Chapter 1
Introduction

The book gives an introduction to the basics of the anomalous magnetic moments of leptons and reviews the current state of our knowledge of the anomalous magnetic moment $(g-2)$ of the muon and related topics. The muon usually is denoted by μ. The last $g-2$ experiment E821 performed at Brookhaven National Laboratory (BNL) in the USA has reached the impressive precision of 0.54 parts per million (ppm) [1]. The anomalous magnetic moment of the muon is now one of the most precisely measured quantities in particle physics and allows us to test relativistic local *Quantum Field Theory* (QFT) in its depth, with unprecedented accuracy. It puts severe limits on deviations from the standard theory of elementary particles and at the same time opens a window to new physics. The book describes the fascinating story of uncovering the fundamental laws of nature to the deepest by an increasingly precise investigation of a single observable. The anomalous magnetic moment of the muon not only encodes all the known but also the as of yet unknown non–Standard-Model physics.[1] The latter, however, is still hidden and is waiting to be discovered on the way to higher precision which allows us to see smaller and smaller effects.

In fact a persisting $3-4\sigma$ deviation between theory and experiment, probably the best established substantial deviation among the many successful SM predictions which have been measured in a multitude of precision experiments, motivated a next generation of muon $g-2$ experiments. A new followup experiment E989 at Fermilab in the US [2–6], will operate very similar as later CERN and the BNL experiments, working with ultrarelativistic magic-energy muons. A second experiment E34 planned at J-PARC in Japan [7–10] will work with ultra-cold muons, and thus can provide an important cross-check between very different experimental setups. While the Fermilab experiment will be able to reduce the experimental

[1] As a matter of principle, an experimentally determined quantity always includes all effects, known and unknown, existing in the real world. This includes electromagnetic, strong, weak and gravitational interactions, plus whatever effects we might discover in future.

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uncertainty by a factor four to 0.14 ppm, the conceptually novel J-PARC experiment is expected to reach the precision of the previous BNL experiment in a first phase.

In order to understand what is so special about the muon anomalous magnetic moment we have to look at leptons in general. The muon (μ^-), like the much lighter electron (e^-) or the much heavier tau (τ^-) particle, is one of the 3 known charged leptons: elementary spin 1/2 fermions of electric charge -1 in units of the positron charge e, as free relativistic one particle states described by the Dirac equation. Each of the leptons has its positively charged antiparticle, the positron e^+, the μ^+ and the τ^+, respectively, as required by any local relativistic quantum field theory [11].[2]

Of course the charged leptons are never really free, they interact electromagnetically with the photon and weakly via the heavy gauge bosons W and Z, as well as very much weaker also with the Higgs boson. Puzzling enough, the three leptons have identical properties, except for the masses which are given by $m_e = 0.511$ MeV, $m_\mu = 105.658$ MeV and $m_\tau = 1776.99$ MeV, respectively. In reality, the lepton masses differ by orders of magnitude and actually lead to a very different behavior of these particles. As mass and energy are equivalent according to Einstein's relation $E = mc^2$, heavier particles in general decay into lighter particles plus kinetic energy. An immediate consequence of the very different masses are the very different lifetimes of the leptons. Within the *Standard Model* (SM) of elementary particle interactions the electron is stable on time scales of the age of the universe, while the μ has a short lifetime of $\tau_\mu = 2.197 \times 10^{-6}$ s and the τ is even more unstable with a lifetime $\tau_\tau = 2.906 \times 10^{-13}$ s only. Also, the decay patterns are very different: the μ decays very close to 100% into electrons plus two neutrinos ($e\bar{\nu}_e\nu_\mu$), however, the τ decays to about 65% into hadronic states $\pi^-\nu_\tau$, $\pi^-\pi^0\nu_\tau$, ,... while the main leptonic decay modes only account for 17.36% $\mu^-\bar{\nu_\mu}\nu_\tau$ and 17.85% $e^-\bar{\nu}_e\nu_\tau$, respectively. This has a dramatic impact on the possibility to study these particles experimentally and to measure various properties precisely. The most precisely studied lepton is the electron, but the muon can also be explored with extreme precision. Since the muon, the much heavier partner of the electron, turns out to be much more sensitive to hypothetical physics beyond the SM than the electron itself, the muon is much more suitable as a "crystal ball" which could give us hints about not yet uncovered physics. The reason is that some effects scale with powers of m_ℓ^2, as we will see below. Unfortunately, the τ is so short lived, that corresponding experiments are not possible with present technology.

A direct consequence of the pronounced mass hierarchy is the fundamentally different role the different leptons play in nature. While the stable electrons, besides protons and neutrons, are everywhere in ordinary matter, in atoms, molecules, gases, liquids, metals, other condensed matter states etc., muons seem to be very rare and their role in our world is far from obvious. Nevertheless, even though we may not be aware of it, muons as cosmic ray particles are also part of our everyday life. They are continuously created when highly energetic particles from deep space, mostly protons, collide with atoms from the Earth's upper atmosphere. The initial collisions

[2]Dirac's theory of electrons, positrons and photons was an early version of what later developed into *Quantum Electrodynamics* (QED), as it is known since around 1950.

create pions which then decay into muons. The highly energetic muons travel at nearly the speed of light down through the atmosphere and arrive at ground level at a rate of about 1 muon per cm^2 and minute. The relativistic *time dilatation* thereby is responsible that the muons have time enough to reach the ground. As we will see later the basic mechanisms observed here are the ones made use of in the muon $g-2$ experiments. Also remember that the muon was discovered in cosmic rays by Anderson & Neddermeyer in 1936 [12], a few years after Anderson [13] had discovered antimatter in form of the positron, a "positively charged electron" as predicted by Dirac, in cosmic rays in 1932.

