Dynamic Performance of Series Parallel Multi-state Systems with Standby Subsystems or Repairable Binary Elements

Gregory Levitin and Liudong Xing

Abstract This chapter presents a method for evaluating dynamic performance of multi-state systems with a general series parallel structure. The system components can be either repairable binary elements with given time-to-failure and repair time distributions, or 1-out-of-*N* warm standby configurations of heterogeneous binary elements characterized by different performances and time-to-failure distributions. The entire system needs to satisfy a random demand defined by a time-dependent distribution. Iterative algorithms are presented for determining performance stochastic processes of individual components. A universal generating function technique is implemented for evaluating the dynamic system performance indices. Examples are provided to demonstrate applications of the proposed methodology.

Keywords Multi-state system · Repair · Warm standby · Stochastic process · Instantaneous availability · Unsupplied demand

Acronyms

cdf	Cumulative distribution function
DSCTP	Discrete-state continuous-time process
MSS	Multi-state system
pdf	Probability density function
RBD	Reliability block diagram
UGF	Universal generating function (*u*-function)

G. Levitin (✉)
The Israel Electric Corporation, P.O. Box 10, 31000 Haifa, Israel
e-mail: levitin@iec.co.il

L. Xing
University of Massachusetts Dartmouth, 285 Old Westport Rd., Dartmouth, MA 02747, USA
e-mail: lxing@umassd.edu

A. Lisnianski et al. (eds.), *Recent Advances in Multi-state Systems Reliability*,
Springer Series in Reliability Engineering, DOI 10.1007/978-3-319-63423-4_8

Nomenclature

T	Mission time
I	Number of system components
N_i	Number of elements within component i
$G_i(t)$	Random performance of component i at time t
$g_{i,k}$	K-th possible performance level of component i
$\boldsymbol{g}_i$	Performance level vector of component i: $g_i = \{g_{i,0}, \ldots, g_{i,N_i}\}$
$p_{i,k}(t)$	Probability that component i operates at level $g_{i,k}$ at time t, *i.e.*, $\Pr(G_i(t) = g_{i,k})$
$\phi(G_1, \ldots, G_I)$	System structure function
$V(t)$	Random system performance at time t
$\boldsymbol{v}$	Vector of possible performance levels of MSS $v = \{v_0, \ldots, v_K\}$
$w_j(t)$	$\Pr(V(t) = v_j)$
D(t)	Random system demand at time t
$\boldsymbol{d}$	Vector of possible system demand levels: $\boldsymbol{d} = \{d_0, \ldots, d_L\}$
$h_l(t)$	$\Pr(D(t) = d_l)$
c(t)	Expected system performance at time t
C(T)	Expected amount of work system can complete in time T
$e(t)$	Expected instantaneous unsupplied demand at time t
E	Expected unsupplied demand over mission time T
$a(t)$	Expected instantaneous system availability at time t
A	Expected system availability over mission time T
θ	Predetermined amount of work to be completed by system,
$\tau\ (\theta)$	Expected time of completing amount of work θ
$s_i(k)$	Index determining type of element that should be activated after the $(k-1)$th element failure in component i
T_k	Random variable representing the time when the last element from sequence $s_i(1), \ldots, s_i(k)$ fails
$q_k(t)$	*pdf* of random variable T_k
$F_{i,j}(t)$, $f_{i,j}(t)$	*cdf, pdf* of lifetime of element j within component i in the operation mode
$\omega_{i,j}$	Nominal performance of element j within standby component i
δ_{ij}	Deceleration factor of element j within component i
$\eta_{i,j}$, $\beta_{i,j}$	Scale, shape parameters of Weibull time-to-failure distribution for element j within component i
$\gamma_i^{\min}$, $\gamma_i^{\max}$	Minimum, maximum repair time of element i
J_i	Maximal number of failures of element i
π_i	Repair efficiency of element i
ω_i	Nominal performance of repairable element i
$\zeta_i(t)$	Hazard rate of element i
X_j	Random time spent by element in operation mode before the j-th failure

$Q_j(t, x)$	Joint distribution of j-th failure event parameters
Γ	Random repair time
$\Psi_i(t)$, $\psi_i(t)$	*Cdf*, *pdf* of random repair time
$r_{ij}(t)$	Probability that element i is under repair after the j-th failure at time t

1 Introduction

Many real-world systems, such as those with shared loads, performance degradation, standby sparing, imperfect coverage, or limited repair resources are multi-state systems (MSSs) [1, 2]. In MSSs, the system and/or its components can exhibit multiple different states or performance levels [3]. MSSs abound in applications including (but not limited to) medical systems [4], power systems [5, 6], computing systems [7], communication systems [8], and transportation systems [9, 10]. Due to their critical applications, the MSS modeling and analysis have attracted lots of research efforts. Diverse types of methods have been developed for MSS analysis including for example, multi-state path/cut-vector based enumerations [3, 11], simulations [12], branch-and-bound technique [13], *Lz*-transform techniques based on Markov processes [14–16], universal generating functions (UGF) [3, 9, 17], and binary or multi-valued decision diagrams [1, 8, 18, 19].

This chapter focuses on a class of MSSs with the general series parallel structure. Different from the traditional structure of multi-state series parallel systems that has been intensively studied [9, 20–23], the system considered in this chapter contains an arbitrary combination of series and parallel configurations of system components. Each system component can be either a warm standby configuration of basic binary functional elements or a repairable binary element. In contrast, the traditional structure contains subsystems connected in a purely series configuration with each subsystem being a purely parallel configuration of functional components.

An iterative algorithm is first presented for determining the performance discrete-state continuous-time process (DSCTP) of an individual component in the considered system. A UGF-based technique is then applied for evaluating system-level performance indices of expected system availability and unsupplied demand. Note that the integrated DSCTP and UGF technique has been applied to model dynamic behavior of MSSs without general standby redundancies in [4–6, 14–16]. These works are based on the Markov process model, thus being limited to exponential element time-to-failure distributions. In this chapter we extend the dynamic MSS model to considering repairable elements with arbitrary repair time distributions and to considering warm standby components (or subsystems) composed of elements with non-identical, arbitrary time-to-failure distributions.

The rest of this chapter is organized as follows. Section 2 presents the generic model and performance metrics of the MSS considered. Section 3 presents algorithms for obtaining the performance DSCTP for system components consisting of either standby elements or repairable binary elements. Section 4 gives examples of the DSCTP evaluation. Section 5 describes the UGF technique for obtaining the DSCTP of the entire system performance based on DSCTPs of its components' performances. Section 6 presents illustrative examples of obtaining the system performance DSCTPs. Section 7 concludes the chapter and outlines the optimization problems that can be solved based on the presented methodology.

