching, learning, and knowledge production if these activities occur in ways aligned with values and features of African American culture? As alluded to above, it can seem unnatural to think about mathematics in a cultural context unless consideration is given to mathematics as a community of practice (more accurately, as communities of practice). Without doing so, one might wonder what is cultural about long division, adding and subtracting fractions, or finding a derivative? To begin to envision and imagine MCRF implications in the context of African American culture, it is helpful to unpack what we mean when we speak of "African American culture."

We almost want to say there is some sense of "you know it when you see it." You know it when you feel it. We share some impressionistic thoughts. To get a feel for African American culture, spend time in Black churches, including hearing (and participating in the call and response of) the sermon, singing the hymns, and eating food prepared by the Church Sisters; hang out in a Black barbershop and a Black beauty salon for a few hours; observe Black girls jumping rope or playing hand games; listen to some Miles Davis and Nina Simone and Billie Holiday and Betty Carter and John Coltrane and Duke Ellington and Sun Ra and Mahalia Jackson and the Last Poets and Aretha Franklin and Jimmi Hendrix and the Fisk Jubilee Singers and McCoy Tyner and Gladys Knight and the Pips and Smokey Robinson and James Brown and Ray Charles and Charley Pride; read and/or listen to some speeches by Dr. Martin Luther King, Jr. and Fannie Lou Hamer and Malcolm X and Frederick Douglass and Michelle Obama and Rev. Garden C. Taylor and Sojourner Truth, etc.; talk to some Black (great)grandmothers and (great)grandfathers over a meal prepared by said (great)grandmother, begun with a few minutes or so "grace"; go to a soul food restaurant and an Ital restaurant and a backyard barbecue; do a review of Black handshakes (including ways of "slappin' five") over the past few decades; attend a CAARMS (Conference for African American Researchers in the Mathematical Sciences) and/or a NSBE (National Society of Black Engineers) and/or a NOBCCE (National Organization for the Professional Advancement of Black Chemists and Chemical Engineers) and/or a NSBP (National Society of Black Physicists) conference; read about George Washington Carver and Benjamin Banneker and Thomas Fuller and Katherine Johnson and Percy Julian and James Edward West and Charles Drew; read about "the Black Wall Street" and about the more than sixty townships founded and settled by African Americans between 1865 and 1915 (Brown, 2015); etc.

Henry Louis Gates, Jr. stated in a 2007 Mother Jones interview, "You know, there are 35 million black people in this country and there are 35 million ways to be black" (Hochschild, Baptiste, Patterson, & Corn, 2007). Along with Professor Gates, we recognize the impossibility of the task of defining African American culture in any way that fully, accurately addresses the multidimensionality of and variability within African American people and the communities of which we are a part. Our impressionistic strokes in the previous paragraph, if anything, underscore this reality. At the same time, we imagine that such a collection of experiences, if undergone by someone not Black, would disabuse any notions of deficits of creativity, innovation, imagination, intellectual capacity, and/or "grit" within African American communities, hence revealing the inherent shortcomings of deficit-based approaches to addressing the underrepresentation of African Americans in the mathematical sciences. For fellow African Americans, contemplating sets of experiences such as the above serves as a reminder of the richness of the wellsprings of cultural resources from which we might draw in order to participate more meaningfully in what remains an inadequately explored frontier for us. We maintain that an MCRF approach provides affirming frames for accessing our cultural resources in mathematics settings.

With caveats to avoid perceptions of monolithic characterizations of African Americans in mind, we believe that it can make sense to speak of African American cultural patterns. Murrell (2002) refers to "a set of enduring cultural patterns in the African American community" that emerges from a "continuous historical struggle for full citizenship and literacy" (p. 31). This set builds upon and/or merges with cultural features traceable back (primarily) to West African societies. One such cultural pattern described by Murrell that pertains to education is intergenerational communication and teaching. Another is "the inseparability of education from all other aspects of social and cultural life" (p. 31); education is viewed as "the process of becoming a capable and full participant in the intellectual, cultural, spiritual, and political life of the community" (p. 32). Jones and Campbell (2011) employ the time, rhythm, improvisation, orality, and spirituality (TRIOS) framework to represent "the critical elements of the cultural psychology of African Americans." According to Jones and Campbell, these elements "provide the means by which African Americans employed African cultural origins in their adaptations to the context of slavery and dehumanization" (p. 10).

Considering the above frames and models, as refrain we speculate on implications for African American participation in the mathematical sciences – in mathematics teaching, learning, and knowledge production – if we, as African Americans, draw and build upon cultural patterns and elements in practicing mathematics. Powerful examples abound of how Black people have drawn upon a cultural value attribute such as improvisation to push boundaries and to be at the cutting edge of the arts and of sports. We, the authors, in this nascent stage of our own mathematics research informed by MCRF principles, have experienced sessions that have felt like playing jazz. Grids, diagrams, music symbols, and mathematics symbols have served as notation for motifs to be played on an assortment of instruments – pencil and paper, marker and whiteboard, drums, wave generator, and oscilloscope. In these moments, we have fallen into exchanges of ideas that have felt like explorations of rhythm and melody played across disciplinary boundaries (what boundaries?). Mathematicians often speak of beauty and elegance in mathematics. We, the authors, have experienced moments of beauty and elegance in our exchanges. We have also experienced moments that have caused one or the other of us to exclaim, "That was funky!" (as James Brown might say).

Diversity as a Means and Not an End

Though the data reflecting underrepresentation of African Americans within the mathematics community remain troubling, we observe that having the diversity discourse driven primarily by these data has been ineffective in leading toward substantive changes to status quo practices. We have come to believe that diversity by the numbers arguments will not compel the quality of action required for transformative impact. Indeed, increasingly, we have come to view diversity as a means and not an end. Our work and life experiences have pushed us to think of strategies that address diversity challenges and opportunities along the way of achieving compelling, transformative aims that resist being packaged as diversity for diversity's sake. In the case of mathematics, we are unconvinced about the efficacy of moral arguments, of arguments about fairness and about social justice, at least insofar as providing rationale to others to change systems and structures that currently suit them perfectly fine. Similarly, diversity by the numbers doesn't serve our communities well either. Just as closing achievement gaps doesn't really lead to rallying cries for inspired action, correspondingly, an aim to produce more PhDs from our African American communities can feel uninspiring, unless tethered to matters that resonate more deeply in these communities. Consequently, much of our work has been organized around building the mathematics capacity of African American communities. What does this look like? What does it look like for a Black community to have mathematics capacity?

As mathematics practitioners, we also are excited and motivated by the potential for transforming mathematics knowledge production. We believe that diversifying the knowledge traditions from which mathematicians draw to inform the mathematics body of knowledge might lead to breakthroughs in how we produce mathematics as well as breakthroughs in mathematics content, including the potential for new areas of mathematics. Wilson (as cited in Herzig, 2004) notes the instability and vulnerability of natural, social, and economic systems that fail to diversify. The mathematics enterprise is likely enriched by opportunities to draw upon an expanded range of thought provided by a more diversely composed mathematics community (Herzig, 2004).

We speculate and assert that drawing upon ways that time, space, pattern, and arrangement are explored and manipulated within African, African American, and other African Diasporic cultural traditions – such as music, dance, visual arts, textile patterns, etc. – provides ripe possibilities for producing new, groundbreaking mathematics knowledge. Thus far, much of the research and educational focus has been on making cultural connections as part of a pedagogical approach while neglecting the potential for transformative impact on mathematics knowledge production. As one illustration of what a cultural impetus for mathematics thought and knowledge production looks like, recent Fields Medal recipient Manjul Bhargava attributes the greater part of his development as a mathematician to his childhood introduction to and subsequent continued involvement with Sanskrit poetry and Indian classical music (he is a master tabla player) (Klarreich, 2014). With respect to African/African Diasporic cultural traditions, ironically but unsurprisingly, there are non-Black researchers who have recognized the mathematical richness of Black arts and cultural traditions and have built significant bodies of research work around this. Unfortunately, many from the Black community have been so conditioned to having our cultural contributions overlooked, neglected, and/or relegated to "less than" status that we walk past gold mines of groundbreaking possibilities accessible to us. We maintain that this is one of the outcomes that arise when "the types of mathematics that are valued and pursued" are dictated through a small subset of available cultural frames of reference (Herzig, 2004, p. 174). As noted in various places in this chapter, in our own work, we seek to draw from the largely untapped mathematics knowledge production potential of African and African Diasporic music traditions. For example, in various African cultures, there are no separate words for music, dance, poetry, and song, each of which is experienced as dynamic expressions of related patterns and forms (Nketia, 1974). What in the West may seem to be disparate activities are perceived as related aspects of one and the same thing in these African societies. It is highly likely that very powerful mathematics ideas are embedded in the cultural weave and its manifold expression of underlying structures. For example, African/African Diasporic drummers understand "two-ness" (or "n-ness") on intellectual, emotional, and kinesthetic levels by virtue of the manner in which cycles and subdivisions of pulses (based on 2 or on n) are manipulated in rhythmic patterns. We speculate that, similarly, transformations and iteration-recursion are embodied in the drumming structures, in the coordination of multiple complex ensemble parts, and in the strategies employed in improvisation. A further example of the potential that exists in using an MCRF approach drawing upon African/African Diasporic cultural traditions has presented itself in our (the authors) own spectral geometry research agenda. A significant body of spectral geometry research has been generated and informed by the question "Can one hear the shape of a drum?" Referencing African and African American drum traditions, one is led naturally to a question such as "Can one hear a family of drums?"

Expanded Definitions of Mathematics Success

In describing factors that influence or contribute to the successful math experiences of African American students, Russell (2013) categorizes what she refers to as "micro-" and "macro-"environmental factors. The "micro-"background refers to a student's present background, which is brought to mathematics educational settings. Micro-factors influence the mathematical achievement of all students. However, as Russell (2013) asserts, Black students also bring to mathematics education settings what she characterizes as "their *macro* milieu – the mathematics experiences, learning and achievement associated with the adults and peers in their networks as well as all the generations that precede them" (p. 297). Echoing Murrell's contextual framing noted earlier, included in the macro-milieu are "the ramifications and circumstances of 300 years of deliberate educational, political and social actions taken to systematically deny Blacks to fully participate in the study of and access to mathematics" (p. 297).

Martin (2006) considers the "ways [in which] mathematics learning, participation, and the struggle for mathematics literacy [can] be conceptualized as *racialized forms of experience* – that is, as experiences where race and the meanings constructed around race become highly salient" (p. 198). In sharing narratives from African American parents advocating for quality mathematics education for their children, Martin highlights "specific instances in [the parents'] mathematical histories where race played a lasting and prominent role" (p. 223). For example, some of the parents related how they had drawn conclusions and internalized beliefs about Black inferiority in mathematics because their mathematics teachers had been almost exclusively White. In the words of one parent, "I saw people that were not Black as my teachers. So that made me self-consciously come on with the thought that… White people are smarter than us. That's why most of my teachers are White" (p. 210).

Martin (2009) contends that it is fundamentally important for mathematics educators to develop "awareness that classroom practices influence the construction of academic and mathematical identities and that these identities are co-constructed with students' racial identities" (p. 299). According to Martin (2006), mathematical identity "encompasses an individual's self understanding in the context of doing mathematics" (p. 206).

McGee (2015) has developed a Fragile and Robust Mathematical Identity Framework that "explores the interplay of mathematical and racial identity in the experiences of Black college students" (p. 601). She notes that as students strive to achieve mathematics success in racialized environments, including "proving one's mathematical talents," "repeated negative racialized experiences can produce unhealthy consequences, even while academic scores remain high" (McGee, 2015, p. 603). McGee and Martin (2011) conclude that Black students "must make meaning, on their own terms, for *being Black* in the context of doing mathematics and for what counts as success" (p. 47).

Speaking to conditions in the professional mathematics community, Walker (2014) observes that "Black mathematicians across generations… although they are convinced that mathematics as a discipline is a worthwhile endeavor… acknowledge that practices and structures within the broader mathematics community do not necessarily support or invite Blacks into the field" (p. 145). "[A]ssumptions and discourses about intellectual ability and merit… rooted in historic and pervasive narratives about Black achievement" (p. 100) contribute to a "spotlight effect" which triggers self-doubts and/or hypervigilance about how one presents oneself (such as dress, such as whether or not to ask a question out of concern that it will be taken as an indication of lack of competence).

What does success look like for Black students throughout their student careers? What does success look like for Black people who are mathematics professionals? What does mathematics success look like for African American communities? Of course, we neither intend nor are able to answer these questions definitively in this chapter. However, they are generative questions that inform and motivate our effort to shape and articulate an MCRF. As we reflect on our personal experiences, we know what one can wind up sacrificing in order to function within settings that are recognized or felt to be unwelcoming. The ability to navigate through "culturally dissonant" environments is a needed competency for African Americans. We assert that among the criteria for determining success is that one's cultural, racial, social identities remain robustly intact as one develops a mathematics identity aligned with these more fundamental identities.

Creating Intentional Culturally Resonant Mathematics Spaces

Walker (2012), in considering how mathematics identities are developed, notes the importance of taking into account experiences that occur across in-school and out-of-school "spaces," including spaces that, along with mathematical meaning, have social and cultural significance, as well. This is consistent with a paradigm that recognizes that all knowledge systems, including mathematics, are experienced and processed through frames of reference provided by our overall social and cultural lives (Nasir et al., 2008). Walker (2012) suggests that "we move from these sometimes 'inadvertent' spaces that foster development for individuals to creating and examining 'intentional' spaces that contribute in strong ways to mathematics socialization and talent development…[and that] reflect the bridging of out-of-school and in-school networks, relationships, and practices" (p. 68). The existence and proliferation of such intentionally created spaces are critically important for achieving the transformative potential of culturally resonant mathematics teaching, learning, and knowledge production.

The Algebra Project and its youth-driven offshoot the Young People's Project (YPP) provide examples of intentionally creating teaching and learning spaces that draw from and "leverage students' everyday social and cultural knowledge" (Nasir et al., 2008, p. 213) to deepen mathematical understanding and develop resonant mathematics identities. The Algebra Project uses a five-step curricular process to enable students to use their own language and intuitive representations to make robust links between concrete experiences and formal, abstract mathematics representations (Moses & Cobb, 2001). What drives the Algebra Project is a goal of "empower[ing] the target population to demand access to literacy for everyone" (Moses & Cobb, 2001, p. 19). A primary aim of YPP is "to train, employ, and support …high school students to become Math Literacy Workers [who]… begin their journey by teaching math to elementary students in their neighborhoods and eventually become engaged citizens prepared to make a difference in their own lives, in the lives of others in their communities, and ultimately in this country" (YPP, n.d.-a). YPP, among other methods, uses "mathematically rich games" to "create a cultural context in which mathematics emerges naturally from students' experience" (YPP, n.d.-b, para. 1). As an example of mathematical richness, in the Flagway Game™, students navigate a course of paths based on the Mobius function.

As we consider an MCRF, we imagine further transformative possibilities for the mathematical sciences that might emerge from African Americans having greater opportunities to practice mathematics in welcoming, culturally affirming spaces (physical, social, psychological). We imagine mathematical spaces created with even more intention to harness and engage cultural resources than has typically been the case, particularly with respect to mathematics knowledge production. For example, we envision efforts to encourage, facilitate, structure, and/or support collaborations between Black mathematical sciences practitioners and Black artists (including artists from Africa and the African Diaspora) and between Black mathematical sciences practitioners and practitioners of various other African/African Diasporic cultural traditions, such as games.[5] We imagine and envision (infra)structures, such as

[5] Kyule (2016), in describing bao, a variant of the mancala group of African board games (Kyule 2016), writes: "Bao is not a game of chance and victory is never a function of luck; rather, foresight is the key to winning the bao. It requires considerable calculative strategy and is completely dependent on one's ability to reason and analyze. According to de Voogt (1995), the speed of the game

cultural resonance-focused institutes and centers that help to make "the net" work more effectively.[6] We contend that ripe and rich opportunities for intentionally creating such culturally resonant mathematics spaces exist in efforts to build the mathematics capacity of African American communities.

Building the STEM[7] Capacity of African American Communities: The WORASI Story (As Told by John Belcher)

We have been engaged in an endeavor to build the mathematics and science capacity of the predominately African American West Ocala community in Ocala, Florida. Because this work has been key to our process for defining and developing an MCRF, we describe in relative detail our work to date in this community. We believe that the West Ocala and Roots and STEM Initiative (WORASI) ultimately can serve as a model for how approaches based on an MCRF might be employed to build the STEM capacity of other African American communities.

WORASI is a community-based effort that emerged out of a partnership involving the Greater Ocala Community Development Corporation (GOCDC), Medgar Evers College, the Second Bethlehem Baptist Association, the Howard Academy Community Center, and other organizations and institutions with a stake in a healthy vibrant future for the West Ocala community. The Greater Ocala Community Development Corporation is coordinating the West Ocala Roots and STEM Initiative as part of its mission to "establish a culture of excellence and self-sufficiency…in the areas of employment, housing, entrepreneurism, and community development" (The Greater Ocala Community Development Corporation, n.d., para. 1). STEM provides powerful lenses for identifying additional opportunities to have meaningful impact in these areas, given the vital importance of STEM capacity to the quality of life in and to the economic, political, and social well-being of communities, particularly communities of color. Further, embracing STEM capacity building as a community-based effort presents opportunities to develop a model based upon the rich histories and cultural traditions of West Ocala.

---

means advanced players must make complicated and highly strategic moves in quick succession, and bao masters are renowned for being able to think tactically up to seven moves ahead" (p. 98).

[6] Oftentimes, networks exist nominally – without thoughtful, planful consideration about what resources exist within the network (i.e., what types of skills, expertise, materials, etc.), how these resources can be accessed, and, generally, how the network "talks to itself" in a more distributed, ongoing fashion. More often than not, the "net" is in place; missing are connection mechanisms and strategies for making the net work.

[7] Though we have written in the chapter about mathematics, the "mathematical sciences," and building "mathematics capacity," the work described in this section has been discussed and organized with members of the West Ocala community as an effort to build "STEM capacity," more generally. For accuracy's sake, we maintain that characterization in describing the West Ocala initiative. Nonetheless, we feel that the effort effectively illustrates MCRF principles.

The seeds for the West Ocala Roots and STEM Initiative were sown during my visits to Ocala, Florida, where my dad has resided since the early 2000s. During visits with my dad and stepmother as they dealt with the ordeal of my stepmother's terminal cancer, I would accompany my dad to the gym that he typically attended 6 days per week. I observed that my dad's gym activity not only kept him physically strong but emotionally strong and supported as well. Having this community of support and genuine played a major role in enabling my dad to move through that very trying, painful, difficult period of the illness and passing of his wife. One of the key members of the gym community of support was Jimmi Griffin.[8]

Across several gym visits, Mr. Griffin and I had conversations about our respective professional interests and experiences. We spoke about my investigations over the years exploring connections between music and mathematics and how this related to my work at TERC, a science and mathematics education research institution, where, at the time, I co-coordinated a STEM and Boys of Color effort. We discussed Mr. Griffin's work in the West Ocala community, particularly through his role as President of the Greater Ocala Community Development Corporation and his experiences over the years focused on the political and economic empowerment of communities of color. We began to consider what it would look like to use STEM lenses to frame and bring additional tools to the economic development and community revitalization and empowerment of Black communities, such as West Ocala. What would those types of lenses and tools bring to the work in which Mr. Griffin and colleagues at GOCDC were engaged in the West Ocala community? Based on this early dialogue, we then considered how to expand the conversation to involve others in determining collaboration possibilities. We met with Henry DeGeneste, the Chair of the GOCDC Board of Directors, a meeting that resulted in further expanding the conversation to include various individuals who had a history of West Ocala community organizing and advocacy over the years.

In parallel with the early dialogue with Mr. Griffin and Mr. DeGeneste, I had multiple and extended conversations with my dad about how the envisioned West Ocala work linked with the responsibility/charge that I felt to build upon our family histories and legacy, including (1) my dad's pioneering experiences as the first Black department head at City College, a prestigious Baltimore high school, as the Coordinator of the National Education Association's newly formed (at the time) Urban Services Division, and as principal of Carver Vocational Technical High School, which received visits from a number of international delegations during his tenure; (2) the legacy of my grandfather Smith, the first Black varsity athlete (football) at what is now Michigan State University and head football coach and assistant athletic director at Hampton Institute (now Hampton University) for many years; (3) the achievements of my grandfather Belcher, high school principal and one of the first African Americans to receive a masters' degree from the University of

[8] At the time of our meeting, Jimmi Griffin was the President of the Greater Ocala Community Development Corporation (GOCDC). He played a linchpin role in the work of organizing and facilitating the WORASI effort. Sadly, Jimmi Griffin passed away on February 12, 2017.

Michigan School of Education; (4) the experiences and achievements of my mother, Mildred Smith Belcher, and her groundbreaking work as an educator using dance, theater, and sports as media for empowering students; and (4) my own education pioneering experiences, beginning as a 5-year-old, my age when I entered the school I attended as the first "Negro" student in the school's history. Recognizing the links of the West Ocala work to our family's histories and traditions, my dad contributed $15,000 to the emerging effort. These funds were used in part to purchase conga drums, a djembe, an oscilloscope, a wave generator, and a mechanical wave driver, along with other supplies. The equipment was intended for use in demonstrating natural frequencies of strings, Chladni patterns, harmonic rhythm (typically called "polyrhythm"), and Lissajous figures, among other ideas, and resonance phenomena embedded within the shape of the drum collaboration.

Terrence, Mr. Griffin, and I co-facilitated a 2-day meeting at the Howard Academy Community Center in January 2014 with a group of West Ocala community stakeholders. We chose to hold the meeting at the Howard Academy Community Center given its historical significance in the West Ocala community.[9] In fact, an aspect of the motivation for undertaking this effort in West Ocala and for taking the approach that we took was the recognition that many residents, particularly, the youth, had lost touch with the powerful history of that community, a history of achievements which were not very well reflected in the community present day.

A key aim of the early conversations, meetings, and presentations was to develop some shared language and a shared vision of what it would mean to build the STEM capacity of the West Ocala community in ways that had value to residents of the community. Terrence and I were intentional in making sure that building *community capacity*, linked meaningfully to other community efforts, remained at the forefront in defining how we proceeded with and built this initiative. In other words, the work wasn't focused on what programs we could bring to schools. It wasn't about developing new school curricula or about creating alternative schools. It wasn't limited to thinking about classroom-based teaching and learning of mathematics and science. The default conversations about mathematics and science typically go to what can be done with children in schools and/or afterschool and summer programs geared toward helping students to become more successful in their school experiences. Mr. Griffin, Terrence, and I envisioned harkening back to the earlier, trailblazing days of the community,[10] when residents were innovating; were breaking ground; were building the community; were establishing schools, businesses,

[9] Howard Academy was the first school for Black students in Marion County. The original building was destroyed by a fire in 1887 (City of Ocala, n.d.). In the 1980s, the school stopped serving students as a result of desegregation. Recently, it reopened as a community center.

[10] "The major outlines of the African American freedom struggle in Florida emerged in the early moments of Reconstruction in Marion County. A Bureau of Refugees, Freedmen, and Abandoned Lands agent, Jacob A. Remley, observed that African Americans in that rural county were organizing themselves 'for religious worship [and] the mutual relief of one another in sickness and pecuniary distress'" (Ortiz, 2005, p. 9).

Fig. 7.2 Founders of Metropolitan Realty and Investment Corporation, the first African American corporation to be granted a charter by the State of Florida (West Ocala Vision and Community Plan, 2011)

churches, and other institutions; and were making history[11] (Figs. 7.2 and 7.3). We considered the type and quality of aims and visions that motivated and sustained those achievements. At the same time, though we had our own ideas that compelled us to take action, we agreed that it was important in the early stages of the West Ocala effort to develop a shared vision with community members and not impose a fully fleshed out imported vision.

In the months that followed, we held meetings and presentations/demonstrations in various locations in West Ocala, making prominent use of the drums and scientific equipment, when and where possible. The primary aims of these events were to provide opportunities for participants to experience links among STEM, African/African Diasporic cultural traditions, and the history of the West Ocala community. In addition, we aimed to shape collaborative efforts that respected and built upon previous and current community-based efforts focused on community empowerment and revitalization. We discussed envisioned work honoring, acknowledging, and showing value for the earlier efforts and emphasized that the dynamic was not meant to be one where some outsider comes in with the latest and greatest idea, as

[11] In 1891, Mr. Gadsden, along with several partners, organized the Metropolitan Realty and Investment Company. It was the first African American corporation to be granted a charter by the State of Florida. By 1914, Black residents in West Ocala were said to be some of the most prosperous in the South (City of Ocala: West Ocala History).

Fig. 7.3 In 1925, Dr. Hughes opened the American National Thrift Association Hospital, which was the only facility to treat African Americans for hundreds of miles (West Ocala Vision and Community Plan, 2011)

if nothing had been attempted or accomplished before. Rather, we needed West Ocala community stakeholders to help figure out how to get to synergy – to combine efforts and to meaningfully collaborate.

Clarifying What It Means to Build STEM Capacity

We held some presentations and demonstrations at a few community centers. In each event, we were struck by the level of engagement of the young people who participated. At a presentation at Howard Academy Community Center, we presented to a group of about 50 young people ranging in age from 5 to approximately 16–17 years old. We presented mathematically and musically related activities. Because of the frames provided by our early conversations with stakeholders, we were able to use these presentation experiences to help provide insight into what it might mean to build STEM capacity in the West Ocala community.

One of the activities we presented involved my playing a pattern on a drum and asking participants to determine how many times I struck the drum. Though we asked the question, “How many times did I strike the drum?” our true interest was in the strategies used to come up with an answer. “How would you solve the problem?” “What strategy did you use?” Not just, “what number did you get?” At one point, the youngest person in the room, a 5-year-old, volunteered to give his answer. He described a strategy that involved recognizing that I played a pattern that repeated a few times, followed by me striking the drum some extra beats at the end. To arrive at an answer, he counted the number of beats in the repeating pattern, multiplying that by the number of repetitions and adding the extra beats at the end. His sophisticated strategy for solving the problem clearly demonstrated his capacity for complex mathematics thought. Again, as we reflected on the experience, we recognized that community capacity around STEM would mean that community resources would be available and employed to nurture this child's talent, to put him in touch with individuals to mentor him and to guide his development over time. In so doing, there would be emphasis on individuals being ambassadors of the community as their talents develop and flourish, communicating messages that extend beyond personal glory and achievement. This type of emphasis, through showing how these talents benefit and advance the community, would help develop the capacity of the community.

Along with the clarity that emerged about compelling means for building community STEM capacity focused on the development of young people in ways that extend beyond success in their classrooms, we also began to consider the implications of building STEM capacity for the kinds of STEM-related, STEM-driven businesses that would benefit the community. We considered STEM-involved ways of equipping/empowering adults in their various roles as caregivers and as citizens. As we considered possibilities, we determined that involving community members in asset inventorying and needs assessment activities would be critical in shaping plans for moving forward.

Community Assets

In one of our early meetings, as we discussed the history of West Ocala and considered questions about building the STEM capacity of the West Ocala community, one of the participants shared that Daphne Smith, the first African American woman mathematics PhD recipient from the Massachusetts Institute of Technology (MIT), hailed from Ocala. The participant related how Dr. Smith had excelled both academically and athletically in high school. The various meetings and presentations turned out to be effective forums for identifying examples of individuals connected with the West Ocala community, who powerfully represented the types of community assets and resources in place and potentially available in developing a visible STEM presence in that community and in building and/or activating STEM capacity.

Over time, we have been able to identify various resources within the West Ocala community potentially available for building the STEM capacity of this community. Among the potential resources is an abandoned building that sits on GOCDC property behind its headquarters. GOCDC leadership has taken some initial steps in pursuing plans to convert this building into a WORASI STEM Cultural Resonance Center.

WORASI STEM Cultural Resonance Center

Among the envisioned activities to occur in the WORASI STEM Cultural Resonance Center are research activities open and visible to the community, presentations-demonstrations, lectures, performances (which integrate arts and STEM disciplines), and a variety of education activities. Envisioned education activities include (1) rhythm, mathematics, and technology percussion ensembles, (2) math circles, (3) coding, and (4) learning about the rich history of the West Ocala community. We have discussed plans for GOCDC housing a server to be managed through the Center. As envisioned, such a server will provide opportunities to host websites and to provide cloud services in support of collaborative activities among WORASI partners and other organizations and individuals involved in building the STEM capacity of West Ocala. We anticipate that these types of services and activities will motivate the need/opportunity for training residents from the community in skills such as website design and maintenance and in managing the server, thereby contributing further to STEM capacity building.

Moving Forward

The WORASI effort is a work and story in progress and in process. This effort, in all its complexity and moving parts, remains key in achieving our aims for understanding and using cultural resonance to build the mathematics capacity of African American communities.

Summary and Conclusion

In this chapter, we propose a mathematics cultural resonance framework (MCRF) to inform mathematics teaching, learning, and knowledge production in ways that affirm and draw upon African American cultural traditions and resources. We argue that defining efforts around closing achievement gaps and/or through making moral arguments, such as has often been the case, is insufficient for compelling the levels of commitment and response needed to address meaningfully issues that contribute to the seeming intractability of Black underrepresentation in the mathematical sciences. The equity and access issues at play are embedded in the histories of oppression and devaluation faced by Black people in this nation. We assert that culturally resonant approaches have the potential to generate transformative patterns of thought and activity that facilitate African Americans developing robust mathematics identities required for meaningful participation in mathematics. Further, we maintain that these approaches provide ripe opportunities for producing new, groundbreaking mathematics knowledge, thereby benefiting the mathematics community (and society) as a whole.

Through work in the West Ocala community of Ocala, Florida, we have begun to investigate implications for using an MCRF to build the mathematics capacity of African American communities. The value to African American communities of having “mathematics capacity” extend far beyond considerations that typically go by default to programs limited to helping students to become more successful in their school experiences.

Overall, we view this chapter as an introduction to a framework we believe supports making some novel connections of “the dots” crucial for achieving the meaningful participation of African Americans in the mathematical sciences. Our eyes are on the prizes to come from the actual work that arises from the frame (alluding to our earlier note on “making the net work”), step by step, step by leap.

References

American Mathematical Society. (n.d.). *What do mathematicians do?* Retrieved from http://www.ams.org/profession/career-info/math-work/math-work

Belcher, J. (2006). *The sound according to John [CD]*. St. Paul, MN: Innova Recordings.

Belcher, J. & Blackman, T. R. (2013). Hearing the drum of the rhythm. In G. Hart & R. Sarhangi (Eds.), *Bridges Enschede, conference proceedings 2013*. Paper presented at Bridges Enschede: Mathematics, Music, Art, Architecture, Culture, University of Twente and Saxion University of Applied Sciences, Enschede, the Netherlands, 27–31 July 2013 (pp. 611–618). Phoenix, AZ: Tessellations Publishing.

Brown, D. L. (2015, March 27). All-black towns across America: Life was hard but full of promise. *The Washington Post*. Retrieved from http://www.washingtonpost.com

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics?. *Educational Studies in Mathematics*, *37*(2), 121–143.

City of Ocala: West Ocala History. (n.d.). Retrieved from http://www.ocalafl.org/living/historic-preservation/west-ocala-history

Conference of African American Researchers in the Mathematical Sciences. (n.d.). *About*. Retrieved from http://www.caarms.net/custom.aspx?qs1=CGhu072513034413ulLA&activeMenu=about

D’Ambrosio, U. (2001). What is ethnomathematics and how can it help children in schools? *Teaching Children Mathematics, 7*(6), 308–310.

Greater Ocala Community Development Corporation. (n.d.). *Our mission*. Retrieved from http://www.gocdc.org

Herzig, A. H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research, 74*(2), 171–214. doi:10.3102/00346543074002171.

Hochschild, A., Baptiste, N., Patterson, B. E., & Corn, D. (2007, March 14). *35 million ways to be black*. Retrieved from http://www.motherjones.com/politics/2007/03/35-million-ways-be-black

Jones, J. M., & Campbell, S. (2011). Cultural psychology of African Americans. *Online Readings in Psychology and Culture, 3*(1). doi:10.9707/2307-0919.1023.

Kac, M. (1966). Can one hear the shape of a drum? *The American Mathematical Monthly, 73*(4), 1–23.

Kenschaft, P. C. (2005). *Change is possible: Stories of women and minorities in mathematics*. Providence, R.I.: American Mathematical Society.

Klarreich, E. (2014, August 12). The musical, magical number theorist. *Quanta Magazine.* Retrieved from https://www.quantamagazine.org/20140812-the-musical-magical-number-theorist/

Kyule, M. (2016). The bao: A board game in Africa’s antiquity. In A. Deisser & M. Njuguna (Eds.), *Conservation of natural and cultural heritage in Kenya: A cross-disciplinary approach* (pp. 93–107). London: UCL Press. Retrieved from http://www.jstor.org/stable/j.ctt1gxxpc6.13.

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal, 32*(3), 465–491.

Lave, J., & Wenger, E. (1998). *Communities of practice*. Cambridge, UK: Cambridge University Press.

Leonard, J., & Martin, D. B. (2013). *The brilliance of black children in mathematics: Beyond the numbers and toward new discourse*. Charlotte, NC: Information Age Publishing, Inc.

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning, 8*(3), 197–229.

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record, 111*(2), 295–338.

McGee, E. O. (2015). Robust and fragile mathematics identities: A framework for exploring racialized experiences and high achievement among Black college students. *Journal of Research in Mathematics Education*, *46* (5), 599–625. Retrieved from http://www.nctm.org/Publications/Journal-for-Research-in-Mathematics-Education/2015/Vol46/Issue5/Robustand-Fragile-Mathematical-Identities-A-Framework-for-Exploring-Racialized-Experiencesand-High-Achievement-Among-Black-College-Students/

McGee, E., & Martin, D. (2011). From the hood to being hooded: A case study of a black male PhD. *Journal of African American Males, 2*(1), 46–65.

Moses, R., & Cobb, C. E. (2001). *Radical equations: Civil rights from Mississippi to the Algebra Project*. Boston: Beacon Press.

Murrell, P. C. (2002). *African-centered pedagogy: Developing schools of achievement for African American children*. Albany, NY: State University of New York Press.

Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of Research in Education, 32*(1), 187–240.

Newell, V. K., Gipson, J. H., Rich, L. W., & Stubblefield, B. (1980). *Black mathematicians and their works*. Ardmore, PA: Dorrance.

Nketia, J. H. K. (1974). *The music of Africa*. New York: W. W. Norton.

Ortiz, P. (2005). *Emancipation betrayed: The hidden history of black organizing and white violence in Florida from reconstruction to the bloody election of 1920*. Berkeley, CA: University of California Press.

Powell, A. B., & Frankenstein, M. (Eds.). (1997). *Ethnomathematics: Challenging eurocentrism in mathematics education*. Albany, NY: State University of New York Press.

Russell, N. M. (2013). Unpacking brilliance: A new discourse for black students and successful mathematics achievement. In J. Leonard & D. B. Martin (Eds.), *The brilliance of black children in mathematics: Beyond the numbers and toward new discourse*. Information Age: Charlotte, NC.

Shanahan, T. (2014). *Raye of light: Jimmy raye, duffy daugherty, the integration of college football and the 1965–66 Michigan state spartans*. Middleton, WI: August Publications.

The Greater Ocala Community Development Corporation. (n.d.). *Welcome to the Greater Ocala Community Development Corporation*. Retrieved from http://www.gocdc.org/

The Young People's Project. (n.d.-a). *About us*. Retrieved from http://www.typp.org/about_us

The Young People's Project. (n.d.-b). *The Flagway™ game*. Retrieved from http://www.typp.org/flagway

Walker, E. N. (2012). Cultivating mathematics identities in and out of school and in between. *Journal of Urban Mathematics Education, 5*(1), 66–83.

Walker, E. N. (2014). *Beyond Banneker: Black mathematicians and the paths to excellence*. Albany, NY: SUNY Press.

Wenger, E. (2009). Communities of practice. *Communities, 22*, 57.

West Ocala Vision and Community Plan. (2011, November 29). Retrieved from http://www.ocalafl.org/government/city-departments/growth-management/ocala-2035-vision/west-ocala-community-plan

You can count on one hand all the black mathematics professors at the highest-ranking American universities. (1998). Winter. *The Journal of Blacks in Higher Education, 22*, 14–16. doi:10.2307/2998808.

Chapter 8
Still Warring After All These Years: Obstacles to a Transdisciplinary Resolution of the Math Wars

Ilona I. Vashchyshyn and Egan J. Chernoff

Abstract Faced with the complex issues of modern society, a growing number of individuals and organizations have embraced a transdisciplinary approach in the attempt to resolve such issues in an ethical, socially responsible way. Such an approach may even prove to be effective in mediating (if not resolving) the math wars, a long-standing, value-laden debate about what (mathematics) children should learn in the twenty-first century and how they should learn it. However, although the math wars have evolved into a conflict involving a wide variety of individuals and groups representing various interests and disciplines, we argue that for this issue, transdisciplinarity is still out of reach. In particular, in reviewing the evolution of the math wars in the United States and in Canada through a transdisciplinary lens, we find that one major obstacle is the reluctance, and sometimes outright refusal, to step outside disciplinary constraints to engage in dialogue and collaboration with diverse stakeholders. We contend that if the attitude of opposition is maintained, we should expect a long and bitter war indeed.

Keywords Canada • Hacker • Math wars • PISA • Transdisciplinarity

> *Math is math. What could there possibly be to fight about?* (Schoenfeld, 2004, p. 254)

The problems of modern society are increasingly complex and interdependent and hence increasingly less isolated to particular disciplines (Thompson Klein, 2004). Faced with such issues, both individuals and organizations have recently started to turn to transdisciplinarity in attempts at reaching resolutions in an ethical, socially responsible way, stepping outside the constraints of disciplines to seek insight from multiple stakeholders both inside and outside of the academic community (Bernstein, 2015; Gibbs, 2015). The transdisciplinary approach has been invoked in attempts to tackle issues as diverse as aging, childcare, health care,

I.I. Vashchyshyn • E.J. Chernoff (✉)
University of Saskatchewan, 28 Campus Drive, Saskatoon, SK S7N 0X1, Canada
e-mail: iiv672@mail.usask.ca; egan.chernoff@usask.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_8

nutrition, sustainable development, and urban land and waste management (Gibbs, 2015; Thompson Klein, 2004). However, we contend that it has yet to be applied in a meaningful way to the long-standing and seemingly intractable debate commonly referred to as the *math wars*, a term used to describe a collection of disparate yet related debates centered on the teaching and learning of mathematics waged on the battlefields of staff rooms, coffee shops, scholarly research journals, newspapers, radio programs, television, online forums, Facebook, and Twitter (to name only a few) and seeming to have left no one without opinions, concerns, and fears related to recent changes in the teaching of mathematics in the schools (Russell & Chernoff, 2016). The issue, which on the surface level is concerned with the questions of *what* (mathematics) children should learn in order to succeed in the twenty-first century and *how* they should learn it, is certainly complex and value-laden, for it involves a multitude of divergent perspectives not only about appropriate curricula, pedagogy, and resources but also about what mathematics content is valuable, which skills will be important in an increasingly technological society, the purposes and goals of (mathematics) education, and so on. (We use the parentheses to emphasize that questions about the value, purpose, and goals of mathematics education are deeply intertwined with questions about the value, purpose, and goals of education in general.) As such, perhaps it is unsurprising that various discipline-based approaches—for example, and as we will see, that which led to the rise (and fall) of the "new math" movement in the 1960s—have thus far been unsuccessful in resolving the three-pronged question of what mathematics children should learn in school, how they should learn it, and why. After all, the values, perspectives, and experiences of actors within particular disciplines may not represent those of the community at large.

Although it may be unlikely that the issue will ever be fully resolved to the satisfaction of any interested party, we contend that a transdisciplinary approach to mediating the conflict may prove to be more effective. Such an approach would focus on the problem and its consequences, rather than its definition or categorization within a particular discipline; moreover, it would involve bringing together as many participants as possible who have a stake in the issue in the effort to negotiate a resolution (Gibbs, 2015; Maguire, 2015). Those interested in the teaching and learning of mathematics represent a wide and growing variety of disciplines and stakeholders, ranging from mathematicians, mathematics educators, and psychologists to parents, policymakers, and economists. However, although it is easier today than ever before to reach out to groups and individuals outside of one's disciplinary domain, many of the actors embroiled in the debate have proven to be reluctant to engage in dialogue and collaboration with other stakeholders and representatives from "outside" disciplines. Perhaps even more worrying than disengagement is a rhetoric that positions some as "experts" and others as "non-experts" in matters of mathematics education, which alienates a majority of the population who has an interest and a stake in children's education and only further entrenches the conflict. We contend that if this attitude of opposition is maintained, we should expect a long and bitter war indeed.

In this chapter, we review the evolution of the math wars and the obstacles to resolution through a transdisciplinary lens, focusing on recent developments in Canada, as the American and international side of the story has been detailed elsewhere (e.g., Schoenfeld, 2004; Kilpatrick, 2012). We begin with an overview of transdisciplinarity and a discussion of why the math wars is an issue that may lend itself well to mediation through a transdisciplinary approach.

