

Chapter 2

Quantum Field Theory and Quantum Electrodynamics

One of the main reasons why quantities like the anomalous magnetic moment of the muon attract so much attention is their prominent role in basic tests of QFT in general and of Quantum Electrodynamics (QED) and the Standard Model (SM) in particular. QED and the SM provide a truly basic framework for the properties of elementary particles and allow us to make unambiguous theoretical predictions which may be confronted with clean experiments which allows one to control systematic errors with amazing precision. In order to set up notation we first summarize some basic concepts. The reader familiar with QED, its renormalization and leading order radiative corrections may skip this introductory section, which is a modernized version of material covered by classical textbooks [1, 2]. Since magnetic moments of elementary particles are intimately related to the spin the latter plays a key role for this book. In a second section, therefore, we will have a closer look at how the concept of spin comes into play in quantum field theory.

2.1 Quantum Field Theory Background

2.1.1 Concepts, Conventions and Notation

We briefly sketch some basic concepts and fix the notation. A relativistic quantum field theory (QFT), which combines special relativity with quantum mechanics [3], is defined on the configuration space of space–time events described by points (**contravariant** vector)

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}); x^0 = t (= \text{time})$$

in Minkowski space with metric

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The metric defines a scalar product¹

$$x \cdot y = x^0 y^0 - \mathbf{x} \cdot \mathbf{y} = g_{\mu\nu} x^\mu y^\nu = x^\mu x_\mu$$

invariant under Lorentz transformations, which include

1. rotations
2. special Lorentz transformations (boosts)

The set of linear transformations (Λ, a)

$$x^\mu \rightarrow x^{\mu'} = \Lambda^\mu_{\nu'} x^\nu + a^{\mu'} \quad (2.1)$$

which leave invariant the **distance**

$$(x - y)^2 = g_{\mu\nu} (x^\mu - y^\mu)(x^\nu - y^\nu) \quad (2.2)$$

between two events x and y from the **Poincaré group** \mathcal{P} . \mathcal{P} includes the Lorentz transformations and the translations in time and space.

Besides the Poincaré invariance, also space reflections (called parity) P and time reversal T , defined by

$$Px = P(x^0, \mathbf{x}) = (x^0, -\mathbf{x}), \quad Tx = T(x^0, \mathbf{x}) = (-x^0, \mathbf{x}), \quad (2.3)$$

play an important role. They are symmetries of the electromagnetic (QED) and the strong interactions (QCD) but are violated by weak interactions. The proper orthochronous transformations \mathcal{P}_+^\uparrow do not include P and T , which requires the constraints on the determinant (orientation of frames) $\det \Lambda = 1$ and the direction of time $\Lambda^0_0 \geq 0$.

Finally, we will need the totally antisymmetric pseudo-tensor

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & (\mu\nu\rho\sigma) \text{ even permutation of } (0123) \\ -1 & (\mu\nu\rho\sigma) \text{ odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases},$$

¹As usual we adopt the summation convention: repeated indices are summed over unless stated otherwise. For Lorentz indices $\mu, \dots = 0, 1, 2, 3$ summation only makes sense (i.e. respects L-invariance) between upper (contravariant) and lower (covariant) indices and is called **contraction**.

which besides $g^{\mu\nu}$ is the second numerically Lorentz-invariant (L-invariant) tensor. Useful relations are

$$\begin{aligned}
\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} &= -24 \\
\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma'} &= -6\delta_{\sigma'}^{\sigma} \\
\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho'\sigma'} &= -2\delta_{\rho'}^{\rho}\delta_{\sigma'}^{\sigma} + 2\delta_{\sigma'}^{\rho}\delta_{\rho'}^{\sigma} \\
\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu'\rho'\sigma'} &= -\delta_{\nu'}^{\nu}\delta_{\rho'}^{\rho}\delta_{\sigma'}^{\sigma} + \delta_{\nu'}^{\nu}\delta_{\sigma'}^{\rho}\delta_{\rho'}^{\sigma} + \delta_{\rho'}^{\nu}\delta_{\nu'}^{\rho}\delta_{\sigma'}^{\sigma} - \delta_{\rho'}^{\nu}\delta_{\sigma'}^{\rho}\delta_{\nu'}^{\sigma} - \delta_{\sigma'}^{\nu}\delta_{\nu'}^{\rho}\delta_{\rho'}^{\sigma} + \delta_{\sigma'}^{\nu}\delta_{\rho'}^{\rho}\delta_{\nu'}^{\sigma}
\end{aligned} \tag{2.4}$$

In QFT relativistic particles are described by quantum mechanical states,² like $|\ell^-(\mathbf{p}, r)\rangle$ for a lepton ℓ^- of momentum \mathbf{p} and 3rd component of spin r [4] (Wigner

²A relativistic quantum mechanical system is described by a state vector $|\psi\rangle \in \mathcal{H}$ in Hilbert space, which transforms in a specific way under \mathcal{P}_+^{\uparrow} . We denote by $|\psi'\rangle$ the state transformed by $(\Lambda, a) \in \mathcal{P}_+^{\uparrow}$. Since the system is required to be invariant, transition probabilities must be conserved

$$|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2 . \tag{2.5}$$

Therefore, there must exist a unitary operator $U(\Lambda, a)$ such that

$$|\psi\rangle \rightarrow |\psi'\rangle = U(\Lambda, a)|\psi\rangle \in \mathcal{H}$$

and $U(\Lambda, a)$ must satisfy the group law:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = \omega U(\Lambda_2\Lambda_1, \Lambda_2a_1 + a_2) .$$

This means that $U(\Lambda, a)$ is a **representation up to a phase** ω (ray representation) of \mathcal{P}_+^{\uparrow} . Without loss of generality one can choose $\omega = \pm 1$ (Wigner 1939).

The generators of \mathcal{P}_+^{\uparrow} are the relativistic energy-momentum operator P_{μ}

$$U(a) \equiv U(1, a) = e^{iP_{\mu}a^{\mu}} = 1 + iP_{\mu}a^{\mu} + \dots \tag{2.6}$$

and the relativistic angular momentum operator $M_{\mu\nu}$

$$U(\Lambda) \equiv U(\Lambda, 0) = e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}} = 1 + \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + \dots \tag{2.7}$$

Since for infinitesimal transformations we have

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu} \quad \text{with} \quad \omega_{\mu\nu} = -\omega_{\nu\mu},$$

the generators $M_{\mu\nu}$ are antisymmetric:

$$M_{\mu\nu} = -M_{\nu\mu} .$$

By unitarity of $U(\Lambda, a)$, P_{μ} and $M_{\mu\nu}$ are Hermitian operators on the Hilbert space. The generator of the time translations P_0 represents the Hamiltonian H of the system ($H \equiv P_0$) and determines the **time evolution**. If $|\psi\rangle = |\psi\rangle_H$ is a Heisenberg state, which coincides with the Schrödinger state $|\psi(0)\rangle_S$ at $t = 0$, then $|\psi(t)\rangle_S = e^{-iHt} |\psi(0)\rangle_S$ represents the state of the system at time t .

states). Spin will be considered in more detail in the next section. These states carry L-invariant mass $p^2 = m^2$ and spin s , and may be obtained by applying corresponding *creation operators* $a^+(\mathbf{p}, r)$ to the ground state $|0\rangle$, called vacuum:

$$|\mathbf{p}, r\rangle = a^+(\mathbf{p}, r) |0\rangle . \quad (2.8)$$

The energy of the particle is $p^0 = \omega_p = \sqrt{\mathbf{p}^2 + m^2}$. The Hermitian adjoints of the creation operators, the *annihilation operators* $a(\mathbf{p}, r) \doteq (a^+(\mathbf{p}, r))^+$, annihilate a state of momentum \mathbf{p} and 3rd component of spin r ,

$$a(\mathbf{p}, r) |\mathbf{p}', r'\rangle = (2\pi)^3 2\omega_p \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{rr'} |0\rangle$$

and since the vacuum is empty, in particular, they annihilate the vacuum

$$a(\mathbf{p}, r) |0\rangle = 0 . \quad (2.9)$$

The creation and annihilation operators for leptons (spin 1/2 fermions), a and a^+ , and the corresponding operators b and b^+ for the antileptons, satisfy the canonical *anticommutation relations* (Fermi statistics)

$$\{a(\mathbf{p}, r), a^+(\mathbf{p}', r')\} = \{b(\mathbf{p}, r), b^+(\mathbf{p}', r')\} = (2\pi)^3 2\omega_p \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{rr'} \quad (2.10)$$

with all other anticommutators vanishing. Note, the powers of 2π appearing at various places are convention dependent. Corresponding creation and annihilation operators for photons (spin 1 bosons) satisfy the *commutation relations* (Bose statistics)

$$[c(\mathbf{p}, \lambda), c^+(\mathbf{p}', \lambda')] = (2\pi)^3 2\omega_p \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{\lambda\lambda'} . \quad (2.11)$$

In configuration space particles have associated fields [5–7]. The leptons are represented by Dirac fields $\psi_\alpha(x)$, which are four-component spinors $\alpha = 1, 2, 3, 4$, and the photon by the real vector potential field $A^\mu(x)$ from which derives the electromagnetic field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The free fields are represented in terms of the creation and annihilation operators

$$\psi_\alpha(x) = \sum_{r=\pm 1/2} \int d\mu(p) \left\{ u_\alpha(\mathbf{p}, r) a(\mathbf{p}, r) e^{-ipx} + v_\alpha(\mathbf{p}, r) b^+(\mathbf{p}, r) e^{ipx} \right\} \quad (2.12)$$

for the fermion, and

$$A_\mu(x) = \sum_{\lambda=\pm} \int d\mu(p) \left\{ \varepsilon_\mu(p, \lambda) c(\mathbf{p}, \lambda) e^{-ipx} + \text{h.c.} \right\} \quad (2.13)$$

for the photon (h.c. = Hermitian conjugation). The Fourier transformation has to respect that the physical state is on the mass-shell and has positive energy (*spectral*

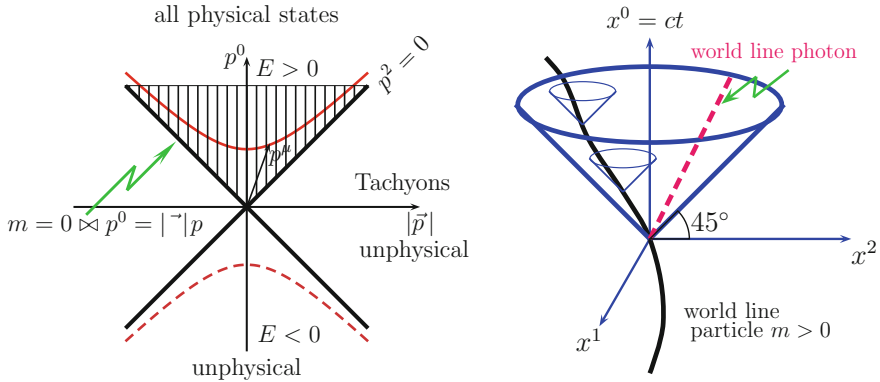


Fig. 2.1 Left the spectral condition: $p^2 = m^2 \geq 0$, $p^0 = E = \sqrt{\mathbf{p}^2 + m^2} \geq 0$. Right Einstein causality: physical signals propagate inside the light-cone $x^2 \geq 0$ (time-like)

condition: $p^2 = m^2$, $p^0 \geq m$, $m \geq 0$ see Fig. 2.1), thus $p^0 = \omega_p = \sqrt{m^2 + \mathbf{p}^2}$ and

$$\int d\mu(p) \cdots \equiv \int \frac{d^3 p}{2\omega_p (2\pi)^3} \cdots = \int \frac{d^4 p}{(2\pi)^3} \Theta(p^0) \delta(p^2 - m^2) \cdots$$

Note that Fourier amplitudes $e^{\mp i p x}$ in (2.12) and (2.13), because of the on-shell condition $p^0 = \omega_p$, are plane wave (free field) solutions of the *Klein–Gordon equation*: $(\square_x + m^2) e^{\mp i p x} = 0$ or the *d’Alembert equation* $\square_x e^{\mp i p x} = 0$ for the photon where $m_\gamma = 0$. Therefore, the fields themselves satisfy the Klein–Gordon or the d’Alembert equation, respectively. The “amplitudes” u , v and ε_μ , appearing in (2.12) and (2.13) respectively, are classical one-particle wave functions (plane wave solutions) satisfying the free field equations in momentum space.³ Thus u the lepton wavefunction and v the antilepton wavefunction are four-spinors, c-number solutions of the Dirac equations,

$$\begin{aligned} (\not{p} - m) u_\alpha(\mathbf{p}, r) &= 0, & \text{for the lepton} \\ (\not{p} + m) v_\alpha(\mathbf{p}, r) &= 0, & \text{for the antilepton.} \end{aligned} \quad (2.15)$$

³Our convention for the four-dimensional Fourier transformation for general (off-shell) fields, reads (all integrations from $-\infty$ to $+\infty$)

$$\tilde{\psi}(p) = \int d^4 x e^{i p x} \psi(x), \quad \tilde{A}^\mu(p) = \int d^4 x e^{i p x} A^\mu(x). \quad (2.14)$$

The inverse transforms then take the form

$$\psi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x} \tilde{\psi}(p), \quad A^\mu(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x} \tilde{A}^\mu(p), \quad \delta^{(4)}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x}$$

and hence the derivative with respect to x^μ turns into multiplication by the *four-momentum* $-i p_\mu$: $\partial_\mu \psi(x) \rightarrow -i p_\mu \tilde{\psi}(p)$ etc.

As usual, we use the short notation $\not{p} \doteq \gamma^\mu p_\mu = \gamma^0 p^0 - \boldsymbol{\gamma} \mathbf{p}$ (repeated indices summed over). Note that the relations (2.15) directly infer that the Dirac field is a solution of the Dirac equation $(i\gamma^\mu \partial_\mu - m) \psi(x) = 0$.

The $\underline{\gamma}$ -matrices are 4×4 matrices which satisfy the **Dirac algebra**⁴:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (2.16)$$

The L-invariant parity odd matrix γ_5 (under parity $\gamma^0 \rightarrow \gamma^0$, $\gamma^i \rightarrow -\gamma^i$ $i = 1, 2, 3$)

$$\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma ; \quad \gamma_5^2 = 1 ; \quad \gamma_5 = \gamma_5^\dagger \quad (2.17)$$

satisfies the anticommutation relation

$$\{\gamma_5, \gamma^\mu\} = \gamma_5 \gamma^\mu + \gamma^\mu \gamma_5 = 0 \quad (2.18)$$

and is required for the formulation of parity violating theories like the weak interaction part of the Standard Model (SM) and for the projection of Dirac fields to left-handed (L) and right-handed (R) chiral fields

$$\psi_R = \Pi_+ \psi ; \quad \psi_L = \Pi_- \psi \quad (2.19)$$

⁴Dirac's γ -matrices are composed from Pauli matrices. In quantum mechanics spacial rotations are described by the group of unitary, unimodular ($\det U = 1$) complex 2×2 matrix transformations $SU(2)$ rather than by classical $O(3)$ rotations. The structure constants are given by ϵ_{ikl} ($i, k, l = 1, 2, 3$) the fully antisymmetric permutation tensor. The generators of $SU(2)$ are given by $T_i = \frac{\sigma_i}{2}$; σ_i ($i = 1, 2, 3$) in terms of the 3 Hermitian and traceless *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one of which (σ_3) is diagonal. The properties of the Pauli matrices are

$$\begin{aligned} [\sigma_i, \sigma_k] &= 2i\epsilon_{ikl}\sigma_l, \quad \{\sigma_i, \sigma_k\} = 2\delta_{ik} \\ \sigma_i^\dagger &= \sigma_i, \quad \sigma_i^2 = 1, \quad \text{Tr } \sigma_i = 0 \\ \sigma_i \sigma_k &= \frac{1}{2} \{\sigma_i, \sigma_k\} + \frac{1}{2} [\sigma_i, \sigma_k] = \delta_{ik} + i\epsilon_{ikl}\sigma_l \end{aligned}$$

As usual we denote by $[A, B] = AB - BA$ the commutator, by $\{A, B\} = AB + BA$ the anticommutator. Dirac's γ -matrices in standard representation (as an alternative to the helicity representation, considered below) are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

where

$$\Pi_{\pm} = \frac{1}{2}(1 \pm \gamma_5) \quad (2.20)$$

are Hermitian chiral projection matrices⁵

$$\Pi_+ + \Pi_- = 1, \quad \Pi_+ \Pi_- = \Pi_- \Pi_+ = 0, \quad \Pi_-^2 = \Pi_- \quad \text{and} \quad \Pi_+^2 = \Pi_+ .$$

Note that $\psi^+ \psi$ or $u^+ u$, which might look like the natural analog of $|\psi|^2 = \psi^* \psi$ of the lepton wave function in quantum mechanics, are not scalars (invariants) under Lorentz transformations. In order to obtain an invariant we have to sandwich the matrix A which implements Hermitian conjugation of the Dirac matrices $A \gamma_{\mu} A^{-1} = \gamma_{\mu}^+$. One easily checks that we may identify $A = \gamma^0$. Thus defining the *adjoint spinor* by $\bar{\psi} \doteq \psi^+ \gamma^0$ we may write $\psi^+ A \psi = \bar{\psi} \psi$ etc.

The standard basis of 4×4 matrices in four-spinor space is given by the 16 elements

$$\Gamma_i = 1, \quad \gamma_5, \quad \gamma^{\mu}, \quad \gamma^{\mu} \gamma_5 \quad \text{and} \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}] . \quad (2.22)$$

The corresponding products $\bar{\psi} \Gamma_i \psi$ are scalars in spinor space and transform as ordinary scalar (S), pseudo-scalar (P), vector (V), axial-vector (A) and tensor (T), respectively, under Lorentz transformations.

⁵Usually, the quantization of a massive particle with spin is defined relative to the z -axis as a standard frame. In general, the direction of polarization ξ , $\xi^2 = 1$ in the rest frame may be chosen arbitrary. For a massive fermion of momentum p

$$\Pi_{\pm} = \frac{1}{2}(1 \pm \gamma_5 \not{n})$$

define the general form of covariant spin projection operators, where n is a space like unit vector orthogonal to p

$$n^2 = -1; \quad n \cdot p = 0 .$$

The general form of n is obtained by applying Lorentz-boost $L_{\mathbf{p}}$ to the polarization vector in the rest frame

$$n = L_{\mathbf{p}}(0, \xi) = \left(\frac{\mathbf{p} \cdot \xi}{m}, \quad \xi + \frac{\mathbf{p} \cdot \xi}{m(p^0 + m)} \mathbf{p} \right) . \quad (2.21)$$

When studying polarization phenomena the polarization vectors n enter as independent additional vectors in covariant decompositions of amplitudes, besides the momentum vectors.

Products of Dirac matrices may be expressed in terms of the basis, as

$$\begin{aligned}\gamma^\mu \gamma^\nu &= \frac{1}{2} \{ \{ \gamma^\mu, \gamma^\nu \} + [\gamma^\mu, \gamma^\nu] \} = g^{\mu\nu} - i \sigma^{\mu\nu} \\ \gamma^\mu \gamma^\nu \gamma^\rho &= (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\rho\nu} - g^{\mu\rho} g^{\nu\sigma}) \gamma_\sigma - i \varepsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma_5 \\ \sigma^{\mu\nu} \gamma_5 &= \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} .\end{aligned}$$

The Dirac spinors satisfy the normalization conditions

$$\begin{aligned}\bar{u}(p, r) \gamma^\mu u(p, r') &= 2 p^\mu \delta_{rr'} , & \bar{v}(p, r) \gamma^\mu v(p, r') &= 2 p^\mu \delta_{rr'} \\ \bar{u}(p, r) v(p, r') &= 0 , & \bar{u}(p, r) u(p, r) &= 2m \delta_{rr'} \\ \bar{v}(p, r) u(p, r') &= 0 , & \bar{v}(p, r) v(p, r) &= -2m \delta_{rr'}\end{aligned}\quad (2.23)$$

and completeness relations

$$\sum_r u(p, r) \bar{u}(p, r) = \not{p} + m , \quad \sum_r v(p, r) \bar{v}(p, r) = \not{p} - m . \quad (2.24)$$

For the photon the *polarization vector* $\varepsilon_\mu(p, \lambda)$ satisfies the normalization

$$\varepsilon_\mu(p, \lambda) \varepsilon^{\mu*}(p, \lambda') = -\delta_{\lambda\lambda'} , \quad (2.25)$$

the completeness relation

$$\sum_{\lambda=\pm} \varepsilon_\mu(p, \lambda) \varepsilon_\nu^*(p, \lambda) = -g_{\mu\nu} + p_\mu f_\nu + p_\nu f_\mu , \quad (2.26)$$

and the absence of a scalar mode requires

$$p_\mu \varepsilon^\mu(p, \lambda) = 0 . \quad (2.27)$$

The “four–vectors” f in the completeness relation are arbitrary gauge dependent quantities, which must drop out from physical quantities. Gauge invariance, i.e. invariance under *Abelian gauge transformations* $A_\mu \rightarrow A_\mu - \partial_\mu \alpha(x)$, $\alpha(x)$ an arbitrary scalar function, amounts to the invariance under the substitutions

$$\varepsilon_\mu \rightarrow \varepsilon_\mu + \lambda p_\mu ; \quad \lambda \text{ an arbitrary constant} \quad (2.28)$$

of the polarization vectors. One can prove that the polarization “vectors” for massless spin 1 fields can not be covariant. The non–covariant terms are always proportional to p_μ , however.

Besides a definite relativistic transformation property, like

$$U(\Lambda, a) \psi_\alpha(x) U^{-1}(\Lambda, a) = D_{\alpha\beta}(\Lambda^{-1}) \psi_\beta(\Lambda x + a) ,$$

for a Dirac field, where $D(\Lambda)$ is a four-dimensional (non-unitary) representation of the group $SL(2, C)$ which, in contrast to L_+^\uparrow itself, exhibits true *spinor representations* (see Sect. 2.2). The fields are required to satisfy Einstein causality: “no physical signal may travel faster than light”, which means that commutators for bosons and anticommutators for fermions must vanish outside the light cone (see Fig. 2.1)

$$[A_\mu(x), A_\nu(x')] = 0, \quad \{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = 0 \quad \text{for } (x - x')^2 < 0 .$$

This is only possible if all fields exhibit two terms, a creation and an annihilation part, and for charged particles this means that to each particle an antiparticle of the same mass and spin but of opposite charge must exist [8]. In addition, and equally important, causality requires spin $1/2, 3/2, \dots$ particles to be fermions quantized with anticommutation rules and hence necessarily have to fulfill the *Pauli exclusion principle* [9], while spin $0, 1, \dots$ must be bosons to be quantized by normal commutation relations [10]. Note that neutral particles only, like the photon, may be their own antiparticle, the field then has to be real. The main consequences of the requirements of locality and causality of a relativistic field theory may be cast into the two theorems: – the **spin–statistics theorem** –

Theorem 2.1 *Bosons quantized with commutation relations must have integer spin. Fermions quantized with anticommutation relations must have half-odd-integer spin.*

– the **particle–antiparticle crossing theorem** –

Theorem 2.2 *Each particle of mass m and spin j must have associated an antiparticle with the same mass and spin, and which transforms under the same representation of \mathcal{P}_+^\uparrow . A particle may be its own antiparticle. If charged, particle and antiparticle have opposite charge.*

For rigorous proofs of the theorems I refer to [11].

2.1.2 C, P, T and CPT

In QED as well as in QCD, not however in weak interactions, interchanging particles with antiparticles defines a symmetry, *charge conjugation* C. It is mapping particle into antiparticle creation and annihilation operators and vice versa:

$$a(\mathbf{p}, r) \xleftrightarrow{C} b(\mathbf{p}, r) , \quad a^+(\mathbf{p}, r) \xleftrightarrow{C} b^+(\mathbf{p}, r) ,$$

up to a phase. For the Dirac field charge conjugation reads (see 2.36)

$$\psi_\alpha(x) \xrightarrow{C} C_{\alpha\beta} \bar{\psi}_\beta^T(x) \tag{2.29}$$

with ($X^T =$ transposition of the matrix or vector X)

$$C = i(\gamma^2 \gamma^0) = -i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} . \quad (2.30)$$

Properties of C are:

$$C^T = -C , \quad C\gamma^\mu C^{-1} = -(\gamma^\mu)^T ,$$

and for the spinors charge conjugation takes the form

$$(Cu)^T = \bar{v} \quad \text{and} \quad (Cv)^T = \bar{u} , \quad (2.31)$$

which may be verified by direct calculation.

As under charge conjugation the charge changes sign, also the electromagnetic current must change sign

$$U(C) j_{\text{em}}^\mu(x) U^{-1}(C) = -j_{\text{em}}^\mu(x) . \quad (2.32)$$

Notice that for any contravariant four-vector j^μ we may write the parity transformed vector ($j^0, -\mathbf{j}$) $\equiv j_\mu$ as a covariant vector. We will use this notation in the following.

Since the electromagnetic interaction $\mathcal{L}_{\text{int}}^{\text{QED}} = e j_{\text{em}}^\mu(x) A_\mu(x)$ respects C-, P- and T-invariance⁶ separately, we immediately get the following transformation properties for the photon field:

$$\begin{aligned} U(C) A^\mu(x) U^{-1}(C) &= -A^\mu(x) \\ U(P) A^\mu(x) U^{-1}(P) &= (PA)^\mu(Px) = A_\mu(Px) \\ \bar{U}(T) A^\mu(x) \bar{U}^{-1}(T) &= -(TA)^\mu(Tx) = A_\mu(Tx) . \end{aligned} \quad (2.35)$$

Notice that the charge parity for the photon is $\eta_C^\gamma = -1$.

⁶Any transformation which involves time-reversal T must be implemented as an anti-unitary transformation $\bar{U}(T)$, because the Hamiltonian cannot be allowed to change sign by the requirement of positivity of the energy (Wigner 1939). **Anti-unitarity** is defined by the properties

$$\bar{U}(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha^* \bar{U}|\psi\rangle + \beta^* \bar{U}|\phi\rangle = \alpha^* |\psi'\rangle + \beta^* |\phi'\rangle \quad (2.33)$$

and

$$\langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle^* . \quad (2.34)$$

The complex conjugation of matrix elements is admitted by the fact that it also preserves the probability $|\langle \psi | \phi \rangle|^2$. Because of the complex conjugation of matrix elements an anti-unitary transformation implies a **Hermitian transposition** of states and operators.

For the Dirac fields C, P and T take the form

$$\begin{aligned}
 U(C) \psi_\alpha(x) U^{-1}(C) &= i (\gamma^2 \gamma^0)_{\alpha\beta} \bar{\psi}_\beta^T(x) \\
 U(P) \psi_\alpha(x) U^{-1}(P) &= (\gamma^0)_{\alpha\beta} \psi_\beta(Px) \\
 \bar{U}(T) \psi_\alpha(x) \bar{U}^{-1}(T) &= i (\gamma^2 \gamma_5)_{\alpha\beta} \bar{\psi}_\beta^T(Tx)
 \end{aligned} \tag{2.36}$$

where the phases have been chosen conveniently. We observe that, in contrast to the boson fields, the transformation properties of the Dirac fields are by no means obvious; they follow from applying C, P and T to the Dirac equation.

A very important consequence of *relativistic local quantum field theory* is the validity of the **CPT–theorem**:

Theorem 2.3 *Any Poincaré (\mathcal{P}_+^\uparrow) [special Lorentz transformations, rotations plus translations] invariant field theory with normal commutation relations [bosons satisfying commutation relations, fermions anticommutation relations] is CPT invariant.*

Let $\Theta = \text{CPT}$ where C, P and T may be applied in any order. There exists an anti–unitary operator $\bar{U}(\Theta)$ which (with an appropriate choice of the phases) is transforming scalar, Dirac and vector fields according to

$$\begin{aligned}
 \bar{U}(\Theta) \phi(x) \bar{U}^{-1}(\Theta) &= \phi^*(-x) \\
 \bar{U}(\Theta) \psi(x) \bar{U}^{-1}(\Theta) &= i\gamma_5 \psi(-x) \\
 \bar{U}(\Theta) A_\mu(x) \bar{U}^{-1}(\Theta) &= -A_\mu(-x) \quad ,
 \end{aligned} \tag{2.37}$$

and which leaves the vacuum invariant: $\bar{U}(\Theta)|0\rangle = |0\rangle$ up to a phase. The CPT–theorem asserts that the transformation $\bar{U}(\Theta)$ under very general conditions is a symmetry of the theory (Lüders 1954, Pauli 1955, Jost 1957) [12].

The basic reason for the validity of the CPT–theorem is the following: If we consider a Lorentz transformation $\Lambda \in L_+^\uparrow$ represented by a unitary operator $U(\chi, \omega = \mathbf{n} \theta)$ (χ parametrizing a Lorentz–boost, ω parametrizing a rotation), then the operator $U(\chi, \mathbf{n} (\theta + 2\pi)) = -U(\chi, \mathbf{n} \theta)$ is representing the same L–transformation. In a local quantum field theory the mapping $\Lambda \rightarrow -\Lambda$ for $\Lambda \in L_+^\uparrow$, which is equivalent to the requirement that $\Theta : x \rightarrow -x$ must be a symmetry: the invariance under four–dimensional reflections.

Consequences of CPT are that modulus of the charges, masses, g –factors and lifetimes of particles and antiparticles must be equal. Consider a one particle state $|\psi\rangle = |e, \mathbf{p}, \mathbf{s}\rangle$ where e is the charge, \mathbf{p} the momentum and \mathbf{s} the spin. The CPT conjugate state is given by $|\tilde{\psi}\rangle = |-e, \mathbf{p}, -\mathbf{s}\rangle$. The state $|\psi\rangle$ is an eigenstate of the Hamiltonian which is describing the time evolution of the free particle:

$$\mathcal{H}|\psi\rangle = E|\psi\rangle \tag{2.38}$$

and the CPT conjugate relation reads $\tilde{\mathcal{H}}|\tilde{\psi}\rangle = E|\tilde{\psi}\rangle$. Since $\tilde{\mathcal{H}} = \mathcal{H}$ by the CPT theorem, we thus have

$$\mathcal{H}|\tilde{\psi}\rangle = E|\tilde{\psi}\rangle . \quad (2.39)$$

At $\mathbf{p} = 0$ the eigenvalue E reduces to the mass and therefore the two eigenvalue equations say that the mass of particle and antiparticle must be the same:

$$\bar{m} = m . \quad (2.40)$$

The equality of the g -factors may be shown in the same way, but with a Hamiltonian which describes the interaction of the particle with a magnetic field \mathbf{B} . Then (2.38) holds with eigenvalue

$$E = m - g \left(\frac{e\hbar}{2mc} \right) \mathbf{s} \cdot \mathbf{B} . \quad (2.41)$$

The CPT conjugate state ($e \rightarrow -e$, $\mathbf{s} \rightarrow -\mathbf{s}$, $m \rightarrow \bar{m}$, $g \rightarrow \bar{g}$, $\mathbf{B} \rightarrow \mathbf{B}$) according to (2.39) will have the same eigenvalue

$$E = \bar{m} - \bar{g} \left(\frac{e\hbar}{2\bar{m}c} \right) \mathbf{s} \cdot \mathbf{B} . \quad (2.42)$$

and since $\bar{m} = m$ we must have

$$\bar{g} = g \quad (2.43)$$

For the proof of the equality of the lifetimes

$$\bar{\tau} = \tau \quad (2.44)$$

we refer to the textbook [13]. Some examples of experimental tests of CPT, relevant in our context, are (see [14])

$$\begin{array}{ll} |q_{e^+} + q_{e^-}|/e & < 4 \times 10^{-8} \\ (m_{e^+} - m_{e^-})/m_{\text{average}} & < 8 \times 10^{-9} \quad 90\% \text{ CL} \\ (g_{e^+} - g_{e^-})/g_{\text{average}} & (-0.5 \pm 2.1) \times 10^{-12} \\ (g_{\mu^+} - g_{\mu^-})/g_{\text{average}} & (-0.11 \pm 0.12) \times 10^{-8} \\ (\tau_{\mu^+} - \tau_{\mu^-})/\tau_{\text{average}} & (2 \pm 8) \times 10^{-5} . \end{array}$$

The best test of CPT comes from the neutral Kaon mass difference

$$\left| \frac{m_{\bar{K}^0} - m_{K^0}}{m_{K^0}} \right| \leq 0.6 \times 10^{-18} \quad \text{at CL} = 90\% .$$

The existence of a possible electric dipole moment we have discussed earlier on p. 9 of the Introduction. An electric dipole moment requires a T violating theory and the CPT theorem implies that equivalently CP must be violated. In fact, CP invariance alone (independently of CPT and T) gives important predictions relating

decay properties of particles and antiparticles. We are interested here particularly in μ -decay, which plays a crucial role in the muon $g - 2$ experiment. Consider a matrix element for a particle a with spin \mathbf{s}_a at rest decaying into a bunch of particles b, c, \dots with spins $\mathbf{s}_b, \mathbf{s}_c, \dots$ and momenta $\mathbf{p}_b, \mathbf{p}_c, \dots$:

$$\mathcal{M} = \langle \mathbf{p}_b, \mathbf{s}_b; \mathbf{p}_c, \mathbf{s}_c; \dots | \mathcal{H}_{\text{int}} | 0, \mathbf{s}_a \rangle . \quad (2.45)$$

Under CP we have to substitute $\mathbf{s}_a \rightarrow \mathbf{s}_{\bar{a}}, \mathbf{p}_a \rightarrow -\mathbf{p}_{\bar{a}},$ etc. such that, provided \mathcal{H}_{int} is CP symmetric we obtain

$$\bar{\mathcal{M}} = \langle -\mathbf{p}_{\bar{b}}, \mathbf{s}_{\bar{b}}; -\mathbf{p}_{\bar{c}}, \mathbf{s}_{\bar{c}}; \dots | \mathcal{H}_{\text{int}} | 0, \mathbf{s}_{\bar{a}} \rangle \equiv \mathcal{M} . \quad (2.46)$$

The modulus square of these matrix-elements gives the transition probability for the respective decays, and (2.46) tells us that the decay rate of a particle into a particular configuration of final particles is identical to the decay rate of the antiparticle into the same configuration of antiparticles with all momenta reversed.

For the muon decay $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu,$ after integrating out the unobserved neutrino variables, the decay electron distribution is of the form

$$\frac{dN_{e^-}}{dx d \cos \theta} = A(x) + B(x) \hat{\mathbf{s}}_\mu \cdot \hat{\mathbf{p}}_{e^-} , \quad (2.47)$$

where $x = 2p_{e^-}/m_\mu$ with p_{e^-} the electron momentum in the muon rest frame and $\cos \theta = \hat{\mathbf{s}}_\mu \cdot \hat{\mathbf{p}}_{e^-}, \hat{\mathbf{s}}_\mu$ and $\hat{\mathbf{p}}_{e^-}$ the unit vectors in direction of \mathbf{s}_μ and $\mathbf{p}_{e^-}.$

The corresponding expression for the antiparticle decay $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ reads

$$\frac{dN_{e^+}}{dx d \cos \theta} = \bar{A}(x) + \bar{B}(x) \hat{\mathbf{s}}_\mu \cdot \hat{\mathbf{p}}_{e^+} , \quad (2.48)$$

and therefore for all angles and all electron momenta

$$A(x) + B(x) \cos \theta = \bar{A}(x) - \bar{B}(x) \cos \theta$$

or

$$A(x) = \bar{A}(x) , \quad B(x) = -\bar{B}(x) . \quad (2.49)$$

It means that the decay asymmetry is equal in magnitude but opposite in sign for μ^- and $\mu^+.$ This follows directly from CP and independent of the type of interaction (V-A, V+A, S, P or T) and whether P is violated or not. In spite of the fact that the SM exhibits CP violation (see the Introduction to Sect. 4.2), as implied by a CP violating phase in the quark family mixing matrix in the charged weak current, in μ -decay CP violation is a very small higher order effect and by far too small to have any detectable trace in the decay distributions, i.e., CP symmetry is perfectly realized in this case. The strong correlation between the muon polarization and charge on the

one side (see Chap. 6) and the decay electron/positron momentum is a key element of tracing spin polarization information in the muon $g - 2$ experiments.

CP violation, and the associated T violation plays an important role in determining the electric dipole moment of electrons and muons. In principle it is possible to test T invariance in μ -decay by searching for T odd matrix elements like

$$\mathbf{s}_e \cdot (\mathbf{s}_\mu \times \mathbf{p}_e) . \quad (2.50)$$

This is very difficult and has not been performed. A method which works is the study of the effect of an electric dipole moment on the spin precession in the muon $g - 2$ experiment. This will be studied in Sect. 6.3.1 on p. 584.

Until recently, the best limit for the electron (1.8) has been obtained by investigating T violation in Thallium (^{205}Tl) where the EDM is enhanced by the ratio $R = d_{\text{atom}}/d_e$, which in the atomic Thallium ground state studied is $R = -585$. Investigated are $\mathbf{v} \times \mathbf{E}$ terms in high electrical fields \mathbf{E} in an atomic beam magnetic–resonance device [15]. A new experiment [16], using the polar molecule Thorium monoxide (ThO), finds

$$d_e = (2.1 \pm 3.7 \text{ stat} \pm 2.5 \text{ syst}) \times 10^{-29} e \cdot \text{cm} .$$

This corresponds to an upper limit of $|d_e| < 8.7 \times 10^{-29} e \cdot \text{cm}$ with 90% confidence, an order of magnitude improvement in sensitivity compared to the previous best limits.

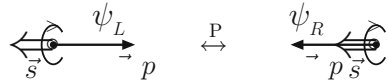
2.2 The Origin of Spin

As promised at the beginning of the chapter the intimate relation of the anomalous magnetic moment to spin is a good reason to have a closer look at how spin comes into play in particle physics. The spin and the magnetic moment of the electron did become evident from the deflection of atoms in an inhomogeneous magnetic field and the observation of the fine structure by optical spectroscopy [17–19].⁷ Spin is the intrinsic “self–angular momentum” of a point–particle and when it was observed by Goudsmit and Uhlenbeck it was completely unexpected. The question about the origin of spin is interesting because it is not obvious how a point–like object can possess its own angular momentum. A first theoretical formulation of spin in quantum mechanics was given by Pauli in 1927 [20], where spin was introduced as a new degree of freedom saying that there are two species of electrons in a doublet.

In modern relativistic terms, in the SM, particles and in particular leptons and quarks are considered to be massless originally, as required by chiral symmetry. All particles acquire their mass due to symmetry breaking via the Higgs mechanism: a

⁷Particle spin has been discovered by Ralph Kronig (well known for the Kramers Kronig relation) in 1925 before the Uhlenbeck and Goudsmit publication.

Fig. 2.2 Massless “electrons” have fixed helicities



scalar neutral Higgs field⁸ H develops a non-vanishing vacuum expectation value v and particles moving in the corresponding Bose condensate develop an effective mass. In the SM, in the physical *unitary gauge* a *Yukawa interaction* term upon a shift $H \rightarrow H + v$

$$\mathcal{L}_{\text{Yukawa}} = \sum_f \frac{G_f}{\sqrt{2}} \bar{\psi}_f \psi_f H \rightarrow \sum_f \left(m_f \bar{\psi}_f \psi_f + \frac{m_f}{v} \bar{\psi}_f \psi_f H \right) \quad (2.51)$$

induces a fermion mass term with mass $m_f = \frac{G_f}{\sqrt{2}} v$ where G_f is the *Yukawa coupling*.

In the massless state there are actually two independent electrons characterized by positive and negative helicities (chiralities) corresponding to right-handed (R) and left-handed (L) electrons, respectively, which do not “talk” to each other. Helicity h is defined as the projection of the spin vector onto the direction of the momentum vector

$$h \doteq \mathbf{S} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \quad (2.52)$$

as illustrated in Fig. 2.2 and transform into each other by space-reflections P (parity). Only after a fermion has acquired a mass, helicity flip transitions as effectively mediated by an anomalous magnetic moment (see below) are possible. In a renormalizable QFT an anomalous magnetic moment term is not allowed in the Lagrangian. It can only be a term induced by radiative corrections and in order not to vanish requires chiral symmetry to be broken by a corresponding mass term.

Angular momentum has to do with rotations, which form the rotation group $O(3)$. Ordinary 3-space rotations are described by orthogonal 3×3 matrices R ($RR^T = R^T R = I$ where I is the unit matrix and R^T denotes the transposed matrix) acting as $\mathbf{x}' = R\mathbf{x}$ on vectors \mathbf{x} of three-dimensional Euclidean position space \mathbf{R}^3 . Rotations are preserving scalar products between vectors and hence the length of vectors as well as the angles between them. Multiplication of the rotation matrices is the group operation and of course the successive multiplication of two rotations is non-commutative $[R_1, R_2] \neq 0$ in general. The rotation group is characterized by the Lie algebra $[\mathcal{J}_i, \mathcal{J}_j] = \varepsilon_{ijk} \mathcal{J}_k$, where the \mathcal{J}_i 's are normalized skew symmetric 3×3 matrices which generate the infinitesimal rotations around the x , y and z axes, labeled by $i, j, k = 1, 2, 3$. By ε_{ijk} we denoted the totally antisymmetric Levi-Civita

⁸The existence of the Higgs boson has been postulated in 1964 by Englert, Brout and Higgs [21, 22] to be a necessary ingredient of minimal renormalizable theory of electroweak interactions, and has been discovered with a mass about 125 GeV 48 years later in 2012 by the ATLAS [23] and the CMS [24] collaborations at the LHC at CERN in Switzerland.

tensor. The Lie algebra may be written in the form of the angular momentum algebra

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \quad (2.53)$$

by setting $\mathcal{J}_i = -iJ_i$, with Hermitian generators $J_i = J_i^\dagger$. The latter form is well known from quantum mechanics (QM). In quantum mechanics rotations have to be implemented by unitary representations $U(R)$ ($UU^\dagger = U^\dagger U = I$ and U^\dagger is the Hermitian conjugate of U) which implement transformations of the state vectors in physical Hilbert space $|\psi\rangle' = U(R)|\psi\rangle$ for systems rotated relative to each other. Let J_i be the generators of the infinitesimal transformations of the group $O(3)$, the angular momentum operators, such that a finite rotation of magnitude $|\omega| = \theta$ about the direction of $\mathbf{n} = \omega/\theta$ may be represented by $U(R(\omega)) = \exp -i\omega\mathbf{J}$ (ω_i , $i = 1, 2, 3$ a real rotation vector). While for ordinary rotations the J_k 's are again 3×3 matrices, in fact the lowest dimensional matrices which satisfy (2.53) in a non-trivial manner are 2×2 matrices. The corresponding Lie algebra is the one of the group $SU(2)$ of unitary 2×2 matrices U with determinant unity: $\det U = 1$. It is a simply connected group and in fact it is the universal covering group of $O(3)$, the latter being doubly connected. Going to $SU(2)$ makes rotations a single valued mapping in parameter space which is crucial to get the right phases in the context of QM. Thus $SU(2)$ is lifting the two-fold degeneracy of $O(3)$. As a basic fact in quantum mechanics rotations are implemented as unitary representations of $SU(2)$ and not by $O(3)$ in spite of the fact that the two groups share the same abstract Lie algebra, characterized by the structure constants ε_{ijk} . Like $O(3)$, the group $SU(2)$ is of order $r = 3$ (number of generators) and rank $l = 1$ (number of diagonal generators). The generators of a unitary group are Hermitian and the special unitary transformations of determinant unity requires the generators to be traceless. The canonical choice is $J_i = \frac{\sigma_i}{2}$; σ_i the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.54)$$

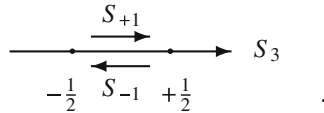
There is one diagonal operator $S_3 = \frac{\sigma_3}{2}$ the 3^{rd} component of spin. The eigenvectors of S_3 are

$$U(r = \frac{1}{2}, -\frac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.55)$$

characterized by the eigenvalues of $\frac{1}{2}$, $-\frac{1}{2}$ of S_3 called spin up [\uparrow] and spin down [\downarrow], respectively. The eigenvectors represent the possible independent states of the system: two in our case. They thus span a two-dimensional space of complex vectors which are called two-spinors. Thus $SU(2)$ is acting on the space of spinors, like $O(3)$ is acting on ordinary configuration space vectors. From the two non-diagonal matrices we may form the two ladder operators: $S_{\pm 1} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$S_{+1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which map the eigenvectors into each other and hence change spin by one unit. The following figure shows the simplest case of a so called *root diagram*: the full dots represent the two states labeled by the eigenvalues $S_3 = \pm \frac{1}{2}$ of the diagonal operator. The arrows, labeled with $S_{\pm 1}$ denote the transitions between the different states, as implied by the Lie algebra:



The simplest non-trivial representation of $SU(2)$ is the so called fundamental representation, the one which defines $SU(2)$ itself and hence has dimension two. It is the one we just have been looking at. There is only one fundamental representation for $SU(2)$, because the complex conjugate U^* of a representation U which is also a representation, and generally a new one, is equivalent to the original one. The fundamental representation describes intrinsic angular momentum $\frac{1}{2}$ with two possible states characterized by the eigenvalues of the diagonal generator $\pm \frac{1}{2}$. The *fundamental representations* are basic because all others may be constructed by taking tensor products of fundamental representations. In the simplest case of a product of two spin $\frac{1}{2}$ vectors, which are called (two component) spinors $u_i v_k$ may describe a spin zero (anti-parallel spins $[\uparrow\downarrow]$) or a spin 1 (parallel spins $[\uparrow\uparrow]$).

In a relativistic theory, described in more detail in the previous section, one has to consider the Lorentz group L^{\uparrow}_+ of proper (preserving orientation of space-time [+]) orthochronous (preserving the direction of time $[\uparrow]$) Lorentz transformations Λ , in place of the rotation group. They include besides the rotations $R(\omega)$ the Lorentz boosts (special Lorentz transformations) $L(\chi)^9$ by velocity χ . Now rotations do not play any independent role as they are not a Lorentz invariant concept. Correspondingly, purely spatial 3-vectors like the spin vector $\mathbf{S} = \frac{\sigma}{2}$ do not have an invariant meaning. However, the three-vector of Pauli matrices σ may be promoted to a four-vector of 2×2 matrices:

$$\sigma_{\mu} \doteq (\mathbf{1}, \boldsymbol{\sigma}) \text{ and } \hat{\sigma}_{\mu} \doteq (\mathbf{1}, -\boldsymbol{\sigma}) \tag{2.57}$$

⁹The special L-transformation $L(p)$ which transforms from a state in the rest frame $(m, \mathbf{0})$ to a state of momentum p^{μ} may be written as

$$\begin{aligned} L^i_j &= \delta^i_j + \hat{p}_i \hat{p}_j (\cosh \beta - 1) \\ L^i_0 &= L^0_i = \hat{p}_i \sinh \beta \\ L^0_0 &= \cosh \beta \end{aligned} \tag{2.56}$$

with $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, $\cosh \beta = \omega_p/m$, $\sinh \beta = |\mathbf{p}|/m$ and $\tanh \beta = |\mathbf{p}|/\omega_p = v$ the velocity of the state.

which will play a key role in what follows. Again, the L -transformations $\Lambda \in L_+^\uparrow$ on the classical level in (relativistic) quantum mechanics have to be replaced by the simply connected universal covering group with identical Lie algebra, which is $SL(2, C)$, the group of unimodular ($\det U = 1$) complex 2×2 matrix transformations U , with matrix multiplication as the group operation. The group $SL(2, C)$ is related to L_+^\uparrow much in the same way as $SU(2)$ to $O(3)$, namely, the mapping $U_\Lambda \in SL(2, C) \rightarrow \Lambda \in L_+^\uparrow$ is two-to-one and the two-fold degeneracy of elements in L_+^\uparrow is lifted in $SL(2, C)$.