2 System Model and Performance Metrics

Two types of MSSs are considered in this chapter, both of which contain I s-independent components composing a general series parallel structure.

In the first type of MSSs, each component i consists of N_i non-repairable binary elements configured as a 1-out-of-N_i warm standby structure, where one element is online and functioning with the remaining elements being kept in a warm standby mode. In the case of the online element failing, according to a pre-defined sequence a standby element is activated to take over the task of the component. If the chosen standby element is not available (fails before it should be activated), the next element in the sequence is checked etc. Elements within the same component can be non-identical, characterized by different time-to-failure distributions and nominal performance rates. Thus, depending on the element functioning at the moment, the performance $G_i(t)$ of each component i can vary dynamically, and be modeled using a DSCTP.

In the second type of MSSs, each component consists of a single binary repairable element with known time-to-failure and repair time distributions as well as nominal performance. Depending on the status of the element at the instant of time t, the performance $G_i(t)$ of component i can dynamically change from zero (failure) to nominal (operation), which constitutes a DSCTP with two discrete states.

The entire system needs to meet a random demand that can be specified by a distribution depending on weather conditions, time of day, season, etc. The demand is also modeled using a DSCTP. In many applications the demand distribution is obtained empirically for specific time periods (times of day, seasons, parts of production cycle etc.)

The considered models are based on the following assumptions.

- The time-to failure and repair time distributions of elements are independent.
- Different components are statistically independent.
- The failure detection is perfect.
- The repair/replacement starts immediately after the failure.
- Specific elements have fixed performance during their operation.
- All the system elements are available in the beginning of the mission.

2.1 Generic Model of Series Parallel MSSs

For modelling behaviour of an MSS, it is necessary to analyse characteristics of its components. In the first type of MSSs, any system component i can assume $N_i + 1$ states, corresponding to different performance levels or rates. Specifically, component i's performance rate at a time instant t can be modelled using a discrete random variable $G_i(t) \in g_i = \{g_{i,0}, g_{i,1}, \ldots, g_{i,N_i}\}$. Let $p_{i,k}(t)$ be the probability that component i operates at level $g_{i,k}$ at time t, i.e., $p_{i,k}(t) = \Pr\{G_i(t) = g_{i,k}\}$. The two vectors $p_i(t) = \{p_{i,0}(t), p_{i,1}(t), \ldots, p_{i,N_i}(t)\}$ and $g_i = \{g_{i,0}, g_{i,1}, \ldots, g_{i,N_i}\}$ can determine the performance distribution of random variable $G_j(t)$ at any time instant t. The second type of MSSs can be considered as a special case of the above model, where $p_i(t) = \{p_{i,0}(t), p_{i,1}(t)\}$ and $g_i = \{g_{i,0}, g_{i,1}\} = \{0, \omega_i\}$ determining the DSCTP $G_i(t)$ of component i consisting of a single repairable binary-state element with nominal performance ω_i.

Performance rates of its constituent components unambiguously determine the performance rate of a system; their mapping relation can be defined by a function called system structure function $\phi(G_1, \ldots, G_I)$. The system structure function and probability mass functions (*pmf*) of performances of system elements at any time instant t give the generic model of the considered MSS, as shown in (1).

$$g_i, p_i(t), 1 \le i \le I, 1 \le t \le T,$$
$$V(t) = \phi(G_1(t), \ldots, G_I(t)), \quad V(t) \in \{v_1, \ldots, v_K\} \tag{1}$$

The DSCTP $V(t)$ in (1) can be determined by *pmf* of the entire system performance at t as

$$w_k(t) = \Pr\{V(t) = v_k\}, \text{ where } 0 \le k \le K. \tag{2}$$

The system must meet a random demand $D(t)$, defined by two vectors $\boldsymbol{d} = \{d_0, \ldots, d_L\}$ and $\boldsymbol{h}(t) = \{h_0(t), \ldots, h_L(t)\}$, where $h_l(t) = \Pr\{D(t) = d_l\}$ for $l = 0, 1, \ldots, L$.

2.2 MSS Dynamic Performance Metrics

Based on the DSCTPs of $V(t)$ and $D(t)$, the following dynamic performance metrics can be defined for the considered MSSs.

- The expected system performance at time t

$$c(t) = \sum_{k=0}^{K} v_k w_k(t) \tag{3}$$

- The expected amount of work the system can complete in time T

$$C(T) = \int_0^T c(t)dt = \int_0^T \sum_{k=0}^{K} v_k w_k(t)dt; \tag{4}$$

- The expected instantaneous system availability at time t

$$a(t) = \sum_{l=0}^{L} \left(h_l(t) \sum_{k=0}^{K} w_k(t) 1(d_l \leq v_k) \right); \tag{5}$$

- The expected system availability during the system mission time T

$$A = \frac{1}{T} \int_0^T a(t)dt = \frac{1}{T} \int_0^T \sum_{l=0}^{L} \left(h_l(t) \sum_{k=0}^{K} w_k(t) 1(d_l \leq v_k) \right) dt; \tag{6}$$

- The expected instantaneous unsupplied demand at time t

$$e(t) = \sum_{l=0}^{L} \left(h_l(t) \sum_{k=0}^{K} w_k(t) \max(0, d_l - v_k) \right) \tag{7}$$

- The expected total unsupplied demand during mission time T

$$E = \int_0^T e(t)dt = \int_0^T \sum_{l=0}^{L} \left(h_l(t) \sum_{k=0}^{K} w_k(t) \max(0, d_l - v_k) \right) dt. \tag{8}$$

- If the system must complete a predetermined amount of work θ, the expected mission time is

$$\tau(\theta) = \arg \left(\int_0^\tau \sum_0^K v_k w_k(t) dt = \theta \right). \tag{9}$$

To evaluate system performance metrics (3)–(9), an iterative algorithm is presented in Sect. 3, which is used for obtaining the description of DSCTP characterizing components' performances in the form of $\boldsymbol{g}_i$, $\boldsymbol{p}_i(t)$, $1 \leq i \leq I, 1 \leq t \leq T$ in (1).

Then a generalized reliability block diagram (RBD) method based on the UGF technique is implemented in Sect. 5 to derive the description of the DSCTP for the entire MSS performance in the form of (2).