A Complex Issue, a Complex Solution

The notion of transdisciplinarity, according to Bernstein (2015), appears to have originated in the 1970s, emerging in response to unsuccessful attempts of "closed system, discipline-based approaches to solve complex social problems" (Gibbs, 2015b, p. 152). By its nature and its recent origins, a variety of definitions of transdisciplinarity exist; we draw our understanding of the notion from Bernstein (2015), Thompson Klein (2004, 2010), Maasen, Lengwiler, & Guggenheim (2006), and Nicolescu (2005). In contrast to *multidisciplinarity* (collected inputs from several disciplines without synthesis) and *interdisciplinarity* (collected inputs from several disciplines aimed at transfer and synthesis of knowledge and methods), *transdisciplinarity* challenges and *trans*cends the framework of disciplinary thinking, rejecting the separation and distribution of topics into disciplines and aiming for overarching synthesis (Bernstein, 2015; Thompson Klein, 2010). The corresponding image is often one of going "beyond boundaries." As Thompson Klein (2004) explains, transdisciplinarity is not a new discipline or a "superdiscipline"; rather, it is "the science and art of discovering bridges between different areas of knowledge and different beings" (Thompson Klein, 2004, p. 516), involving work that "creatively re-imagines the disciplines and the possibilities for combining them" (Bernstein, 2015, p. 7). Nicolescu (2005) and other scholars (e.g., Maguire, 2015) are also careful to point out that disciplinarity and transdisciplinarity are not opposed or antagonist but rather complementary—indeed, the latter cannot exist without the former.

However, disciplinary approaches to tackling many of today's highly complex social problems have often proved to be problematic (Gibbs, 2015). As such, transdisciplinarity reemerged in the 1990s as a new approach to tackling such issues, which range from those in the areas of sustainability, science, health care, and technology, to policy, childcare, and education (to name a few; see Bernstein, 2015; Gibbs, 2015; Thompson Klein, 2004). In general, problems that may benefit from a transdisciplinary perspective tend toward those that are complex, contextual, heterogeneous, value-laden, and involving multiple stakeholders (Bernstein, 2015; Gibbs, 2015; Thompson Klein, 2004). Transdisciplinarity may lend itself well to attempts to solve such many dimensional issues because of its commitment to breaking free of "reductionist and mechanistic assumptions about the way things are related and systems operate" (Thompson Klein, 2004, p. 517) and transcending the either-or, dichotomous mentality that underlies many of today's issues (Bernstein, 2015). Necessarily—and herein lies one of the distinguishing aspects of the transdisciplinary approach to tackling complex issues—transdisciplinarity requires an "engagement with difference across cultural, social and cognitive contexts" (Maguire, 2015, p. 168) and consequently involves meaningful stakeholder and community involvement, including those from outside the academic community who have an interest in resolving the problem and for whom the outcome will be impactful (Bernstein, 2015; Gibbs, 2015; Thompson Klein, 2004; Maguire, 2015).

Although at first glance the math wars may appear to be a disciplinary conflict, upon consideration of the above, it becomes clear that the issue may benefit from being viewed through a transdisciplinary perspective. Certainly, the issue is complex and value-laden: as suggested in the introduction of this chapter and as will become clear shortly, at the deepest level, the math wars involve diverse and sometimes conflicting views about the value and purpose of mathematics education in the twenty-first century and of education in general. Moreover, the act of teaching mathematics is itself "value-full": As Bishop (1999/2008) wrote, "rather than thinking of mathematics teaching as just teaching mathematics to students, we are also teaching students through mathematics. They are learning values through how they are being taught" (p. 236). In line with this reasoning, the expressions "standard algorithm" and "basic skills," may mean different things to different people. If the notion of "standard" is meaningful only in relation to a particular culture or context, an important question arises: Whose mathematics should be learned (and, consequently, valued)? Very quickly, it becomes clear that seemingly pragmatic questions about teaching strategies, curricula, resources, and so on are also ethical considerations with real consequences for the formation of students' identity construction, values, and self-esteem and for society as a whole. The issue also clearly involves a wide range of stakeholders (including children, parents, business and industry leaders, mathematics educators, and mathematicians, to name a few), for *everyone* has a stake in education: The children of today, armed with the knowledge, skills, and values they (partly) develop in school, will shape the world of tomorrow. It is critical, therefore, that their education serves them well.

Note that although fields such as ethnomathematics reveal an increased desire for dialogue between the field of mathematics education and other disciplines, such interdisciplinary approaches still rely on a framework of disciplinary research and, as such, are still constrained to certain methodologies and problem definitions. Although increased interdisciplinarity is a positive development, the approach may still prove to be too limiting when the aim is to resolve an issue as complex as the math wars, for as Maasen et al. (2006) write, "disciplines restrict and shut out certain persons (extra-scientific actors) from becoming part of knowledge production and also certain problems from becoming the focus of research" (p. 395). A transdisciplinary approach, on the other hand, would focus on the problem at hand and its consequences, rather than its definition or categorization (compartmentalization) within a particular discipline. In attempting to mediate the math wars, a transdisciplinary approach would lay bare and examine commonly held assumptions about the value and purpose of mathematics in the twenty-first century that are often hidden when a disciplinary lens is adopted. Necessarily, it would involve bringing together multiple stakeholders into a process of negotiation and collaboration, integrating disciplinary, institutional, and community resources in the attempt to resolve the question of what (mathematics) children should learn, how they should learn it, and—perhaps most importantly—*why*.

As the saying goes, though, some things are easier said than done. In the next sections, we review the math wars through a transdisciplinary lens. As we will see, the barrier to dialogue and collaboration seems to lie less in technology or lack of opportunity—new communication channels have made it easier than ever to engage in dialogue with diverse groups and individuals—but rather in a disciplinary mindset.

The Math Wars: An Abbreviated History

Different authors suggest different points in time as representing the beginning of the math wars, but generally, it is agreed that the conflict first emerged (in North America) in the United States, with its roots in another "crisis": the successful 1957 launch of Sputnik by the Soviet Union, an important victory in the space race. Caught off guard (read: humiliated) and anxious about Soviet threats of world domination, the scientific community on both sides of the Atlantic was spurred into action (Kilpatrick, 2012; Schoenfeld, 2004). An especially noteworthy event in the (pre-)history of the math wars was a 2-week seminar organized by the Organization for European Economic Cooperation (OEEC, now the Organization for Economic Cooperation and Development, OECD) held in 1959 in Asnières-sur-Oise, France, which addressed various proposals for "modernizing" school mathematics curricula, the teaching of mathematics, and the preparation of teachers (Kilpatrick, 2012). (Already, we see an integration of concerns about mathematics education and economics, a diversification of the group of stakeholders in mathematics education.) The session on new thinking in mathematics was led by French mathematician Jean Dieudonné, who recommended that students entering university should be at least somewhat familiar with the logical deduction, the axiomatic method, and the "new language" of sets, mappings, groups, and vector spaces that mathematics had acquired in the twentieth century (Kilpatrick, 2012). Efforts soon began in many OECD countries, including in the United States and in Canada, to reform or "modernize" school mathematics to align with some of the recommendations proposed at the seminar.

For a detailed account of the developments in this era, we direct the reader to Schoenfeld (2004) and Kilpatrick (2012); for the present purpose—and we admit that we are painting with broad strokes—it suffices to say that the movement rather quickly crashed and burned. By Schoenfeld's account, "in a reaction to what were seen as the excesses of the new math" (many critiques pointed to premature abstraction and formalism, arguing that these were not serving the needs of the majority of learners), "the nation's mathematics classrooms went 'back to basics'," where the term "basics" encompassed content, procedures, and pedagogy (2004,

p. 257–58). Again, Canada followed on the heels of the United States in rejecting many aspects of the new math movement that had at one time seemed so promising.

Unsurprisingly, with "back to basics" being the theme of the 1970s, studies in the 1980s revealed that American students showed little aptitude for problem solving (Schoenfeld, 2004). In terms of ideology, the pendulum consequently swung the other way in this decade, but this time, the reform movement was spearheaded by mathematics educators rather than by mathematicians. The National Council of Teachers of Mathematics (NCTM) published several influential reports during the 1980s, but it was the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989), with its focus on mathematical processes—problem solving, communication, reasoning, and connections—that sowed the seeds for the present battle. Advocating an active view of the learning process and challenging the "content-oriented" view of mathematics that had until then predominated in schooling, the *Standards* called for decreased emphasis on memorization and rote practice (Herrera & Owens, 2001). During the coming years, different groups—including the Western and Northern Canadian Protocol (WNCP) (see Russell & Chernoff, 2016)—would produce different materials influenced by or "in the spirit of the *Standards*," some of them dubious, others of high quality but nevertheless unfamiliar and inaccessible to parents, and most of whom had experienced only "traditional" mathematics instruction (Schoenfeld, 2004). Perhaps because of their unfamiliarity, new texts and strategies were soon (and still often are) caricatured as the *new-new math* (Herrera & Owens, 2001).

With the benefit of hindsight and the lens of transdisciplinarity, it is easy to see that one of the major flaws of the new math movement was a disciplinary mindset: Spearheaded by mathematicians concerned about the knowledge that students needed to be prepared for university-level mathematics and by policymakers who associated mathematical prowess with economic success, hastily enforced curricula were accompanied by insufficient professional development for teachers and limited concern for pedagogy and relevance. Perhaps some believed that the math would teach itself. Clearly, we have learned a great deal since then, and a tremendous amount of research in mathematics education is dedicated to the matter of *how* to teach, rather than simply *what* to teach. And yet, debates about the teaching and learning of mathematics have only become more inflamed since the publication of the *Standards*, with the internet serving to democratize and expand the discussion—at least, in the sense that virtually anyone is now easily able to publish and publicize their opinion on the matter. Parents and other interested parties in the United States and in Canada have even formed organizations with catchy names such as Mathematically Correct (United States), Mathematically Sane (United States), and WISE Math (Western Initiative for Strengthening Education in Math; Canada) to build support for their cause. These organizations have been very active, creating websites and petitions, appealing to their local governments, and even organizing rallies; in some cases, supporters number in the thousands. Meanwhile, the controversy has been extensively reported on (and at least in part perpetuated by) in popular media.

Who, What, How, and Why

At this point, before picking up the story in Canada, we pause to identify the major players in the math wars, as well as the main points of contention. Interestingly, although the conflict has swelled, the themes have remained remarkably consistent throughout the years. On the other hand, the adversaries in the conflict (if we choose to soldier on with the martial terminology) have greatly diversified. Although the math wars are often described as being divided into two "camps"—namely, *traditional* and *reformist*—an alternative way to view the controversy is as a conflict involving a large and growing number of disciplines and groups of stakeholders. No longer are questions related to the teaching and learning mathematics considered to be solely within the realm of expertise of mathematicians and teachers; such questions are also addressed by researchers in a range of disciplines (e.g., mathematics education, psychology, cognitive neuroscience, ethnology, policy, and economics). As for the stakeholders, besides the students themselves, they include parents, policymakers, and industry leaders. Of course, as is the case with national politics, the views of any particular individual involved in the debate are nuanced and are not always easily classified as representing a particular discipline or group of stakeholders; moreover, despite surface differences, common goals and values can often be identified. In Schoenfeld's words, "each [camp] can be considered a confederation of strange bedfellows" (2004, p. 281), despite the neat storyline of two dichotomous parties that emerges in the media (see Herbel-Eisenmann et al., 2016) or the storyline of disjoint disciplines and stakeholders that we present. However, as it is impossible to analyze each unique perspective in the math wars, we must inevitably paint the picture with some broad strokes. The reader should keep this caveat in mind as we continue the discussion.

One of the main points of disagreement between various disciplines and groups is curriculum. As noted above, curricula in North America before the 1980s were largely based on content, detailing specific facts and skills that students should learn. Following on the heels of the "new math" curricula of the 1960s, which infamously attempted to incorporate modern mathematics content such as aspects of set theory and modular arithmetic on the recommendation of influential twentieth century mathematicians, the NCTM *Standards* (1989) were radical in that they challenged the "content-oriented" view of mathematics, recommending that curricula focus instead on mathematical processes (Schoenfeld, 2004). While mathematics teachers, mathematics education researchers, and others advocates of process-based curricula argue that they allow teachers to make room for more relevant mathematics and to focus on fostering problem-solving skills, several mathematicians and others have spoken out over the years to assert that they represent "a lowering of real standards" (Schoenfeld, 2004, p. 267) in an artificial attempt to raise success rates (e.g., Klein & Rosen, 1996). In recent years, many parents have aligned with this perspective, advocating content-focused curricula (in Canada, this movement is spearheaded by the public initiative known as WISE Math, which counts more than 1380 signatures on its website calling for increased focus on "standard" algorithms, memorization of facts, and practice of basic mathematical procedures).

The second main point of contention in the math wars is the question of pedagogy. Based on an ever-growing body of research in mathematics education, many researchers in mathematics education in North America today advocate a child-centered view of the learning process and an approach to teaching that gives students opportunities to discover mathematical concepts and procedures on their own and in cooperation with their peers. Such research recommends a decreased emphasis on “rote practice, rote memorization of rules, teaching by telling, relying on outside authority (teacher or an answer key), memorizing rules and algorithms, manipulating symbols, [and] memorizing facts and relationships” (Schoenfeld, 2004, p. 267–68). On the other hand, parents, mathematicians, and some teachers who criticize the notion of “discovery learning” (a blanket term for more child-centered approaches and views of the learning process) typically hold the opposing point of view, namely, that the emphasis in school mathematics should be on individual, rote practice aimed at mastery of basic mathematical facts and conventional algorithms. These are viewed as a foundation upon which understanding can be built, as opposed to a product of understanding. It is not the case that these views are wholly incompatible: as McGarvey and McFeetors (2015) note, the debate often lies in the order in which mastery of basic arithmetic, conceptual understanding, and problem solving should be achieved and on the degree of emphasis that each receives.

Other oft-debated issues in the math wars include technology and resources. Calculators (technology) or no calculators (technology) in the classroom? Should textbook use be minimized so as to discourage students from relying on outside authorities, rather than on their own reasoning? On a deeper level—beneath debates about particular content, curricula, and resources—the conflict is really one of conflicting values and beliefs about the goals and purposes of mathematics education. Who gets to learn mathematics? Whose mathematics should be learned? Why learn mathematics at all? And as Ron Ferguson wrote, “There is nothing quite so violent as a war based on differences in faith” (as cited in Roitman, 1999, p. 130). Perhaps this is why, despite the fact that there appears to be a “large middle ground” (Schoenfeld, 2004, p. 281), the issue has thus far proved to be intractable.

Different authors have proposed different frameworks for organizing beliefs about mathematics education. To give one example, Paul Ernest proposed a typology of five ideologies: old humanist, technological pragmatist, industrial trainer, progressive educator, and public educator (see Ernest, 1991 for more details). It is easy to see how different ideologies may lead to conflicting views about mathematics curricula, teaching strategies, resources, and so on: for example, while an *old humanist* (who believes in maintaining the abstract and rigorous nature of mathematics) may feel that that the circle theorems should be learned by all students because of their inherent value as mathematical abstractions, a *technological pragmatist* (who primarily promotes knowledge and skills that are useful in the workplace and contributes to economic growth) may feel that they are an unnecessary topic in the high school curriculum because they are not directly useful in “real life.” Moreover, even individuals who may be characterized as holding the same ideology may disagree on fundamental aspects of the teaching and learning mathematics. As an example, consider the *technological pragmatist* ideology, which advocates an emphasis on studying knowledge and skills that are relevant to the workplace.

The immediate question is, what knowledge and skills *are* useful in the twenty-first century workplace? Different answers to this question—and there are many, related to one's disciplinary background and values—will lead to different views on appropriate curricula for the twenty-first century. To add to the complexity, "individual educators are not located wholly, exclusively, or unproblematically within one of these ideologies" (Povey, 2003, p. 57). (We add that it is unlikely that *any* individual implicated in the math wars can be located exclusively within one of the ideologies described above.)

Unfortunately, although questions about value and purpose are foundational to the math wars and, as such, need to be confronted directly if the conflict is to be mediated, they seem to be tackled far less often within any discipline or among any group of stakeholders than the question of *how* and *what* mathematics should be taught. Every now and then, however, it does happen that someone (be it a mathematics educator, a mathematician, or, most recently and notoriously, a political scientist by the name of Andrew Hacker, whose case we will examine in a later section) dares to confront these issues directly, typically within the context of a critique of the teaching of mathematics in the primary and secondary schools. As we will see, such overt criticisms of the status quo often cause mathematics educators, mathematicians, and mathematics enthusiasts alike to either scramble for cover or to take the offensive.

This should not be surprising. As mathematics educators, it is too easy take for granted our affinity for mathematics, our ability to "see" mathematics and its value in a variety of situations in our daily lives, and our belief that studying mathematics is inherently a rewarding and worthwhile pursuit. However, it is precisely this bias, coupled with the complexity and seeming intractability of the value-laden conflict that is the math wars, that suggests that the issue may best be mediated from a transdisciplinary perspective. And yet, although a disciplinary mindset was partly to blame for the "new math" debacle of the 1960s, the disciplinary mindset continues prevail in public discussions about mathematics education today. We pick up the story again in Canada.

The Canadian Math Wars

Although popular discussion about the teaching and learning of mathematics in Canada has tended to parallel that of the United States, the "Canadian math wars" really came into force in the 2010s, which saw the 2011 release of Michael Zwaagstra's Frontier Center for Public Policy (FCPP) report entitled "Math Instruction that Makes Sense: Defending Traditional Math Education." The report sparked widespread public debate about the state of mathematics teaching in Canada and thrust into the media the initiative known as WISE Math, a coalition of mathematicians, parents, and other individuals dedicated to lobbying for changes in mathematics curricula and instruction (in particular, for increased focus on "standard" algorithms, memorization of facts, and practice of basic mathematical procedures). Since then, WISE Math has continued to gain support; as of June 2017, its website counts more than 1405 signatures supporting the initiative and has inspired other petitions around the country. Strangely, although WISE Math has been actively advocating its cause in the media and has repeatedly requested meetings with provincial Ministers of Education, invitations for discussion with provincial mathematics teacher associations have either not been extended by either party or have not been well-received. Generally speaking, the lack of engagement with/of mathematics teacher associations, who in theory represent the teaching and learning of mathematics in each of their respective provinces, is a curious gap in the debate. (Worthy of note: Several years ago, the British Columbia Association of Mathematics Teachers [BCAMT] did release a brochure for parents in response to some "frequently asked questions" (BCAMT, n.d.).

The next major development in the Canadian math wars occurred in December 2013, which saw the release of the 2012 PISA results (Brochu, Deussing, Houme, & Chuy, 2013). According to the results of this OECD assessment, Canadian students achieved strong results in each of the three processes assessed by PISA, but the media and many policymakers focused on Canada's ranking relative to other countries, which had decreased by three spots from the previous cycle (McGarvey & McFeetors, 2015). Not shying away from hyperbole, notable newspapers such as *The Globe and Mail* announced that Canada was doing no less than "failing to effectively teach [its] students math" (Editorial, 2013, par. 1), and, memorably, CEO and president of the Canadian Council of Chief Executives John Manley declared that the results were "a national emergency" (as cited in Editorial, 2013, par. 3). As McGarvey and McFeetors (2015) write, public outcry was unprecedented: "Petitions were launched in three provinces [Alberta, British Columbia, and Ontario] and thousands of people petitioned for a 'back to the basics' approach to teaching" (p. 116), denouncing inquiry-based, "discovery" curricula. But the war hasn't only been waged online: For instance, on April 12, 2014, nearly 200 people attended a rally at the Alberta Legislature to protest Alberta's new math curriculum, carrying signs such as "Drills 4 Skills," "Fundamentals First," and "Give Us Education, Not Fads" (Editorial, 2014; Wong, 2014). The group included parents, some current and former teachers, and even former University of Alberta Faculty of Engineering dean Ken Porteous, who contended that he had studied "a good deal of mathematics" and that "there's really nothing to discover" (Editorial, 2014, par. 5). Again, provincial mathematics associations were curiously absent from the debate, as were mathematics education researchers. Is it possible that the debate has simply escaped the radar of our ivory towers? The expansive media coverage of the math wars alone suggests that this cannot be the case.

Spurred on by the PISA powder keg (i.e., the release of the 2012 PISA results), at least 70 newspaper articles on mathematics education were written in Canada between September 2013 and August 2014 alone (Herbel-Eisenmann et al., 2016), including a 44-part series organized by journalist David Staples in the Edmonton Journal entitled "The Great Canadian Math Debate" with titles such as "'This new math is stealing their confidence and their dreams'—educators speaks out against new fuzzy math curriculum" (Staples, 2013). The number of comments on some of these articles numbers in the hundreds. Who was contributing, either directly or indirectly, to this heated public discussion? A brief analysis of the 98 articles about mathematics education published between September 2, 2013 and April 6, 2016 in three Canadian newspapers—*The National Post* (14 articles), *Edmonton Journal* (37 articles), and *The Globe and Mail* (57 articles)—proves to be revealing.

Perhaps unsurprisingly, given the public nature of their office, education ministers and other politicians (such as their spokespeople) were quoted or mentioned in a majority of the articles (57 out of 98). Most often, the ministers were quoted as defending current curriculum or practices and, in some cases, promising improvements or announcing changes; in other cases, opposition party leaders and other members are quoted as criticizing government actions or inactions on certain matters related to mathematics education (e.g., Staples, 2014c, March 8). As even a cursory examination of these articles reveals, provincial—and in some cases, national—politics have become deeply entangled with the math wars, adding another layer of complexity to the issue.

Surely, the next most prolific contributors to the discussion about mathematics education would have been mathematics teachers or mathematics education researchers… alas, this was not the case. Mathematicians were quoted in 35 of the articles, critiquing "discovery math" and demanding a return to "basics" in almost each instance. It is worth noting, however, that in these 35 articles, University of Winnipeg mathematician Anna Stokke was mentioned or quoted in 24 of them, having contributed five of these herself. Stokke, a cofounder of WISE Math and self-proclaimed "numeracy advocate" who advocates greater emphasis on basic skills and standard algorithms in mathematics education, minimizing calculator use in math classes, and 50% representation of "professional mathematicians, and scientists from disciplines that use mathematics regularly […] on committees that shape and make decisions about overall content and the general methodology of mathematics teaching" (among other objectives; see Mission Statement, n.d.), has given over 80 interviews on the topic of mathematics education (Anna Stokke, n.d.). Fellow WISE Math cofounder and University of Manitoba mathematician Robert Craigen, whose views on mathematics education reform parallel those of Stokke, was quoted or mentioned in 14 of 35 the articles, having written one of these himself. Another frequently cited mathematician is John Mighton (5 articles, 1 of which is written by Mighton), founder of the JUMP Math, an innovative curriculum and training program for teachers. Parents were the next group who contributed or were consulted most often, quoted or mentioned (by name) in 29 articles; one article included more than 50 parent comments reprinted from a petition started by medical doctor and parent Nhung Tran-Davies (Staples, 2014a, January 27). Nhung Tran-Davies herself was mentioned or quoted in 24 of these articles (having written four among them), an overwhelming majority. Comments from or references to parents typically express their own or their children's frustration the new math curriculum, their increased use of private learning programs (e.g., Kumon), and their desire to see a greater focus in classrooms on mastery of "basic skills" (e.g., Alphonso & Maki, 2014, January 7; Staples, 2013, December 23, 2014a, January 27; Tran-Davies, 2014, March 24).

Mathematics education researchers were consulted or mentioned in only 15 out of the 98 articles. Three of the articles were contributed by mathematics education researchers (see Bruce, 2013; Friesen, 2014; Wood, 2014), and out of the 15 articles, 10 mentioned Canadian researchers. In the three articles contributed directly by mathematics education researchers, the authors attempt to elucidate the educational research that has led to curriculum changes and denounce the oversimplification of the issues at hand (e.g., Bruce, 2013). In typical cases, however, they were mentioned only in passing, such as in Bennett (2014, par. 8), where Marian Small is described only as a proponent of the "Discovery Learning" ideology and a "purveyor of Nelson mathematics problem-solving books." Among the reporters in this sample of articles who chose to reach out to mathematics education researchers, only one (Erin Anderssen of the *Globe and Mail*) appears to have had a sincere interest in elucidating current research in the field (e.g., Anderssen, 2014); many others, it seems, have only asked researchers to comment in a cursory attempt to provide some "balance" to heavily skewed reports (e.g., MacDonald, 2015). Non-mathematics education researchers were consulted or mentioned in nearly the same number of articles as mathematics education researchers, 14 out of 98; they represented the disciplines of psychology, cognitive neuroscience, psychiatry, biology, and engineering. Retired University of Alberta engineering professor Ken Porteous, a staunch critic of the "discovery approach" and the "new math," was quoted in 8 of the articles. Other individuals and groups given voice to in the 98 articles include elementary or secondary mathematics teachers (9 articles), mathematics coaches and consultants (4 articles), and students (1 article). Although teachers were not frequent contributors to the public conversation, those who did choose to comment tended to express frustration with the new curriculum and a desire to return to a focus on basic skills (e.g., Staples, 2013, December 23); in a few cases, teachers allege that they are reluctant to speak out against curriculum changes for fear of losing their job (e.g., Alphonso & Maki, 2014). Somewhat surprisingly, a representative from a provincial mathematics teachers' association is mentioned only once (see Casey, 2015, where Paul Alves, former president of the Ontario Association for Mathematics Education, discusses how mathematics teachers are using alternative teaching strategies to increase student engagement while acknowledging the need for lecture and practice).

What is to be made of this brief head count? One possible interpretation is that mathematics education researchers are simply not being invited to participate in the conversation. There is undoubtedly some truth to his, and, as such, poor journalism may indeed contribute to the gap. In some cases, those who choose to comment are given considerably less space to do so: to give a representative example, in Macdonald (2015), Stokke's quotes add up to 110 words—not counting the discussion of her C. D. Howe Institute report, which is the focus of the majority of the article—while Ann Kajander (Lakehead University) is given only 34 words near the end of the article. Moreover, many of the articles (inadvertently or otherwise) have advanced the storyline that mathematics education research is unreliable, untrustworthy, or even deliberately misleading; on the other hand, cognitive science research is more often positioned as being more trustworthy and rigorous (Herbel-Eisenmann et al., 2016). Case in point: in a 2015 *National Post* article criticizing discovery learning, Philip Sullivan suggests that "decades of classroom research has not been able to rid itself of uncontrolled influences, making the work unreliable and fruitless," then calls on cognitive psychology research to explain why discovery learning is "inferior to direct instruction" (par. 3). The article goes on to suggest that mathematics educators knowingly "contradict 20 years of research" by critiquing rote practice in the math classroom and have been "sharply criticized for ignoring the insights afforded by cognitive science research" (par. 4–5).

Besides psychology and cognitive neuroscience researchers, often, it is parents and mathematicians who are positioned as having the necessary expertise to make decisions about mathematics education and as having children's best interests at heart: Tran-Davies, for example, is described as an "earnest, honest and dedicated education crusader" (Staples, 2014d, March 12); in another article, Staples notes that Stokke "has a PH.D. [*sic*] in math unlike any of the government consultants who wrote the discovery/inquiry math curriculum" (2014e, April 22, par. 8). In a 2014 article, Tran-Davies lists those whom the signatories of her petition purportedly respect, recognizing teachers, the past, parents, and "the experts: the mathematicians, engineers, computer scientists, accountants, among many others whose successful careers are built upon the deeper understanding and application of mathematics" (par. 8). Mathematics education researchers do not make her list. Teachers' expertise is also sometimes downplayed: in Bennett (2014, par. 10), for instance, Richard Dunne—creator of the textbook series *Math Makes Sense*—is described as "a teacher and math consultant *rather than* a mathematician" (emphasis ours) whose early version of the series "proved popular with teachers who were non-specialists, but was resisted by many university based mathematicians." Faced with this hostile climate, where their expertise is either completely overlooked or mocked with references to "edu-crats" (Wente, 2014) or "education gurus" (McParland, 2014), it is understandable why mathematics education researchers would be reluctant to join the conversation.

However, the onus cannot be entirely on the media. If mathematicians Anna Stokke and Robert Craigen have become the voices of mathematics education in Canada, it is due at least in part to their persistence—Stokke alone has contributed at least a dozen articles advancing her position to various newspapers and has eagerly participated in a far greater number of print and radio interviews. Surely, researchers in the field of mathematics education who read such articles in popular media can easily spot misrepresentations and misunderstandings, such as the assumption that "discovery math" represents a curriculum and pedagogy that is uniformly adopted by all educators in classrooms across Canada or that new curricula advocate *unassisted* discovery (e.g., Hopper, 2014; Mighton, 2013). Moreover, mathematics education researchers are in a strong position to interpret educational research and to speak to the advantages, disadvantages, and contextual aspects of various pedagogical approaches. It is somewhat surprising, then, that many choose to ignore the opportunity to engage in public dialogue. The unintended, but real consequence is that, as Staples wrote (speaking about parents who support "discovery math"), their "continued near silence will speak volumes" (2014b, March 2). This silence can be interpreted in any number of ways, but surely it contributes to the misunderstanding and distrust of mathematics education research, as well as the lack of consultation by reporters, parents, politicians, and other groups. Indeed, the message is loud and clear—from the perspective of the public, mathematics education researchers are, ironically, not interested in a dialogue about the teaching and learning of mathematics and, as they say, if you're not part of the solution….

Meanwhile, in the United States…

One might view the tremendous expansion of interest in the teaching and learning of mathematics as evidence of the transcendence of disciplinary boundaries or at least of a multidisciplinary character to the effort to improve the teaching and learning of mathematics in North America. However, this notion does not seem to accurately describe this effort, for despite the lack of synthesis (Bernstein, 2015), multidisciplinarity as an approach still suggests harmony and appreciation for alternative points of view—in other words, the belief that a topic or a problem resolution will be *enriched* by bringing together several disciplines and stakeholders (Nicolescu, 2005). On the contrary, as the previous discussion suggests, not only do the various actors embroiled in this issue seem to be interested in maintaining boundaries, they sometimes reveal blatant contempt for alternative perspectives, claiming for a particular discipline or group the question of how to teach mathematics and what mathematics to teach. (Recall Tran-Davies, March 24, 2014, who positions representatives of STEM fields, such as mathematicians and engineers, as the experts in the matter of mathematics education by association with mathematics, rather than mathematics education researchers; certain mathematicians, such as Stokke and Craigen, have also advanced this point of view.) We have seen that in many cases, mathematics educators have chosen to simply disengage from the math wars, maintaining a firm disconnect between the various disciplines and stakeholders interested in the teaching and learning of mathematics. However, there are exceptions. For this example, we move outside of Canada to the United States (which is not to say that enthusiasts of the teaching and learning of mathematics here in Canada were not following the controversy with interest).

In 2012, a political scientist and self-proclaimed nonmathematician named Andrew Hacker from Queens College in New York entered the math wars with guns blazing when he published the now notorious article entitled "Is Algebra Necessary?" in *The New York Times*. In the article, Hacker raised concerns about the fact that a high percentage of students in the United States were failing to complete their high school education, qualify for entrance into college or university, and/or complete their tertiary studies, citing the requirement to study higher-level mathematics courses such as algebra, geometry, and calculus as a primary reason. In other words, Hacker transcended the debate about *how* to teach mathematics to ask *why* and to question the oft-unquestioned role of higher-level mathematics as a gatekeeper and its use in fields such as medicine as "a hoop, a badge, a totem to impress outsiders and elevate a profession's status" (par. 21). Hacker also challenged the popular assumption that, beyond basic numeracy, probability, and statistics, mathematics is very useful in the "real world" of the workplace, extending his doubt even to so-called STEM (science, technology, engineering, and mathematics) fields. Much technical training occurs after hiring, argued Hacker; more useful, in his view, would be development of quantitative literacy, "citizen statistics" (familiarizing students "with the kinds of numbers that describe and delineate our personal and public lives," par. 24), and even history and philosophy of mathematics at the university level.

Although a controversial point of view, it should be noted that Hacker is not the first to advance it: see, for example, Dudley (1997) and Smith (1989). To give another example, mathematician Paul Lockhart lamented in his oft-cited (by mathematics educators, no less!) "A Mathematician's Lament" that "people […] are apparently under the gross misconception that mathematics is somehow useful to society!" (2002, p. 7). "Do you think carpenters are out there using trigonometry? How many adults remember how to divide fractions, or solve a quadratic equation? […] I don't see how it's doing society any good," wrote Lockhart, "to have its members walking around with vague memories of algebraic formulas and geometric diagrams, and clear memories of hating them" (p. 7). Lockhart, too, criticized the practice of making higher-level mathematics a mandatory subject in high school. However, it seems that the sections of Lockhart's lament that are most often cited are those that deal with the "heartbreaking beauty" (2002, p. 25) of mathematics, rather than the more controversial passages that challenge the educational system (here's one that we have yet to see on a poster: "TRIGONOMETRY. Two weeks of content are stretched to semester length by masturbatory definitional runarounds" [p. 25]). Hacker was arguably less poetic, but for that, his position was less likely to get buried between the lines.

Curiously, despite the "earthquake" that Hacker's article caused (Baker, 2013, p. 34), few mathematics educators came forth to publicly discuss the issues raised within. Mathematicians like Evelyn Lamb, who wrote a popular response to Hacker in Scientific American (2012) defending higher-level mathematics and its mandatory status, seemed to be more open to discussion. However, Patrick Honner, a mathematics teacher and a frequent speaker, and presenter on mathematics and teaching, as well as a contributor to the *New York Times* himself, did provide a response on his personal blog. The critique focused on Hacker's choice of an example problem for a hypothetical "citizen statistics" course, arguing that the problem was really algebraic in nature and thus not only antithetical to (what Honner took to be) Hacker's main argument but also seemingly offering a reason to dismiss Hacker's other concerns altogether (e.g., about the role of mathematics as gatekeeper). Honner ended the post-contending that "discussions […] about what we are teaching, why we are teaching, and how we are teaching […] should be led by people who really understand what's going on" (2012). In the comments section of the post, Honner added that Hacker is "another non-expert framing and driving dialogue in education," alluding to a "larger public smear campaign against teachers, schools, and public education in general."

History repeated itself in February 2016, when Hacker published another article in the New York Times titled "The Wrong Way to Teach Math" in advance of the publication of his book, *The Math Myth and Other STEM Delusions*. Again, Hacker criticized the mandatory status of advanced mathematics such as calculus in high school, arguing that the focus in school mathematics should be on "quantitative literacy." Again, mathematician Evelyn Lamb (2016) published a response to Hacker that was widely circulated among the mathematics education community, citing some mutual concerns and offering critiques of his arguments. Keith Devlin, also a mathematician, did the same, quick to point out that "Hacker is not a mathematician. He is a retired college professor of political science, who has taught some courses in mathematics to non-majors" (2016, par. 7). Kevin Knudson, yet another mathematician, also criticized Hacker for venturing into a discipline where he is not an "expert" (2016, par. 3). By comparison (again), mathematics educators were relatively silent, at least in the popular media. Again, disciplinary lines were firmly defended, dividing "experts" from "non-experts."

Whether Hacker was being taken more seriously this time around or—if we are more optimistic—because the mathematics education community has become more open-minded and willing to entertain "outside" points of view that challenge the status quo in the last 4 years, a debate was held at the National Museum of Mathematics (MoMath) in New York on May 10, 2016, between Andrew Hacker and James Tanton, the latter a mathematics educator, consultant, and the Mathematical Association of America's "mathematician-at-large." (The MAA describes itself on its website as a "professional society that focuses on mathematics accessible at the undergraduate level," with interests ranging from curriculum, research, and professional development to public policy and public appreciation.) Moderated by John Ewing—former Executive Director of the American Mathematical Society, an MAA partner, and current President of Math for America (a nonprofit organization with a mission of promoting recruitment and retention of high-quality mathematics teachers in New York City)—the debate centered around Hacker's question of why students are required to take a "one size for all," "full menu" of mathematics courses throughout their high school years. Tanton, arguing for mandatory mathematics courses, argued that people were "missing the point of what current high school mathematics is actually about" (National Museum of Mathematics, n.d., par. 2). This public debate seemed to be a positive step in the math wars conflict: As Ewing remarked during his introductory remarks—inadvertently aligning himself with a transdisciplinary perspective—"I believe that finding answers to hard questions is best done through open public discussion" (as quoted by Wees, 2016). As it was the year 2016, several participants tweeted during the debate, allowing many others to follow and participate in the conversation online. Based on our following of the event, a majority of the #MoMathEdTalk tweets were authored by mathematics teachers.

As often happens with these kinds of debates, both sides likely left feeling that they had won. For instance, among the most shared and liked quotations on the #MoMathEdTalk page on Twitter was Tanton's call, "Let's trust teachers." Whatever the original context, the line could easily be interpreted as a slight against Hacker's credentials and was undoubtedly shared by many in this spirit. Indeed, shortly after the debate, Honner (flashback) published an opinion piece on the Math for America website asking, bluntly this time, "Why are we listening to Andrew Hacker?" (2016, par. 4). Elaborating, he wrote:

> Andrew Hacker isn't an expert on mathematics. And he isn't an expert on math teaching, either. […] The fact that Andrew Hacker has such an outsized and undeserved role in steering this conversation is itself one of our problems: we aren't listening to the right people. (par. 5–6)
>
> Understanding and addressing the real, substantial, and complex issues facing math education today depends on hearing from those who understand them best. If we want productive dialogue about how to move mathematics education forward […] Let's start by listening to teachers. (par. 9)

Such rhetoric is clearly problematic and antithetical to a transdisciplinary approach to mediating the math wars conflict. Although it is certainly reasonable, necessary even, to closely examine and critique Hacker's argument—and here, Hacker represents all those "outside" of the mathematics education community who are concerned about the teaching and learning of mathematics—the argument must be judged on its merit, rather than its source. Not only is this a necessity of a transdisciplinary mindset, it is a foundation of good reasoning (and research). On the other hand, resorting to ad hominem attacks and positioning some individuals or groups as "experts" and others as "non-experts"—thereby alienating a majority of the population who has a stake in children's education—only further entrenches the conflict and prevents a necessary open dialogue about the purpose and goals of teaching and learning mathematics in the twenty-first century. (And lest we paint Hacker as a blameless hero, we should note that he, too, is prone to drawing divisions, branding research mathematicians as privileged and powerful, and domineering "mandarins"—an allusion to ancient China's caste and their "complacency and privilege"—who "seek to dictate how a crucial realm of knowledge will be defined, taught, and studied at every level" (Hacker, 2016b, March 12, par. 2).)

Concluding Remarks

Our goal is not to lay the blame on any individual, group, or discipline for the perpetuation of the conflict—this has already been done far too often and has proved to be fruitless many times over. However, the above discussion does suggest that disciplinary approaches to resolving the math wars are unlikely to see success. A broader approach, one that recognizes and takes seriously the perspectives, values, and experiences of all who are interested and have a stake in mathematics education—including those *outside* of academia and *beyond* the disciplines traditionally charged with the responsibility for this issue—may prove to be more effective. Transdisciplinarity, which has already seen some success in tackling complex, value-laden issues in the modern world, offers itself as a promising candidate because of its attempt to transcend reductionist and dichotomous mentalities (such as the idea that mathematics educators and educational policymakers must choose between a "traditional" and a "discovery" approach) and its commitment to meaningful stakeholder and community involvement. Unfortunately, as the many examples of dichotomous, adversarial attitudes on all sides of the debate suggest, transdisciplinarity remains far out of reach. In our view, so long as those who are involved in the debate continue to approach the issue from within disciplinary silos and continue to dismiss "outsiders" based on a lack of certain credentials, the math wars will only continue to rage on. Unfortunately, to paraphrase Schoenfeld (2004), this means that we will continue to see injury caused to innocent parties: children, who deserve to be well served by mathematics education.

Fortunately—as the countless signals (online discussions, articles, books, petitions, protests, and so on) emitted by various groups clearly indicate—all sides of the conflict, perhaps with a few exceptions, are interested in seeing the math wars resolved. Part of the issue certainly lies in misunderstandings of educational research and curriculum recommendations (e.g., the assumption that emphasizing problem solving must come at the expense of procedural fluency, or that "discovery math" is a coherent program that is, or should be, uniformly applied in each and every classroom), which suggests that wider dissemination of information by mathematics educators and educational policymakers is required. This is not to say that the process will be easy or even welcomed. As discussed earlier, researchers in the field of mathematics education are faced with a climate that is hostile to educational research, which is often represented as being untrustworthy and unreliable in the popular media even as its results and recommendations are frequently misinterpreted. A separate analysis would be useful in shedding more light on this issue (see Herbel-Eisenmann et al., 2016, for some discussion), but we feel that it is reasonable to speculate that a major reason is that mathematics education researchers are simply not making enough of an effort to communicate their research beyond the disciplinary boundaries of the mathematics education community—beyond research conferences, beyond closed- and even open-access journals, beyond books such as the one in your hands… perhaps in part because they face a distrustful, hostile public and fear misinterpretation and overgeneralization of results. We can see that a vicious cycle of disengagement, distrust, and hostility is likely at play, and this cycle must be broken. Herbel-Eisenmann et al. (2016) make several suggestions for establishing stronger relationships between mathematics education researchers and other stakeholders, as well as for using additional communication mechanisms and nontraditional outlets, such as social media, to communicate with broader audiences who have an interest and a stake in mathematics education.

However, they also stress that communication between mathematics educators and other researchers and stakeholders should not only be "one way" (i.e., delivering messages) but rather reciprocal (Herbel-Eisenmann et al., 2016). And if meaningful dialogue and collaboration are to be achieved in conversations about the teaching and learning of mathematics, simply bringing people together, whether virtually or physically, is not enough: all interested parties must strive to step beyond disciplinary boundaries and to view the math wars as only one component of the larger, transdisciplinary questions of the purposes and goals of education in the twenty-first century, which can only be tackled collectively. "Outside," "non-expert" perspectives cannot be taken for granted; neither can the bias inherited from a disciplinary connection. Such an effort will demand lateral, creative, and collective thinking and undoubtedly some discomfort—namely, the "pain inherent in abandoning one's intellectual comfort zone by working outside one's home discipline and engaging in new modes of thinking and taking action" (Bernstein, 2015, p. 11). Perhaps it is ironic, then, that this call to action is bound within this book.