The key mapping establishing a linear one-to-one correspondence between real four-vectors and Hermitian 2×2 matrices is the following: with any real four-vector x^μ in Minkowski space we may associate a Hermitian 2×2 matrix

$$x^\mu \rightarrow X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad (2.58)$$

with

$$\det X = x^2 = x^\mu x_\mu \quad , \quad (2.59)$$

while every Hermitian 2×2 matrix X determines a real four vector by

$$X \rightarrow x^\mu = \frac{1}{2} \text{Tr} (X \sigma^\mu) \quad . \quad (2.60)$$

An element $U \in SL(2, C)$ provides a mapping

$$X \rightarrow X' = U X U^+ \quad \text{i.e.} \quad x'^\mu \sigma_\mu = x^\nu U \sigma_\nu U^+ \quad (2.61)$$

between Hermitian matrices, which preserves the determinant

$$\det X' = \det U \det X \det U^+ = \det X \quad , \quad (2.62)$$

and corresponds to the real linear transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (2.63)$$

which satisfies $x'^\mu x'_\mu = x^\mu x_\mu$ and therefore is a Lorentz transformation.

The Lie algebra of $SL(2, C)$ is the one of L_+^\uparrow and thus given by 6 generators: \mathbf{J} for the rotations and \mathbf{K} for the Lorentz boosts, satisfying

$$[J_i, J_k] = i\epsilon_{ikl} J_l \quad , \quad [J_i, K_k] = i\epsilon_{ikl} K_l \quad , \quad [K_i, K_k] = -i\epsilon_{ikl} J_l \quad (2.64)$$

as a coupled algebra of the J_i 's and K_i 's. Since these generators are Hermitian $\mathbf{J} = \mathbf{J}^+$ and $\mathbf{K} = \mathbf{K}^+$ the group elements $e^{-i\omega\mathbf{J}}$ and $e^{i\chi\mathbf{K}}$ are **unitary**.¹⁰ This algebra can be decoupled by the linear transformation

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \quad (2.65)$$

under which the Lie algebra takes the form

$$\mathbf{A} \times \mathbf{A} = i\mathbf{A}, \quad \mathbf{B} \times \mathbf{B} = i\mathbf{B}, \quad [A_i, B_j] = 0 \quad (2.66)$$

of two decoupled angular momentum algebras. Since $\mathbf{A}^+ = \mathbf{B}$ and $\mathbf{B}^+ = \mathbf{A}$, the new generators are not Hermitian any more and hence give rise to **non-unitary** irreducible representations. These are **finite dimensional** and evidently characterized by a pair (A, B) , with $2A$ and $2B$ integers. The dimension of the representation (A, B) is $(2A+1) \cdot (2B+1)$. The angular momentum of the representation (A, B) decomposes into $J = A + B, A + B - 1, \dots, |A - B|$. Massive particle states are constructed starting from the rest frame where J is the spin and the state corresponds to a multiplet of $2J + 1$ degrees of freedom.

The crucial point is that in relativistic QM besides the mass of a state also the spin has an invariant (reference-frame independent) meaning. There exist exactly two **Casimir operators**, invariant operators commuting with all generators (2.6) and (2.7) of the Poincaré group \mathcal{P}_+^\uparrow . One is the mass operator

$$M^2 = P^2 = g_{\mu\nu} P^\mu P^\nu \quad (2.67)$$

the other is

$$L^2 = g_{\mu\nu} L^\mu L^\nu; \quad L^\mu \doteq \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}, \quad (2.68)$$

where L^μ is the Pauli-Lubansky operator. These operators characterize mass m and spin j of the states in an invariant way: $M^2|p, j, j_3; \alpha\rangle = p^2|p, j, j_3; \alpha\rangle$ and $L^2|p, j, j_3; \alpha\rangle = -m^2 j(j+1)|p, j, j_3; \alpha\rangle$.

The classification by (A, B) together with (2.65) shows that for $SL(2, C)$ we have two inequivalent fundamental two-dimensional representations: $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The transformations may be written as a unitary rotation times a Hermitian boost as

¹⁰In $SL(2, C)$ the Lie algebra obviously has the 2×2 matrix representation $J_i = \sigma_i/2$, $K_i = \pm i \sigma_i/2$ in terms of the Pauli matrices, however, $\mathbf{K}^+ = -\mathbf{K}$ is non-Hermitian and the corresponding finite dimensional representation non-unitary. Unitary representations of the Lorentz group, required to implement relativistic covariance on the Hilbert space of physical states, are necessarily infinite dimensional. Actually, the two possible signs of K_i indicated exhibits that there are two different inequivalent representations.

follows¹¹:

$$\begin{aligned} U_\Lambda &= U(\boldsymbol{\chi}, \boldsymbol{\omega}) = D^{(\frac{1}{2})}(\Lambda) = e^{\boldsymbol{\chi} \cdot \frac{\boldsymbol{\sigma}}{2}} e^{-i\boldsymbol{\omega} \cdot \frac{\boldsymbol{\sigma}}{2}} & \text{for } \left(\frac{1}{2}, 0\right) \\ \bar{U}_\Lambda &= U_{\Lambda^{-1}}^+ = \bar{D}^{(\frac{1}{2})}(\Lambda) = e^{-\boldsymbol{\chi} \cdot \frac{\boldsymbol{\sigma}}{2}} e^{-i\boldsymbol{\omega} \cdot \frac{\boldsymbol{\sigma}}{2}} & \text{for } \left(0, \frac{1}{2}\right) . \end{aligned} \quad (2.69)$$

While σ_μ (2.57) is a covariant vector

$$U_\Lambda \sigma_\mu U_\Lambda^+ = \Lambda^\nu{}_\mu \sigma_\nu \quad (2.70)$$

with respect to the representation $U_\Lambda = D^{(\frac{1}{2})}(\Lambda)$, the vector $\hat{\sigma}_\mu$ (2.57) is covariant with respect to $\bar{U}_\Lambda = \bar{D}^{(\frac{1}{2})}(\Lambda)$

$$\bar{U}_\Lambda \hat{\sigma}_\mu \bar{U}_\Lambda^+ = \Lambda^\nu{}_\mu \hat{\sigma}_\nu . \quad (2.71)$$

Note that

$$U(\boldsymbol{\chi}, \mathbf{n}\theta) \text{ and } U(\boldsymbol{\chi}, \mathbf{n}(\theta + 2\pi)) = -U(\boldsymbol{\chi}, \mathbf{n}\theta) \quad (2.72)$$

represent the same Lorentz transformation. U_Λ is therefore a double-valued representation of L_+^\uparrow .

An important theorem [25] says that

Theorem 2.4 *A massless particle of helicity λ may be only in the representations satisfying $(A, B) = (A, A - \lambda)$, where $2A$ and $2(A - \lambda)$ are non-negative integer numbers.*

Thus the simplest representations for massless fields are the spin 1/2 states

$$\begin{aligned} \lambda = +\frac{1}{2} &: \left(\frac{1}{2}, 0\right) & \text{right-handed (R)} \\ -\frac{1}{2} &: \left(0, \frac{1}{2}\right) & \text{left-handed (L)} \end{aligned} \quad (2.73)$$

of helicity $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

The finite dimensional irreducible representations of $SL(2, C)$ to mass 0 and spin j are one-dimensional and characterized by the helicity $\lambda = \pm j$. To a given spin $j > 0$ there exist exactly two helicity states. Each of the two possible states is invariant by itself under L_+^\uparrow , however, the two states get interchanged under parity transformations:

$$U_P h U_P^{-1} = -h . \quad (2.74)$$

Besides the crucial fact of the validity of the spin-statistics theorem (valid in any relativistic QFT), here we notice another important difference between spin in

¹¹Again, these finite dimensional representations U_Λ , U_P (below), etc. should not be confused with the corresponding infinite dimensional unitary representations $U(\Lambda)$, $U(P)$, etc. acting on the Hilbert space of physical states considered in the preceding section.

non-relativistic QM and spin in QFT. In QM spin 1/2 is a system of two degrees of freedom as introduced by Pauli, while in QFT where we may consider the massless case we have two independent singlet states. Parity P, as we know, acts on four-vectors like $Px = (x^0, -\mathbf{x})$ and satisfies¹² $P^2 = 1$. With respect to the rotation group O_3 , P^2 is just a rotation by the angle 2π and thus in the context of the rotation group P has no special meaning. This is different for the Lorentz group. While

$$U_P \mathbf{J} = \mathbf{J} U_P \quad (2.75)$$

commutes

$$U_P \mathbf{K} = -\mathbf{K} U_P \quad (2.76)$$

does not. As a consequence, we learn that

$$U_P U(\boldsymbol{\chi}, \mathbf{n} \theta) = U(-\boldsymbol{\chi}, \mathbf{n} \theta) U_P \quad (2.77)$$

and hence

$$U_P U_\Lambda = \tilde{U}_\Lambda U_P \quad (2.78)$$

Thus under parity a left-handed massless fermion is transformed into a right-handed one and vice versa, which of course is also evident from Fig. 2.2, if we take into account that a change of frame by a Lorentz transformation (velocity $v \leq c$) cannot flip the spin of a massless particle.

The necessity to work with $SL(2, C)$ becomes obvious once we deal with spinors. On a classical level, two-spinors or Weyl spinors w are elements of a vector space V of two complex entries, which transform under $SL(2, C)$ by matrix multiplication: $w' = U w$, $w \in V$, $U \in SL(2, C)$

$$w = \begin{pmatrix} a \\ b \end{pmatrix} ; \quad a, b \in C \quad (2.79)$$

Corresponding to the two representations there exist two local Weyl spinor fields (see (2.12))

$$\begin{aligned} \varphi_a(x) &= \sum_{r=\pm 1/2} \int d\mu(p) \{ u_a(p, r) a(\mathbf{p}, r) e^{-ipx} + v_a(p, r) b^+(\mathbf{p}, r) e^{ipx} \} \\ \chi_a(x) &= \sum_{r=\pm 1/2} \int d\mu(p) \{ \hat{u}_a(p, r) a(\mathbf{p}, r) e^{-ipx} + \hat{v}_a(p, r) b^+(\mathbf{p}, r) e^{ipx} \} , \end{aligned} \quad (2.80)$$

¹²Note that while $P^2 = 1$ the phase η_P of its unitary representation U_P is constrained by $U_P^2 = \pm 1$ only, i.e. $\eta_P = \pm 1$ or $\pm i$.

with two components $a = 1, 2$, which satisfy the Weyl equations

$$\begin{aligned} i (\hat{\sigma}^\mu \partial_\mu)_{ab} \varphi_b(x) &= m \chi_a(x) \\ i (\sigma^\mu \partial_\mu)_{ab} \chi_b(x) &= m \varphi_a(x) . \end{aligned} \quad (2.81)$$

The appropriate one–particle wave functions $u(p, r)$ etc. may be easily constructed as follows: for a massive particle states are constructed by starting in the rest frame where rotations act as ($\omega = |\boldsymbol{\omega}|$, $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$)

$$D^{(\frac{1}{2})}(R(\boldsymbol{\omega})) = \bar{D}^{(\frac{1}{2})}(R(\boldsymbol{\omega})) = e^{-i\boldsymbol{\omega} \cdot \boldsymbol{\sigma} / 2} = \mathbf{1} \cos \frac{\omega}{2} - i \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\omega}} \sin \frac{\omega}{2} . \quad (2.82)$$

Notice that this $SU(2)$ rotation is a rotation by half of the angle, only, of the corresponding classical O_3 rotation. Here the non–relativistic construction of the states applies and the spinors at rest are given by (2.55). The propagating particles carrying momentum \mathbf{p} are then obtained by performing a Lorentz–boost to the states at rest. A boost $L(\mathbf{p})$ (2.56) of momentum \mathbf{p} is given by $D^{(\frac{1}{2})}(L(\mathbf{p})) = e^{\boldsymbol{\chi} \cdot \boldsymbol{\sigma} / 2} = N^{-1} (p^\mu \sigma_\mu + m)$ and $\bar{D}^{(\frac{1}{2})}(L(\mathbf{p})) = e^{-\boldsymbol{\chi} \cdot \boldsymbol{\sigma} / 2} = N^{-1} (p^\mu \hat{\sigma}_\mu + m)$, respectively, in the two basic representations. $N = (2m (p^0 + m))^{-\frac{1}{2}}$ is the normalization factor. The one–particle wave functions (two–spinors) of a Weyl particle and its antiparticle are thus given by

$$u(p, r) = N^{-1} (p^\mu \sigma_\mu + m) U(r) \quad \text{and} \quad v(p, r) = N^{-1} (p^\mu \sigma_\mu + m) V(r) ,$$

respectively, where $U(r)$ and $V(r) = -i\sigma_2 U(r)$ are the rest frame spinors (2.55). The last relation one has to require for implementing the charge conjugation property for the spinors (2.31) in terms of the matrix (2.30). For the adjoint representation, similarly,

$$\hat{u}(p, r) = N^{-1} (p^\mu \hat{\sigma}_\mu + m) U(r) \quad \text{and} \quad \hat{v}(p, r) = -N^{-1} (p^\mu \hat{\sigma}_\mu + m) V(r) .$$

The $-$ sign in the last equation, $(-1)^{2j}$ for spin j , is similar to the $-i\sigma_2$ in the relation between U and V , both are required to make the fields local and with proper transformation properties. We can easily derive (2.81) now. We may write $\hat{\sigma}_\mu p^\mu = \omega_p \mathbf{1} - \boldsymbol{\sigma} \mathbf{p} = 2|\mathbf{p}| (\frac{\omega_p}{2|\mathbf{p}|} \mathbf{1} - h)$ where $h \equiv \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2|\mathbf{p}|}$ is the helicity operator, and for massless states, where $\omega_p = |\mathbf{p}|$, we have $\hat{\sigma}_\mu p^\mu = 2|\mathbf{p}| (\frac{1}{2} - h)$ a projection operator on states with helicity $-\frac{1}{2}$, while $\sigma_\mu p^\mu = 2|\mathbf{p}| (\frac{1}{2} + h)$ a projection operator on states with helicity $+\frac{1}{2}$. Furthermore, we observe that $p^\mu p^\nu \hat{\sigma}_\mu \sigma_\nu = p^\mu p^\nu \sigma_\mu \hat{\sigma}_\nu = p^2 = m^2$ and one easily verifies the Weyl equations using the given representations of the wave functions.

In the massless limit $m \rightarrow 0$: $p^0 = \omega_p = |\mathbf{p}|$ we obtain two decoupled equations

$$\begin{aligned} i(\hat{\sigma}^\mu \partial_\mu)_{ab} \varphi_b(x) &= 0 \\ i(\sigma^\mu \partial_\mu)_{ab} \chi_b(x) &= 0 . \end{aligned}$$

In momentum space the fields are just multiplied by the helicity projector and the equations say that the massless fields have fixed helicities:

$$\left(\frac{1}{2}, 0\right) : \varphi \sim \psi_R \quad \left(0, \frac{1}{2}\right) : \chi \sim \psi_L \quad (2.83)$$

which suggests to rewrite the transformations as

$$\psi_{a_{L,R}}(x) \rightarrow \psi'_{a_{L,R}}(x') = (\Lambda_{L,R})_{ab} \psi_{b_{L,R}}(\Lambda x) \quad (2.84)$$

with

$$(\Lambda_{L,R})_{ab} = \left(e^{\pm i\chi \frac{\sigma}{2}} e^{-i\omega \frac{\sigma}{2}} \right)_{ab} \quad (\Lambda_R^+ = \Lambda_L^{-1}) . \quad (2.85)$$

Using $\sigma_2 \sigma_i \sigma_2 = -\sigma_i^*$ one can show that $\sigma_2 \Lambda_L \sigma_2 = \Lambda_R^*$. Thus, $\psi_L^c \equiv \sigma_2 \psi_L^*$ (up to an arbitrary phase) is defining a charge conjugate spinor which transforms as $\psi_L^c \sim \psi_R$. Indeed $\Lambda_R \psi_L^c = \Lambda_R \sigma_2 \psi_L^* = \sigma_2 \Lambda_L^* \psi_L^* = \sigma_2 \psi_L^{*' } = \psi_L^{c'}$ and thus $\psi_L^c \equiv \sigma_2 \psi_L^* \equiv \varphi \sim \psi_R$. Similarly, $\psi_R^c \equiv \sigma_2 \psi_R^* \equiv \chi \sim \psi_L$. We thus learn, that for massless fields, counting particles and antiparticles separately, we may consider all fields to be left-handed. The second term in the field, the antiparticle creation part, in each case automatically includes the right-handed partners.

The Dirac field is the bispinor field obtained by combining the irreducible fields $\varphi_a(x)$ and $\chi_a(x)$ into one reducible field $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. It is the natural field to be used to describe fermions participating parity conserving interactions like QED and QCD. Explicitly, the Dirac field is given by

$$\psi_\alpha(x) = \begin{pmatrix} \varphi_a \\ \chi_a \end{pmatrix} (x) = \sum_r \int d\mu(p) \left\{ u_\alpha(p, r) a(\mathbf{p}, r) e^{-ipx} + v_\alpha(p, r) b^\dagger(\mathbf{p}, r) e^{ipx} \right\}$$

where

$$u_\alpha = \begin{pmatrix} u_a \\ \hat{u}_a \end{pmatrix} ; \quad v_\alpha = \begin{pmatrix} v_a \\ \hat{v}_a \end{pmatrix} . \quad (2.86)$$

$\psi_\alpha(x)$ satisfies the **Dirac equation**:

$$(i\gamma^\mu \partial_\mu - m)_{\alpha\beta} \psi_\beta(x) = 0$$

where

$$\gamma^\mu \doteq \begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix} \quad (2.87)$$

are the Dirac matrices in the helicity representation (Weyl basis).

The Dirac equation is nothing but the Weyl equations written in terms of the bispinor ψ . Note that a Dirac spinor combines a right-handed Weyl spinor of a particle with a right-handed Weyl spinor of its antiparticle. For $m = 0$, the Dirac operator $i\gamma^\mu\partial_\mu$ in momentum space is $\not{p} = \gamma^\mu p_\mu$. Thus the Dirac equation just is the helicity eigenvalue equation:

$$\gamma^\mu p_\mu \tilde{\psi}(p) \doteq \begin{pmatrix} 0 & \sigma^\mu p_\mu \\ \hat{\sigma}^\mu p_\mu & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} (p) = 2|\mathbf{p}| \begin{pmatrix} 0 & \left(\frac{1}{2} + h\right) \\ \left(\frac{1}{2} - h\right) & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} (p) = 0. \quad (2.88)$$

Under parity $\psi_\alpha(x)$ transforms into itself

$$\psi_\alpha(x) \rightarrow \eta_P (\gamma^0)_{\alpha\beta} \psi_\beta(Px)$$

where γ^0 just interchanges $\varphi \leftrightarrow \chi$ and hence takes the form

$$\gamma^0 \doteq \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

The irreducible components φ and χ are eigenvectors of the matrix

$$\gamma_5 \doteq \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

and the projection operators (2.20) projecting back to the Weyl fields according to (2.19).¹³

The kinetic term of the Dirac Lagrangian decomposes into a L and a R part $\mathcal{L}_{\text{Dirac}} = \bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + \bar{\psi}_L\gamma^\mu\partial_\mu\psi_L$ (4 degrees of freedom). A Dirac mass term $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ breaks chiral symmetry as it is non-diagonal in the Weyl fields and induces helicity flip transitions as required by the anomalous magnetic moment in a renormalizable QFT. A remark concerning hadrons. It might look somewhat surprising that hadrons, which are composite particles made of *colored*

¹³The standard representation of the Dirac field/algebra, described in Sect. 2.1.1, is adapted to a simple interpretation in the rest frame (requires $m \neq 0$). It may be obtained from the ones in the Weyl basis (“helicity” representation) by a similarity transformation \mathcal{S}

$$\psi(x) = \mathcal{S} \psi^{\text{helicity}}(x), \quad \gamma_\mu = \mathcal{S} \gamma_\mu^{\text{helicity}} \mathcal{S}^{-1}, \quad \mathcal{S} = \mathcal{S}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

such that

$$u(0, r) = \sqrt{2m} \begin{pmatrix} U(r) \\ 0 \end{pmatrix}, \quad v(0, r) = \sqrt{2m} \begin{pmatrix} 0 \\ V(r) \end{pmatrix}$$

in the standard basis.

quarks and *gluons*, in many respects look like “elementary particles” which are well described as Wigner particles (if one switches off the electromagnetic interaction which cause a serious IR problem which spoils the naive Wigner state picture as we will describe below), particles of definite mass and spin and charge quantized in units of e and have associated electromagnetic form factors and in particular a definite magnetic moment. However, for the proton for example, the gyromagnetic ratio g_P from the relation $\boldsymbol{\mu}_P = g_P e\hbar/(2m_P c) \mathbf{s}$ turns out to be $g_P \sim 2.8$ or $a_P = (g_P - 2)/2 \sim 0.4$ showing that the proton is not really a Dirac particle and its anomalous magnetic moment indicates that the proton is not a point particle but has internal structure. This was first shown long time ago by atomic beam magnetic deflection experiments [26], before the nature of the muon was clarified. For the latter it was the measurement at CERN which yielded $a_\mu = 0.00119(10)$ [27] and revealed the muon to be just a heavy electron. Within errors at that time the muon turned out to have the same value of the anomalous magnetic moment as the electron, which is known to be due to virtual radiative corrections.

The analysis of the spin structure on a formal level, discussing the quantum mechanical implementation of relativistic symmetry principles, fits very naturally with the observed spin phenomena. In particular the existence of the fundamental spin $\frac{1}{2}$ particles which must satisfy Pauli’s exclusion principle has dramatic consequences for real life. Without the existence of spin as an extra fundamental quantum number in general and the spin $\frac{1}{2}$ fermions in particular, stability of nuclei against Coulomb collapse and of stars against gravitational collapse would be missing and the universe would not be ours.

2.3 Quantum Electrodynamics

The lepton–photon interaction is described by QED, which is structured by local $U(1)$ gauge invariance¹⁴

$$\begin{aligned}\psi(x) &\rightarrow e^{-ie\alpha(x)}\psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \partial_\mu\alpha(x) \ ,\end{aligned}\tag{2.89}$$

with an arbitrary scalar function $\alpha(x)$, implying lepton–photon interaction according to *minimal coupling*, which means that we have to perform the substitution $\partial_\mu \rightarrow$

¹⁴The known elementary particle interactions, the strong, electromagnetic and weak forces, all derive from a local gauge symmetry principle. This was first observed by Weyl [28] for the Abelian QED and later extended to non–Abelian gauge theories by Yang and Mills [29]. The gauge symmetry group governing the Standard Model of particle physics is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$.

$D_\mu = \partial_\mu - ieA_\mu(x)$ in the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ of a free lepton.¹⁵ This implies that the electromagnetic interaction is described by the bare Lagrangian

$$\begin{aligned} \mathcal{L}^{\text{QED}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi^{-1}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ &= \mathcal{L}_{0A}^\xi + \mathcal{L}_{0\psi} + ej_{\text{em}}^\mu(x)A_\mu(x) \ , \end{aligned} \quad (2.90)$$

and the corresponding field equations read¹⁶

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\psi(x) &= -e : A_\mu(x)\gamma^\mu\psi(x) : \\ (\square g^{\mu\nu} - (1 - \xi^{-1})\partial^\mu\partial^\nu)A_\nu(x) &= -e : \bar{\psi}(x)\gamma^\mu\psi(x) : \ . \end{aligned} \quad (2.91)$$

The interaction part of the Lagrangian is

$$\mathcal{L}_{\text{int}} = ej_{\text{em}}^\mu(x)A_\mu(x) \ , \quad (2.92)$$

while the bilinear free field parts \mathcal{L}_{0A}^ξ and $\mathcal{L}_{0\psi}$ define the propagators of the photon and the leptons, respectively (given below). As in classical electrodynamics the gauge potential A^μ is an auxiliary field which exhibits unphysical degrees of freedom, and is not uniquely determined by Maxwell's equations. In order to get a well defined photon propagator a gauge fixing condition is required. We adopt the linear covariant Lorentz gauge : $\partial_\mu A^\mu = 0$, which is implemented via the Lagrange multiplier method, with Lagrange multiplier $\lambda = 1/\xi$, ξ is called gauge parameter.¹⁷ The gauge invariance of physical quantities infers that they do not depend on the gauge parameter.

Above we have denoted by e the charge of the electron, which by convention is taken to be negative. In the following we will explicitly account for the sign of the charge and use e to denote the positive value of the charge of the positron. The charge of a fermion f is then given by $Q_f e$, with Q_f the charge of a fermion in units of the positron charge e . A collection of charged fermions f enters the electromagnetic current as

$$j_{\text{em}}^\mu = \sum_f Q_f \bar{\psi}_f \gamma^\mu \psi_f \ , \quad (2.93)$$

¹⁵The modified derivative $D_\mu = \partial_\mu - ieA_\mu(x)$ is called *covariant derivative*. e is the *gauge coupling*. The minimal substitution promotes the global gauge symmetry of the free Dirac Lagrangian to a local gauge symmetry of the electron–photon system, i.e., the interacting system has more symmetry than the free electron.

¹⁶The prescription $:\dots:$ means *Wick ordering* of products of fields: write the fields in terms of creation and annihilation operators and order them such that all annihilation operators are to the right of all creation operators, assuming the operators to commute (bosons) or to anticommute (fermions). This makes the vacuum expectation value of the field product vanish.

¹⁷The parametrization of the gauge dependence by the inverse of the Lagrange multiplier $\xi = 1/\lambda$ is just a commonly accepted convention.

for the leptons alone $j_{\text{em}}^{\mu \text{lep}} = -\sum_{\ell} \bar{\psi}_{\ell} \gamma^{\mu} \psi_{\ell}$ ($\ell = e, \mu, \tau$). If not specified otherwise $\psi(x)$ in the following will denote a lepton field carrying negative charge $-e$.

The electric charge is a conserved quantity as a consequence of **Noether's theorem**:

Theorem 2.5 *If the Lagrangian $\mathcal{L}(\psi, \partial_{\mu}\psi \dots)$ of a system is invariant under a r -parametric group of global field transformations $\psi(x) \rightarrow \psi(x) + \delta\psi(x), \dots$ then there exist r conserved currents $\partial_{\mu} j_i^{\mu}(x) = 0$, $i = 1, \dots, r$ which imply the existence of r conserved charges*

$$Q_i = \int d^3x j_i^0(t, \mathbf{x}) ; \quad \frac{dQ_i}{dt} = 0, \quad i = 1, \dots, r. \quad (2.94)$$

The global symmetry in our QED case is the global $U(1)_{\text{em}}$ gauge symmetry (i.e. transformations (2.89) with gauge function $\alpha = \text{constant}$).

One important object we need for our purpose is the *unitary* scattering matrix S which encodes the perturbative lepton–photon interaction processes and is given by

$$S = T \left(e^{i \int d^4x \mathcal{L}_{\text{int}}^{(0)}(x)} \right) \Big|_{\otimes}. \quad (2.95)$$

The prescription \otimes says that all graphs (see below) which include vacuum diagrams (disconnected subdiagrams with no external legs) as factors have to be omitted. This corresponds to the proper normalization of the S -operator. Unitarity requires

$$SS^{\dagger} = S^{\dagger}S = 1 \quad \Leftrightarrow \quad S^{\dagger} = S^{-1} \quad (2.96)$$

and infers the conservation of quantum mechanical transition probabilities. The prescription T means time ordering of all operators, like

$$T \{ \phi(x)\phi(y) \} = \Theta(x^0 - y^0)\phi(x)\phi(y) \pm \Theta(y^0 - x^0)\phi(y)\phi(x) \quad (2.97)$$

where the $+$ sign holds for boson fields and the $-$ sign for fermion fields. Under the T prescription all fields are commuting (bosons) or anticommuting (fermions). All fields in (2.95) may be taken to be free fields. With the help of S we may calculate the basic objects of a QFT, the *Green functions*. These are the vacuum expectation values of *time ordered* or *chronological* products of fields like the electromagnetic correlator

$$G_{\mu, \alpha\beta}(x, y, \bar{y}) \doteq \langle 0 | T \{ A_{\mu}(x) \psi_{\alpha}(y) \bar{\psi}_{\beta}(\bar{y}) \} | 0 \rangle. \quad (2.98)$$

2.3.1 Perturbation Expansion, Feynman Rules

The full Green functions of the interacting fields like $A^\mu(x)$, $\psi(x)$, etc. can be expressed completely in terms of corresponding free fields via the *Gell-Mann Low formula* [30] (interaction picture)

$$\begin{aligned} \langle 0|T \{A_\mu(x)\psi_\alpha(y)\bar{\psi}_\beta(\bar{y})\} |0\rangle &= \langle 0|T \left\{ A_\mu^{(0)}(x)\psi_\alpha^{(0)}(y)\bar{\psi}_\beta^{(0)}(\bar{y}) S \right\} |0\rangle_\otimes = \\ \langle 0|T \left\{ A_\mu^{(0)}(x)\psi_\alpha^{(0)}(y)\bar{\psi}_\beta^{(0)}(\bar{y}) e^{i\int d^4x' \mathcal{L}_{\text{int}}^{(0)}(x')} \right\} |0\rangle_\otimes &= \sum_{n=0}^N \frac{i^n}{n!} \int d^4z_1 \cdots d^4z_n \\ \langle 0|T \left\{ A_\mu^{(0)}(x)\psi_\alpha^{(0)}(y)\bar{\psi}_\beta^{(0)}(\bar{y}) \mathcal{L}_{\text{int}}^{(0)}(z_1) \cdots \mathcal{L}_{\text{int}}^{(0)}(z_n) \right\} |0\rangle_\otimes &+ O(e^{N+1}) \quad (2.99) \end{aligned}$$

with $\mathcal{L}_{\text{int}}^{(0)}(x)$ the interaction part of the Lagrangian. On the right hand side all fields are free fields and the vacuum expectation values can be computed by applying the known properties of free fields. Expanding the exponential as done in (2.99) yields the perturbation expansion. The evaluation of the formal perturbation series is not well defined and requires regularization and renormalization, which we will discuss briefly below. In a way the evaluation is simple: one writes all free fields in terms of the creation and annihilation operators and applies the canonical anticommutation (fermions) and the canonical commutation (bosons) relations to bring all annihilation operators to the right, where they annihilate the vacuum $\cdots a(\mathbf{p}, r)|0\rangle = 0$ and the creation operators to the left where again they annihilate the vacuum $0 = \langle 0|b^\dagger(\mathbf{p}, r) \cdots$, until no operator is left over (Wick ordering) [31]. The only non-vanishing contribution comes from the complete contraction of all fields in pairs, where a pairing corresponds to a propagator as a factor. The rules for the evaluation of all possible contributions are known as

The Feynman Rules:

- (1) draw all vertices as points in a plane: external ones with the corresponding external fields $\psi(y_i)$, $\bar{\psi}(\bar{y}_j)$ or $A^\mu(x_k)$ attached to the point, and the internal interaction vertices $-ie\bar{\psi}\gamma_\mu\psi A^\mu(z_n)$ with three fields attached to the point z_n .
- (2) contract all fields in pairs represented by a line connecting the two vertices, thereby fields of different particles are to be characterized by different types of lines. As a result one obtains a Feynman diagram.

The field pairings define the free propagators

$$\overline{\psi(y) \cdots \bar{\psi}(\bar{y})} \Leftrightarrow iS_F(y - \bar{y}) \quad \text{and} \quad \overline{A^\mu(x_1) \cdots A^\nu(x_2)} \Leftrightarrow iD^{\mu\nu}(x_1 - x_2)$$

given by the vacuum expectation values of the pair of time-ordered free fields,

$$\begin{aligned} iS_{F\alpha\beta}(y - \bar{y}) &\doteq \langle 0|T \{ \psi(y)_\alpha \bar{\psi}(\bar{y})_\beta \} |0\rangle \\ iD^{\mu\nu}(x_1 - x_2) &\doteq \langle 0|T \{ A^\mu(x_1) A^\nu(x_2) \} |0\rangle . \end{aligned}$$

(1) Lepton propagator

$$\begin{array}{c} \circ \\ \alpha \end{array} \xrightarrow{p} \begin{array}{c} \circ \\ \beta \end{array} : \quad iS_F(p)_{\alpha\beta} = \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{\alpha\beta}$$

(2) Photon propagator

$$\begin{array}{c} \text{wavy line} \\ \mu \end{array} \xrightarrow{p} \begin{array}{c} \text{wavy line} \\ \nu \end{array} : \quad iD(p, \xi)^{\mu\nu} = -i \left(g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \frac{1}{p^2 + i\epsilon}$$

(3) Lepton–photon vertex

$$\begin{array}{c} \text{wavy line} \\ \mu, p_1 \end{array} \begin{array}{c} \nearrow \\ \alpha, p_3 \\ \searrow \\ \beta, p_2 \end{array} : \quad = \quad -ie (\gamma^\mu)_{\alpha\beta} = ie Q_\ell (\gamma^\mu)_{\alpha\beta}$$

Fig. 2.3 Feynman rules for QED (I)

The latter may easily be calculated using the free field properties.

Feynman diagrams translate into Feynman integrals via the famous Feynman rules given by Fig. 2.3 in momentum space.

In configuration space all interaction vertices in (2.99) are integrated over. The result thus is a Feynman integral. In fact the perturbation expansion is not yet well defined. In order to have a well defined starting point, the theory has to be *regularized* [32] and parameter and fields have to be *renormalized* in order to obtain a well defined set of renormalized Green functions. The problems arise because propagators are singular functions (so called distributions) the products of them are not defined at coinciding space–time arguments (short–distance [coordinate space] or ultra–violet [momentum space] singularities). An example of such an ill–defined product is the Fermion loop contribution to the photon propagator:

$$iS_F(x - y)_{\alpha\beta} (-ie\gamma_\mu)_{\beta\gamma} iS_F(y - x)_{\gamma\delta} (-ie\gamma_\nu)_{\delta\alpha} .$$

The ambiguity in general can be shown to be a local distribution, which for a renormalizable theory is of the form [33]

$$a\delta(x - y) + b^\mu \partial_\mu \delta(x - y) + c \square \delta(x - y) + d^{\mu\nu} \partial_\mu \partial_\nu \delta(x - y)$$

with derivatives up to second order at most, which, in momentum space, is a second order polynomial in the momenta.¹⁸ The regularization we will adopt is dimensional

¹⁸The mathematical problems with the point–like structure of elementary particles and with covariant quantization of the photons hindered the development of QFT for a long time until the break through at the end of the 1940s [34]. In 1965 Tomonaga, Schwinger and Feynman were honored with the Nobel Prize “for their fundamental work in quantum electrodynamics, with deep–ploughing consequences for the physics of elementary particles”. For non–Abelian gauge theories like the modern strong interaction theory *Quantum Chromodynamics* (QCD) [35, 36] and the electroweak *Standard Model* [37], the proper quantization, regularization and renormalization was another obstacle which was solved only at the beginning of the 1970s by ’t Hooft and Veltman [38]. They were awarded the

which correspond to a trace of a product of matrices in spinor space:

$$\text{Tr} \left[S_F(p_1) \gamma_{\mu_1} S_F(p_2) \gamma_{\mu_2} \cdots S_F(p_n) \gamma_{\mu_n} \right] .$$

Closed *fermion loops* actually contribute with two different orientations. If the number of vertices is odd the two orientations yield traces in spinor space of opposite sign such that they cancel provided the two contributions have equal weight. If the number of vertices is even the corresponding traces in spinor space contribute with equal sign, i.e. it just makes a factor of two in the equal weight case. In QED in fact the two orientations have equal weight due to the charge conjugation invariance of QED. An important consequence of C invariance is **Furry's theorem** [40]:

Theorem 2.6 *Fermion loops with an odd number of vector-vertices (i.e. γ^μ type) are vanishing.*

As already mentioned, each Fermion loop carries a factor -1 due the Fermi statistics. All this is easy to check using the known properties of the Dirac fields.¹⁹

For a given set of external vertices and a given order n of perturbation theory (n internal vertices) one obtains a sum over all possible complete contractions, where each one may be represented by a Feynman diagram Γ . The Fourier transform (FT) thus, for each connected component of a diagram, is given by expressions of the form

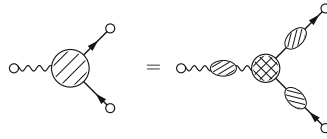
$$\begin{aligned} & \text{FT} \langle 0|T \{ A_\mu(x_1) \cdots \psi_\alpha(y_1) \cdots \bar{\psi}_\beta(\bar{y}_1) \cdots \} |0\rangle_{\text{connected}} = \\ & = (-i)^F (2\pi)^d \delta^{(d)} \left(\sum p_{\text{ext}} \right) \left(\prod_{i=1}^N \int \frac{d^d l_i}{(2\pi)^d} \right) \\ & \times \sum_{\Gamma} \prod_{\ell \in L_\ell, i \in \bar{L}_f} i S_F(p_i) (-ie\gamma_{\mu_i}) \left[\prod_{f \in \bar{L}_f} i S_F(p_f) \right] \prod_{j \in L_\gamma} i D^{\mu_j \nu_j}(q_j) , \end{aligned}$$

where L_ℓ is the set of lepton lines, L_γ the set of photon lines and \bar{L}_f the set of lines starting with an external $\bar{\psi}$ field, N the number of independent closed loops and F the number of closed fermion loops. Of course, spinor indices and Lorentz indices must contract appropriately, and momentum conservation must be respected at each vertex and over all. The basic object of our interest is the Green function associated with the electromagnetic vertex dressed by external propagators:

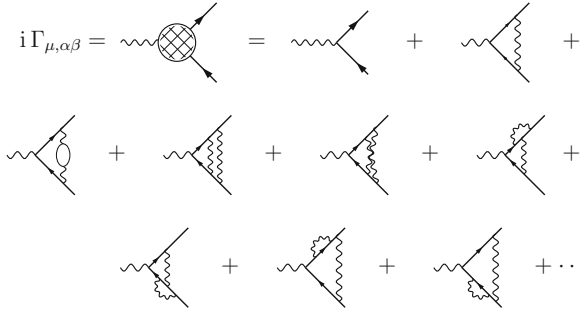
$$\begin{aligned} G_{\mu, \alpha\beta}(x, y, z) & \doteq \langle 0|T \{ A_\mu(x) \psi_\alpha(y) \bar{\psi}_\beta(z) \} |0\rangle = \\ & \int dx' dy' dz' i D'_{\mu\nu}(x' - x) i S'_{F\alpha\alpha'}(y' - y) \left(i \Gamma'_{\alpha'\beta'}(x', y', z') \right) i S'_{F\beta'\beta}(z' - z) \end{aligned}$$

¹⁹Note that in QCD the corresponding closed quark loops with quark–gluon vertices behave differently because of the color matrices at each vertex. The trace of the product of color matrices in general has an even as well as an odd part.

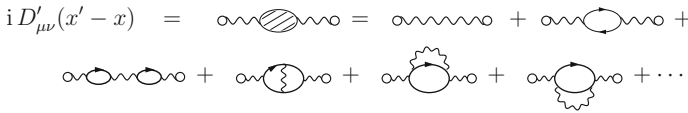
which graphically may be represented as follows



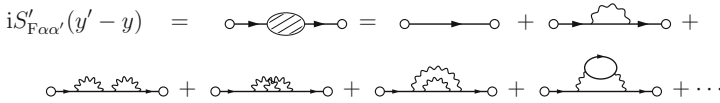
with *one particle irreducible*²⁰ (1PI) dressed vertex



where $iD'_{\mu\nu}(x' - x)$ is a full photon propagator, a photon line dressed with all radiative corrections:



and $iS'_{F\alpha\alpha'}(y' - y)$ is the full lepton propagator, a lepton line dressed by all possible radiative corrections



The tools and techniques of calculating these objects as a perturbation series in lowest non-trivial order will be developed in the next section.

The perturbation series are an iterative solution of the non-perturbative Dyson-Schwinger Equations (DSE) [41], which read: for the full electron propagator $S(p)$

$$S(p) = S_0(p) + S_0(p) \cdot \left(e^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu D^{\mu\nu}(p - k) S(k) \Gamma_\nu(p, k) \right) \cdot S(p),$$

²⁰Diagrams which cannot be cut into two disconnected diagrams by cutting a single line. 1PI diagrams are the building blocks from which any diagram may be obtained as a tree of 1PI “blobs”.

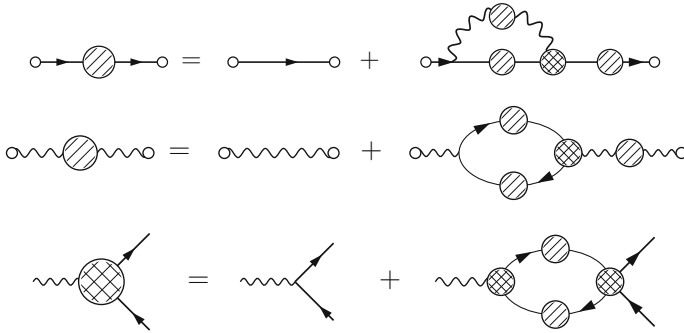


Fig. 2.4 The Dyson–Schwinger integral equations

for the full photon propagator $D^{\mu\nu}(p)$

$$D^{\mu\nu}(p) = D_0^{\mu\nu}(p) + D_0^{\mu\rho}(p) \cdot \left(-e^2 \text{Tr} \left[\int \frac{d^4k}{(2\pi)^4} \gamma_\rho S(k) \Gamma_\sigma(k, k+p) S(k+p) \right] \right) \cdot D^{\sigma\nu}(p),$$

and for the full electron–photon vertex function $\Gamma_\mu(p', p)$

$$\Gamma_\mu(p', p) = \Gamma_{\mu 0}(p', p) + \int \frac{d^4k}{(2\pi)^4} S(p'+k) \Gamma_\mu(p'+k, p+k) S(p+k) K(p+k, p'+k, k)$$

where S_0 is the free electron propagator, $D_0^{\mu\nu}(p)$ the free photon propagator and $\Gamma_{\mu 0}(p', p)$ the free e.m. vertex (see Fig. 2.3). $K(p+k, p'+k, k)$ is the four-electron T -matrix (vanishing at lowest order). The expansion in the free vertex yields the perturbation series. Graphically the SDE are represented in Fig. 2.4.

2.3.2 Transition Matrix–Elements, Particle–Antiparticle Crossing

The Green functions from the point of view of a QFT are building blocks of the theory. However, they are not directly observable objects. The physics is described by quantum mechanical transition matrix elements, which for scattering processes are encoded in the scattering matrix. For QED the latter is given formally by (2.95). The existence of a S -matrix requires that for very early and for very late times ($t \rightarrow \mp\infty$) particles behave as free scattering states. For massless QED, the electromagnetic interaction does not have finite range (Coulomb’s law) and the scattering matrix does not exist in the naive sense. In an order by order perturbative approach the problems manifest themselves as an infrared (IR) problem. As we will see below, nevertheless a suitable redefinition of the transition amplitudes is possible, which allows one a perturbative treatment under appropriate conditions. Usually, one is not directly interested in the S -matrix as the latter includes the identity operator I which describes through–going particles which do not get scattered at all. It is customary

to split off the identity from the S -matrix and to define the T -matrix by

$$S = I + i (2\pi)^4 \delta^{(4)}(P_f - P_i) T , \quad (2.103)$$

with the overall four-momentum conservation factored out. In spite of the fact, that Green functions are not observables they are very useful to understand important properties of the theory. One of the outstanding features of a QFT is the *particle-antiparticle crossing* property which states that in a scattering amplitude an incoming particle [antiparticle] is equivalent to an outgoing antiparticle [particle] and vice versa. It means that the same function, namely an appropriate time-ordered Green function, at the same time describes several processes. For example, muon pair production in electron positron annihilation $e^+e^- \rightarrow \mu^+\mu^-$ is described by amplitudes which at the same time describe electron-muon scattering $e^-\mu^- \rightarrow e^-\mu^-$ or whatever process we can obtain by bringing particles from one side of the reaction balance to the other side as an antiparticle etc. Another example is muon decay $\mu^+ \rightarrow e^+\nu_e\bar{\nu}_\mu$ and neutrino scattering $\nu_\mu e^- \rightarrow \mu^-\nu_e$. For the electromagnetic vertex it relates properties of the electrons [leptons, quarks] to properties of the positron [antileptons, antiquarks].

Since each external free field on the right hand side of (2.99) exhibits an annihilation part and a creation part, each external field has two interpretations, either as an incoming particle or as an outgoing antiparticle. For the adjoint field incoming and outgoing get interchanged. This becomes most obvious if we invert the field decomposition (2.12) for the Dirac field which yields the corresponding creation/annihilation operators

$$a(\mathbf{p}, r) = \bar{u}(\mathbf{p}, r)\gamma^0 \int d^3x e^{ipx} \psi(x) \quad , \quad b^+(\mathbf{p}, r) = \bar{v}(\mathbf{p}, r)\gamma^0 \int d^3x e^{-ipx} \psi(x) .$$

Similarly, inverting (2.13) yields

$$c(\mathbf{p}, \lambda) = -\varepsilon^{\mu*}(\mathbf{p}, \lambda) i \int d^3x e^{ipx} \overleftrightarrow{\partial}_0 A_\mu(x)$$

and its Hermitian conjugate for the photon, with $f(x) \overleftrightarrow{\partial}_\mu g(x) \equiv f(x) \partial_\mu g(x) - (\partial_\mu f(x)) g(x)$. Since these operators create or annihilate scattering states, the above relations provide the bridge between the Green functions, the vacuum expectation values of time-ordered fields, and the scattering matrix elements. This is how the crossing property between different physical matrix elements comes about. The S -matrix elements are obtained from the Green functions by the Lehmann, Symanzik, Zimmermann [42] (LSZ) reduction formula: the external full propagators of the Green functions are omitted (multiplication by the inverse full propagator, i.e. no radiative corrections on external amputated legs) and replaced by an external classical one particle wave function and the external momentum is put on the mass shell. Note that the on-shell limit only exists after the amputation of the external one

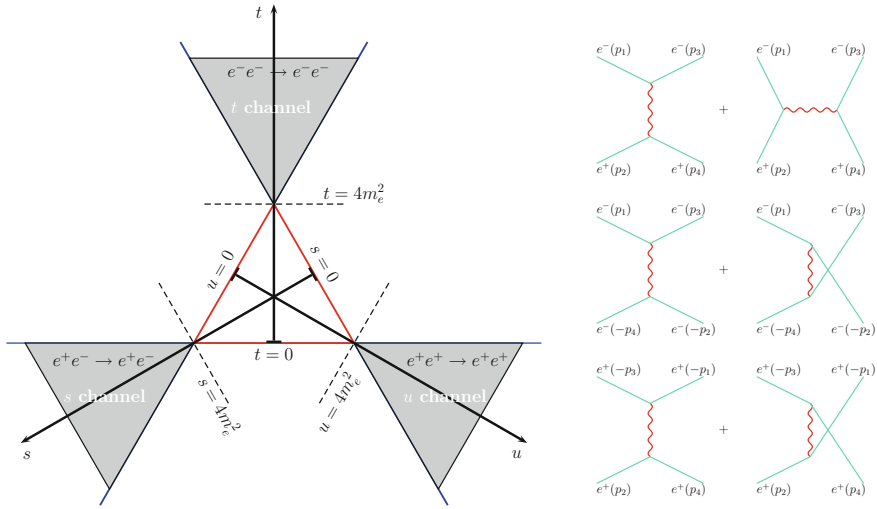


Fig. 2.5 The Mandelstam plane $s + t + u = \sum_{i=1}^4 p_i^2 = \sum_{i=1}^4 m_i^2$. Physical regions are shaded and represent different processes for the appropriate ranges of the Mandelstam variables (s, t, u). The Feynman diagrams shown to be read from *left* (in-state) to *right* (out-state). Light-by-light scattering $\gamma\gamma \rightarrow \gamma\gamma$ is a crossing symmetric process where the different channels represent the same process

$$\begin{aligned}
 e^-(p_1) + e^+(p_2) &\rightarrow e^-(p_3) + e^+(p_4) : & s\text{-channel}; & & s = (p_1 + p_2)^2, \\
 e^-(p_1) + e^-(-p_4) &\rightarrow e^-(-p_2) + e^-(p_3) : & t\text{-channel}; & & t = (p_1 - p_4)^2, \\
 e^+(p_2) + e^+(-p_3) &\rightarrow e^+(-p_1) + e^+(p_4) : & u\text{-channel}; & & u = (p_2 - p_3)^2.
 \end{aligned}$$

Note that $s + t + u = 4m_e^2$ which is the height in a isosceles triangle and gives rise to the Mandelstam plane [43] (see Fig. 2.5).