3 Obtaining Performance DSCTP for Individual Components

The DSCTP $G_i(t)$ for each individual component i is derived for both types of MSSs in this section.

3.1 Performance DSCTP for Warm Standby Components

To determine the DSCTP $G_i(t)$ for 1-out-of-N_i standby component i in the first type of MSS, vector $p_i(t) = \{p_{i,0}(t), p_{i,1}(t), \ldots, p_{i,N_i}(t)\}$ is derived in this subsection while vector $g_i = \{g_{i,0}, g_{i,1}, \ldots, g_{i,N_i}\}$ is given as input parameters, where $g_{i,0} = 0$, $g_{i,k} = \omega_{i,s_i(k)}$ for $k = 1, \ldots, N_i$.

Let the order $s_i(1)$, $s_i(2)$, …, $s_i(N_i)$ determine the predetermined activation sequence of elements composing component i, T_k be a random variable modeling the time when the last element from sequence $s_i(1)$, …, $s_i(k)$ fails during the operation mode, and $q_k(t)$ be the probability density function (*pdf*) of this random variable. For $k = 1$, since only one element $s_i(1)$ belongs to the sequence, $q_1(t) = f_{i,s_i(1)}(t)$, where $f_{i,s_i(1)}(t)$ is the *pdf* of element $s_i(1)$.

With $q_{k-1}(t)$ and $f_{i,s_i(k)}(t)$ $q_k(t)$ can be derived for $k = 2$, …, N_i. Specifically, there exist two scenarios that can cause failure of the last element from sequence $s_i(1)$, …, $s_i(k)$ at time t.

1. Scenario 1: $T_k = T_{k-1} = t$. The last element from sequence $s_i(1)$, …, $s_i(k-1)$ fails at time t; element $s_i(k)$ fails earlier during the standby mode. This scenario can occur with probability $F_{i,s_i(k)}(t)$, where $F_{i,s_i(k)}(t)$ is the *cdf* of element $s_i(k)$.
2. Scenario 2: $T_k = t$, $T_{k-1} = t - \tau$. The last element from sequence $s_i(1)$, …, $s_i(k-1)$ fails at certain time before t, e.g., $t - \tau$ for $0 \le \tau \le t$; element $s_i(k)$ fails after spending $(t - \tau)$ in the standby mode and then working for time τ in the operation mode.

Based on the two scenarios, *pdf* of T_k is

$$q_k(t) = q_{k-1}(t)F_{i,s_i(k)}(t) + \int_0^t q_{k-1}(t-\tau)f_{i,s_i(k)}((t-\tau)\delta_{i,s_i(k)} + \tau)d\tau. \tag{10}$$

In (10), $0 \le \delta_{i,s_i(k)} \le 1$ represents a deceleration factor of element $s_i(k)$ within component i. Such a factor is utilized to reflect lower stresses experienced by the element during the warm standby mode than during the operation mode in the commonly-used cumulative exposure model [24]. Based on (10), $q_k(t)$ can be obtained iteratively for $k = 2$, …, N_i.

Next the derivation of vector $p_i(t) = \{p_{i,0}(t), p_{i,1}(t), \ldots, p_{i,N_i}(t)\}$ is given. The probability $p_{i,1}(t)$ that component i operates with performance level $\omega_{i,s_i(1)}$ provided by element $s_i(1)$ at time t is simply the probability that element $s_i(1)$ does not fail before time t, which is given as:

$$p_{i,1}(t) = 1 - F_{i,s_i(1)}(t). \tag{11}$$

The probability $p_{i,k}(t)$ that component i works with performance level $\omega_{i,s_i(k)}$ (i.e., element $s_i(k)$ is operational) at time t can be evaluated as the probability that $T_{k-1} = t - \tau$ for any $0 \leq \tau \leq t$ and element $s_i(k)$ waiting for time $t - \tau$ in the warm standby mode does not fail before spending at least time τ in the operation mode:

$$p_{i,k}(t) = \int_0^t q_{k-1}(t-\tau)\left[1 - F_{i,s_i(k)}(\delta_{i,s_i(k)}(t-\tau) + \tau)\right] d\tau \tag{12}$$

The probability that component i's performance is zero (i.e., all the elements of component i fail) at time t is thus

$$p_{i,0}(t) = 1 - \sum_{k=1}^{N_i} p_{i,k}(t) \tag{13}$$

3.2 *Performance DSCTP for Repairable Binary Elements*

To determine the DSCTP $G_i(t)$ for component i consisting of a single repairable element in the second type of MSSs, vector $\boldsymbol{p}_i(t) = \{p_{i,0}(t), p_{i,1}(t)\}$ is derived in this subsection, while vector $g_i = \{0, \omega_i\}$ is given as input parameters meaning that element i functions with nominal performance ω_i and has performance 0 while under repair.

It is assumed that the repair starts immediately when an element fails. The repair time of element i is dependent on external factors such as availability and efficiency of repair manpower and equipment. Assume the repair time of element i is randomly distributed within interval $[\gamma_i^{\min}, \gamma_i^{\max}]$ $(0 > \gamma_i^{\min} > \gamma_i^{\max} > \infty)$ with known *cdf* $\Psi_i(t)$ $(\Psi_i(t) \equiv 0$ for $t < \gamma_i^{\min}$, $\Psi_i(t) \equiv 1$ for $t > \gamma_i^{\max})$.

The number of repairs experienced by element i during considered mission time T cannot exceed $T/\gamma_i^{\min}$. Thus, the maximal number of failures that can be experienced by this element is $J_i = 1 + \lfloor T/\gamma_i^{\min} \rfloor$.

According to the repair model in [25], a coefficient π_i can be used to model the repair efficiency of element i. Particularly, π_i can vary from 0 corresponding to perfect repair (the element after repair is as good as new) to 1 corresponding to minimal repair (the element after repair is as bad as old). Under the model of [25], for element i with hazard rate $\zeta_i(t)$ before a repair, its hazard rate after the repair is $\zeta_i(\pi_i t_0 + t)$, where t_0 and t represent operation times of element i before and after the repair, respectively. The *pdf* $f_i^*(t_0, t)$ and *cdf* $F_i^*(t_0, t)$ of the time-to-failure of element i after the repair performed at time t_0 are, respectively,

$$f_i^*(t_0,t) = f_i(\pi_i t_0 + t)/[1 - F_i(\pi_i t_0)] \tag{14}$$

and

$$F_i^*(t_0,t) = [F_i(\pi_i t_0 + t) - F_i(\pi_i t_0)]/[1 - F_i(\pi_i t_0)] \tag{15}$$

For element i, consider a random event denoted by $\langle T_j, X_j \rangle$, meaning that the j-th failure of element i occurs at time T_j from the beginning of the mission after element i has spent time $X_j \leq T_j$ in the operation mode and additional time $T_j - X_j$ in repair. Each event $\langle T_j, X_j \rangle$ corresponds to the initiation of a repair action that takes random time Γ. For any realization of X_j, the time elapsed from the beginning of the mission T_j can range from $X_j + (j-1)\gamma_i^{\min}$ to $X_j + (j-1)\gamma_i^{\max}$, corresponding to the cases where the element spends minimal and maximal time in each of the $j - 1$ repairs, respectively.