Besides charge, spin, mass and lifetime, leptons have other very interesting static (classical) electromagnetic and weak properties like the magnetic and electric *dipole moments*. Classically the dipole moments can arise from either electrical *charges* or *currents*. A well known example is the circulating current, due to an orbiting particle with electric charge e and mass m, which exhibits a magnetic dipole moment $\boldsymbol{\mu}_L = \frac{1}{2c} e\, \mathbf{r} \times \mathbf{v}$ given by

$$\boldsymbol{\mu}_L = \frac{e}{2mc}\,\mathbf{L} \tag{1.1}$$

where $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ is the orbital angular momentum ($\mathbf{r}$ position, $\mathbf{v}$ velocity). An electrical dipole moment can exist due to relative displacements of the centers of positive and negative electrical charge distributions. Thus both electrical and magnetic properties have their origin in the electrical *charges and their currents*. Magnetic charges are not necessary to obtain magnetic moments. This aspect carries over from the basic asymmetry between electric and magnetic phenomena in Maxwell's equations.[3] While electric charges play the fundamental role of the sources of the electromagnetic fields, elementary magnetic charges, usually called magnetic monopoles, are absent. A long time ago, Dirac [14] observed that the existence of magnetic charges would allow us to naturally explain the quantization of both the electric charge e and the magnetic charge m. They would be related by

$$em = \frac{1}{2} n\hbar c \quad , \quad \text{where } n \text{ is an integer.}$$

Apparently, nature does not make use of this possibility and the question of the existence of magnetic monopoles remains a challenge for the future in particle physics.

Whatever the origin of magnetic and electric moments are, they contribute to the electromagnetic interaction Hamiltonian (interaction energy) of the particle with magnetic and electric fields

$$\mathcal{H} = -\boldsymbol{\mu}_m \cdot \mathbf{B} - \mathbf{d}_e \cdot \mathbf{E} \,, \tag{1.2}$$

where $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric field strengths and $\boldsymbol{\mu}_m$ and $\mathbf{d}_e$ the magnetic and electric dipole moment operators. Usually, we measure magnetic moments

[3] It should be noted that a duality $\mathbf{E} \leftrightarrow \mathbf{B}$ of Maxwell electromagnetism is not realized, because the Hamiltonian changes sign and the dual system would be unstable.

in units of the *Bohr magneton*

$$\mu_0 = e\hbar/2mc \tag{1.3}$$

and the *spin operator*

$$\mathbf{S} = \frac{\hbar\boldsymbol{\sigma}}{2} \tag{1.4}$$

is replacing the angular momentum operator **L**. Thus, generalizing the classical form (1.1) of the orbital magnetic moment, one writes (see Sect. 3.1)

$$\boldsymbol{\mu}_m = g\, Q\, \mu_0\, \frac{\boldsymbol{\sigma}}{2} \;\;, \quad \mathbf{d}_e = \eta\, Q\, \mu_0\, \frac{\boldsymbol{\sigma}}{2}\;, \tag{1.5}$$

where σ_i $(i = 1, 2, 3)$ are the Pauli spin matrices, Q is the electrical charge in units of e, $Q = -1$ for the leptons $Q = +1$ for the antileptons. The equations are defining the gyromagnetic ratio g (g-factor) and its electric pendant η, respectively, quantities exhibiting important dynamical information about the leptons as we will see later.

The magnetic interaction term gives rise to the well known *Zeeman effect*: atomic spectra show a level splitting

$$\Delta E = \frac{e}{2mc}\,(\mathbf{L} + g\mathbf{S}) \cdot \mathbf{B} = g_J\, \mu_0\, m_j\, B\;.$$

The second form gives the result evaluated in terms of the relevant quantum numbers. m_j is the 3rd component of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ in units of $\hbar$ and takes values $m_j = -j, -j+1, \ldots, j$ with $j = l \pm \frac{1}{2}$. g_J is Landé's g–factor.[4] If spin is involved one calls it *anomalous Zeeman effect*. The latter obviously is suitable to study the magnetic moment of the electron by investigating atomic spectra in magnetic fields.

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[4]The Landé g_J may be calculated based on the *"vector model"* of angular momentum composition:

$$(\mathbf{L} + g\mathbf{S}) \cdot \mathbf{B} = \frac{(\mathbf{L} + g\mathbf{S}) \cdot \mathbf{J}}{J}\, \frac{\mathbf{J} \cdot \mathbf{B}}{J} = \frac{(\mathbf{L} + g\mathbf{S}) \cdot (\mathbf{L} + \mathbf{S})}{J^2}\, J_z B$$
$$= \frac{L^2 + gS^2 + (g+1)\,\mathbf{L} \cdot \mathbf{S}}{J^2}\, m_j \hbar B = \frac{(g+1)\, J^2 - (g-1)\, L^2 + (g-1)\, S^2}{2J^2}\, m_j \hbar B$$

where we have eliminated $\mathbf{L} \cdot \mathbf{S}$ using $J^2 = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}$. Using $J = j(j+1)\,\hbar$ etc. we find

$$g_J = 1 + (g-1)\, \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\;.$$

With the Dirac value $g = 2$ we find the usual textbook expression.