Given the T matrix-elements, the bridge to the experimental numbers is given by the cross sections and decay rates, which we present for completeness here.

2.3.3 Cross Sections and Decay Rates

The **differential cross section** for a two particle collision

$$A(p_1) + B(p_2) \rightarrow C(p'_1) + D(p'_2) \dots$$

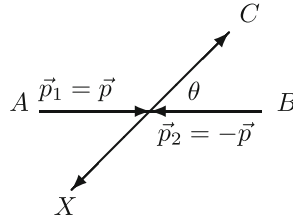
is given by

$$d\sigma = \frac{(2\pi)^4 \delta^{(4)}(P_f - P_i)}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} |T_{fi}|^2 d\mu(p'_1) d\mu(p'_2) \dots$$

$s = (p_1 + p_2)^2$ is the square of the total CM energy and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is a two body phase-space function. In the CM frame (see the figure):

$$\sqrt{\lambda} = \sqrt{\lambda(s, m_1^2, m_2^2)} = 2|\mathbf{p}|\sqrt{s} \quad (2.104)$$

where $\mathbf{p} = \mathbf{p}_i$ is the three-momentum of the initial state particle A.



The total cross section follows by integration over all phase space

$$\sigma = \int d\sigma .$$

Finally, we consider the decay of unstable particles. The differential **decay rate** for $A \rightarrow B + C + \dots$ is given by

$$d\Gamma = \frac{(2\pi)^4 \delta^{(4)}(P_f - P_i)}{2m_1} |T_{fi}|^2 d\mu(p'_1) d\mu(p'_2) \dots$$

By “summing” over all possible decay channels we find the total width

$$\Gamma = \sum d\Gamma = \frac{1}{\tau} , \quad (2.105)$$

where τ is the **lifetime** of the particle, which decays via the exponential **decay law**

$$N(t) = N_0 e^{-t/\tau} . \quad (2.106)$$

Cross sections are measured typically by colliding beams of stable particles and their antiparticles like electrons (e^-), positrons (e^+), protons (p) or antiprotons (\bar{p}). The beam strength of an accelerator or storage ring required for accelerating and collimating the beam particles is determined by the particle flux or *luminosity* L , the number of particles per cm^2 and *seconds*. The energy of the machine determines the resolution

$$\lambda = \frac{hc}{E_{c.m.}} \simeq \frac{1.2 \text{ GeV}}{E_{c.m.}(\text{GeV})} \times 10^{-15} \text{ m} ,$$

while the luminosity determines the collision rate

$$\frac{\Delta N}{\Delta t} = L \cdot \sigma ,$$

and the cross section σ is thus given by dividing the observed event rate by the luminosity

$$\sigma = \frac{1}{L} \frac{\Delta N}{\Delta t} . \quad (2.107)$$

2.4 Regularization and Renormalization

The vertex and self-energy functions, as well as all other Green functions, on the level of the bare theory are well defined order by order in perturbation theory only after smoothing the short distance or ultraviolet (UV) divergences by appropriate regularization. Here we assume QED or the SM to be regularized by dimensional regularization [39]. By going to lower dimensional space-times the features of the theory, in particular the symmetries, remain the same, however, the convergence of the Feynman integrals gets improved. For a renormalizable theory, in principle, one can always choose the dimension low enough, $d < 2$, such that the integrals converge. By one or two partial integrations one can analytically continue the integrals in steps from d to $d + 1$, such that the perturbation expansion is well defined for $d = 4 - \epsilon$ with ϵ a small positive number. For $\epsilon \rightarrow 0$ ($d \rightarrow 4$) the perturbative series in the *fine structure constant* $\alpha = e^2/4\pi$ exhibits poles in ϵ :

$$A = \sum_{n=0}^N \alpha^n \sum_{m=0}^n a_{nm} (1/\epsilon)^{n-m}$$

and the limit $d \rightarrow 4$ to the real physical space-time does not exist, at first. The problems turn out to be related to the fact that the bare objects are not physical ones, they are not directly accessible to observation and require some adjustments. This in particular is the case for the bare parameters, the bare fine structure constant (electric charge) which is modified by vacuum polarization (quantum fluctuations), and the bare masses. Also the bare fields are not the ones which interpolate suitably to the physical states they are assumed to describe. The appropriate entities are in fact obtained by a simple reparametrization in terms of new parameters and fields, which is called *renormalization*.

2.4.1 The Structure of the Renormalization Procedure

Renormalization may be performed in three steps:

- (i) Shift of the mass parameters or *mass renormalization*: replace the bare mass parameters of the bare Lagrangian by renormalized ones

$$\begin{aligned} m_{f0} &= m_{f\text{ren}} + \delta m_f && \text{for fermions} \\ M_{b0}^2 &= M_{b\text{ren}}^2 + \delta M_b^2 && \text{for bosons} \end{aligned} \quad (2.108)$$

(ii) Multiplicative renormalization of the bare fields or *wave function renormalization*: replace the bare fields in the bare Lagrangian by renormalized ones

$$\psi_{f0} = \sqrt{Z_f} \psi_{f\text{ren}} , \quad A_0^\mu = \sqrt{Z_\gamma} A_{\text{ren}}^\mu \quad (2.109)$$

and correspondingly for the other fields of the SM. To leading order $Z_i = 1$ and hence

$$Z_i = 1 + \delta Z_i , \quad \sqrt{Z_i} = 1 + \frac{1}{2} \delta Z_i + \dots \quad (2.110)$$

(iii) Vertex renormalization or *coupling constant renormalization*: substitute the bare coupling constant by the renormalized one

$$e_0 = e_{\text{ren}} + \delta e . \quad (2.111)$$

The **renormalization theorem** (see e.g. [1, 33, 38]) states that

Theorem 2.7 *Order by order in the perturbation expansion all UV divergences showing up in physical quantities (S-matrix elements) get eliminated by an appropriate choice of the counter terms δm_f , δM_b^2 , δe and $\delta Z_i = Z_i - 1$. Physical amplitudes parametrized in terms of physical parameters thus are finite and free of cutoff effects in the large cutoff limit.*

In other words, suitably normalized physical amplitudes expressed in terms of measurable physical parameters are finite in the limit $\epsilon \rightarrow 0$, i.e., they allow us to take away the regularization (cut-off $\Lambda \rightarrow \infty$ if a UV cut-off was used to regularize the bare theory). Note that for Green functions, which are not gauge invariant in general, also the fictitious gauge parameter has to be renormalized in order to obtain finite Green functions. Unitarity requires the counter terms to be real. Therefore the counter terms are determined by the real parts of the location and residues of the one particle poles. Also note: the Z-factors are *gauge dependent* and in order to get gauge invariant S-matrix elements there is no freedom in the choice of the wave function renormalization factors. Only the Z-factors fixed by the LSZ-conditions for the individual fields lead to the physical S-matrix [38, 44]. In fact bare on-shell matrix-elements are not gauge invariant, they become gauge invariant only after wave-function renormalization normalized by the LSZ conditions.

The reparametrization of the bare Lagrangian (2.90) in terms of renormalized quantities reads

$$\begin{aligned}
\mathcal{L}^{\text{QED}} &= -\frac{1}{4} F_{\mu\nu 0}(x) F_0^{\mu\nu}(x) - \frac{1}{2} \xi_0^{-1} (\partial_\mu A_0^\mu(x))^2 + \bar{\psi}_0(x) (i\gamma^\mu \partial_\mu - m_0) \psi_0(x) \\
&\quad - e_0 \bar{\psi}_0(x) \gamma^\mu \psi_0(x) A_{\mu 0}(x) \\
&= \mathcal{L}_{(0)}^{\text{QED}} + \mathcal{L}_{\text{int}}^{\text{QED}} \\
\mathcal{L}_{(0)}^{\text{QED}} &= -\frac{1}{4} F_{\mu\nu \text{ren}}(x) F_{\text{ren}}^{\mu\nu}(x) - \frac{1}{2} \xi_{\text{ren}}^{-1} (\partial_\mu A_{\text{ren}}^\mu(x))^2 \\
&\quad + \bar{\psi}_{\text{ren}}(x) (i\gamma^\mu \partial_\mu - m_{\text{ren}}) \psi_{\text{ren}}(x) \\
\mathcal{L}_{\text{int}}^{\text{QED}} &= -e_{\text{ren}} \bar{\psi}_{\text{ren}}(x) \gamma^\mu \psi_{\text{ren}}(x) A_{\mu \text{ren}}(x) \\
&\quad - \frac{1}{4} (Z_\gamma - 1) F_{\mu\nu \text{ren}}(x) F_{\text{ren}}^{\mu\nu}(x) + (Z_e - 1) \bar{\psi}_{\text{ren}}(x) i\gamma^\mu \partial_\mu \psi_{\text{ren}} \\
&\quad - (m_0 Z_e - m_{\text{ren}}) \bar{\psi}_{\text{ren}} \psi_{\text{ren}}(x) \\
&\quad - (e_0 \sqrt{Z_\gamma} Z_e - e_{\text{ren}}) \bar{\psi}_{\text{ren}}(x) \gamma^\mu \psi_{\text{ren}}(x) A_{\mu \text{ren}}(x) \tag{2.112}
\end{aligned}$$

with $\xi_{\text{ren}} = Z_\gamma \xi_0$ the gauge fixing term remains unrenormalized (no corresponding counter term). The counter terms are now showing up in $\mathcal{L}_{\text{int}}^{\text{QED}}$ and may be written in terms of $\delta Z_\gamma = Z_\gamma - 1$, $\delta Z_e = Z_e - 1$, $\delta m = m_0 Z_e - m_{\text{ren}}$ and $\delta e = e_0 \sqrt{Z_\gamma} Z_e - e_{\text{ren}}$. They are of next higher order in e^2 , either $O(e^2)$ for propagator insertions or $O(e^3)$ for the vertex insertion, in leading order. The counter terms have to be adjusted order by order in perturbation theory by the renormalization conditions which define the precise physical meaning of the parameters (see below).

The Feynman rules Fig. 2.3 have to be supplemented by the rules of including the counter terms as given in Fig. 2.6 in momentum space.

Obviously the propagators (two–point functions) of the photon and of the electron get renormalized according to

$$\begin{aligned}
D_0 &= Z_\gamma D_{\text{ren}} \\
S_{F 0} &= Z_e S_{F \text{ren}} . \tag{2.113}
\end{aligned}$$

Fig. 2.6 Feynman rules for QED (II): the counter terms

- (1) Lepton propagator insertions

$$\begin{array}{c} p \\ \text{---} \otimes \text{---} \\ \alpha \qquad \qquad \beta \end{array} : i (\delta Z_e (\not{p} - m) - \delta m)_{\alpha\beta}$$

- (2) Photon propagator insertion

$$\begin{array}{c} p \\ \text{---} \otimes \text{---} \\ \mu \qquad \qquad \nu \end{array} : -i \delta Z_\gamma (p^2 g^{\mu\nu} - p^\mu p^\nu)$$

- (3) Lepton–photon vertex insertion

$$\begin{array}{c} \alpha, p_3 \\ \text{---} \otimes \text{---} \\ \mu, p_1 \qquad \beta, p_2 \end{array} : = -i \delta e (\gamma^\mu)_{\alpha\beta}$$

The renormalized electromagnetic vertex function may be obtained according to the above rules as

$$\begin{aligned}
 G_{\text{ren}}^\mu &= \frac{1}{\sqrt{Z_\gamma}} \frac{1}{Z_e} G_0^\mu & (2.114) \\
 &= D_{\text{ren}} S_{F \text{ ren}} \Gamma_{\text{ren}}^\mu S_{F \text{ ren}} = \frac{1}{\sqrt{Z_\gamma}} \frac{1}{Z_e} D_0 S_{F 0} \Gamma_0^\mu S_{F 0} \\
 &= \frac{1}{\sqrt{Z_\gamma}} \frac{1}{Z_e} Z_\gamma Z_e^2 D_{\text{ren}} S_{F \text{ ren}} \Gamma_0^\mu S_{F \text{ ren}}
 \end{aligned}$$

and consequently

$$\begin{aligned}
 \Gamma_{\text{ren}}^\mu &= \sqrt{Z_\gamma} Z_e \Gamma_0^\mu = \sqrt{Z_\gamma} Z_e \left\{ e_0 \gamma^\mu + \Gamma_0^{\prime \mu} \right\} \Big|_{e_0 \rightarrow e + \delta e, m_0 \rightarrow m + \delta m, \dots} \\
 &= \sqrt{1 + \delta Z_\gamma} (1 + \delta Z_e) \left\{ e \left(1 + \frac{\delta e}{e} \right) \gamma^\mu + \Gamma_0^{\prime \mu} \right\} \\
 &= \left(1 + \frac{1}{2} \delta Z_\gamma + \delta Z_e + \frac{\delta e}{e} \right) e \gamma^\mu + \Gamma_0^{\prime \mu} + \dots & (2.115)
 \end{aligned}$$

where now the bare parameters have to be considered as functions of the renormalized ones:

$$e_0 = e_0(e, m), \quad m_0 = m_0(m, e) \quad \text{etc.} \quad (2.116)$$

and e, m etc. denote the renormalized parameters. The last line of (2.115) gives the perturbatively expanded form suitable for one-loop renormalization. It may also be considered as the leading n -th order renormalization if $\Gamma_0^{\prime \mu}$ has been renormalized to $n - 1$ -st order for all sub-divergences. More precisely, if we expand the exact relation of (2.115) (second last line) and include all counter terms, including the ones which follow from (2.116), up to order $n - 1$ in $\Gamma_0^{\prime \mu}$, such that all sub-divergences of $\Gamma_0^{\prime \mu}$ are renormalized away, only the overall divergence of order n will be there. After including the wavefunction renormalization factors of order n as well (by calculating the corresponding propagators) the remaining overall divergence gets renormalized away by fixing $\delta e^{(n)}$, according to the last line of (2.115), by the charge renormalization condition:

$$\bar{u}(p_2, r_2) \Gamma_{\text{ren}}^\mu(p_1, p_2) u(p_1, r_1) = e_{\text{ren}} \bar{u}(p_2, r_2) \gamma^\mu u(p_1, r_1)$$

at zero photon momentum $q = p_2 - p_1 = 0$ (classical limit, *Thomson limit*).

2.4.2 Dimensional Regularization

Starting with the Feynman rules of the classical quantized Lagrangian, called bare Lagrangian, the formal perturbation expansion is given in terms of ultraviolet (*UV*) divergent Feynman integrals if we try to do that in $d = 4$ dimensions without a *UV* cut-off. As an example consider the scalar one-loop self-energy diagram and the corresponding Feynman integral

$$\begin{array}{c}
 \text{---} \circlearrowleft \text{---} \\
 \text{---} \text{---} \\
 k
 \end{array}
 = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k+p)^2 - m^2 + i\epsilon} \quad |k| \gg |p|, m \int \frac{d^d k}{k^4}$$

which is logarithmically divergent for the physical space-time dimension $d = 4$ because the integral does not fall-off sufficiently fast at large k . In order to get a well-defined perturbation expansion the theory must be regularized.²¹ The regularization should respect as much as possible the symmetries of the initial bare form of the Lagrangian and of the related *Ward–Takahashi* (*WT*) *identities* of the “classical theory”. For gauge theories like QED, QCD or the SM dimensional regularization [39] (*DR*) is the most suitable regularization scheme as a starting point for the perturbative approach, because it respects as much as possible the classical symmetries of a Lagrangian.²² The idea behind *DR* is the following:

- (i) Feynman rules formally look the same in different space-time dimensions $d = n$ (integer)
- (ii) In the *UV* region Feynman integrals converge the better the lower d is.

The example given above demonstrates this, in $d = 4 - \epsilon$ ($\epsilon > 0$) dimensions (just below $d = 4$) the integral is convergent. Before we specify the rules of *DR* in more detail, let us have a look at convergence properties of Feynman integrals.

²¹Often one simply chooses a cut-off (upper integration limit in momentum space) to make the integrals converge by “brute force”. A cut-off may be considered to parametrize our ignorance about physics at very high momentum or energy. If the cut-off Λ is large with respect to the energy scale E of a phenomenon considered, $E \ll \Lambda$, the cut-off dependence may be removed by considering only relations between low-energy quantities (renormalization). Alternatively, a cut-off may be interpreted as the scale where one expects new physics to enter and it may serve to investigate how a quantity (or the theory) behaves under changes of the cut-off (renormalization group). In most cases simple cut-off regularization violates symmetries badly and it becomes a difficult task to make sure that one obtains the right theory when the cut-off is removed by taking the limit $\Lambda \rightarrow \infty$ after renormalization.

²²An inconsistency problem, concerning the definition of γ_5 for $d \neq 4$, implies that the chiral *WT* identities associated with the parity violating weak fermion currents in the SM are violated in general (see e.g. [45]).

Dyson Power Counting

The action

$$S = i \int d^d x \mathcal{L}_{\text{eff}} \tag{2.117}$$

measured in units of $\hbar = 1$ is dimensionless and therefore $\dim \mathcal{L}_{\text{eff}} = d$ in mass units. The inspection of the individual terms yields the following dimensions for the fields:

$$\begin{aligned} \bar{\psi} \gamma^\mu \partial_\mu \psi & : \dim \psi = \frac{d-1}{2} \\ (\partial_\mu A_\nu - \dots)^2 & : \dim A_\mu = \frac{d-2}{2} \\ \bar{e}_0 \bar{\psi} \gamma^\mu \psi A_\mu & : \dim \bar{e}_0 = \frac{4-d}{2} \Rightarrow \bar{e}_0 = e_0 \mu^{\epsilon/2} \end{aligned} \tag{2.118}$$

where $\epsilon = 4 - d$, e_0 denotes the dimensionless bare coupling constant ($\dim e_0 = 0$) and μ is an arbitrary mass scale. The dimension of time ordered Green functions in momentum space is then given by (the Fourier transformation $\int d^d q e^{-iqx} \dots$ gives $-d$ for each field):

$$\dim G^{(n_B, 2n_F)} = n_B \frac{d-2}{2} + 2n_F \frac{d-1}{2} - (n_B + 2n_F)d$$

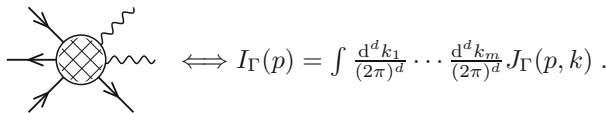
where

$$\begin{aligned} n_B & : \text{\#of boson fields} : G_{i\mu}, \dots \\ 2n_F & : \text{\#of Dirac fields (in pairs)} : \psi \dots \bar{\psi} \end{aligned}$$

It is convenient to split off factors which correspond to external propagators (see p. 52) and four-momentum conservation and to work with 1PI amplitudes, which are the objects relevant for calculating T matrix elements. The corresponding proper amputated vertex functions are of dimension

$$\dim \hat{G}^{\text{amp}} = d - n_B \frac{d-2}{2} - 2n_F \frac{d-1}{2} \tag{2.119}$$

A generic Feynman diagram represents a Feynman integral



$$\iff I_\Gamma(p) = \int \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_m}{(2\pi)^d} J_\Gamma(p, k) .$$

The convergence of the integral can be inspected by looking at the behavior of the integrand for large momenta: For $k_i = \lambda \hat{k}_i$ and $\lambda \rightarrow \infty$ we find

$$\prod_i d^d k_i J_\Gamma(p, k) \rightarrow \lambda^{d(\Gamma)}$$

where

$$d(\Gamma) = d - n_B \frac{d-2}{2} - 2n_F \frac{d-1}{2} + \sum_{i=1}^n (d_i - d)$$

is called the *superficial divergence* of the 1PI diagram Γ . The sum extends over all (n) vertices of the diagram and d_i denotes the dimension of the vertex i . The $-d$ at each vertex accounts for d -momentum conservation. For a vertex exhibiting $n_{i,b}$ Bose fields, $n_{i,f}$ Fermi fields and l_i derivatives of fields we have

$$d_i = n_{i,b} \frac{d-2}{2} + n_{i,f} \frac{d-1}{2} + l_i \quad (2.120)$$

Here it is important to mention one of the most important conditions for a QFT to develop its full predictive power: *renormalizability*. In order that $d(\Gamma)$ in (2.120) is bounded in physical space-time $d = 4$ all interaction vertices must have dimension not more than $d_i \leq 4$. An anomalous magnetic moment effective interaction term (Pauli term)

$$\delta \mathcal{L}_{\text{eff}}^{\text{AMM}} = \frac{ieg}{4m} \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) F^{\mu\nu}(x) , \quad (2.121)$$

has dimension 5 (in $d = 4$) and thus would spoil the renormalizability of the theory.²³ Such a term is thus forbidden in any renormalizable QFT. In contrast, in any renormalizable QFT the anomalous magnetic moment of a fermion is a quantity unambiguously predicted by the theory.

The relation (2.120) may be written in the alternative form

$$d(\Gamma) = 4 - n_B - 2n_F \frac{3}{2} + L(d-4) .$$

The result can be easily understood: the loop expansion of an amplitude has the form

$$A^{(L)} = A^{(0)} [1 + a_1 \alpha + a_2 \alpha^2 + \dots + a_L \alpha^L + \dots] \quad (2.122)$$

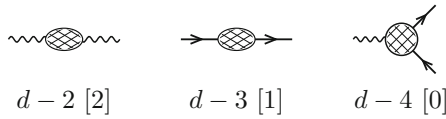
where $\alpha = e^2/4\pi$ is the conventional expansion parameter. $A^{(0)}$ is the tree level amplitude which coincides with the result in $d = 4$.

We are ready now to formulate the convergence criterion which reads:

$$\begin{aligned} I_\Gamma \text{ convergent} &\Leftrightarrow d(\gamma) < 0 \quad \forall \text{ 1PI subdiagrams } \gamma \subseteq \Gamma \\ I_\Gamma \text{ divergent} &\Leftrightarrow \exists \gamma \subseteq \Gamma \text{ with } d(\gamma) \geq 0 . \end{aligned}$$

²³The dimension of $F^{\mu\nu}$ is 2, 1 for the photon field plus 1 for the derivative.

In $d \leq 4$ dimensions, a renormalizable theory has the following types of primitively divergent diagrams (i.e., diagrams with $d(\Gamma) \geq 0$ which may have divergent sub-integrals)²⁴:



$+(L_\Gamma - 1)(d - 4)$ for a diagram with $L_\Gamma (\geq 1)$ loops. The list shows the non-trivial leading one-loop $d(\Gamma)$ to which per additional loop a contribution $(d - 4)$ has to be added (see (2.122)), in square brackets the values for $d = 4$. Thus the dimensional analysis tells us that convergence improves for $d < 4$. For a renormalizable theory we have

- $d(\Gamma) \leq 2$ for $d = 4$.

In lower dimensions

- $d(\Gamma) < 2$ for $d < 4$

a renormalizable theory becomes super-renormalizable, while in higher dimensions

- $d(\Gamma)$ unbounded! $d > 4$

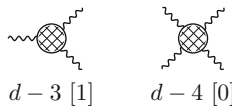
and the theory is non-renormalizable.

Dimensional Regularization

Dimensional regularization of theories with spin is defined in three steps.

1. Start with Feynman rules formally derived in $d = 4$.
2. Generalize to $d = 2n > 4$. This intermediate step is necessary in order to treat the vector and spinor indices appropriately. Of course it means that the UV behavior of Feynman integrals at first gets worse.

²⁴According to (2.122) there are two more potentially divergent structures



with superficial degree of divergence as indicated. However, the triple photon vertex is identically zero by Furry's theorem, C odd amplitudes are zero in the C preserving QED. The four photon light-by-light scattering amplitude, due the transversality of the external physical photons, has an effective dimension $d(\Gamma)_{\text{eff}} = -4$, instead of 0, and is thus very well convergent. For the same reason, transversality of the photon self-energy, actually the photon propagator has $d(\Gamma)_{\text{eff}} = 0$ instead of 2. In both cases it is the Abelian gauge symmetry which makes integrals better convergent than they look like by naive power counting.

(1) For fermions we need the $d = 2n$ -dimensional Dirac algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbf{1}; \quad \{\gamma^\mu, \gamma_5\} = 0 \quad (2.123)$$

where γ_5 must satisfy $\gamma_5^2 = \mathbf{1}$ and $\gamma_5^+ = \gamma_5$ such that $\frac{1}{2}(\mathbf{1} \pm \gamma_5)$ are the chiral projection matrices. The metric has dimension d

$$g^{\mu\nu} g_{\mu\nu} = g_\mu^\mu = d; \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & -1 & \\ \vdots & & \ddots \\ & & & -1 \end{pmatrix}.$$

By $\mathbf{1}$ we denote the unit matrix in spinor space. In order to have the usual relation for the adjoint spinors we furthermore require

$$\gamma^{\mu+} = \gamma^0 \gamma^\mu \gamma^0. \quad (2.124)$$

Simple consequences of this d -dimensional algebra are:

$$\begin{aligned} \gamma_\alpha \gamma^\alpha &= d \mathbf{1} \\ \gamma_\alpha \gamma^\mu \gamma^\alpha &= (2 - d) \gamma^\mu \\ \gamma_\alpha \gamma^\mu \gamma^\nu \gamma^\alpha &= 4g^{\mu\nu} \mathbf{1} + (d - 4) \gamma^\mu \gamma^\nu \\ \gamma_\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\alpha &= -2\gamma^\rho \gamma^\nu \gamma^\mu + (4 - d) \gamma^\mu \gamma^\nu \gamma^\rho \text{ etc.} \end{aligned} \quad (2.125)$$

Traces of strings of γ -matrices are very similar to the ones in 4-dimensions. In $d = 2n$ dimensions one can easily write down $2^{d/2}$ -dimensional representations of the Dirac algebra [46]. Then

$$\begin{aligned} \text{Tr } \mathbf{1} &= f(d) = 2^{d/2} \\ \text{Tr } \prod_{i=1}^{2n-1} \gamma^{\mu_i} (\gamma^5) &= 0 \\ \text{Tr } \gamma^\mu \gamma^\nu &= f(d) g^{\mu\nu} \\ \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= f(d) (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \text{ etc.} \end{aligned} \quad (2.126)$$

One can show that for *renormalized quantities* the only relevant property of $f(d)$ is $f(d) \rightarrow 4$ for $d \rightarrow 4$. Very often the convention $f(d) = 4$ (for any d) is adopted. Bare quantities and the related *minimally subtracted* MS or modified minimally subtracted $\overline{\text{MS}}$ quantities (see below for the precise definition) depend upon this convention (by terms proportional to $\ln 2$).

In anomaly free theories we can assume γ_5 to be fully anticommuting! But then

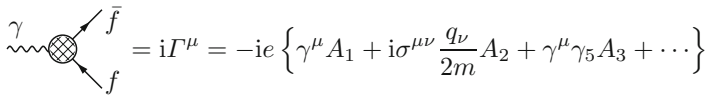
$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 = 0 \text{ for all } d \neq 4! \quad (2.127)$$

The 4-dimensional object

$$4i\varepsilon^{\mu\nu\rho\sigma} = \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \text{ for } d = 4$$

cannot be obtained by dimensional continuation if we use an anticommuting γ_5 [46].

Since fermions do not have self interactions they only appear as closed fermion loops, which yield a trace of γ -matrices, or as a fermion string connecting an external $\psi \cdots \bar{\psi}$ pair of fermion fields. In a transition amplitude $|T|^2 = \text{Tr}(\cdots)$ we again get a trace. Consequently, in principle, we have eliminated all γ 's! Commonly one writes a covariant tensor decomposition into invariant amplitudes, like, for example,



$$= i\Gamma^\mu = -ie \left\{ \gamma^\mu A_1 + i\sigma^{\mu\nu} \frac{q_\nu}{2m} A_2 + \gamma^\mu \gamma_5 A_3 + \cdots \right\}$$

where μ is an external index, q^μ the photon momentum and $A_i(q^2)$ are scalar form factors.

(2) External momenta (and external indices) must be taken $d = 4$ dimensional, because the number of independent “form factors” in covariant decompositions depends on the dimension, with a fewer number of independent functions in lower dimensions. Since four functions cannot be analytic continuation of three etc. we have to keep the external structure of the theory in $d = 4$. The reason for possible problems here is the non-trivial spin structure of the theory of interest. The following rules apply:

External momenta :	$p^\mu = (p^0, p^1, p^2, p^3, 0, \dots, 0)$	4 – dimensional
Loop momenta :	$k^\mu = (k^0, \dots, k^{d-1})$	d – dimensional
	$k^2 = (k^0)^2 - (k^1)^2 - \dots - (k^{d-1})^2$	
	$pk = p^0 k^0 - \mathbf{p} \cdot \mathbf{k}$	4 – dimensional etc.

3. Interpolation in d to complex values and extrapolation to $d < 4$.

Loop integrals now read

$$\mu^{A-d} \int \frac{d^d k}{(2\pi)^d} \cdots \tag{2.128}$$

with μ an arbitrary scale parameter. The crucial properties valid in DR independent of d are: (F.P. = finite part)

- (a) $\int d^d k k_\mu f(k^2) = 0$
 (b) $\int d^d k f(k+p) = \int d^d k f(k)$
 which is not true with UV cut – off's
 (c) If $f(k) = f(|k|)$:

$$\int d^d k f(k) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty dr r^{d-1} f(r)$$

- (d) For divergent integrals, by analytic subtraction:

F.P. $\int_0^\infty dr r^{d-1+\alpha} \equiv 0$ for arbitrary α
 so called *minimal subtraction* (MS). Consequently

$$\text{F.P.} \int d^d k f(k) = \text{F.P.} \int d^d k f(k+p) = \text{F.P.} \int d^d(\lambda k) f(\lambda k) .$$

This implies that **dimensionally regularized integrals behave like convergent integrals** and formal manipulations are justified. Starting with d sufficiently small, by partial integration, one can always find a representation for the integral which converges for $d = 4 - \epsilon$, $\epsilon > 0$ small.

In order to elaborate in more detail how DR works in practice, let us consider a generic one-loop Feynman integral

$$I_\Gamma^{\mu_1 \dots \mu_m}(p_1, \dots, p_n) = \int d^d k \frac{\prod_{j=1}^m k^{\mu_j}}{\prod_{i=1}^n ((k+p_i)^2 - m_i^2 + i\varepsilon)}$$

which has superficial degree of divergence

$$d(\Gamma) = d + m - 2n \leq d - 2$$

where the bound holds for two- or more-point functions in renormalizable theories and for $d \leq 4$. Since the physical tensor and spin structure has to be kept in $d = 4$, by contraction with external momenta or with the metric tensor $g_{\mu_i \mu_j}$ it is always possible to write the above integral as a sum of integrals of the form

$$I_\Gamma^{\hat{\mu}_1 \dots \hat{\mu}_{m'}}(\hat{p}_1, \dots, \hat{p}_{n'}) = \int d^d k \frac{\prod_{j=1}^{m'} \hat{k}^{\mu_j}}{\prod_{i=1}^{n'} ((k+\hat{p}_i)^2 - m_i^2 + i\varepsilon)}$$

where now $\hat{\mu}_j$ and \hat{p}_i are $d = 4$ -dimensional objects and

$$d^d k = d^4 \hat{k} d^{d-4} \bar{k} = d^4 \hat{k} \omega^{d-5} d\omega d\Omega_{d-4} .$$

In the $d - 4$ -dimensional complement the integrand depends on ω only! The angular integration over $d\Omega_{d-4}$ yields

$$\int d\Omega_{d-4} = S_{d-4} = \frac{2\pi^{\epsilon/2}}{\Gamma(\epsilon/2)} ; \quad \epsilon = d - 4 ,$$

which is the surface of the $d - 4$ -dimensional sphere. Using this result we get (discarding the four-dimensional tensor indices)

$$I_{\Gamma}(\{\hat{p}_i\}) = \int d^4\hat{k} J_{\Gamma}(d, \hat{p}, \hat{k})$$

where

$$J_{\Gamma}(d, \hat{p}, \hat{k}) = S_{d-4} \int_0^{\infty} d\omega \omega^{d-5} f(\hat{p}, \hat{k}, \omega).$$

Now this integral can be analytically continued to complex values of d . For the ω -integration we have

$$d^{\omega}(\Gamma) = d - 4 - 2n$$

i.e. the ω -integral converges if

$$d < 4 + 2n.$$

In order to avoid infrared singularities in the ω -integration one has to analytically continue by appropriate partial integration. After p -fold partial integration we have

$$I_{\Gamma}(\{\hat{p}_i\}) = \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2} + p)} \int d^4\hat{k} \int_0^{\infty} d\omega \omega^{d-5+2p} \left(-\frac{\partial}{\partial\omega^2}\right)^p f(\hat{p}, \hat{k}, \omega)$$

where the integral is convergent in $4 - 2p < \text{Re } d < 2n - m = 4 - d^{(4)}(\Gamma) \geq 2$. For a renormalizable theory at most 2 partial integrations are necessary to define the theory.

2.5 Tools for the Evaluation of Feynman Integrals

2.5.1 $\epsilon = 4 - d$ Expansion, $\epsilon \rightarrow +0$

For the expansion of integrals near $d = 4$ we need some asymptotic expansions of Γ -functions:

$$\Gamma(1+x) = \exp \left[-\gamma x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) x^n \right] \quad |x| \leq 1$$

$$\psi(1+x) = \frac{d}{dx} \ln \Gamma(1+x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} \Big|_{|x| \leq 1} = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1}$$

where $\zeta(n)$ denotes Riemann's Zeta function. The defining functional relation is

$$\Gamma(x) = \frac{\Gamma(x+1)}{x},$$

which for $n = 0, 1, 2, \dots$ yields $\Gamma(n+1) = n!$ with $\Gamma(1) = \Gamma(2) = 1$. Furthermore we have

$$\begin{aligned} \Gamma(x) \Gamma(1-x) &= \frac{\pi}{\sin \pi x} \\ \Gamma\left(\frac{1}{2}+x\right) \Gamma\left(\frac{1}{2}-x\right) &= \frac{\pi}{\cos \pi x}. \end{aligned}$$

Important special constants are

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ \Gamma'(1) &= -\gamma; \quad \gamma = 0.577215 \dots \text{ Euler's constant} \\ \Gamma''(1) &= \gamma^2 + \zeta(2); \quad \zeta(2) = \frac{\pi^2}{6} = 1.64493 \dots \end{aligned}$$

As a typical result of an ϵ -expansion, which we should keep in mind for later purposes, we have

$$\Gamma\left(1 + \frac{\epsilon}{2}\right) = 1 - \frac{\epsilon}{2}\gamma + \left(\frac{\epsilon}{2}\right)^2 \frac{1}{2}(\gamma^2 + \zeta(2)) + \dots$$

2.5.2 Bogolubov–Schwinger Parametrization

Suppose we choose for each propagator an independent momentum and take into account momentum conservation at the vertices by δ -functions. Then, for $d = n$ integer, we use

(i)

$$\frac{i}{p^2 - m^2 + i\epsilon} = \int_0^\infty d\alpha e^{-i\alpha(m^2 - p^2 - i\epsilon)} \quad (2.129)$$

(ii)

$$\delta^{(d)}(k) = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} d^d x e^{ikx} \quad (2.130)$$

and find that all momentum integrations are of Gaussian type. The Gaussian integrals yield

$$\int_{-\infty}^{+\infty} d^d k P(k) e^{i(ak^2 + 2b(k \cdot p))} = P\left(\frac{-i}{2b} \frac{\partial}{\partial p}\right) \left(\frac{\pi}{ia}\right)^{d/2} e^{-i b^2/a p^2} \quad (2.131)$$

for any polynomial P . The resulting form of the Feynman integral is the so called Bogolubov–Schwinger representation, also known as α -representation (see e.g. [47]).

2.5.3 Feynman Parametric Representation

Transforming pairs of α -variables in the above Bogolubov–Schwinger parametrization according to (l is denoting the pair (i, k))

$$(\alpha_i, \alpha_k) \rightarrow (\xi_l, \alpha_l) : (\alpha_i, \alpha_k) = (\xi_l \alpha_l, (1 - \xi_l) \alpha_l) \quad (2.132)$$

$$\int_0^\infty \int_0^\infty d\alpha_i d\alpha_k \cdots = \int_0^\infty d\alpha_l \alpha_l \int_0^1 d\xi_l \cdots, \quad (2.133)$$

the integrals are successively transformed into $\int_0^1 d\xi \cdots$ integrals and at the end there remains one α -integration only which can be performed using

$$\int_0^\infty d\alpha \alpha^a e^{-\alpha x} = \Gamma(a + 1) x^{-(a+1)}. \quad (2.134)$$

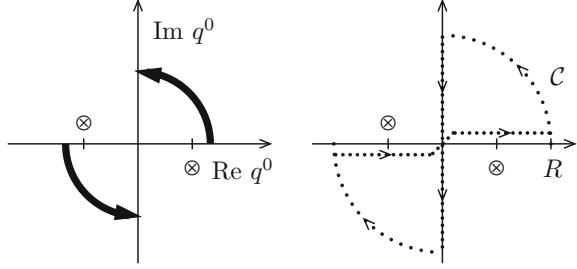
The result is the Feynman parametric representation. If L is the number of lines of a diagram, the Feynman integral is $(L - 1)$ -dimensional.

2.5.4 Euclidean Region, Wick–Rotations

The basic property which allows us to perform a Wick rotation is analyticity which derives from the causality of a relativistic QFT. In momentum space the Feynman propagator

$$\begin{aligned} \frac{1}{q^2 - m^2 + i\varepsilon} &= \frac{1}{q^0 - \sqrt{\mathbf{q}^2 + m^2 - i\varepsilon}} \frac{1}{q^0 + \sqrt{\mathbf{q}^2 + m^2 - i\varepsilon}} \\ &= \frac{1}{2\omega_p} \left\{ \frac{1}{q^0 - \omega_p + i\varepsilon} - \frac{1}{q^0 + \omega_p - i\varepsilon} \right\} \end{aligned} \quad (2.135)$$

Fig. 2.7 Wick rotation in the complex q^0 -plane. The poles of the Feynman propagator are indicated by \otimes 's. C is an integration contour, R is the radius of the arcs



is an analytic function in q^0 with poles at $q^0 = \pm(\omega_p - i\varepsilon)$ ²⁵ where $\omega_p = \sqrt{\mathbf{q}^2 + m^2}$. This allows us to rotate by $\frac{\pi}{2}$ the integration path in q^0 , going from $-\infty$ to $+\infty$, without crossing any singularity. In doing so, we rotate from Minkowski space to Euclidean space

$$q^0 \rightarrow -iq^d \Rightarrow q = (q^0, q^1, \dots, q^{d-2}, q^{d-1}) \rightarrow \underline{q} = (q^1, q^2, \dots, q^{d-1}, q^d)$$

and thus $q^2 \rightarrow -q^2$. This rotation to the Euclidean region is called **Wick rotation**.

More precisely: analyticity of a function $\tilde{f}(q^0, \mathbf{q})$ in q^0 implies that the **contour integral**

$$\oint_{C(R)} dq^0 \tilde{f}(q^0, \mathbf{q}) = 0 \quad (2.136)$$

for the closed path $C(R)$ in Fig. 2.7 vanishes. If the function $\tilde{f}(q^0, \mathbf{q})$ falls off sufficiently fast at infinity, then the contribution from the two ‘‘arcs’’ goes to zero when the radius of the contour $R \rightarrow \infty$. In this case we obtain

$$\int_{-\infty}^{\infty} dq^0 \tilde{f}(q^0, \mathbf{q}) + \int_{+i\infty}^{-i\infty} dq^0 \tilde{f}(q^0, \mathbf{q}) = 0 \quad (2.137)$$

or

$$\int_{-\infty}^{\infty} dq^0 \tilde{f}(q^0, \mathbf{q}) = \int_{-i\infty}^{+i\infty} dq^0 \tilde{f}(q^0, \mathbf{q}) = -i \int_{-\infty}^{+\infty} dq^d \tilde{f}(-iq^d, \mathbf{q}) \quad , \quad (2.138)$$

which is the Wick rotation. At least in perturbation theory, one can prove that the conditions required to allow us to perform a Wick rotation are fulfilled.

²⁵Note that because of the positivity of $\mathbf{q}^2 + m^2$ for any non-vacuum state, we have $\omega_p - i\varepsilon = \sqrt{\mathbf{q}^2 + m^2} - i\varepsilon$ in the limit $\lim_{\varepsilon \rightarrow 0}$, which is always understood. The symbolic parameter ε of the $i\varepsilon$ prescription, may be scaled by any fixed positive number.

We notice that the Euclidean Feynman propagator obtained by the Wick rotation

$$\frac{1}{q^2 - m^2 + i\varepsilon} \rightarrow -\frac{1}{\underline{q}^2 + m^2}$$

has no singularities (poles) and an $i\varepsilon$ -prescription is not needed any longer.

In configuration space a Wick rotation implies going to **imaginary time** $x^0 \rightarrow ix^0 = x^d$ such that $qx \rightarrow -\underline{q}\underline{x}$ and hence

$$x^0 \rightarrow -ix^d \Rightarrow x^2 \rightarrow -\underline{x}^2, \quad \square_x \rightarrow -\Delta_{\underline{x}}, \quad i \int d^d x \cdots \rightarrow \int d^d \underline{x} \cdots$$

While in Minkowski space $x^2 = 0$ defines the light-cone $x^0 = \pm|\mathbf{x}|$, in the Euclidean region $\underline{x}^2 = 0$ implies $\underline{x} = 0$. Note that possible singularities on the light-cone like $1/x^2$, $\delta(x^2)$ etc. turn into singularities at the point $\underline{x} = 0$. This simplification of the singularity structure is the merit of the positive definite metric in Euclidean space.

In momentum space the Euclidean propagators are positive (discarding the overall sign) and any Feynman amplitude in Minkowski space may be obtained via

$$I_M(p) = (-i)^{N_{\text{int}}} (-i)^{V-1} I_E(\underline{p}) \Big|_{p^4=ip^0; m^2 \rightarrow m^2 - i\varepsilon}$$

from its Euclidean version. Here, N_{int} denotes the number of internal lines (propagators) and V the number of vertices if we use the substitutions (convention dependent)

$$\frac{1}{p^2 - m^2 + i\varepsilon} \rightarrow \frac{1}{\underline{p}^2 + m^2}; \quad ig_i \rightarrow i(ig_i) = -g_i; \quad \int d^d k \rightarrow \int d^d \underline{k}$$

to define the Euclidean Feynman amplitudes. By g_i we denote the gauge couplings.

For the dimensionally regularized amplitudes, where potentially divergent integrals are defined via analytic continuation from regions in the complex d -plane where integrals are manifestly convergent, the terms from the arc segments can always be dropped. Also note that dimensional regularization and the power counting rules (superficial degree of divergence etc.) hold irrespective of whether we work in d -dimensional Minkowski space-time or in d -dimensional Euclidean space. The metric is obviously not important for the UV-behavior of the integrals.

The relationship between Euclidean and Minkowski quantum field theory is not only a very basic and surprising general feature of any local relativistic field theory but is a property of central practical importance for the non-perturbative approach to QFT via the Euclidean path-integral (e.g., lattice QCD). In a QFT satisfying the Wightman axioms the continuation of the vacuum-expectation values of time-ordered products of local fields (the time-ordered Green functions) from Minkowski space to four-dimensional Euclidean space is always possible [11]. Conversely, the **Osterwalder-Schrader theorem** [48] ascertains that

Theorem 2.8 *In a local relativistic QFT the time-ordered Green functions exhibit an analytic continuation to Euclidean space. Vice versa, the Euclidean correlation functions of an Euclidean QFT can be analytically continued to Minkowski space, provided we have a local action which satisfies the so-called reflection positivity condition.*

Accordingly, the full Minkowski QFT including its S -matrix, if it exists, can be reconstructed from the knowledge of the Euclidean correlation functions and from a mathematical point of view the Minkowski and the Euclidean version of a QFT are completely equivalent.

2.5.5 The Origin of Analyticity

At the heart of analyticity is the causality. The time ordered Green functions which encode all information of the theory in perturbation theory are given by integrals over products of causal propagators ($z = x - y$)

$$\begin{aligned} iS_F(z) &= \langle 0|T \{ \psi(x) \bar{\psi}(y) \} |0\rangle \\ &= \Theta(x^0 - y^0) \langle 0| \psi(x) \bar{\psi}(y) |0\rangle - \Theta(y^0 - x^0) \langle 0| \bar{\psi}(y) \psi(x) |0\rangle \\ &= \Theta(z^0) iS^+(z) + \Theta(-z^0) iS^-(z) \end{aligned} \quad (2.139)$$

exhibiting a positive frequency part propagating forward in time and a negative frequency part propagating backward in time. The Θ function of time ordering makes the Fourier-transform to be analytic in a half-plane in momentum space. For $K(\tau = z^0) = \Theta(z^0) iS^+(z)$, for example, we have

$$\tilde{K}(\omega) = \int_{-\infty}^{+\infty} d\tau K(\tau) e^{i\omega\tau} = \int_0^{+\infty} d\tau K(\tau) e^{-\eta\tau} e^{i\xi\tau} \quad (2.140)$$

such that $\tilde{K}(\omega = \xi + i\eta)$ is a regular analytic function in the **upper half ω -plane** $\eta > 0$. This of course only works because τ is restricted to be positive.

In a relativistically covariant world, in fact, we always need two terms (see (2.139)), a positive frequency part $\Theta(z^0 = t - t') S^+(z)$, corresponding to the particle propagating forward in time, and a negative frequency part $\Theta(-z^0 = t' - t) S^-(z)$, corresponding to the antiparticle propagating backward in time. The two terms correspond in momentum space to the two terms of (2.135).

Of course, for a free Dirac field we know what the Stückelberg-Feynman propagator in momentum space looks like

$$\tilde{S}_F(q) = \frac{\not{q} + m}{q^2 - m^2 + i\varepsilon}$$

and its analytic properties are manifest. It is an analytic function in q^0 with poles at $q^0 = \pm(\omega_p - i\varepsilon)$ where $\omega_p = \sqrt{\mathbf{q}^2 + m^2}$.

Analyticity is an extremely important basic property of a QFT and a powerful instrument which helps to solve seemingly purely “technical” problems as we will see. For example it allows us to perform a Wick rotation to Euclidean space and in Euclidean space a QFT looks like a classical statistical system and one can apply the methods of statistical physics to QFT [49]. In particular the numerical approach to the intrinsically non-perturbative QCD via lattice QCD is based on analyticity. The objects which manifestly exhibit the analyticity properties and are providing the bridge to the Euclidean world are the *time ordered Green functions*.

Note that by far not all objects of interest in a QFT are analytic. For example, any solution of the homogeneous (no source) Klein–Gordon equation

$$(\square_x + m^2) \Delta(x - y; m^2) = 0 ,$$

like the so called positive frequency part Δ^+ or the causal commutator Δ of a free scalar field $\varphi(x)$, defined by

$$\begin{aligned} \langle 0 | \varphi(x), \varphi(y) | 0 \rangle &= i \Delta^+(x - y; m^2) \\ [\varphi(x), \varphi(y)] &= i \Delta(x - y; m^2) , \end{aligned}$$

which, given the properties of the free field, may easily be evaluated to have a representation

$$\begin{aligned} \Delta^+(z; m^2) &= -i (2\pi)^{-3} \int d^4 p \Theta(p^0) \delta(p^2 - m^2) e^{-ipz} \\ \Delta(z; m^2) &= -i (2\pi)^{-3} \int d^4 p \epsilon(p^0) \delta(p^2 - m^2) e^{-ipz} . \end{aligned}$$

Thus, in momentum space, as solutions of

$$(p^2 - m^2) \tilde{\Delta}(p) = 0 ,$$

only singular ones exist. For the positive frequency part and the causal commutator they read

$$\Theta(p^0) \delta(p^2 - m^2) \quad \text{and} \quad \epsilon(p^0) \delta(p^2 - m^2) ,$$

respectively. The Feynman propagator, in contrast, satisfies an inhomogeneous (with point source) Klein–Gordon equation

$$(\square_x + m^2) \Delta_F(x - y; m^2) = -\delta^{(4)}(x - y) .$$

The δ function comes from differentiating the Θ function factors of the T product. Now we have

$$\langle 0|T \{\varphi(x), \varphi(y)\} |0\rangle = i \Delta_F(x - y; m^2)$$

with

$$\Delta_F(z; m^2) = (2\pi)^{-4} \int d^4 p \frac{1}{p^2 - m^2 + i\varepsilon} e^{-ipz}$$

and in momentum space

$$(p^2 - m^2) \tilde{\Delta}_F(p) = 1 ,$$

obviously has analytic solutions, a particular one being the scalar Feynman propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon} = \mathcal{P} \left(\frac{1}{p^2 - m^2} \right) - i \pi \delta(p^2 - m^2) . \quad (2.141)$$

The $i\varepsilon$ prescription used here precisely correspond to the boundary condition imposed by the time ordering prescription T in configuration space. The symbol \mathcal{P} denotes the principal value; the right hand side exhibits the splitting into real and imaginary part.

