A New Approach for the Non-linear Analysis of the Deflection of Beams Using Lie Symmetry Groups

M. Amin Changizi[1], Ion Stiharu[2](✉), and Davut Erdem Sahin[3]

[1] Intelliquip, Co., Bethlehem, USA
machangizi@gmail.com
[2] Department of Mechanical and Industrial Engineering, Concordia University, 1455 de Maisonneuve Blvd. W, Montreal, QC H3G 1M8, Canada
istih@encs.concordia.ca
[3] Department of Mechanical and Industrial Engineering, Bozok University, Yozgat, Turkey
davuterdemsahin@gmail.com

Abstract. The linear formulation does not yield acceptable results when is used to the beams that experience large deflections. Further, the linear models could accommodate large deflections such as those encountered in some machinery where bending does not exceed three times the thickness of the beam. However, the defection of beams subjected to arbitrary loads that yield non-linear deflection has been solved so far only for two loading conditions; point moment and point force. The present work presents a general method based on Lie symmetry groups that yields an exact solution to the general problem involving any arbitrary loading.

Keywords: Lie group symmetry · Beam structures · General loading conditions · General solution · Static deflection

1 Introduction

Nonlinear deflection of beams under the various forces and boundary conditions has been widely studied. Differential equation of large deflection of cantilever beams under point force at tip was solved on 1944 [1]. In this approach the differential equation expressing the slope of beam versus the length of the deflected curve was formulated and solved based on complete second and first kind elliptic integrals. Differential equation of slope versus length of the deflected curve based on consideration of shear force was numerically solved [2]. The authors used finite difference methods to solve the ODE for distributed force on the cantilever and the simple supported beams. They also used the same method to solve ODE of simple supported beam under a point force. A numerical solution for the tapered cantilever beam under a point force at the tip was presented in 1968 [3]. The author converted the ODE to a non-dimensional ODE and used a computer to solve it. A cantilever beam made from materials exhibiting nonlinear properties and subjected to a point force was also studied [4]. The deflection equation was calculated based on Ludwick experimental strain-stress curve. The integral equation solved

G.I. Gheorghe (ed.), *Proceedings of the International Conference of Mechatronics and Cyber-MixMechatronics - 2017*, Lecture Notes in Networks and Systems 20,
DOI 10.1007/978-3-319-63091-5_17

numerically and the deflection at the end of the beam and the rotation were calculated. The same problem of large deflection of cantilever beams made from nonlinear materials under tip point force was solved by finite deference methods [5]. The authors solved numerically the nonlinear ODE of the curvature for a cantilever made from a material experiencing nonlinear characteristics and subjected to point force at the tip by. Power series and neural network were used to solve large deflection of a cantilever beam subjected to tip force [6]. Nonlinear ODE were decomposed to a system of ODEs and solved by neural networks. Large deflections of cantilever beams made from nonlinear elastic materials under uniform distributed forces and a point force at tip was also studied [7]. In this work a system of nonlinear ODEs was formulated to model the system which was solved by Runge-Kutta method. Researchers used in [8] almost the same method that was used in [1] to solve the large deflection of a cantilever under the point force at the tip and they validated their results with experiments. Also they used non-dimensional formulation to simplify the nonlinear deflection to linear analysis. They showed that nonlinear small deflections are same as those found through the linear analysis. Two dimensional loading of cantilever beams with point force loads at the free end was studied for non-prismatic and prismatic beams [9]. Authors formulated the model for the general loading conditions in beams. The result is a nonlinear PDE which is presented in this paper. Further, the authors numerically solved the non-dimensional equation using a polynomial to define the rotating angle of the beam. They presented few examples which if solved with their method. A cantilever beam subjected to a point moment at the tip made from nonlinear bimorph material was theoretically and numerically studied [10]. The authors used an exact solution for the deflection of a cantilever with a moment applied at the tip. Cantilever beams under uniform and tip point force were numerically and experimentally studied [11]. The authors used a system of ODEs to solve numerically this problem. Finite difference methods for analysis of large deflection of a non-prismatic cantilever beam subjected to different types of continuous and discontinuous loadings was studied [12]. Authors formulated the problem based on [9] and further used quasi-linearization central finite differences method to solve the problem. An explicit solution for large deflection of cantilever beams subjected to point force at the tip was obtained by using the homotopy analysis method (HAM) presented in [13]. Large deflection of a non-uniform spring-hinged cantilever beams under a follower point force at the tip was formulated and solved numerically [14]. In this paper stability of the beam was also studied.

