Chapter 8
Generalized Vector Variational Inequalities

When the objective function involved in the vector optimization problem is not necessarily differentiable, then the method to solve VOP via corresponding vector variational inequality problems is no longer valid. We need to generalize the vector variational inequality problems for set-valued maps. There are several ways to generalize vector variational inequality problems discussed in Chap. 5. The main objective of this chapter is to generalize the vector variational inequality problems for set-valued maps and to present the existence results for such generalized vector variational inequality problems with or without monotonicity assumption. We also present some relations between a generalized vector variational inequality problem and a vector optimization problem with a nondifferentiable objective function. Several results of this chapter also hold in the setting of Hausdorff topological vector spaces, but for the sake of convenience, our setting is Banach spaces.

8.1 Formulations and Preliminaries

When the map T involved in the formulation of vector variational inequality problems and Minty vector variational inequality problems is a set-valued map, then the vector variational inequality problems and Minty vector variational inequality problems, discussed in Chap. 5, are called (more precisely, Stampacchia) generalized vector variational inequality problems and Minty generalized vector variational inequality problems, respectively.

Let X and Y be Banach spaces and K be a nonempty convex subset of X. Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values, and $C : K \to 2^Y$ be a set-valued map such that for all $x \in K$, $C(x)$ is a closed convex pointed cone. We also assume that $\text{int}(C(x)) \neq \emptyset$ wherever $\text{int}(C(x))$ the interior of the set $C(x)$ is involved in the formulation of a problem. For every $l \in \mathcal{L}(X, Y)$, the value of l at x is denoted by $\langle l, x \rangle$.

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We consider the following *generalized vector variational inequality problems* (SGVVIP) and *Minty generalized vector variational inequality problems* (MGVVIP).

(GSVVIP)$_s$: Find $\bar{x} \in K$ such that there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \in C(\bar{x}), \quad \text{for all } y \in K. \tag{8.1}$$

(GSVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \in C(\bar{x}). \tag{8.2}$$

(MGSVVIP)$_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi \in T(y)$, we have

$$\langle \xi, y - \bar{x} \rangle \in C(\bar{x}). \tag{8.3}$$

(MGSVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\xi \in T(y)$ satisfying

$$\langle \xi, y - \bar{x} \rangle \in C(\bar{x}). \tag{8.4}$$

(GVVIP)$_g$: Find $\bar{x} \in K$ such that for all $\bar{\zeta} \in T(\bar{x})$, we have

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{8.5}$$

(GVVIP)$_s$: Find $\bar{x} \in K$ such that there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{8.6}$$

(GVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}. \tag{8.7}$$

(MGVVIP)$_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi \in T(y)$, we have

$$\langle \xi, y - \bar{x} \rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}. \tag{8.8}$$

(MGVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\xi \in T(y)$ satisfying

$$\langle \xi, y - \bar{x} \rangle \notin -C(\bar{x}) \setminus \{\mathbf{0}\}. \tag{8.9}$$

(GWVVIP)$_g$: Find $\bar{x} \in K$ such that for all $\bar{\zeta} \in T(\bar{x})$, we have

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})), \quad \text{for all } y \in K. \tag{8.10}$$

(GWVVIP)$_s$: Find $\bar{x} \in K$ such that there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})), \quad \text{for all } y \in K. \tag{8.11}$$

(GWVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})). \tag{8.12}$$

(MGWVVIP)$_g$: Find $\bar{x} \in K$ such that for all $y \in K$ for all $\xi \in T(y)$, we have

$$\langle \xi, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})). \tag{8.13}$$

(MGWVVIP)$_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exists $\xi \in T(y)$ satisfying

$$\langle \xi, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})). \tag{8.14}$$

In (GSVVIP)$_w$, (GVVIP)$_w$, and (GWVVIP)$_w$, $\bar{\zeta} \in T(\bar{x})$ depends on $y \in K$; Also, in (MGSVVIP)$_w$, (MGVVIP)$_w$, and (MGWVVIP)$_w$, $\xi \in T(y)$ depends on $y \in K$.

We denote by Sol(GSVVIP)$_g^d$, Sol(GSVVIP)$_s^d$, Sol(GSVVIP)$_w^d$, Sol(MGSVVIP)$_g^d$, Sol(MGSVVIP)$_w^d$, Sol(GVVIP)$_g^d$, Sol(GVVIP)$_s^d$, Sol(GVVIP)$_w^d$, Sol(MGVVIP)$_g^d$, Sol(MGVVIP)$_w^d$, Sol(GWVVIP)$_g^d$, Sol(GWVVIP)$_s^d$, Sol(GWVVIP)$_w^d$, Sol(MGWVVIP)$_g^d$, and Sol(MGWVVIP)$_w^d$, the set of solutions of (GSVVIP)$_g$, (GSVVIP)$_s$, (GSVVIP)$_w$, (MGSVVIP)$_g$, (MGSVVIP)$_w$, (GVVIP)$_g$, (GVVIP)$_s$, (GVVIP)$_w$, (MGVVIP)$_g$, (MGVVIP)$_w$, (GWVVIP)$_g$, (GWVVIP)$_s$, (GWVVIP)$_w$, (MGWVVIP)$_g$, and (MGWVVIP)$_w$, respectively.

