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Complex Algebraic Geometry Applied to Integrable Dynamics: Concrete Examples and Open Problems

E. Previato

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WHAT IS. . . ?

In September 2002, the *Notices* of the American Mathematical Society launched a new feature, published in each issue since then, with the following mission statement: "This is the inaugural installment of the "WHAT IS. . . ?" column, which carries short (one- or two-page), nontechnical articles aimed at graduate students. Each article focuses on a single mathematical object, rather than a whole theory"[1].

This is a very popular feature of the *Notices*, and since the School's goal is to introduce an area of research, I tailored my three lectures after it. The original plan was to cover Elliptic and Hyperelliptic Theta Functions, and their generalization – Klein's higher-genus sigma function – specifically to construct solutions to integrable hierarchies such as the Toda Lattice [KMP]; introduce vector bundles over curves and their moduli, with applications to algebraically completely integrable Hamiltonian systems (ACIs) [Hi]; then bring the two topics together through classical theorems of projective geometry, in recent applications, for example, to random-matrix theory (Painlevé equations) [HaS]. As the lectures unfolded, more detail was required and the three lectures reorganized as follows: the first and second are concerned with aspects of elliptic/hyperelliptic curves in

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[1] A list can be found at http://www.ams.org/publications/notices/whatis/noticesarchive.

classical geometry, recently adapted to applications in integrability, in both the contexts of PDE hierarchies and of ACIs. The final lecture covered the Kleinian sigma function, concluding with Baker's striking interpretation in projective geometry of the PDEs that characterize it: this is a tool that brings vector bundles into integrability, but there was no time for specifics

1. Lecture I: What is an elliptic curve?

As Mumford says in [Mum1, Lect. I], "The beginning of the subject is the AMAZING SYNTHESIS, which surely overwhelmed each of us as graduate students", and which he illustrates by the three natures of curves: Algebra (finitely generated field extensions of transcendence degree one over $\mathbb{C}$); Geometry (subvarieties of projective space $\mathbb{P}^n$, locally defined by $n-1$ homogeneous polynomial equations with independent differentials); Analysis (compact Riemann surfaces). I started with the Analysis nature of the elliptic curve, the torus $E = \mathbb{C}/\{n + m\tau, n, m \in \mathbb{Z}\}$, which becomes Algebra by virtue of the ODE satisfied by the Weierstrass $\wp$ function, the doubly periodic meromorphic function whose poles occur at the vertices of the lattice with the smallest possible multiplicity, two. An introduction both accessible and comprehensive, including a proof that the field K of meromorphic functions on E is generated by $\wp$ and $\wp'$, can be found in [DuV].

Two remarks are relevant.

Remark 1. The role of the elliptic curve in integrability. The Korteweg–de Vries (KdV) equation,

$$u_t + \frac{3}{2}uu_x - \frac{1}{4}u_{xxx} = 0$$

was proposed in the 19th century to model waves in a shallow canal (the value of $u(x,t)$ represents the height of the wave, the coordinate x the position in the canal); it was therefore natural to make the 'one-wave ansatz', $u(z,t) = v(x - ct)$, c a constant, where the function $v(z)$ should satisfy the ordinary differential equation $-cv' + \frac{3}{2}vv' - \frac{1}{4}v''' = 0$. By integrating twice, it was originally observed that the general solution is then an elliptic function, $v(x) = 2\wp(z+\alpha)+a$ (α, a two additional constants introduced by integration; when a assumes special values so that the cubic polynomial defining the elliptic curve has repeated roots, the solution becomes an elementary function, given in terms of exponentials or trigonometric/hyperbolic functions.) A modern example arises in statistical mechanics, as a one-dimensional lattice with exponential (nearest-neighbor) interaction, the Toda (differential-difference) system:

$$\frac{d^2 r_n}{dt^2} = a[2\exp(-br_n) - \exp(-br_{n-1}) - \exp(-br_{n+1})],$$

where a, b are arbitrary constants and n is any integer. By the transformation:

$$r = -\frac{1}{b}\ln\left(1 + \frac{f}{a}\right),$$

$$\frac{d^2}{dt^2}\ln\left(1+\frac{f_n}{a}\right) = b(f_{n-1}+f_{n+1}-2f_n),$$

Toda [T] produced exact solutions, expressed algebraically in terms of (Jacobi) elliptic functions,

$$f_n = \frac{(2k\nu)^2}{b}\left[\mathrm{dn}^2\left(2K(\nu t-\frac{n}{\lambda})\right)-\frac{E}{K}\right],$$

where ν is the frequency, λ the wavelength, K and E are complete elliptic integrals of the first and second kind for the modulus k:

$$K=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\quad K=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta;$$

the formula shows that the discrete evolution corresponds to the addition of a point on the elliptic curve: the addition law is arguably the reason for the "unreasonable effectiveness" of elliptic functions in dynamics.