2 Nonlinear Deflection of Beams

Deflection of a cantilever can be described by the following differential equation:

$$\frac{\frac{d^2y}{dx^2}}{(1+\left(\frac{dy}{dx}\right)^2)^{\frac{3}{2}}} = \frac{M(x)}{EI} \tag{1}$$

where:

E is Young modules of elasticity of the homogeneous material of the beam.
I is bending cross-section moment of inertia.
y is the current deflection.
x is the current coordinate along the beam (Fig. 1).

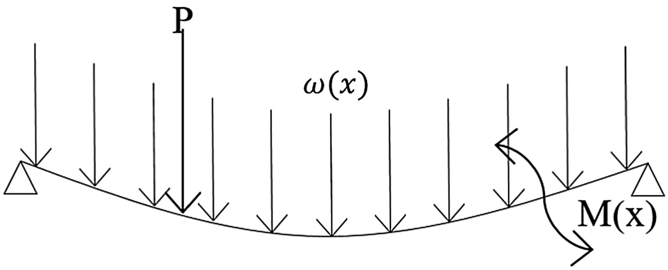

Fig. 1. Deflection of a beam under various kind of loads

3 Lie Symmetry of Large Deflection of Beams

Infinitesimal transformation can be defined according to [20] as following:

$$Xf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} \tag{2}$$

Where:

$$\xi(x, y) = \left.\frac{\partial \varphi}{\partial x}\right|_{\alpha=0} \quad \eta(x, y) = \left.\frac{\partial \varphi}{\partial y}\right|_{\alpha=0} \tag{3}$$

$$f = f(x, y) \tag{4}$$

X is an operator

It can be shown [22] that for a second order differential equation like:

$$\frac{d^2y}{dx^2} = \omega(x, y, \frac{dy}{dx}) \tag{5}$$

The transformation must satisfy the below equation:

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + \left(\eta_y - 2\xi_x - 3\xi_y y'\right)\omega \\ &\quad = \xi\omega_x + \eta\omega_y + (\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2)\omega_{y'} \end{aligned} \tag{6}$$

By decomposing (4) into a system of PDEs, ξ and η can be calculated. Also the transformation φ and ψ can be calculated from (3). One can consider the infinitesimal transformation as of the following form:

$$\begin{aligned} \xi &= C_1 + C_2 x + C_3 y \\ \eta &= C_4 + C_5 x + C_6 y \end{aligned} \tag{7}$$

$C_1, C_2, C_3, C_4, C_5, C_6$ are constant numbers. As most of Lie symmetries including rotation translation and scaling could be found with the above transformations. For Eq. (5) ω in implicit form is given by:

$$\frac{d^2y}{dx^2} = \omega\left(x, y, y'\right) = \frac{M(x)}{EI}(1 + \left(\frac{dy}{dx}\right)^2)^{\frac{3}{2}} \tag{8}$$

Substitution of (4) and (5) in (6) gives:

$$\begin{aligned} -M(x)C_6 + 2M(x)\left(\frac{dy}{dx}\right)^2 C_6 + 2M(x)C_2 - M(x)\left(\frac{dy}{dx}\right)^2 C_2 \\ + 3M(x)\frac{dy}{dx}C_3 \frac{dM(x)}{dx}C_1 + \frac{dM(x)}{dx}C_1\left(\frac{dy}{dx}\right)^2 \\ + \frac{dM(x)}{dx}C_2 + \frac{dM(x)}{dx}C_2 x\left(\frac{dy}{dx}\right)^2 + \frac{dM(x)}{dx}C_3 y \\ + \frac{dM(x)}{dx}C_3 y\left(\frac{dy}{dx}\right)^2 + 3M(x)C_5\frac{dy}{dx} = 0 \end{aligned} \tag{9}$$

This can be further written as:

$$\begin{aligned} &\left(-M(x)C_6 + 2M(x)C_2 + \frac{dM(x)}{dx}C_1 + \frac{dM(x)}{dx}C_2 x\right) \\ &\quad + \left(2M(x)C_6 - M(x)C_2 + \frac{dM(x)}{dx}C_1 + \frac{dM(x)}{dx}C_2 x + \frac{dM(x)}{dx}C_3 y\right) \\ &\quad \left(\frac{dy}{dx}\right)^2 + (3M(x)C_3 + 3M(x)C_5)\frac{dy}{dx} = 0 \end{aligned} \tag{10}$$

As all three parentheses must be zero, in first parenthesis coefficients of x and $\frac{dM(x)}{dx}$ are zero:

$$C_1 = 0, C_2 = 0 \tag{11}$$

$$C_6 = 0 \tag{12}$$

In the second parenthesis coefficient of y must be zero so:

$$C_3 = 0 \tag{13}$$

In the last parenthesis,C_5 must be zero in order to get a zero in both sides.