The anomalous magnetic moment is an *observable*[5] which can be relatively easily studied experimentally from the motion of the lepton in an external magnetic field. The story started in 1925 soon after Kronig, Goudsmit and Uhlenbeck [15] had postulated that an electron had an intrinsic angular momentum of $\frac{1}{2}\hbar$, and that associated with this spin angular momentum there is a magnetic dipole moment equal to $e\hbar/2mc$, which is the Bohr magneton μ_0. The important question "is $(\mu_m)_e$ precisely equal to μ_0", or "is $g = 1$" in our language, was addressed by Back and Landé in 1925 [16]. Their conclusion, based on a study of numerous experimental investigations on the Zeeman effect, was that the magnetic moment of the electron $(\mu_m)_e$ was consistent with the Goudsmit and Uhlenbeck postulate. In fact, the analysis was not conclusive, as we know, since they did not really determine g. Soon after Pauli had formulated the quantum mechanical treatment of the electron spin in 1927 [17], where g remains a free parameter, Dirac presented his relativistic theory in 1928 [18].

The Dirac theory predicted, unexpectedly, $g = 2$ for a free electron [18], twice the value $g = 1$ known to be associated with orbital angular momentum. After first experimental confirmations of Dirac's prediction $g_e = 2$ for the electron (Kinster and Houston 1934) [19], which strongly supported the Dirac theory, yet within relatively large experimental errors at that time, it took about 20 more years of experimental efforts to establish that the electrons magnetic moment actually exceeds 2 by about 0.12%, the first clear indication of the existence of an "anomalous"[6] contribution

$$a_\ell \equiv \frac{g_\ell - 2}{2} \;, \quad (\ell = e, \mu, \tau) \tag{1.6}$$

to the magnetic moment [20]. By end of the 1940's the breakthrough in understanding and handling renormalization of QED (Tomonaga, Schwinger, Feynman, and others around 1948 [21]) had made unambiguous predictions of higher order effects possible, and in particular of the leading (one–loop diagram) contribution to the anomalous magnetic moment

$$a_\ell^{\mathrm{QED}(1)} = \frac{\alpha}{2\pi} \;, \quad (\ell = e, \mu, \tau) \tag{1.7}$$

by Schwinger in 1948 [22] (see Sect. 2.6.3 and Chap. 3). This contribution is due to quantum fluctuations via virtual electron photon interactions and in QED is universal for all leptons. The history of the early period of enthusiasm and worries in the development and first major tests of QED as a renormalizable covariant local quantum field theory is elaborated in great detail in the fascinating book by Schweber [23] (concerning $g - 2$ see Chap. 5, in particular).

[5] A quantity which is more or less directly accessible in an experiment. In general small corrections based on well understood and established theory are necessary for the interpretation of the experimental data.

[6] The anomalous magnetic moment is called anomalous for historic reasons, as a deviation from the classical result. In QED or any QFT higher order effects, so called radiative corrections, are the normal case, which does not make such phenomena less interesting.

In 1947 Nafe, Nelson and Rabi [24] reported an anomalous value by about 0.26% in the hyperfine splitting of hydrogen and deuterium, which was quickly confirmed by Nagle et al. [25], and Breit [26] suggested a possible anomaly $g \neq 2$ of the magnetic moment of the electron. Soon after, Kusch and Foley [27], by a study of the hyperfine–structure of atomic spectra in a constant magnetic field, presented the first precision determination of the magnetic moment of the electron $g_e = 2.00238(10)$ in 1948, just before the theoretical result had been settled. Together with Schwinger's result $a_e^{(2)} = \alpha/(2\pi) \simeq 0.00116$ (which accounts for 99% of the anomaly) this provided one of the first tests of the virtual quantum corrections, usually called radiative corrections, predicted by a relativistic quantum field theory. The discovery of the fine structure of the hydrogen spectrum (Lamb–shift) by Lamb and Retherford [28] and the corresponding calculations by Bethe, Kroll & Lamb and Weisskopf & French [29] was the other triumph of testing the new level of theoretical understanding with precision experiments. These successes had a dramatic impact in establishing quantum field theory as a general framework for the theory of elementary particles and for our understanding of the fundamental interactions. It stimulated the development of QED[7] in particular and the concepts of quantum field theory in general. With the advent of non–Abelian gauge theories, proposed by Yang and Mills (YM) [31] in 1954, and after 't Hooft and Veltman [32] found the missing clues to understanding and handling them on the quantum level, many years later in 1971, the SM [33] (Glashow, Weinberg, Salam 1981/1987) finally emerged as a comprehensive theory of weak, electromagnetic and strong interactions. The strong interactions had emerged as Quantum Chromodynamics (QCD) [34] (Fritzsch, Gell-Mann, Leutwyler 1973), exhibiting the property of Asymptotic Freedom (AF) [35] (Gross, Politzer and Wilczek 1973). All this structure today is crucial for obtaining sufficiently precise predictions for the anomalous magnetic moment of the muon as we will see.

The most important condition for the anomalous magnetic moment to be a useful monitor for testing a theory is its unambiguous predictability within that theory. The predictability crucially depends on the following properties of the theory:

(1) it must be a local relativistic quantum field theory and
(2) it must be renormalizable.

As a consequence $g - 2$ vanishes at tree level. This means that g cannot be an independently adjustable parameter in any renormalizable QFT, which in turn implies that $g - 2$ is a calculable quantity and the predicted value can be confronted with experiments. As we will see $g - 2$ can in fact be both predicted as well as experimentally measured with very high accuracy. By confronting precise theoretical predictions with precisely measured experimental data it is possible to subject the theory to very stringent tests and to find its possible limitation.