If for all $x \in K$, $C(x) = D$ is a fixed closed convex pointed cone with $\operatorname{int}(D) \neq \emptyset$, then the solution set of (GSVVIP)$_g$, (GSVVIP)$_s$, (GSVVIP)$_w$, (MGSVVIP)$_g$, (MGSVVIP)$_w$, (GVVIP)$_g$, (GVVIP)$_s$, (GVVIP)$_w$, (MGVVIP)$_g$, (MGVVIP)$_w$, (GWVVIP)$_g$, (GWVVIP)$_s$, (GWVVIP)$_w$, (MGWVVIP)$_g$, and (MGWVVIP)$_w$, are denoted by Sol(GSVVIP)$_g$, Sol(GSVVIP)$_s$, Sol(GSVVIP)$_w$, Sol(MGSVVIP)$_g$, Sol(MGSVVIP)$_w$, Sol(GVVIP)$_g$, Sol(GVVIP)$_s$, Sol(GVVIP)$_w$, Sol(MGVVIP)$_g$, Sol(MGVVIP)$_w$, Sol(GWVVIP)$_g$, Sol(GWVVIP)$_s$, Sol(GWVVIP)$_w$, Sol(MGWVVIP)$_g$, and Sol(MGWVVIP)$_w$, respectively

Remark 8.1 It is clear that

(a) $\mathrm{Sol(GSVVIP)}_g^d \subseteq \mathrm{Sol(GSVVIP)}_s^d \subseteq \mathrm{Sol(GSVVIP)}_w^d$;
(b) $\mathrm{Sol(MGSVVIP)}_g^d \subseteq \mathrm{Sol(MGSVVIP)}_w^d$;
(c) $\mathrm{Sol(GVVIP)}_g^d \subseteq \mathrm{Sol(GVVIP)}_s^d \subseteq \mathrm{Sol(GVVIP)}_w^d$;
(d) $\mathrm{Sol(MGVVIP)}_g^d \subseteq \mathrm{Sol(MGVVIP)}_w^d$;
(e) $\mathrm{Sol(GWVVIP)}_g^d \subseteq \mathrm{Sol(GWVVIP)}_s^d \subseteq \mathrm{Sol(GWVVIP)}_w^d$;
(f) $\mathrm{Sol(MGWVVIP)}_g^d \subseteq \mathrm{Sol(MGWVVIP)}_w^d$;
(g) $\mathrm{Sol(GSVVIP)}_g^d \subseteq \mathrm{Sol(GVVIP)}_g^d \subseteq \mathrm{Sol(GWVVIP)}_g^d$;
(h) $\mathrm{Sol(GSVVIP)}_s^d \subseteq \mathrm{Sol(GVVIP)}_s^d \subseteq \mathrm{Sol(GWVVIP)}_s^d$;
(i) $\mathrm{Sol(SGVVIP)}_w^d \subseteq \mathrm{Sol(GVVIP)}_w^d \subseteq \mathrm{Sol(GWVVIP)}_w^d$;
(j) $\mathrm{Sol(MGSVVIP)}_g^d \subseteq \mathrm{Sol(MGVVIP)}_g^d \subseteq \mathrm{Sol(MGWVVIP)}_g^d$;
(k) $\mathrm{Sol(MGSVVIP)}_w^d \subseteq \mathrm{Sol(MGVVIP)}_w^d \subseteq \mathrm{Sol(MGWVVIP)}_w^d$.

Definition 8.1 Let K be a nonempty convex subset of X and $x \in K$ be an arbitrary element. The set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *strongly generalized C_x-upper sign continuous* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \in C(x) \text{ implies that there exists } \zeta \in T(x)\\ &\text{such that } \langle \zeta, y-x\rangle \in C(x);\end{aligned}$$

(b) *strongly generalized C_x-upper sign continuous$_+$* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \in C(x) \text{ implies that } \langle \zeta, y-x\rangle \in C(x) \text{ for all } \zeta \in T(x);\end{aligned}$$

(c) *strongly generalized C_x-upper sign continuous$^+$* if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that } \langle \xi_\lambda, y-x\rangle \in C(x)\\ &\text{implies that there exists } \zeta \in T(x) \text{ such that } \langle \zeta, y-x\rangle \in C(x);\end{aligned}$$

(d) *strongly generalized C_x-upper sign continuous$_+^+$* if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that } \langle \xi_\lambda, y-x\rangle \in C(x)\\ &\text{implies that } \langle \zeta, y-x\rangle \in C(x) \text{ for all } \zeta \in T(x);\end{aligned}$$