References

Alphonso, C., & Maki, A. (2014, January 7). Math wrath: Parents and teachers demanding a return to basic skills. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/news/national/education/petitions-press-provinces-to-put-emphasis-on-basic-math-skills/article16240118/

Anderssen, E. (2014, January 23). Solving the math-phobia problem—For both parents and kids. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/life/parenting/solving-the-math-phobia-equation-for-both-parents-and-kids/article16465779/

Anna Stokke. (n.d.). Retrieved from WISE Math website: https://wisemath.org/contributors/anna-stokke/

Baker, N. (2013, September). The case against Algebra II. *Harper's Magazine*, pp. 31–38.

British Columbia Association of Mathematics Teachers. (n.d.). *Mastering basic number facts*. Retrieved from http://www.bcamt.ca/wp-content/uploads/2015/06/BasicNumberFacts1.pdf

Bennett, P. W. (2014, April 29). How much are massive U.S. textbook companies driving Alberta's discovery math agenda? *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/how-much-are-massive-u-s-textbook-companies-driving-albertas-push-to-discovery-math-and-a-discovery-curriculum

Bernstein, J. H. (2015). Transdisciplinarity: A review of its origins, development, and current issues. *Journal of Research Practice, 11*(1).

Bishop, A. J. (2008). Mathematics teaching and values education—An intersection in need of research. In P. Clarkson & N. Presmeg (Eds.), *Critical issues in mathematics education* (pp. 231–238). New York: Springer. (Reprinted from *ZDM, 31*, 1–4, 1999.).

Brochu, P., Deussing, M.-A., Houme, K., & Chuy, M. (2013). *Measuring up: Canadian results of the OECD PISA study. 2012 first results for Canadians aged 15*. Toronto, ON: Council of Ministers of Education.

Bruce, C. (2013, November 1). High expectations will raise math scores. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/news/national/education/students-need-high-expectations-in-math/article15216621/

Casey, L. (2015, July 2). Old school or new? Math teachers debate best methods as Canadian scores fall. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/news/national/old-school-or-new-math-teachers-debate-best-methods-as-canadian-scores-fall/article25224581/

Devlin, K. (2016, February 29). Andrew Hacker and the case for and against algebra. *Huffpost Education.* Retrieved from http://www.huffingtonpost.com/dr-keith-devlin/andrew-hacker-and-the-cas_b_9339554.html

Dudley, U. (1997). Is mathematics necessary? *The College Mathematics Journal, 28*(5), 360–364.

Editorial: Parents, educators rally against Alberta's 'new math.' (2014, April 12). *CBC News Edmonton.* Retrieved from http://www.cbc.ca/news/canada/edmonton/parents-educators-rally-against-alberta-s-new-math-1.2608360

Editorial: Quebec adds, Canada subtracts on its math scores. (2013, December 3). *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/globe-debate/editorials/quebec-adds-canada-subtracts-on-its-math-scores/article15748296/

Ernest, P. (1991). *The philosophy of mathematics education: Studies in mathematics education*. London: Falmer Press.

Friesen, S. (2014, May 28). Sharon Friesen: "Multiple strategies are not taught and learned for the sake of learning multiple strategies." *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/sharon-friesen-multiple-strategies-are-not-taught-and-learned-for-the-sake-of-learning-multiple-strategies

Gibbs, P. (2015). Transdisciplinarity as epistemology, ontology or principles of practical judgment. In P. Gibbs (Ed.), *Transdisciplinary professional learning and practice* (pp. 151–164). Dordrecht: Springer International Publishing.

Hacker, A. (2012, July 28). Is algebra necessary? *The New York Times*. Retrieved from http://www.nytimes.com/2012/07/29/opinion/sunday/is-algebra-necessary.html

Hacker, A. (2016a, February 27). The wrong way to teach math. *The New York Times.* Retrieved from http://www.nytimes.com/2016/02/28/opinion/sunday/the-wrong-way-to-teach-math.html

Hacker, A. (2016b, March 12). The math mandarins. *The Chronicle of Higher Education.* Retrieved from http://chronicle.com/article/The-Math-Mandarins/235718

Herbel-Eisenmann, B., Pape, S. J., Sinclair, N., Stephan, M., Chval, K. B., Wanko, J. J., et al. (2016). Positioning mathematics education researchers to influence storylines. *Journal for Research in Mathematics Education, 47*(2), 102–117.

Herrera, T. A., & Owens, D. T. (2001). The "new new math"? Two reform movements in mathematics education. *Theory Into Practice, 40*(2), 84–92.

Honner, P. (2012, August 1). *Replace algebra with… algebra?* [Web log post]. Retrieved from http://mrhonner.com/archives/8411

Honner, P. (2016, May 11). *When it comes to math teaching, let's listen to math teachers* [Web log post]. Retrieved from http://www.mathforamerica.org/news/when-it-comes-math-teaching-let%E2%80%99s-listen-math-teachers

Hopper, T. (2014, February 28). Does 'discovery learning' prepare Alberta students for the 21st century or will it toss out a top tier education system? *National Post.* Retrieved from http://news.nationalpost.com/news/canada/does-discovery-learning-prepare-alberta-students-for-the-21st-century-or-will-it-toss-out-a-top-tier-education-system

Kilpatrick, J. (2012). The new math as an international phenomenon. *ZDM, 44*(4), 563–571.

Klein, D., & Rosen, J. (1996, May 31). The experts dumb down math education. *Los Angeles Daily News*. Retrieved from https://www.highbeam.com/doc/1P2-25051626.html

Knudson, K. (2016, March 4). 'The math myth' fuels the algebra wars, but what's the fight really about? *The Conversation, U.S. Edition.*. Retrieved from https://theconversation.com/the-math-myth-fuels-the-algebra-wars-but-whats-the-fight-really-about-55740

Lamb, E. (2012, July 30). Abandoning algebra is not the answer. *Scientific American.* Retrieved from http://blogs.scientificamerican.com/observations/abandoning-algebra-is-not-the-answer/

Lamb, E. (2016, March 29). It doesn't add up. *Slate*. Retrieved from http://www.slate.com/articles/health_and_science/education/2016/03/andrew_hacker_s_the_math_myth_is_a_great_example_of_mathematics_illiteracy.html

Lockhart, P. (2002). *A mathematician's lament*. Retrieved from http://www.maa.org/external_archive/devlin/LockhartsLament.pdf

Maasen, S., Lengwiler, M., & Guggenheim, M. (2006). Practices of transdisciplinary research: Close(r) encounters of science and society. *Science and Public Policy, 33*(6), 394–398.

MacDonald, M. (2015, May 27). Decline of Canadian students' math skills the fault of 'discovery learning': C.D. Howe Institute. *National Post.* Retrieved from http://news.nationalpost.com/news/canada/decline-of-canadian-students-math-skills-the-fault-of-discovery-learning-c-d-howe-institute

Maguire, K. (2015). Transdisciplinarity as translation. In P. Gibbs (Ed.), *Transdisciplinary professional learning and practice* (pp. 165–177). Cham: Springer International Publishing.

McParland, K. (2014, February 27). Alberta school learns the price of teaching that adequate is plenty. *National Post.* Retrieved from http://news.nationalpost.com/full-comment/alberta-school-learns-the-price-of-teaching-that-adequate-is-plenty

Mighton, J. (2013, October 28). Kids can't teach themselves math. *The Globe and Mail.*. Retrieved from http://www.theglobeandmail.com/news/national/education/kids-cant-figure-out-math-by-themselves/article15087557/

Mission Statement. (n.d.). Retrieved from WISE Math website. https://wisemath.org/mission-statement/.

McGarvey, L. M., & McFeetors, P. J. (2015). Reframing perceptions of arithmetic learning: A Canadian perspective. In X. Sun, B. Kaur, & J. Novotná (Eds.), *Proceedings of the twenty-third International Commission on Mathematical Instruction: Primary mathematics study on whole numbers* (pp. 113–123). Macau, China: ICMI.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Museum of Mathematics. (n.d.). Course corrections: Is high school math serving society? Can it? Does it? Should it? Retrieved from http://in.momath.org/civicrm/event/info?id=577

Nicolescu, B. (2005). Towards transdisciplinary education. *TD: The Journal for Transdisciplinary Research in Southern Africa, 1*(1), 5–16.

Povey, H. (2003). Teaching and learning mathematics: Can the concept of citizenship be reclaimed for social justice? In L. Burton (Ed.), *Which way social justice in mathematics education?* (pp. 51–64). Westport, CT: Praeger.

Roitman, J. (1999). Beyond the math wars. *Contemporary Issues in Mathematics Education, 36*, 123–134.

Russell, G., & Chernoff, E. (2016). The transreform approach to the teaching and learning of mathematics: Re-viewing the math wars. *Far East Journal of Mathematics Education, 16*(1), 69–109.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.

Smith, M. K. (1989). Why is Pythagoras following me? *The Phi Delta Kappan, 70*(6), 446–454.

Staples, D. (2013, December 23). "This new math is stealing their confidence and their dreams" — Educators speaks out against new fuzzy math curriculum. *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/this-new-math-is-stealing-their-confidence-and-their-dreams-educators-speaks-out-against-new-fuzzy-math-curriculum

Staples, D. (2014a, January 27). Nhung Tran-Davies to take anguished and angry parental concerns about discover math to Alberta Education officials on Tuesday. *Edmonton Journal*. Retrieved from http://edmontonjournal.com/news/local-news/dont-experiment-with-my-children-parents-losing-faith-in-alberta-public-school-system-over-its-voodoo-math-curriculum

Staples, D. (2014b, March 2). More than 6,000 Albertans have signed the math petition. So when will parents on other side speak up? *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/more-than-6000-albertans-have-signed-the-anti-discovery-math-petition-so-when-will-other-side-fight-back

Staples, D. (2014c, March 8). Wildrose leader Danielle Smith accuses Redford government of chasing "new age" educational fads. *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/wildrose-leader-danielle-smith-accuses-redford-government-of-chasing-new-age-educational-fads

Staples, D. (2014d, March 12). Ms. Tran-Davies goes to Washington. *Edmonton Journal*. Retrieved from http://edmontonjournal.com/news/local-news/ms-tran-davies-goes-to-washington

Staples, D. (2014e, April 22). Manitoba Education Minister takes a swipe at Alberta's "not as successful" math curriculum. *Edmonton Journal.* Retrieved from http://edmontonjournal.com/news/local-news/manitobas-education-minister-take-a-swipe-at-albertas-not-as-successful-math-curriclum

Sullivan, P. (2015, June 26). 'Discovery learning' is failing our children. *National Post*. Retrieved from http://news.nationalpost.com/full-comment/philip-sullivan-discovery-learning-is-failing-our-children

Thompson Klein, J. T. (2004). Prospects for transdisciplinarity. *Futures, 36*, 515–526.

Thompson Klein, J. (2010). A taxonomy of interdisciplinarity. In R. Frodeman, J. Thompson Klein, & C. Mitcham (Eds.), *The Oxford handbook of interdisciplinarity* (pp. 15–30). Oxford, UK: Oxford.

Tran-Davies, N. (2014, March 24). The Great Canadian Math Debate, pt. 26: Tran-Davies: "Where the 'new math' model falters is its failure to respect the importance of the indispensable virtues of perseverance, hard work and practice." *Edmonton Journal.* Retrieved from http://edmonton-journal.com/news/local-news/math-petition-calls-for-a-balanced-approach-say-nhung-tran-davies-not-an-over-emphasis-on-inquiry-based-learning

Wees, D. [@davidwees]. (2016, May 10). "I believe that finding answers to hard questions is best done through open public discussion." ~ John Ewing, @MathforAmerica #momathedtalk [*Twitter post*]. Retrieved from https://twitter.com/davidwees/status/730166544882372608

Wente, M. (2014, March 4). Canada's math woes are adding up. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/opinion/canadas-math-woes-are-adding-up/article17226537/

Wood, E. (2014, January 8). Children in Singapore excel at math because parents care. *The Globe and Mail.* Retrieved from http://www.theglobeandmail.com/news/national/education/the-radical-equality-of-singapores-education-system/article16090098/

Wong, L. (2014, April 11). Photos: Math rally at the Alberta Legislatrure. *Edmonton Journal.* Retrieved from http://www.edmontonjournal.com/news/Photos+Math+Rally+Alberta+Legislatrure/9733106/story.html

Zwaagstra, M. (2011). *Math instruction that makes sense: Defending traditional math education*, *FCPP policy series no. 120*. Winnipeg, MB: Frontier Center for Public Policy.

Part IV
Indigenous and Transformational Mathematical Knowledge

Chapter 9
Echoed Rememberings: Considering Mathematics and Science as Reconciliation

Dawn Wiseman and Lisa Lunney Borden

Abstract The editors of this volume have challenged us to consider the concept of transdisciplinarity within our own practices of teaching and learning. We begin by positioning transdisciplinarity as an instance of echoed rememberings, ideas Indigenous peoples have not forgotten. We suggest that such rememberings open up possibilities for transversing, transgressing and transcending what mathematics and science teaching and learning might be, in ways that welcome life and living back into mathematics and science through projects such as Show Me Your Math in Nova Scotia and the Indigenous Teaching and Learning Gardens at the University of Alberta. These illustrative examples suggest how we might create opportunities in K-12 and teacher education to centre Indigenous understandings as places from which learning emerges and provide a means for moving towards reconciliation.

Keywords Remembering • Indigenous ways of knowing, being and doing • Mathematics • Science • K-12 • Teacher education

This chapter considers what it might mean and look like to consider mathematics and science teaching and learning as reconciliation in the Canadian context through the lens of transdisciplinarity. We explore the term as a way of transversing, transgressing and transcending in order to break out of silos of disciplinary understanding and bringing livingness to teaching and learning in mathematics and science. At the same time, we remember to remember that words such as transdisciplinarity

D. Wiseman (✉)
Department of Integrated Studies in Education, McGill University,
3700 McTavish Street, Montreal, QC H3A 1Y2, Canada
e-mail: dawn.wiseman@mcgill.ca

L. Lunney Borden
Faculty of Education, St Francis Xavier University,
PO Box 5000, Antigonish, NS B2G 2W5, Canada
e-mail: lborden@stfx.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_9

are likely only echoed rememberings of ideas Indigenous[1] peoples have not forgotten. As a way of illustrating what we mean, we share experiences from our own work and those of colleagues in other areas of Canada, before using them to question the lens of transdisciplinarity once again.

As is appropriate to the contexts in which we work, we begin by acknowledging the teachings, time, wisdom and humour of the Blackfoot, Sioux, Mohawk and Mi'kmaw[2] Elders we work and have worked alongside. As is appropriate to the places in which and people with whom we teach and learn, we begin with a story that involves both of us. It was originally shared in Dawn's dissertation (Wiseman, 2016).

. . .

In late 2013, we attended a week-long workshop examining the research possibilities of engaging with First Nations, Inuit and Métis communities regarding the relationships between wisdom traditions and mathematics teaching and learning. Participants were university-based researchers, Elders, students, and educators both Indigenous[2] and settler. We have known a number of the participants for many years. On the first day of our meetings, a tenured Indigenous colleague who we have known for almost two decades expressed a sense of frustration with ruttishness, how we seem to need to start over every time we meet. Our colleague feels a sense of urgency related, it seems, to the realities of supporting communities in meeting the priorities they set for themselves and how these priorities are sometimes complicated by ongoing tensions around engaging with mathematics in the community. However, as the week progressed and people shared stories of work undertaken in local places, it became clear that we were neither starting over nor stuck in a rut; that in fact, we were all in our own ways and places caught up in acts of living with our relations in the communities in which we find ourselves and in their engagements with mathematics. Because our work is complex, we sometimes find ourselves in recognizable, familiar places that, if left unexamined, might be mistaken for the same place we have found ourselves in on previous occasions and hence the worry of "starting over" again and again. Instead, while we discovered we stood in a recognizable, familiar place, we were not in exactly the same spot starting over because since our last meeting, we have been engaged in generating understandings. At the same time, we also realized that something was missing in our work that could connect understanding where we had come from and what we had learned in the intervening period; something to ground us in our relations that allows us to both see and read through (Ihde, 1996, 1998) to sustaining "patterns we might hang our hats on" (Little Bear, 2012) and in turn sustain those patterns. Within the context of the academy, this role is in some ways taken up by academic journals, but academic journals have some diffi-

[1] Within the Canadian context, there are three large grouping of Indigenous peoples – First Nations, Métis and Inuit – each that consists of multiple distinct nations and/or communities. The currently accepted term for referring to First Nations, Métis and Inuit as a collective is Indigenous (Vowel, 2016).

[2] Throughout this article, Mi'kmaq is used as a noun and can be either singular or plural. Mi'kmaw is used as an adjective. While the rules for creating adjectival forms of words in Mi'kmaq are considerably more complex, it has been agreed by a working group on Mi'kmaw language learning that, when writing in English, these conventions will be used.

culty accounting for ways of knowing, being and doing beyond the Western. As such, in this case, the chosen means of making that connection was ceremony and a mathematics bundle which will travel with one of our colleagues and return to us – and those who come after us – to tell stories of mathematics in different places. What we came to is ceremony and ritual as a means of returning, remembering, renewing and re/creating in a recursive manner.

. . .

Positioning Ourselves

As white women who work alongside Indigenous people, peoples and communities, it is always necessary to position ourselves in the work. Absolon and Willett (2005) underline how "Identifying, at the outset, the location from which the voice of the researcher emanates is an Aboriginal way of ensuring that those who study, write, and participate in knowledge creation are accountable for their own positionality" (p. 97). While neither of us is Indigenous, we have both lived and worked alongside Indigenous people, peoples and communities for over 40 years combined. This experience leads us to approach teaching, learning and research in ways deeply informed by understandings developed in the places where we live and work.

Lisa's work began in 1995, with a mathematics teaching position in a Mi'kmaw school in We'koqma'q, Cape Breton, Nova Scotia. During 10 years in We'koqma'q, she immersed herself in the community well beyond the school walls and developed a functional use of the Mi'kmaw language. Dawn's work began in 1993, when she was invited to develop a science camp for young Indigenous people by her long-time mentor Corinne Mount Pleasant-Jetté, a member of the Tuscarora Nation from Ohsweken, Ontario, and – at the time – a professor in the Faculty of Engineering and Computer Science at Concordia University (Montréal, Québec). For the next 16 years, they worked together supporting K-12 mathematics teaching and learning for Indigenous children and youth. It was through this work that we met and have had the opportunity to collaborate for over 15 years.

In returning to the academy to explore the complexities of what we have learned in community about teaching and learning mathematics and science, each of us has questioned if it is our place to do such work. We have asked ourselves the questions Kovach (2005) says challenge non-Indigenous people in contexts such as our own, "Am I creating space or taking space?" (p. 26). The intention is always the first, but the second is potential. Seeking advice from community members, each of us has been reminded – with some humour – that the relationships we have engaged in over the years come with obligations. As Kovach (2009) writes,

> Indigenous methodologies, by their nature, evoke collective responsibility… Specific responsibilities will depend upon the particular relationship. They may include guidance, direction, and evaluation. They may include conversation, support, and collegiality. Responsibility implies knowledge and action. It seeks to genuinely serve others, and is inseparable from respect and reciprocity. (p. 178)

People have shared language, culture and ways of knowing, being and doing with us; what we share here is a means of honouring those gifts. One manner in which we do so is by attending to Stewart-Harawira's (2005) call for paying serious attention to "the possibilities inherent in indigenous ontologies" (p. 34) within the academy. As such, we both base research in Indigenist research methodologies, as a means of decolonizing research (Denzin, 2005). The approach "research[es] back to power" (Smith, 2005, p. 90) and holds a "purposeful agenda for transforming the institution of research, the deep underlying structures and taken-for-granted ways of organizing, conducting, and disseminating research and knowledge" (p.88).

Positioning Transdisciplinarity as an Instance of Echoed Remembering

As white academics who work alongside Indigenous people, peoples and communities, it is always necessary to position and question the frameworks within which we are asked to work. The editors of this volume have asked us to consider transdisciplinarity in mathematics education. As we have noted in other places (Lunney Borden & Wiseman, 2016), while we each have disciplinary specialties in mathematics and science, we are more interested in teaching and learning through interesting questions and problems located in places with which learners have some relationship and from which mathematics and science (engineering, art, technology, etc.) may emerge. The sensibilities we bring to our work tend to push against boundaries of Western ways of knowing, being and doing that artificially construct silos of knowledge, defining clear boundaries of belonging that break down kinship and otherwise lead to "epistemic closure" (Rabaka, 2010, p. 13). Thus, we also question ideas and terminology generally used and/or accepted within the academy such as STEM (Lunney Borden & Wiseman, 2016), theory and methodology (Lunney Borden, 2010; Lunney Borden & Wagner, 2013; Wiseman, 2016). In this questioning, our fundamental focus is on how such terms are situated and constructed, how they limit or create space for action and whether they have a place in the contexts in which and for the people with whom we work. Thus, in preparing this chapter, we begun by considering *transdisciplinarity* as a construct emergent from Western ways of knowing, being and doing, to determine whether it is actually useful in describing our work.

We begin in this manner because our work suggests that recently emergent academic terms used to encompass and describe the complexity of living, learning and being in the world still exist in Indigenous languages and thought and that such academic terms are "echoed rememberings" (Wiseman, 2009, p. 4) of concepts, ideas and ways of being Indigenous people and peoples have not forgotten. Thus, research that identifies processes active in the world and names them in English – or other European languages – is not a process of discovery but of remembering ideas that Western "epistemic closure" (Rabaka, 2010, p. 13) has forgotten or repressed. In popular culture the process is referred to as *Columbusing*, a word that "labels acts of cultural reappropriation, typically of something already known to minority cultures, much as Columbus 'discovered' America despite the indigenous (sic) peoples already living there" (Zimmer & Solomon, 2015, p. 85). For us, these ideas that are presented as innovations within the academy resonate with learnings we have had within communities; our commitment to those communities compels us to remember that such ideas may have longer histories and complex relations.

Transdisciplinarity is a term that evokes such remembering. The literature claims it as an approach to breaking down disciplinary barriers, addressing complex problems through multiple perspectives and examining knowledge production and dissemination more holistically (Berstein, 2015; Pohl, 2010). Berstein (2015) argues that transdisciplinarity forces:

> considerations [that] require researchers not only to admit to their own subjectivity but to foreground questions about the ethics of studying populations where a power differential exists between the investigator and the subject of research. This has resulted in research that transcends standard interpretive social science and becomes transdisciplinary in that it brings in the subjects of research participating in the research on an equal footing with the investigators. (Section 2, paragraph 2)

As presented above, the concept of transdisciplinarity shares some alignment with Indigenous research methodologies (Kovach, 2009; Smith, 1999). However, whereas transdisciplinarity is fairly new term (Berstein 2015), the ideas contained within Indigenous methodologies are rooted in very old, community-accepted ways of engaging with one another that extend to the kinship and relations that sustain life and livingness. Erica Violet Lee (2016) terms such livingness "epistemic life giving" and points to the work of Métis scholar Zoe Todd (2016) who explores our deeply entwined bondedness with fish, beavers and long-dead ancestors now present in petroleum.

Epistemic life-giving emerges from and in practices we have come to know in Indigenous communities under multiple names – such as *mawikinumatimk* in Mi'kmaq (Lunney Borden, 2010; Lunney Borden & Wagner, 2013) and *kiskanowapâhkêwin* (Lunney Borden & Wiseman, 2016) or *miyo wichitowin* (Donald, 2013) in Cree – and we wonder if such livingness, or at least the potential for it, is present in transdisciplinarity. In considering this question, Rabaka's (2010) work is helpful. He reminds us that transdisciplinarity is rooted in "transgress[ing], transcend[ing] and transvers[ing]" (p. 13) generally accepted frameworks, to counteract epistemic closure, thus opening up the possibility for epistemic life-giving (Lee, 2016).

In this paper, we attempt to transgress, transcend and transverse accepted notions of mathematics and science teaching and learning to listen for and remember what land teaches. So, rather than throwing away the word transdisciplinarity, we choose to make it live within our work to transcend the boundaries of Western and Indigenous ways of knowing to find alignments, moments of resonance and places for complex conversations, as well as to demonstrate how Indigenous ways of knowing, being and doing can inform a transdisciplinary approach to mathematics and science education. At the same time, we remain uneasy and uncomfortable with the term transdisciplinarity, but we tend to find that such uneasiness and discomfort can be generative and fruitful.

Remembering to Remember

As white women born into and primarily raised in Western ways of knowing, being and doing, in our work alongside Indigenous people, peoples and communities, we have learned that remembering and, more importantly, remembering to remember are essential to living. Remembering to remember is the antithesis of repression which Taubman (2007) explains as "the occurrence of a psychic event that is doubly forgotten" (p. 4), something that is so traumatic or so deeply denied that we have forgotten we forgot it. Tewa scholar Cajete (1999) argues that those of us grounded in Western thought and worldview frequently suffer from a surfeit of repression and that our school systems reflect that unfortunate abundance and that, in fact, they work to (re)produce it.

Examining E. O. Wilson's concept of biophilia, or affinity for nature, Cajete (1999) contends that instead of nurturing young people's curiosity and wonder about the world and introducing them to it through relationship, schools and curricula (re)produce biophobia, or fear of the natural world, by abstracting content from lived experience, people from nature and learning from community, in a sense abstracting all relationship and subjectivity from experience and focusing solely on a (false) objective view. He says, this hidden curriculum leads students to perceive themselves not as a "microcosm of the macrocosm" (Cajete, 2006, p. 249) but instead as distinctly separate and apart from the rest of creation and thus able to control it only through knowledge and expertise, control of nature being one of the key, but largely unconscious, "epistemological underpinnings" (Cajete, 1999, p. 190) of Western institutions. The concept of control runs throughout the curriculum but is most obviously manifested in the teaching of science and mathematics from a Western perspective which he suggests is "the single most powerful paradigm of modern Western culture" (Cajete, 1999, p. 188).

Cajete (1999) links our current ecological predicament in terms of climate change to forgetting what we have forgotten. He writes that "Once people break these cycles of remembering, they begin to forget and start doing the kinds of things that have led to the ecological crisis we are currently experiencing" (p. 197), a crisis that has the potential to end life as we know it. Similar thinking has been presented by Brazilian scholar D'Ambrosio (2015) who argues that Western knowledge has failed us with respect to finding solutions to some of the world's most complex problems such as climate change. D'Ambrosio uses the metaphor of the "epistemological cage" (p. 23) that continuously reproduces "academic sameness" (p. 23) writing that:

> Traditional knowledge is like a birdcage. Birds living in the cage are fed by what is in the cage, they fly only in the space of the cage, they see and feel only what the wires of the cage allow. The birds in the cage communicate among themselves in a language proper to those that live in the cage, they breed and procreate, they repeat themselves. (p. 23)

He underlines that "We need new ideas, new approaches, to face the problems affecting the world. Our generation and our approaches are not producing the global changes to avoid total disaster" (p. 24). He calls for greater creativity and a transcending of boundaries between knowledge systems so that we might escape the epistemological cage to explore different solutions to the world's most critical problems. His call for transcendence emphasizes a means of reconnecting with the world through "really real situations" (p. 27) in mathematics education. The importance of relationship within this process is inferred, but not explicit, and this is where we need to return to Indigenous critiques of Western ways of knowing, being and doing.

Cajete (1999) strongly implies that deep conceptions of relationship once existed in Western ways of knowing, being and doing but have been forgotten in the logical rationalism of Newtonian-Cartesian conceptions of the world. Moreover, he suggests Western forgetting has been a somewhat deliberate, or at least predictable, outcome of breaking "cycles of remembering" (p. 197), and that "when something no longer exists in your perceptual memory, it no longer matters" (2006, p. 255). Papachese Cree scholar Donald (2009) similarly acknowledges that "Indigenous peoples today [do not] hold exclusive copyright on this [relational and interconnected] view of the world" (p. 439). While Donald locates the source of forgetting in the "homogenizing processes of modernity and colonialism" (p. 440), these processes are closely related to Newtonian-Cartesian logical rationalism in their attempts to separate, enclose and abstract rather than to relate, open up and create (Doolittle, Lunney Borden, & Wiseman, 2011). Cajete (1999) contrasts Western epistemic closure and the forgetting it engenders with Indigenous peoples' practices of "remember[ing] to remember" (p. 197) relationships through language, stories, art and ceremony. Blackfoot Elder and scholar, Leroy Little Bear (2012), firmly roots epistemic life-giving with remembering to remember in his description of the flux, the chaos that in Blackfoot thinking underlies all of creation. Within the flux there are regular patterns "you can hang your hat on" – places where living is possible – that are maintained through deliberate and careful renewal of relationships taken up via ceremony (Little Bear, 2000; Wilson, 2008). Doll (1993) also looks to these patterns and, like Little Bear, considers that ritual plays a significant role in their ongoing renewal and recreation.

While we are uneasy claiming any connection to Little Bear's (2012) ceremony, we find ways of being and doing within the teaching and learning that might be akin to Doll's (1993) ritual, ways to acknowledge and enact our responsibility for, obligation to and complicitness in the relationships, processes and places of our work. These rituals remind us of the importance of transversing, transgressing and transcending the privileging of Western ways of knowing, being and doing. Such ritual has tended to fall outside the boundaries of the academy via epistemic closure and/or "epistemic apartheid" which Rabaka (2010) describes as "the process of institutional racism … academic racial colonization and conceptual quarantining of knowledge, anti-imperial thought, and/or radical political praxis produced and presented by non-white...intellectual activists" (p. 11) and hence our opening story of the mathematics bundle.

Given the significance of bundles to community continuity and life (e.g. Crowshoe & Manneschmidt, 2002), the decision to create a bundle was not entered into lightly. There was much discussion within the gathered group, with much support and guidance from Elders who had previous experience with bundles. It is not our place to tell the deeper story of the bundle; we are not the bundle keepers. While we contributed to the ceremony and the bundle, and in many ways it has brought a renewed energy and urgency to our personal and collective work, we acknowledge we neither have the authority nor teachings to share anything further. At the same time, we share the story of the bundle because it reminds us of the importance of returning, renewing and re/creating in an ongoing fashion. It connects us to the importance of ceremony and ritual in our work and brings epistemic life-giving to it by grounding us in our relationships and remembering to remember.

Such remembering to remember is at the heart of the Truth and Reconciliation Commission of Canada's [TRC] (Truth and Reconciliation Commission of Canada, 2015) 94 Calls to Action that focus on establishing renewed relationships between settler and Indigenous people in Canada to "restore what must be restored, repair what must be repaired, and return what must be returned" (p. 6).

Positioning and Problematizing Reconciliation

Canada's residential school system perpetuated an act of cultural genocide on Indigenous peoples, the effects of which are still significantly impacting First Nations, Inuit and Métis communities today (TRC, 2015). In 2015, the TRC stated that the ultimate goal of reconciliation is "to transform Canadian society so that our children and grandchildren can live together in dignity, peace, and prosperity on these lands we now share" (p. 8). The TRC names the education system as having an essential role in repairing the damages caused by residential schools and thus obligates educators at all levels to consider their roles as agents of "unlearning colonialism" (D. Donald, personal communication, December 21, 2016).

We recognize that the TRC (2015) and its recommendations are not unproblematic. The commissioners also recognized the difficulty of the idea of reconciliation within their final report, noting that:

> To some people, reconciliation is the re-establishment of a conciliatory state. However, this is a state that many Aboriginal people assert never has existed between Aboriginal and non-Aboriginal people. To others, reconciliation, in the context of Indian residential schools, is similar to dealing with a situation of family violence. It's about coming to terms with events of the past in a manner that overcomes conflict and establishes a respectful and healthy relationship among people, going forward. It is in the latter context that the Truth and Reconciliation Commission of Canada has approached the question of reconciliation. (p. 6)

We equally recognize that these multiple conceptions of reconciliation exist and circulate together in the same space; that they speak to each other and have conversations that complexify the context in which we work. Reconciliation goes hand in hand with decolonization or at least "unlearning colonialism" (D. Donald, personal communication, December 21, 2016); reconciliation can never be complete without undoing systemic power structures defined by settler governments at multiple levels. All of this is to say that we acknowledge the work is complex and there is often a fear of doing things incorrectly, yet at the same time, we simply cannot continue as we have been – something must change.

In fact, Elders with whom we have relationships have reminded and continue to remind us of the importance of acting differently and being creative as a means of rethinking and reworking relationships between Indigenous and settler peoples. In this instance, we think in particular of Blackfoot Elder Narcisse Blood who once told Dawn that, "The worst thing to do is nothing, and just go with the same, eh. … You know, status quo" (in Wiseman, 2016, p. 107). And so, we remember to be creative, to do things differently, to listen to what land teaches us, to honour ideas rooted in Indigenous languages and to engage with epistemic life-giving that is both emergent from and embedded in all our relations. We thus choose to make the path of reconciliation while walking it; but, we understand that the creation of such a path is also caught up in transversing, transgressing and transcending the status quo. And so, we share some stories of our own.

Stories of Transdisciplinarity: Transversing, Transgressing and Transcending the Status Quo

The following stories emerge from collective and collaborative projects where many people – Elders, K-12 students, pre- and in-service teachers, graduate students, faculty members and school and post-secondary staff – come together to learn together. They are drawn from Lisa's experiences with the Show Me Your Math Project in Nova Scotia, a programme that invites Aboriginal students to explore the mathematics in their own community heritage, and Dawn's experiences with the Indigenous Teaching and Learning Gardens at the University of Alberta (UA) and a similar, developing project at McGill University. Because the projects are collective, the understandings and thinking we share below emerge from those collective endeavours, all the people who have been involved in the projects inform and walk alongside us in this writing.

Transversing

We see transversing as cutting across boundaries; both those boundaries that define curriculum content, and the boundaries that separate school and community. We each begin in our disciplinary specialties of mathematics and science, typically with the goal of supporting greater equity for Indigenous students, yet we have long been aware that these disciplinary boundaries are artificially imposed and often create barriers that cause fragmentation in understanding. Our work has shown us that when we begin with ideas and places where learning can emerge and stay open to what emerges, the learning transverses many boundaries both within and outside the classroom.

Show Me Your Math is a programme that emerged from conversations with Mi'kmaw Elders (Wagner & Lunney Borden, 2015) and was inspired by a particular moment where the late Dianne Toney, a quill box maker from We'koqma'q, explained that to make a ring for her circular box top, she would measure three times across the circular top and add a thumb width to ensure a perfect ring with just a little overlap. This knowledge had been passed to her through Elders and is rooted in a community notion of *tepiaq* (enough) which implies you take only what you need and do not have waste. While Dianne's story provides a clear entry point to help students learn the mathematical concept of π, knowledge of π alone is insufficient to make a quill box. When you begin to think about making a quill box, a multitude of questions emerge: What materials do you use to make the top? How do you gather these materials appropriately? What is the best time of year to gather them? How do you make that wooden strip that will be used for the ring? What type of tree does the strip come from? How do you make strips of wood from a section of a tree? How do you gather quills from a porcupine? Which quills do you use? And so on. There are many interconnected processes that must be understood to create a quill box, and the interconnected nature of inquiring into a phenomenon is what took the Show Me Your Math projects into a direction that focused much more on a holistic inquiry leading to the *mawkina'masultinej* (let's learn together) projects.

One such project focused on creating canoe paddles. While the project involved a need to measure and design, to apply fractions and decimals and to work with concepts of symmetry, the learning that emerged from this project cut across many curricular boundaries and learning contexts. The project was inspired by a video of Mi'kmaw guide Todd Labrador making a traditional birch bark canoe (Levangie, 2012). Although there was insufficient time and resources to make a canoe a conversation Lisa had with an outdoor education colleague and a subsequent conversation with the Mi'kmaw language teacher at the school resulted in a decision to make paddles instead. These two individuals worked with the math teacher and the building technology teacher to bring this project to life. The four educators – two Mi'kmaw and two non-Indigenous – brought together their respective knowledges to begin to restore the knowledge of paddle making that nearly 500 years of colonization had eroded. Questions emerged about how paddle design related to the types of paddles that were used in various places and for various purposes. Questions about the design of these paddles and their relationship to water flow prompted Lisa to call upon Dawn to investigate some potential science connections that might be built into the unit. Each new idea seemed to spark yet another new idea and another new line of inquiry.

The building technology teacher, a Mi'kmaw Elder, who provided the space and support for the paddle making, also provided the students with a connection to their own community history. He brought in binders filled with old newspaper clippings that his mother had collected over the years. These clippings told stories of Mi'kma'ki, stories of canoe races as part of the annual Mi'kmaw summer games, and stories of the canoe trip from Cape Breton to Montreal for Expo '67. As students worked on shaving and sanding their paddles, they also learned about paddling as part of their community heritage. One of these newspaper clippings recounted a canoe race win that involved many men from the community, most of whom were grandfathers of students in the class, and that story proved to be the inspiration for a grandfathers' lunch. Five Elders attended the lunch, and they shared stories with the students as they examined the finished paddles the students had created. There was laughter and remembering, and there was great pride in the eyes of both the students and the Elders. The Elders explained that they had not had such opportunities when they were young and had only learned to paddle as adults when they could afford to purchase a canoe. This remembering highlighted the damaging impact of colonization and demonstrated how the policies and practices of settler governments systematically eroded traditional knowledges. As these Mi'kmaw youth and Elders came together to share stories of paddling and paddle making, stories emerged that spoke back to some of these colonizing practices in simple yet profound ways and in their speaking began the process of remembering and restoring knowledge.

The examples of making quill boxes and canoe paddles both began as ways to connect with mathematical knowledge, yet both practices show how the learning can cut across many content areas and beyond the walls of the school into the community. In fact, if the learning had been limited to only the mathematics, much would have remained lost. Wagner and Lunney Borden (2012) have seen similar examples of cutting across boundaries in other Show Me Your Math projects and have noted the value in the programme is rooted in the wholeness that is maintained in the programme. When we resist the temptations to stay within school defined boundaries, we find that there are many learning opportunities and many ways to cut across disciplines and contexts. Simply put, once you begin one thing, you find all of these other things living there.

Transgressing

That such livingness resides in the places and topics of teaching requires some means of transgressing taken-for-granted ideas about what and how teachers teach within schools and some means of breaking (or breaking out of) the ruttish epistemic closure of curriculum-as-plan (Aoki, 2005) into the deep relationships of epistemic livingness (Lee, 2016; Todd, 2016). One such instance was facilitated by the Indigenous Teaching and Learning Gardens at the UA.

The Gardens were developed in response to provincial/territorial mandates to integrate Indigenous perspectives across K-12 curricula (e.g. Alberta Learning, 2002) and call from the Association of Deans of Canadian Education (ADCE) (Archibald, Lundy, Reynolds, & Williams, 2010) to develop "comprehensive teacher candidate and faculty programs that create meaningful opportunities for learning about and practicing Aboriginal pedagogies and ways of knowing" (p. 6). The gardens were created by undergraduate students in one of the secondary science curriculum and instruction courses working with graduate students, an Elder and faculty members. The Gardens have subsequently been taken up in other courses as a location for exploring the ways in which Indigenous and Western ways of knowing, being and doing might circulate together in teaching and learning. Within a grant focused on teacher education, a special topics course involving the Gardens was developed and offered.

Engaging Aboriginal Perspectives in Mathematics and Science was a cross-listed, graduate/undergraduate, elementary/secondary course offered in spring 2013. The class was evenly split between pre- and in-service teachers, graduate and undergraduate students and elementary and secondary specialists. There were several Indigenous students in the class, both undergraduate and graduate. The in-service teachers had 2–20 years of teaching experience. Some of them worked primarily with Indigenous K-12 students; others did not. The teachers working with Indigenous students were not necessarily Indigenous themselves.

The course was designed such that the place and land were emphasized throughout. One full class focused on explicit discussion and consideration of place and land (Barsh & Marlor, 2003; Hermes, 2005; Watson & Huntingdon, 2008), connection of land to language and understanding and how conceptions of place and land might be taken up within mathematics and science teaching and learning. One of the course assignments also focused on place. It asked students to identify a personally meaningful place, explain why it was meaningful to them and then spend time reflecting on course readings, discussions and activities in that place in an ongoing manner as a means of considering their own thinking and learning in relation to each class. Many of the classes occurred at least partially outside the classroom in the Gardens or the Edmonton River Valley. These places came to have their own voices in the course, telling stories about many things (Wiseman, Onuczko, & Glanfield, 2015) including mathematics, science and how Indigenous and Western ways of knowing, being and doing might circulate together to have conversations about teaching and learning. Over the course of the term, engagement with place and land supported the transversing of disciplinary boundaries with students in the course often considering mathematics and science alongside each other. Time with place and in land also began to break down, or transgress, notions the pre- and in-service teachers had taken-for-granted for a long time.