The particle–antiparticle duality [11], also called crossing or charge conjugation property, which is a basic consequence of any relativistic local QFT, implies first and foremost that particles and antiparticles have identical masses and spins. In

[7] Today we understand QED as an Abelian gauge theory. This important structural property was discovered by Weyl [30] in 1929.

fact, charge conjugation turned out not to be a universal symmetry of the world of elementary particles. Since, in some sense, an antiparticle is like a particle propagating backwards in time, *charge conjugation* C has to be considered together with *time-reversal* T (time-reflection), which in a relativistic theory has to go together with *parity* P (space-reflection). Besides C, T and P are the two other basic discrete transformation laws in particle physics. A well known fundamental prediction which relates C, P and T is the CPT theorem: the product of the three discrete transformations, taken in any order, is a symmetry of any relativistic QFT. Actually, in contrast to the individual transformations C, P and T, which are symmetries of the electromagnetic– and strong–interactions only, CPT is a universal symmetry and it is this symmetry which guarantees that particles and antiparticles have identical masses as well as equal lifetimes.[8] But also the dipole moments are very interesting quantities for the study of the discrete symmetries mentioned.

To learn about the properties of the dipole moments under such transformations we have to look at the interaction Hamiltonian (1.2). In particular the behavior under parity and time-reversal is of interest. Naively, one would expect that electromagnetic (QED) and strong interactions (QCD) are giving the dominant contributions to the dipole moments. However, both preserve P and T and thus the corresponding contributions to (1.2) must conserve these symmetries as well. A glimpse at (1.5) tells us that both the magnetic and the electric dipole moment are proportional to the spin vector $\boldsymbol{\sigma}$ which transforms as an axial vector. Thus, on the one hand, both $\boldsymbol{\mu}_m$ and $\mathbf{d}_e$ are axial vectors. On the other hand, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ transform as a vector (polar vector) and an axial vector, respectively. An axial vector changes sign under T but not under P, while a vector changes sign under P but not under T. We observe that to the extent that P and/or T are conserved only the magnetic term $-\boldsymbol{\mu}_m \cdot \mathbf{B}$ is allowed while an electric dipole term $-\mathbf{d}_e \cdot \mathbf{E}$ is forbidden and hence we must have $\eta = 0$ in (1.5). Since the weak interactions violate parity maximally, weak contributions cannot be excluded by the parity argument. However, T (by the CPT–theorem equivalent to CP) is also violated by the weak interactions, but only via fermion family mixing in the Yukawa sector of the SM (see below). It turns out that, at least for light particles like the known leptons, effects are much smaller. So electric dipole moments are suppressed by approximate T invariance at the level of second order weak interactions (for a theoretical review see [36]).

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[8]In some cases particle and antiparticle although of different flavor (fermion species) may have the same conserved quantum numbers and mix. Examples of such mixing phenomena are $K^0 - \bar{K}^0$–oscillations or $B^0 - \bar{B}^0$–oscillations. The time evolution of the neutral Kaon system, for example, is described by

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = H \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} , \quad H \equiv M - \frac{\mathrm{i}}{2}\Gamma$$

where M and Γ are Hermitian 2×2 matrices, the mass and the decay matrices. The corresponding eigenvalues are $\lambda_{L,S} = m_{L,S} - \frac{\mathrm{i}}{2}\gamma_{L,S}$. CPT invariance in this case requires the diagonal elements of $\mathcal{M}$ to be equal. In fact $|m_{K^0} - m_{\bar{K}^0}|/m_{\mathrm{average}} < 6 \times 10^{-19}$ (90% C.L.) provides the best test of CPT, while the mass eigenstates K_L and K_S exhibit a mass difference $\Delta m = m_{K_L} - m_{K_S} = 3.484 \pm 0.006 \times 10^{-12}$ MeV.

In fact experimental bounds tell us that they are very tiny. The previous best limit $|d_e| < 1.6 \times 10^{-27}\, e \cdot \text{cm}$ at 90% C.L. [37] has been superseded recently by [38][9]

$$|d_e| < 8.7 \times 10^{-29}\, e \cdot \text{cm at } 90\%\, \text{C.L.} \tag{1.8}$$

This will also play an important role in the interpretation of the $g-2$ experiments as we will see later. The planned J-PARC muon $g-2$ experiment will also provide a new dedicated experiment for measuring the muon electric dipole moment [9, 39].

As already mentioned, the anomalous magnetic moment of a lepton is a dimensionless quantity, a pure number, which may be computed order by order as a perturbative expansion in the fine structure constant α in QED, and beyond QED, in the SM of elementary particles or extensions of it. As an effective interaction term an anomalous magnetic moment is induced by the interaction of the lepton with photons or other particles. It corresponds to a dimension 5 operator and since a renormalizable theory is constrained to exhibit terms of dimension 4 or less only, such a term must be absent for any fermion in any renormalizable theory at tree level. It is the absence of such a possible *Pauli term* that leads to the prediction $g = 2 + O(\alpha)$. On a formal level it is the requirement of renormalizability which forbids the presence of a Pauli term in the Lagrangian defining the theory (see Sect. 2.4.2).

In 1956 a_e was already well measured by Crane et al. [40] and Berestetskii et al. [41] pointed out that the sensitivity of a_ℓ to short distance physics scales like

$$\frac{\delta a_\ell}{a_\ell} \sim \frac{m_\ell^2}{\Lambda^2} \tag{1.9}$$

where Λ is a UV cut–off characterizing the scale of new physics. It was therefore clear that the anomalous magnetic moment of the muon would be a much better probe for possible deviations from QED. However, parity violation of weak interaction was not yet known at that time and nobody had an idea how to measure a_μ.