Remark 2. Less famous than the Weierstrass equation, $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, two differential properties that characterize the $\wp$ function were forerunners of the theory of integrable PDEs. On the one hand, I mention $\wp'' = 6\wp^2 - g_2/2$, because the theory of the "higher-genus Kleinian function" σ [B, BEL], to which Lecture III is devoted, and which generalizes the genus-one Weierstrass sigma function, is centered on the search for a complete set of (partial, in higher genus) differential equations satisfied by σ; complete in the sense of differential algebra, for example, namely sets that are bases of differential ideals that define the differential rings of the algebraic varieties where σ is defined. On the other hand, Baker, as pointed out in [EE], wrote an equation for the σ function of a hyperelliptic curve of genus two, using the "bilinear operator" that Hirota rediscovered independently and yields the "Hirota form" of the Kadomtsev–Petviashvili (KP) equation $(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$, namely

$$(D_xD_t + D_x^4 + 3D_y^2)\tau\cdot\tau = 0,$$

for $w(x,y,t) = 2\ln\partial_x^2\tau(x,y,t)$, where two differentiations D_uD_v applied to $\tau\cdot\tau$ signify

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\big(\tau(u+v)\tau(u-v)\big)\mid_{u=v}$$

The Weierstrass equation thus provides the Algebra aspect of E, which Mumford (*loc. cit.*) describes as "field extensions $K\supset\mathbb{C}$, where K is finitely generated and of transcendence 1 over $\mathbb{C}$." I then gave three versions of the Geometry nature of an elliptic curve, all closely related to integrable systems, the third one less known. Briefly: The first, as a smooth cubic in the plane, in Weierstrass normal form, $y^2 = 4x^3 - g_2x - g_3$; The second, as the intersection of two quadrics in 3-dimensional projective space $\mathbb{P}^3$ – these embeddings are images under the divisor map for the linear series of 2∞ and 3∞ respectively, where ∞ is the point $[0,0,1]$ of the Weierstrass cubic in projective coordinates $[x_0,x_1,x_2]$ for which $x = x_1/x_0$, $y = x_2/x_0$. These two projective models can be brought together by the third geometric representation, namely, the incidence correspondence $I \subset C \times D^*$ with a choice of origin for the group law [BKOR]. Here the points of the curve are pairs (P, ℓ), with P a point in a fixed conic C and ℓ one of the two lines through P that are tangents to another fixed conic D; C and D must be in general position, and the limiting cases correspond to cubics that define rational curves. The model provides a beautiful proof of the classical "Poncelet's Porism Theorem", very much relevant to integrable dynamics, such as billiard or geodesic motion [P3].

2. Lecture II: Differential algebra

We now meet another, less known, nature of the elliptic curve.

2.1. Burchnall and Chaundy

A fourth nature of the elliptic curve emerged in the 20th century, in fact surprisingly early. In [BC], the authors pose the following question: what is the structure of a commutative subalgebra of the $\mathbb{C}$-algebra of Ordinary Differential Operators (ODOs) that is not of the form $\mathbb{C}[L]$, with L an ODO? We briefly recall the setting: we choose to work in the formal one of the algebra of Pseudo Differential Operators (ΨDOs), which is the most general, with the disadvantage that no convergence is addressed; for more restrictive (and precise) functional restrictions, cf. Sato's work, e.g., [SS].