Define $Q_j(t, x)$ as the joint distribution of random event parameters T_j and X_j. Because the element spends no time in repair before the first failure and $X_1 = T_1$

$$Q_1(t,x) = \begin{cases} f_i(t) & \text{if } (x = t) \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

The element in the second type of MSSs must have event transition from $\langle T_j, X_j \rangle$ to $\langle T_{j+1}, X_{j+1} \rangle$ with $T_{j+1} \geq T_j + \gamma_i^{\min}$ and $X_{j+1} \geq X_j$. Note that when element i fails immediately after the j-th repair, $X_{j+1} = X_j$ The event transition $\langle T_j, X_j \rangle \to \langle T_{j+1}, X_{j+1} \rangle$ happens when the element has functioned for time $\left(X_{j+1} - X_j\right)$ after a repair that takes time $\Gamma = \left(T_{j+1} - T_j\right) - \left(X_{j+1} - X_j\right)$. Because $\gamma_i^{\min} \leq \Gamma \leq \gamma_i^{\max}$, the condition (17) must hold to make the event transition possible.

$$T_{j+1} + X_j - X_{j+1} - \gamma_i^{\max} \leq T_j \leq T_{j+1} + X_j - X_{j+1} - \gamma_i^{\min} \tag{17}$$

With functions $Q_j(t,x)$, $\psi_i(t)$ and $f_i(t)$, $Q_{j+1}(t,x)$ can be obtained in a recursive manner for $j = 1, \ldots, J_i - 1$ as

$$\begin{aligned} Q_{j+1}(t,x) &= \int_0^x \int_{\max\left(\tilde{x}+(j-1)\gamma_i^{\min},\, t+\tilde{x}-x-\gamma_i^{\max}\right)}^{\min\left(\tilde{x}+(j-1)\gamma_i^{\max},\, t+\tilde{x}-x-\gamma_i^{\min}\right)} Q_j(\tilde{t},\tilde{x})\psi_i(t-\tilde{t}-x+\tilde{x}) f_i^*(\tilde{x}, x-\tilde{x})\, d\tilde{t}\, d\tilde{x} \\ &= \int_0^x \int_{\max\left(\tilde{x}+(j-1)\gamma_i^{\min},\, t+\tilde{x}-x-\gamma_i^{\max}\right)}^{\min\left(\tilde{x}+(j-1)\gamma_i^{\max},\, t+\tilde{x}-x-\gamma_i^{\min}\right)} Q_j(\tilde{t},\tilde{x})\psi_i(t-\tilde{t}-x+\tilde{x}) \frac{f_i(\pi_i\tilde{x}+x-\tilde{x})}{1-F_i(\pi_i\tilde{x})}\, d\tilde{t}\, d\tilde{x}. \end{aligned} \tag{18}$$

Note that for $t < x + j\gamma_i^{\min}$ or $t > x + j\gamma_i^{\max}$, $Q_{j+1}(t,x) = 0$.

The element can be under repair at time t since the mission beginning if the last failure occurred at time $t-\xi$ and the last repair took at least time ξ Hence, the probability that the element is under repair at time t after the occurrence of the j-th failure is

$$r_{ij}(t)=0 \text{ for } t < (j-1)\gamma_i^{\min} \tag{19}$$

$$r_{ij}(t)=\int_0^{\min(t,\gamma_i^{\max})}\int_{\max(0,t-\xi-(j-1)\gamma_i^{\max})}^{t-\xi-(j-1)\gamma_i^{\min}} Q_j(t-\xi,x)(1-\Psi_i(\xi))dxd\xi \text{ for } t \geq (j-1)\gamma_i^{\min} \tag{20}$$

Observe that for any $k \neq j$ $r_{ij}(t)$ and $r_{ik}(t)$ are probabilities of mutually disjoint events corresponding to different numbers of failures occurred before time t. Therefore the overall probability that the element i undergoes repair at time t can be obtained as sum of probabilities $r_{ij}(t)$ for all the possible numbers j of failure/repair events. Because the minimal time when the j-th element failure may occur is $(j-1)\gamma_i^{\min}$, the number of failures that can occur at time not later than t cannot exceed $1+t/\gamma_i^{\min}$. The overall occurrence probability that element i is under repair at time t is thus

$$p_{i,0}(t)=\sum_{j=1}^{\lfloor 1+t/\gamma_i^{\min}\rfloor} r_{ij}(t)=\sum_{j=1}^{\lfloor 1+t/\gamma_i^{\min}\rfloor}\int_0^{\min(t,\gamma_i^{\max})}\int_{\max(0,t-\xi-(j-1)\gamma_i^{\max})}^{t-\xi-(j-1)\gamma_i^{\min}} Q_j(t-\xi,x)(1-\Psi_i(\xi))dxd\xi. \tag{21}$$

For the binary repairable element i, $p_{i,1}(t)=1-p_{i,0}(t)$ which defines the component's instantaneous availability.

4 Examples of Component Performance Evaluation

4.1 Warm Standby Components

Consider a warm standby component denoted by component 1, consisting of three elements characterized by Weibull time-to-failure distributions. The scale (η_j) and shape (β_j) parameters of the distributions, deceleration factor (δ_j) and nominal performance (ω_j) of elements are presented in Table 1.