As already discussed at the beginning of this introduction, the origin of the vastly different behavior of the three charged leptons is due to the very different masses m_ℓ, implying completely different lifetimes $\tau_e = \infty$, $\tau_\ell = 1/\Gamma_\ell \propto 1/G_F^2 m_\ell^5$ ($\ell = \mu, \tau$) and vastly different decay patterns. G_F is the Fermi constant, known from weak radioactive decays. In contrast to muons, electrons exist in atoms which opens the possibility to investigate a_e directly via the spectroscopy of atoms in magnetic fields. This possibility does not exist for muons.[10] However, Crane et al. [40] already used a different method to measure a_e. They produced polarized electrons by shooting high–energy electrons on a gold foil. The part of the electron bunch which is scattered at right angles, is partially polarized and trapped in a magnetic field, where spin precession takes place for some time. The bunch is then released from the trap and allowed to strike a second gold foil, which allows one to analyze the polarization

[9]The unit $e \cdot \text{cm}$ is the dipole moment of an e^+e^-–pair separated by 1cm. Since $d = \frac{\eta}{2}\frac{e\hbar c}{2mc^2}$, the conversion factor needed is $\hbar c = 1.9733 \cdot 10^{-11}$ MeV cm and $e = 1$.

[10]We discard here the possibility to form and investigate muonic atoms.

and to determine a_e. Although this technique is in principle very similar to the one later developed to measure a_μ, it is obvious that in practice handling the muons in a similar way is not possible. One of the main questions was: how is it possible to polarize such short lived particles like muons?

After the proposal of parity violation in weak transitions by Lee and Yang [42] in 1957, it immediately was realized that muons produced in weak decays of the pion ($\pi^+ \to \mu^+ +$ neutrino) should be longitudinally polarized. In addition, the decay positron of the muon ($\mu^+ \to e^+ + 2$ neutrinos) could indicate the muon spin direction. This was confirmed by Garwin, Lederman and Weinrich [43] and Friedman and Telegdi [44].[11] The first of the two papers for the first time determined $g_\mu = 2.00$ within 10% by applying the muon spin precession principle (see Chap. 6). Now the road was free to seriously think about the experimental investigation of a_μ.

It should be mentioned that at that time the nature of the muon was quite a mystery. While today we know that there are three lepton–quark families with identical basic properties except for differences in masses, decay times and decay patterns, at these times it was hard to believe that the muon is just a heavier version of the electron ($\mu - e$–puzzle). For instance, it was expected that the μ exhibited some unknown kind of interaction, not shared by the electron, which was responsible for the much higher mass. So there was plenty of motivation for experimental initiatives to explore a_μ.

The big interest in the muon anomalous magnetic moment was motivated by Berestetskii's argument of dramatically enhanced short distance sensitivity. As we will see later, one of the main features of the anomalous magnetic moment of leptons is that it mediates helicity flip transitions. The *helicity* is the projection of the spin vector onto the momentum vector which defines the direction of motion and the velocity. If the spin is parallel to the direction of motion the particle is right–handed, if it is antiparallel it is called left–handed.[12] For massless particles the helicities would be conserved by the SM interactions and helicity flips would be forbidden. For massive particles helicity flips are allowed and their transition amplitude is proportional to the mass of the particle. Since the transition probability goes with the modulus square of the amplitude, for the lepton's anomalous magnetic moment this implies, generalizing (1.9), that quantum fluctuations due to heavier particles or contributions from higher energy scales are proportional to

$$\frac{\delta a_\ell}{a_\ell} \propto \frac{m_\ell^2}{M^2} \qquad (M \gg m_\ell) \;, \tag{1.10}$$

where M may be

[11] The latter reference for the first time points out that P and C are violated simultaneously, in fact P is maximally violated while CP is to very good approximation conserved in this decay.

[12] Handedness is used here in a naive sense of the "right–hand rule". Naive because the handedness defined in this way for a massive particle is frame dependent. The proper definition of handedness in a relativistic QFT is in terms of the chirality (see Sect. 2.2). Only for massless particles the two different definitions of handedness coincide.

- the mass of a heavier SM particle, or
- the mass of a hypothetical heavy state beyond the SM, or
- an energy scale or an ultraviolet cut–off where the SM ceases to be valid.

On one hand, this means that the heavier the new state or scale the harder it is to see (it decouples as $M \to \infty$). Typically, the best sensitivity we have for nearby new physics, which has not yet been discovered by other experiments. On the other hand, the sensitivity to "new physics" grows quadratically with the mass of the lepton, which means that the interesting effects are magnified in a_μ relative to a_e by a factor $(m_\mu/m_e)^2 \sim 4 \times 10^4$. This is what makes the anomalous magnetic moment of the muon a_μ the predestinated "monitor for new physics". By far the best sensitivity we have for a_τ the measurement of which however is beyond present experimental possibilities, because of the very short lifetime of the τ.

The first measurement of the anomalous magnetic moment of the muon was performed at Columbia in 1960 [45] with a result $a_\mu = 0.00122(8)$ at a precision of about 5%. Shortly after in 1961, the first precision determination was possible at the CERN cyclotron (1958–1962) [46, 47]. Surprisingly, nothing special was observed within the 0.4% level of accuracy of the experiment. It was the first real evidence that the muon was just a heavy electron. In particular this meant that the muon was point–like and no extra short distance effects could be seen. This latter point of course is a matter of accuracy and the challenge to go further was evident.

The idea of a muon storage rings was put forward next. A first one was successfully realized at CERN (1962–1968) [48–50]. It allowed one to measure a_μ for both μ^+ and μ^- at the same machine. Results agreed well within errors and provided a precise verification of the CPT theorem for muons. An accuracy of 270 ppm was reached and an insignificant 1.7 σ (1 σ = 1 Standard Deviation (SD)) deviation from theory was found. Nevertheless the latter triggered a reconsideration of theory. It turned out that in the estimate of the three–loop $O(\alpha^3)$ QED contribution the leptonic light–by–light scattering part (dominated by the electron loop) was missing. Aldins et al. [51] then calculated this and after including it, perfect agreement between theory and experiment was obtained.