Definitions. (i) The ring of formal pseudodifferential operators Ψ is the set

$$\Big\{ \sum_{j=-\infty}^{N} u_j(x)\partial^j ,\ u_j \text{ a formal power series}\Big\}.$$

If we think of these symbols as acting on functions of x by multiplication and differentiation: $(u(x)\partial)f(x) = u\frac{d}{dx}f$, and formally integrate by parts: $\int(uf) = u\int f - \int(u'f)$, we can motivate the composition rules:

$$\partial^{-1}\partial = \partial\partial^{-1} = 1,\quad \partial u = u\partial + u',\quad \partial^{-1}u = u\partial^{-1} - u'\partial^{-2} + u''\partial^{-3} - \ldots$$

and easily check an extended Leibnitz rule for a function f and for $A, B \in \Psi$:

$$\partial^i f\cdot\ = \sum_{j=0}^{\infty}\binom{i}{j}(\partial^j f)\partial^{i-j}\cdot\ ,\quad A\circ B = \sum_{i=0}^{\infty}\frac{1}{i!}\tilde{\partial}^i A * \partial^i B,$$

where $\tilde{\partial}$ is a partial differentiation w.r.t. the symbol ∂ and $*$ has the effect of bringing all functions to the left and powers of ∂ to the right.

(ii) Ψ contains the subring $\mathcal{D}$ of differential operators $A = \sum_0^N u_j\partial^j$ and we denote by $(\ )_+$ the projection $B_+ = \sum_0^N u_j\partial^j$ where $B = \sum_{-\infty}^N u_j\partial^j$. The much studied Weyl algebra in two generators, $\mathbb{C}[p, q]$ with multiplication rule defined by the commutator $[p, q] = 1$ can be viewed as a subring of $\mathcal{D}$, namely the operators with polynomial coefficients, by letting $p = \partial$ and $q = x$.

(iii) The Burchnall–Chaundy (hereafter BC for short) problem asks to find and classify all commutative subrings of $\mathcal{D}$. If we denote by $\mathcal{C}_{\mathcal{D}}(L)$ the centralizer in $\mathcal{D}$ of an element $L \in \mathcal{D}$, we see that the polynomial ring $\mathbb{C}[L]$ is always contained in $\mathcal{C}_{\mathcal{D}}(L)$. We also see that if L has order $n > 0$ then L can be brought to standard form:

$$L = \partial^n + u_{n-2}(x)\partial^{n-2} + u_{n-3}(x)\partial^{n-3} + \cdots + u_0(x)$$

by using change of variable and conjugation by a function, which are the only two automorphisms of $\mathcal{D}$; we shall always assume L to be in standard form, and define a BC solution to be such an L for which $\mathcal{C}_{\mathcal{D}}(L)$ is not a polynomial ring $\mathbb{C}[M]$, $M \in \mathcal{D}$. Notice that any translation in x: $x \mapsto x - a$, transforms a BC solution L into another solution L_a. We refer to this operation as the "x-flow".

(iv) The rank of a subset of $\mathcal{D}$ is the greatest common divisor of the orders of all the elements of $\mathcal{D}$.

Now we can give two new models for the elliptic curve (for references and more examples cf. [P2]):

The classical "Lamé operator" $L = \partial^2 - c\wp(x)$, where $c \in \mathbb{C}$ is a constant, is a BC solution iff $c = n(n+1)$ with n an integer greater than zero; if this is the case, the centralizer $\mathcal{C}_{\mathcal{D}}(L)$ is the affine ring of a hyperelliptic curve of genus n, given by an equation: $\mu^2 = \lambda^{2n+1}+$ lower order, or an elliptic curve when $n = 1$. A singular-cubic example is given by:

$$L = \partial^2 - \frac{2}{x^2}, \quad B = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}$$

which satisfy $B^2 \equiv L^3$.

In the Weyl algebra, define $u = p^3 + q^2 + \alpha$, $v = \frac{1}{2}p$, $L = u^2 + 4v$, $B = u^3 + 3(uv - vu)$; then $\mathcal{C}(L) = \mathbb{C}[L, B]$ and $B^2 - L^3 = -\alpha$, as shown in [Di]. By the assignment $p = \partial$, $q = x$ we obtain $L, B \in \mathcal{D}$ of order 6,9, but notice that the automorphism $\partial \mapsto -x$, $x \mapsto \partial$ will turn the orders into 4,6. Again, $\mathcal{C}_{\mathcal{D}}(L) = \mathbb{C}[L, B]$, the affine ring of the curve $\mu^2 = \lambda^3 - \alpha$; in particular, L is a BC solution, and the rank of this algebra is three, two, respectively.