Figure 1 illustrates the performance level probabilities $p_{1,j}(t)=\Pr\{G_1(t)=\omega_j\}$ for two different element activation sequences 1, 2, 3 and 3, 2, 1. According to Table 1, the component performance can take three different values, $G_1(t) \in (20, 27, 32)$. Thus,

Table 1 Parameters of elements composing standby component 1

Element j	η_j	β_j	δ_j	ω_j
1	280	1.5	0.2	20
2	250	1.1	0.4	27
3	180	2	0.2	32

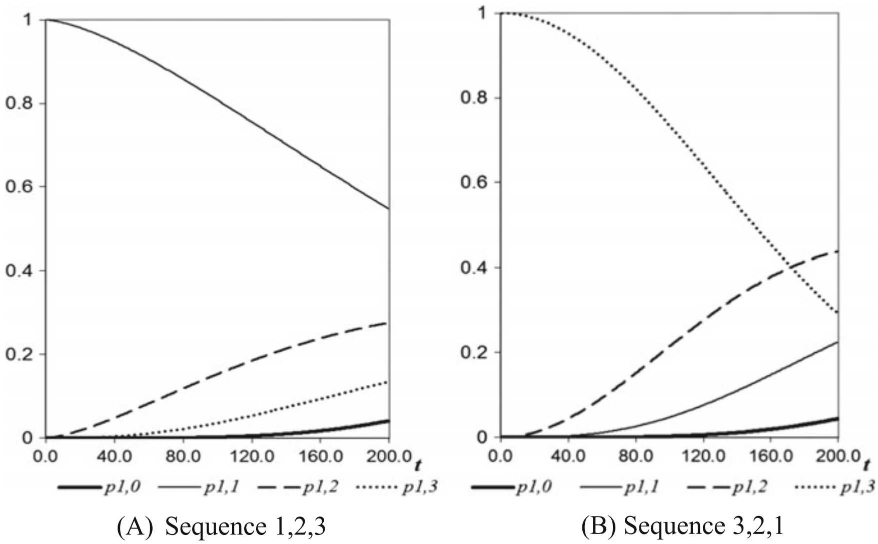

Fig. 1 Performance level probabilities $p_{1,j}(t)$ [27]. **a** Sequence 1, 2, 3 **b** Sequence 3, 2, 1

$$\Pr(G_1(t) \geq 20) = \Pr(G_1(t) > 0) = \Pr(G_1(t) = 20) + \Pr(G_1(t) = 27) + \Pr(G_1(t) = 32),$$
$$\Pr(G_1(t) \geq 27) = \Pr(G_1(t) = 27) + \Pr(G_1(t) = 32), \Pr(G_1(t) \geq 32) = \Pr(G_1(t) = 32).$$

Figure 2 shows the cumulative performance distribution $\Pr(G_1(t) \geq x)$ of component 1 under the two activation sequences. It can be observed that the probability $\Pr(G_1(t) \geq 32)$ is always larger for sequence 3, 2, 1 where element 3 with the greatest performance is activated first; the probability $\Pr(G_1(t) \geq 20)$ is always slightly larger for sequence 1, 2, 3 where the most reliable element 1 is activated first.

4.2 Repairable Binary Element

Consider a repairable element with a Weibull time-to-failure distribution having *cdf* of $F(t) = 1 - \exp\left[(-t/20)^2\right]$. The random repair time follows a truncated normal

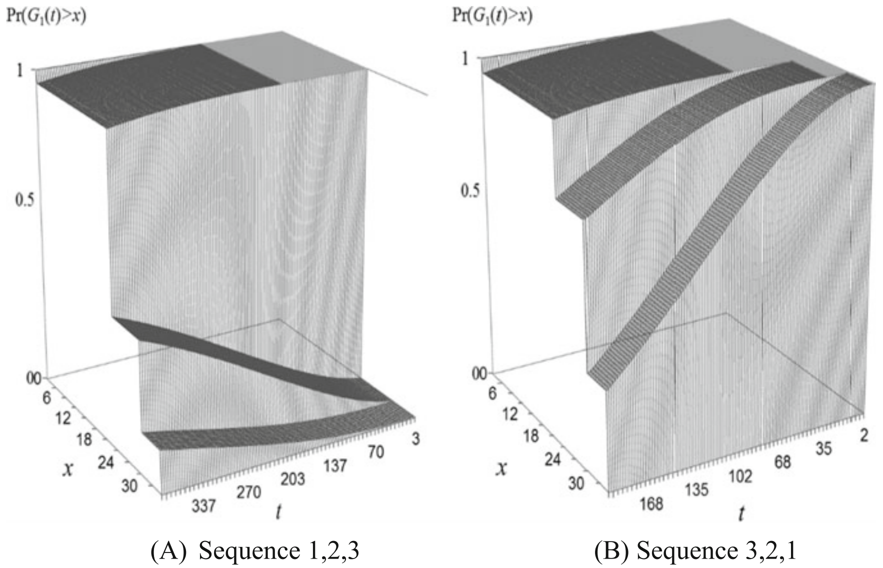

Fig. 2 Cumulative performance distributions $\Pr(G_1(t) \geq x)$ [27]. **a** Sequence 1, 2, 3 **b** Sequence 3, 2, 1

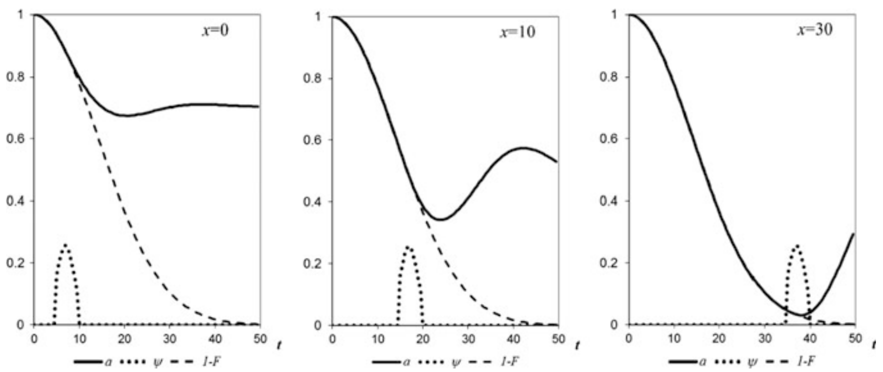

Fig. 3 Performance metrics of the example element for different repair time distributions

distribution with the following parameters: $\gamma^{\min} = x + 5, \gamma^{\max} = x + 10$, $\mu = x + 7$ (mean), $\sigma = 2$ (standard deviation). Figure 3 illustrates the reliability $1 - F(t)$, instantaneous availability $a(t)$ and repair time *pdf* $\psi(t)$ for $x = 0, x = 10$ and $x = 30$ of the element. The perfect repair is assumed, i.e., $\pi = 0$. As the value of x increases (i.e., the repair time increases), the element instantaneous availability reduces significantly.