Karen (pseudonym), a non-Indigenous teacher, chose to reflect in her local neighbourhood, an urban community in which she taught at a public school where the majority of students are Indigenous. One evening she came across two different types of trees whose upper roots were exposed to view. She was taken with how, in the entanglement of roots, she could not tell which roots belonged to what tree but how both trees were thriving in the middle of the city. In this moment the roots spoke to her about the neighbourhood, her school and her students in a deeply pedagogical manner:

> I came into Indigenous Perspectives in Science and Mathematics hoping to build stronger relationships with my students, particularly in such logic-based and concrete subjects [mathematics and science], subjects that held limited appeal for my students unless we were engaged in a game or an experiment. While I believe I have certainly reached that goal, it has not been in the superficial way that I realize now I had expected. Instead I have replaced my

> mere hope of engaging my students a little more than before with a hope and an aim to allow my students to have their stories told, to recognize their own roots so to speak. My students' lives are not just made up of one tree. Their stories are instead the roots, tangled together so that they cannot be recognized as one or the other. They exist in both Western and Indigenous worlds, both essential to their being, and neither can be ignored or invalidated with ignoring and dismissing some part of the child. (Karen, Final reflection, May 23, 2013)

In our final class together, we challenged the students to consider what they had come to understand during the course and apply it to the existing programme of studies for mathematics and/or science. Given a suggestion by one of the co-instructors, Dwayne Donald, in the first class of the term (field notes, May 7, 2013), we asked them to take seriously what we had learned together, even if they were still struggling with some of the ideas, and think about what curriculum looks like when connected to land. We suggested they might use Dwayne's ideas of what a curriculum of berries or buffalo might look like and what this curriculum might say about currently accepted practices in mathematics and/or science teaching and learning. To make the task manageable within the available time, we asked them to choose a range of grades from either 1–3, 4–6, 7–9 or 9–12. One group took on the idea of berries from grades 7–9 and began with the existing science programs. Very quickly, however, they were also looking at the mathematics program and pulling up the social studies and language arts programmes on a computer, again transgressing disciplinary boundaries. Somewhere in the process, one of the students – a vice-principal with many years of teaching experience – said loud enough for everyone to hear, "We even plan wrong!" (field notes, May 25, 2013). Her interjection led to some discussion of what she meant. She spoke about how in connecting to land through thinking about berries they had realized how quickly the science curriculum-as-plan was inadequate to the kind of understanding they wanted students to develop. While their final plan got at most of the prescribed outcomes in the science programme, in thinking about berries, they identified overarching themes connecting to health and food and then discussed how ideas about health and food are prescribed by both worldview and political systems, how everyone in the group had stories and experiences of berries that learning could emerge from and how the mathematics emerged in their plan organically, "always growing" and there to "pick" like the berries (field notes, May 25, 2013). They transgressed not only disciplinary boundaries but accepted ways of and rules about planning to think about teaching in a manner that opened their eyes to new possibilities for teaching and learning.

Transcending

What we find in our practice is that such transversing and transgressing seem necessary to thinking about how Western and Indigenous ways of knowing, being and doing might circulate together in mathematics and science teaching and learning (not that we think of the subjects as singular disciplinary silos anymore). We wonder if, in fact, transversing and transgressing are necessary to moving towards transcending or being able to move beyond current conceptions of what it means to teach mathematics and science towards something more amenable to reconciliation.

This wondering arises not solely from our own work but also from the work of colleagues engaged in similar thinking in different parts of the country. One such example comes from the work of Jim Kreuger, a science and mathematics specialist for Kivalliq School Operations in Nunavut. Jim points to how moving science teaching and learning onto the land – transversing and transgressing generally accepted notions of where and how learning occurs – allows for transcending the notion of teacher:

> It, it taps into something that's in all of us I think, for some of us maybe it's harder to find but, you take people out on to the land and you camp for four, five, six days, and you just treat people differently. A whole different social dynamic develops, and the teacher/learner dichotomy is gone. We all become learners. We all become teachers. And at a science camp you can see students teaching teachers how to light a Coleman[3] lamp, you can see students teaching Elders how to use a GPS, Elders teaching students how to drum dance, Elders teaching teachers how to cache meat, and teachers teaching, you know, everyone how to, what, how to exit a kayak –everyone teaches. It's just sharing, it's natural. (In Wiseman, 2016, p. 281)

Jim also underlines how such transcendence requires letting go of generally received notions about lesson planning and structure:

> Like I say, it's pretty hard to mess up a land-based activity, unless you get way too focused and anal about plans and objectives and goals–because if you truly value the land and the environment the weather is part of that, and it changes, and you have to adapt. And then the Elders are always telling us, you know, 'We're not doing that, we're doing this'. Like a blizzard will produce snow, that means the lesson is now snow, it's not caribou. And, so as long as you are able to be flexible and receptive I guess to the lessons that present themselves. And sometimes that means letting go of something that you planned, or sometimes adjusting it, and doing it in a different way. Yeah, I think you can, for people who are really up tight and not flexible it can be very frustrating. But, ... but if you're patient and you listen, I think you'll hear what needs to be done. (In Wiseman, 2016, p. 283)

In our work, and the work of our colleagues, we recognize transcending in these practices that break down notions of what it means to teach – like planning and authority – but not necessarily of what it means to learn or come to understand in a deep, contextualized manner. As such, we consider how much transcending is about letting go of control to attend to and learn from what is present instead of what is prescribed. In our own work with in-service and pre-service teachers, we have seen that letting go is one of the greatest challenges they experience. Pressures such as curriculum-as-plan, student information systems for recording outcomes, student performance on standardized assessments and other systemic pressures all seem to work towards replicating existing school structures and repressing the memory that there are other ways to engage in teaching. One glimmer of hope we have noticed is that when teachers engage in the kinds of work we have described, it helps them to see other ways of engaging students in learning experiences. "It taps into something

[3] A Coleman lamp is a propane powered, portable light and heat source used for camping and land-based living.

that's in all of us" (J. Kreuger in Wiseman, 2016, p. 281) and helps them to remember there are other ways of teaching and being together in learning with their students. The challenge is how to ensure that these teachers remember to remember these experiences so that they can continue to create opportunities to transverse, transgress and transcend the dominant ways of knowing, being and doing in schools and move towards ways that might further the goals of reconciliation.

Transcending Transdisciplinarity

In the work we have described, we recognize glimpses of healing, of centring Indigenous ways of knowing, being and doing and of letting go of prescribed notions of mathematics and science teaching to see what emerges for and in learning; experiences that seem to provide space for the reclaiming, restoring and returning that are foundational to reconciliation (TRC, 2015). We also recognize that these glimpses – while promising beginnings – are only beginnings and not yet a whole story. The work continues to be fraught with tension because it pushes back against colonizing systems that try to reproduce themselves instead of entering into conversations about what else might be. Still, we remember that these tensions are to be expected and may even be generative, because limited beginnings have a way of growing, unless repressed. To paraphrase Thomas King (2003), once you've heard a story you can't say you didn't know. And so, we look for ways of remembering to remember in order to transcend the status quo (N. Blood, in Wiseman, 2016).

Remembering to remember allows us to transverse by cutting across contexts, ways of knowing, being and doing and opening up spaces for multiple voices in teaching and learning, in science and mathematics. Remembering to remember allows us to transgress by breaking away from the expected and creating new ways of planning for learning. Remembering to remember allows us to transcend by letting go of control and allowing learning to emerge in meaningful contexts and interesting questions. Such remembering allows us as educators to move beyond the imposition of solely Western ideas about school and schooling so that we can get to a place where we might begin to unlearn colonialism and begin to repair, restore and return as a means of moving towards reconciliation.

These ideas seem connected to the idea of transdisciplinarity, but we wonder if we make this connection as white women. And this remembrance brings us back to our questioning of the term transdisciplinarity. It is useful and workable in the senses that we have described in this chapter, but we continue to wonder if it belongs in our work or if it is too much of a Western construct. The canoe paddles, the berries and the buffalo live and emerge in the places and from the land in which we teach, and so we suspect there is a better word (or words) than transdisciplinarity to describe the work we have shared – words that also come from the land and the places in which we teach and likely transcend the notion of transdisciplinarity. While we have not as yet heard or learned the word(s), we hold onto the idea that they might exist because we understand that what we describe is only an echoed remembering of things Indigenous people have not forgotten. It is perhaps the place where we need to be in order to begin the reconciliation process – and so there is still much work to be done, because the worst thing would be to do nothing and just go with the same (N. Blood in Wiseman, 2016).

References

Absolon, K. E. M., & Willett, C. (2005). Putting ourselves forward: Location in Aboriginal research. In L. Brown & S. Strega (Eds.), *Research as resistance: Critical, Indigenous and anti-oppressive approaches* (pp. 97–126). Toronto, ON: Canadian Scholars' Press.

Alberta Learning. (2002). *First Nations, Métis and Inuit education policy framework*. Edmonton, AB: Government of Alberta. Retrieved from http://education.alberta.ca/media/164126/framework.pdf.

Aoki, T. T. (2005). Teaching as indwelling between two curriculum worlds. In W. F. Pinar & R. L. Irwin (Eds.), *Curriculum in a new key: The collected works of Ted T. Aoki* (pp. 159–165). Mahwah, NJ: Lawrence Erlbaum Associates.

Archibald, J., Lundy, J., Reynolds, C., & Williams, L. (2010). *Accord on Indigenous education*. .Association of Canadian Deans of Education. Retrieved from http://www.csse-scee.ca/docs/acde/acde_indigenouseducationaccord_en.pdf

Barsh, R. L., & Marlor, D. (2003). Driving bison and Blackfoot science. *Human Ecology, 31*(4), 571–593.

Berstein, J. H. (2015). Transdisciplinarity: A review of its origins, development, and current issues. *Journal of Research Practice, 11*(1), 1–20. Retrieved from http://jrp.icaap.org/index.php/jrp/article/view/510/412.

Cajete, G. A. (1999). Reclaiming biophilia: Lessons from Indigenous people. In G. A. Smith & D. R. Williams (Eds.), *Ecological education in action: On weaving education, culture and the environment* (pp. 189–206). Albany, NY: State University of New York.

Cajete, G. A. (2006). Western science and the loss of natural creativity. In Four Winds (Ed.), *Unlearning the language of conquest: Scholars expose anti-Indianism in America* (pp. 247–256). Austin, TX: University of Texas Press.

Crowshoe, R., & Manneschmidt, S. (2002). *Akak'stiman: A Blackfoot framework for decision-making and mediation processes*. Calgary, AB: University of Calgary Press.

D'Ambrosio, U. (2015, June). *From mathematics education and society to mathematics education and a sustainable civilization.* Paper presented at the eighth international mathematics education and society conference, Portland, OR.

Denzin, N. K. (2005). Emancipatory discourse and the ethics and politics of interpretation. In N. K. Denzin (Ed.), *Sage handbook of qualitative research* (3rd ed., pp. 933–958). Thousand Oaks, CA: Sage Publications.

Doll, W. E. (1993). *A post-modern perspective on curriculum*. New York: Teachers' College Press.

Donald, D. T. (2013, June). *Miyo wichihtowin*. Paper presented at the Canadian Society for the Study of Education, Victoria, BC.

Doolittle, E., Lunney Borden, L., & Wiseman, D. (2011). Can we be thankful for mathematics?: Mathematical thinking and Aboriginal peoples. In P. Liljedahl, S. Oesterle, & D. Allen (Eds.), *Canadian Mathematics Education Study Group: Proceedings 2010* (pp. 81–94). Burnaby, BC: CMESG.

Hermes, M. (2005). "Ma'iingan is just a misspelling of wolf": A case for teaching culture through language. *Anthropology and Education Quarterly, 36*(1), 43–56.

Ihde, D. (1996). Whole earth measurements: How many phenomenologists does it take to detect a 'greenhouse effect'. *Techné: Research in Technology and Philosophy, 2*(2.) Retrieved from http://scholar.lib.vt.edu/ejournals/SPT/v2n2/ihde.html.

Ihde, D. (1998). *Expanding hermeneutics: Visualism in science*. Evanston, IL: Northwestern University Press.

King, T. (2003). *The truth about stories: A Native narrative*. Toronto: ON: House of Anansi Press Inc.

Kovach, M. (2005). Emerging from the margins: Indigenous methodologies. In L. Brown & S. Strega (Eds.), *Researcher as resistance: Critical, Indigenous, and anti-oppressive approaches* (pp. 19–36). Toronto, ON: Canadian Scholars' Press.

Kovach, M. (2009). *Indigenous methodologies: Characteristics, conversations and context*. Toronto, ON: University of Toronto Press.

Lee, E. V. (2016, November 29). How it feels to experience @ZoeSTodd's Indigenous Feminist scholarship rocking European philosophy to its core: Epistemic lifegiving. https://twitter.Com/ZoeSTodd/status/803741240030175232 … [Tweet]. Retrieved from. https://twitter.com/ericavioletlee/status/803752870981967872?refsrc=email&s=11

Levangie, J. (Director). (2012). Building legends: The Mi'kmaq Canoe Project [Motion picture]. Canada: Viewfinders: International Film Festival for Youth & The Mi'kmaq Liaison Office Nova Scotia Department of Education.

Little Bear, L. (2000). Jagged world views colliding. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 75–85). Vancouver, BC: UBC Press.

Little Bear, L. (2012, August 20). *Keynote lecture.* Presentation to the Canadian prairie theoretical physicists network conference, Regina, SK.

Lunney Borden, L. (2010). Transforming mathematics education for Mi'kmaw students through mawikinutimatimk. Unpublished doctoral dissertation, University of New Brunswick, Fredericton, NB.

Lunney Borden, L., & Wagner, D. (2013). Naming method: "This is it, maybe, but you should talk to...". In R. Jorgensen, P. Sullivan, & P. Grootenboer (Eds.), *Pedagogies to enhance learning for Indigenous students: Evidence-based practice*. Dordrecht: Springer.

Lunney Borden, L., & Wiseman, D. (2016). Considerations from places where indigenous and Western ways of knowing, being, and doing circulate together: STEM as artifact of teaching and learning. *Canadian Journal of Science, Mathematics and technology Education, 16*(2), 140–152. doi:10.1080/14926156.2016.1166292.

Pohl, C. (2010). From transdisciplinarity to transdisciplinary research. *Transdisciplinary Journal of Engineering and Science, 1*, 65–73.

Rabaka, R. (2010). *Against epistemic apartheid: W. E. B. Du Bois and the disciplinary decadence of sociology*. Plymouth, UK: Lexington Books.

Smith, L. T. (1999). *Decolonizing methodologies: Research and Indigenous peoples*. London: Zed Books.

Smith, L. T. (2005). On tricky ground: Researching the native in the age of uncertainty. In N. K. Denzin & Y. S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (pp. 85–107). Thousand Oaks, CA: SAGE Publications.

Stewart-Harawira, M. (2005). *The new imperial order: Indigenous responses to globalization*. London: Zed Books.

Taubman, P. (2007). The beautiful soul of teaching: The contribution of psychoanalytic thought to critical self-reflection and reflective practice. In M. Gordon & T. O'Brien (Eds.), *Bridging theory and practice in teacher education* (pp. 1–25). Rotterdam, The Netherlands: Sense.

Todd, Z. S. (2016, November 10). So long, and thanks for all the fish: Navigating the impact of settler colonial violence and the Anthropocene on human-fish relations in Treaty Six Territory, Alberta, Canada. Invited talk presented to the Department of Architecture, .Eidgenössische Technische Hochschule Zürich, Switzerland. Retrieved from https://cast.switch.ch/vod/clips/1vy6tkcs45/streaming.html

Truth and Reconciliation Commission of Canada. (2015). *Canada's residential schools: Reconciliation* (Vol. 6). Montreal, QC/Kingston, ON: Queen's and McGill University Press.

Vowel, C. (2016). *Indigenous writes: A guide to understanding first nations, Métis and Inuit issues in Canada*. Winnipeg, MB: Highwater Press.

Wagner, D., & Lunney Borden, L. (2012). Aiming for equity in ethnomathematics research. In J. Choppin, B. Herbel-Eisenmann D. Pimm, & D. Wagner (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies,* (pp. 69–87). Netherlands: Springer.
Wagner, D., & Lunney Borden, L. (2015). Common sense and necessity in (ethno)mathematics. In K. Sullenger & S. Turner (Eds.), *CRYSTAL Atlantique: A story of informal learning research in science, mathematics, and technology* (pp. 78–89). Rotterdam, The Netherlands: Sense.
Watson, A., & Huntingdon, O. H. (2008). They're here – I can feel them: The epistemic spaces of Indigenous and Western knowledges. *Social and Cultural Geography, 9*(3), 257–281.
Wilson, S. (2008). *Research is ceremony: Indigenous research methods*. Halifax, NS: Fernwood Publishing.
Wiseman, D. (2009). Greg Cajete: Life and work. Unpublished paper, Department of Secondary Education, University of Alberta, Edmonton, AB.
Wiseman, D. (2016). Acts of living with: Being, doing, and coming to understand indigenous perspectives alongside science curricula*.* Unpublished doctoral dissertation, Department of Secondary Education, University of Alberta, Edmonton, AB.
Wiseman, D., Onuczko, T., & Glanfield, F. A. (2015). Resilience and hope in the garden: Intercropping Aboriginal and Western ways of knowing to inquire into science teacher education. In H. Smits & R. Naqvi (Eds.), *Framing peace: Thinking about and enacting curriculum as "radical hope"* (pp. 237–252). New York: Peter Lang Publishing.
Zimmer, B., & Solomon, J. (2015). Among the new words. *American Speech, 90*(1), 84–105. doi:10.1215/00031283-2914703.

Chapter 10
Gendered? Gender-Neutral? Views of Gender and Mathematics Held by the Canadian General Public

Jennifer Hall

Abstract By investigating the general public's views, we can better understand the cultural milieu in which mathematics teaching and learning take place. In this study, part of an international research project, I investigated the Canadian general public's views of gender and mathematics. Using a brief survey, people on the street and in public spaces in four demographically diverse locations in the Canadian province of Ontario were asked their views on the topic. The findings suggest reasons to be both cautiously optimistic and concerned. While the most common response to the questions examined typically was to see no gender difference, more participants held a gendered view (typically privileging boys) than a gender-neutral view. Interestingly, no age group-related differences occurred in response patterns, but gender-related differences in response patterns were evident.

Keywords Mathematics education • Gender • General public • Beliefs/views

Introduction

Investigating the general public's views about mathematics is essential in order to garner an understanding of the social milieu in which mathematics teaching and learning occur. Unfortunately, as argued by Leder and Forgasz (2010), "attempts to measure directly the general public's views about mathematics, its teaching and its impact on careers are rare" (p. 329). While several studies exist regarding people's views of mathematics, these studies are often conducted with select populations,

J. Hall (✉)
Monash University,
Room 406, Building 6, Clayton Campus, 29 Ancora Imparo Way, Clayton, VIC 3800, Australia
e-mail: jennifer.hall@monash.edu

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_10

such as elementary, secondary and tertiary students (e.g. Hall, 2013; Mendick, Epstein, & Moreau, 2007; Morge, 2006; Towers, Takeuchi, Hall, & Martin, 2015). Only in a few known studies have researchers investigated this topic with the general public, defined as “ordinary people, especially all the people who are not members of a particular organization or who do not have any special type of knowledge” (The general public, 2016). For example, researchers in the United Kingdom (Lim, 1999; Lim & Ernest, 1999, 2000) explored the general public’s images of and opinions about mathematics, in a project involving approximately 550 participants ages 17 and above. Overall, the most negative views of mathematics were found in the youngest group of participants (17–20 years of age) and in university students who were not mathematics majors. Encouragingly, the majority of participants disagreed with the stereotype that mathematics is a male domain. However, the majority of the participants also agreed that mathematics is a difficult subject, only for a select few. Lim and Ernest concluded that the adults’ views were primarily influenced by their school mathematics experiences. More recent research (Lucas & Fugitt, 2009), conducted with more than 1,300 participants in the rural Midwestern United States, explored the general public’s views of mathematics education. The study’s participants tended to hold traditional views, criticizing today’s practices as lacking emphasis on “the basics” and being too focused on technology. Overall, mathematics was seen by the participants as being very important to success in postsecondary education, future careers and everyday life.

Due to concerns about a lack of research in this domain, Leder and Forgasz initiated research in Australia that investigated the general public’s views of mathematics, with a particular focus on gender and mathematics (reported in such publications as Forgasz & Leder, 2011; Forgasz, Leder, & Gómez-Chacón, 2012; Forgasz, Leder, & Tan, 2014; and Leder & Forgasz, 2010, 2011). Using a brief survey, initially conducted on the street and later via Facebook, Leder and Forgasz gathered data from both Australian and international participants. In order to expand the research internationally, a team of researchers was assembled to collect street-level data in a variety of countries. Specifically, street-level data were collected in Australia, Canada, South Korea, Spain and the United Kingdom.[1] The total number of street-level participants in each country is shown in Table 10.1.

Table 10.1 Number of street-level participants in each participating country

Country	Number of participants
Australia	799
Canada	204
South Korea	318
Spain	636
United Kingdom	61

[1] These are the countries in which volunteers agreed to collect data.

In this chapter, I discuss findings from my analysis of the street-level data collected in Canada as part of this large international research project.

Context

The data collection for the Canadian sample took place in the province of Ontario, which is located in Central Canada and contains nearly 40% of the country's population (Statistics Canada, 2015a). In Canada, education falls under the purview of individual provinces and territories (i.e. no national curriculum exists). Ontario's mathematics curriculum (Ontario Ministry of Education, 2005a, 2005b, 2007) addresses a wide variety of mathematical topics in each grade level, and emphasis is placed on diversity in both teaching practices and assessment types. The use of mathematical tools is encouraged, both in class and on large-scale provincial assessments of mathematics. Fundamentally, the Ontario mathematics curriculum is based on the belief that "all students can learn mathematics and deserve the opportunity to do so" (2005a, p. 3).

Ontario students are required to participate in large-scale provincial assessments of mathematics in Grades 3, 6 and 9. These assessments are created and conducted by the Education Quality and Accountability Office (EQAO). The EQAO assessments involve a variety of question types and address the provincial curriculum. My analysis (Hall, 2012) of five years of EQAO data showed that no statistically significant gender differences existed at any grade level in terms of mathematics achievement. In contrast, as demonstrated by my analysis of data from the questionnaires that accompany the assessments, gender differences existed with regard to affective factors. Namely, across all grade levels and across the five years of data examined, a statistically significantly higher percentage of boys, compared to girls, reported liking mathematics and being good at it.

In Ontario, students are required to take three mathematics courses during high school, between Grade 9 and Grade 12 (Ontario Ministry of Education, 2016). At the Grade 12 level, when most students have completed their required mathematics courses, boys have a higher proportion of mathematics courses in their timetables than girls do (Hall, 2012). Additionally, boys are the majority of students in five of the six Grade 12 mathematics courses offered (Hall, 2012). These gender differences persist at the university level, where women are the minority in mathematical fields from the bachelor's to doctoral degree level. Notably, the proportion of women in mathematical fields of study at the bachelor's and master's degree levels has been declining since the early 1990s (Statistics Canada, 2010a, 2010b, 2015b, 2015c).

Theoretical Framework

This study was guided by a social constructivist and feminist epistemological stance, in which gender is viewed as being socially constructed, as well as historically and culturally situated. As suggested by Simon (1995), "we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world" (p. 115). This learning process applies to learning mathematics itself, learning ideas about mathematics (e.g. stereotypes about mathematics and mathematicians) and learning about gender. My views are consistent with Howard and Hollander's (1997) definition of gender as "the culturally determined behaviors and personality characteristics that are associated with, but not determined by, biological sex" (p. 11, as cited in Glasser & Smith, 2008, p. 346). In this definition, the roles that the broader society and culture play in policing behaviours presumed to be "gender-appropriate" are highlighted, which is particularly relevant in mathematics, a field historically viewed as a male domain.

In alignment with scholars such as Butler (1990, 1999) and Fausto-Sterling (2000), I view both gender and sex as performative social constructions that fall on a spectrum, rather than into binary categories. That said, I support the lead researchers' decision to offer "boys" and "girls" as responses and "men" and "women" as coding categories – both in terms of a pragmatic decision and in terms of reflecting current society, in which binaried representations and categorizations are the norm. Gender tends to be a particularly salient aspect of schooling (e.g. grouping students by gender, girls' and boys' teams). While there have been some recent shifts in societal perceptions of gender, our world is still very much gender binaried in most settings. Thus, by having binary categories in the gender-related questions on the survey and analysing the data by binary groups (i.e. men and women), I am reflecting the current cultural milieu in which the participants live. I recognize, however, that in so doing, I may be excluding and/or misrepresenting those individuals whose gender identities do not align with binary categories.

Methodology

As this research is part of a larger, international project instigated by Helen Forgasz and Gilah Leder of Australia, the data collection instrument and methods of data collection followed the guidance of the principal investigators. In the subsequent sections, I begin by describing the data collection instrument and process of data collection. Then, I provide demographic information about the study's participants. I conclude by discussing the methods by which the data were analysed.

Data Collection Instrument

Data were collected using a survey, designed by Forgasz and Leder, comprised of 14 questions. Of the 14 questions, two addressed the participant's school mathematics experiences (i.e. liking mathematics and perceiving themselves as good at mathematics), three addressed mathematics education generally and the remaining nine questions focused on gender and mathematics (or related fields). The three "general" mathematics education questions addressed (1) whether participants thought the teaching of mathematics had changed since they were in school, (2) whether participants thought students should study mathematics when it was no longer compulsory and (3) whether participants thought studying mathematics was important for getting a job.

The nine gender-focused questions addressed the participants' views of who was better in mathematics, girls or boys – both their own stance and their perceptions of teachers' and parents' views. Participants were also asked to reflect on whether there had been a change over time in terms of who was better at mathematics, girls or boys, and whether it was more important for girls or boys to study mathematics. The other gender-focused questions addressed the participants' views of boys' and girls' suitability for jobs in mathematics-related fields (science and the computer industry) and abilities with mathematical tools (computers and calculators). All of the gender-focused questions were worded with "girls or boys" at the end of the question (e.g. "Who do teachers believe are better at mathematics, girls or boys?"). Note that this ordering disrupts the commonly used ordering of "boys and/or girls", which unconsciously privileges boys. In the reporting of findings from this study, I consciously use varied orders when discussing gendered groups, as, like Thorne (1993) in her landmark book, *Gender Play*: *Girls and Boys in School*, I want "both genders to be fully in view" (p. 8).

In addition to the 14 topical survey questions, demographic information about the participant's gender, age (under 20, 20 to 39, 40 to 59 and 60 and older) and home language (English or another language) was collected. Participants had to be at least 18 years of age (i.e. adults under Canadian law) to take part in the study. In addition to responding to the provided questions, participants were offered the chance at the end of the survey to provide further comments.

Data Collection

Data were collected in the Canadian province of Ontario between December of 2012 and August of 2013. Four locations were selected based on their varied demographic make-up, herein referred to by the pseudonyms of Rochester, Thomasville, Upton and Smithburg. To increase the ease of following the findings, I have created pseudonyms that begin with the same letter as each location's demographic classification (e.g. Rochester = rural). Information about each location is shown in Table 10.2.

Table 10.2 Demographic information about the data collection sites

	Rochester	Thomasville	Smithburg	Upton
Classification	Rural	Town	Suburban	Urban
Part of Ontario	Southwestern	Central	Eastern	Eastern
Population	3,000	25,000	110,000	900,000

Data collection took place in grocery stores in Rochester and Smithburg, in a community centre in Thomasville and on a downtown street in Upton. In each location, permission to conduct the research was obtained from the appropriate individuals (e.g. store managers), in addition to the Research Ethics Board permission granted by the Australian and Canadian universities associated with the research. In Thomasville, the initial data collection site, I collected the data by myself, which resulted in an inefficient process (seven hours to complete approximately 50 surveys). That is, while I was speaking to a participant, many other potential participants went by. Thus, for the other three sites, I was assisted by a friend or family member in order to make the data collection process more efficient; in each instance, the requisite number of surveys [~50, the minimum number deemed acceptable for chi-square analyses, as per Muijs' (2004) recommendations] was collected in two hours. We are all young women, so it could be construed that having a woman asking questions about gender and mathematics (a field that has historically been dominated by men) may bias the participants' responses. However, Schaeffer, Dykema and Maynard's (2010) findings from their review of several studies indicated that gender-related effects of the interviewer on the participants' responses are typically minimal or nonexistent. Indeed, when collecting data, we did not feel as though the participants "held back" or otherwise altered their responses because they were being interviewed by a woman about gender and mathematics (e.g. explicitly sexist comments, in boys'/men's favour, were provided).

In each location, data collection occurred on the weekend or on a public holiday, in an attempt to maximize the number and diversity of passersby. Passersby who appeared to be in a hurry, who had small children with them and/or who were wearing headphones were not approached, after initial rejections from these populations and/or difficulty when conducting the survey (in the case of participants with small children). In each instance, the interviewer would approach a passerby, introduce herself and ask if the person would be willing to take part in a brief survey. Participants were then asked if they agreed to be audiotaped; if not, answers were recorded on a hard copy of the survey. In nearly all cases, participants agreed to be audiotaped, particularly upon learning that the purpose of audiotaping was to assist in the quality of data collected and subsequently analysed. Participants were also assured by the fact that I would be the only person to access the audio tapes. Prior to being asked the gender-related questions, participants were informed that, although the questions were worded in a binary manner (i.e. girls or boys), they were welcome to answer as they wished (e.g. "They are equal"). If participants inquired further about the research project, a handout was provided with more information, including a description of the larger project and contact information for the lead researchers, Forgasz and Leder. The majority of participants did not receive a handout.

Participants

In total, 204 people participated in this research project: 52 from Rochester, 53 from Thomasville, 50 from Smithburg and 49 from Upton. In most cases, participants took part in the interviews individually. The exceptions occurred in cases such as a

Table 10.3 Participants in each location, by age group

	Age group			
	Under 20	20–39	40–59	60 and older
Rochester	7.7%	28.8%	25.0%	38.5%
Thomasville	0.0%	49.1%	20.8%	30.2%
Smithburg	0.0%	26.0%	40.0%	34.0%
Upton	0.0%	67.3%	18.4%	14.3%
All participants	1.5%	43.1%	26.0%	29.4%

couple or parents with an adult child agreeing to participate. In total, 17 pairs and two trios completed the interviews, resulting in 40 of the 203 participants (19.6%) completing the survey in a nonindividual situation. In these cases, I attempted to alter the order in which the participants responded, to get a better sense of each participant's individual views, particularly if I noticed a trend of the second respondent simply agreeing with the first respondent's answers, rather than voicing a unique opinion and/or explanation.

In each location, more women than men took part, although the participants were more gender-balanced in Upton and Smithburg (55.1% and 52.0% women, respectively) than in Rochester and Thomasville (67.3% and 62.3% women, respectively). Overall, 59.3% of the participants were women.

Interesting variations in age distribution occurred across the data collection sites, as depicted in Table 10.3 (Percentages apply to each row).

Understandably, few participants in the "under 20" (i.e. 18–19 years of age) age group were involved in the study. Besides the small age range, we may have missed approaching potential participants due to our perceptions of their age: If we thought that a passerby appeared younger than 18 years of age, we would not approach her/him. The high proportion of participants aged 20–39 in Thomasville may be explained by the number of families with children who use the community centre (the data collection site), while the high proportion of participants aged 20–39 in Upton may be explained by the proximity of the street (the data collection site) to postsecondary institutions and neighbourhoods where many young adults live.

With regard to linguistic diversity, great variations occurred across the locations, reflecting their demographic characteristics. In Rochester, a rural town of 3,000 people with very little ethnic diversity, only 4 of the 52 participants (7.7%) reported speaking a language other than English at home. In contrast, in Thomasville, a larger and rapidly expanding town (current population of 24,000 people; the population has doubled in the past five years) located within commuting distance of a large, diverse city, 11 of the 53 participants (20.8%) reported speaking a non-English language at home. Notably, in these two locations, both of which are located in Central/Southwestern Ontario, a wide variety of languages (e.g. German, Tagalog) was reported. In contrast, most of the participants in Upton and Smithburg who reported speaking a non-English language at home spoke French. In Upton, 14 of the 49 participants (28.6%) reported speaking a language other than English at home, compared to 33 of the 50 participants (66.0%) in Smithburg. These findings are not surprising, given that both centres are located in Eastern Ontario (Smithburg is east of Upton), and the primarily French-speaking province of Quebec is located east of Ontario.

Data Analysis

Using the audio or written recordings, the participants' responses to the questions were coded using categories (e.g. "boys", "girls", "same", "don't know" and "depends" for the gender-focused questions) provided by the lead researchers, in order to allow for international comparisons. These data were analysed using descriptive statistics (e.g. counts, percentages). Due to the low number of "don't know" and "depends" responses, these categories were combined into a single "unsure/ambivalent" category.

Chi-square tests for independence were performed in order to determine if there were statistically significant differences (at the $\alpha = 0.05$ level) in the ways that participants' responses were distributed by gender and age group across the response categories. Since there were only three participants in the "under 20" age category, this category was combined with the "20–39" age category, thus resulting in an "18–39" age category. This combined category was used in the chi-square analyses by age group. Hence, the age group categories used in the analysis were "18–39", "40–59" and "60 or older".

If participants provided further explanation for their responses, these comments were transcribed and analysed using emergent coding. That is, the responses for each question were examined through multiple readings to obtain a sense of the data, and then categories were created and used to code the responses.

Findings

For the purposes of this chapter, I focus on the two questions about the participants' school experiences, in order to provide a clearer profile of those who took part in the research, and the five questions that specifically related to gender and mathematics (as opposed to mathematics education in general, electronic tools or related careers, such as being a scientist). Findings are presented for each selected question in the following sections.

Q2: When you were at school, did you like learning mathematics?

Just over half of the participants (54.4%) reported that they enjoyed learning mathematics while they were in school, compared to 33.3% who reported disliking mathematics. Only 12.3% of the respondents reported feeling ambivalent toward mathematics. Unsurprisingly, the explanations provided for positive or negative feelings toward mathematics often related to how strong or weak the participants felt that they were in mathematics. Other reasons provided for liking

Table 10.4 Response distribution by gender for Q2

Group	Like	Dislike	Ambivalent
Men	68.7%	21.7%	9.6%
Women	44.6%	41.3%	14.0%

mathematics included finding the subject interesting and "real world" applicable, as well as appreciating the logic, order and "black and white" nature (i.e. only one right answer) of mathematics. One participant (P41, Upton, man, 18–39) outlined his fascination with finding mathematics in the real world thusly: "I found them [mathematical concepts] amazing and loved that they weren't just made up but also noticeable in nature". Participants who disliked mathematics described it as being boring, reported having poor teachers and labelled themselves as "language people". One participant (P38, Rochester, woman, 18–39) expressed her distaste by simply exclaiming, "Math is dumb!" Those who reported being ambivalent toward mathematics often provided an explanation that related to different feelings for different types of mathematics. For example, one participant (P10, Thomasville, woman, 60 or older) stated, "I enjoyed geometry and stuff, but I didn't enjoy algebra".

Chi-square analyses of the responses to this question revealed statistically significant differences in the response distributions by gender ($\chi^2 = 11.708$, $p = 0.003$) but not by age group ($\chi^2 = 2.117$, $p = 0.714$). In Table 10.4, the spread of the responses by the gender of the participants is shown.

As shown in Table 10.4, the men who participated in the study reported holding far more positive views of mathematics than did the women who participated in the study. More than two-thirds of men reported liking mathematics as students, compared to less than half of the women. While similar percentages of women and men reported feeling ambivalent, nearly twice as many women as men reported disliking mathematics as students. Such findings are not surprising, given the wealth of literature (e.g. Hall, 2012; Lupart, Cannon, & Telfer, 2004) reporting gender differences in boys'/men's favour with regard to feelings toward mathematics.

Q3: Were you good at mathematics?

As discussed, reports of liking mathematics were often linked to reports of being good at mathematics. Consequently, it follows that a similar proportion of participants (52.9%) reported being good at mathematics. However, participants who felt they were average or not good at mathematics were more evenly distributed (20.1% and 27.0%, respectively) than the "dislike" or "ambivalent" responses to the prior question. Explanations for being good at mathematics primarily related to school grades, although a few participants provided other evidence, such as working in a mathematics-focused field, being in gifted classes and understanding mathematics quickly.

Table 10.5 Response distribution by gender for Q3

Group	Strong	Weak	Average
Men	69.9%	13.3%	16.9%
Women	41.3%	36.4%	22.3%

Chi-square analyses of the responses to this question again revealed statistically significant differences in the response distributions by gender ($\chi^2 = 18.063$, $p < 0.001$) but not by age group ($\chi^2 = 4.837$, $p = 0.304$). Given the links between liking mathematics and perceiving oneself to be good at mathematics, this finding is not surprising. In Table 10.5, the spread of the responses by the gender of the participants is shown.

As shown in Table 10.5, the vast majority (nearly 70%) of the men in the study felt that they had been strong mathematics students. The remaining men were distributed fairly evenly between "weak" and "average", with the fewest number of respondents reporting that they were weak students. This distribution was in stark contrast to the responses from women: Nearly as many women reported being weak students as being strong students (approximately two-fifths in both cases). While these claims were often substantiated by reports of poor mathematics marks or placement in "low" streams (e.g. "basic", "workplace") of mathematics classes, it is possible that the men in the study may have over-reported their abilities in the subject area. Reports of boys' and men's overconfidence in mathematics are plentiful in the extant literature (e.g. Bench, Lench, Liew, Miner, & Flores, 2015; Dahlbom, Jakobsson, Jakobsson, & Kotsadam, 2011; Sadker & Sadker, 1994).

Q6: Who are better at mathematics, girls or boys?

Encouragingly, the most common response (37.3%) was that there were no gender differences in mathematics ability. However, this response was only slightly more common than believing that boys are better at mathematics (31.9%). Although a substantial proportion of participants reported that girls are better at mathematics (20.6%), these responses were only two-thirds the number of those who selected boys. In total, over half of the participants held some sort of gendered stance for this question. Few participants (10.3%) reported holding an unsure or ambivalent stance on this topic.

Participants who claimed that there were no gender differences tended to provide an explanation relating to the notion of everyone being equally capable at mathematics. For instance, one participant argued, "Men and women are equal and have the same brain power" (P51, Thomasville, woman, 18–39). Explanations for girls' mathematical superiority often related to girls being stronger students overall, whereas explanations for boys' mathematical superiority tended to relate to innate ability ("mathematical nature"). An example of a comment about the latter is: "I think men have more of a capacity to take that in – math – They're probably better than females" (P2, Smithburg, man, 60 or older). Related, the stereotypical notion of girls being better at language arts and boys being better at mathematics was discussed.

Chi-square analyses of the responses to this question revealed no statistically significant differences in the response distributions by gender ($\chi^2 = 3.037$, $p = 0.386$) or age group ($\chi^2 = 11.402$, $p = 0.077$). Thus, regardless of gender or age group, substantial proportions (approximately 20–40%) of the respondents reported (a) a gender-neutral view, (b) a view of girls as superior or (c) a view of boys as superior, compared to very few participants with an unsure or ambivalent viewpoint.

Q7: Do you think this has changed over time?

Participants' views were quite mixed (40.2% agreed and 44.6% disagreed), which may perhaps be indicative of different interpretations of the question. Less than one-sixth (15.2%) of the participants reported being unsure or ambivalent about this question. Of the participants who either agreed or disagreed, some participants' explanations appeared to indicate that they thought the question referred to ability, whereas others' explanations indicated understanding the question as referring to achievement. In the former cases, participants would explain that girls and boys have always been equally capable of doing mathematics but that societal factors may have held girls back (e.g. sexist teachers, stereotypes). For example, one participant posited that:

> It's not a problem about whether or not boys are better at math than girls; it's a problem of whether or not boys are *encouraged* to be better at math than girls. So, I hesitate to say that, yeah, it has changed. I think what has changed is the perception. Definitely, it's okay now for girls to be good at math and sciences, whereas perhaps 30 years ago, back when I was in high school, it wasn't necessarily perceived that way. If you were a girl and good in math or sciences, you were some kind of grind[2] and you weren't going to get yourself a husband. [Laughter] (P16, Upton, man, 40-59, emphasis in original)

In the case of achievement differences, participants stated that boys used to do better at mathematics but that girls now do equally as well (or, in some cases, better), since they have more opportunities. Participants' discussions often related to the greater proportion of women enrolled in higher education now, compared to in the past: "Girls are encouraged to take math more than they used to just because of the job situation and they're allowed to go on to university, so they get to take math more" (P37, Smithburg, woman, 40–59).

Chi-square analyses of the responses to this question revealed no statistically significant differences in the response distributions by gender ($\chi^2 = 1.384$, $p = 0.709$) or age group ($\chi^2 = 9.962$ $p = 0.126$). Thus, the responses for these subgroups were distributed similarly across the response categories, with large proportions of

[2] According to Urban Dictionary (www.urbandictionary.com), the term "grind" is slang typically used to refer to a group of lesbians.

respondents (approximately 40% in each case) selecting “agree” or “disagree”. As discussed, these mixed views are arguably more related to different interpretations of the question (ability vs. achievement) than differences in views.

Q8: Who do parents believe are better at mathematics, girls or boys?

While the participants’ views of parents’ beliefs about gender and mathematics were quite mixed, the most common response was to purport that parents believe that boys are better than girls at mathematics (30.9%). These participants argued that parents held these views because they believed the stereotypes about gender and mathematics. Nearly as many participants (27.9%) argued that parents held gender-neutral views of their children and mathematics. As with the previous question, the least common gendered view was that parents believed that girls were better at mathematics (21.1%). Again similar to the previous question, one-fifth (20.1%) of the participants reported being unsure or ambivalent about this question. These participants often explained that they either did not have children or that their children were adults.

Such personal “evidence” was often provided, like the following elaboration on an “unsure” response: “My family is all girls, so there was no comparison group” (P44, Upton, woman, 18–39). This comment implies that if there had been boys in the participant’s family, she would have been able to answer the question about parents’ views in general. This type of extrapolation, from a specific personal example (e.g. family, experience in school) to the entire population, was commonly seen for the explanations for the answers to many of the questions in this study, indicating flawed logic (i.e. anecdotal evidence presented as scientific evidence) on the part of many participants. Across all the questions in the survey, very few participants referred to having read/heard about the topic from a broader, more scientific source, such as a news broadcast or article. Paulos (1988) argues that such reasoning is indicative of innumeracy, as “innumerate people characteristically have a strong tendency to personalize – to be misled by their own experiences” (p. 6).

Chi-square analyses of the responses to this question revealed statistically significant differences in the response distributions by gender ($\chi^2 = 11.778$, $p = 0.008$) but not by age group ($\chi^2 = 1.330$, $p = 0.970$). In Table 10.6, the spread of the responses by the gender of the participants is shown.