It can be shown that centralizers $\mathcal{C}_{\mathcal{D}}(L)$ are maximal-commutative subalgebras of $\mathcal{D}$. How large can they be? Not very: since their quotient fields are function fields of one variable (cf. Th. 3), they are affine rings of curves, and in a formal sense these are indeed spectral curves; the algebras that correspond to a fixed curve make up the (generalized) Jacobian of that curve, and the x-flow is a holomorphic vector field on it. We may (formally) view this as a "direct" spectral problem; the "inverse" spectral problem allows us to reconstruct the coefficients of the operators (in terms of theta functions) from the data of a point on the Jacobian. The x-flow is tangent to the Abel image of the curve in its Jacobian, at a specific point. The higher osculating flows form a sequence (essentially finite): $x = t_1, t_2, \ldots, t_s, \ldots$ and the corresponding operators depend on these parameters in such a way as to satisfy the KP hierarchy. The higher-rank algebras are still much of a mystery.

In Ψ any (normalized) L has a unique nth root, n =ord L, of the form $\mathcal{L} = \partial + u_{-1}(x)\partial^{-1} + u_{-2}(x)\partial^{-2} + \cdots$. By a dimension count based on the orders, I. Schur showed that

Theorem 3. $\mathcal{C}_{\mathcal{D}}(L) = \{\sum_{-\infty}^{N} c_j \mathcal{L}^j, c_j \in \mathbb{C}\} \cap \mathcal{D}$.

This shows that the quotient field of $\mathcal{C}_{\mathcal{D}}(L)$ is a function field of one variable; indeed, a B which commutes with L must satisfy an algebraic equation $f(L, B) = 0$ (identically in x). In the case that the algebra can be generated by two elements $L,\ B$, the curve has a plane model, where $L,\ B$ can be viewed as affine coordinates $x,\ y$. I offered a little-known algorithm for computing the equation of the curve, the "differential resultant" ([BP, P1]). Since the algebra $\mathbb{C}[L, B]$ has no zero-divisors, it can be viewed as the affine ring $\mathbb{C}[X, Y]/(h)$ of a plane curve, with $h(X, Y)$ an irreducible polynomial. The BC curve $= \{(\lambda, \mu) \mid L, B$ have a joint eigenfunction $Ly = \lambda y,\ By = \mu y\}$ is included in the curve Spec $\mathbb{C}[L, B]$ and since the latter is irreducible, they must coincide; this shows in particular that the BC polynomial is some power of an irreducible polynomial h: $f(\lambda, \mu) = h^{r_1}$. In addition, each point of the spectral curve has a solution space: this gives a vector bundle over the curve. More precisely, let $r_2 =$ rank $\mathbb{C}[L, B]$, and $r_3 = \dim V_{(\lambda,\mu)}$ where $V_{(\lambda,\mu)}$ is the vector space of common eigenfunctions at any smooth point (λ, μ) of the BC curve. Then $r_1 = r_2 = r_3$. Moreover, this integer is the order of $G = gcd(L - \lambda, B - \mu)$, the operator (found by the Euclidean algorithm) of highest order for which a factorization holds, $B - \mu = T_1 G,\ L - \lambda = T_2 G$.

In theory, higher-rank algebras are classified by vector bundles over curves [Mul], but there is no explicit dictionary between the vector bundles and the coefficients of the operators; a recent paper [BZ] completed the result in [PW], covering the genus-one case of the spectral curve.

Lastly, we introduce the KP deformations, following [SS].

Definitions. (i) In Ψ, it is possible to conjugate any $\mathcal{L} = \partial + u_{-1}(x)\partial^{-1} + \cdots$ into ∂ by a $K \in \Psi,\ K = 1 + v_{-1}(x)\partial^{-1} + \cdots$, determined up to elements of $\mathbb{C}[\partial] = \mathcal{C}_{\mathcal{D}}(\partial)$. From now on we assume that $K^{-1}\mathcal{L}K = \partial$.

(ii) We define a formal Baker function for $\mathcal{L}$ as the element of the module of formal eigenfunctions such that $\mathcal{L}\psi = z\psi$; notice that $\psi = Ke^{xz}$.