(e) *generalized C_x-upper sign continuous* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\} \text{ implies that there exists } \zeta \in T(x) \text{ such that}\\ &\langle \zeta, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\};\end{aligned}$$

(f) *generalized C_x-upper sign continuous$_+$* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\} \text{ implies that } \langle \zeta, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\}\\ &\text{for all } \zeta \in T(x);\end{aligned}$$

(g) *generalized C_x-upper sign continuous$^+$* if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\} \text{ implies that there exists } \zeta \in T(x) \text{ such that}\\ &\langle \zeta, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\};\end{aligned}$$

(h) *generalized C_x-upper sign continuous$^+_+$* if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\} \text{ implies that } \langle \zeta, y-x\rangle \notin -C(x)\setminus\{\mathbf{0}\}\\ &\text{for all } \zeta \in T(x);\end{aligned}$$

(i) *weakly generalized C_x-upper sign continuous* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -\operatorname{int}(C(x)) \text{ implies that there exists } \zeta \in T(x) \text{ such that}\\ &\langle \zeta, y-x\rangle \notin -\operatorname{int}(C(x));\end{aligned}$$

(j) *weakly generalized C_x-upper sign continuous$_+$* if for all $y \in K$,

$$\begin{aligned}&\text{there exists } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -\operatorname{int}(C(x)) \text{ implies that } \langle \zeta, y-x\rangle \notin -\operatorname{int}(C(x))\\ &\text{for all } \zeta \in T(x);\end{aligned}$$

(k) *weakly generalized C_x-upper sign continuous*$^+$ if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -\operatorname{int}(C(x)) \text{ implies that there exists } \zeta \in T(x)\\ &\text{such that } \langle \zeta, y-x\rangle \notin -\operatorname{int}(C(x));\end{aligned}$$

(l) *weakly generalized C_x-upper sign continuous*$^+_+$ if for all $y \in K$,

$$\begin{aligned}&\text{for all } \xi_\lambda \in T(x+\lambda(y-x)) \text{ for } \lambda \in]0,1[\text{ such that}\\ &\langle \xi_\lambda, y-x\rangle \notin -\operatorname{int}(C(x)) \text{ implies that } \langle \zeta, y-x\rangle \notin -\operatorname{int}(C(x))\\ &\text{for all } \zeta \in T(x).\end{aligned}$$

Example 8.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K =]0,1]$ and $C(x) = \mathbb{R}^2_+$ for all $x \in K$. Consider the map $T(x) := \{(y_1, y_2) \in \mathbb{R}^2 : |y_1| \le x,\ |y_2| \le x\}$. Then T is strongly generalized C_x-upper sign continuous, strongly generalized C_x-upper sign continuous$^+$, generalized C_x-upper sign continuous, generalized C_x-upper sign continuous$^+$, weakly generalized C_x-upper sign continuous, and weakly generalized C_x-upper sign continuous$^+$. However, T is not strongly generalized C_x-upper sign continuous$_+$, strongly generalized C_x-upper sign continuous$^+_+$, generalized C_x-upper sign continuous$_+$, generalized C_x-upper sign continuous$^+_+$, weakly generalized C_x-upper sign continuous$_+$, or weakly generalized C_x-upper sign continuous$^+_+$ (Fig. 8.1).

Definition 8.2 Let K be a nonempty convex subset of X. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be *generalized v-hemicontinuous* if for all $x, y \in K$, the set-valued map $F : [0,1] \to 2^Y$, defined by $F(\lambda) = \langle T(x+\lambda(y-x)), y-x\rangle$, is upper semicontinuous at 0^+, where $\langle T(x+\lambda(y-x)), y-x\rangle = \{\langle \zeta, y-x\rangle : \zeta \in T(x+\lambda(y-x))\}$.

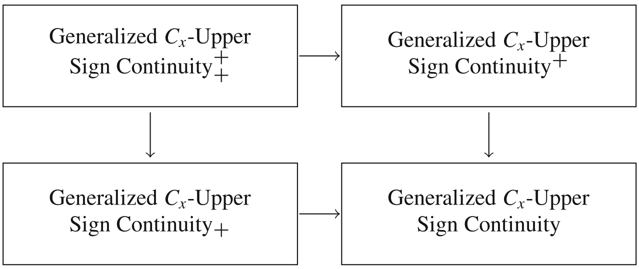

Fig. 8.1 Relations among different kinds of generalized C_x-upper sign continuities. The similar diagram also holds for weak as well as for strong cases

Lemma 8.1 *Let K be a nonempty convex subset of X and $x \in K$ be an arbitrary element. If the set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is generalized υ-hemicontinuous, then it is strongly generalized C_x-upper sign continuous as well as weakly generalized C_x-upper sign continuous.*