Analyticity will play a crucial role later on and is the basic property from which dispersion relations derive (see Sect. 3.7).

Digression on the configuration space representation of Lorentz invariant distributions

Usually particle physics is practiced in momentum space, perturbative calculations are performed using momentum space Feynman rules and one calculates Feynman integrals and cross sections etc. as functions of energies and momenta (see below). This is in contrast to non- perturbative lattice field theory, where calculations have to be performed on a discretized finite Euclidean space–imaginary-time lattice in configuration space, by numerically evaluating (2.99), reformulated as a path integral, without expanding the exponential (see Sect. 5.3 below). It is therefore instructive to do a short excursion considering the properties of free fields in configuration space. For later reference we consider here the singularity structure of the solutions of the Klein–Gordon equation in configuration space. We first list some one–dimensional Fourier transforms of distributions as boundary values $\lim_{\varepsilon \rightarrow 0}$ of analytic functions:

$$\begin{aligned}\delta(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp e^{-ipx} ; & 1 &= \int dx \delta(x) e^{ipx} \\ e^{-\varepsilon x} \Theta(x) &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \frac{e^{-ipx}}{p+i\varepsilon} ; & \frac{i}{p+i\varepsilon} &= \int dx \Theta(x) e^{-\varepsilon x} e^{ipx} \\ e^{\varepsilon x} \Theta(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \frac{e^{-ipx}}{p-i\varepsilon} ; & \frac{-i}{p+i\varepsilon} &= \int dx \Theta(-x) e^{\varepsilon x} e^{ipx}\end{aligned}$$

where $\lim_{\varepsilon \rightarrow 0}$ is understood. The solutions of the Klein–Gordon equation:

$$\begin{aligned}(\square + m^2) G(x) &= -\delta(x) ; & \text{inhomogeneous case} \\ (\square + m^2) F(x) &= 0 & ; \text{homogeneous case}\end{aligned}$$

exhibit several special solutions:

$$F(x) = \Delta^+ , \quad \Delta^- , \quad \Delta \text{ and } \Delta^{(1)} ,$$

the positive frequency part Δ^+ , the negative frequency part Δ^- , the causal commutator $\Delta = \Delta^+ + \Delta^-$ and $\Delta^{(1)} = \Delta^+ - \Delta^-$, and

$$G(x) = \Delta_R , \quad \Delta_A , \quad \Delta_P \text{ and } \Delta_F ,$$

the retarded (future time) $\Delta_R = \Theta(x^0) \Delta$, the advanced (past time) $\Delta_A = -\Theta(-x^0) \Delta$, the principal value Δ_P and the Feynman propagator $\Delta_F = \Theta(x^0) \Delta^+ - \Theta(-x^0) \Delta^-$. The general homogeneous solution is

$$F(x) = \alpha \Delta^+(x) + \beta \Delta^-(x)$$

and the general inhomogeneous one

$$G(x) = \Delta_P(x) + F(x) ; \quad \Delta_P(x) = -\frac{1}{2} \varepsilon(x_0) \Delta(x) ,$$

where $\Delta_P(x)$ is the particular principle value solution. All these solutions are L_+^\uparrow invariant, where the invariant pieces in configuration space are:

$$\begin{aligned}L_+ &: x^0 > 0 , \quad x^2 > 0 ; & \text{future cone} \\ L_- &: x^0 < 0 , \quad x^2 > 0 ; & \text{past cone} \\ L_0 &: & x^2 < 0 ; & \text{space – like region} \\ C_+ &: x^0 > 0 , \quad x^2 = 0 ; & \text{forward light cone} \\ C_- &: x^0 < 0 , \quad x^2 = 0 ; & \text{backward light cone}\end{aligned}$$

This implies that a general invariant Green function must be of the form

$$\begin{aligned} \Delta_{\text{inv}} = & \Theta(x^0) \Theta(x^2) f(x^2) + \Theta(-x^0) \Theta(x^2) g(x^2) + \Theta(-x^2) h(-x^2) \\ & + \Theta(x^0) \delta(x^2) a + \Theta(-x^0) \delta(x^2) b \end{aligned}$$

and applying the Klein–Gordon operator $\partial_0^2 - \Delta + m^2$ one obtains a set of differential equations of the form

$$z^2 \frac{d^2 w^\pm}{dz^2} + z \frac{dw^\pm}{dz} \pm z^2 w^\pm - \nu^2 w^\pm = 0$$

with $\nu^2 = 1$, $z = m\sqrt{|\lambda|}$ and $\lambda \equiv x^2$. The functions $f(x^2)$ and $g(x^2)$ are of type $w^+(z)$, which represents a Bessel function $J_{\pm\nu}(z)$, a Neumann function $N_\nu(z)$ or one of the Hankel functions $H^{(1)}(z) = J_\nu(z) + iN_\nu(z)$ or $H^{(2)}(z) = J_\nu(z) - iN_\nu(z)$ (see [50]). The function $h(-x^2)$ is of type $w^-(z)$, which represents a modified Bessel functions $I_{\pm\nu}(z)$ or $K_\nu(z)$. As $\nu^2 = 1$, only index $\nu = 1$ functions play a role here. With the appropriate boundary condition, which fixes the right species of solution one finds

$$\begin{aligned} \Delta^\pm(x) = & \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}} \Theta(\lambda) \left\{ \varepsilon(x^0) J_1(m\sqrt{\lambda}) \pm i N_1(m\sqrt{\lambda}) \right\} \\ & \pm i \frac{m}{4\pi^2\sqrt{-\lambda}} \Theta(-\lambda) K_1(m\sqrt{-\lambda}) \\ \simeq & \frac{1}{4\pi} \varepsilon(x^0) \delta(\lambda) \pm i \frac{1}{4\pi^2\lambda} \mp i \frac{m^2}{8\pi^2} \ln \frac{m\sqrt{|\lambda|}}{2} - \frac{m^2}{16\pi} \varepsilon(x^0) \Theta(\lambda) \\ & + O(\sqrt{|\lambda|} \ln |\lambda|), \quad (\lambda \rightarrow 0), \end{aligned}$$

which reveals the light cone singularities $\delta(x^2)$, $\Theta(x^2)$, $1/x^2$ and $\ln|x^2|$. Interesting is also the causal commutator function $\Delta(x)$ which is vanishing for $x^2 < 0$:

$$\begin{aligned} \Delta(x) = \Delta^+(x) + \Delta^-(x) = & \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m}{4\pi\sqrt{\lambda}} \Theta(\lambda) \varepsilon(x^0) J_1(m\sqrt{|\lambda|}) \\ \simeq & \frac{1}{2\pi} \varepsilon(x^0) \delta(\lambda) - \frac{m}{8\pi} \Theta(\lambda) \varepsilon(x^0) + O(\lambda), \quad (\lambda \rightarrow 0). \end{aligned}$$

For the Feynman propagator we have

$$\begin{aligned} \Delta_F(x) = & \frac{1}{4\pi} \delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}} \Theta(\lambda) \left\{ J_1(m\sqrt{\lambda}) - i N_1(m\sqrt{\lambda}) \right\} \\ & + i \frac{m}{4\pi^2\sqrt{-\lambda}} \Theta(-\lambda) K_1(m\sqrt{-\lambda}) \end{aligned}$$

$$\begin{aligned} &\simeq \frac{1}{4\pi} \delta(\lambda) - i \frac{1}{4\pi^2 \lambda} - i \frac{m^2}{8\pi^2} \ln \frac{m\sqrt{\lambda}}{2} - \frac{m^2}{16\pi} \Theta(\lambda) \\ &+ O(\sqrt{|\lambda|} \ln |\lambda|), \quad (\lambda \rightarrow 0) \end{aligned}$$

It is instructive to evaluate

$$\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ipx}; \quad \frac{i}{p^2 - m^2 + i\varepsilon} = \int_0^\infty d\alpha e^{-i\alpha[m^2 - p^2 - i\varepsilon]}$$

directly, using the Bogolubov–Schwinger representation (2.129):

$$\Delta_F(x) = \int_0^\infty d\alpha \int \frac{d^4 p}{(2\pi)^4} e^{-i(px + \alpha[m^2 - p^2 - i\varepsilon])}$$

together with (2.131)

$$\int \frac{d^4 p}{(2\pi)^4} e^{i(\alpha p^2 - px)} = \frac{1}{(2\pi)^4} \left(\frac{\pi}{i\alpha}\right)^2 e^{-ix^2/4\alpha}$$

such that

$$\Delta_F(x) = \frac{-1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha} e^{-i\alpha m^2} e^{-\alpha\varepsilon}$$

which upon a change of the integration variable $\alpha \rightarrow \omega = 1/4\alpha$ takes the form

$$\Delta_F(x) = \frac{1}{4\pi^2} \int_0^\infty d\omega e^{-i(\omega x^2 + m^2/4\omega)} e^{-\varepsilon/4\omega}$$

always understood that $\lim_{\varepsilon \rightarrow 0}$ is to be taken. Now, using the integral representation [51] of the Hankel function (for properties see [50])

$$\int_0^\infty \frac{d\omega}{\omega} \omega^\nu e^{i(a\omega + b/4\omega)} = 2 \left(\frac{b}{4a}\right)^{\nu/2} i \frac{\pi}{2} e^{i\pi\nu/2} H_\nu^{(1)}(\sqrt{ab}) \quad \text{and} \quad \left(H_\nu^{(1)}(z)\right)^* = H_\nu^{(2)}(z^*),$$

we obtain

$$\Delta_F(x) = i \frac{m^2}{8\pi} \frac{H_1^{(2)}(m\sqrt{x^2})}{m\sqrt{x^2}}; \quad x^2 > 0.$$

If $x^2 < 0$ we may continue

$$\sqrt{x^2} \rightarrow -i\sqrt{-x^2}, \quad H_1^{(2)}(z) = H_1^{(1)}(-z) \quad \text{and} \quad \frac{i\pi}{2} \frac{H_\nu^{(1)}(iz)}{(iz)^\nu} = \frac{K_\nu(z)}{z^\nu}$$

in order to find

$$\Delta_F(x) = \frac{m^2}{4\pi^2} \frac{K_1(m\sqrt{-x^2})}{m\sqrt{-x^2}}; \quad x^2 < 0.$$

It is interesting to see what happens upon a Wick rotation $p, x \rightarrow p_E, x_E$ to the Euclidean region. The Euclidean version will be central for the non-perturbative lattice QCD approach considered in Sect. 5.3 later. Which of the light-cone sectors in configuration space will take over? The Euclidean correlation function of the scalar field is the Wick rotated Feynman propagator as mentioned above. Again we may use the representation

$$\Delta_F(x)_E = \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m^2} e^{+i(p x)_E}; \quad \frac{1}{p_E^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(p_E^2 + m^2)}$$

to obtain

$$\Delta_F(x)_E = \int_0^\infty d\alpha \int \frac{d^4 p_E}{(2\pi)^4} e^{-\alpha(p_E^2 + m^2) + i(p x)_E},$$

and a quadratic completion achieved by the shift $p_E \rightarrow p'_E = p_E - i x_E/2\alpha$ leads to a simple Gaussian p_E integration. The integration measure being invariant under the translation, with $\int_{-\infty}^{+\infty} dp_i e^{-\alpha p_i^2} = \sqrt{\frac{\pi}{\alpha}}$ (of each of the components) we arrive at

$$\Delta_F(x)_E = \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-x_E^2/4\alpha} e^{-\alpha m^2} = \frac{-1}{16\pi^2} \int_0^\infty d\omega e^{-(x_E^2 \omega + m^2/4\omega)}$$

Again this is related to a Bessel type integral, namely

$$\int_0^\infty \frac{d\omega}{\omega} \omega^\nu e^{-(a\omega + b/4\omega)} = 2 \left(\frac{b}{4a} \right)^{\nu/2} K_\nu(\sqrt{ab}),$$

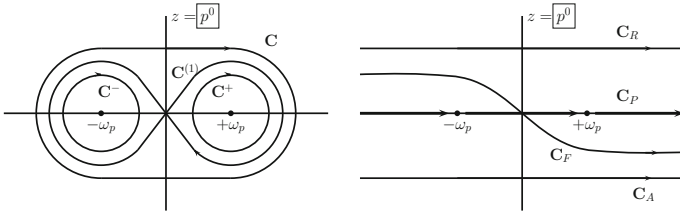


Fig. 2.8 Analytic plane contours and solutions of the Klein–Gordon equation. The two simple poles lie on the real axis at $\pm\omega_p$. *Left* homogeneous cases $C^i \rightarrow \Delta^i$ ($i = +, -$ and (1)) and $C \rightarrow \Delta$. *Right* inhomogeneous cases $C_i \rightarrow \Delta_i$ ($i = R, A, P$ and F)

which defines the spherical Bessel function $K_\nu(z)$. This leads to

$$\Delta_F(x)_E = \frac{m^2}{4\pi^2} \frac{K_1(m\sqrt{x_E^2})}{m\sqrt{x_E^2}}, \tag{2.142}$$

in agreement with the result for the $x^2 < 0$ sector in Minkowski space.

In momentum space a free scalar field L–invariant two point function

$$\tilde{\Delta}_{\text{inv}}(p) = \int d^4x e^{+ipx} \Delta_{\text{inv}}(x),$$

satisfies

$$(m^2 - p^2) \tilde{G}(p) = -1 \text{ or } (m^2 - p^2) \tilde{F}(p) = 0,$$

and the corresponding Green functions are the possible distribution valued singular function of p . The possibilities may be characterized by contours (path) C_{inv} in the complex p^0 –plane as illustrated in Fig.2.8. In fact a general representation of $\Delta_{\text{inv}}(x)$ is

$$\Delta_{\text{inv}}(x) = (2\pi)^{-4} \int_{C_{\text{inv}}} d^4p e^{-ipx} \frac{1}{m^2 - p^2}.$$

Key behind is the residue theorem

$$\frac{1}{2\pi i} \oint_C f(z) dz = \text{Res}[f(z); z_0] = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

in case the oriented path C encloses simple poles of $f(z)$.

As $m^2 - p^2 = (\omega_p - p^0)(\omega_p + p^0)$; $\omega_p = \sqrt{m^2 + \mathbf{p}^2}$ has two simple zeros, the inverse

$$\frac{1}{m^2 - p^2} = \frac{1}{(\omega_p - p^0)(\omega_p + p^0)} = \frac{-1}{2\omega_p} \left(\frac{1}{p^0 - \omega_p} - \frac{1}{p^0 + \omega_p} \right)$$

has two simple poles. For the evaluation of the contour integrals one uses the fact that the contour can be closed in one of the half planes at infinity, depending on whether $x^0 > 0$ or $x^0 < 0$, as discussed before.

In Euclidean space a more direct calculation shows how Bessel functions emerge from a Fourier transform of a radial function on \mathbf{R}^n

$$\tilde{f}(P) = \int d^n X f(X) e^{-iPX}$$

with $X, P \in \mathbf{R}^n$ and $r = |X|$. We first remember that the area of the unit sphere $S_{n-1} \subseteq \mathbf{R}^n$ is

$$S_{n-1} = 2\pi^{n/2} / \Gamma(n/2) .$$

Let $f(X) = F(r)$, then, in polar coordinates we can choose axes such that $PX = sr \cos \theta$. Then

$$\tilde{f}(P) = \tilde{F}_n(s) = \int_0^\infty \int_0^\pi e^{-isr \cos \theta} F(r) S_{n-2}(\sin \theta)^{n-2} d\theta r^{n-1} dr .$$

The angular integral is related to a Bessel function by

$$J_{\frac{n-2}{2}}(t) = \frac{t^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}} S_{n-2} \int_0^\pi e^{-it \cos \theta} (\sin \theta)^{n-2} d\theta$$

such that the Fourier transformation of a radial function takes the form

$$\tilde{F}_n(s) = (2\pi)^{\frac{n}{2}} s^{-\frac{n-2}{2}} \int_0^\infty J_{\frac{n-2}{2}}(sr) F(r) r^{-\frac{n-2}{2}} r^{n-1} dr .$$

Thus the n dimensional Fourier transform of a radial function is a radial function too. These results will be useful later when discussing the lattice QCD evaluation of the hadronic light-by-light scattering in $n = 4$ Euclidean space, where

$$\tilde{F}(s = |P|) = \int d^4 X F(r = |X|) e^{-iPX} = (2\pi)^2 s^{-1} \int_0^\infty J_1(sr) F(r) r^2 dr . \quad (2.143)$$

End of the Digression.

2.5.6 Scalar One-Loop Integrals

Here we apply our tools to the simplest scalar one-loop integrals (p.i. = partial integration).²⁶

$$\begin{aligned}
 \text{Diagram: } \frac{m}{p} \text{ (circle with } m \text{ on top, } p \text{ on bottom)} &= \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \frac{1}{k^2+m^2} = \mu^{4-d} (4\pi)^{-d/2} \int_0^\infty d\alpha \alpha^{-d/2} e^{-\alpha m^2} \\
 &\text{convergent for } d < 2 \quad * * * \quad ^{26} \\
 \stackrel{\text{p.i.}}{=} & -\frac{2m^2}{d-2} \mu^{4-d} (4\pi)^{-d/2} \int_0^\infty d\alpha \alpha^{1-d/2} e^{-\alpha m^2} \\
 &\text{convergent for } d < 4 \\
 &= -2m^2 (4\pi)^{-d/2} \frac{\Gamma(2-d/2)}{d-2} \left(\frac{m^2}{\mu^2}\right)^{d/2-2} \\
 &= -2m^2 (4\pi)^{-2} \frac{2}{\epsilon} \Gamma\left(1 + \frac{\epsilon}{2}\right) \frac{1}{2-\epsilon} e^{\frac{\epsilon}{2}(\ln 4\pi - \ln \frac{m^2}{\mu^2})} \\
 \stackrel{\epsilon \rightarrow +0}{\simeq} & -m^2 (4\pi)^{-2} \left\{ \frac{2}{\epsilon} - \gamma + 1 + \ln 4\pi - \ln \frac{m^2}{\mu^2} \right\} + O(\epsilon)
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram: } \frac{m_1}{p} \text{ (circle with } m_1 \text{ on top, } m_2 \text{ on bottom, } p \text{ on left)} &= \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \frac{1}{k^2+m_1^2} \frac{1}{(k+p)^2+m_2^2} \\
 &= \mu^{4-d} (4\pi)^{-d/2} \int_0^\infty d\alpha_1 d\alpha_2 (\alpha_1 + \alpha_2)^{-d/2} e^{-(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2)} \\
 &\quad \alpha_1 = x\lambda; \quad \alpha_2 = (1-x)\lambda \\
 &= \mu^{4-d} (4\pi)^{-d/2} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx (x m_1^2 + (1-x) m_2^2 + x(1-x) p^2)^{d/2-2} \\
 &\quad \text{convergent for } d < 4 \\
 &= (4\pi)^{-2} \frac{2}{\epsilon} \Gamma\left(1 + \frac{\epsilon}{2}\right) e^{\frac{\epsilon}{2} \ln 4\pi} \int_0^1 dx e^{-\frac{\epsilon}{2} \ln \frac{x m_1^2 + (1-x) m_2^2 + x(1-x) p^2}{\mu^2}} \\
 \stackrel{\epsilon \rightarrow +0}{\simeq} & (4\pi)^{-2} \left\{ \frac{2}{\epsilon} - \gamma + \ln 4\pi - \int_0^1 dx \ln \frac{x m_1^2 + (1-x) m_2^2 + x(1-x) p^2}{\mu^2} \right\} + O(\epsilon)
 \end{aligned}$$

$$\begin{aligned}
 \text{Diagram: } \frac{m_1}{p_3} \text{ (triangle with } m_1 \text{ on top, } m_2 \text{ on right, } m_3 \text{ on left, } p_1 \text{ on top-left, } p_2 \text{ on bottom-left)} &= \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \frac{1}{k^2+m_1^2} \frac{1}{(k+p_1)^2+m_2^2} \frac{1}{(k+p_1+p_2)^2+m_3^2} \\
 &\quad \text{convergent for } d = 4 \\
 \stackrel{\epsilon \rightarrow +0}{\simeq} & (4\pi)^{-2} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{(\alpha_1 + \alpha_2 + \alpha_3)^2} e^{-(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2)} \\
 &\quad \times e^{-\frac{\alpha_1 \alpha_2 p_1^2 + \alpha_2 \alpha_3 p_2^2 + \alpha_3 \alpha_1 p_3^2}{\alpha_1 + \alpha_2 + \alpha_3}} \\
 &\quad \alpha_1 = xy\lambda; \quad \alpha_2 = x(1-y)\lambda; \quad \alpha_3 = (1-x)\lambda; \quad \alpha_1 + \alpha_2 + \alpha_3 = \lambda \\
 &= (4\pi)^{-2} \int_0^1 dy dx x \frac{1}{N}
 \end{aligned}$$

$$N = x^2 y (1-y) p_1^2 + x (1-x) (1-y) p_2^2 + x (1-x) y p_3^2 + x y m_1^2 + x (1-y) m_2^2 + (1-x) m_3^2$$

²⁶A direct integration here yields

$$m^2 (4\pi)^{-d/2} \Gamma(1-d/2) \left(\frac{m^2}{\mu^2}\right)^{d/2-2}$$

which by virtue of $\Gamma(1-d/2) = -2\Gamma(2-d/2)/(d-2)$ is the same analytic function as the one obtained via the partial integration method.

Standard Scalar One-Loop Integrals ($m^2 \hat{=} m^2 - i\varepsilon$).

$$\frac{\text{Diagram}}{p} = \mu_0^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \hat{=} -\frac{i}{16\pi^2} A_0(m),$$

defines the standard *tadpole type integral*, where

$$A_0(m) = -m^2(\text{Reg} + 1 - \ln m^2) \quad (2.144)$$

with

$$\text{Reg} = \frac{2}{\varepsilon} - \gamma + \ln 4\pi + \ln \mu_0^2 \equiv \ln \mu^2. \quad (2.145)$$

The last identification defines the $\overline{\text{MS}}$ *scheme of (modified) minimal subtraction*.

$$\frac{\text{Diagram}}{p} = \mu_0^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)((k+p)^2 - m_2^2)} \hat{=} \frac{i}{16\pi^2} B_0(m_1, m_2; p^2),$$

defines the standard *propagator type integral*, where

$$B_0(m_1, m_2; s) = \text{Reg} - \int_0^1 dz \ln(-sz(1-z) + m_1^2(1-z) + m_2^2z - i\varepsilon). \quad (2.146)$$

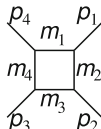
$$\begin{aligned} \frac{\text{Diagram}}{p_3} &= \mu_0^\varepsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)((k+p_1)^2 - m_2^2)((k+p_1+p_2)^2 - m_3^2)} \\ &= -\frac{i}{16\pi^2} C_0(m_1, m_2, m_3; p_1^2, p_2^2, p_3^2), \end{aligned}$$

defines the standard *form factor type integral*, where

$$C_0(m_1, m_2, m_3; s_1, s_2, s_3) = \int_0^1 dx \int_0^x dy \frac{1}{ax^2 + by^2 + cxy + dx + ey + f} \quad (2.147)$$

with

$$\begin{aligned} a &= s_2, & d &= m_2^2 - m_3^2 - s_2, \\ b &= s_1, & e &= m_1^2 - m_2^2 + s_2 - s_3, \\ c &= s_3 - s_1 - s_2, & f &= m_3^2 - i\varepsilon. \end{aligned}$$



$$= \frac{i}{16\pi^2} D_0(m_1, m_2, m_3, m_4; p_1^2, p_2^2, p_3^2, p_4^2),$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)((k + p_1)^2 - m_2^2)((k + p_1 + p_2)^2 - m_3^2)((k + p_1 + p_2 + p_3)^2 - m_4^2)}$$

defines the standard *box type integral*, where

$$D_0(m_1, m_2, m_3, m_4; s_1, s_2, s_3, s_4) = \int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{[ax^2 + by^2 + gz^2 + cxy + hxz + jyz + dx + ey + kz + f]^2} \quad (2.148)$$

with

$$\begin{aligned} a &= s_3 = p_3^2, & b &= s_2 = p_2^2, & g &= s_1 = p_1^2, \\ c &= 2(p_2 p_3), & h &= 2(p_1 p_3), & j &= 2(p_1 p_2), \\ d &= m_3^2 - m_4^2 - s_3, & e &= m_2^2 - m_3^2 - s_2 - 2(p_2 p_3), & k &= m_1^2 - m_2^2 - s_1 - 2(p_1 p_2) - 2(p_1 p_3), \\ f &= m_4^2 - i\varepsilon \end{aligned}$$

Remark: the regulator term Reg in (2.145) denotes the UV regulated pole term $\frac{2}{\epsilon}$ supplemented with $O(1)$ terms which always accompany the pole term and result from the ϵ -expansion of the d -dimensional integrals. While in the $\overline{\text{MS}}$ scheme just the poles $\frac{2}{\epsilon}$ are subtracted, in the modified $\overline{\text{MS}}$ scheme $\overline{\overline{\text{MS}}}$ also the finite terms included in (2.145) are subtracted. The dependence on the UV cut-off $\frac{2}{\epsilon}$ in the $\overline{\overline{\text{MS}}}$ scheme defined by $\text{Reg} \equiv \ln \mu^2$ is reflected in a dependence on the $\overline{\overline{\text{MS}}}$ renormalization scale μ .

The UV -singularities (poles in ϵ at $d = 4$) give rise to *finite extra contributions* when they are multiplied with d (or functions of d) which arise from contractions like $g_\mu^\mu = d$, $\gamma^\mu \gamma_\mu = d$ etc. For $d \rightarrow 4$ we obtain:

$$dA_0(m) = 4A_0(m) + 2m^2, \quad dB_0 = 4B_0 - 2. \quad (2.149)$$

The explicit evaluation of the scalar integrals (up to the scalar four-point function) is discussed in Ref. [52] (see also [53, 54]). The analytic structure of scalar functions is analyzed in [52, 55].

2.5.7 Tensor Integrals

In dimensional regularization also the calculation of tensor integrals is rather straightforward. Sign conventions are chosen in accordance with the Passarino–Veltman convention [56] (see also [57]). Invariant amplitudes are defined by performing covariant decompositions of the tensor integrals, which then are contracted with external vectors or with the metric tensor. A factor $i/16\pi^2$ is taken out for simplicity of notation, i.e.

$$\int_k \cdots = \frac{16\pi^2}{i} \int \frac{d^d k}{(2\pi)^d} \cdots \quad (2.150)$$

(1) One point integrals:

By eventually performing a shift $k \rightarrow k + p$ of the integration variable we easily find the following results:

$$\begin{aligned} \int_k \frac{1}{(k+p)^2 - m^2} &= -A_0(m) \\ \int_k \frac{k^\mu}{(k+p)^2 - m^2} &= p^\mu A_0(m) \\ \int_k \frac{k^\mu k^\nu}{(k+p)^2 - m^2} &= -p^\mu p^\nu A_{21} + g^{\mu\nu} A_{22} \end{aligned} \quad (2.151)$$

$$\begin{aligned} A_{21} &= A_0(m) \\ A_{22} &= -\frac{m^2}{d} A_0(m) \stackrel{\epsilon \rightarrow 0}{\simeq} -\frac{m^2}{4} A_0(m) + \frac{m^4}{8} \end{aligned} \quad (2.152)$$

(2) Two point integrals: the defining equations here are

$$\begin{aligned} \int_k \frac{1}{(1)(2)} &= B_0(m_1, m_2; p^2) \\ \int_k \frac{k^\mu}{(1)(2)} &= p^\mu B_1(m_1, m_2; p^2) \\ \int_k \frac{k^\mu k^\nu}{(1)(2)} &= p^\mu p^\nu B_{21} - g^{\mu\nu} B_{22}, \end{aligned} \quad (2.153)$$

where we denoted scalar propagators by (1) $\equiv k^2 - m_1^2$ and (2) $\equiv (k + p)^2 - m_2^2$. The simplest non-trivial example is B_1 . Multiplying the defining equation with $2p_\mu$ we have

$$2p^2 B_1 = \int_k \frac{2pk}{k^2 - m_1^2 + i\epsilon} \frac{1}{(p+k)^2 - m_2^2 + i\epsilon}$$

and we may write the numerator as a difference of the two denominators plus a remainder which does not depend on the integration variable:

$$2pk = (p+k)^2 - k^2 - p^2 = [(p+k)^2 - m_2^2] - [k^2 - m_1^2] - (p^2 + m_1^2 - m_2^2)$$

After canceling the square brackets against the appropriate denominator we obtain

$$B_1(m_1, m_2; p^2) = \frac{1}{2p^2} \{A_0(m_2) - A_0(m_1) - (p^2 + m_1^2 - m_2^2) B_0(m_1, m_2; p^2)\} \quad (2.154)$$

A further useful relation is

$$B_1(m, m; p^2) = -\frac{1}{2} B_0(m, m; p^2) .$$

In a similar way, by contracting the defining relation with p_ν and $g_{\mu\nu}$ we find for arbitrary dimension d

$$\begin{aligned} B_{21} &= \frac{1}{(d-1)p^2} \{(1-d/2)A_0(m_2) - d/2(p^2 + m_1^2 - m_2^2)B_1 - m_1^2 B_0\} \\ B_{22} &= \frac{1}{2(d-1)} \{A_0(m_2) - (p^2 + m_1^2 - m_2^2)B_1 - 2m_1^2 B_0\} . \end{aligned}$$

Expansion in $d = 4 - \epsilon$, $\epsilon \rightarrow 0$ yields

$$\begin{aligned} B_{21} &= \frac{-1}{3p^2} \{A_0(m_2) + 2(p^2 + m_1^2 - m_2^2)B_1 + m_1^2 B_0 + 1/2(m_1^2 + m_2^2 - p^2/3)\} \\ B_{22} &= \frac{1}{6} \{A_0(m_2) - (p^2 + m_1^2 - m_2^2)B_1 - 2m_1^2 B_0 - (m_1^2 + m_2^2 - p^2/3)\} \end{aligned}$$

where the arguments of the B -functions are obvious.

Note the appearance of $1/p^2$ terms, which represent a kinematical singularity. Kinematical singularities unavoidably show up when working with covariant decompositions of tensor amplitudes. Observables are always scalars and are obtained from tensor structures via contractions with numerical tensors and the external momenta in our simplest case with p_μ . Factors p^2 arising from the contraction eliminate/compensate the kinematic singularity of the scalar amplitudes in the contracted object. The higher the tensor the higher the singularity: in general B_1 exhibit a $1/p^2$, B_{21} a $(1/p^2)^2$ etc.

(3) Three point integrals: for the simplest cases we define the following invariant amplitudes

$$\begin{aligned} \int_k \frac{1}{(1)(2)(3)} &= -C_0(m_1, m_2, m_3; p_1^2, p_2^2, p_3^2) \\ \int_k \frac{k^\mu}{(1)(2)(3)} &= -p_1^\mu C_{11} - p_2^\mu C_{12} \\ \int_k \frac{k^\mu k^\nu}{(1)(2)(3)} &= -p_1^\mu p_1^\nu C_{21} - p_2^\mu p_2^\nu C_{22} - (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) C_{23} + g^{\mu\nu} C_{24} \end{aligned} \quad (2.155)$$

where $p_3 = -(p_1 + p_2)$, (1) $\equiv k^2 - m_1^2$, (2) $\equiv (k + p_1)^2 - m_2^2$ and (3) $\equiv (k + p_1 + p_2)^2 - m_3^2$.

The C_{1i} 's can be found using all possible independent contractions with $p_{1\mu, \nu}$, $p_{2\mu, \nu}$ and $g_{\mu\nu}$. This leads to the equations

$$\underbrace{\begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}}_X \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

with

$$\begin{aligned} R_1 &= \frac{1}{2}(B_0(m_2, m_3; p_2^2) - B_0(m_1, m_3; p_3^2) \\ &\quad - (p_1^2 + m_1^2 - m_2^2)C_0) \\ R_2 &= \frac{1}{2}(B_0(m_1, m_3; p_3^2) - B_0(m_1, m_2; p_1^2) \\ &\quad + (p_1^2 - p_3^2 - m_2^2 + m_3^2)C_0) . \end{aligned}$$

The inverse of the kinematic matrix of the equation to be solved is

$$X^{-1} = \frac{1}{\det X} \begin{pmatrix} p_2^2 & -p_1 p_2 \\ -p_1 p_2 & p_1^2 \end{pmatrix}, \quad \det X \doteq p_1^2 p_2^2 - (p_1 p_2)^2$$

and the solution reads

$$\begin{aligned} C_{11} &= \frac{1}{\det X} \{p_2^2 R_1 - (p_1 p_2) R_2\} \\ C_{12} &= \frac{1}{\det X} \{-(p_1 p_2) R_1 + p_1^2 R_2\} . \end{aligned} \quad (2.156)$$

The same procedure applies to the more elaborate case of the C_{2i} 's where the solution may be written in the form

$$C_{24} = -\frac{m_1^2}{2}C_0 + \frac{1}{4}B_0(2, 3) - \frac{1}{4}(f_1 C_{11} + f_2 C_{12}) + \frac{1}{4} \quad (2.157)$$

$$\begin{pmatrix} C_{21} \\ C_{23} \end{pmatrix} = X^{-1} \begin{pmatrix} R_3 \\ R_5 \end{pmatrix}; \quad \begin{pmatrix} C_{23} \\ C_{22} \end{pmatrix} = X^{-1} \begin{pmatrix} R_4 \\ R_6 \end{pmatrix} \quad (2.158)$$

with

$$\begin{aligned} R_3 &= C_{24} - \frac{1}{2}(f_1 C_{11} + B_1(1, 3) + B_0(2, 3)) \\ R_5 &= -\frac{1}{2}(f_2 C_{11} + B_1(1, 2) - B_1(1, 3)) \\ R_4 &= -\frac{1}{2}(f_1 C_{12} + B_1(1, 3) - B_1(2, 3)) \\ R_6 &= C_{24} - \frac{1}{2}(f_2 C_{12} - B_1(1, 3)) \end{aligned}$$

and

$$f_1 = p_1^2 + m_1^2 - m_2^2; \quad f_2 = p_3^2 - p_1^2 + m_2^2 - m_3^2 .$$

The notation used for the B -functions is as follows: $B_0(1, 2)$ denotes the two point function obtained by dropping propagator $\frac{1}{(3)}$ from the form factor i.e. $\int_k \frac{1}{(1)(2)}$ and correspondingly for the other cases.

As we mentioned at the end of the paragraph on the two point tensor integrals above, the tensor decomposition leads to kinematical singularities. In the case of the three point tensor integrals they show up in form of powers of the factors $\frac{1}{\det X}$ (in place of the simple $1/p^2$ in case of the two point integrals). The determinant $\det X = p_1^2 p_2^2 - (p_1 p_2)^2$ is called *Gram determinant* and exhibits a zero at points of degenerate momenta i.e. $p_2 \propto p_1$. After contracting the tensor integral with an external tensor structure in the two independent moments $p_{1\mu}$ and $p_{2\mu}$ and the possible numerical tensors when forming an observable the singularities cancel.

In the following sections we present an introduction to the calculation of the perturbative higher order corrections, also called *radiative corrections*, for the simplest QED processes. For extensions to electroweak SM processes I refer to my TASI lectures [58].

2.6 One-Loop Renormalization

2.6.1 The Photon Propagator and the Photon Self-Energy

We first consider the full photon propagator

$$iD_{\gamma}^{\mu\nu'}(x-y) = \langle 0|T \{A^{\mu}(x)A^{\nu'}(y)\} |0\rangle ,$$

which includes all electromagnetic interactions, in momentum space. It is given by repeated insertion of the *one-particle irreducible* (1PI) self-energy function

$$-i\Pi_{\gamma}^{\mu\nu}(q) \equiv \text{wavy line with a shaded loop} = \text{wavy line with a loop} + \dots$$

also called the vacuum polarization tensor. Since the external photon couples to the electromagnetic current via the vertex $ie j_{\text{em}}^{\mu}(x)A_{\mu}(x)$, the latter may also be represented as a correlator of two electromagnetic currents (2.93):

$$-i\Pi_{\gamma}^{\mu\nu}(q) = (ie)^2 \int d^4x e^{iqx} \langle 0|T \{j_{\text{em}}^{\mu}(x) j_{\text{em}}^{\nu}(y)\} |0\rangle . \quad (2.159)$$

Because the electromagnetic current is conserved $\partial_{\mu} j_{\text{em}}^{\mu} = 0$ the non-trivial part of the self-energy function is transversal

$$\Pi^{\mu\nu} = - (q^{\mu} q^{\nu} - q^2 g^{\mu\nu}) \Pi'(q^2) \quad (2.160)$$

which implies $q_{\nu} \Pi^{\mu\nu} = 0$ automatically. Note however, that the free propagator, because of the required gauge fixing does not satisfy the transversality condition. The left over terms are gauge fixing artifacts and will drop out from physical matrix

elements. An external real photon, for example, is represented by a polarization vector $\varepsilon^\mu(q, \lambda)$ which satisfy $q_\mu \varepsilon^\mu(q, \lambda) = 0$ and thus nullifies all terms proportional to q^μ .

In any case, we will need to consider the transverse part only in the following. In order to see how the splitting into transverse and longitudinal parts works, we introduce the projection tensors

$$T^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \quad (\text{transverse projector}), \quad L^{\mu\nu} = \frac{q^\mu q^\nu}{q^2} \quad (\text{longitudinal projector})$$

which satisfy


$$T_\nu^\mu + L_\nu^\mu = \delta_\nu^\mu, \quad T_\rho^\mu T_\nu^\rho = T_\nu^\mu, \quad L_\rho^\mu L_\nu^\rho = L_\nu^\mu, \quad T_\rho^\mu L_\nu^\rho = L_\rho^\mu T_\nu^\rho = 0.$$

Then writing

$$\Pi^{\mu\nu}(q) = \left(T_{\mu\nu} \Pi(q^2) + L_{\mu\nu} L(q^2) \right) = \left(g_{\mu\nu} \Pi_1(q^2) + q_\mu q_\nu \Pi_2(q^2) \right) \quad (2.161)$$

we have $L = q^2 \Pi_2 + \Pi_1$ and $\Pi \equiv \Pi_1$. Thus the transverse amplitude Π is uniquely given by the $g_{\mu\nu}$ -term in the propagator and the longitudinal amplitude L does not mix with the transverse part.

This allows us to calculate the full or dressed photon propagator by simply considering it in the *Feynman gauge* $\xi=1$, for which the free propagator takes the simple form $iD_\gamma^{\mu\nu} = -i g^{\mu\nu} / (q^2 + i\epsilon)$. The so called *Dyson series* of self-energy insertions then takes the form (we omit the metric tensor $g^{\mu\nu}$ which acts as a unit matrix)



$$\begin{aligned}
 i D'_\gamma(q^2) &\equiv \frac{-i}{q^2} + \frac{-i}{q^2} (-i\Pi_\gamma) \frac{-i}{q^2} + \frac{-i}{q^2} (-i\Pi_\gamma) \frac{-i}{q^2} (-i\Pi_\gamma) \frac{-i}{q^2} + \dots \\
 &= \frac{-i}{q^2} \left\{ 1 + \left(\frac{-i\Pi_\gamma}{q^2} \right) + \left(\frac{-i\Pi_\gamma}{q^2} \right)^2 + \dots \right\} \\
 &= \frac{-i}{q^2} \left\{ \frac{1}{1 + \frac{i\Pi_\gamma}{q^2}} \right\} = \frac{-i}{q^2 + \Pi_\gamma(q^2)}. \quad (2.162)
 \end{aligned}$$

The fact that the series of self-energy insertions represents a geometrical progression allows one for a closed resummation and is called a Dyson summation. The result is very important. It shows that the full propagator indeed has a simple pole in q^2 only, as the free propagator, and no multi-poles as it might look like before the resummation has been performed.

In a more general form the dressed propagator, including an auxiliary photon mass term for a moment, reads

$$iD'_{\gamma}{}^{\mu\nu}(q) = \frac{-i}{q^2 - m_{0\gamma}^2 + \Pi_{\gamma}(q^2)} \left(g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} \right) + \frac{q^{\mu}q^{\nu}}{q^2} \dots \quad (2.163)$$

and we observe that in general the position of the pole of the propagator, at the tree level given by the mass of the particle, gets modified or renormalized by higher order corrections encoded in the self-energy function Π . The condition for the position $q^2 = s_P$ of the pole is

$$s_P - m_{0\gamma}^2 + \Pi_{\gamma}(s_P) = 0. \quad (2.164)$$

By $U(1)_{\text{em}}$ gauge invariance the photon necessarily is massless and must remain massless after including radiative corrections. Besides $m_{0\gamma} = 0$ this requires $\Pi_{\gamma}(q^2) = \Pi_{\gamma}(0) + q^2 \Pi'_{\gamma}(q^2)$ with $\Pi_{\gamma}(0) \equiv 0$, in agreement with the transversality condition (2.160). As a result we obtain

$$iD'_{\gamma}{}^{\mu\nu}(q) = -ig^{\mu\nu} D'_{\gamma}(q^2) + \text{gauge terms} = \frac{-ig^{\mu\nu}}{q^2 (1 + \Pi'_{\gamma}(q^2))} + \text{gauge terms} . \quad (2.165)$$

The inverse full bare photon propagator is of the form

$$\begin{aligned} -iD_{\gamma}{}^{\mu\nu\prime-1} &= \text{diagram with a photon line and a self-energy loop} = \text{diagram with a photon line} + \text{diagram with a photon line and a loop} + \dots \\ &= i \left\{ g^{\mu\nu} (q^2 - m_{0\gamma}^2) - \left(1 - \frac{1}{\xi} \right) q^{\mu}q^{\nu} \right\} - i\Pi_{\gamma}{}^{\mu\nu}. \end{aligned} \quad (2.166)$$

After these structural considerations about the photon propagator we are ready to calculate the one-loop self-energy and to discuss the renormalization of the photon propagator. We have to calculate²⁷

$$\begin{aligned} -i\Pi^{\mu\nu}(q) &= \text{diagram of a photon line with a fermion loop} \\ &= (-1)^F i^4 e^2 \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left\{ \gamma^{\mu} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon} \gamma^{\nu} \frac{\not{p} + \not{k} + m}{(q+k)^2 - m^2 + i\varepsilon} \right\} \\ &= -e^2 \int_k \frac{\text{Tr} \{ \gamma^{\mu} \not{k} \gamma^{\nu} (\not{q} + \not{k}) \}}{(1)(2)} - e^2 m^2 \int_k \frac{\text{Tr} \{ \gamma^{\mu} \gamma^{\nu} \}}{(1)(2)}. \end{aligned}$$

²⁷Fermion propagators are represented either as an inverse matrix $\frac{1}{\not{k} - m + i\varepsilon}$ or as a matrix $\frac{\not{k} + m - i\varepsilon}{k^2 - m^2 + i\varepsilon}$ with a scalar denominator. This second form is obtained from the first one by multiplying numerator and denominator from the left or from the right with $\not{k} + m - i\varepsilon$. In the denominator we then have $(\not{k} + m - i\varepsilon)(\not{k} - m + i\varepsilon) = \not{k}\not{k} - (m - i\varepsilon)^2 = k^2 - m^2 + i\varepsilon + O(\varepsilon^2)$ where the $O(\varepsilon^2)$ order term as well as the $O(\varepsilon)$ in the numerator in ε may be dropped as the limit $\varepsilon \rightarrow 0$ is always understood.

We have used already the property that the trace of an odd number of γ -matrices is zero. F is the number of closed fermion loops, $F = 1$ in our case. As a convention the string of γ -matrices is read against the direction of the arrows. We again use the short notation

$$(1) = k^2 - m^2 + i\varepsilon, \quad (2) = (q + k)^2 - m^2 + i\varepsilon$$

and

$$\int_k \dots = \int \frac{d^d k}{(2\pi)^d} \dots$$

Gauge invariance or transversality of the photon field requires

$$q_\mu \Pi^{\mu\nu} = 0$$

where $\Pi^{\mu\nu}$ is the symmetric vacuum polarization tensor. We may check transversality directly as follows

$$\begin{aligned} & q_\nu \text{Tr} \gamma^\mu \frac{1}{\not{k} - m} \gamma^\nu \frac{1}{(\not{q} + \not{k}) - m} = \text{Tr} \gamma^\mu \frac{1}{\not{k} - m} \not{q} \frac{1}{(\not{q} + \not{k}) - m} \\ &= \text{Tr} \gamma^\mu \frac{1}{\not{k} - m} [(\not{q} + \not{k} - m) - (\not{k} - m)] \frac{1}{(\not{q} + \not{k}) - m} \\ &= \text{Tr} \gamma^\mu \left(\frac{1}{\not{k} - m} - \frac{1}{(\not{q} + \not{k}) - m} \right) \end{aligned}$$

which upon integration should be zero. Indeed, in dimensional regularization, we may shift the integration variable in the second integral $q + k = k'$, and by integrating we find

$$\int_k \text{Tr} \gamma^\mu \frac{1}{\not{k} - m} - \int_k \text{Tr} \gamma^\mu \frac{1}{(\not{q} + \not{k}) - m} = 0.$$

It is understood that d is chosen such that the integrals converge to start with. The result is then analytically continued to arbitrary d . This then explicitly proves the transversality (2.160). We may exploit transversality and contract the vacuum polarization tensor with the metric tensor and consider the resulting scalar quantity

$$\begin{aligned} & i g_{\mu\nu} \Pi^{\mu\nu} = -i g_{\mu\nu} (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi'(q^2) = i q^2 (d - 1) \Pi'(q^2) \\ &= e^2 \int_k \frac{\text{Tr} (\gamma^\alpha \not{k} \gamma_\alpha (\not{q} + \not{k}))}{(1)(2)} + e^2 m^2 \int_k \frac{\text{Tr} (\gamma^\alpha \gamma_\alpha)}{(1)(2)}. \end{aligned}$$

Using the d -dimensional Dirac algebra relations (2.125) or, directly the trace relations (2.126), we have $\gamma^\alpha \not{k} \gamma_\alpha = (2 - d) \not{k}$ and thus the trace in the first integral is $(2 - d) \text{Tr} (\not{k} (\not{q} + \not{k})) = (2 - d) k(q + k) \text{Tr} 1$. The scalar products

$k^2 + kq$ in the numerator may be written as a difference of the two denominators (1) and (2) plus a term which does not depend on the integration variable k : $k^2 = (1) + m^2$ and $2kq = (q+k)^2 - m^2 - k^2 + m^2 - q^2 = (2) - (1) - q^2$ and hence $k^2 + kq = \frac{1}{2}[(2) + (1) - q^2 + 2m^2]$. The terms proportional to (1) and (2) each cancel against one of the denominators and give a momentum independent *tadpole* integral.