5 Obtaining Performance DSCTP for Entire MSS5

5.1 UGF (U-Function) Technique

The polynomials in (22) give the u-function modeling the DSCTPs of random performance of *s*-independent component *i* at time *t*.

$$u_i(z,t) = \sum_{n_i=0}^{N_i} p_{i,n_i}(t) z^{g_{i,n_i}}. \tag{22}$$

The composition operator of (23) is used to obtain the *u*-function modeling the DSCTP of the system random performance $V(t)$ at time *t*.

$$\begin{aligned} U(z,t) &= \otimes_\phi(u_1(z,t), \ldots, u_I(z,t)) = \otimes_\phi\Big(\sum_{n_1=0}^{N_1} p_{1,n_1}(t) z^{g_{1,n_1}}, \ldots, \sum_{n_I=0}^{N_I} p_{I,n_I}(t) z^{g_{I,n_I}}\Big) \\ &= \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \cdots \sum_{n_n=0}^{N_I} \left(\prod_{i=1}^{I} p_{i,n_i}(t) z^{\phi(g_{1,n_1}, \ldots, g_{I,n_I})} \right) \end{aligned} \tag{23}$$

For each time instance *t*, the polynomial *U*(*z*, *t*) models all the possible disjoint combinations of realizations of *s*-independent random variables $G_1(t), \ldots, G_I(t)$ by relating the occurrence probability of each combination to the value of the structure function $\phi(G_1(t), \ldots, G_I(t))$ for this particular combination. This polynomial can eventually take the form of (24), representing performance distribution of the entire system at time *t*.

$$U(z,t) = \sum_{k=0}^{K} w_k(t) z^{v_k}, \tag{24}$$

With the MSS generic model in the form of (1), the system performance metrics (3)–(9) can be obtained through the following 3-step procedure.

1. Apply the *u*-function (22) to represent the *pmf* of random performance distribution of each component *i* at time *t*.
2. Apply the composition operator $\otimes_\phi$ (23) to obtain the *u*-function *U*(*z*, *t*) of the entire system random performance distribution *V*(*t*).
3. Evaluate metrics (3)–(9) based on *pmf* (2) modeled by the u-function *U*(*z*, *t*) (24).

Steps 1 and 3 are straightforward. Step 2 often involves complex computations because it can be difficult to derive the system structure functions. According to studies in [17], representing the structure function recursively can be beneficial for computation simplicity and derivation clarity. For a system with complex series parallel structure, its structure function can be represented as a composition of structure functions of the system's s-independent subsystems. Those subsystems contain only components configured in purely series or in purely parallel. During the aggregation process, the RBD method is commonly applied to distinguish recurrent subsystems and replace them with equivalent single components in a graphical representation of system structure, as detailed in Sect. 5.2.

5.2 Generalized RBD Method for Multi-state Series-Parallel System

For obtaining the u-function of a series parallel system, the composition operators is applied recursively to obtain u-functions of intermediate purely series or purely parallel subsystems using the following procedure.

1. Identify any pair of components (i and j) that are connected in parallel or in series in the considered MSS.
2. Obtain the u-function of each pair (i and j) by applying the composition operator $\otimes_\varphi$ over the u-functions of these two components:

$$U_{\{i,j\}}(z,t) = u_i(z,t) \otimes_\phi u_j(z,t) = \sum_{n_i=0}^{N_i} \sum_{n_j=0}^{N_j} p_{i,n_i}(t) p_{j,n_j}(t) z^{\phi(g_{i,n_i}, g_{j,n_j})}, \tag{25}$$

The function ϕ in (25) depends on the interaction nature between the two s-independent components' performances. For example for a production system with throughput being its performance metric, if components i and j operate in parallel, the sum of throughputs of the two components gives the total throughput, as determined in (26).

$$\phi\big(G_i(t), G_j(t)\big) = G_i(t) + G_j(t) \tag{26}$$

If two components process some material consecutively (i.e., forming a series connection), the performance of the component with the minimal performance (i.e., the bottleneck) determines the entire system throughput, as shown in (27).

$$\phi\big(G_i(t), G_j(t)\big) = \min\big(G_i(t), G_j(t)\big) \tag{27}$$

3. Replace the component pair with a single component that has the u-function determined in step 2.
4. If there are more than one component remained in the system, then return to step 1.

The final u-function obtained from the above algorithm models performance distribution of the entire series parallel system.

6 Examples of System Performance Evaluation

6.1 Systems with Warm Standby Components

Figure 4 illustrates an example of MSS with each component consisting of several binary elements configured in a warm standby subsystem. All elements have Weibull time-to-failure distributions with parameters presented in Table 2.

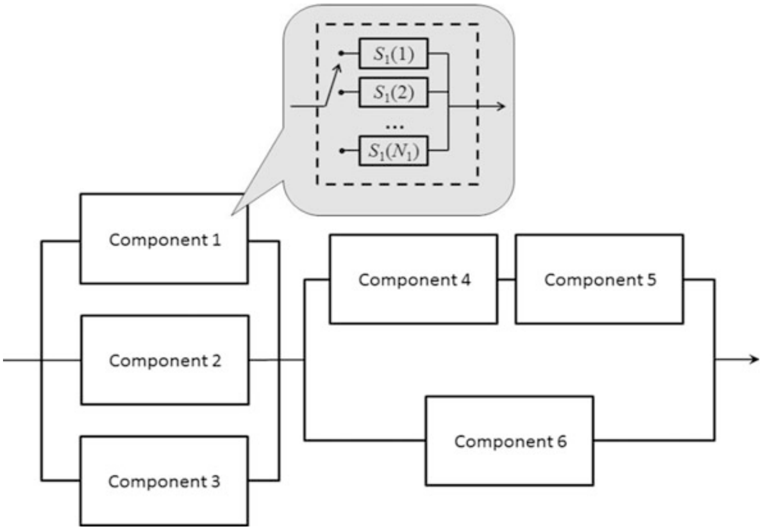

Fig. 4 Example series parallel MSS with warm standby components

Table 2 Parameters of elements composing components

Component i	Element j	$\eta_{i,j}$	$\beta_{i,j}$	$\delta_{i,j}$	$\omega_{i,j}$
1	1	280	1.5	0.2	20
	2	250	1.1	0.4	27
	3	180	2	0.2	32
2	1	300	1	0	18
	2	200	1.4	0	25
3	1	380	2.2	0.5	10
	2	360	1.8	0.7	12
	3	270	1.1	0.5	15
	4	210	1.1	0.6	17
4	1	400	1	0.2	37
	2	400	1	0.2	37
5	1	400	1.4	0.3	30
	2	540	1.2	0.1	40
6	1	380	1.1	0.2	35
	2	340	1.1	0.1	40
	3	280	1.4	0.1	45