The KP hierarchy is determined by the Lax equations ($\partial_n = \partial/\partial t_n$),

$$\partial_n \mathcal{L} = [B_n, \mathcal{L}] := B_n \mathcal{L} - \mathcal{L} B_n,$$

where $B_n = (\mathcal{L}^n)_+$. Motivated by an algebraic conjugation,

$$\partial e^{x\lambda} = \lambda e^{x\lambda},\ \ \mathcal{L}\psi = \lambda\psi,\ \psi = We^{x\lambda},$$

$$W\partial W^{-1}We^{x\lambda} = \lambda We^{x\lambda},\ W = 1 + \sum_{1}^{\infty} w_n \partial^{-n}$$

set: $\mathcal{L} = W\partial W^{-1}$, then the KP hierarchy is given by the Sato equations:

$$\partial_n W\, W^{-1} = -(\mathcal{L})^n_- = (\mathcal{L})^n_+ - (\mathcal{L})^n.$$

The "inverse spectral construction", which holds for any number of variables and yields explicit, exact solutions to the KP equations, is largely due to Krichever [Kr]:

Inverse spectral problem. The following choices: (i) A Riemann surface X of genus g; (ii) A point $\infty \in X$; (iii) A local parameter λ^{-1} near ∞; (iv) A generic divisor $P_1 + \cdots + P_g = D$ (the condition is that $h^0(P_1 + \cdots + P_g - \infty) = 0$, equivalently, there are no meromorphic functions on X with a zero at ∞ and poles bounded by $P_1 + \cdots + P_g$); determine uniquely a function $\psi(\underline{t}, P)$, the "Baker–Akhiezer (BA) function," such that near ∞, $\psi \sim \exp(\sum_{i\geq 1} t_i \lambda^i)(1 + \sum \xi_i(\underline{t})\lambda^{-i})$ and at finite points P of the curve, ψ has poles bounded by D and is analytic elsewhere.

For such a ψ there exist unique operators K_j such that $K_j\psi = \partial_{t_j}\psi$ and these operators are a solution to the KP hierarchy, in particular $\mathcal{L}\psi = \lambda\psi$ gives $\mathcal{L} \in \Psi$ as above. All statements are local in $\underline{t}$. Explicitly,

$$\psi(\underline{t}) = e^{(\sum_{i\geq 1} t_i (\int_{P_0}^{P} \eta_i - c_i))} \cdot \frac{\vartheta(A(P) + \sum_{i\geq 1} U_i t_i + \delta)\vartheta(A(\infty) + \delta - A(D))}{\vartheta(A(P) + \delta - A(D))\vartheta(A(\infty) + \sum_{i\geq 1} U_i t_i + \delta)},$$

where δ is Riemann's constant so that $\vartheta(A(P)+\delta-A(D))$ vanishes for $P = P_j, j = 1, \ldots, g$; η_i are suitably normalized meromorphic differentials; $U_i \in \mathbb{C}^g$ are suitable vectors that make ψ into a function of P independent of the path of integration; $c_i \in \mathbb{C}$ are constants that normalize ψ as above.

In conclusion, the general (algebro-geometric) solution of KP is:

$$u(\underline{t}) = 2\partial_x^2 \log \vartheta\left(\sum_{j\geq 1} t_j U_j + A(P) + \delta\right) + \text{const.}$$

Most strikingly, Novikov conjectured that a theta function which satisfies the KP hierarchy arises from a Jacobian, and this was shown to be true, thus settling the "Shottky Problem" [BD].

To conclude the lecture on differential algebra, I mentioned a second major still largely open problem: what is the answer to the Burchnall–Chaundy question if we consider the algebra of Partial Differential Operators (PDOs)? Is there an analog of the spectral curve, such as, in two variables, a surface, and are its equations given by a differential resultant? Much work has been done, but concrete results are scarce, and the answer to simple questions is not known; for example, given that the multivariate resultant, a multivariate polynomial, vanishes identically when evaluated on a set of PDOs that have a common eigenfunction (cf. [KP] for a precise formulation and references), is the differential resultant independent of the differential variables? This is what happens in the ODO case, where the resultant if the equation of the spectral curve.