Therefore only $C_4 \neq 0$, so:

$$\eta = C_4 = 1 \tag{14}$$

Therefore:

$$Xf = \frac{\partial f}{\partial y} \tag{15}$$

Canonical coordinates can be calculated as [21]:

$$s(r,x) = \left(\int \frac{dx}{\xi(x, y(r,x))} \right) \Big|_{r=r(x,y)} \tag{16}$$

$r(x,y)$ is the solution of:

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} \tag{17}$$

One can show that:

$$\begin{aligned} r(x,y) &= x \\ s(x,y) &= y \end{aligned} \tag{18}$$

Any canonical coordinates must satisfy the following conditions [19]:

$$\begin{gathered} \xi(x,y)r_x + \eta(x,y)r_y = 0 \\ \xi(x,y)s_x + \eta(x,y)s_y = 1 \\ \frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y} \\ \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix} \neq 0 \end{gathered} \tag{19}$$

It is easy to show that (16) satisfies (17). Reduced form of (19) becomes:

$$\begin{aligned} v &= x \\ u(v) &= \frac{dy(x)}{dx} \end{aligned} \tag{20}$$

Substituting (18) in (19) gives:

$$\frac{du(v)}{dv} = \frac{M(v)}{EI}(1 + u(v)^2)^{\frac{3}{2}} \tag{21}$$

This is a first order ODE and it is possible to further solve it by Lie symmetry method.
It can be shown that [19] for a first order differential equation like:

$$\frac{dy}{dx} = \omega(x, y) \tag{22}$$

where:

$$\frac{du(v)}{dv} = \omega(u, v) = \frac{M(v)}{EI}(1 + u(v)^2)^{\frac{3}{2}} \tag{23}$$

ξ and η in (21) must satisfy the below equation:

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 = \xi\omega_x + \eta\omega_y \tag{24}$$

Substituting (23) in (24) yields:

$$\begin{aligned} \eta_v + (\eta_u - \xi_v)\frac{M(v)}{EI}(1 + u(v)^2)^{\frac{3}{2}} - \xi_u\left(\frac{M(v)}{EI}\right)^2\left(1 + u(v)^2\right)^{\frac{3}{2}} \\ = \xi\frac{1}{EI}\frac{dM(v)}{dv}\left(1 + u(v)^2\right)^{\frac{3}{2}} \\ + 3\eta\frac{M(v)}{EI}u(v)(1 + u(v)^2)^{\frac{3}{2}} \end{aligned} \tag{25}$$

There is no term of $u(v)\left(1 + u(v)^2\right)^{\frac{1}{2}}$ in left hand side of equation, so the equation yields:

$$\eta = 0 \tag{26}$$

Therefore (25) can be written as:

$$\begin{aligned} -\frac{M(v)}{EI}\left(\xi_v(1 + u(v)^2)^{\frac{3}{2}} + \xi_u\frac{M(v)}{EI}\left(1 + u(v)^2\right)^3\right) \\ = \xi\frac{1}{EI}\frac{dM(v)}{dv}(1 + u(v)^2)^{\frac{3}{2}} \end{aligned} \tag{27}$$

Comparing the moment in both sides of equation one can show that $\xi_u = 0$, so:

$$\xi = \xi(v) \tag{28}$$

Equation (27) will simplify, by considering (28), to:

$$-\frac{1}{M(v)}\frac{dM(v)}{dv} = \frac{1}{\xi(v)}\frac{d\xi(v)}{dv} \tag{29}$$

Its solution is as following:

$$\xi(v) = \frac{C}{M(v)} \tag{30}$$

C is a constant that can be considered as 1. Hence:

$$\xi(v) = \frac{1}{M(v)} \tag{31}$$

Therefore:

$$Xf = \frac{1}{M(v)}\frac{\partial f}{\partial x} \tag{32}$$

Canonical coordinates can be calculated as:

$$\begin{aligned} r(u,v) &= u(v) \\ s(u,v) &= \int M(v)dv \end{aligned} \tag{33}$$