As evidenced by the data in Table 10.6, approximately the same proportion of men felt that parents held gendered views in favour of girls or in favour of boys. This is a very different distribution from the responses from women, where nearly

Table 10.6 Response distribution by gender for Q8

Group	Girls	Boys	Same	Unsure or ambivalent
Men	30.1%	34.9%	18.1%	16.9%
Women	14.9%	28.1%	34.7%	22.3%

twice as many felt that parents held gendered views in boys' favour compared to girls' favour. Notably, for both women and men, more people selected a gendered view in boys' favour rather than girls' favour. Another interesting comparison occurs for the "same" responses: More than one-third of women participants thought that parents held gender-neutral views, whereas less than one-fifth of men participants reported this stance.

Q9: Who do teachers believe are better at mathematics, girls or boys?

In contrast to perceptions of parents, the most common perception of teachers was that they held gender-neutral views of their students and mathematics (33.8%). Participants explained that teachers would have more knowledge about this topic than the "average person", plus they would have exposure to many children doing mathematics, so would form a less biased view than parents (who, the participants argued, may base their opinions solely on their own children – which, as mentioned, was indeed the case for some of the participants in the study). As one participant (P46, Smithburg, man, 40–59) suggested, "Teachers don't pick a side… They're neutral… They're always looking to help somebody who's struggling". Perceptions of teachers holding gendered views were fairly equally distributed: 18.6% of participants reported boys, compared to 20.1% reporting girls. Explanations provided were similar to those discussed with regard to being better at mathematics in general, such as arguments about boys' "natural" abilities with mathematics and girls' preferences for language arts. A large proportion of the participants (27.5%) reported being unsure about teachers' feelings. These participants typically explained that they had no contact with teachers at the present time, either because they did not have school-aged children or because they did not know any teachers personally. Such responses are again indicative of participants extrapolating their personal experiences to make a general claim.

Chi-square analyses of the responses to this question revealed no statistically significant differences in the response distributions by gender ($\chi^2 = 1.427$, $p = 0.699$) or age group ($\chi^2 = 7.352$, $p = 0.290$). Hence, the most common perception, regardless of gender or age group, was to believe that teachers hold gender-neutral views of children's mathematics abilities. However, substantial proportions argued that teachers held gendered views, with responses fairly evenly distributed between "girls" and "boys" responses (approximately 20% for each category).

Q11: Is it more important for girls or boys to study mathematics?

Of all the questions regarding gender and mathematics, this question had the most consistency in the participants' responses: 94.6% of the participants argued that it was equally important for boys and girls to study mathematics, an encouraging finding. In fact, many participants were incredulous that the survey would even include such a question. Only 2.5% of participants reported a gendered stance (0.5% for girls; 2.0% for boys). Additionally, only 3.0% reported being unsure or ambivalent toward this question. The overwhelmingly most common explanation provided was that everyone needs to know mathematics – for school, everyday life and future occupations. Some participants also discussed how mathematics was helpful to thinking in a more general sense: "Studying and learning math helps develop thinking and reasoning processes that contribute to the overall ability to make decisions" (P52, Townsville, woman, 20–39).

Chi-square analyses of the responses to this question revealed no statistically significant differences in the response distributions by gender ($\chi^2 = 2.311$, $p = 0.510$) or age group ($\chi^2 = 5.741$, $p = 0.453$). Thus, the vast majority of respondents argued that it was equally important for girls and boys to study mathematics, a heartening finding.

Conclusions

The findings from my analysis of data from over 200 participants from the Canadian province of Ontario suggest that gendered views of mathematics (and of others' views of mathematics) tend to be the norm. Although "no difference" was typically the modal category for the questions examined, the combination of "girls" and "boys" categories (i.e. the gendered responses) was almost always a higher proportion. The only question for which the majority of participants reported holding a gender-neutral view (rather than a gendered view) addressed the importance of studying mathematics, for girls and boys. For the questions regarding superiority in mathematics, more participants held a gendered view (i.e. selecting either boys or girls as their response) than a gender-neutral view. In most cases involving gendered views, more participants selected boys than girls, indicating a more favourable view of boys and mathematics. This finding suggests that gender stereotypes regarding mathematics persist, even in a very gender-equitable society like Ontario, wherein equity is inscribed in the mathematics curriculum.

Similar outcomes occurred in a study in the United Kingdom, where Lim and Ernest (1999) found that 20% of their participants subscribed to the stereotype that mathematics is a male domain. Additionally, in the responses to this questionnaire by Canadian Facebook participants (35 participants, of whom 62.9% were women, which is a similar gender distribution to my sample) in this international gender and mathematics study (Forgasz et al., 2014), response patterns emerged that were similar to those found with my street-level sample. The modal response for the Facebook participants was to report a gender-neutral view of girls' and boys' mathematics abilities, but of the gendered responses, more participants selected "boys". For the questions about parents and teachers, the Facebook responses had the same modal categories ("boys" and "same", respectively) as the street-level sample, although the Facebook sample had a far lower proportion of the participants who reported holding gender-neutral perceptions of parents' views (12.5%, compared to 27.9% in the street-level sample) and who selected "girls" as the response for their perception of teachers' views (8.3%, compared to 20.1% in the street-level sample). Encouragingly, the vast majority of participants in both the Facebook (91.7%) and street-level (94.6%) Canadian samples purported that it was equally important for boys and girls to study mathematics.

When considering the findings by the gender of the participants, statistically significant differences in response distribution only occurred for one of the gender-focused questions, regarding the participants' perceptions of parents' views. In this instance, women participants tended to hold gender-neutral perceptions of parents' views, while men participants tended to think that parents held gendered views (approximately balanced between "boys" and "girls" responses). However, when considering gendered views, twice as many women claimed that parents favoured boys compared to those who thought that parents favoured girls when considering mathematics ability. While it is encouraging that, for the other gender-related questions (general views of ability, views of changes over time, perceptions of teachers' views and the importance of studying mathematics), men and women responded in similar ways, the general response patterns were still troubling, as discussed above. Slightly different patterns were found with Lim and Ernest's (1999) participants, where the men tended to believe that men are better at mathematics while women tended to hold a gender-neutral view of women's and men's abilities. However, it is important to note that these participants were discussing views of adults, while my participants were discussing views of children, which may have altered the response patterns.

Notably, there were no statistically significant differences in response distribution by age group for any of the gender-related questions examined. This was a somewhat surprising finding, given my assumption (and hope) that subsequent generations would become progressively more gender-neutral in their views. However, this assumption/hope is not necessarily supported by research: General research about gendered views provides conflicting results, with some studies supporting my assumption, while others challenge it (Leder & Forgasz, 2011). Varied patterns of age-related views have also been found in studies about gender and mathematics. For instance, in Lim and Ernest's (1999) research, the youngest (17–20 years old) and oldest (50 and older) participants tended to hold less gender-egalitarian views than those in the middle age groups. In an early report (Leder & Forgasz, 2010) on the Australian portion of the large international research project of which my research was a part (i.e. the street-level data), none of the seven questions examined (two questions about the participant's experiences with school mathematics and five questions about gender and mathematics) had any statistically significant differences in response distribution by age group. However, with a larger Australian sample (689, compared to 203 in the early report), including respondents from Facebook, age-related differences in response patterns were found to be present for several questions. Leder and Forgasz (2011) compared "younger" (under 40) and "older" (40 and over) participants' responses, which is a different strategy than I employed when making age-based comparisons.

Nonetheless, of the same questions examined, age-related differences were found with regard to the response distributions for the general question about gendered abilities, the question regarding perceptions of teachers' views and the question regarding perceptions of parents' views but not for the question regarding the importance of studying mathematics. Leder and Forgasz (2011) found that, while the younger participants believed that "parents and teachers were more likely to be more egalitarian, they themselves hold more strongly than those in the older group to the traditional gender-stereotyped view that boys are more suited to and more successful in mathematics than girls" (p. 453), a troubling finding.

The findings from this Canadian research project, while somewhat encouraging, should also raise concerns for those involved in mathematics education. Since the majority of the adults surveyed tended to hold gendered views (with more of these gendered views favourable toward boys than girls), these messages are ostensibly being disseminated to young people, particularly by their parents. Additionally, even in a volunteer-based study like this, which arguably leads to positively skewed results, one-third of the respondents – particularly women – reported disliking mathematics and not doing well in mathematics as students. It is very likely that such negative views and experiences may impact the mathematical interactions that these adults have with children in their lives. In another research project (Hall, 2013), I found that children's views of mathematics are indeed impacted by their parents' views of the subject matter, a finding that is supported by prior research (e.g. Jacobs & Bleeker, 2004; Tiedemann, 2000). Thus, targeting parents' understandings of gender and mathematics, by both the educational system and the media (in which mathematics education researchers can play a key role, in both cases), in both cases, should be a focus. More generally, parents, teachers and – particularly in today's technology-focused world – media all play key socializing roles in children's lives (Arnon, Shamai, & Ilatov, 2008; Roberts & Foehr, 2004), so it is important that mathematics educators strive to target the messages that are being disseminated to children about gender and mathematics so that all children are exposed to positive, gender-neutral messages that encourage positive relationships with mathematics and participation at non-mandatory levels of study.

Looking Ahead

As elaborated earlier, I do not view gender as a binary construct. However, I adhered to the lead researchers' decision to offer binaried options for response and coding categories, for pragmatic reasons, and to reflect a society that continues to be very binaried, particularly in school settings. During data collection, only one participant (representing less than 0.5% of the sample) challenged the "girls or boys" wording of the questions. This participant (P16, Upton, man, 40–59) argued that "I think it's a more nuanced – Your survey seems to be binary, and I think it's a lot more nuanced than that". He suggested that the survey's wording should be changed to better reflect this complexity. During the interview, I agreed with this participant that "it" (meaning gender) is indeed not a binary and that it is more nuanced than the "girls or boys" options suggest. However, at the time, I was not sure how a survey of the general public on such a topic could be constructed without including binary options as prompts.

Since completing this project, I have spent time pondering what a non-binary "gender and mathematics" survey written for the general public might look like. Guidance is provided by GLTBQIA organizations for wording "gender" questions (i.e. questions about the participant's gender identity) on surveys (e.g. Kellerman 2016; Miller & Weingarten, 2005), but, to my knowledge, no suggestions exist – particularly within mathematics education – regarding writing survey questions *about* gender. After discussing this topic with a colleague, we have come up with ideas for a parallel survey to the one used in the research reported here. In this proposed survey, all of the questions will be reworded to allow participants complete freedom (without any priming) in their responses regarding gender and mathematics. None of the questions will be worded in such a way that binary categories (i.e. "girls" and "boys") will be mentioned. For instance, rather than asking "Who are better at mathematics, girls or boys?", we would ask "Do you think that there are any gender differences in mathematics ability? Please explain". Using the latter wording, participants would have the freedom to address the same topic in a less-guided manner. Certainly, rewording some of the questions is challenging, but we think, with some creativity, parallels can be found for all the survey's existing questions.

We believe that conducting such a survey has great potential for shifting the manner in which "gender issues" research in mathematics education is conducted, moving away from the rarely questioned binaries and other issues that are seen in most research of this type. Indeed, analyses of general education publications (Glasser & Smith, 2008) and mathematics education publications (Damarin & Erchick, 2010) highlight common issues: a lack of operational definitions for "sex" and "gender" provided by researchers and the problematic use of these (and related) terms interchangeably. As I have discussed elsewhere (Hall, 2014), I have become particularly cognizant of these issues and strive to write in a manner that aligns with my views about gender (e.g. avoiding "sex" language). Thus, in this reconceptualised survey, we will particularly focus on the language choices provided by the participants in the absence of any examples of "gender" language. Due to the revised wording of the questions, we anticipate that much richer data will be collected, as not only will gendered views be shared, similar to those attained in the current version of the survey, but we will also gain insight into the general public's use of gendered language when discussing mathematics. Our hope is that this proposed research project will instigate a shift and provide a challenge to the manner in which much "gender issues" research is conducted in mathematics education. While the way forward is uncharted territory, it is also an exciting prospect for a field with a long history.

Acknowledgements I gratefully acknowledge the opportunity provided to me by Helen Forgasz and Gilah Leder in being part of this international research project. I would also like to thank Heather Rogers, Stéphanie Chiasson and Michelle Kania for their assistance with data collection.

This chapter builds on a paper presented at the 38th Conference of the International Group for the Psychology of Mathematics Education (PME 38) and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education (PME-NA 36).

References

Arnon, S., Shamai, S., & Ilatov, Z. (2008). Socialization agents and activities of young adolescents. *Adolescence, 43*(170), 373–397.

Bench, S. W., Lench, H. C., Liew, J., Miner, K., & Flores, S. A. (2015). Gender gaps in overestimation of math performance. *Sex Roles, 72*(11–12), 536–546. doi:10.1007/s11199-015-0486-9.

Butler, J. (1990). *Gender trouble: Feminism and the subversion of identity*. New York, NY: Routledge.

Butler, J. (1999). Subjects of sex/gender/desire. In S. During (Ed.), *The cultural studies reader* (2nd ed., pp. 340–358). London, England: Routledge.

Dahlbom, L., Jakobsson, A., Jakobsson, N., & Kotsadam, A. (2011). Gender and overconfidence: Are girls really overconfident? *Applied Economics Letters, 18*(4), 325–327. doi:10.1080/13504851003670668.

Damarin, S. K., & Erchick, D. B. (2010). Toward clarifying the meanings of gender in mathematics education research. *Journal for Research in Mathematics Education, 41*(4), 310–323. Retrieved from http://www.nctm.org/publications/journal-for-research-in-mathematics-education/

Fausto-Sterling, A. (2000). *Sexing the body: Gender politics and the construction of sexuality*. New York, NY: Basic Books.

Forgasz, H., Leder, G., & Gómez-Chacón, I. M. (2012). Young pedestrians' gendering of mathematics: Australia and Spain. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons: Proceedings of the 35th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 298–305). Singapore: MERGA.

Forgasz, H., Leder, G., & Tan, H. (2014). Public views on the gendering of mathematics and related careers: International comparisons. *Educational Studies in Mathematics, 87*(3), 369–388. doi:10.1007/s10649-014-9550-6.

Forgasz, H. J., & Leder, G. C. (2011). Mathematics, computers in mathematics, and gender: Public perceptions in context. *PNA, 6*(1), 29–39. Retrieved from http://www.pna.es/index2Eng.html

Glasser, H. M., & Smith, J. P., III. (2008). On the vague meaning of "gender" in education research: The problem, its sources, and recommendations for practice. *Educational Researcher, 37*(6), 343–350. doi:10.3102/0013189x08323718.

Hall, J. (2012). Gender issues in mathematics: An Ontario perspective. *Journal of Teaching and Learning, 8*(1), 59–72. Retrieved from http://ojs.uwindsor.ca/ojs/leddy/index.php/JTL

Hall, J. (2013). *Societal views of and experiences with mathematics and mathematicians and their influence on elementary students* (Unpublished doctoral dissertation). University of Ottawa, Ottawa, Canada.

Hall, J. (2014). Unpacking "gender issues" research. *Philosophy of Mathematics Education Journal, 28,* Article 5. Available at http://people.exeter.ac.uk/PErnest/pome28/index.html

Jacobs, J. E., & Bleeker, M. M. (2004). Girls' and boys' developing interests in math and science: Do parents matter? *New Directions for Child and Adolescent Development, 2004*(106), 5–21. doi:10.1002/cd.113.

Kellerman, S. (2016). *How you can make the gender question on an application more inclusive.* Retrieved from http://itspronouncedmetrosexual.com/2012/06/how-can-i-make-the-gender-question-on-an-application-form-more-inclusive/

Leder, G. C., & Forgasz, H. J. (2010). I liked it till Pythagoras: The public's views of mathematics. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 328–335). Fremantle, Australia: MERGA.

Leder, G. C., & Forgasz, H. J. (2011). The public's views on gender and the learning of mathematics: Does age matter? In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and new practices: Proceedings of the AAMT-MERGA Conference* (pp. 446–454). Adelaide, Australia: MERGA.

Lim, C. S. (1999). Using metaphor analysis to explore adults' images of mathematics. *Philosophy of Mathematics Education Journal, 12*, Article 9. Retrieved from http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome12/default.htm

Lim, C. S., & Ernest, P. (1999). Public images of mathematics. *Philosophy of Mathematics Education Journal, 11*, Article 6. Retrieved from http://socialsciences.exeter.ac.uk/education/research/centres/stem/publications/pmej/pome11/art6.htm

Lim, C. S., & Ernest, P. (2000). A survey of public images of mathematics. *Research in Mathematics Education, 2*(1), 193–206. doi:10.1080/14794800008520076.

Lucas, D. M., & Fugitt, J. (2009). The perceptions of math and math education in Midville, Illinois. *The Rural Educator, 31*(1), 38–54. Retrieved from http://www.nrea.net/index.cfm?pID=7925

Lupart, J. L., Cannon, E., & Telfer, J.-A. (2004). Gender differences in adolescent academic achievement, interests, values, and life-role expectations. *High Ability Studies, 15*(1), 25–42. doi:10.1080/1359813042000225320.

Mendick, H., Epstein, D., & Moreau, M.-P. (2007). *Mathematical images and identities: Education, entertainment, social justice*. Swindon, England: Economic and Social Research Council.

Miller, & Weingarten. (2005). *Creating GLBTQIA-inclusive forms: Suggestions for policy and implementation*. Retrieved from http://www.glhv.org.au/files/CreatingGLBTQIAForms.pdf

Morge, S. P. (2006). *College students' beliefs about mathematics, gender, and popular media* (Unpublished doctoral dissertation). University of Indiana, Bloomington, IN.

Muijs, D. (2004). *Doing quantitative research in education with SPSS*. London, England: SAGE.

Ontario Ministry of Education. (2005a). *The Ontario curriculum grades 1–8: Mathematics* (*Revised*). Toronto, Canada: Queen's Printer for Ontario.

Ontario Ministry of Education. (2005b). *The Ontario curriculum grades 9 and 10: Mathematics* (*Revised*). Toronto, Canada: Queen's Printer for Ontario.

Ontario Ministry of Education. (2007). *The Ontario curriculum grades 11 and 12: Mathematics* (*Revised*). Toronto, Canada: Queen's Printer for Ontario.

Ontario Ministry of Education. (2016). *What do you need to graduate?* Retrieved from http://www.edu.gov.on.ca/extra/eng/ppm/graduate.html

Paulos, J. A. (1988). *Innumeracy: Mathematical illiteracy and its consequences*. New York, NY: Vintage Books.

Roberts, D. F., & Foehr, U. G. (2004). *Kids and media in America*. Cambridge, England: Cambridge University Press.

Sadker, M., & Sadker, D. (1994). *Failing at fairness: How our schools cheat girls*. New York, NY: Touchstone.

Schaeffer, N. C., Dykema, J., & Maynard, D. W. (2010). Interviewers and interviewing. In P. V. Marsden & J. D. Wright (Eds.), *Handbook of survey research* (2nd ed., pp. 437–470). Bingley, England: Emerald Group.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. doi:10.2307/749205.

Statistics Canada. (2010a). *Table 477-0013 – University enrolments, by registration status, program level, Classification of Instructional Programs, Primary Grouping (CIP_PG) and sex, annual (number),* CANSIM (database). Retrieved from http://www5.statcan.gc.ca/cansim/a26?lang=eng&retrLang=eng&id=4770013&&pattern=&stByVal=1&p1=1&p2=-1&tabMode=dataTable&csid=

Statistics Canada. (2010b). *Table 477-0014 – University degrees, diplomas and certificates granted, by program level, Classification of Instructional Programs, Primary Grouping (CIP_PG) and sex, annual (number),* CANSIM (database). Retrieved from http://www5.statcan.gc.ca/cansim/a26?lang=eng&retrLang=eng&id=4770014&&pattern=&stByVal=1&p1=1&p2=-1&tabMode=dataTable&csid=

Statistics Canada. (2015a). *Table 051-0001 – Estimates of population, by age group and sex for July 1, Canada, provinces and territories, annual (persons unless otherwise noted),* CANSIM (database). Retrieved from http://www5.statcan.gc.ca/cansim/a26?lang=eng&retrLang=eng&id=0510001&&pattern=&stByVal=1&p1=1&p2=-1&tabMode=dataTable&csid=

Statistics Canada. (2015b). *Table 477-0029 – Postsecondary enrolments, by program type, credential type, Classification of Instructional Programs, Primary Grouping (CIP_PG), registration status and sex, annual (number),* CANSIM (database). Retrieved from http://www5.statcan.gc.ca/cansim/a26?lang=eng&retrLang=eng&id=4770029&&pattern=&stByVal=1&p1=1&p2=-1&tabMode=dataTable&csid=

Statistics Canada. (2015c). *Table 477-0030 - Postsecondary graduates, by program type, credential type, Classification of Instructional Programs, Primary Grouping (CIP_PG) and sex, annual (number),* CANSIM (database). Retrieved from http://www5.statcan.gc.ca/cansim/a26?lang=eng&retrLang=eng&id=4770030&&pattern=&stByVal=1&p1=1&p2=-1&tabMode=dataTable&csid=

The general public. (2016). *Cambridge dictionary*. Retrieved from http://dictionary.cambridge.org/dictionary/english/general-public

Thorne, B. (1993). *Gender play: Girls and boys in school*. New Brunswick, NJ: Rutgers University Press.

Tiedemann, J. (2000). Parents' gender stereotypes and teachers' beliefs as predictors of children's concept of their mathematical ability in elementary school. *Journal of Educational Psychology, 92*(1), 144–151. doi:10.1037/0022-0663.92.1.144.

Towers, J., Takeuchi, M., Hall, J., & Martin, L. C. (2015). Exploring the culture of school mathematics through students' images of mathematics. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 570–573). East Lansing, MI: Michigan State University.

Part V
Re-formulating, Re-presenting, and Re-defining Mathematical Knowledge and the Curricula

Chapter 11
Borrow, Trade, Regroup, or Unpack? Revealing How Instructional Metaphors Portray Base-Ten Number

Julie Nurnberger-Haag

Abstract This chapter uses embodied cognition to reveal unintended consequences for learning due to the processes that students enact with manipulatives. Base-ten block manipulatives and terms educators used for whole number arithmetic and place value are examples of ubiquitous "hands-on" instructional and assessment practices. Yet, the theoretical perspectives used to research this learning have not considered how students' actual physical movements represent intended ideas of arithmetic. The students whom educational researchers serve need us to better understand these practices in order to select and improve the design of such tools. Thus, this chapter examines how the language that educators use in combination with manipulatives influences students' understanding of addition and subtraction. This is the focus of the chapter for at least two reasons. First, it is crucial for elementary students to build procedural fluency and conceptual understanding of the base-ten number system. Second, these specific examples reveal the broader implications for any manipulative-based learning experiences for any topic across preK-16+ mathematics. Due to the physical motions students make during "hands-on" learning, it is critical to investigate these common practices through a lens of embodied mathematics learning. That is, research must attend to implications of how students move during instruction with "hands-on" materials as well as any metaphors educators orally express that imply motions even when students do not put their hands on materials.

Keywords Embodied cognition • Base-ten blocks • Metaphor • Physical movement • Dienes blocks • Digi-Blocks

J. Nurnberger-Haag (✉)
School of Teaching, Learning & Curriculum Studies, Kent State University,
800 E Summit St., Kent, OH 44240, USA
e-mail: jnurnber@kent.edu

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_11

The intentions of a tool are what it does. A hammer intends to strike, …a lever intends to lift. They are what it is made for. … Sometimes in doing what you intend, you also do what the knife intends, without knowing.
—— Philip Pullman

The words and materials that educators choose to use to teach arithmetic are instructional tools intended to foster learning. Research is needed to understand how student learning outcomes with such tools reflect educators' intentions as well as how students' experiences and learning reveal unintended consequences. As the quote above implies, the focus of this chapter is to show that instructional tools used for base-ten number concepts, while in some ways accomplish the intended goals, may actually cut like a knife, that is, interfere with intended learning in ways and for reasons that until now have been unexplored. The goal of this chapter is to spark recognition of issues with using such tools through the lens of embodied cognition.

To understand the intended and unintended results of instruction with base-ten materials, the chapter first considers the intended learning, that is, base-ten number structure and meanings of addition and subtraction. Then some common instructional tools that educators have used to accomplish these ends will be shared before discussing how empirical and theoretical perspectives of embodied cognition can be used to posit potential unintended consequences of the ways students experience base-ten number with such tools.

Base-Ten Number Operations and Structure

The structure of the Hindu-Arabic or base-ten number system requires conceptual structures that are difficult for elementary students to develop. Using the position of numbers to represent different units of quantity where 0 represents none of a given unit was a significant societal advancement (West, Griesbach, Taylor, & Taylor, 1982). When children first learn to write numerals, they are unaware of this positional system. They simply understand that if a person means the quantity orally said as "twenty-six," they know it should be written as 26 (Fuson, 1992). This is not much different from knowing that if their name is Sara, they write it as *S-a-r-a* before understanding phonics. Consequently, when students begin writing larger numbers, they often write one hundred twenty-six, for example, as 10026 (Labinowicz, 1985). This written symbol reflects a logical understanding that they composed the quantities 100 and 26, but does not reflect the positional nature of the established written nomenclature.

It is only those with mathematically developed perspectives who see the base-ten number structure in the numeral 126 or 342, for example. Although many adults in the United States consider addition and subtraction to be basic math, consider the complexity of the mathematics underlying the base-ten number system:

- The system uses ten digits (0 through 9).
- The position of the digit determines its value (3 in 342 is different than 3 in 234).

- The face value of a digit is multiplied by its place value to determine its complete value (e.g., in 342, the face value of 3 is 3, so its complete value is 3 times its place value of 100, or 300 is its complete value).
- The system is multiplicative and additive ($342 = 3 \times 100 + 4 \times 10 + 2$).
- Each place value is ten times greater or less than the next.
- Each place value is determined by a power of base ten (e.g., 10^4).

This system is built upon and can only be fully understood by grasping all of these complexities. Yet, elementary students are expected to develop toward this full understanding long before they have even been introduced to multiplication or exponents, let alone mastered such topics. Due to the exponential structure of the base-ten number system, instructional practices that help students experience this structure are essential.

To accomplish this, many educators and researchers have investigated processes to help students restructure their conceptions of numbers as singular objects to see the units of quantities as higher-level units (Verschaffel, Greer, & Corte 2007). The units that can be expressed as powers of ten such as tens units, hundreds units, thousands units are composite units (Steffe & Cobb, 1988). In other words, drawing on psychological ideas of categorization, these are higher-level units, in that they are superordinate units in relation to a basic level unit (Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976). In this case the basic level units are the ones. Different researchers have used a myriad of terms and labels for levels of base-ten thinking (Sarama & Clements, 2009). Consequently, I explain generally here the ways students think about numbers as they develop base-ten number understanding and emphasize the units students think about or see at each level that differs from the way adults may see these units. Students first think about quantities as values of single objects or ones units, although the students themselves at this point do not use a word like "ones," because this is a term that only becomes necessary as part of the larger base-ten place value system (Labinowicz, 1985). As adults who understand the place value system, however, it can be helpful to characterize students' thinking at this level as thinking only of the ones units. Through instruction, students begin to group quantities for efficiency and organization (e.g., counting 26 objects collectively by two as 2-4-6, etc.) but still think of the individual singles or ones units. Students can also learn to group objects into sets of ten, and additional ones such as two groups of 10 objects and 6 additional objects are 26 objects. Adults, however, often overestimate this ability to group, seeing it through their adult perspective as 2 tens units and 6 ones units. The students, however, need extensive time to develop that way of viewing hierarchical units to see the group itself as a unit the way adults may see as "a ten." What students first see in the same scenario are simply 10 objects, 10 objects, and 6 more objects. Even if students are able to parrot the language of "tens" at appropriate times, this does not mean they really think in terms of both the ten units and 10 ones units.

Students, who conceptually understand this positional place value system, can think flexibly about units to solve problems. Some examples of combinations of units for the number 342 can be thought about as:

(a) 342 ones
(b) 300 ones and 40 ones and 2 ones
(c) 3 hundreds units, 4 tens units, and 2 ones units (positional place value)
(d) 3 hundreds units, 3 tens unit, and 12 ones
(e) 34 tens and 2 ones

These ways portray just a few of the many ways these quantities can be composed and decomposed. Whereas both (a) and (b) use single units of thought as the item to be counted, (c), (d), and (e) all coordinate multiple levels of units. Formal positional place value is reflected in the (c) way of thinking about 342, yet example (d) shows how students should think of 342, if they need to subtract a number with more than 2 in the ones place using a traditional algorithm. The ability to think of such quantities structured as the ways shown in example (e) would mean they would not need to algorithmically divide 342 by 10 or use a memorized rule to move the decimal.

Base-Ten Manipulatives

To help students learn the culturally determined structure of the base-ten number system, many manipulative tools have been developed and are commonly used in schools. Some authors working within the tradition of radical constructivism suggest that the students should not be required to use manipulatives (Kamii, Lewis, & Kirkland 2001), whereas others suggest students can use such available tools as one of many student-determined ways to solve problems, which they consider consistent with constructivist approaches (Carpenter, Fennema, Franke, Levi, & Empson, 1999). In contrast, others claim such tools are crucial to learning (Fuson & Briars, 1990). These debates largely stem from and reflect differing theoretical perspectives of learning applied to issues of using manipulatives in general. In contrast, this chapter focuses on revealing the intended and unintended ways that specific manipulatives influence how students and teachers represent mathematical ideas.

The multiple materials used for teaching base-ten concepts can be categorized as *ungrouped* or *pre-grouped* and *proportional* or *nonproportional* models (Reys, Lindquist, Lambdin, & Smith, 2014; Van de Walle, Karp & Bay-Williams, 2010). Ungrouped or groupable models are individual objects (e.g., blocks, beans, sticks, and straws) that could be grouped in sets of ten but are not yet grouped and nothing inherent in the material structures that they be grouped this way (Fraivillig, 2017; Reys et al., 2014). Educators commonly use these ungrouped models during calendar math (Fraivillig, 2017). Even the phrasing of the standard 1.NBT.2a of the US Common Core State Standards for Mathematics implies proficiency with such ungrouped materials which are a learning goal by stating that “10 can be thought of as a bundle of ten ones — called a ‘ten’” (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010, para. 1). Here I claim that this is an example of an instructional metaphor for how students will learn (bundling ungrouped materials) conflated with the intended mathematics (understanding tens units as composed of or containing 10 ones), because the intended learning is not that students know the particular context of a bundle but place value units.

The term pre-grouped models refer to how these individual models were prepared for instruction. For example, someone has already grouped some of these materials into sets of 10 for students to use during instruction along with the ungrouped items (Reys et al., 2014). Pre-grouped items could also refer to blocks that manufacturers molded to represent base-ten structure, such as blocks that are commonly referred to as "base-ten blocks" (Reys et al., 2014). Thus, these base-ten blocks might be more specifically referred to as prestructured, rather than simply grouped. Since this chapter will later show that "group" is an instructional metaphor, in the rest of the chapter, such materials will be referred to as prestructured.

The representations most researchers, teachers, and even national educational assessments (e.g., Warfield & Meier 2007) mean when referring to "base-ten blocks" are the specific most common type, which are Dienes blocks. The blocks were named after Zoltan Dienes, the mathematician who created them to help students represent arithmetic in multiple bases, including base ten (Web Minder, 2014). These blocks consist of a single cube to represent ones units, a fused stick in which etched lines indicate 10 single units, a fused block of ten of these ten sticks, as well as a cube with etchings intended to represent one-thousand units. In other words, these blocks by design intend to provide physical representations of multiple units at once. A single hundreds block (1 hundreds unit) is typically scored to show 100 ones unit blocks, and this scoring is done in such a way to be equivalent to 10 tens unit blocks. Although it should be noted that only the 600 squares etched on each face of a thousand cube are visible, so students typically misunderstand the intent that this cube actually contains 1000 cubes, rather than 600 (Labinowicz, 1985).

Elon Kohlberg, another professor with a PhD in mathematics, developed a commercial base-ten block manipulative called Digi-Blocks after using rocks in containers to help his nephew understand the base-ten number system (Digi-Block Inc., 2017a). The ones unit of Digi-Blocks are the only solid blocks. Each larger place value is a container that is proportional to the original unit and can hold exactly ten of them (Digi-Block Inc., 2017b), such that all the larger place values are simply *containers* or *holders* until filled with the smaller place value blocks. This means that a collection of 10 units fits inside the ten container. Then, once students collect and fill 10 ten holders, they can pack them into the hundred container and then follow the same pattern for the thousand container. Such blocks or ten frames that students can fill provide feedback signaling students when to make a new group of ten (Fraivillig, 2017). When completely separated, this tool might be considered unstructured; however, the structure of the containers requires that the only grouping that can occur is in nested sets of ten, so in effect, this tool might be considered prestructured, which they are when they are full.

All of these materials discussed thus far are considered proportional models in that an adult or child knowledgeable about base-ten structure might see or build progressively higher place value units with smaller units contained within each higher unit (Reys et al., 2014). Regardless of the type of proportional manipulative used, even if students can name the block as instructed such as “one hundred,” this does not mean that the child understands or sees this block as representing a single unit of hundreds (Labinowicz, 1985). During an extended period of time, students simply see this hundred block as a convenient fusion of 100 individual blocks (Labinowicz, 1985). For as Labinowicz stated “we see what we understand” (1985, p.301).

The proportional Dienes block brand seems to be universally seen as equivalent to the generic term “base-ten block,” yet Digi-Blocks are also base-ten block manipulatives. Thus, for clarity in this chapter, the term *multiunit blocks* (*MUBs*) will refer to the class of proportional blocks that include Dienes blocks, Digi-Blocks, and any other similar materials that prestructure single units and higher-order composite units.

Examples of nonproportional models are colored counters (i.e., each color represents a different unit value), coins, or abacuses (Reys et al., 2014). Mathematically proficient students and adults need to work with nonproportional models that require trading values, because they need to understand, for example, that a single hundred-dollar bill could be traded for ten 10 dollar bills. Such nonproportional models of quantities, however, cannot model the idea of groupings of groupings, composite units, or containment. It is widely accepted that such nonproportional models, which are not a focus of this chapter, are more abstract and should only come after students gain a conception of quantity through proportional models (Reys et al., 2014).

“Hands-On” Learning with MUBs

To set the stage for the investigation of “hands-on” learning experiences with the most common MUB, consider what you see when looking at Fig. 11.1 and how this is influenced by what you understand about mathematics that a novice does not. Figure 11.1a shows a Dienes block representation of 1040 on a workspace for a problem (purposefully withheld from the reader at the moment). In Fig. 11.1b, pay attention to the student’s movement and what it models or represents about arithmetic. What is the students’ hand doing? Do you agree that the hand removes, takes out, or takes away one-thousand cube? What does this physical movement represent arithmetically?

Physical manipulatives are a common way to support students to learn intended or targeted ideas. Such materials have been acknowledged as metaphors, microworlds, or models of abstract mathematical ideas (Nesher, 1989; Pimm, 1981). Moreover, student use of such manipulative materials has been referred to for decades as “hands-on” learning. Paradoxically, research and practice have not attended to what students’ hands mathematically represent during “hands-on” learning such as the questions I raised in relation to Fig. 11.1. Thus far, research about what happens during learning experiences with such materials has focused on the visual arrangements of the blocks after students move blocks. In other words, these student movements have been treated as a necessary step to get the physical

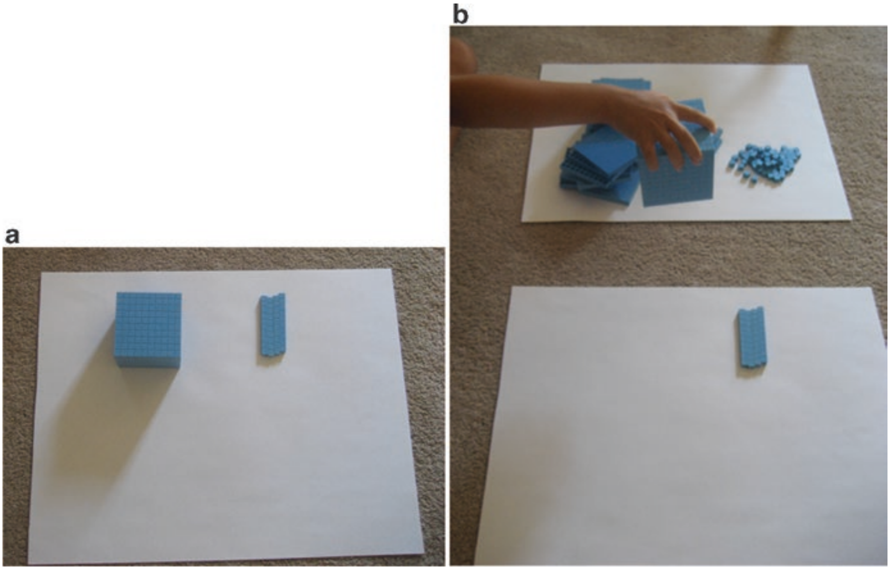


Fig. 11.1 Quantity 1040 modeled with Dienes blocks (**a**) before student's hand moves the one-thousand cube in (**b**)

arrangements to visually represent numerical quantities. However, this is a static perspective of students' entire experience. This implies students' experiences consist of a compilation of still-frame photos. Research must use a video approach (both literally and metaphorically) to view students' experiences in order to understand the actual process and potential causes of learning outcomes. To reveal what has gone unnoticed about the processes, I will focus on what happens between each resulting photo. That is, within a dynamic process of solving a problem with MUBs, I ignore the commonly portrayed resulting photos readers might expect to see in order to attend to students' physical motions that moved the blocks and what those motions (and seeing the blocks move) arithmetically represent. To provide justification as to why prior still-frame perspectives have limited the field's understanding of the learning process and why new perspectives are needed, research on how physical motions influence cognition and evoke metaphorical concepts will be discussed before examining cases of how students move particular types of MUBs.

Embodied Cognitive Perspectives

Embodied cognition encompasses a variety of research foci such as investigations of how existing ideas are grounded in prior physical experiences with the world, how real-time interactions with the world influence cognition, and how verbally expressed metaphors reveal embodied bases of cognition (Glenberg, 2010).

According to embodied cognitive perspectives, physical motions evoke concepts even if the language used does not communicate this idea (Antle, 2013; Goldin-Meadow, Cook, & Mitchell, 2009). Much research has been conducted with adults that supports the claim that the influence of human movements on thinking is more than a developmental stage of childhood. For example, Antle (2013) found that adults who watched images of humans with inequitable resources were more likely to not only notice the inequity (abstract imbalance of resources) but also express a desire to correct the imbalance, if while watching the images they had to work to keep their whole body physically balanced on a platform compared to those who rotated a joystick (Antle, 2013). In other words, the physical motions of balancing influenced people's concepts to see the same images with different meaning.

Another implication of the Antle (2013) study is evidence that the consistency between concepts and motions matters, in this case, consistency of the underlying concepts of balance and imbalance in an abstract metaphorical sense with people's physical movements. In another study the physical motion of adults moving objects from one bowl to another that was away from them or toward them evoked the underlying idea of away and toward, which affected their comprehension of literal and metaphorical written sentences (Glenberg & Kaschak, 2002).

All of these examples provide evidence that how humans move influences how they think. Moreover, these studies also evince the importance of consistency of physical motions with the intended ideas or concepts. Additional examples have emerged that show consistency of physical motions matter for third- to sixth-grade students learning mathematics (Goldin-Meadow et al., 2009; Nurnberger-Haag, 2015). That is, that students' physical motions serve a metaphorical purpose, and sometimes their motions led them to verbally express this metaphor. Specifically, in Goldin-Meadow and colleagues when the students were taught to put two fingers of one hand on the two addends of an equation that should be simplified and point one finger of the other hand to the relevant number on the other side of the equation, this evoked for the students the concept of putting together two addends (Goldin-Meadow et al., 2009). Moreover, those who were most successful articulated this metaphor verbally as "grouping." The students who were in another condition who made the same motion with irrelevant numbers did not do as well and did not verbally express this grouping idea even though the motion was the same (Goldin-Meadow et al., 2009). Thus, the relationship of the movements with mathematical objects, such as written symbols, also matters.

Such research suggests that the areas of arithmetic in which educators already use "hands-on" instructional metaphors, such as MUBs, warrant research with embodied perspectives of cognition. Analyses of how students move MUBs are needed in order to understand how their motions may be influencing their thinking and consequently their learning. The idea of grouping is a necessary but insufficient aspect of understanding base-ten arithmetic. Consequently, results about grouping numerical symbols in prior studies (e.g., Goldin-Meadow et al., 2009) suggest that analyses of the ways that students group manipulative materials would provide critical insights about student understanding of base-ten arithmetic.

First, let us consider the common ways such materials have been viewed and then show what has been previously overlooked. For example, the action of substituting blocks with the equivalent value (e.g., 1 ten block for 10 ones) is often referred to as “a trade,” and the blocks that are fairly traded are usually circled within the static problem diagrams (e.g., Fuson & Briars, 1990; Labinowicz, 1985). Where these blocks that were traded came from and the four separate motions required for students to perform each trade have not been attended to in theory or in the drawn representations of the block arrangements shown in researcher nor teacher publications. Such still-frame perspectives limit what can be noticed about students physically moving tools to represent arithmetic. Just as it is already commonly understood in mathematics education that research on social interactions in classrooms must be captured with video cameras rather than photographs, investigations of “hands-on” learning experiences must also use these same methods. Given these perspectives on how movement influences cognition, what follows is the analysis of students' movements of two prestructured proportional MUB blocks (i.e., Dienes blocks and Digi-Blocks). This will be followed by a brief discussion of embodied perspectives on metaphors in relation to the instructional metaphors educators (including educational researchers) orally express.