The point of these manipulations is that we got rid of the polynomial in k in the numerator and thus were able to reduce the integrals to a set of basic integrals of a scalar theory. In our example, with the definitions (2.151) and (2.153), we get

$$\int_k \frac{k^2 + kq}{(1)(2)} = \frac{i}{16\pi^2} \frac{1}{2} ((2m^2 - q^2) B_0(m, m; q^2) - 2 A_0(m)) .$$

For the one-loop vacuum polarization as a result we then have²⁸

$$q^2 \Pi'(q^2) = \frac{e^2}{16\pi^2} \frac{1}{(d-1)} \left\{ 4(2-d) \left(m^2 - \frac{q^2}{2}\right) B_0(m, m; q^2) - 4(2-d) A_0(m) + 4dm^2 B_0(m, m; q^2) \right\} .$$

Now we have to expand the result in $d = 4 - \epsilon$. At the one-loop level at most simple poles in ϵ are expected, thus a bare one-loop amplitude in the vicinity of $d = 4$ is of the form

$$A = a_{-1} \frac{1}{\epsilon} + a_0 + a_1 \epsilon + \dots$$

The expansions for the standard scalar integrals A_0 and B_0 are given in (2.144) and (2.146), respectively, and the singular terms read

$$A_0(m) = -m^2 \frac{2}{\epsilon} + O(1) , \quad B_0(m_1, m_2; q^2) = \frac{2}{\epsilon} + O(1)$$

which leads to (2.149). In addition, we have to expand

$$\frac{1}{d-1} = \frac{1}{3-\epsilon} = \frac{1}{3(1-\frac{\epsilon}{3})} \simeq \frac{1}{3} + \frac{\epsilon}{9} + O(\epsilon^2) .$$

²⁸We adopt the scheme setting the trace of the unit matrix in spinor space $\text{Tr } \mathbf{1} = 4$; it is of course mandatory to keep this convention consistently everywhere. While bare quantities obviously depend on this convention, one can prove that quantities finite in the limit $d \rightarrow 4$, like the renormalized ones, are unambiguous.

As a result for the bare amplitude we obtain

$$q^2 \Pi'(q^2) = \frac{e^2}{16\pi^2} \frac{8}{3} \left\{ m^2 - \frac{q^2}{6} + A_0(m) + \left(m^2 + \frac{q^2}{2} \right) B_0(m, m; q^2) \right\} \quad (2.167)$$

an expression which exhibits regularized UV singularities, represented by the poles in ϵ present in A_0 and B_0 .

We now have to discuss the renormalization of the photon propagator. Concerning mass renormalization, we first go back to the general form (2.161) of the vacuum polarization tensor and identify $\Pi_2 = -\Pi'$ and $\Pi_1 = -q^2 \Pi_2 = q^2 \Pi'(q^2)$ due to transversality. As we have shown earlier in this section, electromagnetic gauge invariance requires:

$$\lim_{q^2 \rightarrow 0} \Pi_1(q^2) = 0 \quad (2.168)$$

and we may check now explicitly whether the calculated amplitude satisfies this condition. For $q^2 = 0$ we have

$$B_0(m, m; 0) = -1 - \frac{A_0(m)}{m^2} = \text{Reg} - \ln m^2 \quad (2.169)$$

and hence, as it should be,

$$\lim_{q^2 \rightarrow 0} q^2 \Pi'(q^2) = \frac{e^2}{16\pi^2} \frac{8}{3} \{ m^2 + A_0(m) + m^2 B_0(m, m; 0) \} = 0 .$$

This proves the absence of a photon mass renormalization at this order as a consequence of $U(1)_{\text{em}}$ gauge invariance.

Next we consider the wavefunction renormalization. The renormalized photon propagator is $D'_{\text{ren}} = Z_\gamma^{-1} D'_0$, where the renormalized physical propagator is required to have residue unity of the pole at $q^2 = 0$. This infers that the interacting photon propagator in the vicinity of the pole behaves like a free photon (asymptotically free scattering state). From (2.165) we learn that the residue of the pole $q^2 = 0$ in the bare propagator is given by $1/(1 + \Pi'_\gamma(0))$ such that the wave function renormalization condition for the photon reads $Z_\gamma(1 + \Pi'_\gamma(0)) = 1$ or

$$Z_\gamma = [1 + \Pi'_\gamma(0)]^{-1} \simeq 1 - \Pi'_\gamma(0) . \quad (2.170)$$

We thus have to calculate

$$\begin{aligned} \lim_{q^2 \rightarrow 0} \Pi'_\gamma(q^2) &= \frac{e^2}{16\pi^2} \frac{8}{3q^2} \left\{ \frac{m^2 - q^2}{6} + A_0(m) + \left(m^2 + \frac{q^2}{2} \right) B_0(m, m; q^2) \right\} \Big|_{q^2 \rightarrow 0} \\ &= \frac{e^2}{16\pi^2} \frac{8}{3} \left\{ -\frac{1}{6} + m^2 \dot{B}_0(m, m; 0) + \frac{1}{2} B_0(m, m; 0) \right\} \end{aligned}$$

where we have used the expansion

$$B_0(m, m; q^2) = B_0(m, m; 0) + q^2 \dot{B}_0(m, m; 0) + O(q^4) .$$

Using the integral representation (2.146) it is easy to find

$$\dot{B}_0(m, m; 0) = \frac{1}{6} \frac{1}{m^2} , \quad (2.171)$$

and together with (2.169) we obtain the simple result

$$\begin{aligned} Z_\gamma - 1 &= \frac{e^2}{12\pi^2} B_0(m, m; 0) \\ &= \frac{\alpha}{3\pi} \ln \frac{\mu^2}{m^2} . \end{aligned} \quad (2.172)$$

where the last expression is given in the $\overline{\text{MS}}$ scheme with $\text{Reg} = \ln \mu^2$. We finally may write down the renormalized photon vacuum polarization which takes the form

$$\begin{aligned} \Pi'_{\gamma \text{ ren}}(q^2) &= \Pi'_\gamma(q^2) - \Pi'_\gamma(0) \\ &= \frac{e^2}{6\pi^2} \frac{1}{q^2} \left\{ m^2 - \frac{q^2}{6} + A_0(m) + \left(m^2 + \frac{q^2}{2} \right) B_0(m, m; q^2) - \frac{q^2}{2} B_0(m, m; 0) \right\} . \end{aligned}$$

Evaluating the integrals one obtains

$$B_0(m, m; q^2) = \text{Reg} + 2 - \ln m^2 + 2(y - 1) G(y) \quad (2.173)$$

where

$$y = \frac{4m^2}{q^2}$$

and

$$G(y) = \begin{cases} -\frac{1}{\sqrt{y-1}} \arctan \frac{1}{\sqrt{y-1}} & (y > 1) \\ \frac{1}{2\sqrt{1-y}} \ln \frac{\sqrt{1-y}+1}{\sqrt{1-y}-1} & (y < 1) . \end{cases} \quad (2.174)$$

For $0 < y < 1$, which means $q^2 > 4m^2$, the self-energy function is complex, given by

$$G(y) = \frac{1}{2\sqrt{1-y}} \left(\ln \frac{1 + \sqrt{1-y}}{1 - \sqrt{1-y}} - i\pi \right) . \quad (2.175)$$

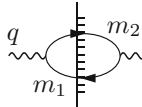
The imaginary part in the *time-like region* $q^2 > 0$ for $\sqrt{q^2} > 2m$ is a consequence of the fact that an electron–positron pair can be actually produced as real particles when the available energy exceeds the sum of the rest masses of the produced particles. The vacuum polarization function is thus an analytic function in the complex q^2 -plane with a cut along the positive real axis starting at $q^2 = 4m^2$, which is the *threshold* for pair-creation.²⁹

The final result for the renormalized vacuum polarization then reads

$$\Pi'_{\gamma \text{ ren}}(q^2) = \frac{\alpha}{3\pi} \left\{ \frac{5}{3} + y - 2 \left(1 + \frac{y}{2}\right) (1 - y) G(y) \right\} \tag{2.176}$$

which in fact is a function of q^2/m^2 . This renormalized vacuum polarization function will play a crucial role in different places later. For later purposes it is useful to note that it may be written in compact form as the following integral³⁰

²⁹As a rule, a cut diagram



contributes to the imaginary part if the cut diagram kinematically allows physical intermediate states: $q^2 \geq (m_1 + m_2)^2$. In place of the virtual photon (a real photon requires $q^2 = 0$ and does not decay) let us consider the massive charged weak gauge boson W . The W is an unstable particle and decays predominantly as $W^- \rightarrow \ell^- \bar{\nu}_\ell$ ($\ell = e, \mu, \tau$) leptonically, and $W^- \rightarrow d\bar{u}, b\bar{c}$ hadronically. Looking at the transversal self-energy function $\Pi_W(q^2)$ of the W on the mass shell $q^2 = M_W^2$ we have

$$\text{Im } \Pi_W(q^2 = M_W^2) = M_W \Gamma_W \neq 0$$

defining the finite width Γ_W of the W -particle. Note that $W^- \rightarrow b\bar{t}$ is not allowed kinematically because the top quark t is heavier than the W ($M_W = 80.385 \pm 0.015 \text{ GeV}, m_t = 173.21 \pm 0.87 \text{ GeV}, m_b = 4.18 \pm 0.03 \text{ GeV}$) for an on-shell W and hence does not contribute to the width.

Cutting lines means applying the substitution (see (2.141))

$$\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow -i \pi \delta(p^2 - m^2)$$

for the corresponding propagators. In general the imaginary part is given by cutting sets of lines of a diagram in all possible ways such that the diagram is cut into two disconnected parts. A cut contributes if the cut lines can be viewed as external lines of a real physical subprocess. Note that the imaginary part of an n -loop amplitude is given by cut diagrams exhibiting $n - 1$ closed loops at most. The imaginary part therefore is less UV divergent in general. In particular, the imaginary part of a one-loop diagram is always finite.

³⁰Which derives from

$$B_0(m, m; q^2) = \text{Reg} - \ln m^2 - \int_0^1 dz \ln(1 - z(1 - z)q^2/m^2)$$

$$\begin{aligned}\Pi'_{\gamma \text{ ren}}(q^2/m^2) &= -\frac{\alpha}{\pi} \int_0^1 dz \, 2z(1-z) \ln(1-z(1-z)q^2/m^2) \\ &= \frac{\alpha}{\pi} \int_0^1 dt \, t^2(1-t^2/3) \frac{1}{4m^2/q^2 - (1-t^2)} .\end{aligned}\quad (2.177)$$

The result (2.176) may be easily extended to include the other fermion contributions. In the $\overline{\text{MS}}$ scheme, defined by setting $\text{Reg} = \ln \mu^2$ in the bare form, we have

$$\Pi'_\gamma(q^2) = \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} \left[\ln \frac{\mu^2}{m_f^2} + \hat{G} \right] \quad (2.178)$$

where f labels the different fermion flavors (fermion species), Q_f is the charge in units of e and N_{cf} the *color factor*, $N_{cf} = 3$ for quarks and $N_{cf} = 1$ for the leptons. We have introduced the auxiliary function

$$\hat{G} = \frac{5}{3} + y - 2 \left(1 + \frac{y}{2}\right) (1-y) G(y) \simeq \begin{cases} \hat{G} = 0, & q^2 = 0 \\ \text{Re } \hat{G} = -\ln \frac{|q^2|}{m_f^2} + \frac{5}{3}, & |q^2| \gg m_f^2 \end{cases}$$

which vanishes at $q^2 = 0$. The imaginary part is given by the simple formula

$$\text{Im } \Pi'_\gamma(q^2) = \frac{\alpha}{3} \sum_f Q_f^2 N_{cf} \left(\left(1 + \frac{y}{2}\right) \sqrt{1-y} \right) . \quad (2.179)$$

Using the given low and high energy limits we get

$$\Pi'_\gamma(0) = \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} \ln \frac{\mu^2}{m_f^2} \quad (2.180)$$

and

$$\text{Re } \Pi'_\gamma(q^2) = \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} \left(\ln \frac{\mu^2}{|q^2|} + \frac{5}{3} \right) ; \quad |q^2| \gg m_f^2 . \quad (2.181)$$

(Footnote 30 continued)

(see (2.146)). The second form is obtained from the first one by a transformation of variables $z \rightarrow t = 2z - 1$, noting that $\int_0^1 dz \dots = 2 \int_{\frac{1}{2}}^1 dz \dots$, and performing a partial integration with respect to the factor $z(1-z) = (1-t^2)/4 = \frac{d}{dt} t(1-t^2/3)/4$ in front of the logarithm.

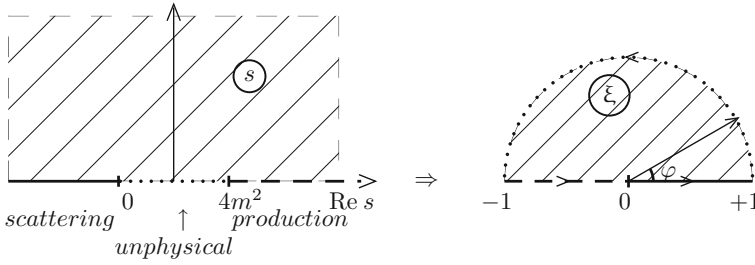


Fig. 2.9 Conformal mapping of the upper half s -plane into a half unit-circle

This concludes our derivation of the one-loop photon vacuum polarization, which will play an important role also in the calculation of the anomalous magnetic moment of the muon.

Conformal Mapping

For numerical evaluations and for working with asymptotic expansions, it is often a big advantage to map the physical upper half $s = q^2$ -plane into a bounded region as, for example, the interior of a half unit-circle as shown in Fig. 2.9. Such a conformal mapping is realized by the transformation of variables (ξ should not be confused with the gauge parameter ξ)

$$s \rightarrow \xi = \frac{\sqrt{1-y} - 1}{\sqrt{1-y} + 1} ; \quad y = \frac{4m^2}{s} \quad (2.182)$$

or

$$\frac{s}{m^2} = -\frac{(1-\xi)^2}{\xi} ; \quad \sqrt{1-y} = \frac{1+\xi}{1-\xi} .$$

If we move along the real s axis from $-\infty$ to $+\infty$ we move on the half unit-circle from 0 to +1, then on the arc segment counter clockwise and from -1 back to 0. We distinguish the following regions:

$$\begin{array}{lll} \text{scattering} & s < 0 & : 0 \leq \xi \leq 1 , \quad \ln \xi \\ \text{unphysical} & 0 < s < 4m^2 & : \xi = e^{i\varphi} , \quad \ln \xi = i\varphi \\ \text{production} & 4m^2 < s & : -1 \leq \xi \leq 0 , \quad \ln \xi = \ln |\xi| + i\pi \end{array}$$

where

$$\varphi = 2 \arctan \frac{1}{\sqrt{y-1}} ; \quad 0 \leq \varphi \leq \pi .$$

On the arc holds $1/y = \sin^2 \frac{\varphi}{2}$. The function $G(y)$ has now the representation

$$G(y) = \begin{cases} -\frac{1}{2} \frac{1-\xi}{1+\xi} \ln \xi, & 0 > s \\ -\frac{1}{2} \varphi \tan \frac{\varphi}{2}, & 4m^2 > s > 0 \\ -\frac{1}{2} \frac{1-\xi}{1+\xi} (\ln |\xi| + i\pi), & s > 4m^2. \end{cases}$$

As an application we may write the photon vacuum polarization amplitude (2.176) in the form

$$\begin{aligned} \Pi_{\gamma \text{ ren}}(s) &= q^2 \Pi'_{\gamma \text{ ren}}(s) \\ &= \frac{\alpha m^2}{3\pi} \begin{cases} -\frac{22}{3} + \frac{5}{3} (\xi^{-1} + \xi) + (\xi^{-1} + \xi - 4) \frac{1+\xi}{1-\xi} \ln \xi, & s < 0 \\ -\frac{20}{3} \sin^2 \frac{\varphi}{2} - 4 + 2(1 + 2 \sin^2 \frac{\varphi}{2}) \varphi \cot \frac{\varphi}{2}, & 0 < s < 4m^2. \end{cases} \end{aligned}$$

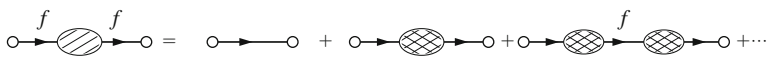
For $s > 4m^2$ the first form holds with $\ln \xi = \ln |\xi| + i\pi$. Corresponding representations are used for the vertex function as well as for the kernel function of the vacuum polarization integral contributing to $g - 2$ (see Sect. 5.1.7).

2.6.2 The Electron Self-Energy

Next we study the full propagator of a Dirac fermion f

$$iS'_f(x-y) = \langle 0|T \{ \psi_f(x) \bar{\psi}_f(y) \} |0\rangle$$

in momentum space. Again, the propagator has the structure of a repeated insertion of the 1PI self-energy $-i\Sigma_f(p)$



$$\begin{aligned} iS'_f(p) &\equiv \frac{i}{\not{p} - m_f} + \frac{i}{\not{p} - m_f} (-i\Sigma_f) \frac{i}{\not{p} - m_f} \\ &\quad + \frac{i}{\not{p} - m_f} (-i\Sigma_f) \frac{i}{\not{p} - m_f} (-i\Sigma_f) \frac{i}{\not{p} - m_f} + \dots \\ &= \frac{i}{\not{p} - m_f} \left\{ 1 + \left(\frac{\Sigma_f}{\not{p} - m_f} \right) + \left(\frac{\Sigma_f}{\not{p} - m_f} \right)^2 + \dots \right\} \\ &= \frac{i}{\not{p} - m_f} \left\{ \frac{1}{1 - \frac{\Sigma_f}{\not{p} - m_f}} \right\} = \frac{i}{\not{p} - m_f - \Sigma_f}. \end{aligned} \tag{2.183}$$

The Dyson series here is a geometric progression of matrix insertions which again can be summed in closed form and the inverse full fermion propagator reads

$$\begin{aligned}
 -iS_f'^{-1} &= \text{---} \rightarrow \text{---} \text{---} \rightarrow \text{---} = \text{---} \rightarrow \text{---} + \text{---} \rightarrow \text{---} + \dots \\
 &= -i \left\{ \not{p} - m_f - \Sigma_f(p) \right\} .
 \end{aligned} \tag{2.184}$$

The self-energy is given by an expansion in a series of 1PI diagrams

$$-i\Sigma_f(p) \equiv \text{---} \rightarrow \text{---} \text{---} \rightarrow \text{---} = \text{---} \rightarrow \text{---} + \dots .$$

The covariant decomposition of $\Sigma_f(p)$ for a massive fermion takes the form

$$\Sigma(p) = \not{p} (A(p^2, m_f, \dots)) + m_f (B(p^2, m_f, \dots)) , \tag{2.185}$$

where A and B are Lorentz scalar functions which depend on p^2 and on all parameters (indicated by the dots) of a given theory. In vector-like theories, like QED and QCD, no parity violating γ_5 terms are present, and the pole of the propagator, or, equivalently, the zero of the inverse propagator, is given by a multiple of the unit matrix in spinor space:

$$\not{p} = \tilde{m} , \quad \text{where} \quad \tilde{m}^2 = s_P \tag{2.186}$$

defines the ‘‘pole mass’’ of the fermion in the p^2 -plane

$$\not{p} - m_f - \Sigma_f(p) \Big|_{\not{p}=\tilde{m}} = 0 . \tag{2.187}$$

Among the charged leptons only the electron is stable, and hence $\tilde{m}_e = m_e$ is real and given by the physical electron mass. For the unstable fermions $s_P = \tilde{m}^2 = m^2 - im\Gamma$ is the complex *pole mass*, where the real part defines the physical mass m and the imaginary part the width Γ , which is the inverse of the life time. Looking at the full propagator

$$S_f'(p) = \frac{1}{\not{p} - m_f - \Sigma_f(p)} = \frac{\not{p} (1 - A) + m_f (1 + B)}{p^2 (1 - A)^2 - m_f^2 (1 + B)^2} . \tag{2.188}$$

the pole condition may written in a form (2.164)

$$s_P - m_0^2 - \Omega(s_P, m_0^2, \dots) = 0 , \tag{2.189}$$

where

$$\Omega(p^2, m_0^2, \dots) \equiv p^2 (2A - A^2) + m_0^2 (2B + B^2) .$$

One easily checks that the numerator matrix is non–singular at the zero of the denominator of the full Dirac propagator. Thus the solution may be obtained by iteration of (2.189) to a wanted order in perturbation theory.

Now the fermion wave function renormalization has to be considered. The renormalized propagator is obtained from the bare one by applying the appropriate wave function renormalization factor $S'_{f\text{ren}} = Z_f^{-1} S'_{f0}$ (see (2.109)), where the renormalized physical propagator is required to have residue unity at the pole $\not{p} = \tilde{m}$. The interacting fermion propagator in the vicinity of the pole is supposed to behave like a free fermion (asymptotically free scattering state). In fact, this naive requirement cannot be satisfied in massless QED due to the long range nature of the electromagnetic interaction. Charged particles never become truly free isolated particles, they rather carry along a cloud of soft photons and this phenomenon is known as the infrared problem of QED. Strictly speaking the standard perturbation theory breaks down if we attempt to work with one–electron states. While the off–shell Green functions are well defined, their on–shell limit and hence the S –matrix does not exist. A way out is the so called Bloch–Nordsieck construction [59] which will be discussed below.

At intermediate stages of a calculation we may introduce an IR regulator like a tiny photon mass, which truncates the range of the electromagnetic interaction and thus allows one for a perturbative treatment to start with.

In vector–like theories the fermion wave function renormalization factor $\sqrt{Z_f} = 1 + \delta Z_f$ is just a number, i.e., it is proportional to the unit matrix in spinor space.³¹ Working now with a finite photon mass we may work out the on–shell wave function renormalization condition (LSZ asymptotic condition). For this purpose, we have to perform an expansion of the inverse bare propagator (2.184) about the pole $\not{p} = \tilde{m}$.

$$\begin{aligned} \not{p} - m_0 - \Sigma &= \tilde{m} + (\not{p} - \tilde{m}) - m_0 - \tilde{m}A(\tilde{m}^2, m_0, \dots) - m_0B(\tilde{m}^2, m_0, \dots) \\ &- \tilde{m} (p^2 - \tilde{m}^2) \left. \frac{\partial A(p^2, m_0, \dots)}{\partial p^2} \right|_{p^2=\tilde{m}^2} - m_0 (p^2 - \tilde{m}^2) \left. \frac{\partial B(p^2, m_0, \dots)}{\partial p^2} \right|_{p^2=\tilde{m}^2} \\ &+ \dots \end{aligned}$$

where \tilde{m} is the pole solution (2.187):

$$\not{p} - m_0 - \Sigma|_{\not{p}=\tilde{m}} = \tilde{m} - m_0 - \tilde{m}A(\tilde{m}^2, m_0, \dots) - m_0B(\tilde{m}^2, m_0, \dots) = 0$$

³¹In the unbroken phase of the SM the left–handed and the right–handed fermion fields get renormalized independently by c–number renormalization factors $\sqrt{Z_L}$ and $\sqrt{Z_R}$, respectively. In the broken phase, a Dirac field is renormalized by $\sqrt{Z_f} = \sqrt{Z_L}\Pi_- + \sqrt{Z_R}\Pi_+$ where $\Pi_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ are the chiral projectors. Hence, the wave function renormalization factor, becomes a matrix $\sqrt{Z_f} = 1 + \alpha + \beta\gamma_5$ and the bare fields are related to the renormalized one's by $\psi_0(x) = \sqrt{Z_f}\psi_r(x)$, which for the adjoint field reads $\bar{\psi}_0(x) = \bar{\psi}_r(x)\gamma^0\sqrt{Z_f}\gamma^0$.

We consider the first term, applying relations (2.125) we find

$$\begin{aligned} T_1 &= \int_k \frac{1}{k^2 - m_\gamma^2 + i\varepsilon} \frac{md + (2-d)(\not{p} + \not{k})}{(p+k)^2 - m^2 + i\varepsilon} \\ &= \frac{i}{16\pi^2} \left\{ (md + (2-d)\not{p}) B_0(m_\gamma, m; p^2) + (2-d)\not{p} B_1(m_\gamma, m; p^2) \right\} \end{aligned}$$

where B_1 is defined in (2.153) and may be expressed in terms of B_0 via (2.154). The limit of vanishing photon mass is regular and we may set $m_\gamma = 0$. Furthermore, expanding d about 4 using (2.149) we find

$$T_1 = \frac{i}{16\pi^2} \left\{ m(4B_0 - 2) + \not{p} \left(1 - \frac{A_0(m)}{p^2} - \frac{p^2 + m^2}{p^2} B_0 \right) \right\} \quad (2.193)$$

with

$$B_0 = B_0(0, m; p^2) = \text{Reg} + 2 - \ln m^2 + \frac{m^2 - p^2}{p^2} \ln \left(1 - \frac{p^2 + i\varepsilon}{m^2} \right).$$

We note that the first term T_1 is gauge independent. In contrast, the second term of (2.192) is gauge dependent. In the Feynman gauge $\xi = 1$ the term vanishes. In general,

$$T_2 = \int_k \frac{(1-\xi)}{(k^2 - m_\gamma^2)(k^2 - \xi m_\gamma^2)} \not{k} \frac{1}{\not{p} + \not{k} - m} \not{k}$$

where we may rewrite

$$\begin{aligned} \not{k} \frac{1}{\not{p} + \not{k} - m} \not{k} &= [(\not{p} + \not{k} - m) - (\not{p} - m)] \frac{1}{\not{p} + \not{k} - m} [(\not{p} + \not{k} - m) - (\not{p} - m)] \\ &= \not{k} - (\not{p} - m) + (\not{p} - m) \frac{1}{\not{p} + \not{k} - m} (\not{p} - m). \end{aligned}$$

The first term being odd in the integration variable yields a vanishing result upon integration, while the remaining one's vanish on the mass shell $\not{p} = m$ and hence will not contribute to the mass renormalization. We obtain

$$\begin{aligned} T_2 &= -(\not{p} - m) \int_k \frac{(1-\xi)}{(k^2 - m_\gamma^2)(k^2 - \xi m_\gamma^2)} \\ &\quad + (\not{p} - m) \int_k \frac{(1-\xi)}{(k^2 - m_\gamma^2)(k^2 - \xi m_\gamma^2)} \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i\varepsilon} (\not{p} - m), \end{aligned}$$

a result which affects the residue of the pole and thus contributes to the wave function renormalization. To proceed, we may use the pole decomposition

$$(1 - \xi) \frac{1}{k^2 - m_\gamma^2} \frac{1}{k^2 - \xi m_\gamma^2} = \frac{1}{m_\gamma^2} \left(\frac{1}{k^2 - m_\gamma^2} - \frac{1}{k^2 - \xi m_\gamma^2} \right).$$

Then all integrals are of the type we already know and the result may be worked out easily. Since these terms must cancel in physical amplitudes, we will not work them out in full detail here. Note that the second term is of order $O((\not{p} - m)^2)$ near the mass shell and hence does not contribute to the residue of the pole and hence to the wave function renormalization. The first term is very simple and given by

$$T_2 = (\not{p} - m) \left\{ -(1 - \xi) \frac{i}{16\pi^2} B_0(m_\gamma, \sqrt{\xi} m_\gamma; 0) \right\} + O((\not{p} - m)^2). \quad (2.194)$$

We now consider the mass renormalization. The latter is gauge invariant and we may start from $\Sigma = -ie^2 T_1 + ie^2 T_2$ in the Feynman gauge

$$\begin{aligned} \Sigma^{\xi=1} &= -ie^2 T_1 = A(p^2) \not{p} + B(p^2) m \\ &= \frac{e^2}{16\pi^2} \left\{ \not{p} \left(1 - \frac{A_0(m)}{p^2} - \frac{p^2 + m^2}{p^2} B_0 \right) + m (4B_0 - 2) \right\}. \end{aligned}$$

The physical on-shell mass renormalization counter term is determined by

$$\not{p} - m_0 - \Sigma|_{\not{p}=m} = \not{p} - m - \delta m - \Sigma|_{\not{p}=m} = 0 \quad \text{or} \quad \delta m = -\Sigma|_{\not{p}=m}$$

and hence

$$\begin{aligned} \frac{\delta m}{m} &= - (A(p^2) + B(p^2))|_{p^2 \rightarrow m^2} \\ &= \frac{e^2}{16\pi^2} \left\{ 1 + \frac{A_0(m)}{m^2} - 2B_0(m_\gamma, m; m^2) \right\} = \frac{e^2}{16\pi^2} \left\{ 3 \frac{A_0(m)}{m^2} - 1 \right\} \end{aligned}$$

where we have used

$$B_0(0, m; m^2) = 1 - \frac{A_0(m)}{m^2} = \text{Reg} + 2 - \ln m^2.$$

As a result the mass renormalization counter term is gauge invariant and infrared finite for $m_\gamma = 0$. The gauge dependent amplitude T_2 does not contribute. Using (2.144) we may write

$$\frac{\delta m}{m} = \frac{\alpha}{2\pi} \left\{ \frac{3}{2} \ln \frac{m^2}{\mu^2} - 2 \right\}. \quad (2.195)$$

The wave function renormalization at one-loop order is given by³³

$$\begin{aligned} Z_f - 1 &= \left(A(p^2) + 2m^2 \frac{\partial(A+B)(p^2)}{\partial p^2} \right) \Big|_{p^2 \rightarrow m^2} \\ &= \frac{e^2}{16\pi^2} \left\{ 1 + \frac{A_0(m)}{m^2} + 4m^2 \dot{B}_0(m_\gamma, m; m^2) + (1 - \xi) B_0(m_\gamma, \sqrt{\xi}m_\gamma; 0) \right\}. \end{aligned}$$

A calculation of \dot{B}_0 in the limit of a small photon mass yields

$$\dot{B}_0(m_\gamma, m; m^2) \stackrel{m_\gamma \rightarrow 0}{\simeq} -\frac{1}{m^2} \left(1 + \frac{1}{2} \ln \frac{m_\gamma^2}{m^2} \right)$$

a result which exhibits an IR singularity and shows that in massless QED the residue of the pole does not exist. An asymptotically small photon mass m_γ is used as an IR regulator here. In IR regularized QED we may write the result in the form

$$Z_f - 1 = \frac{\alpha}{2\pi} \left\{ \frac{1}{2} \ln \frac{m^2}{\mu^2} - 2 + 2 \ln \frac{m}{m_\gamma} + \frac{1}{2} (1 - \xi) \left(1 - \ln \frac{m_\gamma^2}{\mu^2} \right) + \frac{1}{2} \xi \ln \xi \right\}. \quad (2.197)$$

The important message here is that the residue of the pole of the bare fermion propagator is gauge dependent and infrared singular. What it means is that the LSZ asymptotic condition for a charged particle cannot be satisfied. The cloud of soft photons accompanying any charged state would have to be included appropriately. However, usually in calculating cross sections the Bloch–Nordsieck construction is applied. This will be elaborated on below.

³³Note that with T_2 from (2.194) we have

$$\Sigma^{\xi \neq 1} = ie^2 T_2 = (\not{p} - m) A^{\xi \neq 1}$$

where

$$A^{\xi \neq 1} = (1 - \xi) \frac{e^2}{16\pi^2} B_0(m_\gamma, \sqrt{\xi}m_\gamma; 0)$$

and $B^{\xi \neq 1} = -A^{\xi \neq 1}$, such that $A^{\xi \neq 1} + B^{\xi \neq 1} = 0$. This leads to a contribution

$$\begin{aligned} \delta Z_f^{\xi \neq 1} &= \frac{e^2}{16\pi^2} (1 - \xi) B_0(m_\gamma, \sqrt{\xi}m_\gamma; 0) \\ &= \frac{e^2}{16\pi^2} \left\{ (1 - \xi) \left(\text{Reg} + 1 - \ln m_\gamma^2 \right) + \xi \ln \xi \right\} \end{aligned} \quad (2.196)$$

to the wave function renormalization.

The renormalized fermion self-energy is given by

$$\begin{aligned}\Sigma_{f \text{ ren}} &= \Sigma_f + \delta m_f - (Z_f - 1) (\not{p} - m_f) \\ &= A_{\text{ren}} (\not{p} - m_f) + C_{\text{ren}} m_f\end{aligned}\quad (2.198)$$

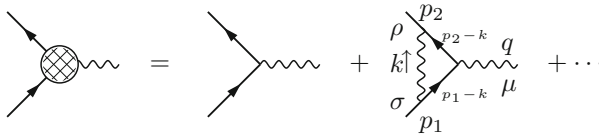
with

$$\begin{aligned}A_{\text{ren}} &= A - (Z_f - 1) \\ C_{\text{ren}} &= A + B + \frac{\delta m}{m}.\end{aligned}$$

In the context of $g-2$ the fermion self-energy plays a role as an insertion into higher order diagrams starting at two loops.

2.6.3 Charge Renormalization

Besides mass and wave function renormalization as a last step we have to perform a renormalization of the coupling constant, which in QED is the electric charge, or equivalently, the fine structure constant. The charge is defined via the electromagnetic vertex. The general structure of the vertex renormalization has been sketched in Sect. 2.4.1, already. Up to one-loop the diagrams to be considered are



Let us first consider the impact of *current conservation* and the resulting *Ward-Takahashi identity*. Current conservation, $\partial_\mu j_{\text{em}}^\mu(x) = 0$ translates into a consideration of

$$iq_\mu \Gamma^\mu = -ie \not{q} - i^6 e^3 \int \frac{d^d k}{(2\pi^d)} D_{\rho\sigma}(k) \gamma^\rho S_F(p_2 - k) \not{q} S_F(p_1 - k) \gamma^\sigma + \dots$$

with $q = p_2 - p_1$. First we note that

$$\not{q} = \not{p}_2 - \not{p}_1 = [\not{p}_2 - \not{k} - m] - [\not{p}_1 - \not{k} - m] = S_F^{-1}(p_2 - k) - S_F^{-1}(p_1 - k)$$

and thus

$$\begin{aligned}S_F(p_2 - k) \not{q} S_F(p_1 - k) &= S_F(p_2 - k) (S_F^{-1}(p_2 - k) - S_F^{-1}(p_1 - k)) S_F(p_1 - k) \\ &= S_F(p_1 - k) - S_F(p_2 - k),\end{aligned}$$

which means that contracted with q_μ the tree-point function reduces to a difference of two two-point functions (self-energies). Therefore, for the non-trivial one-loop part, using (2.192) we obtain

$$\begin{aligned} iq_\mu \Gamma^\mu(1) &= +e^3 \int_k D_{\rho\sigma}(k) \gamma^\rho S_F(p_1 - k) \gamma^\sigma - e^3 \int_k D_{\rho\sigma}(k) \gamma^\rho S_F(p_2 - k) \gamma^\sigma \\ &= ie \{ \Sigma^{(1)}(p_2) - \Sigma^{(1)}(p_1) \} \end{aligned}$$

which yields the electromagnetic Ward-Takahashi (WT) identity

$$\begin{aligned} q_\mu \Gamma^\mu(p_2, p_1) &= -e ([\not{p}_2 - m - \Sigma(p_2)] - [\not{p}_1 - m - \Sigma(p_1)]) \\ &= -e \left(S_F'^{-1}(p_2) - S_F'^{-1}(p_1) \right) \end{aligned} \quad (2.199)$$

which is the difference of the full inverse electron propagators. This relation can be shown easily to be true to all orders of perturbation theory. It has an important consequence for the renormalization of QED since it relates the vertex renormalization to the one of the charge (factor e) and the multiplicative wave function renormalization of the electron propagator. Combining the general form of the vertex renormalization (2.115) and $S_{F0}' = Z_e S_{F\text{ren}}'$ with the bare form of the WT identity we obtain the relationship

$$\begin{aligned} \sqrt{Z_\gamma} Z_e q_\mu \Gamma_0^\mu(p_2, p_1) &= -e_0 \sqrt{Z_\gamma} Z_e \left(S_{F0}'^{-1}(p_2) - S_{F0}'^{-1}(p_1) \right) \\ &= q_\mu \Gamma_{\text{ren}}^\mu(p_2, p_1) = -e_0 \sqrt{Z_\gamma} \left(S_{F\text{ren}}'^{-1}(p_2) - S_{F\text{ren}}'^{-1}(p_1) \right) \\ &= -e_{\text{ren}} \left(S_{F\text{ren}}'^{-1}(p_2) - S_{F\text{ren}}'^{-1}(p_1) \right) . \end{aligned}$$

We note that Z_e dropped out from the renormalized relation and we obtain the Ward-Takahashi identity

$$e_0 \sqrt{Z_\gamma} = e_{\text{ren}} \quad \text{or} \quad 1 + \frac{\delta e}{e} = \frac{1}{\sqrt{1 + \delta Z_\gamma}} = \sqrt{1 + \Pi'_\gamma(0)} . \quad (2.200)$$

The WT identity thus has the important consequence that the charge gets renormalized only by the photon vacuum polarization! This fact will play a crucial role later, when we are going to evaluate the hadronic contributions to the effective fine structure constant.

Another important consequence of the WT identity (2.199) we obtain by taking the limit $q_\mu \rightarrow 0$:

$$\begin{aligned}\Gamma^\mu(p, p) &= -e \lim_{p_2 \rightarrow p_1 = p} \frac{(S_F'^{-1}(p_2) - S_F'^{-1}(p_1))}{(p_2 - p_1)_\mu} \\ &= -e \frac{\partial S_F'^{-1}(p)}{\partial p_\mu} = e\gamma^\mu \left(1 - \frac{\partial \Sigma}{\partial \not{p}}\right).\end{aligned}$$

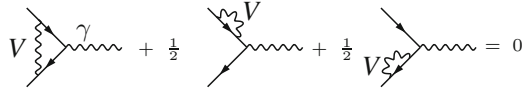
For on-shell leptons $\not{p} = \tilde{m}$ (see (2.187)) we arrive at the electromagnetic WT identity in the form

$$\Gamma^\mu(p, p)|_{\text{on-shell}} = -e\gamma^\mu \left(1 - \frac{\partial \Sigma}{\partial \not{p}} \Big|_{\not{p}=\tilde{m}}\right) = -e\gamma^\mu Z_f^{-1}.$$

Alternatively, we may write $Z_f \Gamma^\mu(p, p)|_{\text{on-shell}} = -e\gamma^\mu$ or

$$-e\gamma^\mu \delta Z_f + \Gamma'^\mu(p, p) \Big|_{\text{on-shell}} = 0 \quad (2.201)$$

where the prime denotes the non-trivial part of the vertex function. This relation tells us that some of the diagrams directly cancel. For example, we have ($V = \gamma$)



$$\frac{1}{2} \left[\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right] = 0 \quad (2.202)$$

The diagrams with the loops sitting on the external legs are contributions to the wave function renormalization and the factor $\frac{1}{2}$ has its origin in Eq. (2.110). This cancellation is the reason why the charge renormalization in QED is given by the simple relation (2.200).

We are now ready to calculate the vertex function at one-loop order. The Feynman diagram shown above translates into the Feynman integral

$$i\Gamma^\mu(p_2, p_1) = -i^6 e^3 \int \frac{d^d k}{(2\pi^d)} D_{\rho\sigma}(k) \frac{\gamma^\rho (\not{p}_2 - \not{k} + m) \gamma^\mu (\not{p}_1 - \not{k} + m) \gamma^\sigma}{((p_2 - k)^2 - m^2)((p_1 - k)^2 - m^2)}. \quad (2.203)$$

Actually, we are only interested here in the physical on-shell matrix element

$$\bar{u}(p_2, r_2) \Gamma^\mu(p_2, p_1) u(p_1, r_1),$$

$p_1^2 = m^2, p_2^2 = m^2$, the photon being still off-shell, however. For notational simplicity we omit writing down the spinors explicitly in most cases, however, always take advantage of simplifications possible if $\Gamma^\mu(p_2, p_1)$ would be sandwiched between

spinors. The first term of $D_{\rho\sigma}(k)$ (see (2.191)) produces a term proportional to

$$\gamma^\rho (\not{p}_2 - \not{k} + m) \gamma^\mu (\not{p}_1 - \not{k} + m) \gamma_\rho$$

and applying the Dirac algebra (2.123) and (2.125) in arbitrary dimension d together with the Dirac equation we can bring this string of γ -matrices to standard form. We anticommute \not{p}_2 to the left and \not{p}_1 to the right such that the Dirac equation $\bar{u}(p_2, r_2) (\not{p}_2 - m) \cdots = 0$ at the left end of the string of Dirac matrices may be used and $\cdots (\not{p}_1 - m) u(p_1, r_1) = 0$ at the right end. We denote $q = p_2 - p_1$ and $P = p_1 + p_2$. Furthermore we may write scalar products like $2kP = 2[k^2] - [(p_1 - k)^2 - m^2] - [(p_2 - k)^2 - m^2]$ in terms of the inverse scalar propagators which cancel against corresponding terms in the denominators. We thus obtain

$$\begin{aligned} & \gamma^\mu \{ (d-6)k^2 + 2[(p_1 - k)^2 - m^2] + [(p_2 - k)^2 - m^2] + 4p_1 p_2 \} \\ & + 4k^\alpha (P^\mu \gamma_\alpha - m g^\mu_\alpha) + 2(2-d)k^\alpha k^\mu \gamma_\alpha. \end{aligned}$$

In order to stick to the definitions (2.155) we have to replace the momentum assignments as $k \rightarrow -k$, $p_1 \rightarrow p_1$ and $p_2 \rightarrow p_2 - p_1$, and we obtain

$$\begin{aligned} T_1^\mu = \frac{i}{16\pi^2} & \left\{ \gamma^\mu \{ (d-6) B_0(m, m, q^2) + 4B_0(0, m; m^2) \right. \\ & + 2(q^2 - 2m^2) C_0(m_\gamma, m, m) + 2(2-d) C_{24} \} \\ & \left. + \frac{P^\mu}{2m} m^2 \{ 4C_{11} - 2(2-d) C_{21} \} \right\}. \end{aligned}$$

An unphysical amplitude proportional to q^μ also shows up at intermediate stages of the calculation. After reduction of the tensor integrals to scalar integrals this term vanishes. On the mass shell $p_1^2 = p_2^2 = m^2$ and for $m_\gamma = 0$ the three point tensor integrals in fact are completely expressible in terms of two point functions. Evaluating the C -integrals using (2.156), (2.157) and (2.158)) we find

$$\begin{aligned} C_{11}(m_\gamma, m, m) &= 2C_{12} \\ C_{12}(m_\gamma, m, m) &= -1/(sz) (B_0(m, m; s) - B_0(0, m; m^2)) \\ C_{21}(m_\gamma, m, m) &= -1/(sz) (B_0(0, m; m^2) - B_0(m, m; s)) \\ C_{22}(m_\gamma, m, m) &= -1/(sz) \left[\frac{m^2}{s} (1 + A_0(m)/m^2 + B_0(m, m; s)) \right. \\ & \quad \left. - \frac{1}{2} (A_0(m)/m^2 + B_0(m, m; s)) \right] \\ C_{23}(m_\gamma, m, m) &= -1/(sz) \frac{1}{2} (B_0(0, m; m^2) - B_0(m, m; s)) \\ C_{24}(m_\gamma, m, m) &= \frac{1}{4} (1 + B_0(m, m; s)) \end{aligned}$$

with $z = 1 - y$ where

$$y = 4m^2/q^2$$

is the kinematic variable we have encountered earlier in connection with the photon vacuum polarization.

Given the above relations we arrive at fairly simple expressions for the one-loop form factors in the Feynman gauge $\xi = 1$:

$$i\Gamma^{\mu \xi=1(1)} = -e^3 T_1^{\mu} = -ie \left\{ \gamma^{\mu} A_1 + \frac{P^{\mu}}{2m} A_2 \right\}$$

with

$$\begin{aligned} A_1 &= \frac{e^2}{16\pi^2} \left\{ 2(s - 2m^2) C_0(m_{\gamma}, m, m) \right. \\ &\quad \left. - 3B_0(m, m; s) + 4B_0(0, m; m^2) - 2 \right\} \\ A_2 &= \frac{e^2}{16\pi^2} \left\{ \frac{-y}{1-y} (B_0(m, m; s) - B_0(0, m; m^2)) \right\}. \end{aligned} \quad (2.204)$$

The only true vertex structure is the scalar three-point function C_0 in A_1 , which may be calculated from (2.147) (see [52] Appendix E) with the result

$$C_0(m_{\gamma}, m, m; m^2, q^2, m^2) = -\frac{2}{q^2} \ln \frac{-q^2}{m_{\gamma}^2} G(y) + \frac{1}{q^2} F(y) \quad (2.205)$$

with

$$\begin{aligned} G(y) &= -\frac{1}{2\sqrt{1-y}} \ln \xi \\ F(y) &= \frac{1}{2\sqrt{1-y}} \left\{ \frac{\pi^2}{3} + 4 \operatorname{Sp}(-\xi) + \ln^2 \xi + 4 \ln \xi \ln \frac{1+\xi}{1-\xi} \right\}. \end{aligned}$$

The variable

$$\xi = \frac{\sqrt{1-y} - 1}{\sqrt{1-y} + 1}, \quad (2.206)$$

used in this representation, was introduced in Sect. 2.6.1. The Spence function³⁴ or dilogarithm $\text{Sp}(x)$ is defined by

$$\text{Sp}(x) \equiv \text{Li}_2(x) = - \int_0^1 \frac{dt}{t} \ln(1 - xt) . \quad (2.208)$$

Looking at the standard form factor integral (2.205) for on-shell electrons, once more, we are confronted with an IR singular object. In massless QED the off-shell vertex function is regular, however, the on-shell limit does not exist. We thus again have to resort to an IR regularization by taking a small photon mass if we insist in calculating the on-shell amplitude.

Together with (2.173) the bare amplitudes may be written in a more explicit manner as in the $\overline{\text{MS}}$ scheme

$$A_1 = \frac{\alpha}{2\pi} \left\{ -\frac{1}{2} \ln \frac{m^2}{\mu^2} - 2 \left(1 - \frac{y}{2}\right) G(y) \ln \frac{-q^2}{m_\gamma^2} + 3 \left(1 - y\right) G(y) + \left(1 - \frac{y}{2}\right) F(y) \right\}$$

$$A_2 = \frac{\alpha}{2\pi} \left\{ y G(y) \right\} .$$

The second term of the photon propagator in (2.203) yields a contribution

$$T_2^\mu = - (1 - \xi) \int_k \frac{1}{k^2 - m_\gamma^2} \frac{1}{k^2 - \xi m_\gamma^2} \not{k} \frac{1}{\not{p}_2 - \not{k} - m} \gamma^\mu \frac{1}{\not{p}_1 - \not{k} - m} \not{k}$$

³⁴The Spence function is an analytic function with the same cut as the logarithm. Useful relations are

$$\begin{aligned} \text{Sp}(x) &= - \text{Sp}(1 - x) + \frac{\pi^2}{6} - \ln x \ln(1 - x) \\ \text{Sp}(x) &= - \text{Sp}\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2} \ln^2(-x) \\ \text{Sp}(x) &= - \text{Sp}(-x) + \frac{1}{2} \text{Sp}(x^2) . \end{aligned} \quad (2.207)$$

For $|x| \leq 1$ it has a series expansion

$$\text{Sp}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} .$$

Special values are:

$$\text{Sp}(0) = 0 , \quad \text{Sp}(1) = \frac{\pi^2}{6} , \quad \text{Sp}(-1) = -\frac{\pi^2}{12} , \quad \text{Sp}\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 .$$

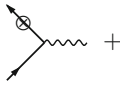
and for the on-shell vertex, applying the Dirac equation, one easily verifies that

$$\bar{u}_2 \not{k} \frac{1}{\not{p}_2 - \not{k} - m} \gamma^\mu \frac{1}{\not{p}_1 - \not{k} - m} \not{k} u_1 = \bar{u}_2 \gamma^\mu u_1$$

and hence this gauge dependent and UV divergent but q^2 independent term only contributes to the amplitude A_1 and is given by

$$i\delta\Gamma^{\mu\xi\neq 1(1)} = -e^3 T_2^\mu = -ie\gamma^\mu A_1^{\xi\neq 1} = -ie\gamma^\mu \left(-\frac{e^2}{16\pi^2} (1-\xi) B_0(m_\gamma, \sqrt{\xi}m_\gamma; 0) \right). \quad (2.209)$$

This term exactly cancels against the gauge parameter dependent lepton part of the wave function renormalization (2.196):



$$+ \quad = -ie\gamma^\mu \delta Z_e = -ie\gamma^\mu \left(\frac{e^2}{16\pi^2} (1-\xi) B_0(m_\gamma, \sqrt{\xi}m_\gamma; 0) \right).$$

In view of the discussion after (2.201), this cancellation is again a consequence of the WT identity. As it should be the gauge dependent term does not contribute to any physical amplitude after the appropriate wave function renormalization has been applied, i.e., the terms do not appear in the renormalized Dirac form factor A_1 . The Pauli form factor in any case is not affected, it is gauge invariant and UV finite and is not subject to renormalization.