Proof Let x be an arbitrary but fixed element. Suppose to the contrary that T is not weakly generalized C_x-upper sign continuous. Then for some $y \in K$ and all $\xi_\lambda \in T(x + \lambda(y - x))$, $\lambda \in]0, 1[$, we have

$$\langle \xi_\lambda, y - x\rangle \notin -\operatorname{int}(C(x)) \tag{8.15}$$

implies

$$\langle \zeta, y - x\rangle \in -\operatorname{int}(C(x)), \quad \text{for all } \zeta \in T(x).$$

Since T is generalized υ-hemicontinuous, the set-valued map $F : [0, 1] \to 2^Y$, defined in Definition 8.2, is upper semicontinuous at 0^+, and $F(0) = \langle T(x), y - x\rangle \subseteq -\operatorname{int}(C(x))$, we have that there exists an open neighborhood $V =]0, \delta[\subseteq [0, 1]$ such that $F(\lambda) = \langle T(x + \lambda(y - x)), y - x\rangle \subseteq -\operatorname{int}(C(x))$ for all $\lambda \in]0, \delta[$, that is, for all $\xi_\lambda \in T(x + \lambda(y - x))$ and all $\lambda \in]0, \delta[$, we have $\langle \xi_\lambda, y - x\rangle \in -\operatorname{int}(C(x))$, a contradiction of (8.15). Hence, T is weakly generalized C_x-upper sign continuous.

Since $W(x) = Y \setminus \{C(x)\}$ is an open set for all $x \in K$, the proof for strong case is similar, and therefore, we omit it. □

Remark 8.2 The generalized υ-hemicontinuity does not imply the generalized C_x-upper sign continuity.

Definition 8.3 Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty compact values. Then T is said to be *$\mathscr{H}$-hemicontinuous* if for all $x, y \in K$, the set-valued map $F : [0, 1] \to 2^Y$, defined by $F(\lambda) = \mathscr{H}(T(x + \lambda(y - x)), T(x))$, is $\mathscr{H}$-continuous at 0^+, where $\mathscr{H}$ denotes the Hausdorff metric on the family of all nonempty closed bounded subsets of $\mathcal{L}(X, Y)$.

Lemma 8.2 *Let K be a nonempty convex subset of X and $x \in K$ be an arbitrary element. If the set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty compact valued and $\mathscr{H}$-hemicontinuous, then it is strongly generalized C_x-upper sign continuous$^+$ as well as weakly generalized C_x-upper sign continuous$^+$.*

Proof Let x be an arbitrary but fixed element and suppose that T is strongly generalized C_x-upper sign continuous$^+$. Let $x_\lambda := x + \lambda(y - x)$ for all $y \in K$ and $\lambda \in]0, 1[$. Assume that for all $y \in K$ and all $\xi_\lambda \in T(x_\lambda)$, $\lambda \in]0, 1[$, we have

$$\langle \xi_\lambda, y - x\rangle \in C(x).$$

Since $T(x_\lambda)$ and $T(x)$ are compact, from Lemma 1.13, it follows that for each fixed $\xi_\lambda \in T(x_\lambda)$, there exists $\zeta_\lambda \in T(x)$ such that

$$\|\xi_\lambda - \zeta_\lambda\| \leq \mathscr{H}(T(x_\lambda), T(x)).$$

Since $T(x)$ is compact, without loss of generality, we may assume that $\zeta_\lambda \to \zeta \in T(x)$ as $\lambda \to 0^+$. Since T is $\mathscr{H}$-hemicontinuous, $\mathscr{H}(T(x_\lambda), T(x)) \to 0$ as $\lambda \to 0^+$. Thus,

$$\begin{aligned}\|\xi_\lambda - \zeta\| &\leq \|\xi_\lambda - \zeta_\lambda\| + \|\zeta_\lambda - \zeta\| \\ &\leq \mathscr{H}(T(x_\lambda), T(x)) + \|\zeta_\lambda - \zeta\| \to 0 \text{ as } \lambda \to 0^+.\end{aligned}$$

This implies that $\xi_\lambda \to \zeta \in T(x)$. Since $C(x)$ is closed, we have that there exists $\zeta \in T(x)$ such that $\langle \zeta, y - x \rangle \in C(x)$ for all $y \in K$. Hence, T is strongly generalized C_x-upper sign continuous$^+$.

Since $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$ is closed for all $x \in K$, by using the similar argument, it is easy to show that T is weakly generalized C_x-upper sign continuous$^+$. □

Remark 8.3 The $\mathscr{H}$-hemicontinuity does not imply the generalized C_x-upper sign continuity$^+$.