Analysis of MUBs from Embodied Perspectives

In a different publication, I used the term *model-movements* to refer to the ways that students and educators typically move their bodies or physical materials due to the affordances and constraints of those models (Nurnberger-Haag, 2015), so this term will be used to describe movements with MUBs. For educators familiar with elementary mathematics, Dienes blocks as classroom materials may be as commonly accepted as any other tools such as chairs whose purpose and function no longer require effortful attention. Thus, in order to see something so common from a new perspective, it is often necessary to hide aspects of a context that reinforce existing understandings. Consequently, to focus on how the physical and visual experiences represent ideas, let's imagine for a moment that classroom instruction with these blocks occurred without oral language or sign language. What do students' physical model-movements represent about arithmetic? Due to their model-movements, what might we hypothesize students would verbally express if teachers did not insert their own language to this process?

MUBs that Require Trading If proportional models like Dienes blocks or nonproportional models are used, students must physically trade one place value unit for a different unit in order to calculate with these materials. Next, I trouble what these trading requirements could conceptually mean or mathematically represent in order to reveal potential reasons that such materials may fail to support student learning of conceptual structures in intended ways.

Trading Model-Movement Model-Unintended Operations Trading and equivalence are key themes in mathematics, particularly for solving equations; however, in this context of multi-digit calculation and place value, trading is an unnecessary metaphor. Trading is actually composed of physical giving and taking movements that a student needs to consider together as an abstracted fair trade. Based on research about how physical movements subconsciously influence human thinking (Glenberg & Kaschak, 2002), even if students agree that they completed a fair trade, the students' physical model-movements of putting in (add) quantities and then taking out (subtract) quantities likely activate ideas of addition and subtraction at unintended times that may interfere with learning.

Revisit Fig. 11.1, which shows a student taking a thousand block away from a representation of the quantity 1040. This model-movement could be representing the subtraction problem 1040 − 1000, because taking something away is one way to model subtraction. An educator might also recognize such movements as the first step of processes with Dienes blocks to trade 1 thousand for 10 hundreds; however, the physical model-movement to perform this first step of a trade is the same as students' movement to subtract 1000. Regardless of how an educator might intend that students see or think about the action as part of a trade, students' actual motions model taking away or removing. In other words, students' physical model-movements with Dienes blocks model subtraction operations even when unintended. In Fig. 11.1b, notice also the collection of extra blocks, which is where the student is moving the thousand cube to. This student was actually demonstrating a first step of the problem 1040–463.

Table 11.1 explains student model-movements to calculate 1040 − 463 with Dienes blocks using the trade-first left-to-right subtraction algorithm Fuson and Briars (1990) used and subsequently found in elementary textbooks (e.g., The University of Chicago School Mathematics Project, 2012). Although in practice I encourage students to use methods that make sense to them, for space and illustrative purposes, this chapter explains the problem using the particular algorithm Fuson and Briars (1990) indicated students find more beneficial than a traditional algorithm. The second column of Table 11.1 shows each action and quantity using numerical expressions to illustrate the unintended arithmetic similar to the method Vig, Murray, and Star (2014) used to illustrate how a chip model represents addition and subtraction of negative numbers.

In order to trade 10, students may not be able to instantly grab 10 and only 10 of a certain sized unit block. This means students may have more than four separate movements in order to prepare objects for trading (count out and gather each set of 10). To focus attention on how all students would move to trade, the table focuses on the four main trading actions for the sake of argument.

Note the processes needed to enact a single trading metaphor require at least four separate movements (see Table 11.1). Each of these movements is indistinguishable from how students move to represent intended operations. In the rightmost column of Table 11.1, notice that more of the student's movements represent unintended operations of both addition and subtraction in unintended situations than intended subtractions. I hypothesize that such unintended operations could interfere with elementary student learning of whole number operations as prior research had found for older students learning integer operations. For example, this interference was found with fifth- and sixth-grade students using color-coded counters to represent positive and negative numbers (Nurnberger-Haag, 2015). Students who experienced

Table 11.1 Descriptions of Dienes block model-movements to reveal unintended and intended operations, using example of 1040-463

Verbal description of movements	Numerical representation of model-movements in each space	Operation model-movement meaning	Pedagogical intent	Model-movement operation match
Take away 1 thousand cube	1040 – 1000 = 40	Subtract	First quarter of trading action	Unintended
Put thousand cube with blocks external to problem	E + 1000	Add	Second quarter of trading action	Unintended
Take 10 hundred blocks away from external blocks	(E + 1000) – 1000	Subtract	Third quarter of trading action	Unintended
Put 10 hundred blocks with problem blocks	40 + 1000 = 1040	Add	Fourth quarter of trading action	Unintended
Take 1 hundred block away from problem blocks	1040 – 100 = 940	Subtract	First quarter of trading action	Unintended
Put 1 hundred block with external blocks	E + 100	Add	Second quarter of trading action	Unintended
Take 10 ten sticks away from external blocks	(E + 100) – 100	Subtract	Third quarter of trading action	Unintended
Put 10 ten sticks with problem blocks	940 + 100 = 1040	Add	Fourth quarter of trading action	Unintended
Take away 1 ten stick from problem blocks	1040 – 10 = 1030	Subtract	First quarter of trading action	Unintended
Put 1 ten stick with external blocks	E + 10	Add	Second quarter of trading action	Unintended
Take away 10 single blocks from the external blocks	(E + 10) – 10	Subtract	Third quarter of trading action	Unintended
Put 10 single blocks with problem blocks	1030 + 10 = 1040	Add	Fourth quarter of trading action	Unintended
Take away 4 hundred blocks	1040 – 400 = 640	Subtract	Subtract	Intended

(continued)

Table 11.1 (continued)

Verbal description of movements	Numerical representation of model-movements in each space	Operation model-movement meaning	Pedagogical intent	Model-movement operation match
Take away 6 ten sticks	640 – 60 = 580	Subtract	Subtract	Intended
Take away 3 single blocks	580 – 3 = 577	Subtract	Subtract	Intended

Note: E = the unknown value of blocks represented in the extraneous or external trading zone

integer instruction with counters compared to a number line model did worse on problems for which they had to put in extra counters (addition model-movement) in order to subtract as the problem required, compared to those problems that did not require these unintended addition operations (Nurnberger-Haag, 2015). Moreover, there are at least two other related problems this trading constraint of the materials creates that have potentially unintended consequences: opening a closed system and failing to model base-ten ideas of containment.

Trading Violates Closed System If students solve a sum of 14 and 28, for example, there are many ways students could conceptually use ones units or a combination of ones and tens units as promoted with Number Talks (Parrish 2011). All of these ways of thinking allow students to think about combining the quantities 28 and 14 within a closed system of those quantities. The physical limitations of Dienes blocks or any other materials that require an exchange of ten of one thing for another pose another potential issue that may impact students' conceptual structures. These blocks require students to treat a given arithmetic task as an open system, which is inconsistent with base-ten ideas. When students use Dienes blocks, they must introduce additional blocks from outside the system of the given problem, in other words open the system to include extra blocks that do not directly model the problem. These extra blocks are irrelevant to the arithmetic problem at hand but necessitated by the particular instructional metaphors. In other words, the trading model-movements students must enact open the system to include this trading zone of extra blocks. In this way, it creates an "otherness" that is unnecessary and potentially confusing (similar to Table 11.1 columns 1 and 2). That is, students need to leave the block representation of the problem at hand to go to this other source of blocks that becomes conflated with the blocks intended to represent the problem. For example, to use Dienes blocks to calculate 28 plus 14, after collecting 2 tens, 8 ones, 1 ten, and 4 ones or 42 total ones, students temporarily reduce the quantity modeled in the problem space from 42 to 32. Students remove ten of these ones from the quantity being considered and go to a trading pool of blocks external to the quantity to get this "other" ten to exchange. This means a student works with a total unintended quantity of 52 ones during the course of solving the intended problem (42 from the original system and 10 additional from the external stash of blocks). Research on consistency of movements with cognition (Glenberg & Kaschak, 2002; Goldin-Meadow et al., 2009) would suggest that it may be counterproductive for students to imply that one has to externally trade some other values with the values in the problem to which we want students to attend. Do students mathematically categorize in their mind and distinguish between the blocks meant to represent the problem and the extra blocks that serve only as a repository to make trading blocks possible? From psychological perspectives, is the additional cognitive load useful or a source of interference?

In contrast, the Digi-Blocks support students to combine 28 blocks (which they could represent as two containers of 10 blocks and 8 additional blocks) with one container of 10 blocks and 4 additional blocks. The 42 total blocks remain together as part of a closed system. The only external objects students bring to the problem system are containers to organize or structure the single quantities into units of ten. Thus, these containers serve an organizing function, not a block in of itself. The higher-level unit of tens does not exist without the basic level unit; adding a container is not the same as adding the thing it holds. This differs from changing the number of blocks in the problem system in the ways Dienes blocks require. Such comparisons of affordances and constraints of these various materials warrant investigation for the potential intended learning and, with respect to Dienes blocks, unintended interferences of learning.

Trading Fails to Connote Containment At least one other unintended consequence of proportional materials such as Dienes blocks that require opening what should be a closed system is that they fail to model the successive containment of units of the base-ten number system. It is crucial that students develop understanding of the same quantity in terms of different sized units (Steffe & Cobb, 1988). In regard to linear or other forms of measurement, it is more productive for a person to understand that 1 kilometer contains 1000 meters than needing to think that 1 kilometer must be traded for 1000 meters. This idea is equally important for thinking about base-ten number units. The learning objective is not for students to think that 1 one-thousand unit must be traded for 1000 ones or 10 hundreds units rather that each unit contains those values. Some MUBs can model this containment idea such as Digi-Blocks, although modern Dienes blocks do not. So if a sum of ones were 42 units, for example, mathematically knowledgeable people can see this same quantity of 42 using different ways of categorizing or decomposing the units: 42 individual ones, containing 4 tens and 2 ones, 3 tens and 12 ones, and many other ways.

Theoretically, metaphors that support ideas of containment should better support student learning, because ideas of containment reflect the intended mathematical ideas in ways that also build on innate cognitive mechanisms. Research on how people learn and think about categories has identified base level and superordinate as well as subordinate categories (Rosch et al., 1976). Mathematically, place value units are categories with a base level (ones units), superordinate levels (tens, hundreds, etc.), and subordinate levels (tenths, hundreds, etc.). Research supports the claim that humans think of categories metaphorically as though they are containers (Boot & Pecher, 2011; Johnson, 1987). Consequently, it is important to consider how metaphors could influence learning of a category-based topic such as base-ten arithmetic structure. This should lead us to test how materials that encourage students to move in ways that put in and remove objects from containers or physically build structures of contained or nested quantities might afford building conceptual understandings of category units. Thus far I identified the ways students move and see physical materials used for base-ten number that do and do not support these ideas as well as metaphors educators orally express. These theoretical analyses warrant studies that use a range of methods from psychology-based experiments to investigations of classroom-based instruction.

Deconstructing Orally Expressed Instructional Metaphors

In the previous section, I asked the reader to suspend knowledge of the reality of classrooms to ignore student and educator use of language in order to focus on what students' movements would physically represent (i.e., metaphors they might physically enact with various tools). Now consider the real classrooms in which teachers and textbooks use language to explain what they intend students perceive. Consider whether and how the metaphorical meaning students may experience by physically moving those tools relate to the following discussion of the terms textbooks and educators use orally and aurally. Educators (including educational researchers) have recognized and discussed the use of analogy and metaphor to teach content areas including mathematics (English, 2013; Pimm, 1981). Yet, to my knowledge, the particular terms for base-ten arithmetic have not been discussed as metaphors in prior work, so I analyze them here in terms of their intended and unintended mappings to addition and subtraction operations. Several terms have been used such as *carry*, *borrow*, *trade*, *group*, *ungroup*, *regroup, pack*, and *unpack* (Digi-Block Inc., 2017c; My Math, 2013; SRA Concepts, 2013; The University of Chicago School Mathematics Project, 2012). Elsewhere detailed mappings will illustrate how each of these terms maps from source to target in intended and unintended ways. Due to space and for clarity of the general framing of this chapter on instructional metaphors, the following focuses on revealing the primary issues with such metaphors.

Carry and Borrow When people use the terms *carry* and *borrow* in the context of addition and subtraction calculations, unlike the rest of the terms analyzed here, people may not think of the typical meanings of the terms *carry* or *borrow*. That is, due to the specific mathematical context in which adults previously practiced the terms, when adding, they may associate *carry* as meaning literally to inscribe a 1 or 2 as needed to the left of a place value or, when subtracting, *borrow* as meaning to cross out, reduce, and place a 1 next to the ones digit. If, when using these terms in this context, people only associate it with that particular meaning instead of other or original metaphorical meanings of the terms *carry* and *borrow*, then these would be considered "dead metaphors" (Lakoff & Johnson, 2003/1980). In other words, what began as metaphors to facilitate understanding between adults of a known idea to an unfamiliar idea adults now think of as having a literal meaning (Lakoff & Johnson, 2003/1980). Due to these extensive experiences, even if adults conceive of these terms as names for literal algorithmic procedures, whether students novice to base-ten arithmetic expect these *carry* and *borrow* terms to help them learn arithmetic using the meanings they already understand warrants research. Consequently, this theoretical analysis deconstructs the meaning of *carry* and *borrow* to trouble these instructional metaphors.

Carry means "to transfer from one place (as a column) to another" (Carry, 2017). The meaning of carry implies that the position of the same item is simply transferred or moved. Yet, *carry* fails to connote the intended mathematical idea that students should conceive of a carried value as a different unit. In the case of adding, for example, a written notation of a 1 may be procedurally transferred; however, once moved, it becomes ten times the value.

Moreover, the term *borrow* fails to reflect conceptual meaning and procedures. Borrow means "to receive with the implied or expressed intention of returning the same or an equivalent" (Borrow, 2017). Thus, this term is a misnomer because when teachers or other adults say "borrow from the tens place," for example, there is never an intention of returning the equivalent value of ones back to tens. The term or phrase *gifting* or *taking* might more accurately reflect this written mathematical procedure. Yet none of these terms reflect the intended mathematical ideas or procedures of converting a large composite unit into ten times the next smaller unit.

Furthermore, the pair of terms *carry* and *borrow* are meant to represent processes for addition and subtraction, respectively. Given that addition and subtraction are inverse operations, an effective instructional metaphorical pair would likely communicate the inverse relationship. However, using the definitions above, it is clear that the term to *borrow* is not the inverse of to *carry*.

Trade Given that the terms *carry* and *borrow* were in use long before the term *trade* became part of school mathematics, were it not because of the popularity of Dienes blocks and some research using these terms because of these blocks (e.g., Fuson & Briars, 1990), then we should have seen the term arise much earlier. For as Labinowicz (1985) explained, with prestructured materials such as Dienes blocks, students can only decompose blocks "indirectly by trading" (p. 273). This term "trade" like "bundle" referred to earlier has been treated as though it describes or is the conceptual and literal meaning of an arithmetic process (e.g., Fuson, 1990; Saxton and Cakir 2006), which this analysis aims to reveal is really an instructional metaphor that fails to reflect the processes.

Although the physical actions the verbal metaphor trade implies are consistent with how to physically use the Dienes block material, this term is inconsistent with ideas of base-ten numbers or even written procedures. Even if students do not use the physical Dienes blocks, if an educator were to use the term "trade" verbally with written symbols, it is important to consider the limitations of this verbal metaphor. The idea of trading one set of values for another is crucial in mathematics; however, the term trade fails to communicate the hierarchical structure or nesting of units and composite units. Consequently, next let's deconstruct the meaning of this verbal metaphor.

The vernacular term *trade* means "the act of exchanging one thing for another" (Trade, 2017). This idea of trading implies some degree of perceived equivalence, in that children, for example, might trade different numbers of valued treasures based on their perceived values (e.g., three trading cards for one necklace). The term *trading*, however, gives no indication of a change in level of these units, which is an essential characteristic of the base-ten number system. An educator could encourage students to articulate the units they are trading to compensate for this limitation of the term (e.g., trade 10 ones for 1 ten). Yet, consider that even for the written procedures, moving 10 ones in the ones place to 1 ten in the tens place does not convey the meaning of trade in either the childhood or commercial sense. In order for arithmetic operations to reflect the denoted meaning of *trade*, the ones and tens values would need to switch places. An exchange or transaction in life means each person has something different than before, which does not occur arithmetically. In the algorithm, the reason 10 ones are changed into 1 tens unit is because a ten contains 10 ones.

Educators use the term trade to refer to both directions of processes, meaning the term does not reveal if one is converting a unit into the next higher-level or lower-level unit. The verbal metaphor fails to represent the direction of the intended action and thus fails to reflect the inverse nature of addition and subtraction operations.

Grouping Metaphors One basis of the Hindu-Arabic number system is grouping by ten. Thus, terms related to the idea of grouping might seem to be productive verbal metaphors to communicate base-ten number structure. Unlike the other terms discussed here that have single forms, educators use multiple variations of the term *group* as metaphors for the arithmetic processes: group, regroup, and ungroup. Let us compare each of the terms used in practice to how they may or may not facilitate the base-ten number structure with various manipulatives and then summarize these as related collection of terms.

Group The meaning of "to group" that would be most common for students would be "to combine in a group" (Group, 2017). Although putting objects together into groups of ten is necessary to build base-ten structure, this is insufficient. Successive groupings of those groupings are required (Labinowicz, 1985).

Regroup Decades ago, standards and textbooks classified problems as addition or subtraction with and without borrowing or carrying and then shifted to classifying such problems simply with the new term regrouping, as in "the student will subtract two-digit numbers with regrouping." Given the critiques of the terms *carry* and *borrow* shared earlier, the change to a "grouping" metaphor may more accurately represent the underlying arithmetic ideas, but let us deconstruct the term *regroup*. The prefix "re" means "again." Thus, regroup means "to form into a group again" or in practice "to form into a new grouping" (Group, 2017). This term could represent well the mathematical actions of regrouping a quantity such as 8 into 4 and 4 and then 3 and 5. Similarly, the quantity 14 can be grouped as 7 and 7 for a doubles strategy or 10 ones and 4 more ones. These examples, I argue, reflect meanings of the term regroup that are consistent with mathematical ideas. These groupings, however, are different arrangements within the same unit size or level. The idea of regrouping or to form into a new grouping fails to connote constructing superordinate or subordinate units. When students obtain 10 groups of tens either strictly with written symbols in an algorithm, with popsicle stick bundles, or some other materials such as Dienes blocks, some textbooks tell them they need to "regroup" into one hundred (e.g., My Math, 2013). This could simply mean to change the group size for efficiency as when counting by twos or fives, so this term "regroup" may not support the intended learning goal of reorganizing student thinking to a higher-order unit.

Another issue with the term *regroup* is that it is used for both addition and subtraction. Thus, *to regroup* does not indicate to students whether to make a quantity into a larger or smaller unit. Consequently, it cannot support the idea of inverse operations.

Ungroup The terms group and ungroup when used together could convey the inverse nature of how to move materials such as straws, sticks, and individual blocks to do and undo or put together and take apart. In other words, the pair of terms *group* and *ungroup* could connote the inverse operations of addition and subtraction.

Summaries of Grouping Metaphors All of these variations of groupings could support expanded notation algorithms or student thinking and invented strategies about individual units or ones. Consequently, this may be a useful initial verbal metaphor. A related phrase that may better, albeit awkwardly, describe the hierarchical structure of base-ten number system would be "groups of groups" (Labinowicz, 1985, p.273). Yet, in practice, such uses seem to be rare; instead educators who express metaphors of grouping use terms that reflect a single-level unit or moving from one type of grouping to another, rather than the building of higher-order units.

Pack and Unpack Some classic problems, such as the candy-packing problem in which students are given a task to pack candy into boxes that hold ten candies and then into shipping boxes that hold ten of each box (Heuser, 2005), have been used in practice and in research. The terms *pack* or *unpack,* however, seem to have been used only when this literal meaning of packing motions applied to the problem context. Yet, the term *pack* may be a potentially powerful verbal metaphor for abstract base-ten number structure, because it could promote the idea of units contained within other units. When researchers such as Kamii have provided diagrams to encourage researchers and educators to think of individual units as a collection, they draw a loop around the collection of individual units to refer to that ring as the new collection or container (Kamii, 1986). Although such researchers have not invoked the term packing in these scenarios, the ideas are about *containment*. Consider that the term *pack* at a minimum implies the idea of multiple objects contained within some other type of object that serves as a container. Once a student has 20 ones (single objects), for example, the student has two full containers of 10 ones, the containers of which are the composite unit "2 tens."

Consequently, I claim that the term pack is not a synonym for other terms used as instructional metaphors for base-ten number. Notice that these definitions of *pack* and *unpack* refer to multiple levels of objects at once. These are the objects and a unit that contains those objects (container). Other terms such as *borrow*, *trade*, or *group* connote only working within the same level categories, so they do not communicate the idea or need for higher-level units or superordinate categories. At least in theory, the terms *pack* and *unpack* better reflect this nested unit structure of the base-ten number system.

Definitions of *to pack* include "to fill completely" and "to put items into a container" (Pack, 2017a, 2017b). This meaning of *pack* can serve as an instructional metaphor for teaching base-ten structure using Digi-Blocks (Digi-Block Inc., 2017c). Consider arithmetically that ones units cannot form "a ten" until the idea of a ten unit is completely filled. The number system involves this ten structure that students must learn when and how to fill or pack each successively higher unit if and only if completely filled. The container represents the idea of a different unit.

The definition of the term *unpack* makes even more explicit the need for a container: "to remove the contents of" or "to remove or undo from packing or a container" (Unpack, 2017). Such an analysis of the instructional metaphors *pack* and *unpack* opens many questions for future research. For example, how might verbally describing arithmetic processes with the pedagogical metaphors *pack* and *unpack* support students to think about each place value as being contained within successively larger place values by a factor of 10, irrespective of whether students physically pack objects to model quantities?

Conclusions

Intended and unintended meanings of many common instructional metaphors for base-ten arithmetic have been analyzed in this chapter, both those that might be evoked through students' physical motions and those that educators verbally express. The following concludes by summarizing the single metaphors analyzed here as a hypothetical exercise and then discusses potential issues with mixing these metaphors, which reflects potential issues of real classroom instruction.

Single Metaphors

The metaphors discussed are all tools used with the intent to facilitate students' conceptual and procedural development of base-ten number. Regardless of the form in which the metaphors might be evoked, whether verbal, visual, or physically enacted, some metaphors insufficiently map to the targeted base-ten number structure, whereas others contradicted or were inconsistent with this structure. Thus, most of these tools may be ineffective for the intended job. *Group*, *regroup*, and *ungroup* are in theory insufficient metaphors in that they addressed part but not all of the essential ideas of base-ten numbers. Whereas, materials that promote physical motions or oral terms such as *carry, borrow*, or *trade* promote several unintended meanings that are inconsistent with what educators intend students learn.

In particular, the pervasive *trade* metaphor may serve the unintended function of the knife in the quote that began this chapter. Even if educators avoid trade as a verbal metaphor in favor of a variation of the term *group*, the materials educators provide such as Dienes blocks would still encourage students to experience arithmetic by physically enacting a trading metaphor. Whether verbal metaphors or enacted model-movements, trading violates the intended mathematical ideas and procedures, potentially distracting, interrupting, or causing inconsistencies when students experience these metaphors during instruction. This analysis revealed that one primary issue is that when students trade blocks for multi-digit calculations to model the intended operation (e.g., taking away blocks to model subtraction problems), their model-movements actually represent a greater number of contradictory addition and subtraction operations. These unintended inconsistencies between the model-movements and mathematics may interfere with learning base-ten number, because Nurnberger-Haag (2015) empirically found the same interference when students learning integer operations with chips had to put in or add chips when such addition operations were unintended operations.

The theoretical analysis in this chapter suggests that empirical investigation is needed to test the assertions that the materials that would encourage physical model-movements most consistent with the targeted mathematical ideas are materials that afford packing and unpacking groups of groups of ten and verbal metaphors that reflect these packing model-movements. For decades, methods textbooks for elementary mathematics have mentioned packing objects (Reys et al., 2014; Van de Walle et al., 2010), but aside from Labinowicz (1985) who recommended that grouping objects should come before Dienes blocks, such approaches were suggested simply as one of many potential groupable manipulatives that educators could offer students. This is understandable since the physical motions students enact to use these materials had largely been ignored, which this analysis used embodied cognition to reveal. The enacted and verbal metaphors pack/unpack reflect inverse operations consistent with addition and subtraction. Materials that encourage packing and the verbal terms pack/unpack maintain a closed system that reflects containment of multiple unit levels (i.e., within the given quantity of the problem, there are enough hundreds, tens, or ones to subtract or add whatever is needed without opening the system to an external source of blocks to find these sufficient quantities). Moreover, when students take away or put in blocks with these tools that promote a packing metaphor, each student motion represents intended arithmetic operations. Thus, bringing embodied cognition and other disciplinary perspectives to bear on the problem of how typical tools foster students' base-ten number understanding and how to design and choose better tools could help the field notice when pedagogical practices cut like a knife, in favor of tools that better serve the intended job.

Limited Metaphors Limit Conceptual Categories

Lest someone might argue that the limitations of verbal metaphors described here may not be crucial, consider that research from embodied perspectives has shown that oral terms that have a basis in prior physical motions prime those same ideas by neurally reactivating much of the pathways of those movements (see Kontra, Goldin-Meadow, & Beilock, 2012). Moreover, evidence from cognitive research that does not draw on embodied perspectives has shown that terms adults use influence both what children notice and do not notice in environmental stimuli leading to changes in how children categorize concepts (Plunkett, Hu, & Cohen, 2008). Such evidence indicates that the words educators use for base-ten number would likely influence how students' concepts are structured. Consequently, research that investigates these nuances of verbal instructional metaphors is warranted.

Mixing Metaphors

A single metaphor or representation will provide certain information and lack others (Johnson, 1987/1990). The response to limitations of representations in mathematics education has been to promote multiple representations as beneficial for learning (Goldin, 2003). In the United States, the use of multiple models is encouraged rather than making sure that students have a deep understanding of a single model, which should lead us to recognize that metaphors may be mixed. An example of mixing metaphors during instruction could be an educator who verbally expresses a grouping metaphor yet encourages students to physically enact a trading metaphor with Dienes blocks. Investigations are needed to test intended and unintended outcomes of mixing metaphors during instruction. There is evidence that mixing valid but incongruent metaphors interferes with comprehension of concepts even when adults already understood each metaphor and the target concepts (Gentner, Bowdle, Wolff, & Boronat, 2001). Consequently, how mixing metaphors influences children's thinking when learning and developing complex concepts of base-ten number structure is a crucial understanding for the field to investigate. Although multiple metaphors may be needed over time, because no metaphor can fully convey targeted ideas, questions for research include which metaphors should be used, in what ways, in what sequences, and how to connect these meanings for robust concept development.

Call for Transdisciplinary Research

Research that transcends disciplinary boundaries is needed to understand the effects of single instructional metaphors used for base-ten arithmetic as well as how mixing particular metaphors influence student experiences and learning. One approach could be for multiple studies each from divergent individual disciplinary perspectives to be conducted and encourage researchers across disciplinary boundaries to learn from and compile this collective knowledge rather than citing primarily within particular disciplines. Moreover, studies that merge perspectives within individual designs could be conducted to reflect transdisciplinary contributions to apply their current perspectives to the study of this problem of how to help elementary students develop understanding of base-ten number structure and operations.

References

Antle, A. N. (2013). Balancing justice: Comparing whole body and controller-based interaction for an abstract domain. *International Journal of Arts and Technology, 6*, 388–409. doi:10.1504/IJART.2013.058285.

Boot, I., & Pecher, D. (2011). Representation of categories: Metaphorical use of the container schema. *Experimental Psychology, 58*, 162–170.

Borrow. (2017). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Carry. (2017). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary

Digi-Block, Inc. (2017a). *About is*. Retrieved from http://www.digiblock.com/pages/about-us

Digi-Block, Inc. (2017b). *Blocks*. Retrieved from http://www.digiblock.com/pages/better-base-ten-blocks

Digi-Block, Inc. (2017c). *Teacher resources*. Retrieved from http://www.digiblock.com/pages/lessons

English, L. D. (Ed.). (2013). *Mathematical reasoning: Analogies, metaphors, and images*. New York: Routledge.

Fraivillig, J. (2017). Enhancing established counting routines to promote place-value understanding: An empirical study in early elementary classrooms. *Early Childhood Education Journal* (online first). doi:10.1007/s10643-016-0835-5

Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 243–275). Inc.: Macmillan Publishing Co.

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction, 7*(4), 343–403.

Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *Journal for Research in Mathematics Education, 21*(3), 180–206.

Gentner, D., Bowdle, B., Wolff, P., & Boronat, C. (2001). Metaphor is like analogy. In D. Gentner, K. J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 199–253). Cambridge, MA: MIT Press.

Glenberg, A. M. (2010). Embodiment as a unifying perspective for psychology. *Wiley Interdisciplinary Reviews: Cognitive Science, 1*, 586–596.

Glenberg, A. M., & Kaschak, M. P. (2002). Grounding language in action. *Psychonomic Bulletin & Review, 9*(3), 558–565.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275–285). Reston, VA: National Council of Teachers of Mathematics.

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. (2009). Gesturing gives children new ideas about math. *Psychological Science, 20*(3), 1–6.

Group. (2017). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary

Heuser, D. (2005). Teaching without telling: Computational fluency and understanding through invention. *Teaching Children Mathematics, 11*(8), 404–412.

Johnson, M. (1987/1990). *The body in the mind*. Chicago, IL: The University of Chicago Press.

Kamii, C. (1986). Place value: An explanation of its difficulty and educational implications for the primary grades. *Journal of Research in Childhood Education, 1*(2), 75–86.

Kamii, C., Lewis, B. A., & Kirkland, L. (2001). Manipulatives: When are they useful? *The Journal of Mathematical Behavior, 20*, 21–31.
Kontra, C., Goldin-Meadow, S., & Beilock, S.L. (2012). Embodied learning across the life span. *Topics in Cognitive Science*, 731–739.
Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking*. Reading, MA: Addison-Wesley Publishing Company.
Lakoff, G., & Johnson, M. (2003/1980). *Metaphors we live by*. Chicago, IL: The University of Chicago Press.
My Math teacher edition. (2013). Columbus, OH: The McGraw-Hill Companies.
National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Retrieved from http://www.corestandards.org/Math/Content/1/NBT/B/2/
Nesher, P. (1989). Microworlds in mathematical education: A pedagogical realism. In R. Glaser & L. B. Resnick (Eds.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 187–215). Hillsdale, NJ: Lawrence Erlbaum Associates.
Nurnberger-Haag, J. (2015, November). How students' integer arithmetic learning depends on whether they walk a path or collect chips. Research Report in Bartell, T. G., Bieda, K. N., Putnam, R. T., Bradfield, K., & Dominguez, H. (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 165–172). East Lansing, MI: Michigan State University.
Pack. (2017a). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary
Pack. (2017b). In *Cambridge dictionary*. Retrieved from http://dictionary.cambridge.org/us/dictionary/english/pack
Parrish, S. D. (2011). Number talks build numerical reasoning. *Teaching Children's Mathematics, 18*, 198–206.
Pimm, D. (1981). Metaphor and analogy in mathematics. *For the Learning of Mathematics, 1*(3), 47–50.
Plunkett, K., Hu, J., & Cohen, L. B. (2008). Labels can override perceptual categories in early infancy. *Cognition, 106*, 665–681. doi:10.1016/j.cognition.2007.04.003.
Pullman, P. (2007). *The amber spyglass*. Quote retrieved on August 1, 2014, from http://www.goodreads.com/quotes/55075-the-intentions-of-a-tool-are-what-it-does-a
Reys, R., Lindquist, M., Lambdin, D., & Smith, N. (2014). *Helping children learn mathematics* (11th ed.). Hoboken, NJ: Wiley.
Rosch, E., Mervis, C. B., Gray, W. D., Johnson, D. M., & Boyes-Braem, P. (1976). Basic objects in natural categories. *Cognitive Psychology, 8*, 382–439.
Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.
Saxton, M., & Cakir, K. (2006). Counting-on, trading and partitioning: Effects of training and prior knowledge on performance on base-10 tasks. *Child Development, 77*, 767–785.
SRA Connecting Math Concepts. (2013). *Level D teacher's guide: A direct instruction program* (comprehensive ed.). Columbus, OH: The McGraw-Hill Companies.
Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer.
The University of Chicago School Mathematics Project. (2012). *Everyday mathematics: Common core state* (standards ed.). Chicago, IL: The McGraw-Hill Companies.
Trade. (2017). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary
Unpack. (2017). In *Merriam-Webster's dictionary*. Retrieved from http://www.merriam-webster.com/dictionary
Van de Walle, J., Karp, K., & Bay-Williams, J. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). New York: Allyn and Bacon.
Verschaffel, L., Greer, B., & Corte, D. (2007). Whole number concepts and operations. In F. K. J. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Reston, VA: National Council of Teachers of Mathematics.

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review, 26*, 73–90.

Warfield, J., & Meier, S. L. (2007). Student performance in whole-number properties and operations. In P. Kloosterman & F. K. J. Lester (Eds.), *Results and interpretations of the 2003 mathematics assessment of the national assessment of educational progress* (pp. 43–66). Reston, VA: National Council of Teachers of Mathematics.

Web Minder. (2014, January 13). Zoltan Dienes Web Site. Retrieved on June 30, 2016 from http://www.zoltandienes.com

West, B. H., Griesbach, E. N., Taylor, J. D., & Taylor, L. T. (1982). *The prentice-hall encyclopedia of mathematics*. Englewood Cliffs, NJ: Prentice-Hall.

Chapter 12
Mathematics and Movement

Susan Gerofsky

Abstract The author introduces aspects of the problematic around the idea of mathematics teaching learning through an integration of movement (dance, gesture) with other instructional modes. There is a discussion of working at various scales and using a variety of movement types. Through three examples from her collaborative, transdisciplinary research, the author suggests design principles for working across and among disciplines to create new and surprising spaces for research and pedagogy integrating mathematics and movement.

Keywords Mathematics learning • Movement • Dance • Gesture • Pedagogy • Embodiment • Scale • Integration • Graphs • Number theory • Geometry

I research embodied mathematics learning, via gesture, movement, dance, and voice. When I tell people about my research, I encounter two quite different immediate responses. The first response is one of disbelief and incomprehension. For many, including some mathematics educators, the very nature of mathematics is to be radically *disembodied* and *static*. Mathematics is treated as an activity of mind disconnected from the body, physical objects, and movement. The goal of increasingly sophisticated mathematics education is seen, in this view, to be 'pure abstract cognition', entirely cut off from contextual features of human life (emotion, social interaction, physicality) and the material world (the greater-than-human natural world, sensory ways of knowing, environments and objects). This view of mathematics, based on Platonic and Cartesian philosophies of a mind-body split, has been the basis for many practices in mathematics pedagogy for over 100 years (see Gerofsky, 2016, for further elaboration on this).

Most people who have been through school anywhere in the world have been enculturated through mathematical experiences based on disembodiment and

S. Gerofsky (✉)
Department of Curriculum and Pedagogy, University of British Columbia,
2125 Main Mall, Vancouver, BC V6T 1Z4, Canada
e-mail: susan.gerofsky@ubc.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_12

prohibition of movement. In (stereo)typical secondary school mathematics classes, students sit still and silent in rows of individual desks and receive lectures from teachers, moving only eyes and fingers as they copy down notes, do textbook exercises and worksheets, and ‘pay careful attention’ (Boaler, 2014). Classrooms are typically bare and grey, without colourful displays of student work and with minimal sensory input. The implicit message embedded in this pedagogy (and understood by students) is that mathematics learning comprises silent, rather monastic individual work and contemplation, engaging the mind, in isolation from the body, senses, social interactions, and movement. This is in stark contrast to students’ experiences in other secondary school classes like language arts, social studies, science, and any of the fine, performing, or practical arts, where classrooms typically display current student work and include interactive activities like discussions, project work, debates, and dramatic and ‘maker’ activities as a well-established part of pedagogy.

Elementary school mathematics is more likely to include physical materials, manipulatives, colour, and social and physical activities, but, nonetheless, even mathematics in the early years is affected by assumption of a Cartesian/Platonic dualism. The younger the learners are, the more acceptable it is to include sensory experiences, physicality, movement, and manipulatives as part of mathematics learning. However, as students move from kindergarten and primary school into the upper years of elementary education and secondary school, physicality in mathematics learning is considered increasingly unacceptable. Embodied math learning is fine for ‘the little ones’, but for older learners, it is considered coarse, primitive, and babyish. Increased mathematical sophistication is implicitly associated with decreased physicality – with the stereotype of solely mental, abstract, disembodied, and static modalities for learning. Many mathematics educators who make these kinds of assumptions even feel embarrassment at the idea of incorporating any kind of physicality in their teaching, and if the teacher begins with negative feelings about using a particular pedagogical intervention, there is little likelihood of that intervention being successfully adopted.

The second, contrasting initial response I encounter on the topic of embodied mathematics usually comes from people who are fearful of mathematics or unsure about their knowledge of the subject (often teachers whose interests lie elsewhere). This second response is a kind of gleeful assertion that all we need to do now is have students move around, dance, or make gestures, and we will have finished with mathematics instruction. Neither of these two responses is particularly helpful in terms of developing embodied ways of teaching and learning mathematics through movement. The first response assumes that bodily movement is childish and embarrassing and has no place in upper levels of mathematics and the second that moving one’s body in random ways might substitute for learning mathematics.

What I am advocating is something different from both of these: first, that bodily movement, gesture, voice, and sensory experiences are necessary experiential components in developing new mathematical ideas at any age or level and, second, that embodied pedagogical experiences in mathematics need to be thoughtfully designed and folded into teaching if they are to help students learn about new mathematical relationships and ideas. My research and experience support an integration of movement and sensory experiences, along with metaphors, drawings, and diagrams, explanations, and mathematical problem explorations to help learners build robust mathematical conceptualizations. What is more, these various modalities can ideally inform one another, so that one might move back and forth among movement, gesture, explanation, written definitions and solutions, diagrams, and physical materials, in a social, emotionally engaged, embodied, multimodal, and multifaceted exploration of mathematical ideas.

A number of recent studies analyse research mathematicians' use of gesture and movement in their own ideation and communication in mathematics (e.g. Sinclair & Tabaghi, 2010; Soto-Johnson, Hancock, & Oehrtman, 2016), and other studies document undergraduate mathematics students' embodied gesture and movement in understanding and explaining mathematical relationships (e.g. Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Yoon, Thomas, & Dreyfus, 2011). These offer clear evidence that mathematics, gesture, and movement are not only helpful for young children but are used spontaneously and in an integrated way by sophisticated learners and research mathematicians as they develop and communicate mathematical understandings. Other researchers in embodied mathematics have approached the idea of designing kinesthetic activities to support specific areas of mathematics learning (Abrahamson & Trninic, 2015; Nurnberger-Haag, 2015; Wright, 2001), focusing on mathematical topics including proportionality, the number line and integers, order of operations, and physical relationships like distance, time, and velocity. There is room for a great deal more work in this area as many of the 'big ideas' of mathematics, particularly those typically included in the secondary and post-secondary curriculum, have not been considered from the point of view of integrating an embodied pedagogy.

In this chapter, I will discuss several projects highlighting different aspects of design with mathematics and movement as transdisciplinary phenomena. Each of these projects has a collaborative element across the disciplines of mathematics and dance, and each involves transdisciplinary negotiations that open up new spaces for both mathematics and dance. The effects of transdisciplinary work, over time, and with openness to surprise and learning, can be to create unexpected, fruitful new practices and ideas that would never have arisen without this collaboration.

The three projects I will highlight here (Graphs & Gestures, number theory dances, and the geometry of longsword locks) are some of the math and movement and dance collaborations I have been involved with (Gerofsky, 2013). There are others working in this area too, and any account of mathematics and movement must acknowledge the important contributions of Schaffer, Stern, and Kim (2016) who have been working with mathematical ideas in their professional dance company and producing mathematics and movement curricular materials for many years. Schaffer, who has expertise both as a mathematician and a modern dancer, is a regular contributor to the Bridges Math and Art conference (see, e.g., Schaffer 2014, 2015) and continues to push the mathematical boundaries of math/dance.

Work in the pedagogy of mathematics and movement more generally includes work on the scale of bodily movement and mathematical noticing (Knoll, Landry, Taylor, Carreiro, & Gerofsky, 2015; Noble, Nemirovsky, Wright, & Tierney, 2001), small-scale hand movements and with dynamic computer displays as part of mathematics learning (Abrahamson, Lee, Negrete, & Gutiérrez, 2014; Gerofsky, Savage, & Maclean, 2009; Sinclair & Pimm, 2014), and studies of unconsciously produced speech-accompanying gesture as a way to understand learners' and teachers' meaning-making in mathematics (Alibali & Nathan, 2012; Arzarello, Paola, Robutti, & Sabena, 2009; Goldin-Meadow, Cook, & Mitchell 2009). Other related work on embodied mathematics includes studies of multisensory mathematics (e.g. Rains, Kelly, & Durham, 2008) and mathematics with learners with sensory impairments (Figueiras & Arcavi, 2014; Healy & Fernandes, 2011; Zebehazy & Wilton, 2014). There is also a flourishing of theoretical works that question the nature of embodied knowing in more fundamental philosophical ways (for some examples, see Arzarello, 2006; De Freitas & Ferrara, 2015; De Freitas & Sinclair, 2013; Edwards 2003; Lakoff & Núñez, 2000; Radford, 2009; Thom & Roth, 2011). All this work is relatively new, mutually enriching, entangled, and still unsettled – that is to say, this is a productive new area of scholarship and praxis – but this chapter will focus only on transdisciplinary collaborative work on mathematics and dance and bodily movement. Interested readers may want to follow up with further exploration of the burgeoning field of embodied mathematics learning more generally as well.