In order to discuss charge renormalization, we have to write the form factors in terms of the Dirac (electric) plus a Pauli (magnetic) term. This we may do with the help of the *Gordon identity*

$$\bar{u}(p_2) \frac{i\sigma^{\mu\nu} q_\nu}{2m} u(p_1) = \bar{u}(p_2) \left(\gamma^\mu - \frac{P^\mu}{2m} \right) u(p_1).$$

Starting from our form factor decomposition, which is more convenient from a calculational point of view, we obtain

$$\begin{aligned} i\Gamma^\mu(p_2, p_1) &= -ie \left\{ \gamma^\mu A_{10}(q^2) + \frac{P^\mu}{2m} A_{20}(q^2) \right\} \\ &= -ie \left\{ \gamma^\mu (A_{10} + A_{20})(q^2) - i\sigma^{\mu\alpha} \frac{q_\alpha}{2m} A_{20}(q^2) \right\} \\ &= -ie \left\{ \gamma^\mu \delta F_E(q^2) + i\sigma^{\mu\alpha} \frac{q_\alpha}{2m} F_M(q^2) \right\}. \end{aligned}$$

Charge renormalization, according to (2.115), is fixed by the condition that $e_{\text{ren}} = e$ at $q^2 = 0$ (classical charge). We therefore have to require

$$\delta F_{\text{E ren}}(0) = A_{10}(0) + A_{20}(0) + \delta Z_e + \frac{1}{2}\delta Z_\gamma + \frac{\delta e}{e} = 0 .$$

The complete Dirac form factor, including the tree level value is given by

$$F_{\text{E ren}}(q^2) = 1 + \delta F_{\text{E ren}}(q^2) \quad (2.210)$$

and satisfies the charge renormalization condition

$$F_{\text{E ren}}(0) = 1 . \quad (2.211)$$

However, the electromagnetic Ward–Takahashi identity (2.201) infers

$$A_{10} + A_{20} + \delta Z_e = 0$$

such that, in agreement with (2.200), the charge renormalization condition fixes the charge counter term to the wave function renormalization constant of the photon

$$\frac{\delta e}{e} = -\frac{1}{2}\delta Z_\gamma = \frac{1}{2}\Pi'_\gamma(0) = -\frac{\alpha}{2\pi} \frac{1}{3} \ln \frac{m^2}{\mu^2} \quad (2.212)$$

with the explicit result given in the $\overline{\text{MS}}$ scheme $\text{Reg} = \ln \mu^2$.

As a result the renormalized one-loop virtual photon contributions to the lepton electric (E) and magnetic (M) form factors read

$$\begin{aligned} \delta F_{\text{E}} &= (A_{10} + A_{20} + \delta Z_e) \\ &= \frac{\alpha}{2\pi} \left\{ \ln \frac{m^2}{m_\gamma^2} - (2-y) G(y) \ln \frac{-q^2}{m_\gamma^2} - 2 + (3-2y)G(y) + \left(1 - \frac{y}{2}\right) F(y) \right\} \\ F_{\text{M}} = -A_{20} &= \frac{\alpha}{2\pi} \{-y G(y)\} . \end{aligned} \quad (2.213)$$

In the scattering region $q^2 < 0$ ($y < 0$) with $0 \leq \xi \leq 1$ the form factors are real; in the production region $q^2 > 4m^2$ ($0 < y < 1$) with $-1 \leq \xi \leq 0$ we have an imaginary part (using $\ln(\xi) = \ln(-\xi) + i\pi$, $\ln(-q^2/m^2 - i\varepsilon) = \ln(q^2/m^2) - i\pi$)

$$\begin{aligned} \frac{1}{\pi} \text{Im} F_{\text{E}} &= \frac{\alpha}{4\pi} \frac{1}{\sqrt{1-y}} \left\{ (2-y) \ln \frac{q^2 - 4m^2}{m_\gamma^2} - 3 + 2y \right\} \\ \frac{1}{\pi} \text{Im} F_{\text{M}} &= \frac{\alpha}{4\pi} \frac{y}{\sqrt{1-y}} \end{aligned} \quad (2.214)$$

The Dirac form factor for $q^2 \neq 0$ (on-shell electron, off-shell photon) at this stage is still IR singular in the limit of vanishing photon mass and cannot be physical. Before

we continue the discussion of the result we have to elaborate on the infrared problem in massless QED and the difficulties to define scattering states for charged particles.

However, the Pauli form factor, of primary interest to us turns out to be IR save. It is a perturbatively calculable quantity, which seems not to suffer from any of the usual problems of gauge dependence, UV divergences and the related renormalization scheme dependence. We thus are able to calculate the leading contribution to the anomalous magnetic moment without problems. The anomalous magnetic moment of a lepton is given by $F_M(0)$ where $F_M(q^2)$ is given in (2.213). We hence have to calculate $-y G(y)$ for $Q^2 = -q^2 > 0$ and $Q^2 \rightarrow 0$ or $y < 0$ and $|y| \rightarrow \infty$. Let $z = -y = |y|$ and z be large; the expansion yields

$$\begin{aligned} \sqrt{1-y} &= \sqrt{z+1} \simeq \sqrt{z} \left(1 + \frac{1}{2z} + \dots\right) \\ \ln \frac{\sqrt{1-y}-1}{\sqrt{1-y}+1} &= \ln \frac{\sqrt{z+1}-1}{\sqrt{z+1}+1} \simeq -\frac{2}{\sqrt{z}} + \dots \end{aligned}$$

and therefore

$$\begin{aligned} -y G(y)|_{-y \rightarrow \infty} &= -\frac{z}{2\sqrt{z+1}} \ln \frac{\sqrt{z+1}-1}{\sqrt{z+1}+1} \Big|_{z \rightarrow \infty} \\ &\simeq 1 + O\left(\frac{1}{\sqrt{|y|}}\right). \end{aligned}$$

We thus arrive at

$$F_M(0) = \frac{\alpha}{2\pi} \simeq 0.0011614\dots \quad (2.215)$$

which is Schwinger's classic result for the anomalous magnetic moment of the electron and which is universal for all charged leptons.

An important cross check of our calculation of F_E is also possible at this stage. Namely, we may check directly the WT identity (2.201), which now reads $\delta F_E(0) = 0$. Taking the limit $q^2 \rightarrow 0$ for space-like momentum transfer $q^2 < 0$, we may use the expansion just presented for calculating $F_M(0) = \alpha/2\pi$. For $y < 0$ and $|y| \rightarrow \infty$ we have $\xi \sim 1 - 2/\sqrt{|y|}$ and the somewhat involved expansion of $F(y)$ in (2.213) yields that $yF(y) \rightarrow 0$ in this limit. Since $-yG(y) \rightarrow 1$ we get precisely the cancellations needed to prove $\delta F_E(q^2) \rightarrow 0$ for $q^2 \rightarrow 0$.³⁵ The leading term for $|q^2| \ll 4m^2$ reads

³⁵One also may check this directly on the level of the standard scalar integrals A_0 , B_0 and C_0 . Denoting by $AA(m) = A_0(m)/m^2$ we have

$$\begin{aligned} \delta F_E(q^2) \stackrel{q^2 \rightarrow 0}{\sim} &\propto ([-4m^2 C_0 - 3B_0(m, m; 0) + 4B_0(0, m; m^2) - 2]_{A_1} \\ &+ [B_0(m, m; 0) - B_0(0, m; m^2)]_{A_2} + [1 + AA(m) + 4m^2 \dot{B}_0(m_\gamma, m; m^2)]_{\delta z_\epsilon}). \end{aligned}$$

$$\delta F_E(q^2) = \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{m}{m_\gamma} - \frac{3}{8} \right) + O(q^4/m^4)$$

and is IR singular and hence non-physical without including soft real photon emission. The leading behavior of the form factors for large $|q^2| \gg m^2$ reads

$$\begin{aligned} \delta F_E(q^2) &\sim -\frac{\alpha}{2\pi} \left(\frac{1}{2} \ln^2 \frac{|q^2|}{m^2} + 2 \ln \frac{m}{m_\gamma} \ln \frac{|q^2|}{m^2} - 2 \ln \frac{m}{m_\gamma} - \frac{3}{2} \ln \frac{|q^2|}{m^2} + 2 - \frac{\pi^2}{6} \right. \\ &\quad \left. - \Theta(q^2 - 4m^2) \frac{\pi^2}{2} \right) + \Theta(q^2 - 4m^2) i \frac{\alpha}{2} \left(\ln \frac{q^2}{m_\gamma^2} - \frac{3}{2} \right) \\ F_M(q^2) &\sim -\frac{\alpha}{\pi} \frac{m^2}{q^2} \ln \frac{|q^2|}{m^2} + \Theta(q^2 - 4m^2) i \alpha \frac{m^2}{q^2} . \end{aligned}$$

As in the examples discussed so far, often we will need to know the behavior of Feynman amplitudes for large momenta or equivalently for small masses. The tools for estimating the asymptotic behavior of amplitudes are discussed next.

2.6.4 Dyson- and Weinberg-Power-Counting Theorems

Since, in momentum space, any amplitude may be obtained as a product of 1PI building blocks, the vertex functions $\Gamma(p_1, \dots, p_n)$, it is sufficient to know the asymptotic behavior of the latter. This behavior may be obtained by considering the contributions from individual Feynman integrals $\Gamma_G(p_1, \dots, p_n)$, the index G denoting the corresponding Feynman graph. As we know already from Sect. 2.4.2, power counting theorems play an important role for evaluating

1. the convergence of Feynman integrals (UV divergences),
2. the behavior of Feynman amplitudes for large momenta.

Weinberg's power-counting theorem is an extension of Dyson's power-counting theorem, and describes the off-shell behavior of vertex functions (amputated n -point functions with $n \geq 2$)

(Footnote 35 continued)

Using the relations

$$\begin{aligned} C_0(m_\gamma, m, m; m^2, 0, m^2) &= \frac{-1}{4m^2} (B_0(0, m; m^2) - 1 - AA(m) + 2AA(m_\gamma)) \\ B_0(m, m; 0) &= -1 - AA(m) \\ B_0(0, m; m^2) &= 1 - AA(m) \\ m^2 \dot{B}_0(m_\gamma, m; m^2) &= -1 - \frac{1}{2} AA(m_\gamma) + \frac{1}{2} AA(m) \end{aligned}$$

one easily finds that indeed $\delta F_E(q^2) \stackrel{q^2 \rightarrow 0}{\sim} 0$. This kind of approach is usually utilized when working with computer algebra methods.

$$\Gamma(p_1, \dots, p_n) = \sum_G \Gamma_G(p_1, \dots, p_n)$$

for large p_i ($i = 1, \dots, m$) in a subspace of the momenta

$$\Gamma(\lambda p_1, \dots, \lambda p_m, p_{m+1}, \dots, p_n) \xrightarrow{\lambda \rightarrow \infty} ?$$

where (p_1, \dots, p_n) is a fixed set of momenta, $2 \leq m \leq n$ and λ a real positive stretching (dilatation) factor, which we are taking to go to infinity. The sum is over all possible Feynman graphs G which can contribute.

We first introduce some notions and notation. A set of external momenta (p_1, \dots, p_m) is called *non-exceptional* if no subsum of momenta vanishes, i.e., the set is generic. The set of external lines which carry momenta going to infinity is denoted by \mathcal{E}_∞ . By appropriate relabeling of the momenta we may always achieve that the first m of the momenta are the ones which go to infinity. Primarily the power counting theorems hold in the Euclidean region (after Wick-rotation) or in the Minkowski region for space-like momenta, which will be sufficient for our purpose. Also for massless theories there may be additional complications [60].

Dyson's power-counting theorem [61] states that

Theorem 2.9 *For all non-exceptional sets of momenta when all momenta are going to infinity a vertex function behaves as*

$$\Gamma(\lambda p_1, \dots, \lambda p_n) = \mathcal{O}(\lambda^{\alpha_F} (\ln \lambda)^{\beta_F}); \quad \lambda \rightarrow \infty,$$

where $\alpha_F = \max_{G \in \mathcal{G}} d(G)$ with $d(G)$ the superficial degree of divergence of a diagram G , and \mathcal{G} the set of diagrams which contribute to $\Gamma(p_1, \dots, p_n)$.

$d(G)$ has been introduced in Sect. 2.4.2. The asymptotic coefficient β_F giving the leading power of the logarithm may also be characterized in terms of diagrams [62], but will not be discussed here as we will need the asymptotic behavior modulo logarithms only. For an individual 1PI diagram G the Dyson power-counting theorem says that provided all momenta go to infinity, and the set of momenta is non-exceptional the behavior is determined by the superficial degree of divergence $d(G)$ of the corresponding diagram. The crucial point is that in a renormalizable theory $d(G)$ is independent of the particular graph G and given by the dimension of the vertex function $\dim \Gamma$ which only depends on type and number of external legs as discussed before in Sect. 2.4.2. In fact, in $d = 4$ dimensions,

$$\Gamma(\lambda p_1, \dots, \lambda p_n) = \mathcal{O}(\lambda^{4-b-\frac{3}{2}f} (\ln \lambda)^\ell).$$

with $b = n_B$ the number of boson lines and $f = n_F$ the number of fermion lines. ℓ is a non-negative integer depending on the order of perturbation theory. Its maximum possible value $\ell \leq L$ is given by the number L of loops.

Weinberg’s power-counting theorem [63] generalizes Dyson’s theorem and answers the question what happens when a subset only of all momenta is scaled to infinity. We first consider an individual Feynman integral G and 1PI subdiagrams $H \supset \mathcal{E}_\infty$ which include all lines \mathcal{E}_∞ tending to infinity. A subset $H \subset G$ here is a set of lines from G (external and internal) such that at each vertex there is either no line or two or more lines.³⁶ Then

$$\Gamma_G(\lambda p_1, \dots, \lambda p_m, p_{m+1}, \dots, p_n) = \mathcal{O}(\lambda^{d(H_0)} (\ln \lambda)^{\beta(H_0)})$$

where H_0 has maximal superficial degree of divergence $d(H)$. For a characterization of the logarithmic coefficient $\beta(H)$ see [62]. The result simplifies considerably if we consider the complete vertex function. When a non-exceptional set \mathcal{E}_∞ of external lines have momenta tending to infinity, then the total vertex function has as its asymptotic power a quantity $\alpha(\mathcal{E}_\infty)$

$$\Gamma(\lambda p_1, \dots, \lambda p_m, p_{m+1}, \dots, p_n) = \mathcal{O}(\lambda^{\alpha(\mathcal{E}_\infty)} (\ln \lambda)^\ell)$$

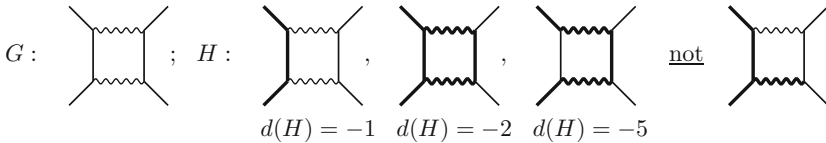
which depends only on the numbers and type of lines in \mathcal{E}_∞ , and is given by

$$\alpha(\mathcal{E}_\infty) = 4 - \frac{3}{2}f(\mathcal{E}_\infty) - b(\mathcal{E}_\infty) - \min_{\mathcal{E}'} \left[\frac{3}{2}f(\mathcal{E}') + b(\mathcal{E}') \right]. \quad (2.216)$$

Here $b(\mathcal{E})$, $f(\mathcal{E})$ are the number of bosons or fermions in the set \mathcal{E} . The minimum in (2.216) is taken over all sets \mathcal{E}' of lines such that the virtual transition $\mathcal{E}_\infty \leftrightarrow \mathcal{E}'$ is not forbidden by selection rules (charge, fermion number etc.). \mathcal{E}' is the set of external lines of H which are not in \mathcal{E}_∞ . Again, $\ell \leq L$.

Besides the high energy expansion (UV behavior) equally important is the low momentum expansion (IR behavior), which in a theory with massive particle fields is equivalent to a large mass expansion. Interestingly, in QED as well as in QCD (see below) masses are independent parameters of the theory, not related with the coupling constants. It means that on the level of the bare theory, masses only appear in propagators, which behave like $1/M^2$ for a heavy boson of mass $M \gg p$ and

³⁶The following example (electrons = full lines and photons = wavy lines) may illustrate this: fat lines carry the flow of large momentum (subgraph H)



The first graph in the set H determines the leading behavior $\mathcal{O}(\lambda^{-1} \ln^x \lambda)$. Note that all subgraphs H are connected and have no *dead end* lines (like the last diagram above, which is not a subgraph in the sense the term is used here). Thin lines attached to vertices of a subgraph H figure as external lines \mathcal{E}' , such that $\mathcal{E}_H = \mathcal{E}_\infty + \mathcal{E}'$ is the set of all external lines of H and $d(H) = 4 - \frac{3}{2}f(\mathcal{E}_H) - b(\mathcal{E}_H)$.

like $1/M$ for a heavy Dirac fermion of mass $M \gg p$. However, in loop integrals we cannot simply interchange limits $M \rightarrow \infty$ with $p \rightarrow \infty$ as the $O(1/p^2)$ behavior of a boson propagator or a $O(1/p)$ behavior of a fermion propagator are crucial for the convergence of the loop integrals. Indeed masses in general affect renormalization counterterms as we have seen in our one-loop renormalization calculations above. However, these residual mass effects drop out after renormalization (subtraction of the potential UV singularities). The property that very heavy particles do not affect the physics at much lower scales is called “decoupling” (of the heavy states), which looks to be a natural property of physics in general. Surprisingly, in the weak interaction sector of the electroweak SM decoupling is no longer true as masses and couplings are interrelated (mass generation via the Higgs mechanism, see below). Thus in the broken phase of the SM decoupling only holds in the QCD and QED sectors, and there is controlled by the Appelquist-Carazzone **decoupling theorem** [64].

Theorem 2.10 *If all external momenta of a process or in the corresponding amplitude are small relative to the mass M of a heavy state, then the “light fields only” Green functions of the full theory differ from the theory which has no heavy fields at all, only by finite renormalizations of couplings, masses and fields of the light theory, up to terms which are suppressed by inverse powers of the heavy mass. Thus further corrections are of the form $(\mu/M)^x$ with $x \geq 1$.*

It means that only the renormalization subtraction constants are dependent on M (logarithms) and this M -dependence gets renormalized away by physical subtraction conditions. The decoupling theorem is the root of the famous $\delta a_\ell \propto m_\ell^2/M^2$ behavior (1.9) of the lepton anomalies, and plays an important role in the classification of the various types of contributions to a_e and a_μ , as we will see.

For useful refinements of asymptotic expansion theorems in momenta and masses see e.g. [65] and references therein. Another tool to study the asymptotic behavior of Green- or vertex-functions is the renormalization group which we will consider next and in particular allows us to control effects due to the large UV logarithms.

2.6.5 The Running Charge and the Renormalization Group

Charge renormalization is governed by a *renormalization group* [66] (RG), which controls the response of the theory with respect to a change of the renormalization scale parameter μ in the $\overline{\text{MS}}$ scheme, like for example in the charge renormalization according to (2.212). It gives rise to the definition of an effective or running charge $\alpha(\mu)$ and running mass $m(\mu)$ as a function of the renormalization scale μ . However, the RG not only governs the dependence of a renormalized QFT on the renormalization scale, it yields the behavior of the theory with respect to dilations, the simultaneous stretching of all momenta, and hence allows one to discuss the asymptotic behavior for small and large momenta. The RG serves as a tool to systematically include large logarithmic radiative corrections, in fact, it permits the resummation to all orders of the perturbation expansion, of *leading logarithms* (LL),

next to leading logarithms (NLL) etc. It thus allows us to estimate leading radiative corrections of higher order without the need to actually perform elaborate calculations, under the condition that *large scale changes* are involved. Besides the all orders Dyson summation of self–energy corrections and the soft photon exponentiation to be discussed in the next section, the RG is a third method which allows us to predict leading higher order corrections from low order calculations. The RG generalizes the classical concept of dimensional analysis to QFT, where renormalization anomalies of the dilatation current [67] lead to a breaking of dilatation invariance by quantum effects (see Sect. 5.1.6 footnote on p. 375).

The RG may be obtained by starting from the bare vertex functions (the amputated Green functions) mentioned already briefly in Sect. 2.4.2. Note that the renormalization scale parameter μ is entering in DR by the fact that in the d –dimensional QFT the bare coupling constant \bar{e}_0 must have a dimension $\frac{4-d}{2}$, i.e., $\bar{e}_0 = e_0 \mu^{\varepsilon/2}$ with e_0 dimensionless (see (2.118)). This gives rise to the factors μ^{4-d} in the definitions of the standard integrals in Sect. 2.5.6 when working with the dimensionless bare coupling e_0 . As a result the μ dependence formally comes in via the UV regulator term (2.145). Since μ only enters via the bare coupling \bar{e}_0 all bare quantities, like the vertex function Γ_0 , at fixed \bar{e}_0 are independent of μ :

$$\mu \left. \frac{d\Gamma_0}{d\mu} \right|_{\bar{e}_0} \equiv 0 . \quad (2.217)$$

The bare vertex functions in $d = 4 - \varepsilon$ dimensions

$$\Gamma_0^{(n_A, 2n_\psi)}(\{p\}; \bar{e}_0, m_0, \xi_0)_\varepsilon$$

are homogeneous under simultaneous dilatation of all momenta and all dimensionful parameters including the scale μ . According to (2.119) we have

$$\Gamma_0^{(n_A, 2n_\psi)}(\{\kappa p\}; e_0 (\kappa \mu)^{\varepsilon/2}, \kappa m_0, \xi_0) = \kappa^{\dim \Gamma} \Gamma_0^{(n_A, 2n_\psi)}(\{p\}; e_0 (\mu)^{\varepsilon/2}, m_0, \xi_0) \quad (2.218)$$

with

$$\dim \Gamma = d - n_A \frac{d-2}{2} - 2n_\psi \frac{d-1}{2} .$$

The renormalized vertex functions are obtained by renormalizing parameters and fields: $A_0 = \sqrt{Z_A} A_r$, $\psi_0 = \sqrt{Z_\psi} \psi_r$, $e_0 = Z_g e_r$ and $m_0 = Z_m m_r$ and thus

$$\Gamma_0^{(n_A, 2n_\psi)}(\{p\}; \bar{e}_0, m_0, \xi_0)_\varepsilon = (Z_A)_\varepsilon^{-\frac{n_A}{2}} (Z_\psi)_\varepsilon^{-n_\psi} \Gamma_{\text{ren}}^{(n_A, 2n_\psi)}(\{p\}; e_r, m_r, \xi_r, \mu)_\varepsilon$$

where the wave function renormalization factors have the property to make the limit $\lim_{\varepsilon \rightarrow 0} \Gamma_{\text{ren}}(\{p\}; e_r, m_r, \xi_r, \mu)_\varepsilon$ exist. The trivially looking bare RG (2.217) becomes highly non-trivial if rewritten as an equation for Γ_{ren} as a function of the renormalized parameters. By applying the chain rule of differentiation we find the RG equation

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e_r} + \omega \frac{\partial}{\partial \xi_r} + \gamma_m m_r \frac{\partial}{\partial m_r} - n_A \gamma_A - 2n_\psi \gamma_\psi \right\} \Gamma_{\text{ren}} = 0 \quad (2.219)$$

where the coefficient functions are given by

$$\begin{aligned} \beta &= D_{\mu, \varepsilon} e_r = e_r \left(-\frac{\varepsilon}{2} + \frac{\varepsilon}{2} e_0 \frac{\partial}{\partial e_0} \ln Z_g \right) \\ \gamma_m m_r &= D_{\mu, \varepsilon} m_r = \frac{\varepsilon}{2} m_0 e_0 \frac{\partial}{\partial e_0} \ln Z_m \\ \gamma_A &= D_{\mu, \varepsilon} \ln Z_A = -\frac{\varepsilon}{4} e_0 \frac{\partial}{\partial e_0} \ln Z_A \\ \gamma_\psi &= D_{\mu, \varepsilon} \ln Z_\psi = -\frac{\varepsilon}{4} e_0 \frac{\partial}{\partial e_0} \ln Z_\psi \\ \omega &= D_{\mu, \varepsilon} \xi_r = -\frac{\varepsilon}{2} e_0 \frac{\partial}{\partial e_0} \xi_r = -2\xi_r \gamma_A. \end{aligned} \quad (2.220)$$

We have used

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} F(\bar{e}_0 = e_0 \mu^{\varepsilon/2})|_{\bar{e}_0} &= \left(\mu \frac{\partial}{\partial \mu} - \frac{\varepsilon}{2} e_0 \frac{\partial}{\partial e_0} \right) F(e_0, \mu) \doteq D_{\mu, \varepsilon} F(e_0, \mu) \\ \text{and} \quad F^{-1} D_{\mu, \varepsilon} F(e_0, \mu) &= D_{\mu, \varepsilon} \ln F(e_0, \mu) \end{aligned}$$

and the relation $\xi_0 = Z_A \xi_r$, i.e., $Z_\xi = Z_A$, which is a consequence of a WT identity, and implies $\omega = -2\xi_r \gamma_A$. Note that $\beta = \beta(e_r)$ and $\gamma_m = \gamma_m(e_r)$ are gauge invariant. In the *Landau gauge* $\xi_r = 0$ the coefficient function $\omega \equiv 0$ and $\gamma_i = \gamma_i(e_r)$ ($i = A, \psi$). The right hand sides of (2.220) have to be rewritten in terms of the renormalized parameters by inversion of the formal power series. The renormalization factors Z_i are of the form

$$Z_i = 1 + \sum_{n=1}^{\infty} \frac{Z_{i,n}(e_r, \xi_r)}{\varepsilon^n} \quad (2.221)$$

and applying the chain rule, we observe that the coefficient functions are uniquely determined by $Z_{i,1}(e_r, \xi_r)$ alone:

$$\begin{aligned}
\beta(e) &= \frac{e}{2} e \frac{\partial}{\partial e} Z_{g,1}(e) = \frac{\alpha}{\pi} \frac{e}{3} + \dots \\
\gamma_m(e) &= \frac{1}{2} e \frac{\partial}{\partial e} Z_{m,1}(e) = \frac{\alpha}{\pi} \frac{3}{2} + \dots \\
\gamma_A(e, \xi) &= \frac{1}{4} e \frac{\partial}{\partial e} Z_{A,1}(e, \xi) = \frac{\alpha}{\pi} \frac{2}{3} + \dots \\
\gamma_\psi(e, \xi) &= \frac{1}{4} e \frac{\partial}{\partial e} Z_{\psi,1}(e, \xi) = \frac{\alpha}{\pi} \frac{\xi}{2} + \dots
\end{aligned} \tag{2.222}$$

These are the residues of the simple ε -poles of the renormalization counter terms. The one–loop contributions we calculated above: $Z_A = Z_\gamma$ (2.172), $Z_\psi = Z_f$ (2.197), $Z_g = 1 + \frac{\delta e}{e}$ (2.212) and $Z_m = 1 + \frac{\delta m}{m}$ (2.195) with $\text{Reg} = \ln \mu^2 \rightarrow \frac{2}{\varepsilon}$ (see (2.145)). Note that in QED the WT identity (2.200) implies $Z_g = 1/\sqrt{Z_\gamma}$, which is very important because it says that charge renormalization is governed by photon vacuum polarization effects. The latter will play a crucial role in calculations of $g - 2$. The UV singular parts of the counter terms read

$$\begin{aligned}
Z_e &= 1 + \frac{e^2}{4\pi^2} \frac{1}{3} \frac{1}{\varepsilon}, & Z_m &= 1 - \frac{e^2}{4\pi^2} \frac{3}{2} \frac{1}{\varepsilon}, \\
Z_A &= 1 + \frac{e^2}{4\pi^2} \frac{2}{3} \frac{1}{\varepsilon}, & Z_\psi &= 1 + \frac{e^2}{4\pi^2} \frac{\xi}{2} \frac{1}{\varepsilon},
\end{aligned}$$

from which the leading terms of the RG coefficient functions given in (2.222) may be easily read off. The RG equation is a partial differential equation which is homogeneous and therefore can be solved easily along so called characteristic curves. Let s parametrize such a curve, such that all quantities become functions of a the single parameter s : $e = e(s)$, $m = m(s)$, $\mu = \mu(s)$ and

$$\frac{d\Gamma}{ds}(\{p\}; e(s), m(s), \mu(s)) = n\gamma \Gamma$$

with

$$\frac{d\mu}{ds} = \mu, \quad \frac{de}{ds} = \beta(e), \quad \frac{dm}{ds} = m\gamma_m(e),$$

which is a set of ordinary differential equations the solution of which is solving the RG equation (2.220). For simplicity of notation and interpretation we have assumed the Landau gauge $\xi = 0$ and we abbreviated $n_A \gamma_A + n_\psi \gamma_\psi = n\gamma$. The successive integration then yields

(1)

$$\frac{d\mu}{ds} = \mu \triangleright \ln \mu = s + \text{constant} \triangleright \mu = \mu_0 e^s = \mu_0 \kappa$$

where $\kappa = e^s$ is a scale dilatation parameter

(2)

$$\frac{de}{ds} = \beta(e) \triangleright \frac{de}{\beta(e)} = ds = \frac{d\mu}{\mu} \triangleright$$

$$\ln(\mu/\mu_0) = \ln \kappa = \int_e^{e(\kappa)} \frac{de'}{\beta(e')} \quad (2.223)$$

which is the implicit definition of the running coupling $e(\kappa)$ with $e = e(1)$ the coupling at reference scale μ_0 and $e(\kappa) = e(\mu/\mu_0)$ the coupling at scale μ .

(3)

$$\frac{dm}{ds} = m\gamma_m \triangleright \frac{dm}{m} = \gamma_m(e) ds = \gamma_m(e) \frac{de}{\beta(e)} \triangleright$$

$$m(\kappa) = m \exp \int_e^{e(\kappa)} \frac{\gamma(e') de'}{\beta(e')} \quad (2.224)$$

(4)

$$\frac{d\Gamma}{ds} = n\gamma(e) ds = n\gamma(e) \frac{d\mu}{\mu} = n\gamma(e) \frac{de}{\beta(e)} \triangleright$$

$$\Gamma(\kappa) = \Gamma \exp \left\{ n \int_e^{e(\kappa)} \frac{\gamma(e') de'}{\beta(e')} \right\} = \Gamma z_A(e, \kappa)^{n_A} z_\psi(e, \kappa)^{2n_\psi} \quad (2.225)$$

with $\Gamma = \Gamma(1)$, and

$$z_A(e, \kappa) = \exp \int_e^{e(\kappa)} \frac{\gamma_A(e') de'}{\beta(e')}, \quad z_\psi(e, \kappa) = \exp \int_e^{e(\kappa)} \frac{\gamma_\psi(e') de'}{\beta(e')}.$$

Altogether, we may write this as an equation which describes the response of the theory with respect to a change of the scale parameter μ :

$$\Gamma(\{p\}; e, m, \mu/\kappa) = z_A(e, \kappa)^{-n_A} z_\psi(e, \kappa)^{-2n_\psi} \Gamma(\{p\}; e(\kappa), m(\kappa), \mu) \quad (2.226)$$

Thus a **change of the scale parameter μ is equivalent to a finite renormalization of the parameters and fields** and together with the homogeneity relation we have for the vertex functions with scaled momenta

$$\begin{aligned}\Gamma(\{\kappa p\}; e, m, \mu) &= \kappa^{\dim\Gamma} \Gamma(\{p\}; e(\kappa), m(\kappa)/\kappa, \mu/\kappa) \\ &= \kappa^{\dim\Gamma} z_A(e, \kappa)^{-n_A} z_\psi(e, \kappa)^{-2n_\psi} \Gamma(\{p\}; e(\kappa), m(\kappa)/\kappa, \mu)\end{aligned}\quad (2.227)$$

which is the basic relation for a discussion of the asymptotic behavior.

Asymptotic Behavior

Two regimes are of interest, the high energy (ultraviolet) behavior and the low energy (infrared) behavior. For the general discussion we consider a generic gauge coupling g (in place of e in QED).

(1) UV behavior

The ultraviolet behavior, which determines the short distance properties, is obtained by choosing $\kappa|p| \gg m, \mu$ thus

$$\ln \kappa = \int_g^{g(\kappa)} \frac{dg'}{\beta(g')} \rightarrow +\infty ; \quad \kappa \rightarrow \infty .$$

However, the integral can only become divergent for finite $g(\kappa)$ if $\beta(g)$ has a zero at $\lim_{\kappa \rightarrow \infty} g(\kappa) = g^*$: more precisely, in the limit $\kappa \rightarrow \infty$ the effective coupling has to move to a fixed point $g(\kappa) \rightarrow g^*$ if finite, and the fixed point coupling is characterized by $\beta(g^*) = 0, \beta'(g^*) < 0$. Thus g^* is an ultraviolet fixed point coupling. Note that by dilatation of the momenta at fixed m and μ , the effective coupling is automatically driven into a fixed point, a zero of the β -function with negative slope, if it exists. If $g^* = 0$ we have asymptotic freedom. This is how QCD behaves, which has a β -function

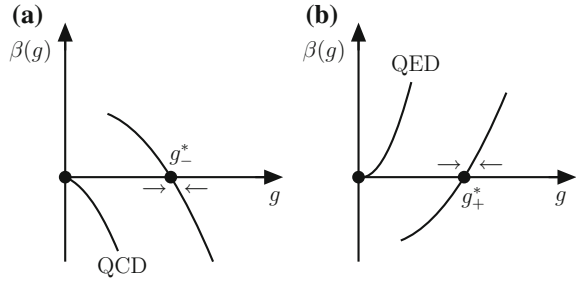
$$\beta_{\text{QCD}}(g_s) = -g_s \left(\beta_0 \left(\frac{g^2}{16\pi^2} \right) + \beta_1 \left(\frac{g^2}{16\pi^2} \right)^2 + \dots \right) \quad (2.228)$$

with $\beta_0 > 0$ (see Fig. 2.10a). QCD will be considered in more detail later on.

A possible fixed point is accessible in perturbation theory provided g^* is sufficiently small, such that perturbation theory is sufficiently “convergent” as an asymptotic series. One may then expand about g^* :

$$\begin{aligned}\beta(g) &= (g - g^*) \beta'(g^*) + \dots \\ \gamma(g) &= \gamma^* + (g - g^*) \gamma'(g^*) + \dots\end{aligned}$$

Fig. 2.10 RG fixed points are zeros of the β -function: **a** UV fixed points, **b** IR fixed points



and provided $\beta'(g_-^*) \neq 0$ we have

$$\begin{aligned} a(g, \kappa) &= \exp \int_g^{g(\kappa)} \frac{\gamma(g')}{\beta(g')} dg' = \exp \int_g^{g(\kappa)} \frac{\gamma(g_-^*)}{\beta(g')} dg' \cdot r(g, \kappa) \\ &= \kappa^{\gamma^*} r(g, \kappa) \end{aligned}$$

where

$$r(g, \kappa) = \exp \int_g^{g(\kappa)} \frac{(\gamma(g') - \gamma^*)}{\beta(g')} dg'$$

in the limit of large κ yields a finite scale independent wave function renormalization

$$\lim_{\kappa \rightarrow \infty} r(g, \kappa) = r(g, \infty) .$$

We thus find the asymptotic form

$$\Gamma(\{p\}; g, m, \mu) \xrightarrow{\kappa} \kappa^d (\kappa^{d_A} r_A(g, \infty))^{-n_A} (\kappa^{d_\psi} r_\psi(g, \infty))^{-2n_\psi} \Gamma(\{p\}; g_-^*, 0, \mu) \quad (2.229)$$

which exhibits asymptotic *scaling*. As naively expected it is given by the vertex functions of a *massless theory*. Indeed, at high energies masses may be neglected, however on the expense that another mass scale remains in the game, the scale parameter μ . The first factor κ^d is trivial and is due to the d -momentum conservation which was factored out. Then each field exhibits a homogeneous (power-like) behavior in the dilatation factor κ , the exponent of which exhibits an *anomalous dimension* as a consequence of the dynamics of the theory:

$$d_A = \frac{d-2}{2} + \gamma_A^* , \quad d_\psi = \frac{d-1}{2} + \gamma_\psi^* . \quad (2.230)$$

The first term is the naive or engineers dimension the second part is the anomalous part which is a quantum effect, a relict of the breaking of scale invariance, when $g \neq g^*$. While naively we would expect that in $d = 4$ dimensions the massless theory has scaling: for example a scalar two-point function, the only dimensionful physical quantity being the momentum, one would expect $G(p; g) \sim 1/p^2$ as G has dimension 2. However, if there would be a non-trivial UV fixed point one would have $G(p, g, \mu) \sim (\mu^2)^{\gamma^*}/(p^2)^{1+\gamma^*}$ ($\gamma^* > 0$) which shows the role and unavoidability of the scale parameter μ , which has to eat up the extra dimension γ^* induced by the dynamics of the theory. Otherwise only truly free theories could have scaling, called *canonical scaling* in this case. The discovery of asymptotic freedom of QCD [36] is the prime example of a dynamical theory, nota bene of the theory of strong interactions, exhibiting asymptotic canonical scaling (Bjorken scaling) of liberated quarks (quark parton model) [68]. The latter was discovered before in the pioneering investigations concerning *Deep Inelastic Scattering* (DIS) [69] of electrons on protons and bound neutrons by Friedman, Kendall and Taylor (Nobel prize 1990). These experiments have been of essential importance for the development of the quark model and to the discovery of QCD as the theory of the strong interactions.

(2) IR behavior

The infrared behavior corresponds to the long distance properties of a system. Here the regime of interest is $\kappa|p| \ll m, \mu$ and the discussion proceeds essentially as before: now as $\kappa \rightarrow 0$ the effective $g(\kappa) \rightarrow g_+^*$ where g_+^* is a zero of the β -function with positive slope, see Fig. 2.10b, $\beta(g_+^*) = 0$ and $\beta'(g_+^*) > 0$. This is the typical situation in the construction of low energy effective theories, particularly in the discussion of critical phenomena of statistical systems (keywords: critical behavior, critical exponents, scaling laws, universality). If $g_+^* = 0$ the effective theory is infrared free (the opposite of *asymptotic freedom*), also called Gaussian (Gaussian fixed point). Here the well known examples are QED

$$\beta_{\text{QED}}(e) = \frac{e^3}{12\pi^2} \sum_f N_{cf} Q_f^2 + \dots \quad (2.231)$$

or the self-interacting scalar field ϕ^4 -theory

$$\beta(\lambda) = -\varepsilon\lambda + \frac{3\lambda^2}{16\pi^2} + \dots$$

in $d = 4$ dimensions. For QED the running coupling to leading order thus follows from

$$\ln \kappa = \int_e^{e(\kappa)} \frac{1}{\beta(e')} de' = \frac{12\pi^2}{\sum_f N_{cf} Q_f^2} \int_e^{e(\kappa)} \frac{1}{(e')^3} de' = \frac{24\pi^2}{\sum_f N_{cf} Q_f^2} \left(\frac{1}{e^2} - \frac{1}{e(\kappa)^2} \right)$$

where the sum extends over all light flavors $f : m_f < \mu$.³⁷ The running fine structure constant thus at leading order is given by

$$\alpha(\mu) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi} \sum_f N_{cf} Q_f^2 \ln \mu/\mu_0} \quad (2.232)$$

where μ_0 is the scale where the lightest particle starts to contribute, which is the electron $\mu_0 = m_e$. We then may identify $\alpha(\mu_0) = \alpha$ the classical low energy value of the fine structure constant, with the proviso that only logarithmic accuracy is taken into account (see below). The running α is equivalent to the Dyson summation of the transversal part of the photon self-energy to the extent that only the logs are kept. The RG running takes into account the leading radiative corrections in the case the logs are dominating over constant terms, i.e., provided large scale changes are involved.

In the calculation of the contributions from electron loops in photon propagators to the muon anomaly a_μ , such large scale changes from m_e to m_μ are involved and indeed one may calculate such two-loop contributions starting from the lowest order result

$$a_\mu^{(2)} = \frac{\alpha}{2\pi} \text{ via the substitution } \alpha \rightarrow \alpha(m_\mu) \quad (2.233)$$

where

$$\alpha(m_\mu) = \frac{\alpha}{1 - \frac{2}{3} \frac{\alpha}{\pi} \ln \frac{m_\mu}{m_e}} = \alpha \left(1 + \frac{2}{3} \frac{\alpha}{\pi} \ln \frac{m_\mu}{m_e} + \dots \right) \quad (2.234)$$

such that we find

$$a_\mu^{(4)\text{LL}}(\text{vap}, e) = \frac{1}{3} \ln \frac{m_\mu}{m_e} \left(\frac{\alpha}{\pi} \right)^2$$

which indeed agrees with the leading log result obtained in [70] long time ago by a direct calculation. The method has been further developed and refined by Lautrup and de Rafael [71]. In the calculation of a_μ only the electron VP insertions are governed by the RG and the corresponding one-flavor QED β -function has been calculated to three loops

$$\beta(\alpha) = \frac{2}{3} \left(\frac{\alpha}{\pi} \right) + \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^2 - \frac{121}{144} \left(\frac{\alpha}{\pi} \right)^3 + \dots \quad (2.235)$$

³⁷This latter restriction takes into account the decoupling of heavy flavors, valid in QED and QCD. Since in the $\overline{\text{MS}}$ scheme, i.e., renormalization by the substitution $\text{Reg} \rightarrow \ln \mu^2$, which we are considering here, decoupling is not automatic, one has to impose it by hand. At a given scale one is thus considering an effective theory, which includes only those particles with masses below the scale μ .

long time ago by [72], which thus allows us to calculate leading $\alpha^n (\ln m_\mu/m_e)^n$, next-to-leading $\alpha^n (\ln m_\mu/m_e)^{n-1}$ and next-to-next-to-leading $\alpha^n (\ln m_\mu/m_e)^{n-2}$ log corrections. At present $\beta(\alpha)$ is known to five loops [73, 74] which allows one to calculate leading log a_μ contributions to six loops [75].

As $\alpha(\mu)$ is increasing with μ , at some point this resummed effective coupling (2.232) exhibits a pole, the so called *Landau pole* at which the coupling becomes infinite: $\lim_{\mu \rightarrow \mu_L} \alpha(\mu) = \infty$. The “fixed point” very likely is an artifact of perturbation theory, which of course ceases to be valid when the one-loop correction approaches 1. What this tells us is that we actually do not know what the high energy asymptotic behavior of QED is. This is in contrast to QCD, which exhibits the high energy asymptotic behavior of a free (non-interacting) field theory, which means that perturbation theory gets the better the higher the energy,

α in the on-shell versus α in the $\overline{\text{MS}}$ scheme

In our discussion of renormalizing QED we were considering originally the *on-shell renormalization scheme*, while the RG provides α in the $\overline{\text{MS}}$ scheme. Here we briefly discuss the relationship between the OS and the $\overline{\text{MS}}$ fine structure constants $\alpha_{\text{OS}} = \alpha$ and $\alpha_{\overline{\text{MS}}}$, respectively. Since the bare fine structure constant

$$\alpha_0 = \alpha_{\overline{\text{MS}}} \left(1 + \frac{\delta\alpha}{\alpha} \Big|_{\overline{\text{MS}}} \right) = \alpha_{\text{OS}} \left(1 + \frac{\delta\alpha}{\alpha} \Big|_{\text{OS}} \right) \quad (2.236)$$

is independent of the renormalization scheme. The one-loop calculation in the SM yields (including the charged W contribution for completeness)

$$\begin{aligned} \frac{\delta\alpha}{\alpha} \Big|_{\overline{\text{MS}}} &= \frac{\alpha}{3\pi} \sum Q_f^2 N_{cf} \ln \frac{\mu^2}{m_f^2} - \frac{\alpha}{3\pi} \frac{21}{4} \ln \frac{\mu^2}{M_W^2} \\ \frac{\delta\alpha}{\alpha} \Big|_{\text{OS}} &= \Pi'_\gamma(0) + \frac{\alpha}{\pi} \ln \frac{M_W^2}{\mu^2} \\ &= \frac{\delta\alpha}{\alpha} \Big|_{\overline{\text{MS}}} - \frac{\alpha}{6\pi} \end{aligned}$$

and thus

$$\alpha_{\overline{\text{MS}}}^{-1}(0) = \alpha^{-1} + \frac{1}{6\pi} \quad (2.237)$$

as a low energy matching condition. The α -shift in the $\overline{\text{MS}}$ scheme is very simple, just the UV logs,

$$\Delta\alpha_{\overline{\text{MS}}}(\mu) = \frac{\alpha}{3\pi} \sum Q_f^2 N_{cf} \ln \frac{\mu^2}{m_f^2} - \frac{\alpha}{3\pi} \frac{21}{4} \ln \frac{\mu^2}{M_W^2} \quad (2.238)$$

such that

$$\Delta\alpha_{\overline{\text{MS}}}(\mu) = \Delta\alpha_{\text{OS}}(\mu) + \frac{\alpha}{3\pi} \frac{5}{3} \sum Q_f^2 N_{cf} \quad (2.239)$$

where the sum goes over all fermions f with $N_{cf} = 1$ for leptons and $N_{cf} = 3$ for quarks.

In perturbation theory, the leading light fermion ($m_f \ll M_W, \sqrt{s}$) contribution in the OS scheme is given by

$$\Delta\alpha(s) = \frac{\alpha}{3\pi} \sum_f Q_f^2 N_{cf} \left(\ln \frac{s}{m_f^2} - \frac{5}{3} \right). \quad (2.240)$$

We distinguish the contributions from the leptons, for which the perturbative expression is appropriate, the five light quarks (u, d, s, c, b) and the top

$$\Delta\alpha = \Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{had}} + \Delta\alpha_{\text{top}}. \quad (2.241)$$

Since the top quark is heavy we cannot use the light fermion approximation for it. A very heavy top in fact decouples like

$$\Delta\alpha_{\text{top}} \simeq -\frac{\alpha}{3\pi} \frac{4}{15} \frac{s}{m_t^2} \rightarrow 0$$

when $m_t \gg s$. Since pQCD does not apply at low energies, $\Delta\alpha_{\text{had}}$ has to be evaluated via dispersion relations from e^+e^- -annihilation data.

Note that in $d = 4$ dimensions both for QCD and QED very likely there is no RG fixed point at finite value of g except $g = 0$, which always is a fixed point, either a UV one (QCD) or an IR one (QED). In QCD this could mean that $\alpha_s(\mu) \rightarrow \infty$ for $\mu \rightarrow 0$ (infrared slavery, confinement). In perturbation theory a Landau pole shows up at finite scale Λ_{QCD} when coming from higher energy scales, where $\alpha_s \rightarrow \infty$ for $\mu \xrightarrow{\sim} \Lambda_{\text{QCD}}$. In QED likely $\alpha(\mu) \rightarrow \infty$ for $\mu \rightarrow \infty$.

It is important to emphasize that the RG only accounts for the UV logarithms, which in DR are related to the UV poles in $d = 4 - \varepsilon$ dimensions. Large logs may also be due to IR singular behavior, like the terms proportional to $\ln m_\gamma$ which we have regulated with an infinitesimally small photon mass in the on-shell lepton wave function renormalization factor $Z_\psi = Z_f$ (2.197). In spite of the fact that this term appears in the UV renormalization counter term, it has nothing to do with a UV singularity and does not contribute in the RG coefficients. In DR also IR singularities may be regularized by analytic continuation in d , however, by dimensional continuation to $d = 4 + \varepsilon_{\text{IR}}$, and corresponding IR poles at negative ε_{UV} . Also the terms proportional to $\ln \frac{-q^2}{m_z^2}$ showing up in the electric form factor (2.213) is not covered by the RG analysis. As will be explained in the next section, the IR singularities have their origin in the attempt to define free charged particle states as simple iso-

lated poles in the spectrum (by trying to impose an on-shell condition). In reality, the Coulomb potential mediated by the massless photon has infinite range and the charged states feel the interaction whatever the spatial separation in corresponding scattering states is.