These canonical coordinates satisfy the conditions (19). Equation (19) can be written as:

$$\frac{ds}{dr} = \frac{EI}{(1+r^2)^{\frac{3}{2}}} \tag{34}$$

Its solution is:

$$s(r) = r\frac{EI}{(1+r^2)^{\frac{1}{2}}} + C_1 \tag{35}$$

or,

$$r = \frac{s(r) + C_1}{\sqrt{-s(r)^2 - 2s(r)C_1 - C_1^2 + (EI)^2}} \tag{36}$$

Substituting (33) in (36) yields to:

$$u(v) = \frac{\int M(v)dv + C_1}{\sqrt{-(\int M(v)dv)^2 - 2C_1\int M(v)dv - C_1^2 + (EI)^2}} \tag{37}$$

Further, substituting (20) in (37) results in:

$$\frac{dy}{dx} = \frac{\int M(x)dx + C_1}{\sqrt{-(\int M(x)dx)^2 - 2C_1 \int M(x)dx - C_1^2 + (EI)^2}} \tag{38}$$

Therefore, y becomes:

$$y(x) = \int \frac{\int M(x)dx + C_1}{\sqrt{-(\int M(x)dx)^2 - 2C_1 \int M(x)dx - C_1^2 + (EI)^2}} dx + C_2 \tag{39}$$

This solution is expressed in term of two constants $C_1$and C_2 that could be evaluated from the boundary conditions.

4 Validation

Below a close form solution of large deflection of a cantilever beam subjected to a point moment at the tip will be compared with the solution obtained the proposed method.

If a moment applies at the tip of cantilever beam as Fig. 2, Eq. (1) becomes:

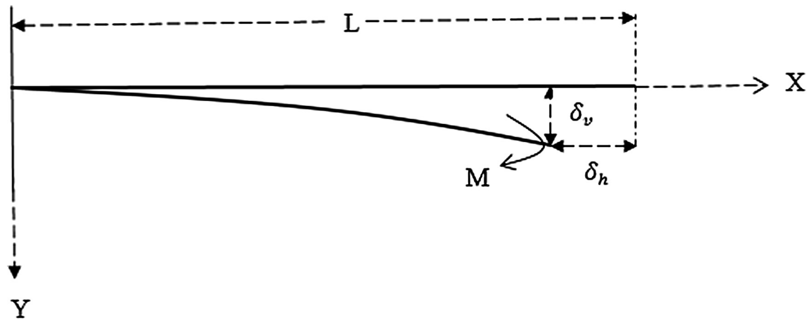

Fig. 2. Cantilever beam subjected to an end moment.

$$\frac{\frac{d^2y}{dx^2}}{(1 + \left(\frac{dy}{dx}\right)^2)^{\frac{3}{2}}} = \frac{M}{EI} \tag{40}$$

According to [10] deflection can be calculated as a function of x:

$$y(x) = \frac{EI}{M}(1 - \sqrt{1 - \left(\frac{M}{EI}\right)^2 x^2}) \tag{41}$$

By considering the boundary condition as the deflection at the fixed point:

$$y(0) = \left.\frac{dy}{dx}\right|_{x=0} = 0 \tag{42}$$

(42) satisfies Eq. (41).

One can write Eq. (38) as following by considering Eq. (38) and the boundary conditions:

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{Mx + C_1}{\sqrt{-(Mx)^2 - 2MxC_1 - C_1^2 + (EI)^2}}\right|_{x=0} = 0 \tag{43}$$

hence:

$$C_1 = 0 \tag{44}$$

Equation (19) becomes:

$$y(x) = \int \frac{Mx}{\sqrt{-(Mx)^2 + (EI)^2}} dx + C_2 = \frac{(EI - Mx)(EI + Mx)}{M\sqrt{-(Mx)^2 + (EI)^2}} + C_2 \tag{45}$$

By considering B.C, C_2 Eq. (45) becomes:

$$C_2 = \frac{EI}{M} \tag{46}$$

Finally Eq. (45) becomes:

$$y(x) = \frac{(EI - Mx)(EI + Mx)}{M\sqrt{-(Mx)^2 + (EI)^2}} + \frac{EI}{M} = \frac{EI}{M}(1 - \sqrt{1 - \left(\frac{M}{EI}\right)^2 x^2}) \tag{47}$$

As one can see (47) is same as (41).