The Graphs & Gestures Project

The Graphs & Gestures project started with an observation from my own experiences teaching my secondary math classes about the graphs of mathematical functions. I noticed that when my students and I used a gesture to represent the horizontal line $y = 4$, my gesture was considerably lower in relation to my body than the gestures of many of my students. I began to wonder why this might be so and to speculate about where different people imagined the x- and y-axes in relation to their own bodies. For example, did some people imagine the crossing point of x- and y-axes (the 'origin') to be at their navel, while others pictured it at their nose? Did some track the shape of the graph from left to right or right to left, while others used two hands to emphasize symmetrical patterns? What difference might these individual variations in gesture mean in terms of mathematical understanding – if they meant anything? What significance might they possibly have for mathematics teaching and learning?

From these small observations, questions and 'hunches' (Bavelas, 1987) have grown a larger research project into ways of incorporating elicited gestures, whole-body movement, objects, vocalization, metaphors, narrative, and imagery, integrated with short episodes of direct instruction, in the teaching and learning of mathematical functions and their graphs, a topic central to several years of secondary school precalculus algebra curricula worldwide. A fuller description of the progress of Graphs & Gestures can be found in Gerofsky (2016), and research findings emerging from the original hunch and other questions that followed on from it are available in a number of publications, including Gerofsky (2010, 2011a, 2011b). For the purposes of this chapter, I will describe the particular ways that mathematics and movement/dance have developed as part of this project and how a transdisciplinary collaboration between a mathematics educator and dance choreographer/arts educator made possible new questions and opened up new spaces in both fields.

The Project

Graphs & Gestures began with two initial pilot projects that undertook basic research into students' and teachers' gestures describing the graphs of mathematical functions. One of the key findings of the pilot projects in Graphs & Gestures was that learners who were ranked by their teachers as 'top' mathematics students (in terms of their depth of mathematical understanding) were also the ones who tended to place the origin lower against their bodies (at heart, navel, or hip level), to use larger gestures that engaged their spines and the core of their bodies, and to bend their knees, rise to their toes, and put themselves off their centre of balance in gesturing the graphs. Students who were ranked by their teachers as 'average students' (who worked hard but depended on rote learning rather than deep mathematical understanding) were more likely to gesture the graphs higher, with the origin at throat or nose level, and to use smaller gestures of just hand and arm, treating a finger like a pencil. These patterns in the data have been analysed in Gerofsky (2010) using theoretical constructs from gesture theory in terms of an in-depth identification and experience with the graph (character viewpoint or CVP) vs. an arms-length third-person experience of the graph (observer viewpoint or OVP) (see McNeill, 1992). These findings led to several iterations of pedagogical design experiments (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) with secondary and upper elementary school math classes. The overarching idea of these design experiments was to help *all* students in a class take up embodied, imaginative, metaphorical, and narrative approaches that the 'top' students were already using spontaneously to understand graphs.

I enlisted the collaboration of a research colleague in education, Kathryn Ricketts, who also has a distinguished international career as a modern dancer and choreographer. My collaborative work with Ricketts with school classes over the past 10 years has generally comprised five or six one hour videotaped class sessions with students and teachers over the course of 6 months (i.e. one session per month), plus pre- and post-tests, and focus group and individual interviews with students. We have worked with a wide variety of classes of students, including those categorized as gifted, average, reluctant learners, dyslexic, and blind or visually impaired. Our work has taken us to secondary and elementary schools in Vancouver and North Vancouver, BC; Regina, Saskatchewan; and Torino, Italy. Our collaboration over the past 10 years has created new spaces for mathematics through movement in ways that neither of us could have foreseen at the start.

Contributions from the Field of Dance

Rickett's extensive understanding of bodies in motion from her knowledge as a dancer helped me to be able to observe and name features of movement in the pilot study videotapes that I might not otherwise have noticed and later to work on incorporating them into our collaboratively planned design experiment lessons. Some of the insights we have arrived at together about the qualities of bodily movement that are most effective in supporting mathematics learning are summarized in Gerofsky and Ricketts (2014). In our work together on the project, Ricketts was able to notice qualities of physical movement and engagement that I would not have seen if working alone. For example, she identified movement of the body core and spine, changing levels, moving across the centre line of the body, moving off the vertical, and locomotion around the space as effective movement modalities that stimulate mathematical noticing, especially when contrasted to static poses, or movement of limbs and extremities alone.

Ricketts has a great deal of experience in working with groups of untrained dancers in community workshops (including multi-age groups, people with disabilities, and people from multicultural backgrounds dancing together). Her knowledge from this work in dance was key to designing and leading highly effective warm-ups with classes of mathematics students to prepare them for the specific work we would be doing with mathematics and movement. As a choreographer in dance theatre, she was aware of physical aspects of a warm-up, including body alignment, warming up large muscle groups, mobilizing joints, warming up the cardiovascular system, engaging voice, and 'warming down' to achieve calm and centred movement. Her work with community groups had developed her expertise in using familiar metaphors, vocalizations, and movements to help people feel comfortable with undertaking whole-body movement in mathematics learning. For example, she uses sports metaphors (throwing and catching a ball, arriving at home plate), images from popular culture ('Ninja point'), and everyday actions (looking over each shoulder to see 'who's there?') to build a movement repertoire that students feel comfortable with. In our work together, there has never been a problem with students feeling embarrassed or awkward about combining bodily movement and mathematics learning. It is helpful as well that Ricketts brings an atmosphere of respect and acceptance and a relaxed but highly enthusiastic attitude towards movement to her work with groups – in contrast with those 'traditional' mathematics educators who might begin from a feeling of embarrassment to be moving at all in a math class.

Ricketts is also expert in working with objects and props as components of performative dance theatre, and her focus on 'thing theory' (the phenomenology of everyday material life and the way that objects speak to us through their very materiality) (Ricketts, 2011; Brown, 2004) informed our choices of objects to use in our experimental classes with students. For example, we have used pleasingly coloured lengths of wide sewing elastic, stretched across the room, to represent the *x*-axis; a beautiful brass plumb bob for a slow-motion tug-of-war in our absolute value olympics; and smooth gold and silver cardboard discs (originally designed as bakery cake trays) to help learners use both hands and their body core to gesture the shapes of graphs. Rickett's knowledge of the importance of aesthetically pleasing, evocative, and sensual objects as tools to support learning informed our choices, and helped engage learners in the experiential, movement-oriented math learning activities.

Contributions from the Field of Mathematics Education

My own contributions as a mathematics educator kept our work connected with goals in mathematics learning and mathematics curriculum. Throughout the project, I have always been aware of the aims and pressures that mathematics teachers feel in terms of enculturating learners in the knowledge, methods, and traditions of mathematics as a field and its articulation into grade-level goals in school mathematics curricula. If our work with movement were not helping to support mathematics learning towards these goals, it would not be seen as useful for the school mathematics education community.

So my role has often been to say "not that, but this" in terms of the mathematics – in other words, to keep our collaborative work focused on the goals of in-depth understanding of topics important to the learning of mathematics at the secondary level and particularly to the understanding of aspects of the graphs of mathematical functions. Over the years, I have selected particular aspects of this learning that we wanted to highlight: in some years, the graphs of polynomial functions, their roots, slopes, and maximum/minimum points (extrema) and in others, the graphs of absolute value functions (or the absolute value graphs of other functions) and the concept of absolute value; maximum/minimum regions of quadratic functions; and, at present, transformations of function graphs. For each of these aspects or topics, different kinds of movement, voice, props, and learning activities would help draw students' attention to salient features of the graphs. Part of my role was to keep our focus on those salient features and to ensure that we helped learners notice these.

It has also been part of my role in our collaboration to keep the mathematical work moving forward in ways that reflected the project's pedagogical intentions. We take care to sequence lessons, building on past experiences and making connections between different kinds of activities in aid of building a broader and deeper understanding of the mathematical topic at hand. It is also my responsibility to introduce mathematical terminology and to teach very brief (5–10 min) 'typical math lessons' where helpful and necessary in the course of our design experiments. I also design and carry out assessments of learning (including pretest/post-test and group written and oral content tests) to support analysis and evaluation of the effects of the design experiments.

Throughout the planning and carrying out of these design experiments, there is a constant need to keep clear what we want to represent mathematically via embodied movement, and this responsibility has been mine. For example, at an early stage, we began to move in the direction of representing covariation of the x and y variables in mathematical functions, as others in this area have done (viz. Noble, Nemirovsky, Wright & Tierney 2001; Jackiw & Sinclair 2009) and as I have done in other embodied mathematics projects using devices like Etch-a-Sketches (Gerofsky & Marchand 2006). However, the mathematical aims of Graphs & Gestures have been to represent graphs holistically, through movement and metaphor, in contrast to a covariation approach, which splits the behaviours of *x* and *y* elements. It was important for me to step in before we had gone too far in a direction we did not really want to take and to say, "not that, but this" to keep our mathematical intentions clear.

Similarly, it has been important for me, as the mathematics educator on the research team, to decide when further exploration or exploration is needed as when senior elementary students asked *why* absolute value had any importance or was worth learning. Before we could move forward with exploring the behaviour of the absolute value of functions, we had to be able to work with more basic activities, narratives, images, and metaphors that would help learners make sense of the purposes of absolute value functions. I see it as my role to keep the connections with mathematical meaning-making clear, even when taking a different path might result in a particularly interesting result from a dance or movement point of view.

Transdisciplinary Surprises and Discoveries

Collaborations are always more than the sum of their parts, and while arts educator/choreographer and mathematics educator each contributed ideas and practices from our respective fields, we were also surprised and delighted by discoveries that could only happen by working together across disciplines. Both of us became attuned to one another's interests and concerns and began to notice ways that movement and mathematics learning worked (or didn't work) together. Some of our surprises included the following:

- *Cognition lives in the body-mind*, not in the mind separated from the body; embodied experiences and abstract concepts nourish and inform one another in a continual oscillation. We watched students use bodily experiences (and memories of those experiences) as cognitive resources, or 'experiences to think with', as they developed, and later reconstructed sophisticated mathematical ideas. Learners need and deserve embodied, mindful experiences to think and act with as they are introduced to new patterns and relationships in mathematics.
- *Visceral experiences – those that involve the body core, internal organs, and spine – are far more salient* and easy to notice and attend to than physical experiences that only involve peripheral parts of the body (e.g. only fingers and hands). One cannot ignore movements and sensations that affect the core of the body. If educators want learners to notice features of graphs (e.g. slope, symmetry, intersections with axes or other lines, reflections, or continuity and discontinuity), it is far more effective to stimulate attentiveness through visceral movements that

engage the body fully, rather than just by looking at a diagram or tracing its shape with a finger alone.

- *'Sonification' is as important as visualization*, and vocalization of sounds that originate and resonate within the body is far more effective in mathematical noticing than sounds created by an external source (e.g. the midi function on a computer). At a number of points in the project, we tried working with machine-generated sounds through game systems platforms like Kinect (in collaboration with computer science colleagues), but we kept returning to the sheer visceral power of the human voice. As with movement of the body core, the engagement of voice to create mathematical representations of slope, intercepts, maximum/minimum points, and other mathematical features made it far easier to be attentive to these features.
- *'Kinesthetic playfulness' is a prerequisite* for imaginative exploration of new mathematical ideas – and it works counter to many of the perfectionist traditions of both mathematics and dance. We discovered that both our fields have a potential bias towards 'flawless' performance, correct moves or answers, and a potentially stultifying seriousness arising from fears of being wrong and vulnerable. These fearful, perfectionist tendencies can prevent learners from taking on new ideas and experimenting with them in a generative, improvisational mode. By fostering a kinesthetically playful atmosphere in the classroom where learners can feel free to try out new, movement-oriented experiments to explore mathematical ideas, we can work against rigidity and encourage creative, more flexible approaches. These are complemented by eventual arrival at certainties and rigorous proofs and definitions, but certainty and rigor *arise* from extensive experience and ought not to be imposed before learners have the chance to play and experiment.
- *Movements and gestures can help learners enter into the discipline (and let the discipline enter into them).* When mathematics lives only in textbooks, notes, and lectures, it is easy to hold the discipline at arm's length and treat it as something external and superfluous. However, when mathematics is available to be explored as part of one's own body, it becomes necessary to 'let it in' and/or to 'enter into' the disciplinary field. Learners who identified with mathematics in this way were consistently better able to develop better in-depth understandings than those who held mathematics 'at arm's length'.

The Graphs & Gestures project has opened up new spaces primarily within mathematics education (and to a lesser extent within dance). The other two examples I describe offer new spaces to mathematics, to math education, and to dance itself.

Chase's Number Theory Dances

I have collaborated in a different way with another internationally known modern dancer based in British Columbia, Sarah Chase. Our collaboration began when we were introduced by mutual friends who knew we were both interested in 'dancing mathematics'. For Chase, mathematics has been a medium for generating new dances and dance practices; for me, collaborations with Chase's work has offered the means for exploring new embodied understandings and dynamic representations of aspects of mathematics.

When I met Chase, she had already developed some of her number theory dances. These were based on the idea of representing several cycles simultaneously with different parts of the body – for example, the cycles of moon, sun, and earth that govern the tides, or the cycles of 12 animals, 5 elements, and yin-yang that govern the 60-year Chinese lunar horoscope calendar. Chase 'uses her body as a calculator', marking places in space and time by dancing cyclic patterns of different periods – periods that coincide only when all the cycles have repeated many times. The effect is quite astonishing, as when she dances the combinatorics of 11 different movements with one arm, 13 movements with the other arm, and 7 distinct movements with legs and feet. It takes $11 \times 13 \times 7 = 1001$ moves to complete all the cycles simultaneously, with many interesting combinations of movements occurring throughout the 45-minute dance as every element meets every other element of each of the cycles. This dance, *A Thousand and One*, takes a tremendous concentration and engagement to achieve, and some of the sequencing must become automated or be undertaken at the unconscious level, as the conscious mind cannot process the combinations fast enough. Amplifying that effect, Chase accompanies this dance with the dancer undertaking improvised storytelling, in the spirit of the traditional tales of *A Thousand and One Nights* (Byatt & Burton 2004). The resulting dance and stories are fascinating explorations of embodied mathematical combinatorics and moving meditations (see Dickinson, 2014).

Working with Chase and seeing her mathematically inspired dances, I wondered whether a simpler version of her number theory dances might be helpful to mathematics learners in elementary and secondary school learning about factorization, least common multiple (LCM) and greatest common factor (GCF), prime, relatively prime numbers, and other related ideas in number theory. Chase had workshopped a pared-down version of these combinatoric dances with community non-professional dancers and shared a two-against-three pattern (one side of the body doing a pattern with a sequence of two and the other a sequence of three, which takes six combinations to complete). We talked about using other number combinations as well, including different pairs of numbers that are relatively prime (e.g. 5 and 6 or 8 and 9) and pairs that are not relatively prime because they share a factor (e.g. 4 and 6 or 9 and 12). Could learners experiment with their bodies to see where the patterns started to repeat – and could they work out how to predict this? Could learners in pairs or threes document which combinations of movements were and weren't enacted with particular pairs of numbers? Could they make sense of, and explain, why some pairs (*n,m*) went through all possible combinations and repeated only after $n \times m$ moves while others 'skipped' some possible combinations and ended after fewer moves?

These embodied mathematical explorations connected with work a group of us had been doing representing factorization and relatively prime numbers using spirographs, musical rhythms, and circle/modular diagrams and polygons (Gerofsky, Gomez, Rappaport, & Toussaint, 2009). Chase's whole-body, highly engaging mathematics in movement offered a different way of approaching this topic that allowed 'learners to enter into the discipline and the discipline to enter the learners' in a way that drawing diagrams and working with physical tools like the Spirograph did not, although work at these different scales and levels of physical involvement were complementary in interesting educational ways (Knoll, Landry, Taylor, Carreiro, & Gerofsky 2015). The deeply involving dance/movement approach at a large, full-body scale has promise of working synergistically with medium-scale body-mind engagement of clapping, singing, or playing musical rhythms, using a large spirograph (viz. Sayers, 2013) and drawing circle/modular diagrams, perhaps at large scale with sidewalk chalk outdoors, so that the positions could be 'hopped' or locomoted. From a mathematics education point of view, it seemed that an effective pedagogical approach could bring together all these arts-infused representations (dance, musical rhythms, aesthetically pleasing spirographic star patterns, circle modular representations, and polygons), along with some more 'traditional' direct instruction and exercises where these would be helpful.

At time of writing, Chase has experimented with using simpler versions of the number theory dances with some young mathematics learners that she tutors, and I have piloted workshops using aspects of this integrated arts-based approach with groups of experienced mathematics teachers and teacher candidates, with promising results. Chase and I have also collaborated on a short film about the relationship between combinatorics and her number theory dances (Gerofsky & Chase, 2013). Next steps may include a full-scale design experiment working with senior elementary students on factorization, LCM, and GCF – and/or with undergraduate mathematics students on 'necklace' and 'bracelet' combinatoric patterns in university-level mathematics (see Wolfram Mathworld, n.d.).

Chase's dances originated from her interest and inspiration from mathematics and the combinatorial patterns observable in tides, horoscopes, and stories. Mathematics was part of generating these dances. In turn, the dances are inspiring new modalities of mathematics pedagogy, based in dance and movement (and possibly combining with music and graphic arts) to generate new ways that learners can understand and be inspired by mathematics. New spaces open up in all these areas – dance, mathematics education, and possibly even in mathematics as a field – through these transdisciplinary collaborations. We cannot predict all the effects of these collaborations but experience surprise and delight as new possibilities appear over long-term work together.

The Geometry of Longsword Locks

The third example comes from a form of traditional dance that I have been involved with for years: Morris dancing and, the related form, longsword dance (Allsop, 1996). Morris dance is a traditional ritual dance form from England going back more than 600 years, and longsword dance, which likely reached England with the Vikings around 800 CE, is a tradition often performed by Morris dance groups or ‘sides’, together with mummers plays and associated dance and song traditions.

Longsword dances are performed by six or more dancers, each carrying a fairly rigid ‘sword’ made of wood or metal about the dimensions of a metre stick. Throughout most of the dance, the dancers are joined into a ring, holding the hilt of their own sword and the (blunt) point of their neighbour’s. A distinctive feature of longsword dancing is the formation of ‘locks’, or woven stick patterns in the form of polygonal stars, double polygons, and other geometric shapes. These shapes are formed quickly by the dancers through what I have named a ‘physical algorithm’ of dance moves; the lock is displayed to the audience and then quickly dismantled as the dance continues.

I was curious to explore these longsword lock formations from the point of view of geometry and mathematical thinking, with mathematics education in mind. Thinking in terms of variance and invariance, scale, algorithms, and angles, I asked questions like the following:

- Is there a minimum and/or a maximum number of swords that can be used to make a lock due to mathematical and/or physical constraints?
- Is there an upper and/or lower limitation to the size of the angles formed? If so, why?
- Is it easier to make these locks individually, on a small scale (with coffee stir sticks), or as a group, on a large scale, through the physical algorithms of the dance? What might account for the differences in these two kinds of physical processes?
- How could one generalize the physical algorithms that produce different kinds of longsword locks? Could these be represented algebraically?
- What is the minimal number of crossings the swords must have to hold the lock in place?
- Are there new possible locks that have not yet been discovered? Could a mathematical analysis help to produce new (and beautiful) lock shapes for the dance?
- How might the geometry of longsword locks connect with other areas of mathematics like knot theory and abstract algebra?

To move this exploration forward, I led an experimental longsword lock workshop at the Banff International Research Centre in 2009, and later that year, my Morris side led a demonstration and workshop on longsword locks at the Bridges Math and Art conference (Gerofsky, 2009). Participants made small locks with wooden stir sticks and then learned to make large-scale locks through the physical algorithms of the dance. We addressed some of the questions posed above and created a 31-sword star the size (and weight) of a large gate, conjecturing that, with more dancers and sticks, we could likely go to a number larger than 31.

For the Bridges workshop, my Morris side created a new dance that featured every kind of longsword lock we knew of, including some that we learned from Allsop (1996). This new dance then became the impetus for two short films on mathematics and art, the first directed by McCague (McCague & Gerofsky, 2013) for a subsequent Bridges conference and the second directed by Hart (2014) as part of the Simons Foundation Mathematical Impressions short film series. Hart's film references a related mathematical paper on the geometry of tensegrity frameworks (aka 'popsicle stick bombs') (Whiteley, 1989). Hart issues a call to research mathematicians to take up some of the unanswered questions about the geometry of longsword locks and to explore the limits of these physical algorithms and their algebraic representations.

I have used the films and the activity of creating longsword locks in workshops with experienced and preservice mathematics teachers and with classes of secondary school mathematics students, as an example of thinking mathematically using small- and large-scale, individual, and collective mathematical experimentation. This transdisciplinary work has opened up space for exploration of new mathematics at the research level, new pedagogical resources and approaches for teaching geometry, reasoning and problem-solving, and new dances, by focusing on categorizing and representing whole classes of different types of geometric figures (and possibly discovering new ones).

Conclusions

In the three examples discussed above, transdisciplinary work involving movement/dance, mathematics, and mathematics education has generated unexpected new forms and spaces for development in all three fields. These spaces begin with curiosity and a willingness to collaborate among people who have a firm commitment to and grounding in their own particular discipline. To develop further, there must also be an openness and attentiveness to unexpected, surprising results, a willingness to follow up on emergent ideas and new directions that present themselves, and the energy to take up something still somewhat undefined and unproven.

A precondition to these kinds of productive collaborations is a radical openness to move beyond unexamined cultural commonplaces like the Platonic/Cartesian assumption of a mind-body split, and the assumption that mathematics is or should be a solely mental, nonphysical activity. Once these deep-seated cultural assumptions become open to questioning, there is space created for meaningful, potentially beneficial transdisciplinary innovations.

References

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM, 46*(1), 79–93.

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM, 47*(2), 295–306.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *The Journal of the Learning Sciences, 21*(2), 247–286.

Allsop, I. (1996). *Longsword dances from traditional and manuscript sources*. Marshfield, VT: Northern Harmony.

Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME. Revista latinoamericana de investigación en matemática educativa, 9*(1), 267–300.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Bavelas, J. B. (1987). Permitting creativity in science. In D. Jackson (Ed.), *Scientific excellence: Origins and assessment* (pp. 307–327). Beverly Hills, CA: SAGE Publications.

Boaler, J. (2014). *Beautiful maths: How successful school approaches change students' lives.* Videotaped talk given at the Ontario Institute for Studies in Education (OISE). Retrieved February 2017 from https://www.youtube.com/watch?v=2OgKnrPNZiI

Brown, B. (Ed.). (2004). *Things*. Chicago: University of Chicago Press.

Byatt, A.S. (Ed.) & Burton, R. (Trans.) (2004). *The Arabian nights: Tales from 1001 nights.* New York: Modern Library.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

De Freitas, E., & Ferrara, F. (2015). Movement, memory and mathematics: Henri Bergson and the ontology of learning. *Studies in Philosophy and Education, 34*(6), 565–585.

De Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics, 83*(3), 453–470.

Dickinson, P. (2014). Edge1 at DOTE. Blog post from *Performance, Place, and Politics.* Retrieved March 2017 from http://performanceplacepolitics.blogspot.ca/2014/07/edge-1-at-dote.html

Edwards, L. D. (2003, February). The nature of mathematics as viewed from cognitive science. In M. A. Mariotti (Ed.), *Proceedings of the third congress of the European Society for Research in Mathematics (CERME 3), Bellaria, Italy*, Thematic group 1, pp. 1–10. Retrieved February 2017 from http://www.mathematik.uni-dortmund.de/~erme/CERME3/Groups/TG1/TG1_edwards_cerme3.pdf

Figueiras, L., & Arcavi, A. (2014). A touch of mathematics: Coming to our senses by observing the visually impaired. *ZDM, 46*(1), 123–133.

Gerofsky, S. (2009). The geometry of longsword and rapper sword locks. In C. S. Kaplan, & R. Sarhangi (Eds.), *Proceedings of Bridges Mathematics and Art 2009 Banff, AB: Mathematics, Music, Art, Architecture, Culture,* pp. 383–385. Retrieved February 2017 from http://bridgesmathart.org/2009/2009proceedings.pdf

Gerofsky, S. (2010). Mathematical learning and gesture: Character viewpoint and observer viewpoint in students' gestured graphs of functions. *Gesture, 10*(2), 321–343.

Gerofsky, S. (2011a). Seeing the graph vs. being the graph: Gesture, engagement and awareness in school mathematics. In G. Stam & M. Ishino (Eds.), *Integrating gestures: The interdisciplinary nature of gesture* (pp. 245–256). Amsterdam: John Benjamins.

Gerofsky, S. (2011b). Ancestral genres of mathematical graphs. *For the Learning of Mathematics, 31*(1), 14–19.

Gerofsky, S. (2013). Learning mathematics through dance. In G. Hart, & R. Sarhangi (Eds.), *Proceedings of bridges math and art,* Enschede, Netherlands, pp. 337–344. Retrieved February 2017 from http://archive.bridgesmathart.org/2013/bridges2013-337.pdf

Gerofsky, S. (2016). Approaches to embodied learning in mathematics. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 60–97). New York: Routledge.

Gerofsky, S. & Chase, S. (2013). *Sarah Chase: Dancing combinatorics, phases and tides.* Short film for Bridges Math and Art conference. Retrieved March 2017 from https://vimeo.com/68811119

Gerofsky, S., Gomez, F., Rappaport, D. & Toussaint, G. (2009, July). Spirograph patterns and circular representations of rhythm: Exploring number theory concepts through visual, tangible and audible representations. In C. S. Kaplan, & R. Sarhangi (Eds.), *Proceedings of Bridges Mathematics and Art 2009 Banff, AB: Mathematics, Music, Art, Architecture, Culture,* pp. 279–286. Retrieved February 2017 from http://bridgesmathart.org/2009/2009proceedings.pdf

Gerofsky, S., & Marchand, P. (2006). The body, the senses and mathematics learning. In P. Liljedahl (Ed.), *Proceedings of the Canadian Mathematics Education Study Group 2006, University of Calgary,* pp. 59–75. Retrieved February 2017 from http://www.cmesg.org/wp-content/uploads/2015/01/CMESG2006.pdf

Gerofsky, S., & Ricketts, K. (2014, April). The need for discernment on the qualities of bodily movement to inform mathematics education research. Paper presented as part of interactive symposium, Theorizing movement and movement-based methods in embodied mathematics learning. AERA (American Educational Research Association) annual meeting, Philadelphia.

Gerofsky, S., Savage, M., & MacLean, K. E. (2009, July). 'Being the graph': Using haptic and kinesthetic interfaces to engage students learning about functions. Paper presented at the 9th international conference on Technology in Mathematics Teaching (ICTMT '09), Metz, France.

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological Science, 20*(3), 267–272.

Hart, G. (2014). Mathematical impressions: Longsword dancing. *Simons Foundation Mathematical Impressions Films.* New York: Simons Foundation. Retrieved February 2017 from https://www.simonsfoundation.org/multimedia/mathematical-impressions-multimedia/mathematical-impressions-long-sword-dancing/

Healy, L., & Fernandes, S. H. A. A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics, 77*(2–3), 157–174.

Jackiw, N., & Sinclair, N. (2009). Sounds and pictures: Dynamism and dualism in dynamic geometry. *ZDM, 41*(4), 413–426.

Knoll, E., Landry, W., Taylor, T., Carreiro, P., & Gerofsky, S. (2015). The aesthetics of scale: Weaving mathematical understandings. In K. Delp, C. S. Kaplan, D. McKenna, & R. Sarhangi (Eds.), *Proceedings of bridges 2015: Mathematics, music, art, Architecture, culture* (pp. 533–540). Tesselations: Phoenix, AZ.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

McCague, C. (Dir.) & Gerofsky, S. (Prod.) (2013). *Steel Phoenix: The geometry of longsword locks.* In Hart, G., & Sarhangi, R. (Eds.), *Proceedings of Bridges Math and Art, Enschede, Netherlands: Mathematical art gallery.* Retrieved February 2017 from http://gallery.bridgesmathart.org/exhibitions/2013-bridges-conference-short-movie-festival/susan-gerofsky

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *The Journal of the Learning Sciences, 21*(2), 287–323.

Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education, 32, Part 1*, 85–108.

Nurnberger-Haag, J. (2015). How students' integer arithmetic learning depends on whether they walk a path or collect chips. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter*

of the International Group for the Psychology of Mathematics Education (pp. 165–172). East Lansing, MI: Michigan State University.
Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111–126.
Rains, J. R., Kelly, C. A., & Durham, R. L. (2008). The evolution of the importance of multisensory teaching techniques in elementary mathematics: Theory and practice. *Journal of Theory and Practice in Education, 4*(2), 239–252.
Ricketts, K. (2011). *The suitcase, the map, and the compass: An expedition into embodied poetic narrative and its application toward fostering optimal learning spaces.* Unpublished doctoral thesis, Simon Fraser University, Burnaby, BC, Canada. Retrieved February 2017 at http://summit.sfu.ca/item/11561
Sayers, N. (2013). Mathematical art gallery: Bicycle spirograph. In G. Hart, & R. Sarhangi (Eds.), *Proceedings of Bridges Math and Art, Enschede, Netherlands.* Retrieved February 2017 from <http://gallery.bridgesmathart.org/exhibitions/2013-bridges-conference/nicksayers>
Schaffer, K. (2014). Dancing deformations. In G. Greenfield, G. Hart, & R. Sarhangi (Eds.), *Proceedings of bridges math and art 2014: Mathematics, music, art, Architecture* (pp. 253–260). Tesselations: Phoenix, AZ.
Schaffer, K. (2015). A skeleton key for the Platonic solids. In K. Delp, C. S. Kaplan, D. McKenna, & R. Sarhangi (Eds.), *Proceedings of bridges math and art 2015: Mathematics, music, art, Architecture* (pp. 199–206). Tesselations: Phoenix, AZ.
Schaffer, K., Stern, E., & Kim, S. (2016). *Math dance with Dr. Schaffer and Mr. Stern: Whole body math and movement activities for the K-12 classroom*. San Francisco: CreateSpace.
Sinclair, N., & Pimm, D. (2014, July). Number's subtle touch: Expanding finger gnosis in the era of multi-touch technologies. In *Proceedings of the 38th conference of the International Group for the Psychology of Mathematics Education and the 36th conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 5, pp. 209–216).
Sinclair, N., & Tabaghi, S. G. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics, 74*(3), 223–240.
Soto-Johnson, H., Hancock, B., & Oehrtman, M. (2016). The interplay between mathematicians' conceptual and ideational mathematics about continuity of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education, 2*(3), 362–389.
Thom, J. S., & Roth, W. M. (2011). Radical embodiment and semiotics: Toward a theory of mathematics in the flesh. *Educational Studies in Mathematics, 77*(2–3), 267–284.
Whiteley, W. (1989). Rigidity and polarity II: Weaving lines and tensegrity frameworks. *Geometriae Dedicata, 30*(3), 255–279.
Wolfram Mathworld. (n.d.). *Combinatorics, permutations, necklaces*. Web page retrieved February 2017 from http://mathworld.wolfram.com/Necklace.html>(blunt)
Wright, T. (2001). Karen in motion: The role of physical enactment in developing an understanding of distance, time, and speed. *The Journal of Mathematical Behavior, 20*(2), 145–162.
Yoon, C., Thomas, M. O., & Dreyfus, T. (2011). Gestures and insight in advanced mathematical thinking. *International Journal of Mathematical Education in Science and Technology, 42*(7), 891–901.
Zebehazy, K. T., & Wilton, A. P. (2014). Straight from the source: Perceptions of students with visual impairments about graphic use. *Journal of Visual Impairment & Blindness, 108*(4), 275–286.

Chapter 13
Examining the Development of a Transdisciplinary Collaboration

Limin Jao, Melissa Proietti, and Marta Kobiela

Abstract The urban arts play a significant role in the lives of many of today's youth and may thus be a powerful vehicle for students to engage in and with mathematics. The Urban Arts Project (UAP) aims to incorporate the urban arts across all curricular subjects in one secondary school in Eastern Canada. A key feature of the project is to bring in artists to partner with school teachers in designing and delivering lessons that integrate urban arts. In this paper, we describe the case of one urban street artist and one mathematics teacher and the development of their collaboration to teach two interdisciplinary units of study. Although both the artist and teacher articulated initial hesitations to work across disciplinary boundaries and to collaborate, interview data suggest that by the end of the collaboration, both members had developed mutual respect, an important component in a successful collaboration. In the paper, we describe three factors that were important in supporting the development of this mutual respect: (a) opportunities for the artist to watch the teacher teach prior to the unit, (b) a pivotal moment where the artist's respect for the teacher was made explicit, and (c) an artist-teacher liaison who was able to help facilitate collaboration between the artist and teacher. Given the success of this transdisciplinary collaboration, we encourage (mathematics) educators to think more broadly about opportunities and options for collaboration. By extending beyond disciplinary boundaries, collaborators can learn and benefit from varied and diverse perspectives.

Keywords Arts integration • Collaboration • Community partnerships • Interdisciplinary teaching • Mathematics education • Urban arts

---

L. Jao (✉) • M. Proietti • M. Kobiela
Department of Integrated Studies in Education, McGill University,
3700 McTavish Street, Montreal, QC H3A 1Y2, Canada
e-mail: limin.jao@mcgill.ca; melissa.proietti@mail.mcgill.ca; marta.kobiela@mcgill.ca

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_13

In this chapter, we describe the case of one urban graffiti street artist and one mathematics teacher and the development of their collaboration to teach two transdisciplinary units of study. This collaboration is part of a larger project aimed at incorporating the urban arts into Central High School[1] located in a large city in Eastern Canada. Historically, the school struggles with low academic performance (less than half of the senior class graduated in 2014) and high student absenteeism rates. Many of the students require considerable pedagogic support as the majority of students have been identified with having learning disabilities. Central High School is the first recorded attempt at incorporating the urban arts across all curricular subjects, in order to attempt to increase student engagement. The decision to become an urban arts school was first taken by the school's administration when students were noticeably attending after-school programs which were focused around the urban arts, including a graffiti club, a rhyme writing and production program, and a street dance program. The move toward including the urban arts as a learning tool is being supported by a tripartite research effort called the Urban Arts Project (UAP)[2] which attempts to bridge the gaps between school, community, and university hoping to develop best practices for a sustainable school model.

The goal of the project is to draw upon students' interests in the urban arts within subject area courses to increase their motivation and engagement within their coursework. A key feature of the project is to bring in artists to partner with school teachers in designing and delivering lessons that integrate urban arts. By bringing in artists, teachers were able to draw upon their experiences to create projects more authentic to the practices and techniques of the urban arts. However, such an approach relies also heavily on developing a collaboration that bridges across disciplines and cultures to reach common goals and shared understandings. We aim to describe factors that appeared to support this unlikely collaboration to be successful. We first explain what urban arts entails and why the urban arts, and in particular graffiti, is a rich, yet understudied, context for transdisciplinary education. We then highlight what the research literature suggests about successful collaboration – aspects we drew upon within our analysis of the collaboration.

The Urban Arts

The urban arts include a range of aesthetic practices that are closely tied to, but also extend beyond, the aesthetic forms and values of hip-hop culture. These forms include rapping, DJing, breakdancing and other hip-hop dance forms, and graffiti but also more recent forms such as hip-hop theater, photography, journalism, and fashion (Chang,

[1] The names of the school, teacher, and urban artist have been replaced with pseudonyms.

[2] This research stems from the SSHRC-funded study, "The urban arts as tool for transforming a disadvantaged high-school: Building partnership synergy between school, university, and community artists" (SSHRC Partnership Development Grant: principal investigator Bronwen Low, co-investigators Mindy Carter, Elizabeth Wood, and Claudia Mitchell).

2007). Current research trends in hip-hop education show the positive potential outcomes with regard to students' participation and engagement in after-school activities as well as individual classrooms (Emdin, 2010; Hill, 2009; Low, 2011; Morrell & Duncan-Andrade, 2008). According to Emdin (2010), participation and engagement are identified with communication and cooperation (Duit & Treagust, 2003), values which are intrinsically linked to foundational principles of graffiti culture.

Although the field of hip-hop education has grown and become increasingly recognized as contributing to student engagement and learning as valid and sound in educational practice (Emdin, 2009; Hall & Diaz, 2007; Hill, 2009; Hill & Petchauer, 2013; Low, 2011; Morrell & Duncan-Andrade, 2008), studies of graffiti and street art as tools to reach students and build on their level of classroom- and school-wide engagement are practically nonexistent. While there has been research on hip-hop music successfully being used as a method of teaching, there is a minimal amount of research documenting the pedagogical potential of graffiti and street art culture as pedagogically sound tools and only a few programs which exist formally and incorporate these visual arts methodically. Yet the visual urban arts are just as viable as hip-hop music as options to reach students and to validate their knowledge and experience in a formal setting given the interest that many students express about the culture.

When graffiti first came to the public's attention, as documented in the American mass media in the 1970s and 1980s, the people participating in the culture were young, marginalized Americans who used their ingenuity, creativity, and talent to create a space for themselves and declare their presence in an undeniably bold manner. In a society where they had little or no expressive outlets available and were faced with grim futures of low-paying jobs, unemployment, or the probability of ending up involved in street gang life, graffiti became an outlet (Austin, 2001; Castleman, 1982; Ferrell, 1993; Powers, 1999). Graffiti gave these emerging artists and young activists the opportunity to become kings and queens of their own domain and claim or reclaim space without having the economic potential to buy it.

The graffiti writers who pioneered the movement did so at an enormous risk to themselves, including threat of legal action and physical harm, and long-term damages to health as spray paint is toxic and carcinogenic. Historically, graffiti culture valued the mentorship of younger writers with the older crew members; respect was something which was earned over time and certain rules of conduct were almost universally followed by those participating in the culture. The concept of ownership and belonging is pivotal in graffiti culture as the majority of writers belong to a crew and associate their work to it (Powers, 1999; Wimsat-Upksi, 1994). Respect within the graffiti community is given to those who get the most exposure, the ones who *get up* the most. Gaining recognition and notoriety, particularly from within the community of writers, is sought after; those who are the best artists will often gain the most visibility and have the longest careers. Historically speaking, before the days of increased accessibility through the Internet, the only way for younger writers to learn techniques and improve their skill was to be mentored by older, more experienced writers. This kind of mentorship and eventual collaboration in painting was a cornerstone in the development of graffiti culture and a way in which traditions and knowledge were passed down.

Graffiti culture has since changed and evolved under heavy media exposure and the notable support and financial backing from art collectors and marketing companies. However, at the root of what has developed into current graffiti and street art culture are the values and expectations upon which these cultures were founded such as respect, freedom of expression, empowerment, a sense of belonging and accomplishment, as well as recognition and pride. These values can be linked to attributes of positive, collaborative learning spaces including cooperative learning, mentorship, and community. It follows that collaboration parallels the values of the urban arts culture. Thus, a project bringing together urban artists and school teachers to foster transdisciplinary collaborations may yield positive potential outcomes.

Collaboration

Collaboration is commonly defined as a mutually beneficial relationship with members sharing responsibility to work toward a common goal (e.g., Chrislip & Larson, 1994; Friend & Cook, 1990). Teacher collaboration can have many benefits. These include teacher change (Bolam, McMahon, Stoll, Thomas, & Wallace, 2005; Clement & Vandenberghe, 2000), teacher motivation (Calderhead & Gates, 1993), improved school climate (Gable & Manning, 1997; Park, Oliver, Johnson, Graham, & Oppong, 2007), and student academic success (Goddard, Goddard, & Tschannen-Moran, 2007; Vescio, Ross, & Adams, 2008).

Collaboration extending beyond disciplinary boundaries may yield additional advantages. Effective cross-disciplinary collaborations occur when members are able to draw on each other's strengths to achieve a shared goal (Dooner, Mandzuk, & Clifton 2008). Research describes three forms of cross-disciplinary collaborations: multidisciplinary, interdisciplinary, and transdisciplinary. Multidisciplinary collaborations involve members working independently but across disciplinary boundaries (Mallon & Bunton 2005). By contrast, in interdisciplinary collaborations, members work jointly but from each of their respective disciplinary perspectives (Rosenfield 1992). Similar to interdisciplinary collaborations, members of a transdisciplinary collaboration also work jointly but use a shared conceptual framework that draws together discipline-specific theories, concepts, and approaches (Slatin, Galizzi, Melillo, Mawn, & Phase In Healthcare Team, 2004). The shared framework found in transdisciplinary collaborations provides a mutually constructed basis for collaboration drawing on the knowledge and perspectives of all members of the group. Thus, while still reaping the benefits of other cross-disciplinary collaborations (e.g., bringing together collaborators from different fields with varied experiences and expertise), transdisciplinary collaborations both inherently acknowledge the contributions of all members and develop a sense of positive interdependence.

While the benefits may be numerous and worthwhile, collaboration has its fair share of challenges as well. Welch (1998) describes four categories of challenges for collaboration: (1) conceptual, rigid definitions that members hold for their roles in the collaborative process; (2) pragmatic, logistical considerations including time and resources; (3) attitudinal, beliefs and expectations that members bring to the collaboration; and (4) professional, interpersonal and communication skills required to work with other members. For transdisciplinary collaborations, conceptual and attitudinal challenges may be particularly challenging as members may hold varied disciplinary norms for collaboration. Additionally, for many teachers who are used to autonomy, collaboration can be a struggle. Yet, breaking down the professional norms of isolation and collaborating with colleagues can yield many positive outcomes.