2.6.6 Bremsstrahlung and the Bloch–Nordsieck Prescription

As we have seen the on-shell form factor A_1 is IR singular in the limit of physical zero mass photons at the one-loop level and beyond. As already mentioned, the problem is that we try to work with scattering states with a fixed number of free particles, while in QED due to the masslessness of the photon and the related infinite interaction range of the electromagnetic forces soft photons are emitted and eventually reabsorbed at any distance from the “interaction region”, i.e. the latter extends to ∞ . The basic problem in this case is the proper definition of a charged particle state as obviously the order by order treatment of a given scattering amplitude breaks down. Fortunately, as Bloch and Nordsieck [59] have observed, a simple prescription bring us back to a quasi perturbative treatment. The basic observation was that virtual and soft real photons are not distinguishable beyond the resolution of the measuring apparatus. Thus besides the virtual photons we have to include the soft real photons of energies below the resolution threshold. For a given tree level process, the Bloch–Nordsieck prescription requires to include photonic corrections at a given order $O(e^n)$ irrespective of whether the photons are virtual or real (soft). We thus are led back to a perturbative order by order scheme, on the expense that, at the given order, all possible final states which only differ by (soft) photons have to be summed over.

Thus in order to obtain a physics-wise meaningful observable quantity, in the case of the electromagnetic form factor

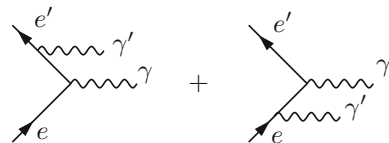
$$e^-(p_1) + \gamma(q) \rightarrow e'-(p_2) ,$$

at one-loop order $O(e^2)$, we have to include the corresponding process

$$e^-(p_1) + \gamma(q) \rightarrow e'-(p_2) + \gamma'(k) ,$$

with one additional real (soft) photon attached in all possible ways to the tree diagram as shown in Fig. 2.11. The second photon is assumed to be soft, i.e. having energy $E_{\gamma'} = |\mathbf{k}| < \omega$, where ω is the threshold of detectability of the real photon. Since the photon cannot be seen, the event looks like an “elastic” event, i.e. like one of

Fig. 2.11 Bremsstrahlung in $e^-(p_1) + \gamma(q) \rightarrow e'-(p_2)$



the same final state as the tree level process. The soft photons thus factorize into the Born term of the original process times a soft photon correction, with the soft photons integrated out up to energy ω . The correction given by the bremsstrahlung cross section is proportional to the square $|T_{\text{bre}}|^2$ of the sum of the matrix elements of the two diagrams which reads

$$T_{\text{bre}} = i^3 e^2 \bar{u}(p_2) \left\{ \gamma^\rho \frac{\not{p}_2 + \not{k} + m}{(p_2 + k)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k} + m}{(p_1 - k)^2 - m^2} \gamma^\rho \right\} u(p_1) \varepsilon_\rho^*(k, \lambda). \quad (2.242)$$

In the soft photon approximation $k \sim 0$ and hence $p_1 + q = p_2 + k \simeq p_2$ we may neglect the \not{k} terms in the numerator. Using the Dirac–algebra and the Dirac equation we may write, in the first term, $\bar{u}(p_2) \not{\varepsilon}^*(\not{p}_2 + m) = \bar{u}(p_2) [2\varepsilon^* p_2 + (-\not{p}_2 + m) \not{\varepsilon}^*] = \bar{u}(p_2) 2\varepsilon^* p_2$, in the second term, $(\not{p}_1 + m) \not{\varepsilon}^* u(p_1) = [2\varepsilon^* p_1 + \not{\varepsilon}^*(-\not{p}_1 + m)] u(p_1) = 2\varepsilon^* p_1 u(p_1)$. Furthermore, in the bremsstrahlung integral the scalar propagators take a very special form, which comes about due to the on–shellness of the electrons and of the bremsstrahlung photon: $(p_2 + k)^2 - m^2 = p_2^2 + 2(kp_2) + k^2 - m^2 = 2(kp_2)$ and $(p_1 - k)^2 - m^2 = p_1^2 - 2(kp_1) + k^2 - m^2 = -2(kp_1)$ as $p_1^2 = p_2^2 = m^2$ and $k^2 = 0$. Therefore, the soft bremsstrahlung matrix element factorizes into the Born term times a radiation factor

$$T_{\text{bre}}^{\text{soft}} \simeq -ie \bar{u}(p_2) \gamma^\mu u(p_1) \left\{ -2e \left(\frac{\varepsilon^* p_1}{kp_1} - \frac{\varepsilon^* p_2}{kp_2} \right) \right\}$$

and one obtains

$$d\sigma = d\sigma_0 \frac{4e^2}{(2\pi)^3} \left| \frac{\varepsilon p_1}{kp_1} - \frac{\varepsilon p_2}{kp_2} \right|^2 \frac{d^3k}{2\omega_k}$$

where $d\sigma_0$ denotes the lowest order cross section for the absorption of a virtual photon by an electron. If we sum over the two photon polarizations λ indexing the polarization vector and use the completeness relation (2.26) we find

$$d\sigma = -d\sigma_0 \frac{4e^2}{(2\pi)^3} \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2 \frac{d^3k}{2\omega_k}. \quad (2.243)$$

Actually, the integral for massless photons does not exist as it is logarithmically IR singular

$$\int_{|\mathbf{k}| < \omega} \frac{d^3k}{|\mathbf{k}|^3} \cdots = \infty.$$

Again an IR regularization is required and we introduce a tiny photon mass such that $\omega_k = \sqrt{\mathbf{k}^2 + m_\gamma^2}$. As a correction to the cross section, we may write the *inclusive cross section* for

$$e^-(p_1) + \gamma(q) \rightarrow e'^-(p_2), \quad e'^-(p_2) + \gamma'(k, \text{soft})$$

as

$$d\sigma_{\text{inc}} = d\sigma_0 (1 + C_{\text{bre}})$$

which, for the vertex on the amplitude level reads

$$i\Gamma_{\text{inc}}^\mu = -ie\gamma^\mu \left(1 + \frac{1}{2}C_{\text{bre}} + \dots \right) \simeq -ie\gamma^\mu + i\delta\Gamma_{\text{bre}}^\mu$$

where

$$i\delta\Gamma_{\text{bre}}^\mu = -ie\gamma^\mu \frac{1}{2}C_{\text{bre}} \quad (2.244)$$

with

$$C_{\text{bre}} = \frac{e^2}{2\pi^3} \int_{|\mathbf{k}| < \omega} \frac{d^3k}{2\omega_k} \left\{ \frac{2(p_1 p_2)}{(k p_1)(k p_2)} - \frac{m^2}{(k p_1)^2} - \frac{m^2}{(k p_2)^2} \right\} \quad (2.245)$$

is the $O(\alpha)$ contribution to the Dirac form factor due to bremsstrahlung. The first term is the interference from the two diagrams, the second and third correspond to the squares of the first and the second diagram, respectively. For a finite photon mass the integral is finite and may be worked out (see e.g. [52] Sect. 7). The result may be written in the form

$$C_{\text{bre}} = \frac{\alpha}{\pi} \left\{ \left(1 - \frac{y}{2} \right) \left(4G'(y) \ln \frac{2\omega}{m_\gamma} - F'(y) \right) - 2 \ln \frac{2\omega}{m_\gamma} + 2G'(y) \right\}$$

with $(\xi = (\sqrt{1-y} - 1)/(\sqrt{1-y} + 1))$ as defined in (2.182))

$$G'(y) = -\frac{1}{4\sqrt{1-y}} \ln(\xi^2)$$

$$F'(y) = \frac{1}{2\sqrt{1-y}} \left\{ \frac{2\pi^2}{3} - 4\text{Sp}(-\xi) + \ln^2(-\xi) - 4 \ln(-\xi) \ln(1 + \xi) \right\}$$

where, for simplicity, F' is given for the production channel

$$\gamma(q) \rightarrow e^-(-p_1) + e'^-(p_2), \quad e^-(-p_1) + e'^-(p_2) + \gamma'(k, \text{soft})$$

where $0 < y < 1$ ($-1 < \xi < 0$). In spite of the fact that the soft bremsstrahlung factor (2.245) looks universal, the result of the evaluation of the integrals is *process dependent*: apart from the universal terms, which in particular include the IR singular ones, the function $F'(y)$ depends on the channel considered. Note that, in contrast to

the form factors, like $A_{E \text{ ren}}$, which are analytic in q^2 , C_{bre} is not analytic in the same variable, because it is the integral over the absolute square $|T|^2$ of a transition matrix element. It must be real and positive. Above, we have chosen to present $F'(y)$ for the production channel as it allows us to discuss the main points of the Bloch–Nordieck prescription, keeping the notation substantially simpler.³⁸ The leading behavior in this case reads

$$C_{\text{bre}} = \frac{\alpha}{\pi} \left\{ 2 \ln \frac{q^2}{m^2} \ln \frac{2\omega}{m_\gamma} - \frac{1}{2} \ln^2 \frac{q^2}{m^2} - 2 \ln \frac{2\omega}{m_\gamma} + \ln \frac{q^2}{m^2} + \dots \right\}.$$

Now, we are able to calculate the form factor for soft photon dressed electrons. The real part of the Dirac form factor gets modified to

$$\begin{aligned} \text{Re } A_{E \text{ ren}} + \frac{1}{2} C_{\text{bre}} &= \frac{\alpha}{2\pi} \left\{ -2 \ln \frac{2\omega}{m} + 4 \left(1 - \frac{y}{2} \right) G'(y) \ln \frac{2\omega}{\sqrt{q^2}} \right. \\ &+ 2 \left(1 - \frac{y}{2} \right) \frac{\pi^2}{2\sqrt{1-y}} - 2 + (5 - 2y) G'(y) + \left. \left(1 - \frac{y}{2} \right) (\text{Re } F - F')(y) \right\} \end{aligned} \quad (2.246)$$

where

$$(\text{Re } F - F')(y) = \frac{1}{2\sqrt{1-y}} \left\{ -\frac{4\pi^2}{3} + 8\text{Sp}(-\xi) + 4 \ln(-\xi) (2 \ln(1 + \xi) - \ln(1 - \xi)) \right\}.$$

³⁸In the scattering region the result is more complicated, because, there is one more kinematic variable, the scattering angle Θ , or equivalently, the electron velocity β_e . Considering, elastic scattering $|\mathbf{p}_1| = |\mathbf{p}_2|$, $E_1 = E_2$ the finite function $F'(y)$, now for $y < 0$ ($0 < \xi < 1$), reads

$$\begin{aligned} F'(y) &= \frac{1}{\sqrt{1-y}} \left\{ -\text{Sp} \left(1 + \frac{2}{1+\xi} \frac{1}{1-\beta_e} \right) - \text{Sp} \left(1 + \frac{2}{1+\xi} \frac{1}{1+\beta_e} \right) \right. \\ &\quad \left. + \text{Sp} \left(1 + \frac{2\xi}{1+\xi} \frac{1}{1-\beta_e} \right) + \text{Sp} \left(1 + \frac{2\xi}{1+\xi} \frac{1}{1+\beta_e} \right) \right\} \end{aligned}$$

where $\beta_e = \sqrt{1 - 4m^2/s}$ is the velocity of the electron. s and $Q^2 = -q^2 > 0$ are related by $Q^2 = s \frac{1 - \cos \Theta}{2}$. The asymptotic behavior $Q^2 \gg m^2$ at fixed angle requires $s \gg m^2$ with $r \equiv Q^2/s = (1 - \cos \Theta)/2$ fixed. The arguments of the Spence functions behave like $1 + \frac{2}{1+\xi} \frac{1}{1-\beta_e} \simeq \frac{s}{m^2} - r^{-1} + \dots$, $1 + \frac{2}{1+\xi} \frac{1}{1+\beta_e} \simeq 2 - \frac{m^2}{Q^2} + \frac{m^2}{s} + \dots$, $1 + \frac{2\xi}{1+\xi} \frac{1}{1-\beta_e} \simeq 1 + r^{-1} - (1 + 3r^{-1}) \frac{m^2}{Q^2} + \dots$, and $1 + \frac{2\xi}{1+\xi} \frac{1}{1+\beta_e} \simeq 1 + \frac{m^2}{Q^2} + \dots$. Utilizing the relations (2.207), one may work out the leading behavior

$$C_{\text{bre}}^{\text{scattering}} = \frac{\alpha}{\pi} \left\{ 2 \ln \frac{Q^2}{m^2} \ln \frac{2\omega}{m_\gamma} - \frac{1}{2} \ln^2 \frac{s}{m^2} - 2 \ln \frac{2\omega}{m_\gamma} + \ln \frac{Q^2}{m^2} + \dots \right\}$$

which, with $\ln^2 s/m^2 = -\ln^2 Q^2/m^2 + 2 \ln Q^2/m^2 \ln s/m^2 + \ln^2 s/Q^2$ and after neglecting the last (sub leading) term, is in agreement with [2]. In the production channel with $q^2 = -Q^2 > 0$, in the center of mass frame of the produced lepton pair, the leptons are back-to-back and hence $\Theta = \pi$, or $\cos \Theta = -1$, such that s may be identified as $s = q^2$.

This is the result for the time-like region (production or annihilation) where $-1 \leq \xi \leq 0$. Here the photon mass has dropped out and we have an IR finite result, at the expense that the form factor is dependent on the experimental resolution ω , the threshold detection energy for soft photons. This is the Bloch–Nordsieck [59] solution of the IR problem. The Pauli form factor is not affected by real photon radiation. In general, as a rule, soft and collinear real photon radiation is always integral part of the radiative corrections.

When combining virtual and soft photon effects one typically observes the cancellations of large or potentially large radiative correction and the range of validity of the perturbative results must be addressed. To be more specific, the calculation has revealed terms of different type and size: typically IR sensitive soft photons logarithms of the type $\ln(m/2\omega)$, or collinear logarithms $\ln(q^2/m^2)$ show up. The latter come from photons traveling in the direction of a lepton, which again cannot be resolved in an experiment with arbitrary precision. This is the reason why the limit $m \rightarrow 0$, in which photon and lepton would travel in the same direction at the same speed (the speed of light) is singular. These logarithms can be very large (high resolution, high energy) and if the corrections $\frac{\alpha}{\pi} \ln(q^2/m^2)$ tend to be of $O(1)$ one cannot trust the perturbative expansion any longer. Even more dangerous are the double logarithmic corrections like the so called *Sudakov logarithms* $\frac{\alpha}{\pi} \ln^2(q^2/m^2)$ or the mixed IR sensitive times collinear terms $\frac{\alpha}{\pi} \ln(m/2\omega) \ln(q^2/m^2)$. There are several possibilities to deal with the large logs:

(a) the leading large terms are known also in higher orders and may thus be resummed. The resummation leads to more reliable results. A typical example here is the soft photon exponentiation according to Yennie–Frautschi–Suura [76].

(b) UV sensitive large logs may be resummed by the renormalization group, as discussed above.

(c) Some observable quantities may have much better convergence properties in a perturbative approach than others. A typical example is the attempt of an *exclusive* measurement of a lepton, which because of the soft photon problematic per se is not a good object to look for. In fact, increasing the exclusivity by choosing the IR cut-off ω smaller and smaller, the correction becomes arbitrary large and the perturbative result becomes meaningless. Somehow the experimental question in such a situation is not well posed. In contrast, by choosing ω larger the correction gets smaller. The possibility to increase ω in the formula given above is kinematically constraint by the requirement of soft radiation *factorization*. Of course photons may be included beyond that approximation. Indeed, there is a famous theorem, the **Kinoshita–Lee–Nauenberg theorem** (KLN) [77] which infers the cancellations of mass singularities and infrared divergences for observables which are defined to include summation over all degenerate or quasi degenerate states:

Theorem 2.11 *After a summation over all possible degenerate states has been performed for the initial (i) and the final (f) states, the squared transition amplitude*

$$\sum_{i,f} |T_{fi}|^2 \quad (2.247)$$

and the corresponding cross section is free of all infrared singularities in the limit of all masses vanishing.

Such observables typically are “all inclusive” cross-sections averaged over the initial spin.

In our example, the inclusive cross section is obtained by adding the hard photons of energy $E_\gamma > \omega$ up to the kinematic limit $E_{\gamma \max} = \sqrt{q^2 - 4m^2}/2$. To illustrate the point, let us consider the lepton pair creation channel $\gamma^*(q) \rightarrow \ell^-(p_-) + \ell^+(p_+) + \gamma(k)$, where the * denote that the corresponding state is virtual, i.e. off-shell, with an additional real bremsstrahlung photon $\gamma(k)$ emitted from one of the final state leptons. We thus include the so called *final state radiation* (FSR). The “heavy” virtual photon γ^* of momentum $q = p_- + p_+ + k$, we may think to have been created previously in e^+e^- -annihilation, for example.³⁹ The center of mass energy is $E_{\text{cm}} = E_- + E_+ + E_\gamma = \sqrt{q^2}$. Let $\lambda = 2\omega/E_{\text{cm}}$ and $1 - \lambda \gg y$ such that we may work in the approximation up to terms of order $O(\alpha \frac{m^2}{q^2})$, i.e., neglecting power corrections in m^2/q^2 . Relaxing from the soft photon approximation which defined C_{bre} in Eq. (2.245), the hard bremsstrahlung integral of interest is

$$\int_{\omega}^{E_{\text{cm}}/2} dE_\gamma \dots$$

with the spectral density (integrand)

$$\begin{aligned} & \frac{1}{\Gamma_0(\gamma^* \rightarrow \ell\bar{\ell})} \frac{d^2\Gamma(\gamma^* \rightarrow \ell\bar{\ell}\gamma)}{dudv} = P(u, v) \\ & = \frac{\alpha}{2\pi} \left\{ \left(2 \frac{u}{1-u} + 1 - u \right) \left(\frac{1}{v} + \frac{1}{1-u-v} \right) \frac{a}{2} \left(\frac{1}{v^2} + \frac{1}{(1-u-v)^2} \right) - 2 \right\}. \end{aligned} \quad (2.248)$$

where $a = 4m^2/q^2$, $u = (p_- + p_+)^2/q^2$ and $v = (q - p_-)^2/q^2$. In the rest frame of the heavy photon we have $u = 1 - 2E_\gamma/M_\gamma$, $v = 1 - 2E_-/M_\gamma$ and $1 - u - v = 1 - 2E_+/M_\gamma$. In the center of mass frame of the lepton pair

$$v = \frac{1}{2} (1 - u) (1 - \sqrt{1 - y} \cos \Theta_+); \quad 1 - u - v = \frac{1}{2} (1 - u) (1 - \sqrt{1 - y} \cos \Theta_-)$$

³⁹The factorization into $e^+e^- \rightarrow \gamma^*$ production and subsequent $e^- \gamma^* \rightarrow \ell^+\ell^-$ only makes sense at relatively low q^2 , when the one-photon exchange approximation can be used. In the SM the γ^* may also be a “heavy light” particle Z of mass about $M_Z \simeq 91 \text{ GeV}$ which is unstable and thus is described well by a Breit-Wigner resonance. Near the resonance energy again factorization is an excellent approximation and the following discussion applies. In e^+e^- -annihilation, the radiation of additional photons from the initial state electron or positron (Fig. 2.11 with e' an incoming e^+) is called *initial state radiation* (ISR). In the soft approximation (2.243) still holds. For details see (5.11) in Sect. 5.1.3.

with $y = a/u$ and Θ_{\pm} the angle between the final state photon and the lepton with momentum p_{\pm} ($\Theta_- = \pi - \Theta_+$). We have to integrate the distribution over the angles $0 \leq \Theta_{\pm} \leq \pi/2$ and over the hard photon $E_{\gamma} \geq \omega = \lambda (M_{\gamma}/2)$ with $1 - a > \lambda > 0$ yields [78] up to $O(\alpha y)$ precision

$$\Delta C_{>\omega} = \frac{\alpha}{2\pi} \left\{ \left(4 \ln \frac{1}{\lambda} - (1 - \lambda)(3 - \lambda) \right) \ln \frac{q^2}{m^2} - 4 \ln \frac{1}{\lambda} \right. \\ \left. + 4 \text{Sp}(\lambda) - \frac{2}{3} \pi^2 - (1 - \lambda)(3 - \lambda) \ln(1 - \lambda) + \frac{1}{2} (1 - \lambda)(11 - 3\lambda) \right\}$$

or for $\omega \ll E_{\text{cm}}/2$

$$\Delta C_{>\omega} = \frac{\alpha}{2\pi} \left\{ \left(4 \ln \frac{\sqrt{q^2}}{2\omega} - 3 \right) \ln \frac{q^2}{m^2} - 4 \ln \frac{\sqrt{q^2}}{2\omega} - \frac{2}{3} \pi^2 + \frac{11}{2} \right\}. \quad (2.249)$$

In this approximation the complementary soft plus virtual part (see (2.246))

$$C_{<\omega} = C_{\text{QED}}^{\text{virtual}} + C_{\omega}^{\text{soft}} \\ = \frac{\alpha}{2\pi} \left\{ - \left(4 \ln \frac{\sqrt{q^2}}{2\omega} - 3 \right) \ln \frac{q^2}{m^2} + 4 \ln \frac{\sqrt{q^2}}{2\omega} + \frac{2}{3} \pi^2 - 4 \right\}. \quad (2.250)$$

The total inclusive sum is

$$C^{\text{total}} = C_{<\omega} + \Delta C_{>\omega} = \frac{\alpha}{2\pi} \frac{3}{2} \simeq 1.74 \times 10^{-3} \quad (2.251)$$

a truly small perturbative correction. No scale and no log involved, just a pure number. This is the KLN theorem at work. It will play a crucial role later on in this book.

The two separate contributions become large when the cut energy ω is chosen very small and in fact we get a negative cross section, which physics wise makes no sense. The reason is that the correction gets large and one has to include other relevant higher order terms. Fortunately, the multi soft γ emission can be calculated to all orders. One can prove [76] that the IR sensitive soft photon exponentiates: Thus,

$$1 + C_{\text{IR}} + \frac{1}{2!} C_{\text{IR}}^2 + \dots = e^{C_{\text{IR}}} \\ = \exp \frac{\alpha}{2\pi} \left\{ -4 \ln \frac{\sqrt{q^2}}{2\omega} \ln \frac{q^2}{m^2} + 4 \ln \frac{\sqrt{q^2}}{2\omega} + \dots \right\} = \left(\frac{2\omega}{\sqrt{q^2}} \right)^{\frac{2\alpha}{\pi} \left(\ln \frac{q^2}{m^2} - 1 \right)}$$

and the result is

$$1 + C_{<\omega} + \dots = e^{C_{\text{IR}}} + \Delta C^{v+s} + \dots \quad (2.252)$$

with

$$\Delta C^{v+s} = C_{<\omega} - C_{\text{IR}} = \frac{\alpha}{2\pi} \left\{ 3 \ln \frac{q^2}{m^2} + \frac{2}{3} \pi^2 - 4 \right\}$$

a correction which is small if q^2/m^2 is not too large. Otherwise higher order collinear logs have to be considered as well. They do not simply exponentiate. By the resummation of the leading IR sensitive terms we have obtained a result which is valid much beyond the order by order perturbative result. Even the limit $\omega \rightarrow 0$ may be taken now, with the correct result that the probability of finding a naked lepton of mass m tends to zero. In contrast $1 + C_{<\omega} \rightarrow -\infty$ as $\omega \rightarrow 0$, a nonsensical result.

For our consideration of soft photon dressed states the inspection of the complementary hard photon part is important as far as the expression (2.249) tells us which are the logs which have to be canceled for getting the log free inclusive result. Namely, the IR sensitive log terms appear with the center of mass energy scale $\sqrt{q^2}$ not with the lepton mass m . This observation allows us to write the virtual plus soft result in a slightly different form than just adding up the results.

Another consideration may be instructive about the collinear mass singularities (terms $\propto \ln(q^2/m^2)$), which are a result of integrating the propagators $2|\mathbf{k}|(E_i - |\mathbf{p}_i| \cos \Theta_i)^{-1}$ in the distribution (2.243) or (2.248). If we integrate the angular distribution over a cone $\Theta_1, \Theta_2 \leq \delta$ only, instead of over the full angular range and add up the contributions

$$C_{<\omega, <\delta} = C_{\text{QED}}^{\text{virtual}} + C_{\omega}^{\text{soft}} + \Delta C_{>\omega, <\delta}^{\text{hard, collinear}} \quad (2.253)$$

the collinear singularities exactly cancel in the limit $m \rightarrow 0$, provided $\delta > 0$. The result reads

$$C_{<\omega, <\delta}^{m=0} = \frac{\alpha}{2\pi} \left\{ \left(4 \ln \frac{1}{\lambda} - (1-\lambda)(3-\lambda) \right) \ln \frac{1-\rho}{1+\rho} \frac{3}{2} + \rho(1-\lambda^2) \right\}$$

with $\rho = \cos \delta$, $\lambda = \frac{2\omega}{M_\gamma}$ and we have assumed $\frac{1-\rho}{2} \gg \frac{m^2}{M_\gamma^2}$. Thus, in addition to the virtual plus soft photons we have included now the hard collinear photons traveling with the leptons within a cone of opening angle δ . Here the collinear cone has been defined in the c.m. frame of the lepton pair, where the two cones are directed back to back and non overlapping for arbitrary cuts $\delta \leq \pi/2$. In an experiment one would rather define the collinear cones in the c.m. frame of the incoming virtual photon. In this case a slightly more complicated formula Eq. 14 of Ref. [78] is valid, which simplifies for small angles δ_0 and $\lambda = 2\epsilon = 2\omega/M_\gamma \ll 1$ to

$$C_{<\omega, <\delta_0}^{m=0} = -\frac{\alpha}{\pi} \left\{ (4 \ln 2\epsilon + 3) \ln \frac{\delta_0}{2} + \frac{\pi^2}{3} - \frac{5}{2} \right\} \quad (2.254)$$

which is the QED analog of the famous Serman-Weinberg (SW) formula [79]

$$C_{\text{SW}} = -\frac{4}{3} \frac{\alpha_s}{\pi} \left\{ (4 \ln 2\epsilon + 3) \ln \frac{\delta_0}{2} + \frac{\pi^2}{3} - \frac{5}{2} \right\} \quad (2.255)$$

for the two-jet event rate in QCD. The extra factor $\frac{4}{3}$ is an $SU(3)$ Casimir coefficient and α_s is the $SU(3)$ strong interaction coupling constant. The physical interpretation of this formula will be considered in Sect. 5.1.5.

Some final remarks are in order here: the IR problem of QED is a nice example of how the “theory reacts” if one is not asking the right physical questions. The degeneracy in the energy spectrum which manifests itself in particular kinematic regions (soft and/or collinear photons), at first leads to ill-defined results in a naive scattering picture approach, misleadingly assuming forces to be of finite range. At the end one learns that in QED the S -matrix as defined by the Gell-Mann Low formula does not exist, because the physical state spectrum is modified by the dynamics and is not the one suggested by the free part of the Lagrangian. Fortunately, a perturbative calculation of cross sections is still possible, by modifications of the naive approach by accounting appropriately for the possible degeneracy of states.

As we have observed in the above discussion, the radiatively induced Pauli form factor is not affected by the IR problem. The Pauli form factor is an example of a so called *infrared safe* quantity, which does not suffer from IR singularities in the naive scattering picture approach. As the anomalous magnetic moment is measured with extremely high accuracy, it nevertheless looks pretty much like a miracle how it is possible to calculate the anomalous magnetic moment in the naive approach to high orders (five loops at the moment) and confront it with an experimental result which is also measured assuming such a picture to be valid. But the states with which one formally is operating do not exist in nature. For a careful investigation of the problem we refer to the article by Steinmann [80].

We have discussed the IR problem for the simplest case, the electromagnetic form factor. In general the problem is more complicated, but the Bloch–Nordsieck prescription works and provides an order by order rule to overcome infrared singularities. The principle behind the “Bloch–Nordsieck solution” is the focus on “truly observable quantities”, which take into account detection problems in the measuring process, when state degeneracies in phase space come into play. One should ask the right questions in order to get useful and testable answers. In contrast, over-idealized formal quantities may be plagued by singularities. Let me sketch the procedure for the simplest case of a two-to-two fermion reaction, the process $e^+e^- \rightarrow \mu^+\mu^-$, which exhibits the radiative corrections depicted in Fig. 2.12. The amplitudes are considered to be the renormalized on-shell ones, which exist only after IR regularization. Since off-shell amplitudes are IR finite, the off-shellness $\mu_{\text{IR}}^2 = m^2 - p^2$, p the four-momentum of an external particle of mass m , can be used as an IR regulator, in principle. Actually such a regularization may be the most physical choice.⁴⁰

⁴⁰The limits $m_\gamma^2 \rightarrow 0$ for $p^2 = m^2$ and $\mu^2 = m^2 - p^2 \rightarrow 0$ for $m_\gamma^2 = 0$ coincide upon identifying $\mu^2 = m m_\gamma$ at least in one-loop calculations.

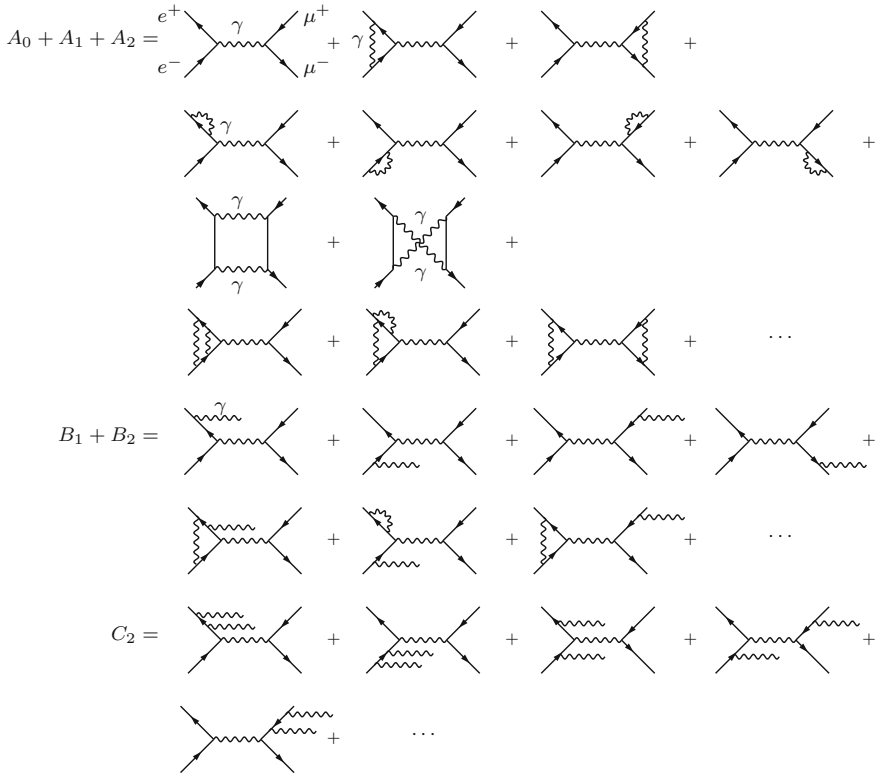


Fig. 2.12 Diagrams for muon pair production in electron-positron annihilation at lowest order $O(\alpha)$ (LO), next-to-leading order $O(\alpha^2)$ (NLO) and next-to-next-to-leading order $O(\alpha^3)$ (NNLO) together with the relevant real photon corrections. For the $O(\alpha^3)$ case only a sample of typical diagrams is shown. Corresponding amplitudes are denoted by A_0 at LO, A_1 , B_1 at NLO and A_2 , B_2 , C_2 at NNLO, where A_i , B_i and C_i ($i = 0, 1, 2$) refer to zero, one and two emitted real photons, respectively

Practical aspects usually let it look easier to use a small photon mass as a regulator or to apply dimensional regularization by going to $d = 4 + \varepsilon$ dimensions with an associated scale parameter μ_{IR} , which should be distinguished from the $\overline{\text{MS}}$ scale parameter μ , standing for μ_{UV} , related to the $d = 4 - \varepsilon$ expansion.

A_0 denotes the tree level amplitude, A_1 is the corresponding 1-loop virtual photon correction, which is IR singular unless we apply a soft photon infrared cutoff μ_{IR} , which may be chosen to be a tiny photon mass. Tiny means smaller than any other relevant physics scale, like the electron mass, for example. The crucial point is that the μ_{IR} -dependent IR sensitive part of the IR regularized amplitude is proportional to A_0 : $A_1 = A_0 \delta_v^{(1)}(\mu_{\text{IR}}) + \delta^{\text{fin}} A_1$, i.e. the IR dangerous part is *factorizable*, while the non-factorizable part is finite, independent of the IR regulator. The radiative amplitude splits into $B_1 = B_1^{\text{soft}}(E_\gamma < \omega) + \delta B_1^{\text{hard}}(E_\gamma \geq \omega)$ corresponding to the

$\mu\mu(\gamma)$ and $\mu\mu\gamma$ final states. The soft part includes the unresolved hidden photon $[(\gamma)]$ part, from photons too soft to be detectable, which indeed look like $\mu\mu$ “elastic event”,⁴¹ and hence is factorizable $B_1^{\text{soft}} = A_0 \rho^{\text{soft}}(\mathbf{k})$ with a soft photon radiation density $\rho^{\text{soft}}(\mathbf{k})$ (\mathbf{k} the photon momentum), which has to be integrated over and yields the bremsstrahlung correction $C_{\text{bre}}^{(1)}(\mu_{\text{IR}}, \omega)$, where $\mu_{\text{IR}} \ll \omega \ll E_{\gamma\text{max}}$ (see (2.245) for the initial state part). The soft photon integral again only exists after IR regularization, by a, relative to the virtual part, commensurate cutoff μ_{IR} . Again a tiny photon mass provides such a cutoff. The soft photon integral should include the soft photons of energy $E_\gamma < \omega \ll E_{\gamma\text{max}}$, where ω has to be chosen such that the factorization is within the numerical accuracy of the attempted calculation, ideally it can be identified with the photon detection threshold of the detector utilized to measure the cross section of the process.

In order to get the NLO correction, we have to evaluate

$$|A_0 + A_1|^2 = |A_0|^2 + A_0 A_1^* + A_1 A_0^* + \dots \simeq |A_0|^2 \cdot \left(1 + 2\text{Re} \delta_v^{(1)}(\mu_{\text{IR}})\right) + A_0 \delta A_1^* + \delta A_1 A_0^* .$$

The omitted higher order terms are to be included in the NNLO correction. These also exhibit further IR sensitive contributions, which will cancel against other NNLO IR sensitive terms. Altogether, we then get the physical “soft photon dressed” Born transition probability amplitude at NLO

$$|A_0|^2{}^{\text{dressed}} = |A_0|^2 \cdot \left(1 + 2\text{Re} \delta_v^{(1)}(\mu_{\text{IR}}) + C_{\text{bre}}^{(1)}(\mu_{\text{IR}}, \omega)\right) ,$$

in which the IR cutoff μ_{IR} cancels and the result depends on ω only. The ω dependence disappears if we include the hard photon part from $|B_1(E_\gamma \geq \omega)|^2$ as well. The total inclusive cross section, which includes non-factorizable terms as well as hard photon contributions, is a sum of a 2 to 2 and a 2 to 3 cross section, the explicit form of which is beyond the scope of this discussion (see, however, the corresponding results for the process $e^+e^- \rightarrow \pi^+\pi^-$ presented in Chap. 5, Sect. 5.1.3).

At NNLO, including the 2-loop correction A_2 the procedure follows the same line. We have to collect all $\mathcal{O}(\alpha^2)$ contributions by including real photon radiation up to two photons now. Starting from the IR regularized amplitude $A = A_0 + A_1 + A_2$, we have to include the 1-loop virtual correction to $B = B_1 + B_2$ as well as the double real photon contribution $e^+e^- \rightarrow \mu^+\mu^-\gamma\gamma$ from “undetectable” soft photons $C = C_2$. Beyond $\mu\mu$ factorizable soft photon effects in addition $\mu\mu\gamma$ factorizable soft photon effects come into play etc.

The ω -dependent virtual plus soft photon corrected Born cross section, gets negative if ω is taken too small and the order-by-order treatment breaks down as the correction blows up. The limit $\omega \rightarrow 0$ only can be taken after infinite resummation of the leading soft photon effects, the $\omega \rightarrow 0$ limit is then vanishing. The probability to find just two naked muons in the final state is zero as a charged particle constantly radiates soft photons. Here another point comes into consideration: we never measure perturbative quantities, and whether a perturbatively calculated quantity

⁴¹i.e. particle number conserving, when looked at from the crossed t -channel $e^-\mu^+ \rightarrow e^-\mu^+$.

approximates a measured object well depends on the experimental conditions. For a certain range of ω 's the prediction may well be perfect within experimental precision, while going to smaller ω 's convergence of the perturbative series breaks down. Quantities sharing such behavior are called *infrared unsafe*. Good observables preferably should be defined such that they are *infrared safe* and *collinear safe*. Infrared safe means that the quantity, in order to be a true observable, should not change discontinuously if one adds a soft particle to the final state. Similarly, collinear safe means that the quantity should not change abruptly if one splits one final state particle into two particles with equal momentum. A more concrete account will be presented in Chap. 5, Sect. 5.1.3 for the process $e^+e^- \rightarrow \pi^+\pi^-$, which plays an important role in evaluation the leading hadronic contribution to a_μ .

One more aspect has to be mentioned here: what is “detectable” is device dependent and therefore not what is of primary theoretical interest. It is therefore common practice to unfold experimental data from radiative effects. In our case, this amounts to “undress” the physical cross section, by comparison with the theoretical prediction, in order to recover the “bare” cross section as the quantity of interest. This in any case allows to extract the relevant parameters like couplings and masses which enter the undressed cross section. In many cases undressing attempts to separate for example strong interaction effects from electromagnetic ones.

Concluding remarks: we note that the problem with the non-existence of electrically charged one-particle states imply that the S -matrix in the naive LSZ sense in QED does not exist. In perturbation theory the Bloch–Nordsieck approximation and its Yennie–Frautschi–Suura improvement provides an acceptable perturbatively improvable framework for making well defined predictions which can be confronted with experimental data of a given precision. Interestingly, such infrared type problem is absent for charged particles in atoms or molecules, because radiation in bound systems is subject to quantum mechanics with a discrete spectrum and soft or collinear degeneracies of states are not an issue. For what concerns the non-Abelian part of the SM: the physical state space exhibits no other massless particle besides the photon. The weak $SU(2)$ gauge bosons get masses via the Higgs mechanism and are actually very heavy and therefore very unstable such that they never can show up as true LSZ Wigner states, because the track they leave in a real world detector is by far too short to ever be resolved as a particle track. Nevertheless the neutral Z boson shows up as a very pronounced resonance as $\Gamma_Z/M_Z \simeq 0.0274$, such that its quasi on-shell properties can be investigated very precisely, and as performed at the LEP ring at CERN at the beginning of the 90ies. This information, however is only accessible via the decay products which are seen in the detector. Similarly for the W boson, except that the W as a charged state in addition exhibits the same types of problems as the charged leptons in QED. In principle one could integrate out the W and the Z fields, which however would result in a very complicated non-local effective Lagrangian. Certainly one better sticks to the standard SM approach, treating the weak gauge bosons as quasi LSZ states, in a production and decay chain, which can be implemented order by order in perturbation theory. The strong interaction sector solves its problems with the massless gluons in its own way: by confinement. All fields in the QCD Lagrangian have no asymptotic states themselves but form color

singlet hadrons of short ranged effective interactions. For the hadrons a S -matrix is well defined, besides the issues connected with unstable particles (often due to weak and electromagnetic decays) and the problem of the electrically charged states as elaborated above.

2.7 Pions in Scalar QED and Vacuum Polarization by Vector Mesons

The strong interaction effects in $(g - 2)$ are dominated by the lightest hadrons, the isospin $SU(2)$ triplet (π^+, π^0, π^-) of pions, pseudoscalar spin 0 mesons of masses: $m_{\pi^\pm} = 139.75018(35)$ MeV, $m_{\pi^0} = 134.9766(6)$ MeV. Pions are quark-antiquark color singlet bound states $(u\bar{d}, \frac{1}{\sqrt{2}}[u\bar{u} - d\bar{d}], d\bar{u})$ and their electromagnetic interaction proceeds via the charged quarks. This is particularly pronounced in the case of the neutral π^0 which decays electromagnetically via $\pi^0 \rightarrow \gamma\gamma$ and has a much shorter life time $\tau_{\pi^0} = 8.4(6) \times 10^{-17}$ s than the charged partners which can decay by weak interaction only according to $\pi^\pm \rightarrow \mu^\pm \nu_\mu$ and hence live longer by almost 10 orders of magnitude $\tau_{\pi^\pm} = 2.6033(5) \times 10^{-8}$ s. However, at low energies, in many respects the pions behave like point particles especially what concerns soft photon emission and the Bloch-Nordsieck prescription. The effective Lagrangian for the electromagnetic interaction of a charged point-like pion described by a complex scalar field φ follows from the free Lagrangian

$$\mathcal{L}_\pi^{(0)} = (\partial_\mu \varphi)(\partial^\mu \varphi)^* - m_\pi^2 \varphi \varphi^*$$

via *minimal substitution* $\partial_\mu \varphi \rightarrow D_\mu \varphi = (\partial_\mu + ieA_\mu(x)) \varphi$ (also called *covariant derivative*), which implies the scalar QED (sQED) Lagrangian

$$\mathcal{L}_\pi^{\text{sQED}} = \mathcal{L}_\pi^{(0)} - ie(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) A^\mu + e^2 g_{\mu\nu} \varphi \varphi^* A^\mu A^\nu . \quad (2.256)$$

Thus gauge invariance implies that the pions must couple via two different vertices to the electromagnetic field, and the corresponding Feynman rules are given in Fig. 2.13.

- (1) Pion propagator

$$\text{O} \cdots \overset{p}{\cdots} \cdots \text{O} \quad : \quad i\Delta_\pi(p) = \frac{i}{p^2 - m_\pi^2 + i\epsilon}$$

- (2) Pion-photon vertices

$$\begin{array}{cc}
 \begin{array}{c} \text{A}^\mu \\ \text{wavy} \\ \text{p} \end{array} \begin{array}{c} \nearrow \varphi^+ \\ \searrow \varphi^- \end{array} & : = -ie(p + p')^\mu , &
 \begin{array}{c} \text{A}^\mu \\ \text{wavy} \\ \text{A}^\nu \end{array} \begin{array}{c} \nearrow \varphi^+ \\ \searrow \varphi^- \end{array} & : = 2ie^2 g^{\mu\nu}
 \end{array}$$

Fig. 2.13 Feynman rules for sQED. p is incoming, p' outgoing

The bound state nature of the charged pion is taken care off by introducing a pion form factor $e \rightarrow e F_\pi(q^2)$, $e^2 \rightarrow e^2 |F_\pi(q^2)|^2$.

In sQED the contribution of a pion loop to the photon VP is given by

$$-i \Pi_\gamma^{\mu\nu}(\pi)(q) = \text{diagram 1} + \text{diagram 2}$$

The bare result for the transversal part defined by (2.160) reads

$$\Pi_\gamma^{(\pi)}(q^2) = \frac{e^2}{48\pi^2} \left\{ B_0(m, m; q^2) (q^2 - 4m^2) - 4 A_0(m) - 4m^2 + \frac{2}{3}q^2 \right\} \quad (2.257)$$

with $\Pi_\gamma(0) = 0$. We again calculate the renormalized transversal self-energy $\Pi'_\gamma(q^2) = \Pi_\gamma(q^2)/q^2$ which is given by $\Pi'_{\gamma\text{ren}}(q^2) = \Pi'_\gamma(q^2) - \Pi'_\gamma(0)$. The subtraction term

$$\Pi'_{\gamma}(\pi)(0) = \frac{-e^2}{48\pi^2} \left\{ \frac{A_0(m)}{m^2} + 1 \right\}$$

is the π^\pm contribution to the photon wavefunction renormalization and the renormalized transversal photon self-energy reads

$$\Pi'_{\gamma\text{ren}}(\pi)(q^2) = \frac{\alpha}{6\pi} \left\{ \frac{1}{3} + (1-y) - (1-y)^2 G(y) \right\} \quad (2.258)$$

where $y = 4m^2/q^2$ and $G(y)$ given by (2.174). For $q^2 > 4m^2$ there is an imaginary or absorptive part given by substituting

$$G(y) \rightarrow \text{Im } G(y) = -\frac{\pi}{2\sqrt{1-y}}$$

according to (2.175)

$$\text{Im } \Pi'_{\gamma}(\pi)(q^2) = \frac{\alpha}{12} (1-y)^{3/2} \quad (2.259)$$

and for large q^2 is 1/4 of the corresponding value for a lepton (2.179). According to the *optical theorem* the absorptive part may be written in terms of the $e^+e^- \rightarrow \gamma^* \rightarrow \pi^+\pi^-$ production cross section $\sigma_{\pi^+\pi^-}(s)$ as

$$\text{Im } \Pi'_{\gamma}{}^{\text{had}}(s) = \frac{s}{4\pi\alpha} \sigma_{\text{had}}(s) \quad (2.260)$$

which hence we can read off to be

$$\sigma_{\pi^+\pi^-}(s) = \frac{\pi\alpha^2}{3s} \beta_\pi^3 \quad (2.261)$$

with $\beta_\pi = \sqrt{(1 - 4m_\pi^2/s)}$ the pion velocity in the CM frame. Often, one writes hadronic cross sections as a ratio

$$R(s) \doteq \sigma_{\text{had}}(s) / \frac{4\pi\alpha^2}{3s} \quad (2.262)$$

in units of the high energy asymptotic form of the cross section $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-)$ for muon pair production in e^+e^- -annihilation. Given the cross section or imaginary part, conversely, the real part of the renormalized vacuum polarization function may be obtained by integrating the appropriate dispersion relation (see Sect. 3.7), which reads

$$\text{Re } \Pi'_{\gamma\text{ren}}{}^{\text{had}}(s) = \frac{s}{4\pi^2\alpha} \oint_{s_1}^{s_2} ds' \frac{\sigma_{\text{had}}(s')}{s' - s} = \frac{\alpha}{3\pi} \oint_{s_1}^{s_2} ds' \left\{ \frac{1}{s' - s} - \frac{1}{s'} \right\} R(s'). \quad (2.263)$$

This is another way, the *dispersive approach*, to get the result (2.258) via the easier to calculate imaginary part, which here is just given by the tree level cross section for $\gamma^* \rightarrow \pi^+\pi^-$.

As already mentioned, sometimes one has to resort to sQED in particular in connection with the soft photon radiation problem of charged particles, where sQED provides a good description of the problem. However, the photon vacuum polarization due to an elementary charged spin 0 pion, we just have been calculating, includes hard photons in the region of interest above the $\pi^+\pi^-$ production threshold to about 1 GeV, say. As we will see sQED in this case gives a rather bad approximation. In reality $e^+e^- \rightarrow \gamma^* \rightarrow \pi^+\pi^-$ is non-perturbative and exhibits a pronounced resonance, the neutral spin 1 meson ρ^0 , and the hadron production cross section is much better parametrized by a Breit-Wigner (BW) resonance shape. The relevant parameters are M_R the mass, Γ the width and $\Gamma_{e^+e^-}/\Gamma$ the branching fraction for $\rho \rightarrow e^+e^-$. We briefly present the different possible parametrizations and how a BW resonance contributes to the renormalized photon vacuum polarization when integrated over a range (s_1, s_2) with $4m_\pi^2 \leq s_1 < s_2 \leq \infty$ [81]:

- **Narrow width resonance**

The contribution from a zero width resonance

$$\sigma_{\text{NW}}(s) = \frac{12\pi^2}{M_R} \Gamma_{e^+e^-} \delta(s - M_R^2) \quad (2.264)$$

is given by

$$\Pi'_{\gamma\text{ren}}{}^{\text{NW}}(s) = \frac{-3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s}{s - M_R^2} \quad (2.265)$$

which in the limit $|s| \gg M_R^2$ becomes

$$\Pi'_{\gamma\text{ren}}{}^{\text{NW}}(s) \simeq \frac{-3\Gamma_{e^+e^-}}{\alpha M_R} . \quad (2.266)$$

• Breit–Wigner resonance

The contribution from a classical Breit–Wigner resonance

$$\sigma_{BW}(s) = \frac{3\pi}{s} \frac{\Gamma \Gamma_{e^+e^-}}{(\sqrt{s} - M_R)^2 + \frac{\Gamma^2}{4}} \quad (2.267)$$

is given by

$$\Pi'_{\gamma\text{ren}}{}^{\text{BW}}(s) = \frac{-3\Gamma \Gamma_{e^+e^-}}{4\pi\alpha} \{I(0) - I(W)\} \quad (2.268)$$

where

$$I(W) = \frac{1}{2ic} \left\{ \frac{1}{W - M_R - ic} \left(\ln \frac{W_2 - W}{W_1 - W} - \ln \frac{W_2 - M_R - ic}{W_1 - M_R - ic} \right) - \frac{1}{W + M_R + ic} \left(\ln \frac{W_2 + W}{W_1 + W} - \ln \frac{W_2 - M_R - ic}{W_1 - M_R - ic} \right) - \text{h.c.} \right\}$$

with $c = \Gamma/2$. For $W_1 \ll M_R \ll W_2$ and $\Gamma \ll M_R$ this may be approximated by

$$\Pi'_{\gamma\text{ren}}{}^{\text{BW}}(s) \simeq \frac{-3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s(s - M_R^2 + 3c^2)}{(s - M_R^2 + c^2)^2 + M_R^2 \Gamma^2} \quad (2.269)$$

which agrees with (2.265) and (2.266) in the limits $\Gamma^2 \ll |s - M_R^2|$, M_R^2 and $|s| \gg M_R^2$, respectively.