Lemma 8.3 *Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values. Then*

(a) $\text{Sol(MGSVVIP)}_w^d \subseteq \text{Sol(GSVVIP)}_s^d$ *if T is strongly generalized C_x-upper sign continuous;*
(b) $\text{Sol(MGVVIP)}_w^d \subseteq \text{Sol(GVVIP)}_s^d \subseteq \text{Sol(GVVIP)}_w^d$ *if T is generalized C_x-upper sign continuous;*
(c) $\text{Sol(MGWVVIP)}_w^d \subseteq \text{Sol(GVVIP)}_s^d \subseteq \text{Sol(GWVVIP)}_w^d$ *if T is weakly generalized C_x-upper sign continuous.*

Proof (a) Let $\bar{x} \in \text{Sol(MGSVVIP)}_w^d$. Then for all $y \in K$, there exists $\xi \in T(y)$ such that

$$\langle \xi, y - \bar{x} \rangle \in C(\bar{x}).$$

Since K is convex, for all $\lambda \in]0, 1[$, $y_\lambda := x + \lambda(y - \bar{x}) \in K$. Therefore, for $y_\lambda \in K$, there exists $\xi_\lambda \in T(y_\lambda)$ such that

$$\langle \xi_\lambda, \bar{x} + \lambda(y - \bar{x}) - \bar{x} \rangle \in C(\bar{x}),$$

equivalently,

$$\lambda \langle \xi_\lambda, y - \bar{x} \rangle \in C(\bar{x}).$$

Since $C(x)$ is a convex cone, we have

$$\langle \xi_\lambda, y - \bar{x} \rangle \in C(\bar{x}).$$

By strong generalized C_x-upper sign continuity of T, there exists $\bar{\zeta} \in T(\bar{x})$ such that

$$\left\langle \bar{\zeta}, y - \bar{x} \right\rangle \in C(\bar{x}), \quad \text{for all } y \in K.$$

Hence, $\bar{x} \in \mathrm{Sol(GSVVIP)}_s^d$.

Since $W(x) = Y \setminus \{-C(x) \setminus \{\mathbf{0}\}\}$ and $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$ are cones, the proof of the part (b) and (c) lies on the lines of the proof of part (a). □

Similarly, we can prove the following lemma.

Lemma 8.4 *Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values. Then*

(a) $\mathrm{Sol(MGSVVIP)}_g^d \subseteq \mathrm{Sol(GSVVIP)}_g^d$ *if T is strongly generalized C_x-upper sign continuous$_+^+$;*
(b) $\mathrm{Sol(MGSVVIP)}_g^d \subseteq \mathrm{Sol(GSVVIP)}_s^d$ *if T is strongly generalized C_x-upper sign continuous$^+$;*
(c) $\mathrm{Sol(MGSVVIP)}_w^d \subseteq \mathrm{Sol(GSVVIP)}_g^d$ *if T is strongly generalized C_x-upper sign continuous$_+$;*
(d) $\mathrm{Sol(MGVVIP)}_g^d \subseteq \mathrm{Sol(GVVIP)}_g^d$ *if T is generalized C_x-upper sign continuous$_+^+$;*
(e) $\mathrm{Sol(MGVVIP)}_g^d \subseteq \mathrm{Sol(GVVIP)}_s^d$ *if T is generalized C_x-upper sign continuous$^+$;*
(f) $\mathrm{Sol(MGVVIP)}_w^d \subseteq \mathrm{Sol(GVVIP)}_g^d$ *if T is generalized C_x-upper sign continuous$_+$;*
(g) $\mathrm{Sol(MGWVVIP)}_g^d \subseteq \mathrm{Sol(GWVVIP)}_g^d$ *if T is weakly generalized C_x-upper sign continuous$_+^+$;*
(h) $\mathrm{Sol(MGWVVIP)}_g^d \subseteq \mathrm{Sol(GWVVIP)}_s^d$ *if T is weakly generalized C_x-upper sign continuous$^+$;*
(i) $\mathrm{Sol(MGWVVIP)}_w^d \subseteq \mathrm{Sol(GWVVIP)}_g^d$ *if T is weakly generalized C_x-upper sign continuous$_+$.*

We introduce the following set-valued maps:

- $S_g^S(y) = \{x \in K : \forall \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \in C(x)\}$;
- $S_w^S(y) = \{x \in K : \exists \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \in C(x)\}$;
- $M_g^S(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \in C(x)\}$;
- $M_w^S(y) = \{x \in K : \exists \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \in C(x)\}$;
- $S_g(y) = \{x \in K : \forall \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$;
- $S_w(y) = \{x \in K : \exists \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$;

- $M_g(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$;
- $M_w(y) = \{x \in K : \exists \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$;
- $S_g^W(y) = \{x \in K : \forall \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \notin -\operatorname{int}(C(x))\}$;
- $S_w^W(y) = \{x \in K : \exists \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \notin -\operatorname{int}(C(x))\}$;
- $M_g^W(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -\operatorname{int}(C(x))\}$;
- $M_w^W(y) = \{x \in K : \exists \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -\operatorname{int}(C(x))\}$.

From the above definition of set-valued maps, the following result can be easily derived.