5 Conclusions

The above proposed method represents a general approach in solving the large deflections of beams when subjected to point or distributed loads. The method makes use of Lie symmetry to solve the non-linear differential equation that describes the deflection of the beam at a current point on its length. The method could be applied for any type of loading or combinations of it. The computation behind the problem is light in comparison with solving elliptic integrals. The proposed procedure was validated on a standard problem, this is a cantilever beam subjected to point moment at the tip for which there is an available formulation of the exact solution. The proposed method yield same result.

Lie symmetry method is a powerful mathematical approach that could be successfully used to reduce the order of non-linear ordinary differential equations and ultimately, to express an exact analytical solution of the describing equation. The method is requires lighter computation but the existence of the solution is dependent on finding a symmetry of the equation.

References

1. Bisshopp, K.E., Drucker, D.C.: Large deflection of cantilever beams. J. Appl. Math. **2**, 168–171 (1944)
2. Wang, T.M., Lee, S.L., Zienkiewicz, O.C.: A numerical analysis of large defections of beams. Int. J. Mech. Sci. **3**, 219–228 (1961)
3. Kemper, J.D.: Large deflections of tapered cantilever beams. Int. J. Mech. Sci. **10**, 469–478 (1968)
4. Lewis, G., Monsana, F.: Large deflection of cantilever beams of nonlinear Materials. Comput. Struct. **14**(5–6), 357–360 (1981)
5. Monasa, F., Lewis, G.: Large deflection of point load cantilevers with nonlinear behavior. J. Appl. Math. Phys. **34**, 124–130 (1983)
6. Ang Jr., M.H., Wei, W., Teck-Seng, L.: On the estimation of the large deflection of a cantilever. In: Proceedings of the IECON 1993 (1993)
7. Lee, K.: Large deflections of cantilever beams of non-linear elastic material under a combined loading. Int. J. Nonlinear Mech. **37**, 439–443 (2002)
8. Belendez, T., Neipp, C., Belendez, A.: Large and small defections of a cantilever beam. Eur. J. Phys. **23**, 371–379 (2002)
9. Dado, M., Al-Sadder, S.: A new technique for large deflection analysis of non-prismatic cantilever beams. Mech. Res. Commun. **32**, 692–703 (2005)
10. Baykara, C., Guven, U., Bayer, I.: Large deflections of a cantilever beam of nonlinear bimodulus material subject to an end moment. J. Reinf. Plast. Compos. **24**, 1321 (2005)
11. Belendez, T., Perez-Polo, M., Neipp, C.: Belendez: numerical and experimental analysis of large deflections of cantilever beams under a combined load. Phys. Scr. **118**, 61–65 (2005)
12. Al-Sadder, S., Al-Rawi, R.A.O.: Finite difference scheme for large-deflection analysis of non-prismatic cantilever beams subjected to different types of continuous and discontinuous loadings. Arch. Appl. Mech. **75**, 459–473 (2006)
13. Wang, J., Chen, J., Liao, S.: An explicit solution of the large deformation of a cantilever beam under point load at the free tip. J. Comput. Appl. Math. **212**(2), 320–330 (2008)
14. Shvartman, B.S.: Large deflections of a cantilever beam subjected to a follower force. J. Sound Vib. **304**, 969–973 (2007)
15. Scott, W.R.: Group Theory, 1st edn. Dover Publications Inc., Mineola (1987)
16. Munkres, J.R.: Analysis on Manifolds, 1st edn. Addison-Wesley Publishing Company, Boston (1990)
17. Olver, P.J.: Applications of Lie Groups to Differential Equations, 2nd edn. Springer, Heidelberg (1993)
18. Olver, P.J.: Equivalence, Invariants, and Symmetry. Cambridge University Press, Cambridge (1995)

19. Hydon, P.E.: Symmetry Methods for Differential Equations, 1st edn. Cambridge University Press, Cambridge (2000)
20. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations, vol. 81, 1st edn. Springer, Heidelberg (1989)
21. Stephani, H.: Differential Equations, Their Solution Using Symmetries. Cambridge University Press, Cambridge (1989)
22. Emanuel, G.: Solutions of ODEs by Continuos Groups, 1st edn. Chapman, Edinburgh (2001)