One also should keep in mind that the first theoretical successes of QED predictions and the growing precision of the a_e experiments challenged theoreticians to tackle the much more difficult higher order calculations for a_e as well as for a_μ. Soon after Schwinger's result Karplus and Kroll 1949 [52] calculated the two–loop term for a_e. In 1957, shortly after the discovery of parity violation and a first feasibility proof in [43], dedicated experiments to explore a_μ were discussed. This also renewed the interest in the two–loop calculation which was reconsidered, corrected and extended to the muon by Sommerfield [53] and Petermann [54], in the same year. Vacuum polarization insertions with fermion loops with leptons different from the external one were calculated in [55, 56]. About 10 years later with the new generation of $g-2$ experiments at the first muon storage ring at CERN $O(\alpha^3)$ calculations were started by Kinoshita [57], Lautrup and de Rafael [58] and Mignaco and Remiddi [59]. It then took about 30 years until Laporta and Remiddi [60] found a final analytic result in 1996. Many of these calculations would not have been possible without the pioneering *computer algebra* programs, like `ASHMEDAI` [61], `SCHOONSHIP`

[62, 63] and REDUCE [64]. More recently Vermaseren's FORM [65] package evolved into a standard tool for large scale calculations. Commercial software packages like MACSYMA or the more up–to–date ones MATEMATICA and MAPLE, too, play an important role as advanced tools to solve difficult problems by means of computers. Of course, the dramatic increase of computer performance and the use of more efficient computing algorithms have been crucial for the progress achieved. In particular calculations like the ones needed for $g-2$ had a direct impact on the development of these computer algebra systems.

In an attempt to overcome the systematic difficulties of the first a second *muon storage ring* was built (1969–1976) [66, 67]. The precision of 7 ppm reached was an extraordinary achievement at that time. For the first time the m_μ^2/m_e^2–enhanced hadronic contribution came into play. Again no deviations were found. With the achieved precision the muon $g-2$ remained a benchmark for beyond the SM theory builders ever since. Only 20 years later the BNL experiment E821, again a muon storage ring experiment, was able to set new standards in precision. Now, at the present level of accuracy the complete SM is needed in order to be able to make predictions at the appropriate level of precision. As already mentioned, at present further progress is hampered by the difficulties to include properly the non–perturbative strong interaction part. At a certain level of precision *hadronic effects* become important and we are confronted with the question of how to evaluate them reliably. At low energies QCD gets strongly interacting and a perturbative calculation is not possible. Fortunately, analyticity and unitarity allow us to express the leading hadronic vacuum polarization (HVP) contributions via a dispersion relation (analyticity) in terms of experimental data [68]. The key relation here is the optical theorem (unitarity) which determines the imaginary part of the vacuum polarization amplitude through the total cross section for electron–positron annihilation into hadrons. First estimations were performed in [69–71] after the discovery of the ρ– and the ω–resonances,[13] and in [74], after first e^+e^- cross–section measurements were performed at the colliding beam machines VEPP-2 and ACO in Novosibirsk [75] and Orsay [76], respectively. One drawback of this method is that now the precision of the theoretical prediction of a_μ is limited by the accuracy of experimental data. We will say more on this later on.

The success of the CERN muon anomaly experiment and the progress in the consolidation of the SM, together with given possibilities for experimental improvements, were a good motivation for Vernon Hughes and other interested colleagues to push for a new experiment at Brookhaven. There the intense proton beam of the Alternating Gradient Synchrotron (AGS) was available which would allow to increase the statistical accuracy substantially [77]. The main interest was a precise test of the electroweak contribution due to virtual W and Z exchange, which had been calculated immediately after the renormalizability of the SM had been settled

[13]The ρ is a $\pi\pi$ resonance which was discovered in pion nucleon scattering $\pi^- + p \to \pi^-\pi^0 p$ and $\pi^- + p \to \pi^-\pi^+ n$ [72] in 1961. The neutral ρ^0 is a tall resonance in the $\pi^+\pi^-$ channel which may be directly produced in e^+e^-–annihilation and plays a key role in the evaluation of the hadronic contributions to $a_\mu^{\rm had}$. The ρ contributes about 70% to $a_\mu^{\rm had}$ which clearly demonstrates the non–perturbative nature of the hadronic effects. Shortly after the ρ also the ω–resonance was discovered as a $\pi^+\pi^0\pi^-$ peak in proton–antiproton annihilation $p\bar{p} \to \pi^+\pi^+\pi^0\pi^-\pi^-$ [73].

in 1972 [78]. An increase in precision by a factor of 20 was required for this goal. On the theory side the ongoing discussion motivated, in the early 1980's already, Kinoshita and his collaborators to start the formidable task to calculate the $O(\alpha^4)$ contribution with 891 four–loop diagrams. The direct numerical evaluation was the only promising method to get results within a reasonable time. Early results [79, 80] could be improved continuously [81] and culminated in 2012 with the first complete $O(\alpha^5)$ calculation for both the electron [82] and the muon [83] $g-2$ (involving 12672 five-loop diagrams). Very recently Laporta [84] has been able to obtain a quasi–exact 4–loop result for the 891 universal diagrams, which improves the electron $g-2$ essentially. Increasing computing power was and still is a crucial factor in this extreme project. Beyond the full analytic $O(\alpha^3)$ calculation, only a subset of diagrams are known analytically (see Sect. 4.1 for many more details and a more complete list of references). The size of the $O(\alpha^4)$ contribution is about 6 σ's in terms of the present experimental accuracy and thus mandatory for the interpretation of the experimental result. The improvement achieved with the evaluation of the $O(\alpha^5)$ term, which itself is about 0.07 σ's only, resulted in a substantial reduction of the uncertainty of the QED contribution.