Proposition 8.1

(a) $\text{Sol(GSVVIP)}_g^d = \bigcap_{y \in K} S_g^S(y)$ *and* $\text{Sol(GSVVIP)}_w^d = \bigcap_{y \in K} S_w^S(y)$;
(b) $\text{Sol(MGSVVIP)}_g^d = \bigcap_{y \in K} M_g^S(y)$ *and* $\text{Sol(MGSVVIP)}_w^d = \bigcap_{y \in K} M_w^S(y)$;
(c) $\text{Sol(GVVIP)}_g^d = \bigcap_{y \in K} S_g(y)$ *and* $\text{Sol(GVVIP)}_w^d = \bigcap_{y \in K} S_w(y)$;
(d) $\text{Sol(MGVVIP)}_g^d = \bigcap_{y \in K} M_g(y)$ *and* $\text{Sol(MGVVIP)}_w^d = \bigcap_{y \in K} M_w(y)$;
(e) $\text{Sol(GWVVIP)}_g^d = \bigcap_{y \in K} S_g^W(y)$ *and* $\text{Sol(GWVVIP)}_w^d = \bigcap_{y \in K} S_w^W(y)$;
(f) $\text{Sol(MGWVVIP)}_g^d = \bigcap_{y \in K} M_g^W(y)$ *and* $\text{Sol(MGWVVIP)}_w^d = \bigcap_{y \in K} M_w^W(y)$.

Proposition 8.2

(a) *If the set-valued map $C : K \to 2^Y$ is closed, then for each $y \in K$, $M_g^S(y)$ is a closed set.*
(b) *If the set-valued map $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$, is closed, then for each $y \in K$, $M_g^W(y)$ is a closed set.*
(c) *If K is compact and the set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty compact valued and the set-valued map $C : K \to 2^Y$ is closed, then for each $y \in K$, $M_w^S(y)$ is a closed set.*
(d) *If K is compact and the set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty compact valued and the set-valued map $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$, is closed, then for each $y \in K$, $M_w^W(y)$ is a closed set.*

(e) *If the set-valued map* $T : K \to 2^{\mathcal{L}(X,Y)}$ *is lower semicontinuous and the set-valued map* $C : K \to 2^Y$ *is closed, then for each* $y \in K$, $S_g^S(y)$ *is a closed set.*
(f) *If the set-valued map* $T : K \to 2^{\mathcal{L}(X,Y)}$ *is lower semicontinuous and the set-valued map* $W : K \to 2^Y$*, defined by* $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$*, is closed, then for each* $y \in K$, $S_g^W(y)$ *is a closed set.*
(g) *If the set-valued map* $W : K \to 2^Y$*, defined by* $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$*, is concave, then for each* $y \in K$, $M_g^W(y)$ *is a convex set.*
(h) *If the set-valued map* $C : K \to 2^Y$ *is concave, then for each* $y \in K$, $M_g^S(y)$ *is a convex set.*

Proof The proof of part (a) is similar to that of (b), therefore, we prove only part (b).

(b) For any fixed $y \in K$, let $\{x_m\}$ be a sequence in $M_g^W(y)$ such that $\{x_m\}$ converges to $x \in K$. Since $x_m \in M_g^W(y)$, for all $\xi \in T(y)$, we have

$$\langle \xi, y - x_m \rangle \in W(x_m) = Y \setminus \{-\operatorname{int}(C(x_m))\}, \quad \text{for all } m.$$

Since $\xi \in \mathcal{L}(X, Y)$, ξ is continuous, and so, the sequence $\{\langle \xi, y - x_m \rangle\}$ converges to $\langle \xi, y - x \rangle \in Y$. Since W is closed, so its graph $\mathcal{G}(W)$ is closed, and therefore, we have $(x_m, \langle \xi, y - x_m \rangle)$ converges to $(x, \langle \xi, y - x \rangle) \in \mathcal{G}(W)$. Thus,

$$\langle \xi, y - x \rangle \in W(x) = Y \setminus \{-\operatorname{int}(C(x))\},$$

so that $x \in M_g^W(y)$. Consequently, $M_g^W(y)$ is a closed subset of K.

The proof of part (c) is similar to that of (d), therefore, we prove only part (d).

(d) For any fixed $y \in K$, let $\{x_m\}$ be a sequence in $M_w^W(y)$ such that $\{x_m\}$ converges to $x \in K$. Since $x_m \in M_w^W(y)$, there exists $\xi_m \in T(y)$ such that

$$\langle \xi_m, y - x_m \rangle \in W(x_m) = Y \setminus \{-\operatorname{int}(C(x_m))\}, \quad \text{for all } m.$$

Since $T(y)$ is compact, we may assume that $\{\xi_m\}$ converges to some $\xi \in T(y)$. Besides, since K is compact, $\{x_m\}$ is bounded. Therefore, $\langle \xi_m - \xi, y - x_m \rangle$ converges to $\mathbf{0}$, but $\langle \xi, y - x_m \rangle$ converges to $\langle \xi, y - x \rangle \in Y$ due to $\xi \in \mathcal{L}(X, Y)$. Hence, $\langle \xi_m, y - x_m \rangle$ converges to $\langle \xi, y - x \rangle \in Y$. Therefore, $(x_m, \langle \xi_m, y - x_m \rangle)$ converges to $(x, \langle \xi, y - x \rangle) \in \mathcal{G}(W)$ since $\mathcal{G}(W)$ is closed. Thus, for $\xi \in T(y)$,

$$\langle \xi, y - x \rangle \in W(x) = Y \setminus \{-\operatorname{int}(C(x))\},$$

so that $x \in M_w^W(y)$. Consequently, $M_w^W(y)$ is a closed subset of K.