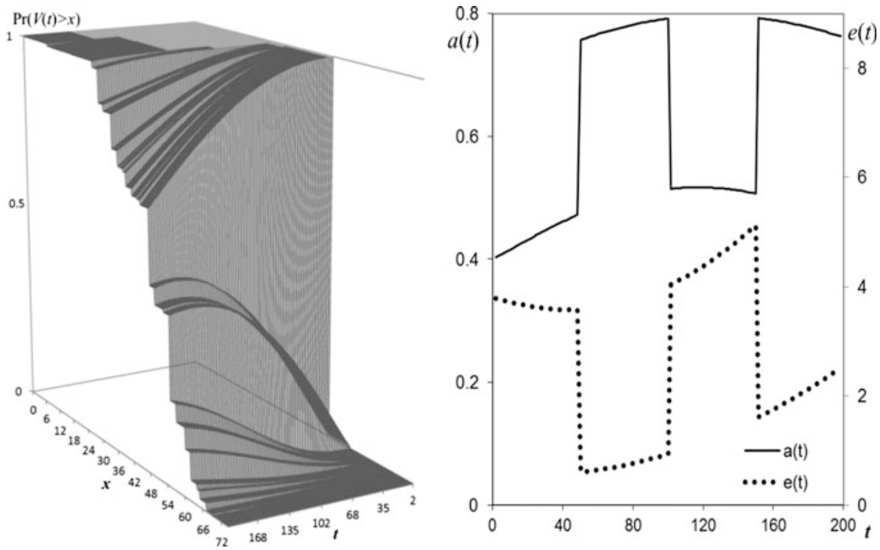

Fig. 5 Performance metrics of the example series parallel system with standby components

A random demand should be supplied by the system, which changes periodically with distribution as follows: $\boldsymbol{d} = \{63, 50, 46, 22\}$, for $t < 50$ and $100 < t < 150$ $\boldsymbol{h}(t) = \{0.2, 0.4, 0.4, 0\}$; for $50 \le t \le 100$ and $t \ge 150$, $\boldsymbol{h}(t) = \{0, 0.3, 0.2, 0.5\}$. The components' interaction corresponds to functions (26) and (27). The time of replacement by standby elements is negligible compared to the mission time $T = 200$ (days).

Assume the elements within each component are activated according to their numerical order. For mission time $T = 200$, the expected system availability is obtained as $A(T) = 0.629$ and the expected unsupplied demand is obtained as $E(T) = 539.9$. Figure 5 illustrates the system cumulative performance distribution $\Pr(V(t) \ge x)$, instantaneous availability $a(t)$ and unsupplied demand $e(t)$.

6.2 Systems with Repairable Binary Elements

Assume that the system presented in Fig. 4 consists of repairable binary elements with Weibull time-to-failure distribution parameters and performances presented in Table 3. Table 3 also gives repair efficiency coefficients π_j and parameters related to the truncated normal distributions of elements' repair time. The time horizon of

Table 3 Parameters of repairable elements

Element j	η_j	β_j	$\omega_{,j}$	π_j	$d_j^{\min}$	$d_j^{\max}$	μ_j	σ_j
1	60	2.0	33	0.30	15	20	17	2
2	78	1.1	22	0.70	10	40	25	100
3	90	1.0	19	0.80	28	48	32	5
4	75	1.1	48	0.00	30	40	35	2
5	60	1.0	45	0.20	5	15	10	6
6	80	2.0	33	0.00	10	15	12	100

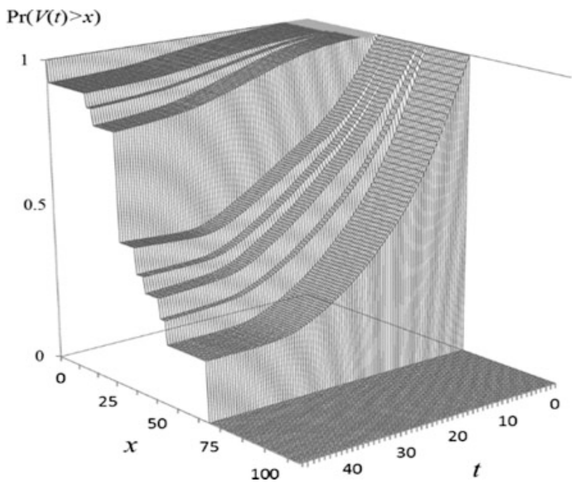

Fig. 6 Cumulative performance distribution Pr($V(t) \geq x$) of example series parallel system with repairable elements

interest is $T = 50$. The random system demand can take four different values $\boldsymbol{d} = \{70, 50, 40, 20\}$ and its distribution is $\boldsymbol{h}(t) = \{0.05, 0.25, 0.45, 0.25\}$, which does not change during T.

The performance metrics obtained for the considered system are $A = 0.6704$, $C = 2565$, $E = 267.3$ and the expected time needed to complete the amount of work $\theta = 1700$ is $\tau = 29.75$.

Figure 6 presents the system cumulative performance distribution $\Pr(V(t) \geq x)$ as a function of time. Figure 7 presents the instantaneous system performance metrics $\alpha(t), c(t)$ and $e(t)$.

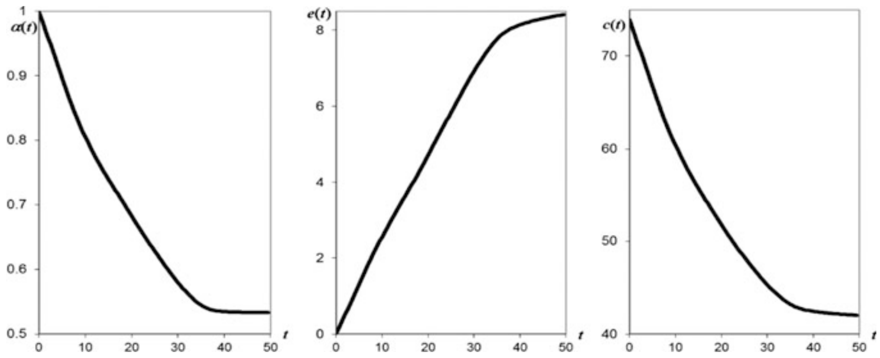

Fig. 7 System instantaneous performance metrics $\alpha(t)$, $c(t)$ and $e(t)$

7 Summary

This chapter demonstrates a methodology that extends the state-of-the-art in system modeling by considering multi-state series parallel systems with components subject to dynamic performance. Each system component can be either a heterogeneous warm standby configuration of binary elements or a repairable binary element. The entire system is considered being available if it can meet a pre-specified random system demand distribution. Iterative algorithms are described for determining dynamic stochastic performances of individual components. A generalized RBD method based on UGFs is implemented for analyzing expected system availability, performance and unsupplied demand over a specific mission time for the entire series parallel MSS.