A general problem in electroweak precision physics are the higher order contributions from hadrons (quark loops) at low energy scales. While leptons primarily exhibit the fairly weak electromagnetic interaction, which can be treated in perturbation theory, the quarks are strongly interacting via confined gluons where any perturbative treatment breaks down. Considering the lepton anomalous magnetic moments one distinguishes three types of non-perturbative corrections: **(a)** Hadronic Vacuum Polarization (HVP) of order $O(\alpha^2)$, $O(\alpha^3)$, $O(\alpha^4)$; **(b)** Hadronic Light-by-Light (HLbL) scattering at $O(\alpha^3)$; **(c)** hadronic effects at $O(\alpha G_F m_\mu^2)$ in 2-loop electroweak (EW) corrections, in all cases quark-loops appear as hadronic "blobs". The hadronic contributions are limiting the precision of the predictions.

As mentioned already before, the evaluation of non-perturbative hadronic effects is possible by using experimental data in conjunction with Dispersion Relations (DR), by low energy effective modeling via a Resonance Lagrangian Approach (RLA) (Vector Meson Dominance (VMD) implemented in accord with chiral structure of QCD) [85–87], like the Hidden Local Symmetry (HLS) or the Extended Nambu Jona-Lasinio (ENJL) models, or by lattice QCD. Specifically: **(a)** HVP via a dispersion integral over $e^+e^- \to \text{hadrons}$ data (1 independent amplitude to be determined by one specific data channel) (see e.g. [88, 89]), by the HLS effective Lagrangian approach [90], or by lattice QCD [91–95]; **(b)** hadronic Light-by-Light (HLbL) scattering effects via a RLA together with operator product expansion (OPE) methods [96–99], by a dispersive approach using $\gamma\gamma \to \text{hadrons}$ data (19 independent amplitudes to be determined by as many independent data sets in principle) [100, 101] or by lattice QCD [102]; **(c)** EW quark-triangle diagrams are well under control, because the possible large corrections are related to the Adler-Bell-Jackiw (ABJ) anomaly which is perturbative and non-perturbative at the same time. Since VVV $= 0$ by the Furry theorem, only VVA (of $\gamma\gamma Z$ -vertex, V $=$ vector, A $=$ axialvector) contributes. In fact leading effects are of short distance type

(M_Z mass scale) and cancel against lepton-triangle loops (anomaly cancellation) [103, 104].

In the early 1980's the hadronic contributions were known with rather limited accuracy only. Much more accurate e^+e^-–data from experiments at the electron positron storage ring VEPP-2M at Novosibirsk allowed a big step forward in the evaluation of the leading hadronic vacuum polarization effects [80, 105, 106] (see also [107]). A more detailed analysis based on a complete up–to–date collection of data followed about 10 years later [88]. Further improvements were possible thanks to new hadronic cross section measurements by BES-II [108] (BEPC ring) at Beijing and by CMD-2 [109] at Novosibirsk. A new approach of cross section measurements via the radiative return or initial state radiation (ISR) mechanism, pioneered by the KLOE Collaboration [110] (DAΦNE ring) at Frascati, started to provide high statistics data at about the time when Brookhaven stopped their muon $g-2$ experiment. The results are in fair agreement with the later CMD-2 and SND data [111, 112]. In the meantime ISR data for the dominating $\pi^+\pi^-$ channel have been collected by KLOE [113–115] at the ϕ factory by BaBar at the B factory [116] and a first measurement by BES-III [117] at the BEPCII collider. Still one of the main issue in HVP are hadronic cross-sections in the region 1.2 to 2.4 GeV, which actually has been improved dramatically by the exclusive channel measurements by BaBar in the past decade (see [118] and references therein). The most important 20 out of more than 30 channels are measured, many known at the 10 to 15% level. The exclusive channel therefore has a much better quality than the very old inclusive data from Frascati. Attempts to include τ spectral functions via isospin relations will be discussed in Sect. 5.1.10.

The physics of the anomalous magnetic moments of leptons has challenged the particle physics community for more than 60 years now and experiments as well as theory in the meantime look rather intricate. For a long time a_e and a_μ provided the most precise tests of QED in particular and of relativistic local QFT as a common framework for elementary particle theory in general.

Of course it was the hunting for deviations from theory and the theorists speculations about "new physics around the corner" which challenged new experiments again and again. The reader may find more details about historical aspects and the experimental developments in the interesting review: "The 47 years of muon g-2" by Farley and Semertzidis [119].

Until about 1975 searching for "new physics" via a_μ in fact essentially meant looking for physics beyond QED. As we will see later, also standard model hadronic and weak interaction effect carry the enhancement factor $(m_\mu/m_e)^2$, and this is good news and bad news at the same time. Good news because of the enhanced sensitivity to many details of SM physics like the weak gauge boson contributions, bad news because of the enhanced sensitivity to the hadronic contributions which are very difficult to control and in fact limit our ability to make predictions at the desired precision. This is the reason why quite some fraction of the book will have to deal with these hadronic effects (see Chap. 5).

The pattern of lepton anomalous magnetic moment physics which emerges is the following: a_e is a quantity which is dominated by QED effects up to very high precision, presently at the .66 parts per billion (ppb) level! The sensitivity to hadronic and weak effects as well as the sensitivity to physics beyond the SM is very small. This allows for a very solid and model independent (essentially pure QED) high precision prediction of a_e [82, 84]. The very precise experimental value [120, 121] (at 0.24 ppb) and the very good control of the theory part in fact allows us to determine the fine structure constant α with the highest accuracy [121–123] in comparison with other methods (see Sect. 3.2.2). A very precise value for α of course is needed as an input to be able to make precise predictions for other observables like a_μ, for example. While a_e, theory wise, does not attract too much attention, although it required to push the QED calculation to $O(\alpha^5)$, a_μ is a much more interesting and theoretically challenging object, sensitive to all kinds of effects and thus probing the SM to much deeper level (see Chap. 4). Note that in spite of the fact that a_e has been measured about 2250 times more precisely than a_μ the sensitivity of the latter to "new physics" is still about 19 times larger. However, in order to use a_e as a monitor for new physics one requires the most precise a_e independent determination of α which comes from atomic interferometry [124] and is about a factor 5.3 less precise than the one based on a_e. Taking this into account a_μ is about a factor 43 more sensitive to new physics at present.