3. Lecture III: The Kleinian sigma function

Why switch from theta, which yields exact KP solutions, to sigma? I offer three reasons, of which the previous leads to the next: Modular invariance, thus a stronger relationship with the Jacobian (briefly put, in the following sense: the symplectic group $Sp(2g,\mathbb{Z})$ acts in the standard way [Mum2, II.5, (5.3)] on the two variables of ϑ, the Abelian variable $z \in \mathbb{C}^g$ and the period lattice Λ; the action produces a multiplicative non-zero factor, whereas σ is invariant); an explicit transliteration between meromorphic and transcendental functions; More useful formulas for solutions of integrable equations, since meromorphic functions lend themselves more clearly to a qualitative analysis.

Recall the definition of the Weierstrass sigma function (genus one):

$$\wp(u) = -\frac{d^2}{du^2}\ln\sigma(u),\ \ (\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

Recall σ is an odd function (ϑ is even), with expansion:

$$\sigma(u) = u - \frac{1}{240}g_2u^5 - \frac{1}{840}g_3u^7 - \cdots .$$

Klein defined σ for two variables, then for any hyperelliptic curve, and for a trigonal curve [KS].

The sigma function for a hyperelliptic curve X of genus $g \geq 2$ defined in the affine plane by:

$$y^2 = f(x) := x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0$$

(where λ_j's are generic complex numbers so that X, completed by ∞ at infinity, is smooth), is easy to define, because there is an explicit basis of differentials of the first kind:

$$\omega_i := \frac{x^{i-1}dx}{2y} \qquad (i = 1, \dots, g),$$

and differentials of the second kind,

$$\eta_i := \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx, \quad (j = 1, \dots, g)$$

so that when taking the periods around a symplectic homology basis $\{\alpha_i, \beta_j\}$, $1 \leq i, j \leq g$, the matrices $\omega = \begin{bmatrix}\omega' \\ \omega''\end{bmatrix}$ where

$$\omega' = \frac{1}{2}\left[\oint_{\alpha_j}\omega_i\right], \quad \omega'' = \frac{1}{2}\left[\oint_{\beta_j}\omega_i\right],$$
$$\eta' = \frac{1}{2}\left[\oint_{\alpha_j}\eta_i\right], \quad \eta'' = \frac{1}{2}\left[\oint_{\beta_j}\eta_i\right],$$

satisfy the generalized Legendre relation

$$\mathfrak{M}\begin{pmatrix}0 & -1_g \\ 1_g & 0\end{pmatrix}\mathfrak{M}^T = \frac{\imath\pi}{2}\begin{pmatrix}0 & -1_g \\ 1_g & 0\end{pmatrix}, \tag{1}$$

where $\mathfrak{M} = \begin{pmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{pmatrix}$. We let Λ be the lattice in $\mathbb{C}^g$ spanned by the column vectors of $2\omega'$ and $2\omega''$. The Jacobian variety of X is identified with $\mathbb{C}^g/\Lambda$. We let κ be the projection $\mathbb{C}^g \to \mathbb{C}^g/\Lambda$. For a non-negative integer k, we define the Abel map from the kth symmetric product $\mathrm{Sym}^k X$ of the curve X to $\mathbb{C}^g$ by first choosing any (suitable) path of integration[2]:

$$w : \mathrm{Sym}^k X \to \mathbb{C}^g, \quad w((x_1, y_1), \dots, (x_k, y_k)) = \sum_{i=1}^{k} \int_{\infty}^{(x_i,y_i)} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}.$$

We denote the image by W_k. Let $\mathbb{T} = \omega'^{-1}\omega''$. The theta function on $\mathbb{C}^g$ with "modulus" $\mathbb{T}$ and characteristics $\mathbb{T}a + b$ for $a, b \in \mathbb{C}^g$ is given by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z; \mathbb{T}) = \sum_{n \in \mathbb{Z}^g} \exp \left[2\pi i \left\{ \frac{1}{2} \, {}^t(n+a)\mathbb{T}(n+a) + \, {}^t(n+a)(z+b) \right\} \right].$$