• Breit–Wigner resonance: field theory version

Finally, we consider a field theoretic form of a Breit–Wigner resonance obtained by the Dyson summation of a massive spin 1 transversal part of the propagator in the approximation that the imaginary part of the self–energy yields the width by $\text{Im}\Pi_V(M_V^2) = M_V \Gamma_V$ near resonance.

$$\sigma_{BW}(s) = \frac{12\pi}{M_R^2} \frac{\Gamma_{e^+e^-}}{\Gamma} \frac{s\Gamma^2}{(s - M_R^2)^2 + M_R^2 \Gamma^2} \quad (2.270)$$

which yields

$$\begin{aligned} \Pi'_{\gamma\text{ren}}{}^{\text{BW}}(s) = & \frac{-3\Gamma_{e^+e^-}}{\pi\alpha M_R} \frac{s(s - M_R^2 - \Gamma^2)}{(s - M_R^2)^2 + M_R^2\Gamma^2} \left\{ \left(\pi - \arctan \frac{\Gamma M_R}{s_2 - M_R^2} \right. \right. \\ & \left. \left. - \arctan \frac{\Gamma M_R}{M_R^2 - s_1} \right) - \frac{\Gamma}{M_R} \frac{s}{(s - M_R^2 - \Gamma^2)} \left(\ln \left| \frac{s_2 - s}{s_1 - s} \right| - \ln \left| \frac{s_2 - M_R^2 - iM_R\Gamma}{s_1 - M_R^2 - iM_R\Gamma} \right| \right) \right\} \end{aligned} \quad (2.271)$$

and reduces to

$$\Pi'_{\gamma\text{ren}}{}^{\text{BW}}(s) \simeq \frac{-3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s(s - M_R^2 - \Gamma^2)}{(s - M_R^2)^2 + M_R^2\Gamma^2} \quad (2.272)$$

for $s_1 \ll M_R^2 \ll s_2$ and $\Gamma \ll M_R$. Again we have the known limits for small Γ and for large $|s|$.

For broad resonances the different parametrizations of the resonance in general yield very different results. Therefore, it is important to know how a resonance was parametrized to get the resonance parameters like M_R and Γ . For narrow resonances, which we will have to deal with later, results are not affected in a relevant way by using different parametrizations. A finite width BW resonance is related to the NW resonance via identity

$$\delta(s - M^2) = \frac{1}{\pi} \lim_{\gamma \rightarrow 0} \frac{\gamma}{(s - M^2)^2 + \gamma^2} \quad (2.273)$$

with $\gamma = \Gamma M$. Note that for the broad non-relativistic ρ meson only the classical BW parametrization works. In fact, due to isospin breaking of the strong interactions ($m_d - m_u$ mass difference as well as electromagnetic effects $Q_u = 2/3 \neq Q_d = -1/3$) the ρ and ω mix and more sophisticated parametrizations must be applied, like the Gounaris-Sakurai (GS) parametrization [82] based on the vector meson dominance (VMD) model (see Sect. 5.2). Actually, the GS model is missing to take into account $\rho^0 - \gamma$ mixing and it is not electromagnetically gauge invariant and therefore should be replaced by a manifestly gauge invariant VMD (so called type II) plus sQED Lagrangian approach [83]. For the strong interaction part (undressed from electromagnetic effects) most appropriate is a parametrization which relies on first principle concepts only, the description by unitarity, analyticity and constrained by chiral perturbation theory (CHPT), which is the low energy effective form of QCD (see [84] and references therein).

We will use the results presented here later for the evaluation of the contributions to $g-2$ from hadron-resonances. In e^+e^- -annihilation a large number of resonances, like ρ , ω , ϕ , J/ψ series and the Υ series, show up and will have to be taken into account.

2.8 Note on QCD: The Feynman Rules and the Renormalization Group

Quantum Chromodynamics, the modern theory of the strong interactions, is a non-Abelian gauge theory with gauge group $SU(3)_c$ consisting of unitary 3×3 matrices of determinant unity. The corresponding internal degrees of freedom are called *color*. The generators are given by the basis of Hermitian traceless 3×3 matrices T_i , $i = 1, \dots, 8$. Quarks transform under the fundamental 3-dimensional representation 3 (quark triplets) antiquarks under the complex conjugate 3^* (antiquark anti-triplets). The requirement of local gauge invariance with respect to $SU(3)_c$ transformations implies that quark fields $\psi_i(x)$ must couple to an octet of gauge fields, the gluon fields $G_{\mu j}$, $j = 1, \dots, 8$, and together with the requirement of renormalizability this fixes the form of the interactions of the quarks completely: in the free quark Dirac-Lagrangian we have to replace the derivative by the *covariant derivative*

$$\partial_\mu \psi(x) \rightarrow D_\mu \psi(x), \quad (D_\mu)_{ik} = \partial_\mu \delta_{ik} - i g_s \sum_j (T_j)_{ik} G_{\mu j}(x) \quad (2.274)$$

where g_s is the $SU(3)_c$ gauge coupling constant. The dynamics of the gluon fields is controlled by the non-Abelian *field strength tensor*

$$G_{\mu\nu i} = \partial_\mu G_{\nu i} - \partial_\nu G_{\mu i} + g_s c_{ijk} G_{\mu j} G_{\nu k} \quad (2.275)$$

where c_{ijk} are the $SU(3)$ structure constants obtained from the commutator of the generators $[T_i, T_j] = i c_{ijk} T_k$. The locally gauge invariant Lagrangian density is then given by

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} \sum_i G_{i\mu\nu} G_i^{\mu\nu} + \bar{\psi} (i \gamma^\mu D_\mu - m) \psi. \quad (2.276)$$

We split \mathcal{L}_{inv} into a free part \mathcal{L}_0 and an interaction part \mathcal{L}_{int} which is taken into account as a formal power series expansion in the gauge coupling g_s . The perturbation expansion is an expansion in terms of the free fields described by \mathcal{L}_0 . The basic problem of quantizing massless spin 1 fields is familiar from QED. Since \mathcal{L}_{YM} is gauge invariant, the gauge potentials $G_{i\mu}$ cannot be uniquely determined from the gauge invariant field equations. Again one has to break the gauge invariance, now, for a $SU(n)$ gauge group, by a sum of $r = n^2 - 1$ gauge fixing conditions

$$C_i(G) = 0, \quad i = 1, \dots, r.$$

It is known from QED that the only relativistically invariant condition linear in the gauge potential which we can write is the Lorentz condition. Correspondingly we require

$$C_i(G) = -\partial_\mu G_i^\mu(x) = 0, \quad i = 1, \dots, r. \quad (2.277)$$

It should be stressed that a covariant formulation is mandatory for calculations beyond the tree level. We are thus lead to break the gauge invariance of the Lagrangian by adding the **gauge fixing** term

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} \sum_i (\partial_\mu G_i^\mu(x))^2 \quad (2.278)$$

with ξ a free gauge parameter. Together with the term \mathcal{L}_0^G from \mathcal{L}_{inv} we obtain for the bilinear gauge field part

$$\mathcal{L}_{0,i}^{G,\xi} = -\frac{1}{4} (\partial_\mu G_{i\nu} - \partial_\nu G_{i\mu})^2 - \frac{1}{2\xi} (\partial_\mu G_i^\mu(x))^2 \quad (2.279)$$

which now uniquely determines a free gauge field propagator. Unlike in QED, however, \mathcal{L}_{GF} breaks local gauge invariance explicitly and one has to restore gauge invariance by a compensating **Faddeev-Popov term** (Faddeev and Popov 1967). The Faddeev-Popov trick consists in adding further charged ghost fields $\bar{\eta}_i(x)$ and $\eta_i(x)$, the so called **Faddeev-Popov ghosts**, which conspire with the other ghosts in such a way that physical matrix elements remain gauge invariant. Unitarity and renormalizability are then restored. The FP-ghosts must be **massless spin 0 fermions**. For the unphysical ghosts this wrong spin-statistics assignment is no obstacle. The Faddeev-Popov term must be of the form

$$\mathcal{L}_{\text{FP}} = \bar{\eta}_i(x) M_{ik} \eta_k(x)$$

where

$$\begin{aligned} M_{ik} &= \frac{\partial C_i(G)}{\partial G_{j\mu}(x)} (D_\mu)_{jk} = -\partial^\mu (\partial_\mu \delta_{ik} - g c_{ikj} G_{j\mu}(x)) \\ &= -\square \delta_{ik} + g c_{ikj} G_{j\mu}(x) \partial^\mu + g c_{ikj} (\partial^\mu G_{j\mu}(x)) \quad . \end{aligned}$$

By partial integration of $S_{FP} = \int d^4x \mathcal{L}_{\text{FP}}(x)$ we may write

$$\mathcal{L}_{\text{FP}} = \partial_\mu \bar{\eta}_i \partial^\mu \eta_i - g c_{ikj} (\partial^\mu \bar{\eta}_i) C_{j\mu} \eta_k \quad (2.280)$$

which describes massless scalar fermions in interaction with the gauge fields. The complete Lagrangian for a quantized Yang-Mills theory is

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} \quad . \quad (2.281)$$

The free (bilinear) part

$$\mathcal{L}_0 = \mathcal{L}_0(G) + \mathcal{L}_0(\psi) + \mathcal{L}_0(\eta)$$

with

$$\begin{aligned}\mathcal{L}_0(G) &= \frac{1}{2} G_{i\mu} \left[\left(\square g^{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right) \delta_{ik} \right] G_{k\nu} \\ \mathcal{L}_0(\psi) &= \bar{\psi}_{\alpha a} \left[\left((i\gamma^\mu)_{\alpha\beta} \partial_\mu - m \delta_{\alpha\beta} \right) \delta_{ab} \right] \psi_{\beta b} \\ \mathcal{L}_0(\eta) &= \bar{\eta}_i \left[(-\square) \delta_{ik} \right] \eta_k\end{aligned}$$

determines the free propagators, the differential operators in the square brackets being the inverses of the propagators. By Fourier transformation the free propagators are obtained in algebraic form (i.e. the differential operators are represented by c–numbers) in momentum space. Inverting these c–number matrices we obtain the results depicted in Fig. 2.14.

The interaction part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{int}} &= g_s \bar{\psi} \gamma^\mu T_i \psi G_{i\mu} - \frac{1}{2} g_s c_{ikl} \left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu \right) G_{k\mu} G_{l\nu} \\ &\quad - \frac{1}{4} g_s^2 c_{ikl} c_{ik'l'} G_k^\mu G_l^\nu G_{k'\mu} G_{l'\nu} - g_s c_{ikj} \left(\partial^\mu \bar{\eta}_i \right) G_{j\mu} \eta_k\end{aligned}\quad (2.282)$$

with a single coupling constant g_s for the four different types of vertices.

While the formal argumentation which leads to the construction of local gauge theories looks not too different for Abelian and non–Abelian gauge groups, the physical consequences are very different and could not be more dramatic: in contrast to Abelian theories where the gauge field is neutral and exhibits no self–interaction, non–Abelian gauge fields necessarily carry non–Abelian charge and must be self–interacting. These Yang–Mills self–interactions are responsible for the *anti–screening* of the non–Abelian charge, known as *asymptotic freedom* (AF) (see end of section). It implies that the strong interaction force gets weaker the higher the energy, or equivalently, the shorter the distance. While it appears most natural to us that particles interact the less the farther apart they are, non–Abelian forces share the opposite property, the forces get the stronger the farther away we try to separate the quarks. In QCD this leads to the *confinement* of the constituents within hadrons. The latter being quark bound states which can never be broken up into free constituents. This makes QCD an intrinsically non–perturbative theory, the fields in the Lagrangian, quarks and gluons, never appear in scattering states, which define the physical state space and the S –matrix. QED is very different, it has a perturbative S –matrix, its proper definition being complicated by the existence of the long range Coulomb forces (see Sect. 2.6.6 above). Nevertheless, the fields in the QED Lagrangian as interpolating fields are closely related to the physical states, the photons and leptons. This extends to the electroweak SM, where the weak non–Abelian gauge bosons, the W^\pm and the Z particles, become massive as a consequence of the breakdown of the $SU(2)_L$ gauge symmetry by the Higgs mechanism. Also the weak gauge bosons cannot be seen as scattering states in a detector, but this time because of their very short lifetime.

are symmetric quark – antiquark states

$$\mathcal{M} = \sum_{ik} \delta^{ik} (q_i \bar{q}_k) \quad (2.283)$$

and have baryon number $B = 0$. Baryons (like proton (uud) and neutron (ddu)) are antisymmetric three quark states

$$\mathcal{B} = \sum_{ikl} \varepsilon^{ikl} (q_i q_k q_l) \quad (2.284)$$

and have baryon number $B = 1$. Sums are over color indices. The quarks (u, d, s) are in the fundamental representation 3, the antiquarks ($\bar{u}, \bar{d}, \bar{s}$) in the representation 3^* of the color $SU(3)_c$. First principles calculations of the spectrum and properties of hadrons are possible only by non-perturbative methods as lattice QCD because the theory is strongly coupled at low energies. Here we are interested primarily in the spectrum of light hadrons, which is accessible to a different non-perturbative approach: chiral perturbation theory, which exploits the symmetries of the QCD Lagrangian. As the three light flavors the u, d and s quarks are much lighter than the c, b and t quarks the chiral limit of vanishing light quark masses $m_u = m_d = m_s = 0$ is a good approximation for setting up a perturbative chiral expansion, with momenta and light quark masses as expansion parameters. The QCD Hamiltonian then commutes with the global chiral flavor group

$$U(3)_L \otimes U(3)_R \sim SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes \cancel{U(1)}_A$$

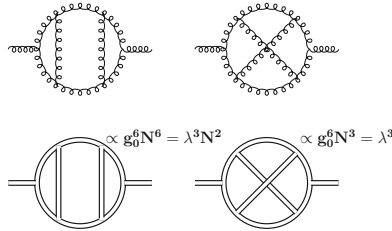
of the left and right handed massless quark fields, i.e. QCD exhibits *chiral symmetry* broken softly by small quark masses. The chiral group $SU(3)_L \otimes SU(3)_R$ is equivalent to $SU(3)_V \otimes SU(3)_A$ of which the axial $SU(N_f)_A$ subgroup turns out to be broken spontaneously in nature. In the isospin limit $N_f = 2, m_u = m_d = 0$, this implies the existence of a triplet of massless pions (Nambu-Goldstone bosons) and in the $SU(3)$ limit $m_u = m_d = m_s = 0$, the existence of an octet of massless pseudoscalars, the pions, Kaons and the η meson [87]. The $U(1)_V$ symmetry is exact beyond the chiral approximation and is responsible for *baryon number conservation*, which in particular guarantees the stability of the proton, whereas in contrast $U(1)_A$ is always broken by quantum corrections, the *Adler-Bell-Jackiw anomaly* (see p. 299 below).

A second approach to learn about the hadron spectrum is to consider QCD from the point of view of the large- N_c limit, i.e. $SU(N_c)$ non-Abelian gauge theory where the number of colors goes to infinity as a starting point and use $1/N_c$ as an expansion parameter. The $1/N_c$ expansion provides counting rules for hadronic processes. In large- N_c QCD [88–90] all hadrons become infinitely narrow, since all widths are suppressed by powers of $1/N_c$, and the VMD model becomes exact with an infinite number of narrow vector meson states, the lowest states corresponding to $\rho, \omega, \phi \dots$

According to 't Hooft 1974 a $SU(N_c)$ generalization of QCD exhibits amplitudes

$$\mathcal{A} \propto (g_0^2 N_c)^F [(g_0^2 N_c)^{-1} N_c]^{2-2H} \xrightarrow{\lambda=g_0^2 N_c \text{ fixed}} \mathcal{A} \propto N_c^{2-2H}$$

where $\chi_E = 2 - 2H - B$ is the Euler characteristic and depends only on the topology of the graph, with H the number of handles and B the number of boundaries (or holes). A trick allows to visualize the topological genus of $SU(N_c)$ Yang-Mills theory by replacing a gluon line by a pair of quark anti-quark lines as illustrated in the figure:



The first planar graph grows with N_c^2 , the second non-planar one remains constant. A closed quark loop is a boundary and brings a $1/N_c$. Each vertex has a factor N_c , each propagator a factor $1/N_c$ and each color index loop gives an extra factor N_c as it represents a sum over N_c colored copies.

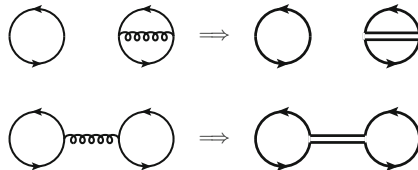
For finite and large N_c , planar diagrams dominate the dynamics. Each quark loop is suppressed by one factor of $1/N_c$ and non-planar gluon exchange is suppressed by two factors of $1/N_c$.

Some consequences of the large- N_c counting rules:

- Only planar diagrams ($H = 0$) dominate in the large- N_c limit.
- Quark loop effects are suppressed by $1/N_c$
- Phenomenology: theory of stable non-interacting mesons, the η' meson mass, OZI rule, etc. find simpler explanations
- Factorization for correlators of gauge invariant operators

$$\langle O_1 \cdots O_n \rangle = \langle O_1 \rangle \cdots \langle O_n \rangle + O(1/N_c^2)$$

follows from large- N_c counting rules order by order in perturbation theory:



The planar approximation contains no quark-anti-quark pair creation and annihilation and thus has the symmetry $U(1)_{q_i} \otimes U(1)_{\bar{q}_i}$, which allows to transform

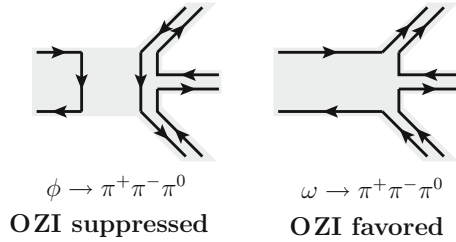


Fig. 2.15 Quark flavor disconnected processes are suppressed relative to quark flavor connected processes. As ϕ is essentially a pure $s\bar{s}$ state and the final state is made of u and d quarks only the process can only be mediated by gluon exchange. For the ω quark flavors are preserved. The gray shading indicates gluonic dressing

locally each quark and each anti-quark separately, which implies the conservation of each quark flavor and each anti-quark flavor light or heavy. Another important consequence of the planar flavor symmetry include the Okubo-Zweig-Iizuka (OZI) rule [91]⁴² (see Fig. 2.15 and the formation of ideally mixed meson nonets (in the $SU(3)$ flavor limit) at leading order in $1/N_c$). The η' is then the ninth pseudoscalar which would be massless in the chiral limit. Planar flavor symmetry is often called *nonet symmetry*.

The combined use of chiral perturbation theory and the $1/N_c$ expansion can constrain the low-energy interactions of hadrons with the pion nonet π , K , η and η' more effectively than either method alone. For later reference we remind the meson composition here. They are the $\bar{q}q'$ bound states, differing by flavor composition and spin. A $\bar{q}q'$ state with orbital angular momentum L has Parity $P = (-1)^{L+1}$. For $q' = q$ we have a $\bar{q}q$ bound state which is also an eigenstate of charge conjugation C with $C = (-1)^{L+S}$, where S is the spin 0 or 1. The $L = 0$ states are the pseudoscalar mesons, $J^P = 0^-$, and the vectors mesons, $J^P = 1^-$.

In the limit of exact $SU(3)$ the pure states would read

$$\pi^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}; \quad \eta_1 = (\bar{u}u + \bar{d}d + \bar{s}s)/\sqrt{3}; \quad \eta_8 = (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}, \quad (2.285)$$

$$\rho^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}; \quad \omega_1 = (\bar{u}u + \bar{d}d + \bar{s}s)/\sqrt{3}; \quad \omega_8 = (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}. \quad (2.286)$$

In fact $SU(2)_{\text{flavor}}$ breaking by the quark mass difference $m_d - m_u$ leads to $\rho - \omega$ -mixing [mixing angle $\sim 10^\circ$] (Glashow 1961) [92]:

⁴²This basically says that diagrams that destroy the initial quark and antiquark are strongly suppressed with respect to those that do not. As an example, while $\phi \rightarrow \pi^+ \pi^-$ is “Zweig” forbidden, $\rho^0 \rightarrow \pi^+ \pi^-$ is allowed.

$$\begin{aligned}\rho^0 &= \cos \theta \rho' + \sin \theta \omega' \\ \omega &= -\sin \theta \rho' + \cos \theta \omega'\end{aligned}\quad (2.287)$$

Similarly, the substantially larger $SU(3)_{\text{flavor}}$ breaking by the quark masses, leads to large $\omega - \phi$ -mixing [mixing angle $\sim 36^\circ$ close to so called ideal mixing where $\phi \sim$ is a pure $\bar{s}s$ state] (Okubo 1963) [93]:

$$\begin{aligned}\phi &= \cos \theta \omega_8 + \sin \theta \omega_1 \\ \omega &= -\sin \theta \omega_8 + \cos \theta \omega_1\end{aligned}\quad (2.288)$$

The angle in case of *ideal* mixing is given by $\tan \theta = 1/\sqrt{2}$ or $\theta = 35.3^\circ$.

In the isospin limit and in absence of e.m. interaction the pion triplet (π^+ , π^0 , π^-) exhibits G-parity as a symmetry. It represents a generalization of charge conjugation (C-parity) to strong interactions. The strong interaction does not distinguish the charges of the pions, therefore a rotation by 180° about the 3rd axis in isospin space $\mathcal{G} = \exp i\pi I_3$, which rotates $\pi^\pm \leftrightarrow \pi^\mp$ up to a phase, does not change the triplet field modulo a phase $\eta_G = \pm 1$. For particles (u, d mesons) of isospin I , the G-parity number is given by $G = (-1)^I C$, where C is the charge conjugation number of the neutral member of the multiplet. As π^0 has charge conjugation parity $C = +1$ the e.m. decay $\pi^0 \rightarrow \gamma\gamma$ requires $G_{\pi^0} = -1$ and thus also $G|\pi^\pm\rangle = -|\pi^\pm\rangle$ while $C|\pi^\pm\rangle = -|\pi^\mp\rangle$. Therefore, non-electromagnetic decays of flavor $SU(2)$ resonances in the isospin limit can decay either into an even or an odd number of pions only: $\rho^0 (I = 1) \rightarrow 2\pi$, $\omega (I = 0) \rightarrow 3\pi$. As isospin is broken by the small quark mass difference $m_d \neq m_u$ actually π^\pm and π^0 have different masses and G-parity is broken accordingly, and $\omega \rightarrow 2\pi$ is allowed with a small branching fraction.

The RG of QCD in Short

The renormalization group, introduced in Sect. 2.6.5, for QCD plays a particularly important role for a quantitative understanding of AF as well as a tool for improving the convergence of the perturbative expansion [36, 94]. For QCD the RG is given by

$$\begin{aligned}\mu \frac{d}{d\mu} g_s(\mu) &= \beta(g_s(\mu)) \\ \mu \frac{d}{d\mu} m_i(\mu) &= -\gamma(g_s(\mu)) m_i(\mu)\end{aligned}\quad (2.289)$$

with

$$\begin{aligned}\beta(g) &= -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + O(g^7) \\ \gamma(g) &= \gamma_0 \frac{g^2}{4\pi^2} + \gamma_1 \frac{g^4}{(4\pi^2)^2} + O(g^6)\end{aligned}\quad (2.290)$$

where, in the $\overline{\text{MS}}$ scheme (Sect. 2.5.6),

$$\begin{aligned}\beta_0 &= 11 - \frac{2}{3}N_f ; \gamma_0 = 2 \\ \beta_1 &= 102 - \frac{38}{3}N_f ; \gamma_1 = \frac{101}{12} - \frac{5}{18}N_f\end{aligned}\quad (2.291)$$

and N_f is the number of quark flavors. The RG for QCD is known to 5 loops [95–97]. It allows one to define effective parameters in QCD, which incorporate the summation of leading logarithmic (1-loop), next-to-leading logarithmic (2-loop), \dots corrections (RG improved perturbation theory). The solution of (2.289) for the running coupling constant $\alpha_s(\mu) = g_s^2(\mu)/(4\pi)$ yields (see (2.223))

$$\begin{aligned}\frac{4\pi}{\beta_0\alpha_s(\mu)} - \frac{\beta_1}{\beta_0^2} \ln\left(\frac{4\pi}{\beta_0\alpha_s(\mu)} + \frac{\beta_1}{\beta_0^2}\right) = \\ \ln \mu^2/\mu_0^2 + \frac{4\pi}{\beta_0\alpha_s(\mu_0)} - \frac{\beta_1}{\beta_0^2} \ln\left(\frac{4\pi}{\beta_0\alpha_s(\mu_0)} + \frac{\beta_1}{\beta_0^2}\right) \equiv \ln \mu^2/\Lambda^2\end{aligned}\quad (2.292)$$

with reference scale (integration constant)

$$\Lambda_{\text{QCD}} = \Lambda_{\overline{\text{MS}}}^{(N_f)} = \mu \exp\left\{-\frac{4\pi}{2\beta_0\alpha_s(\mu)}\left(1 + \frac{\alpha_s(\mu)}{4\pi} \frac{\beta_1}{\beta_0} \ln \frac{\beta_0\alpha_s(\mu)}{4\pi + \frac{\beta_1}{\beta_0}\alpha_s(\mu)}\right)\right\}\quad (2.293)$$

which can be shown easily to be independent of the reference scale μ . It is RG invariant

$$\mu \frac{d}{d\mu} \Lambda_{\text{QCD}} = 0 ,$$

and thus QCD has its own intrinsic scale Λ_{QCD} which is related directly to the coupling strength (dimensional transmutation). This is most obvious at the one-loop level where we have the simple relation

$$\alpha_s(\mu) = \frac{1}{\frac{\beta_0}{4\pi} \ln \frac{\mu^2}{\Lambda^2}} .\quad (2.294)$$

Thus Λ_{QCD} incorporates the reference coupling $\alpha_s(\mu_0)$ measured at scale μ_0 in a scale invariant manner, i.e., each experiment measures the same Λ_{QCD} irrespective of the reference energy μ_0 at which the measurement of $\alpha_s(\mu_0)$ is performed.

The solution of (2.289) for the effective masses $m_i(\mu)$ reads (see (2.224))

$$m_i(\mu) = m_i(\mu_0) \frac{r(\mu)}{r(\mu_0)} \equiv \bar{m}_i r(\mu)\quad (2.295)$$

with

$$r(\mu) = \exp -2 \left\{ \frac{\gamma_0}{\beta_0} \ln \frac{4\pi}{\beta_0 \alpha_s(\mu)} + \left(\frac{\gamma_0}{\beta_0} - \frac{4\gamma_1}{\beta_1} \right) \ln \left(1 + \frac{\beta_1}{\beta_0} \frac{\alpha_s(\mu)}{4\pi} \right) \right\}. \quad (2.296)$$

Note that also the \bar{m}_i are RG invariant masses (integration constants) and for the masses play a role similar to Λ_{QCD} for the coupling. The solution of the RG equation may be expanded in the large $\log L \equiv \ln \frac{\mu^2}{\Lambda^2}$, which of course only makes sense if L is large ($\mu \gg \Lambda$),

$$\begin{aligned} \alpha_s(\mu) &= \frac{4\pi}{\beta_0 L} \left(1 - \frac{\beta_1}{\beta_0^2} \frac{\ln(L + \frac{\beta_1}{\beta_0^2})}{L} + \dots \right) \\ m_i(\mu) &= \bar{m}_i \left(\frac{L}{2} \right)^{-\frac{\gamma_0}{\beta_0}} \left(1 - \frac{2\beta_1\gamma_0}{\beta_0^3} \frac{\ln L + 1}{L} + \frac{8\gamma_1}{\beta_0^2 L} + \dots \right). \end{aligned} \quad (2.297)$$

If L is not large one should solve (2.292) or its higher order version numerically by iteration for $\alpha_s(\mu)$. For the experimental proof of the running of the strong coupling constant [98] see Fig. 3.3 in Sect. 3.2.1 and the most actual update presented in Fig. 9.3 in [99]. The non-perturbative calculations in lattice QCD are able to demonstrate a surprisingly good agreement with perturbative results (see [100–103] and references therein). Most interestingly the non-perturbative strong coupling persists being monotonically increasing at very low scales, in clear contrast to speculations about a possible IR freezing $\lim_{\mu \rightarrow 0} \alpha_s(\mu) \rightarrow \text{const.}$

Note on the RG of the SM

The electroweak sector of the SM will be introduced in Sect. 4.2. But a comment on the RG of the full SM is in order here. After the discovery of the Higgs boson all SM couplings are known via the mass–coupling relations (4.46) and so are the β -functions. The main couplings are the gauge couplings of the SM local gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$: g_1 , g_2 and g_3 and the top quark Yukawa coupling y_t and the Higgs boson self-coupling λ . The SM renormalization group in the $\overline{\text{MS}}$ scheme is known to three loops. The key point concerning the behavior of the effective parameters we may understand when we look at the leading terms of the β -functions. At the Z boson mass scale the couplings are given by $g_1 \simeq 0.350$, $g_2 \simeq 0.653$, $g_3 \simeq 1.220$, $y_t \simeq 0.935$ and $\lambda \simeq 0.807$. While the gauge couplings behave as expected, g_1 is infrared (IR) free, g_2 and g_3 are asymptotically (ultraviolet) free (AF), with leading coefficients exhibiting the related coupling only,

$$\beta_1 = \frac{41}{6} g_1^3 c \simeq 0.00185; \quad \beta_2 = -\frac{19}{6} g_2^2 c \simeq -0.00558; \quad \beta_3 = -7 g_3^3 c \simeq -0.08049,$$

with $c = \frac{1}{16\pi^2}$, the leading top Yukawa β -function given by

$$\begin{aligned}\beta_{y_t} &= \left(\frac{9}{2} y_t^3 - \frac{17}{12} g_1^2 y_t - \frac{9}{4} g_2^2 y_t - 8 g_3^2 y_t \right) c \\ &\simeq 0.02327 - 0.00103 - 0.00568 - 0.07048 \\ &\simeq -0.05391\end{aligned}$$

not only depends on y_t , but also on mixed terms with the gauge couplings which have a negative sign. In fact the QCD correction is the leading contribution and determines the behavior. Notice the critical balance between the dominant strong and the top Yukawa couplings: QCD dominance requires $g_3 > \frac{3}{4} y_t$ in the “gaugeless” limit $g_1 = g_2 = 0$.

Similarly, the β -function of the Higgs self-coupling, given by

$$\begin{aligned}\beta_\lambda &= (4\lambda^2 - 3g_1^2\lambda - 9\lambda g_2^2 + 12y_t^2\lambda + \frac{9}{4}g_1^4 + \frac{9}{2}g_1^2g_2^2 + \frac{27}{4}g_2^4 - 36y_t^4) c \\ &\simeq 0.01650 - 0.00187 - 0.01961 + 0.05358 + 0.00021 + 0.00149 + 0.00777 \\ &\quad - 0.17401 \simeq -0.11595\end{aligned}$$

is dominated by the top Yukawa contribution and not by the λ coupling itself. At leading order it is not subject to QCD corrections. Here, the y_t dominance condition reads $\lambda < \frac{3(\sqrt{5}-1)}{2} y_t^2$ in the gaugeless limit. The top Yukawa coupling is turned from an intrinsically IR free to an AF coupling by the QCD term and similarly the Higgs self-coupling is transmuted from IR free to AF by the dominating top Yukawa term. Including known higher order terms, except from β_λ , which exhibits a zero at about $\mu_\lambda \sim 10^{17}$ GeV, all other β -functions do not exhibit a zero in the range from $\mu = M_Z$ to $\mu = M_{\text{Planck}}$. So, apart from the $U(1)_Y$ coupling g_1 , which increases moderately only, all other couplings decrease and perturbation theory works well up to the Planck scale. Actually, at $\mu = M_{\text{Planck}}$ gauge couplings are all close to $g_i \sim 0.5$, while $y_t \sim 0.35$ and $\sqrt{\lambda} \sim 0.36$ (see [104] and references therein).

References

1. J.D. Bjorken, S.D. Drell, *Relativistic Quantum Mechanics*, 1st edn. (McGraw-Hill, New York 1964), 300 p; *Relativistic Quantum Fields*, 1st edn. (McGraw-Hill, New York, 1965), 396 p
2. V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii, Quantum electrodynamics, *Landau and Lifshitz Course of Theoretical Physics*, vol. 4, 2nd edn. (Pergamon, London, 1982), 652 p
3. P.A.M. Dirac, Proc. R. Soc. A **114**, 243 (1927); Proc. R. Soc. A **126**, 360 (1930); Proc. R. Soc. A **136**, 453 (1932)
4. E.P. Wigner, Ann. Math. **40**, 149 (1939)
5. P. Jordan, E.P. Wigner, Zeits. Phys. **47**, 631 (1928)
6. W. Heisenberg, W. Pauli, Zeits. Phys. **56**, 1 (1929); Zeits. Phys. **59**, 168 (1930); P.A.M. Dirac, V.A. Fock, B. Podolsky, Phys. Zeits. Sowjetunion **3**, 64 (1932)
7. H. Joos, Fortsch. Phys. **10**, 65 (1962); S. Weinberg, Phys. Rev. **133**, B1318 (1964); Phys. Rev. **134**, B882 (1964)
8. P.A.M. Dirac, Proc. R. Soc. A **114**, 243 (1927); A **117**, 610 (1928)
9. W. Pauli, Zeits. Physik **31**, 765 (1925)

10. W. Pauli, Phys. Rev. **58**, 716 (1940)
11. R.F. Streater, A.S. Wightman, *CPT, Spin and Statistics and All That* (Benjamin, New York, 1964)
12. G. Lüders, K. Danske Vidensk. Selsk. Mat.-Fys. Medd. **28**(5) (1954); W. Pauli, Exclusion principle, Lorentz group and reflection of space-time and charge, in *Niels Bohr and the Development of Physics*, ed. by W. Pauli (Pergamon Press, London, 1955, reissued 1962), pp. 30–51; W. Pauli, II Nuovo Cim. **6**, 204 (1957); G. Lüders, Ann. Phys. N. Y. **2**, 1 (1957); G. Lüders, B. Zumino, Phys. Rev. **106**, 345 (1957); R. Jost, Helv. Phys. Acta **30**, 409 (1957)
13. S. Gasiorovicz, *Elementary Particle Physics* (Wiley, New York, 1966)
14. S. Eidelman et al. [Particle Data Group], Phys. Lett. B **592**, 1 (2004); K.A. Olive et al. [Particle Data Group], Chin. Phys. C **38**, 090001 (2014) and 2015 update
15. B.C. Regan, E.D. Commins, C.J. Schmidt, D. DeMille, Phys. Rev. Lett. **88**, 071805 (2002)
16. J. Baron et al. [ACME Collab.], Science **343**, 269 (2014)
17. W. Gerlach, O. Stern, Zeits. Physik **8**, 110 (1924)
18. R. Kronig (1925 unpublished), https://en.wikipedia.org/wiki/Ralph_Kronig
19. G.E. Uhlenbeck, S. Goudsmit, Naturwissenschaften **13**, 953 (1925); Nature **117**, 264 (1926)
20. W. Pauli, Zeits. Phys. **43**, 601 (1927)
21. F. Englert, R. Brout, Phys. Rev. Lett. **13**, 321 (1964)
22. P.W. Higgs, Phys. Lett. **12**, 132 (1964)
23. G. Aad et al. [ATLAS Collab.], Phys. Lett. B **716**, 1 (2012); Science **338**, 1576 (2012)
24. S. Chatrchyan et al. [CMS Collab.], Phys. Lett. B **716**, 30 (2012); Science **338**, 1569 (2012)
25. S. Weinberg, Phys. Rev. **134**, B882 (1964)
26. R. Frisch, O. Stern, Zeits. Physik **85**, 4 (1933); I. Estermann, O. Stern, *ibid.* **85**, 17 (1933)
27. G. Charpak, F.J.M. Farley, R.L. Garwin, T. Muller, J.C. Sens, A. Zichichi, Phys. Lett. **1B**, 16 (1962)
28. H. Weyl, Zeits. Phys. **56**, 330 (1929)
29. C.N. Yang, R.L. Mills, Phys. Rev. **96**, 191 (1954)
30. M. Gell-Mann, F. Low, Phys. Rev. **95**, 1300 (1954)
31. G.C. Wick, Phys. Rev. **80**, 268 (1950)
32. W. Pauli, F. Villars, Rev. Mod. Phys. **21**, 434 (1949)
33. N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, 1st & 2nd edn. (Wiley, New York, 1957, 1980), 720 p
34. S. Tomonaga, R. Iho, Progr. Theor. Phys. **1**, 27 (1946); J. Schwinger, Phys. Rev. **74**, 1439 (1948); R.P. Feynman, Phys. Rev. **76**, 749 (1949); F. Dyson, Phys. Rev. **75**, 486 (1949), *ibid.* 1736
35. H. Fritzsch, M. Gell-Mann, H. Leutwyler, Phys. Lett. **47B**, 365 (1973)
36. H.D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); D. Gross, F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973)
37. S.L. Glashow, Nucl. Phys. B **22**, 579 (1961); S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967); A. Salam, Weak and electromagnetic interactions, in *Elementary Particle Theory*, ed. by N. Svartholm (Amquist and Wiksells, Stockholm, 1969), pp. 367–377
38. G. 't Hooft, Nucl. Phys. B **33**, 173 (1971); **35**, 167 (1971); G. 't Hooft, M. Veltman, Nucl. Phys. B **50**, 318 (1972)
39. G. 't Hooft, M. Veltman, Nucl. Phys. B **44**, 189 (1972)
40. W.H. Furry, Phys. Rev. **51**, 125 (1937)
41. F. Dyson, Phys. Rev. **75**, 1736 (1949); J.S. Schwinger, Proc. Nat. Acad. Sci. **37**, 452–459 (1951)
42. H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cim. **1**, 205 (1955); Nuovo Cim. **6**, 319 (1957)
43. S. Mandelstam, Phys. Rev. **112**, 1344 (1958)
44. I. Białyński-Birula, Phys. Rev. D **2**, 2877 (1970)
45. F. Jegerlehner, Eur. Phys. J. C **18**, 673 (2001)
46. D. Akyeamong, R. Delbourgo, Nuovo Cim. **17A**, 47 (1973); W.A. Bardeen, R. Gastmans, B. Lautrup, Nucl. Phys. B **46**, 319 (1972); M. Chanowitz, M. Furman, I. Hinchliffe, Nucl. Phys. B **159**, 225 (1979)

47. M.C. Bergère, F. David, J. Math. Phys. **20**, 1244 (1979)
48. K. Osterwalder, R. Schrader, Commun. Math. Phys. **31**, 83 (1973); *ibid.* **42**, 281 (1975)
49. K.G. Wilson, Phys. Rev. D **10**, 2445 (1974)
50. M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965) (Chap. 9 and formulas 9.1.39/9.1.40/9.6.4 there)
51. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, eds. by A. Jeffery, D. Zwanziger (Academic Press, London, 2015[1943]) (Formulas 3.471.9/11)
52. G. t' Hooft, M. Veltman, Nucl. Phys. B **153**, 365 (1979)
53. A.I. Davydychev, M.Y. Kalmykov, Nucl. Phys. B **605**, 266 (2001)
54. J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. B **672**, 303 (2003)
55. J.S. Frederiksen, W.S. Woolcock, Nucl. Phys. B **28**, 605 (1971); Aust. J. Phys. **26**, 691 (1973); Ann. Phys. **75**, 503 (1973); *ibid.* **80**, 86 (1973)
56. G. Passarino, M. Veltman, Nucl. Phys. B **160**, 151 (1979)
57. K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B **192**, 159 (1981)
58. F. Jegerlehner, Renormalizing the standard model, in *Testing the Standard Model*, eds. by M. Cvetič, P. Langacker (World Scientific, Singapore, 1991), pp. 476–590, <http://www-com.physik.hu-berlin.de/~fjeger/books.html>
59. F. Bloch, A. Nordsieck, Phys. Rev. D **52**, 54 (1937)
60. Y. Hahn, W. Zimmermann, Commun. Math. Phys. **10**, 330 (1968); W. Zimmermann, Commun. Math. Phys. **11**, 1 (1968); J.H. Lowenstein, W. Zimmermann, Commun. Math. Phys. **44**, 73 (1975)
61. F.J. Dyson, Phys. Rev. **75**, 1736 (1949)
62. J.P. Fink, J. Math. Phys. **9**, 1389 (1968); E.B. Manoukian, J. Math. Phys. **19**, 917 (1978)
63. S. Weinberg, Phys. Rev. **118**, 838 (1960)
64. T. Appelquist, J. Carazzone, Phys. Rev. D **11**, 2856 (1975)
65. V.A. Smirnov, Mod. Phys. Lett. A **10**, 1485 (1995) (and references therein)
66. E.C.G. Stueckelberg, A. Petermann, Helv. Phys. Acta **26**, 499 (1953); M. Gell-Mann, F.E. Low, Phys. Rev. **95**, 1300 (1954)
67. G. Mack, Nucl. Phys. B **5**, 499 (1968)
68. J.D. Bjorken, Phys. Rev. **179**, 1547 (1969); J.D. Bjorken, E.A. Paschos, Phys. Rev. **185**, 1975 (1969)
69. D.H. Coward et al., Phys. Rev. Lett. **20**, 292 (1968); E.D. Bloom et al., Phys. Rev. Lett. **23**, 930 (1969); G. Miller et al., Phys. Rev. D **5**, 528 (1972)
70. H. Suura, E. Wichmann, Phys. Rev. **105**, 1930 (1957); A. Petermann, Phys. Rev. **105**, 1931 (1957)
71. B.E. Lautrup, E. de Rafael, Nucl. Phys. B **70**, 317 (1974)
72. E. de Rafael, J.L. Rosner, Ann. Phys. (N.Y.) **82**, 369 (1974)
73. A.L. Kataev, S.A. Larin, Pisma Zh. Eksp. Teor. Fiz. **96**, 64 (2012) [JETP Lett. **96**, 61 (2012)]
74. P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, C. Sturm, Nucl. Phys. B **867**, 182 (2013)
75. P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, J. Rittinger, JHEP **1207**, 017 (2012)
76. D.R. Yennie, S.C. Frautschi, H. Suura, Ann. Phys. **13**, 379 (1961); D.R. Yennie, Phys. Lett. **34B**, 239 (1975); J.D. Jackson, D.L. Scharre, Nucl. Instr. **128**, 13 (1975); M. Greco, G. Pancheri, Y. Srivastava, Nucl. Phys. B **101**, 234 (1975)
77. T. Kinoshita, J. Math. Phys. **3**, 650 (1962); T.D. Lee, M. Nauenberg, Phys. Rev. D **133**, B1549 (1964)
78. J. Fleischer, F. Jegerlehner, Z. Phys. C **26**, 629 (1985)
79. G. Serman, S. Weinberg, Phys. Rev. Lett. **39**, 1436 (1977)
80. O. Steinmann, Commun. Math. Phys. **237**, 181 (2003)
81. F. Jegerlehner, Nucl. Phys. B (Proc. Suppl.) **51C**, 131 (1996)
82. G.J. Gounaris, J.J. Sakurai, Phys. Rev. Lett. **21**, 244 (1968); A. Quenzer et al., Phys. Lett. B **76**, 512 (1978)
83. F. Jegerlehner, R. Szafron, Eur. Phys. J. C **71**, 1632 (2011)
84. H. Leutwyler, Electromagnetic form factor of the pion, in *Continuous Advances in QCD 2002: Proceedings*, eds. by K.A. Olive, M.A. Shifman, M.B. Voloshin (World Scientific, Singapore, 2002), 646 p, [arXiv:hep-ph/0212324](http://arxiv.org/abs/hep-ph/0212324)

85. J. Gasser, H. Leutwyler, *Ann. Phys.* **158**, 142 (1984); *Nucl. Phys. B* **250**, 465 (1985)
86. K.G. Wilson, *Phys. Rev. D* **10**, 2445 (1974); M. Creutz, *Phys. Rev. D* **21**, 2308 (1980)
87. Y. Nambu, *Phys. Rev. Lett.* **4**, 380 (1960)
88. G. 't Hooft, *Nucl. Phys. B* **72**, 461 (1974); *ibid.* **75**, 461 (1974); E. Witten, *Nucl. Phys. B* **160**, 57 (1979)
89. A.V. Manohar, *Hadrons in the $1/N$ expansion*, in *At the frontier of Particle Physics*, vol. 1, ed. M. Shifman (World Scientific, Singapore, 2001)pp. 507–568
90. H. Leutwyler, *Nucl. Phys. Proc. Suppl.* **64**, 223 (1998); R. Kaiser, H. Leutwyler, *Eur. Phys. J. C* **17**, 623 (2000)
91. S. Okubo, *Phys. Lett.* **5**, 1975 (1963); G. Zweig, CERN Report No.8419/TH412 (1964); J. Iizuka, *Prog. Theor. Phys. Suppl.* **37**, 38 (1966)
92. S.L. Glashow, *Phys. Rev. Lett.* **7**, 469 (1961)
93. S. Okubo, *Phys. Lett.* **5**, 165 (1963)
94. W.E. Caswell, *Phys. Rev. Lett.* **33**, 244 (1974); D.R.T. Jones, *Nucl. Phys. B* **75**, 531 (1974); E. Egorian, O.V. Tarasov, *Theor. Math. Phys.* **41**, 863 (1979) [*Teor. Mat. Fiz.* **41**, 26 (1979)]
95. O.V. Tarasov, A.A. Vladimirov, A.Y. Zharkov, *Phys. Lett. B* **93**, 429 (1980); S.A. Larin, J.A.M. Vermaseren, *Phys. Lett. B* **303**, 334 (1993)
96. T. van Ritbergen, J.A.M. Vermaseren, S.A. Larin, *Phys. Lett. B* **400**, 379 (1997); M. Czakon, *Nucl. Phys. B* **710**, 485 (2005); K.G. Chetyrkin, *Nucl. Phys. B* **710**, 499 (2005)
97. P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, *Phys. Rev. Lett.* **118**, 082002 (2017)
98. S. Bethke, *Phys. Rep.* **403–404**, 203 (2004)
99. K.A. Olive et al. (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014); *Quantum Chromodynamics* eds. by S. Bethke, G. Dissertori, G.P. Salam (therein)
100. M. Della Morte, R. Frezzotti, J. Heitger, J. Rolf, R. Sommer, U. Wolff, *Nucl. Phys. B* **713**, 378 (2005)
101. F. Tekin et al. [ALPHA Collab.], *Nucl. Phys. B* **840**, 114 (2010)
102. M. Dalla Brida et al. [ALPHA Collab.], *Phys. Rev. D* **95**, 014507 (2017)
103. M. Bruno et al., *Nucl. Part. Phys. Proc.* **285–286**, 132 (2016); *PoS Lattice* **2016**, 197 (2016), [arXiv:1701.03075](https://arxiv.org/abs/1701.03075) [hep-lat]
104. F. Jegerlehner, *Acta Phys. Polon. B* **45**, 1167 (2014)