The experimental accuracy achieved in the past few years at BNL is at the level of 0.54 parts per million (ppm) and better than the accuracy of the theoretical predictions which are still obscured by hadronic uncertainties. A discrepancy at the 2 to 3 σ level persisted [125–127] since the first new measurement in 2000 up to the one in 2004 (four independent measurements during this time), the last for the time being (see Chap. 7). Again, the "disagreement" between theory and experiment, suggested by the first BNL measurement, rejuvenated the interest in the subject and entailed a reconsideration of the theory predictions. The most prominent error found this time in previous calculations concerned the problematic hadronic light–by–light scattering contribution which turned out to be in error by a sign [128]. The change improved the agreement between theory and experiment by about 1 σ. Problems with the hadronic e^+e^-–annihilation data used to evaluate the hadronic vacuum polarization contribution led to a similar shift in opposite direction, such that a discrepancy persists.

Speculations about what kind of effects could be responsible for the deviation will be presented in Sect. 7.2. With the advent of the Large Hadron Collider (LHC) the window of possibilities to explain the observed deviation by a contribution from a new heavy particle have substantially narrowed, such that the situation is rather puzzling at the time. No real measurement yet exists for a_τ. Bounds are in agreement with SM expectations[14] [129]. Advances in experimental techniques one day could promote a_τ to a new "telescope" which would provide new perspectives in exploring the short distance tail of the unknown real world, we are continuously hunting for. The point is that the relative weights of the different contributions are quite different for the τ in comparison to the μ.

[14]Theory predicts $(g_\tau - 2)/2 = 117721(5) \times 10^{-8}$; the experimental limit from the LEP experiments OPAL and L3 is $-0.052 < a_\tau < 0.013$ at 95% C.L.

In the meantime activities are expected to go on to improve the impressive level of precision reached by the muon $g-2$ experiment E821 at BNL. Since the error was still dominated by statistical errors rather than by systematic ones, further progress is possible in any case. But also new ideas to improve on sources of systematic errors play an important role for future projects. Plans for an upgrade of the Brookhaven experiment lead to a new experiment which presently is realized at Fermilab. The muon storage ring will be the same and has been moved to the new location some time ago, most of the other elements like production and injection of the polarized muons as well as the detection of the muon decay electrons will be new. An alternative project designed to work with ultra-cold muon is being buildup at J-PARC in Japan. The new experiments are expected to be able to improve the accuracy by a factor of 5 or so [2–5]. For the theory such improvement factors are a real big challenge and require much progress in our understanding of non–perturbative strong interaction effects. In addition, challenging higher order computations have to be pushed further within the SM and beyond. Another important aspect: the large hadron collider LHC now in operation at CERN will certainly provide important hints about how the SM has to be completed by new physics. Progress in the theory of a_μ will come certainly in conjunction with projects to measure hadronic electron–positron annihilation cross–sections with substantially improved accuracy (see Sect. 7.4). These cross sections are an important input for reducing the hadronic vacuum polarization uncertainties which yield the dominating source of error at present. Although progress is slow, there is evident progress in reducing the hadronic uncertainties, most directly by progress in measuring the relevant hadronic cross-sections. Near future progress we expect from BINP Novosibirsk/Russia and from IHEP Beijing/China. Energy scan as well as ISR measurement of cross-sections in the region from 1.4 to 2.5 GeV are most important to reduce the errors to a level competitive with the factor 4 improvement achievable by the upcoming new muon $g-2$ experiments at Fermilab/USA and at J-PARC/Japan [5, 7–9]. Also BaBar data are still being analyzed and are important for improving the results. Promising is that lattice QCD evaluations come closer to be competitive. In any case there is good reason to expect also in future interesting promises of physics beyond the SM from this "crystal ball" of particle physicists.

Besides providing a summary of the status of the physics of the anomalous magnetic moment of the muon, the aim of this book is an introduction to the theory of the magnetic moments of leptons also emphasizing the fundamental principles behind our present understanding of elementary particle theory. Many of the basic concepts are discussed in details such that physicists with only some basic knowledge of quantum field theory and particle physics should get the main ideas and learn about the techniques applied to get theoretical predictions of such high accuracy, and why it is possible to measure anomalous magnetic moments so precisely.

Once thought as a QED test, today the precision measurement of the anomalous magnetic moment of the muon is a test of most aspects of the SM with the electromagnetic, the strong and the weak interaction effects and beyond, maybe supersymmetry is responsible for the observed deviation.

There are many excellent and inspiring introductions and reviews on the subject [130–148], which were very helpful in writing this book. A topical workshop held in 2014 at the Mainz Institute for Theoretical Physics (MITP) has been gathering people with new ideas to work on the improvement of the predictions of the hadronic contributions, in particular on the challenging hadronic light-by-light scattering problem. A short account of the topics discussed the reader may find in the "mini proceedings" [149]. It addresses the next steps required on the theory side to compete with the experimental progress to come.

For further reading I also recommend the reviews [150, 151], which are focusing on theory issues and the article [152], which especially reviews the experimental aspects in much more depth than this book. For a recent brief view into the future also see [153].

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