The proof of part (f) is similar to that of (e), therefore, we prove only part (e).

(e) For any fixed $y \in K$, let $\{x_m\}$ be a sequence in $S_g^S(y)$ converging to $x \in K$. By lower semicontinuity (see Lemma 1.9) of T, for any $\zeta \in T(x)$, there exists $\zeta_m \in T(x_m)$ for all m such that the sequence $\{\zeta_m\}$ converges to $\zeta \in \mathcal{L}(X, Y)$. Since

$x_m \in S_g^S(y)$ for all m, we have

$$\langle \zeta_m, y - x_m \rangle \in C(x_m).$$

Moreover,

$$\begin{aligned} \|\langle \zeta_m, y - x_m \rangle - \langle \zeta, y - x \rangle\| &= \|\langle \zeta_m, y - x_m \rangle - \langle \zeta_m, x \rangle + \langle \zeta_m, x \rangle - \langle \zeta, y - x \rangle\| \\ &= \|\langle \zeta_m, x - x_m \rangle + \langle \zeta_m, y - x \rangle - \langle \zeta, y - x \rangle\| \\ &= \|\langle \zeta_m, x - x_m \rangle + \langle \zeta_m - \zeta, y - x \rangle\| \\ &\leq \|\zeta_m\| \, \|x - x_m\| + \|\zeta_m - \zeta\| \, \|y - x\|. \end{aligned}$$

Since $\{\zeta_m\}$ is bounded in $\mathcal{L}(X, Y)$, $\{\langle \zeta_m, y - x_m \rangle\}$ converges to $\langle \zeta, y - x \rangle$. By the closedness of C, we have $\langle \zeta, y - x \rangle \in C(x)$. Hence, $x \in S_g^S(y)$, and therefore, $S_g^S(y)$ is closed.

(g) Let $y \in K$ be any fixed element and let $x_1, x_2 \in M_g^W(y)$. Then for all $\xi \in T(y)$, we have

$$\langle \xi, y - x_1 \rangle \in W(x_1) \quad \text{and} \quad \langle \xi, y - x_2 \rangle \in W(x_2).$$

By concavity of W, for all $\lambda \in [0, 1]$, we have

$$\begin{aligned} \langle \xi, y - (\lambda x_1 + (1 - \lambda) x_2) \rangle &= \lambda \, \langle \xi, y - x_1 \rangle + (1 - \lambda) \, \langle \xi, y - x_2 \rangle \\ &\in \lambda W(x_1) + (1 - \lambda) W(x_2) \\ &\subseteq W(\lambda x_1 + (1 - \lambda) x_2). \end{aligned}$$

Therefore, $\lambda x_1 + (1 - \lambda) x_2 \in M_g^W(y)$, and hence, $M_g^W(y)$ is convex.

Similarly, we can prove part (h). □

Remark 8.4 The set-valued maps S_g, S_w, M_g, and M_w fail to have the property that $S_g(y)$, $S_w(y)$, $M_g(y)$, and $M_w(y)$ are closed for all $y \in K$.

Example 8.2 Consider $X = Y = \mathbb{R}$, $K =]0, 1]$, $C(x) = R_+$ for all $x \in K$ and $T(x) = [0, 1]$. Then the set

$$\begin{aligned} S_g(y) &= \{x \in K : \forall \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\} \\ &= \{x \in]0, 1] : x \leq y\} \end{aligned}$$

is not closed.

Proposition 8.3 *Let K be a nonempty convex subset of X. The set-valued maps S_w and S_w^W are KKM-maps.*

Proof Let $\hat{x}$ be in the convex hull of any finite subset $\{y_1, y_2, \ldots, y_p\}$ of K. Then $\hat{x} = \sum_{i=1}^{p} \lambda_i y_i$ for some nonnegative real number λ_i, $1 \leq i \leq p$, with $\sum_{i=1}^{p} \lambda_i = 1$. If $\hat{x} \notin \bigcup_{i=1}^{p} S_w(y_i)$, then for all $\zeta \in T(\hat{x})$, we have

$$\langle \zeta, y_i - \hat{x} \rangle \in -C(\hat{x}) \setminus \{\mathbf{0}\}, \quad \text{for each } i = 1, 2, \ldots, p.$$