Trust and respect are necessary components of any strong collaboration (Bronstein, 2003; Marlow, Kyed, & Connors, 2005; Russell, 2002), and positive personal relationships are the first step toward a trusting collaboration (Jao & McDougall, 2016; Tschannen-Moran, 2001). Productive collaborations take time to develop (Marlow et al., 2005), yet this time is important as it provides an opportunity for the group to develop a shared history and culture (Selznick, 1992). When a member recognizes what is important to another member, a more trusting relationship is forged (Marlow et al., 2005). This is important because research has shown that a trusting school climate results in more positive student academic achievement (e.g., Goddard, Tschannen-Moran, & Hoy 2001; Uline, Miller, & Tschannen-Moran, 1998). Additionally, trust is essential for individuals to share their thoughts, feelings, or ideas (Tschannen-Moran, 2001). Collaboration requires an investment of time and energy as well as a willingness for collaborators to share responsibility and rewards and to give up a certain amount of one's autonomy. Thus, without a level of trust and respect, individuals tend not to take the risk and fully commit to a collaborative process (Mattessich & Monsey, 1992; McLaughlin & Talbert, 2006).

In addition to developing a sense of trust, time is required to develop a shared familiarity with the context. Researchers assert that a shared base understanding of the teaching context can focus collaborative efforts (Buysse, Sparkman, & Wesley, 2003; Wenger, 1998). Thus, in school-based transdisciplinary collaborations, it is important for members who are not teachers to develop a familiarity with the teaching context. In what follows, we highlight the types of challenges that the secondary mathematics teacher and artist experienced and how a foundation of trust and mutual respect contributed to the emergence of their collaboration.

The Context of the Transdisciplinary Collaboration

This chapter focuses on the collaboration between a secondary mathematics teacher, Anthony, and an urban street artist, Koopa. As a teacher at Central High School, Anthony was aware of the Urban Arts Project and chose to be an active member of the project. As a member of the local urban artist community, Koopa was approached (to which he readily agreed) to be part of the project. Anthony and Koopa collaborated on two units of study for the two grade 8 mathematics classes at the school: (1)

Geometry: Dilatations and similar figures and (2) Geometry: Regular polygons and surface area of three-dimensional objects. In the first unit, the students learned the targeted mathematics concept and saw its applications in art through mural making. For the culminating task for this unit, students drew and enlarged an image of their choice using a designated scale factor. Students demonstrated their mathematical knowledge by first calculating the dimensions of the enlarged image before creating their final piece of art. In the second unit, the students developed their fluency in measuring surface area of three-dimensional objects that they decorated. After seeing preconstructed models of the objects, the students were provided with the nets of the objects, which they first decorated and calculated the surface area, before constructing the three-dimensional form. Anthony and Koopa's collaboration spanned approximately 3 months. This included a week of planning prior to implementing the first unit of study and 3 weeks for each unit of study. There were approximately 5 weeks between the units of study comprising of school holidays, midyear exams, and some planning for the second unit of study.

We conducted artist and teacher interviews throughout the collaboration. Pre-interview were conducted prior to the commencement of the first unit of study to find out background information about Anthony and Koopa, any previous experiences with collaboration and/or the urban arts, and their expectations and goals for the collaboration. A mid-collaboration interview was conducted between the two units of study to have Anthony and Koopa reflect on the process thus far and set additional goals for the remainder of the collaboration. A third set of interviews was conducted at the end of the collaboration, serving as a final reflection of the collaboration.[3] To analyze the interviews, we identified comments made by Anthony and Koopa where they discussed the collaboration. Through an iterative process, we then categorized those comments into themes related to their perspectives on the collaboration. Finally, we corroborated our analysis of the interview with field notes of classroom observations and planning meetings.

The collaboration between Anthony and Koopa made an appropriate case for this study given that both were passionate about the potential to work on the project and about their professions; yet, as we describe later, they both entered the project with reservations about collaborating. Although this was Anthony's first year teaching at Central High School, he had already taught for 9 years. Initially certified to teach physical education, Anthony felt comfortable teaching secondary level mathematics having taken mathematics content courses as his electives during his teaching degree. Anthony started his teaching career at a school with similar student demographics as Central High School. Subsequently, Anthony taught in two alternative schools (the first for students with severe behavior difficulties) before returning back to traditional public school contexts. Prior to assuming his position at Central High School, he taught at an elementary school in an affluent community. He was keen to take a position at Central High School where he said, "I'm at home again" (A1). Having grown up in a community similar to that of Central High School,

[3] In the following sections, participant quotes are referred to by participant name and the interview number (e.g., A1: Anthony, pre-interview).

Anthony says that he can appreciate the challenges and experiences of students at Central High School. Additionally, although only in his first year at the school, Anthony already felt a connection to his students as he said that, "Every class that I teach has at least one younger sibling of a student I've taught in the past in the alternative schools" (A1). Anthony's dedication to his students at Central High School stemmed from his personal commitment to the profession. He explained, "I became a teacher to take people from the fringes of society, from the lower socioeconomic echelons of society and, through education, bring them closer to the middle" (A1). His drive to excel as a mathematics teacher is a result of the challenges that he faced as a mathematics student:

> I had a lot of struggles with math myself in early high school and late elementary school. And what used to frustrate me is that a lot of my math teachers (strong math students themselves) couldn't put themselves in the shoes of a person who doesn't get the material [snaps fingers] right away...there's a lot of empathy lacking. (A1)

Thus, Anthony was keen to engage in the Central High School community, where, historically, students have struggled academically and seemed to be disengaged from their academic experience.

Although Koopa had been a graffiti writer since he was a teenager, he had not made art his professional focus until just 5 years prior to the start of the collaboration. Koopa described his early experiences in the urban arts community as one of apprenticeship: "I was always like, the younger guy growing up with guys four or five years older than me that were really good (at graffiti). So they kind of apprenticed me in" (K1). After completing undergraduate degrees in English and Art, Koopa went into business. During his time owning a successful company, Koopa still painted. As a true artist, Koopa spoke to how art, although not his priority, was still an important part of his life: "You can't stop it. All of these years of painting on the side. Just doing it for myself. I do these really cool canvases and then nobody would see them. They'd just be in my house, in my room" (K1). Slightly removed from the graffiti scene, Koopa was unaware that younger artists whom he had mentored were now finding international recognition. After being told of one mentee's success, Koopa found himself both shocked with how much the graffiti world had changed and questioning his personal choices. "[H]e's making $40 000 a canvas? Man, why didn't I go for that when I had the chance? Why did I go into English?...I was still doing art and graffiti on the side, but I wasn't looking at the professional side of where things were changing" (K1). Koopa continued by describing how this changed the direction of his life by saying:

> When I saw that kid. Man, then I starting watching him a little bit...And I was looking at him like, "Man, I made the wrong decision. I shouldn't have listened to my dad and went with my gut." I'm 34 or something at the time and I'm like, if I don't try now, I'm going to be one of those guys who has lots of money sitting around with some boring average life doing my job, going on vacation twice a year and cycle, rinse, repeat, over and over and over and over. And I just didn't want that anymore. I didn't want all the stuff I had. I had a big TV and $5000 leather couch imported from Italy. I didn't want any of that anymore. It didn't mean anything. I was like, if I don't try it, I'm going to be that old man regretting it. (K1)

Afterward, Koopa sold his business and decided to turn his professional focus toward his art. In addition to staying true to his roots in street graffiti culture, Koopa has found commercial success creating art for major international corporations. Koopa's artwork has been (and continues to be) seen in galleries and outdoor spaces across Canada and the United States.

The Development of the Transdisciplinary Collaboration

Initially, Koopa and Anthony showed openness to the project but also articulated hesitations to work across disciplinary boundaries and to collaborate. These hesitations created conceptual and attitudinal challenges (Welch, 1998) to building a successful collaboration. As described previously, one important aspect for overcoming such challenges and creating productive collaboration is to have mutual respect. Our analysis of the interview data showed that by the end of the collaboration, Anthony and Koopa had developed mutual respect for one another, and this mutual respect appeared to serve an important role in facilitating their transdisciplinary collaboration. We found that three factors were important in supporting the development of mutual respect: (a) opportunities for the artist to watch the teacher teach prior to the unit, (b) a pivotal moment where the artist's respect for the teacher was made explicit, and (c) an artist-teacher liaison who was able to help facilitate collaboration between the artist and teacher. In what follows, we first describe Koopa and Anthony's initial impressions and then illustrate how these three features helped to shift those initial impressions.

Initial Impressions

Initial Impressions About the Urban Arts and the Urban Arts Project

Before the start of the instructional unit, both Anthony and Koopa believed that the urban arts could be a meaningful way to increase student engagement in the mathematics classroom. For Anthony, he saw that his students had an interest in the urban arts. In addition to engaging with the urban arts outside of the school, students had enthusiastically participated in various extracurricular activities at the school within the urban arts. Thus, Anthony felt that there would be positive outcomes if he were able to bring the urban arts into his classroom, including that students would "enjoy math more" (A1). For Koopa, his beliefs came from his personal experiences. Drawing from his own negative experiences as a mathematics student, Koopa hypothesized that he would have enjoyed mathematics more "if I had some cool street artist from like France come in like, and I'm like 14 years old and he's like look math isn't all just numbers and boring it's actually a little fun sometimes" (K1).

Although neither Anthony nor Koopa part of the inception of the UAP, both were committed to it. Anthony described that the project was intriguing and that he was willing to engage in integrating the urban arts into his teaching. He made this clear even in his interview for the position saying that he told the principal that he was "more than open" (A1) to the project. Koopa saw the UAP as an opportunity to give back to the community and to help at-risk youth. In addition, Koopa explained that he was committed to the project because of the respect that he had for Melissa, the liaison. In particular, he was inspired, "just seeing the amount of work and time that she puts in" (K1).

Initial Hesitations to Participate in the Urban Arts Project

Hesitations About Working in a New Domain Although both Anthony and Koopa were committed to the project, each entered with reservations. They both entered the collaboration confident within their own domain (mathematics teaching and the urban arts, respectively) but uncertain about their potential to contribute to the other.

For Anthony, his hesitation with the urban arts meant that he was uncertain whether he could meaningfully integrate it into his mathematics teaching. He described that his challenge would be "to make sure that I make [the integration] relevant, and meaningful and that I connect the two (mathematics and the urban arts). So that would be the challenge, as I said, to make it kind of smooth and streamlined" (A1). He related this to an overall teaching goal: "I want them to see that various aspects of their life are not as siloed as they like to think they are…So math isn't over here and hip hop isn't over there…There are elements of math in hip hop" (A1). Anthony's uncertainty of how to meaningfully integrate was perhaps linked to his lack of experience with integrating mathematics and art. When asked about previous experiences combining the urban arts into his teaching, Anthony responded that he had done "very little" (A1) integration of the two. Moreover, he admitted that previous experiences were relatively superficial, stating that his attempts were "in hackneyed, played out ways…sort of paying it lip service but… not really integrating it" (A1). To illustrate, he gave an example of a word problem that used an art context: "…So and so buys a certain number of hip hop records and his uncle gives him seven, then he buys three more every month and how many does he have at the end of a certain number of months" (A1). While Anthony believed that this approach was "better than doing nothing" (A1), he admitted that he struggled to find possibilities for meaningful integration of the urban arts and mathematics.

While enthusiastic to lend his support to the project, Koopa also entered unsure of his role in a mathematics class. As previously mentioned, Koopa had negative experiences as a mathematics student and described that his lack of mathematics content knowledge might hinder his ability to meaningfully contribute to the project: "[I'm] terrible at math so when [Melissa] was telling me math class I was like I don't know man if I'm even gonna be any good, to do the kids any good" (K1). Only considering his strengths within disciplinary boundaries, Koopa continued to explain that he felt that he would only be able to contribute as "the artist" and was skeptical of his role within a mathematics context saying, "Sure, like I can draw but that's about as far as I am willing to pretend" (K1). Koopa perceived that disciplines were discrete describing himself to be in a "different world" than mathematics.

Although Koopa felt disconnected to mathematics, he was able to identify mathematics within the artistic process of mural making:

> You like I know the grid system and that stuff. The grid system is like if I have ten boxes you know ten squares or whatever and I draw a circle on there I get another grid on a larger scale like say on the wall, and that's a hundred. So I have to multiply that image by ten and match every box where the image is, like each line, so if there is like three lines in the corner and it's one box up there I know that's the ten boxes in the corner and that line there. (K1)

In his description, Koopa highlighted the use of a grid system and scaling to produce a larger image. Moreover, Koopa recognized the mathematics in the unit that the students would be engaged in, explaining "...it's drawing. It's ratios. It's proportions" (K1). Although Koopa acknowledged these topics, "It is math" (K1), he retained his perspective of being from a "different world" stating "I don't use it as math" (K1).

Hesitations to Collaborate Anthony was also apprehensive of the collaborative component of the UAP. He attributed this to "a very negative experience" (A1) that he had collaborating with another teacher early in his teaching career. Although the intent was to have Anthony and his colleague team teach lessons, Anthony believed that the other teacher lacked the necessary content knowledge to teach the material. Anthony found himself teaching his colleague the material prior to each lesson and grew to resent the situation and his colleague. Additionally, differing teaching styles hindered the collaboration. Anthony said of the situation, "We had to go back to the drawing board a number of times. It went from co-teaching to, 'Okay, you know what, I'll handle the academics and you handle the classroom management'" (A1). Rather than co-teaching and sharing responsibilities in the classroom, the two teachers took on separate roles and limited their interaction. Anthony also described that the relationship lacked "trust." This was especially evident after Anthony disagreed with his colleague's classroom management style. Anthony said of this, "I was afraid for the children" (A1). Anthony also felt that this collaboration "had no impact on (students') academic results" (A1). Anthony summarized this experience as such: "I didn't leave (the experience) with a very comfortable feeling" (A1). With seemingly no positive outcomes from this collaboration, it is understandable that Anthony was skeptical of collaboration as part of the UAP.

In addition to this, Anthony voiced his concerns about the time necessary for collaboration. Anthony elaborated on this by stating **"**That's probably why a lot of us don't collaborate with each other more often. It's very difficult to find time in the day to do it" (A1). Additionally, Anthony acknowledged that over time, teachers tend to become isolated after spending many years honing their preferred teaching methods. Perhaps also partly reflecting on his previous negative experience with collaboration, Anthony admitted to having some resistance to change and being able to adapt his teaching approach to complement another individual's approach: "A lot of us have been teaching for long enough that we have things, kind of, set up in a certain way where it's very comfortable and…for better or worse we're comfortable doing what we're doing" (A1). To this end, Anthony believed that teacher collaboration was especially challenging as ego can play a factor in hindering these relationships:

> There's a lot of pride that has to be put aside. And each teacher is used to being the authority in their classroom, each teacher is used to having all of the answers in their classroom, each teacher is used to always being right and being able to punish the person who is not right. And so these two, kind of, working together … it can be difficult, right? (A1)

Yet, when asked if he believed that collaborating with another teacher would be different than collaborating with an artist, Anthony said, "Yes…If I'm collaborating with an artist…it's very clear that I'm out of my depth and that I need them" (A1). Thus, while Anthony discussed some resistance to the process and adapting his teaching approach, given the context in which he felt less comfortable (urban arts), Anthony acknowledged that the collaborating artist had skills and expertise that he himself was lacking necessary for the integration to be successful. He believed that the artist would also have similar feelings about an artist-artist collaboration versus an artist-teacher collaboration. Anthony shared that he felt that an artist would appreciate the pedagogical expertise that the teacher would bring to the collaboration.

Similarly, Koopa believed that he would bring a different skill set to the collaboration stating that his "approach will be different" (K1). He attributed this difference to not being a mathematics teacher. Returning back to his sentiment that mathematics and the arts are different, Koopa said: "One of the things that I don't like about math is that it's so exact and there is no freedom to it" (K1). He continued by describing how as a result he believed that Anthony, as a mathematics teacher, would probably follow routine procedures in his instructional approach. Koopa spoke of his prior perceptions of mathematics teachers in general: "[A math teacher] has a set format. He has this planned out. Like, he probably taught that class ten times already, ten different years, ten different schools, ten different venues. You know and like his formula works every time" (K1). Koopa thus believed that his approach (as an artist) would be different than Anthony's (as a mathematics teacher). Additionally, with the traditional impression that classroom teachers are authoritative figures, Koopa claimed that he would be able to connect with students saying "I'm not as authoritative as he is you know?…But when you get down to the level of a 13 year old they respond to you a little better because like, I can let the fart jokes slide" (K1).

Keys to the Success of the Collaboration: Developing Mutual Respect

Developing Impressions That Countered Past Experiences

Prior to meeting Anthony, Koopa expected all mathematics teachers to be similar to his own teachers, which he described as being “some boring guy up there explaining (mathematics) to me” (K1). Thus, he was pleasantly surprised after observing Anthony teach for the first time. Koopa described his initial impression of Anthony: “This math teacher seems really cool and hip and fun. He seems likeable to the kids. But he's a rare case. Somebody's been working 20 years in the school system, he'd be burnt out. They don't care” (K1). With initial hesitations related to his negative impressions of mathematics, mathematics classrooms, and mathematics teachers, Koopa was hesitant about how much he would enjoy working in a mathematics context. Yet, his positive impression of Anthony shows that Koopa may have had a more optimistic attitude about this collaboration even saying “[Anthony’s] so energetic and so fun, you know I want to take his class…Like this guy made me be like, ‘Shit maybe I want to take math again’” (K1).

In addition to watching Anthony teach, we set up an initial meeting for Koopa and Anthony to get to know each other as individuals. This proved to be fruitful as Koopa described his positive impressions of Anthony:

> [J]ust hanging out with that guy was like you could see how like he could keep a class’ attention…You know but like I’m pretty sure he has a good success rate with his students that guy like wherever he goes. Because he has a good attitude. (K1)

Koopa’s comments show the positive impact of members spending time with one another early in the collaboration to get to know each other. We assert that these opportunities were an important component leading to the success of the collaboration of Anthony and Koopa. In addition to becoming comfortable in each other’s presence and approach to working with students, it was critical for both to value the students and each other’s expertise and commitment to the project. Through this, Anthony and Koopa developed the mutual respect necessary to work together as true partners. Next, we describe how Anthony and Koopa further developed this mutual respect through a pivotal moment that cemented their collaboration.

Making Respect Explicit: A Pivotal Moment

A key moment in developing Anthony and Koopa’s mutual respect was when Koopa publically vocalized his respect for Anthony during class time. After two classes of sparse successes in engaging the students to work on their projects, Koopa had developed a heightened respect and sense of appreciation for not only the teaching profession but for Anthony in particular, as his energy and humor never seemed to dim in the face of students who resisted. In an unprovoked response to speaking to a classroom of students, at least half of whom had heads on desks, Koopa became visibly frustrated and addressed the students as a whole, commenting on their lack of understanding of how lucky they were that their teacher cared so much about their success. Addressing the class, he stated:

> When I was growing up... you have a teacher here who cares, my teachers never cared. This guy cares take that to heart. Anything you learn here is to help you, it's not for him he knows this, it's not for me - I paint giant buildings, but it's up to you. Take what you can get out of this. (*paraphrased from field notes*)

This moment was pivotal for the working collaboration and relationship that Anthony and Koopa shared and allowed an increased awareness for the importance of having a mutually existing respect between two individuals who were both similar and different in many ways. From Koopa's perspective, the moment arose because he had become increasingly frustrated with how the students were acting in Anthony's class. Students were sleeping and talking during class, and for Koopa, this was unacceptable. As he described in his second interview, "Kids were like just really getting unruly and I'd passed the point of like being understanding" (K2). Koopa felt that he needed to support Anthony because no one else was. He reflected, "I was just fed up, you know. Like Anthony's still like laughing it up and like making jokes and dealing with it but nobody's sticking up for him, you know" (K2). Koopa's frustration with the student's treatment of their teacher was notably different from the way in which he had first envisioned himself and his role within this classroom context. In the pre-interview, Koopa described the way he imagined reacting to students who were disengaged, with disengagement himself:

> You know if he doesn't wanna learn, I'm not gonna make him learn. Like I'm ok with you sittin' there just don't bother anybody else. You don't wanna pay attention? Don't. I'll encourage you not to pay attention. If you're not gonna get anything out of this, and you think your time is better off reading Of Mice and Men, I'll let you sit in class and completely ignore me. (K1)

For Anthony, this moment was important because it was in this moment that he saw that Koopa had developed an appreciation for him. In a very public way, Koopa showed his support and mutual respect by defending Anthony. As Anthony reflected:

> I know Koopa was surprised. He seemed to be someone who was very turned off by the idea of even being in the math classroom in the first place. I kept a lot of things to myself, but I was very unsure on how this was actually going to play out. And that's a good thing. And he has a lot more respect for what I do for a living. And has told that. He blasted them (the students). "You don't understand what most math teachers are like. You don't understand what I had when I was your age…" It was awesome. I was like, "I didn't pay him to say that. Thank you though." It was totally off the cuff too, because he was actually getting annoyed. (A2)

Anthony's reaction to that moment was notable given his understanding of how Koopa initially perceived him.

As previously discussed, respect is an integral component of the graffiti/urban arts culture/community. As a member of that community, Koopa deeply valued the notion of respect, and this came out in both interviews and classroom interactions. For example, during an interview, in recounting his frustration toward the students' behavior toward Anthony, Koopa said:

> So it was like one or two times like I sort of like told everybody all right, be quiet right now, I've got to tell you something, you know. And like this guy (Anthony) right here cares about you. For you guys to be sitting in class talking **is totally disrespectful.** Whatever term you want to use that to relate to but dis me on the street, **disrespectful**, the same thing. **You're being disrespectful** to this guy and you're going to learn in life and a lot of you probably even know this now that it's really hard to find people that care about you in life and this guy does so give him your attention, all right? And I'm not just saying that because I'm here. I'm like I came from New York to do this for you too, you know. Like think this is a cakewalk for me? I'd rather be home with my family doing life, you know, but I'm here because I know this is important. So like please, just this guy cares. **Give him some respect** because you're not going to find people like this, you know *(bold added for emphasis).* (K2)

As evidenced by Koopa defending Anthony to the students and voicing his appreciation for the effort put forth by Anthony in his teaching, it is clear that Koopa respected Anthony. Being core to Koopa's perspective, this respect helped to cement Koopa's buy-in to the collaboration.

The relationship was able to grow from Koopa's respect for Anthony's hard work and dedication to his students, which tipped the balance in his proclivity to identify with the disengaged students who did not care about mathematics. This also enabled the development of a united teaching front between Anthony and Koopa, both playing to each other's strengths and being able to respond the strengths and interests of the students' learning.

*Facilitating Collaboration: The Role of the Liaison (*Written from the Perspective of the Liaison*)*

The transdisciplinary collaboration of Anthony and Koopa brought together two individuals from two disciplines (mathematics education and the urban arts, respectively). A liaison, Melissa, proved to be critical in supporting and mitigate the challenges of this transdisciplinary collaboration, thereby leading to its success. In this section, we present the experiences and attributes that positively contribute to Melissa's role as liaison and the considerations and challenges that she faced in her role. We have chosen to present this section written from Melissa's perspective (thus, written in first person) to provide a personal narrative to highlight the significant and complex role played by the liaison to support the success of the transdisciplinary collaboration of Anthony and Koopa.

***

What do mathematics and graffiti have in common? Why are we even asking the question? These were questions I (Melissa – the liaison) had been asking myself since the first day we introduced graffiti writer turned street artist Koopa to secondary mathematics teacher Anthony. Being a liaison between the artist and the teacher meant that a certain amount of knowledge and level of comfort was required to be able to help move the project forward. In addition to having facilitated graffiti projects for over 10 years, I have also been offering a graffiti-based program at the school. Thus, I felt as though I had the necessary background knowledge to understand the needs of the students as well as the timelines required for completing art projects.

Selecting an artist for a mathematics class was the first challenge, given that the students at our school typically struggle a great deal with mathematics, and it has represented a serious barrier for student success. The most success experienced at the school has been when the focus was on the students. The artist needed to be someone who could not only speak from the experience of having accomplished his or her professional art practices using mathematics concepts but someone who the students would relate to. As an artist coming from a graffiti background, Koopa had not only faced similar challenges to our students but also could understand the value of mentorship and of encouraging youth to gain the confidence and voice found in practicing art freely. Koopa had also been chosen for this project because of his past success in relating to the students he had worked with in the school. His manner of speaking with the students, informal but still clearly in control, and his never-ending stories seemed to pique the interest of the students in a previous large-scale mural project he had completed a year earlier with a group of students in an art class. Moreover, I knew we needed to have buy-in from the artist and teacher for what we were attempting. Both Koopa and Anthony agreed that incorporating art into a mathematics class would probably help increase student engagement and general interest.

However, what I had not carefully considered in selecting the artist was the relationship they would have to the subject matter and to the teacher. After the first interactions, I began to worry that Koopa might not have respect for the teaching profession and that his past negative experiences with school mathematics would outweigh his desire and ability to help the students succeed. Respect, particularly as documented in the history of the urban arts, is something earned through honest commitment to one's practice. The negative experiences of Koopa's past confirmed the truth of the old saying "those who can't do teach." Koopa asserted he was there to make mathematics interesting, a self-declared hater of mathematics, having only negative memories of his high school mathematics teachers in particular; he and Anthony thus made an unlikely pair.

After nearly a decade of experience working through graffiti in different school and community projects, one of the most important factors for success has been the matching of artists with the participants. Younger children in elementary school need a different kind of interaction than high school students or senior citizens; interests and life experiences always need to be considered and respected before any kind of collaboration can ensue. In my experience, what is often prevalent in graffiti writers' attitudes about school is that their past has been so negative they do not want to have the one thing they have had positive experiences with (graffiti) be associated with or available through a school system which is negative, oppressive, and domineering. Koopa was no exception; however he was able to see the bigger picture in what we were trying to accomplish through this project.

An important factor in this situation is that I have been successful in selecting and integrating artists into school settings, which are beneficial for both the students and the adults involved. Koopa echoed this sentiment in the pre-interview before the project began:

> More schools need to be like that but I think what's different, I mean besides the fact that she (Melissa) chose me to teach it, it's more like the people you bring. They have to be effective. Cuz you can't just grab any old street artist and throw 'em in this environment and think that's gonna work. It's not you know? She's, Melissa's really selective of who she brings in, she's made great choices you know? (K1)

Having put full confidence in Koopa to try something so different, I was nervous about the curricular material given that I have little understanding of most mathematics concepts myself. Trying to understand how arts could be integrated into this unit, I also took some time to consult with a mathematics education PhD colleague of mine, furthering the importance of collaboration in this project. The phrase "It takes a village" started to feel very pertinent as this kind of project took buy-in from a cross section of adults before the kids were even introduced into the equation. After understanding the concept of dilations, I realized that ratios is something which graffiti writers need to understand well, given they sketch endlessly in their black books before putting up their pieces large scale. Once we agreed on the art project, and knowing how long art projects generally take to advance, we settled on an alphabet which the students would use and be able to reproduce themselves, increasing their ownership over the letters and putting the emphasis on their ability to scale the measurements properly.

Once the content was agreed upon, the first class happened, and I became aware of how foreign the traditional classroom was for Koopa. The circumstances were quite different from his prior experience at the school, which involved taking a selected group of five students out of their art class for 2 months time to work on developing and producing a mural (Proietti, 2015). For the artist-teacher collaboration with Anthony, the two were going to be working with the entire class together every day for the entire unit, which meant classroom management and concept learning were now the focus, differing from the mural project described earlier, in which the art process/product was the objective.

There were definite growing pains in the first class, going from associating with the students to now being the teacher. Koopa was unprepared for how the difference would affect him; he dressed differently, read out of the textbook for most of the class, and had less jokes and casual remarks than usual. Given the importance of the role of the mentor and understanding how important personal relationships with our students are, I thought it best to check in with Koopa as soon as the class was over. In a little informal debrief session on a bench in the hallway, Koopa was visibly flustered and stated, "That was rough." We talked about what was difficult and why. He expressed his level of discomfort being in front of the class, at which point we discussed how he could fulfill his teaching role while still staying true to his identity, which the students seemed to connect better with. It seemed like, along with every other aspects of this project, the knowledge sharing, mentorship, and collaboration between us as colleagues were what truly enabled the actual curriculum delivery possible. Koopa reflected in an interview midway through the project about how the development of the project was happening.

> So Melissa thought it would be good to like show them a little bit of my website once a day and show them how things relate to what we're doing. So you know, like the first time that we put that into theory I brought in a little sketch of a turtle that I did on a building that I drew and I drew the grid system on it and then I pulled up my website the actual two storey building I painted it on and I could show them from here to here, how did I do that. By using that formula we learned this week, you know. So just this like few classes that we did that gave them a better grasp of like this is not a waste of my time, you know. (K2)

Conclusion

The case of Anthony and Koopa illustrates the positive potential of extending beyond disciplinary boundaries to form a transdisciplinary collaboration. With the support of Melissa (a liaison), Anthony (a mathematics educator) and Koopa (an urban street artist) came together in the Urban Arts Project with the common goal to support generally disengaged and academically struggling students at Central High School. Grounded in trust and mutual respect, each member allowed the other to bring in their personal and disciplinary approaches to the collaboration. Through their joint work, Anthony and Koopa developed a shared conceptual framework (Slaton et al., 2004) that consisted of (a) goals for supporting students' success, both artistically and mathematically, (b) a pedagogical approach that drew together wisdom about street art (e.g., lettering, work ethic, etc.) and knowledge of geometric procedures, and (c) an approach to co-teaching that drew upon their disciplinary knowledge. It is this shared framework that identifies Anthony and Koopa's transdisciplinary collaboration (Slatin et al., 2004).

Several factors suggested that this transdisciplinary collaboration was successful. The increased respect, as evidenced by the interviews, was an important indication that Koopa and Anthony valued the collaboration. Moreover, students seemed to enjoy the interdisciplinary units and attendance in class increased. Anthony and Koopa noted that there was a noticeable increase in student engagement and autonomy. Mathematics learning seemed to be enhanced as there was an improvement in students' mathematics achievement, as evidenced by their performance on summative assignments.

We do not naively assume that any transdisciplinary collaboration will be successful. As evidenced by early comments from both Anthony and Koopa, members may enter the collaboration with assumptions and hesitations developed from previous experiences. For the collaboration of Anthony and Koopa, we assert that mutual respect was a critical component to support the success of their collaboration. Bringing previous negative experiences as a mathematics student, Koopa had doubts about working with a mathematics teacher. Coming from the urban arts culture, where respect is of utmost importance, Koopa's early lack of respect for mathematics teachers hindered his full commitment to the project. Yet, an opportunity to see Anthony teach prior to the interdisciplinary unit showed Koopa that this mathematics teacher was animated, engaged, and dedicated to his students. Koopa respected Anthony for his approach and continued on with the collaboration with a new found commitment. However, this success was also due in large part to the work of the liaison in selecting an artist that would pair well with the teacher and students. Although not intentional, the liaison selected an artist whose personality complemented the teacher and whose previous hated mathematics experiences differed significantly from the teaching approach of the teacher.

For Anthony, his skepticism of the benefits of a collaboration (resulting from a previous negative collaborative experience) made it difficult for him to trust Koopa in the early phases of the collaborative process. Through the pivotal moment, in recognizing Koopa's commitment to the project and respect for Anthony as a teacher, Anthony cemented his trust in his partner, and the collaboration could begin to flourish (Mattessich & Monsey, 1992; McLaughlin & Talbert, 2006). This pivotal moment and the effect it had on the collaboration highlight the power of collaborations that happen authentically. We acknowledge that this pivotal moment occurred organically and in no way did we prompt the events to unfold. Yet, we humbly suggest that the liaison's (Melissa's) thoughtful selection of a passionate and committed artist encouraged a respectful collaboration to emerge. We also assert that this was a moment of true serendipity and may very well have had the profound effect on both Anthony and Koopa due to the spontaneity of the exchange. At a time when both members were still tentatively moving forward with the collaboration, the sincere emotion of the moment spoke volumes and forged a relationship with trust (Tschannen-Moran, 2001). Additionally, this moment also suggests the importance of having spaces where partners can articulate their respect for one another.

As Anthony and Koopa's collaboration occurred within a classroom context, the students also played a role in the collaboration. In the pivotal moment, while Koopa's remarks had a profound impact on Anthony, the students in the classroom were in fact Koopa's intended audience. Pedagogically, it seemed that Koopa's focus was to set expectations for students' participation, both showing his role in the classroom and explicitly articulating his expectations and norms for participation to all stakeholders (both his collaborator and the students). We encourage teachers and collaborators to make clear to students their expectations for how they should interact with each partner resulting in a common understanding for all.

Happening over a relatively short timespan, this transdisciplinary collaboration faced a challenge of not having ample time to forge a comfort with each other and the context (Marlow et al., 2005). In addition to playing a key role in selecting the artist, through purposeful interventions, the liaison provided suggestions for how to shape the roles of the teacher and artist to increase their comfort when co-teaching (e.g., by suggesting that Koopa show a bit of his artwork at the start of each class). The liaison's expertise with the culture of graffiti and with the culture of the school helped her navigate how she supported the artist and teacher. However, in addition, drawing upon additional supports (such as her PhD mathematics education colleague) was important for the liaison to further her understanding in order to help support the planning between the artist and teacher. Our study not only shows the importance of a liaison in facilitating transdisciplinary collaboration but also considerations for how the liaison approaches his or her role.

While collaboration may have its challenges, its potential to yield positive outcomes cannot be ignored. As such, collaborative models for professional development are becoming more and more common in education contexts, for example, peer coaching (Hargreaves & Dawe, 1990; Jao, 2013; Showers & Joyce, 1996), co-teaching (Cook & Friend, 1995; Friend, Cook, Hurley-Chamberlain, & Shamberger, 2010), and professional learning communities (DuFour & Eaker, 2005; Stoll, Bolam, McMahon, Wallace, & Thomas, 2006). In these initiatives, teachers typically collaborate with other members of the education community (e.g., principals, consultants, education researchers, other teachers). In contrast, as part of the Urban Arts Project, teachers were partnered with members of the urban arts community to integrate the urban arts into their teaching. Given the success of this transdisciplinary collaboration, we encourage (mathematics) educators to think more broadly about opportunities and options for collaboration. By extending beyond disciplinary boundaries, collaborators can learn and benefit from varied and diverse perspectives.

References

Austin, J. (2001). *Taking the train: How graffiti art became an urban crisis in New York City*. New York: Columbia University Press.

Bolam, R., McMahon, A., Stoll, L., Thomas, S., & Wallace, M. (2005). *Creating and sustaining professional learning communities*, *Research Report Number 637*. London: General Teaching Council for England, Department for Education and Skills.

Bronstein, L. R. (2003). A model for interdisciplinary collaboration. *Social Work, 48*(3), 297–306.

Buysse, V., Sparkman, K. L., & Wesley, P. W. (2003). Communities of practice: Connecting what we know with what we do. *Exceptional children, 69*(3), 263–277.

Calderhead, J., & Gates, P. (1993). Introduction. In J. Calderhead & P. Gates (Eds.), *Conceptualizing reflection in teacher development* (pp. 1–10). London: Falmer.

Castleman, C. (1982). *Getting up: Subway graffiti in New York*. London: MIT Press.

Chang, J. (2007). *Can't stop, won't stop : A history of the hip-hop generation* (4th ed.). London: Ebury Press.

Chrislip, D. D., & Larson, C. E. (1994). *Collaborative leadership: How citizens and civic leaders can make a difference*. San Francisco: Jossey-Bass.

Clement, M., & Vandenberghe, R. (2000). Teachers' professional development: A solitary or collegial (ad)venture? *Teaching and Teacher Education, 16*(1), 81–101.

Cook, L., & Friend, M. (1995). Co-teaching: Guidelines for creating effective practices. *Focus on Exceptional Children, 28*(3), 1–16.

Dooner, A. M., Mandzuk, D., & Clifton, R. A. (2008). Stages of collaboration and the realities of professional learning communities. *Teaching and Teacher Education, 24*(3), 564–574.

DuFour, R., & Eaker, R. (2005). *Professional learning communities at work: Best practices for enhancing students' achievement*. Bloomington, IN: Solution Tree Press.

Duit, R., & Treagust, D. F. (2003). Conceptual change: A powerful framework for improving science teaching and learning. *International Journal of Science Education, 25*(6), 671–688.

Emdin, C. (2009). Rethinking student participation: A model from hip-hop and urban science education. *Edge, 5*(1), 6–18.

Emdin, C. (2010). Affiliation and alienation: Hip-hop, rap, and urban science education. *Journal of Curriculum Studies, 42*(1), 1–25.

Ferrell, J. (1993). *Crimes of style: Urban graffiti and the politics of criminality*. New York: Garland Publishing.

Friend, M., & Cook, L. (1990). Collaboration as a predictor for success in school reform. *Journal of Educational and Psychological Consultation, 1*(1), 69–86.

Friend, M., Cook, L., Hurley-Chamberlain, D., & Shamberger, C. (2010). Co-teaching: An illustration of the complexity of collaboration in special education. *Journal of Educational and Psychological Consultation, 20*(1), 9–27.

Gable, R. A., & Manning, M. L. (1997). The role of teacher collaboration in school reform. *Childhood Education, 73*(4), 219–223.

Goddard, R. D., Tschannen-Moran, M., & Hoy, W. K. (2001). Teacher trust in students and parents: A multilevel examination of the distribution and effects of teacher trust in urban elementary schools. *Elementary School Journal, 102*(1), 3–17.

Goddard, Y. L., Goddard, R. D., & Tschannen-Moran, M. (2007). A theoretical and empirical investigation of teacher collaboration for school improvement and student achievement in public elementary schools. *Teachers College Record, 109*(4), 877–896.

Hall, M. R., & Diaz, M. (2007). *The hip hop education guidebook: Volume 1*. New York: Hip Hop Association.

Hargreaves, A., & Dawe, R. (1990). Paths of professional development: Contrived collegiality, collaborative culture, and the case of peer coaching. *Teaching and Teacher Education, 6*(3), 227–241.

Hill, M., & Petchauer, E. (2013). *Schooling hip-hop: Expanding hip-hop based education across the curriculum*. New York: Teachers College Press.

Hill, M. L. (2009). *Beats, rhymes and classroom life: Hip-hop pedagogy and the politics of identity*. New York: Teachers College Press.

Jao, L. (2013). Peer coaching as a model for professional development in the elementary mathematics context: Challenges, needs and rewards. *Policy Futures in Education, 11*(3), 290–297.

Jao, L., & McDougall, D. (2016). Moving beyond the barriers: Supporting meaningful teacher collaboration to improve secondary school mathematics. *Teacher Development, 20*(4), 557–573.

Low, B. E. (2011). *Slam school: Learning through conflict in the hip-hop and spoken word classroom*. Stanford, CA: Stanford University Press.

Mallon, W. T., & Bunton, S. A. (2005). The functions of centers and institutes in academic biomedical research. *Analysis in Brief, 5*(1), 1–2.

Marlow, M. P., Kyed, S., & Connors, S. (2005). Collegiality, collaboration and kuleana: Complexity in a professional development school. *Education, 125*(4), 557.

Mattessich, P. W., & Monsey, B. R. (1992). *Collaboration: What makes it work?* St Paul, MN: Amherst H. Wilder Foundation.

McLaughlin, M. W., & Talbert, J. E. (2006). *Building school-based teacher learning communities: Professional strategies to improve student achievement*. New York: Teachers College Press.

Morrell, E., & Duncan-Andrade, J. (2008). *The art of critical pedagogy: Possibilities for moving from theory to practice in urban schools*. New York: Peter Lang Publishing.

Park, S., Oliver, J. S., Johnson, T. S., Graham, P., & Oppong, N. K. (2007). Colleagues' roles in the professional development of teachers: Results from a research study of National Board certification. *Teaching and Teacher Education, 23*(4), 368–389.

Powers, S. (1999). *The art of getting over: Graffiti at the millennium*. New York: St Martin's Press.

Proietti, M. (2015). The teaching artist as cultural broker. In D. Garbett & A. Ovens (Eds.), *Teaching for tomorrow today* (pp. 204–212). Auckland, NZ: Edify Ltd..

Rosenfield, P. L. (1992). The potential of transdisciplinary research for sustaining and extending linkages between the health and social sciences. *Social Sciences and Medicine, 35*(11), 1343–1357.

Russell, S. (2002). Teachers and library media specialists: Collaborative relationships. *Teacher Library Media Specialist, 29*(5), 35–38.

Selznick, P. (1992). *The moral commonwealth: Social theory and the promise of community*. Berkeley, CA: University of California Press.

Showers, B., & Joyce, B. (1996). The evolution of peer coaching. *Educational Leadership, 53*, 12–16.

Slatin, C., Galizzi, M., Melillo, K. D., Mawn, B., & Phase In Healthcare Team. (2004). Conducting interdisciplinary research to promote healthy and safe employment in health care: Promises and pitfalls. *Public Health Reports, 119*, 60–72.

Stoll, L., Bolam, R., McMahon, A., Wallace, M., & Thomas, S. (2006). Professional learning communities: A review of the literature. *Journal of Educational Change, 7*(4), 221–258.

Tschannen-Moran, M. (2001). Collaboration and the need for trust. *Journal of Educational Administration, 39*(4), 308–331.

Uline, C., Miller, D., & Tschannen-Moran, M. (1998). Underlying dimensions of school effectiveness. *Educational Administration Quarterly, 34*, 462–483.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. *Teaching and Teacher Education, 24*(1), 80–91.

Welch, M. (1998). Collaboration: Staying on the bandwagon. *Journal of Teacher Education, 49*(1), 26–37.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, UK: Cambridge University Press.

Wimsat-Upksi, W. (1994). *Bomb the suburbs*. Chicago: The Subway and Elevated Press Company.

Index

L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5