The presented algorithms allow fast determination of system dynamic performance metrics. Based on these algorithms different optimization problems can be solved. For example, as shown in Sect. 4.1, standby element activation sequence can have significant impacts on component and thus system performance metrics. Therefore solving the following two problems can considerably improve system performance. The optimal operation problem finds the element activation sequence of each component maximizing system availability or minimizing unsupplied system demand. The optimal design problem finds both component structures and element activation sequences minimizing the total cost consisting of design and operation expenses. In addition, increased elements loading can on one hand improve the system performance; on the other hand, it can cause failures that are more frequent and, thus reduces the system availability. Hence, the element loading can be optimized to achieve a proper balance among different system performance metrics. Examples of solving the optimization problems based on the suggested methodology can be found in [26, 27].

References

1. Amari S, Xin L, Shrestha A, Akers J, Trivedi K (2010) Performability analysis of multi-state computing systems using multi-valued decision diagrams. IEEE Trans Comput 59(10):1419–1433
2. Peng R, Zhai Q, Xing L, Yang J (2014) Reliability of demand-based phased-mission systems subject to fault level coverage. Reliab Eng Syst Saf 121:18–25
3. Lisnianski A, Levitin G (2003) Multi-state system reliability. In: Assessment, optimization and applications. World Scientific, NJ
4. Frenkel I, Lisnianski A, Khvatskin L (2014) On the *Lz*-transform application for availability assessment of an aging multi-state water cooling system for medical equipment. In: Frenkel I, Lisnianski A, Karagrigoriou A, Kleiner A (eds) Applied reliability and risk analysis: probabilistic models and statistical inference. Wiley, New York, pp 60–77
5. Lisnianski A, Ben Haim H (2013) Short-term reliability evaluation for power stations by using *Lz*-transform. J Mod Pow Syst Clean Energy 1(2):110–117
6. Lisnianski A, Ding Y (2016) Using inverse *Lz*-transform for obtaining com-pact stochastic model of complex power station for short-term risk evaluation. Reliab Eng Syst Saf 145:19–27
7. Veeraraghavan M, Trivedi KS (1994) A combinatorial algorithm for performance and reliability analysis using multi-state models. IEEE Trans Comput 43:229–234
8. Zang X, Wang D, Sun H, Trivedi KS (2003) A BDD-based algorithm for analysis of multistate systems with multistate components. IEEE Trans Comput 52:1608–1618
9. Li Y, Peng R (2014) Availability modeling and optimization of dynamic multi-state series-parallel systems with random reconfiguration. Reliab Eng Syst Saf 127:47–57
10. Mo Y, Xing L, Amari S, Dugan J (2015) Efficient analysis of multi-state k-out-of-n systems. Reliab Eng Syst Saf 133:95–105
11. Ramirez-Marquez J, Coit D, Tortorella M (2006) A generalized multi-state-based path vector approach to multistate two-terminal reliability. IIE Trans 38(6):477–488
12. Zio E, Podofillini L (2003) Monte-Carlo simulation analysis of the effects on different system performance levels on the importance on multi-state components. Reliab Eng Syst Saf 82 (1):63–73
13. Levitin G, Amari S, Xing L (2013) Algorithm for reliability evaluation of non-repairable phased-mission systems consisting of gradually deteriorating multi-state elements. IEEE Trans Syst Man Cybern Syst 43(1):63–73
14. Frenkel I, Lisnianski A, Khvatskin L (2012) Availability assessment for aging refrigeration system by using *Lz*-transform. J Reliab Stat Stud 5(2):33–43
15. Lisnianski A (2012) *Lz*-transform for a discrete-state continuous-time Markov process and its applications to multi-state system reliability. In: Lisnianski A, Frenkel I (eds) Recent advances in system reliability. Signatures, multi-state systems and statistical inference. Springer, London, pp 79–96
16. Lisnianski A, Frenkel I, Khvatskin L (2015) On Birnbaum importance assessment for aging multi-state system under minimal repair by using *Lz*-transform method. Reliab Eng Syst Saf 142:258–266
17. Levitin G (2005) Universal generating function in reliability analysis and optimization. Springer, London
18. Xing L, Dai Y (2009) A new decision diagram based method for efficient analysis on multi-state systems. IEEE Trans Dependable Sec Comput 6(3):161–174
19. Xing L, Levitin G (2011) Combinatorial algorithm for reliability analysis of multi-state systems with propagated failures and failure isolation effect. IEEE Trans Syst Man Cybern Part A Syst Hum 41(6):1156–1165
20. Dao C, Zuo M, Pandey M (2014) Selective maintenance for multi-state series-parallel systems under economic dependence. Reliab Eng Syst Saf 121:240–249

21. Levitin G, Lisnianski A (1999) Joint redundancy and maintenance optimization for multistate series-parallel systems. Reliab Eng Syst Saf 64(1):33–42
22. Tian Z, Levitin G, Zuo M (2009) A joint reliability–redundancy optimization approach for multi-state series-parallel systems. Reliab Eng Syst Saf 94(10):1568–1576
23. Xiao H, Peng R (2014) Optimal allocation and maintenance of multi-state elements in series-parallel systems with common bus performance sharing. Comput Ind Eng 72:143–151
24. Levitin G, Xing L, Dai Y (2013) Optimal sequencing of warm standby elements. Comput Ind Eng 65(4):570–576
25. Lindqvist BH (2006) On the statistical modeling and analysis of repairable systems. Stat Sci 21(4):532–551
26. Levitin G, Xing L, Dai Y (2017) Optimal loading of series parallel systems with arbitrary element time-to-failure and time-to-repair distributions. Reliab Eng Syst Saf 164:34–44
27. Levitin G, Xing L, Dai Y (2016) Optimizing dynamic performance of multi-state systems with heterogeneous 1-out-of-N warm standby components". IEEE Trans Syst Man Cybern Syst (to appear). doi:10.1109/TSMC.2016.2633808