The σ-function, an analytic function on the space $\mathbb{C}^g$ and a theta-series having modular invariance of a given weight with respect to $\mathfrak{M}$, is given by the formula

$$\sigma(u) = \gamma_0 \exp \left\{ -\frac{1}{2} \, {}^t u \eta' \omega'^{-1} u \right\} \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \left(\frac{1}{2} \omega'^{-1} u \, ; \, \mathbb{T} \right),$$

where δ' and δ'' are half-integer characteristics giving the vector of Riemann constants with basepoint at ∞ and γ_0 is a non-zero constant. Computing γ_0 is again possible because the curve is hyperelliptic: the result is based on a normalization, thus it is achieved by expanding the function at ∞ as a power series in the Abelian variables:

$$\gamma_0 = \frac{\epsilon_4}{\vartheta(0; \mathbb{T})} \prod_{r=1}^{g} \frac{\sqrt{P'(a_r)}}{\sqrt[4]{f'(a_r)}} \frac{1}{\prod_{k<l} \sqrt{e_k - e_l}}.$$

Since this constant plays no role in this paper, we have retained the slightly different notation of [BEL], where the curve is written as

$$y^2 = \sum_{i=0}^{2g+1} \lambda_i x^i = \lambda_{2g+1} \prod_{k=1}^{2g+1} (x - e_k) = 4P(x)Q(x)$$

with:

$$P(x) = \prod_{i=1}^{g} (x - a_i), \quad Q(x) = (x - b) \prod_{i=1}^{g} (x - b_i),$$

for the homology basis whose loops correspond to the branch cuts beginning at a_i and ending at b_i, with an additional one beginning at $a = \infty$ and ending at b. The fourth root of unity ϵ_4 is difficult to compute, but clearly does not depend on the moduli of the curve, since it is a discrete parameter and σ depends holomorphically on the moduli parameters. In genus one, the formula reduces to Weierstrass' σ,

[2] The results presented are independent of the particular choice.

and in that case this root of unity is related to the eight root of unity appearing in the functional equation of ϑ under the action of the congruence subgroup

$$\Gamma := \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - bc = 1, \; cd \; \text{ even} \right),$$

which is calculated in [Mum2, Vol. I, II.5] and involves the Jacobi symbol [Mum2, Vol. I, I.7, Th. 7.1].

The σ-function vanishes to the first order on $\kappa^{-1}(W_{g-1})$. The Kleinian $\wp$ and ζ functions are defined by

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \zeta_i = \frac{\partial}{\partial u_i} \log \sigma(u).$$

I concluded returning full circle to Mumford's vision, but now for surfaces: indeed, Baker generalized Weierstrass' equation to cut out the Kummer surface in $\mathbb{P}^3$, the linear series of $|2\Theta|$, where Θ is the canonical theta divisor for a curve of genus two (hence also, up to translation, the zero locus of σ). This is Analysis turning into Geometry; Algebra is the field of meromorphic functions of the Kummer surface, but for surfaces the perfect synthesis no longer holds, since two surfaces may have isomorphic fields without being isomorphic, such as $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

The Kummer surface is the image of the $|2\Theta|$-divisor map $\mathrm{Jac}(X) \to \mathbb{P}^3$, using the basis $1, \wp_{11}, \wp_{12}, \wp_{22}$, and a quartic in these coordinates:

$$\det \begin{bmatrix} -\lambda_0 & \frac{1}{2}\lambda_1 & 2\wp_{11} & -2\wp_{12} \\ \frac{1}{2}\lambda_1 & -(\lambda_2 + 4\wp_{11}) & \frac{1}{2}\lambda_3 + 2\wp_{12} & 2\wp_{22} \\ 2\wp_{11} & \frac{1}{2}\lambda_3 + 2\wp_{12} & -(\lambda_4 + 4\wp_{22}) & 2 \\ -2\wp_{12} & 2\wp_{22} & 2 & 0 \end{bmatrix} = 0$$

is an algebraic differential equation that holds identically exactly on the Kummer surface.

This was generalized to all hyperelliptic Kummer varieties in [BÉ], and to trigonal Kummer varieties in [BLÉ]. For non-hyperelliptic curves, the Kummer variety is the singular locus of a projective model for the moduli space of rank-two, trivial-determinant vector bundles over X, a key ingredient in the construction of Hitchin-type ACIs [vGP]. This is one more area of intense study centered on the role of σ function in integrability.

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E. Previato
Department of Mathematics and Statistics
Boston University
Boston, MA 02215, USA
e-mail: ep@bu.edu