Since $-C(\hat{x})$ is a convex cone and $\lambda_i \geq 0$ with $\sum_{i=1}^{p} \lambda_i = 1$, we have

$$\sum_{i=1}^{p} \lambda_i \langle \zeta, y_i - \hat{x} \rangle \in -C(\hat{x}) \setminus \{\mathbf{0}\}.$$

It follows that

$$\begin{aligned} \mathbf{0} = \langle \zeta, \hat{x} - \hat{x} \rangle &= \left\langle \zeta, \sum_{i=1}^{p} \lambda_i y_i - \sum_{i=1}^{p} \lambda_i \hat{x} \right\rangle \\ &= \left\langle \zeta, \sum_{i=1}^{p} \lambda_i (y_i - \hat{x}) \right\rangle = \sum_{i=1}^{p} \lambda_i \langle \zeta, y_i - \hat{x} \rangle \in -C(\hat{x}) \setminus \{\mathbf{0}\}. \end{aligned}$$

Thus, we have $\mathbf{0} \in -C(\hat{x}) \setminus \{\mathbf{0}\}$, a contradiction. Therefore, we must have

$$\operatorname{co}(\{y_1, y_2, \ldots, y_p\}) \subseteq \bigcup_{i=1}^{p} S_w(y_i),$$

and hence, S_w is a KKM map on K.

Since $-C(x)$ is a convex cone, by using the similar argument, we can easily prove that S_w^W is a KKM map on K. □

Remark 8.5 The above argument cannot be applied for S_g^S and S_w^S. In general, S_g^S and S_w^S are not KKM maps.

Example 8.3 Let $X = K = \mathbb{R}$, $Y = \mathbb{R}^2$ and let the operator $T : K \to 2^{\mathcal{L}(X,Y)}$ be the single-valued map $T(x) := (x, -x)$. Then the sets S_g^S and S_w^S coincide, and it can be easily seen that they are not KKM maps: Consider, for instance, the points $y_1 = 0$ and $y_2 = 1$. Then $S_g^S(y_1) = S_w^S(y_1) = \{0\}$ and $S_g^S(y_2) = S_w^S(y_2) = \{0, 1\}$. However, $\frac{1}{2} \in \operatorname{co}(y_1, y_2)$ and $S_g^S\left(\frac{1}{2}\right) = S_w^S\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, but $\frac{1}{2} \notin \{0, 1\}$.

8.2 Existence Results under Monotonicity

Let X and Y be Banach spaces and K be a nonempty convex subset of X. Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values, and $C : K \to 2^Y$ be a set-valued map such that for all $x \in K$, $C(x)$ is a closed convex pointed cone with $\operatorname{int}(C(x)) \neq \emptyset$.

Definition 8.4 Let $x \in K$ be an arbitrary element. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *strongly generalized C_x-monotone* on K if for every $y \in K$ and for all $\zeta \in T(x)$, $\xi \in T(y)$, we have

$$\langle \zeta - \xi, x - y \rangle \in C(x);$$

(b) *strongly generalized C_x-monotone$^+$* on K if for every $y \in K$ and for all $\zeta \in T(x)$, there exists $\xi \in T(y)$ such that

$$\langle \zeta - \xi, x - y \rangle \in C(x);$$

(c) *strongly generalized C_x-monotone$_+$* on K if for every $y \in K$ and for all $\xi \in T(y)$, there exists $\zeta \in T(x)$ such that

$$\langle \zeta - \xi, x - y \rangle \in C(x);$$

(d) *strongly generalized C_x-pseudomonotone* on K if for every $y \in K$ and for all $\zeta \in T(x)$ and $\xi \in T(y)$, we have

$$\langle \zeta, y - x \rangle \in C(x) \quad \text{implies} \quad \langle \xi, y - x \rangle \in C(x);$$

(e) *strongly generalized C_x-pseudomonotone$^+$* on K if for every $y \in K$ and for all $\zeta \in T(x)$, we have

$$\langle \zeta, y - x \rangle \in C(x) \quad \text{implies} \quad \langle \xi, y - x \rangle \in C(x), \text{ for some } \xi \in T(y);$$

(f) *strongly generalized C_x-pseudomonotone$_+$* on K if for every $y \in K$, we have for some $\zeta \in T(x)$,

$$\langle \zeta, y - x \rangle \in C(x) \text{ implies } \langle \xi, y - x \rangle \in C(x), \text{ for all } \xi \in T(y).$$

Definition 8.5 Let $x \in K$ be an arbitrary element. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *generalized C_x-monotone* on K if for every $y \in K$ and for all $\zeta \in T(x)$, $\xi \in T(y)$, we have

$$\langle \zeta - \xi, x - y \rangle \notin -C(x) \setminus \{\mathbf{0}\};$$

(b) *generalized C_x-monotone$^+$* on K if for every $y \in K$ and for all $\zeta \in T(x)$, there exists $\xi \in T(y)$ such that

$$\langle \zeta - \xi, x - y \rangle \notin -C(x) \setminus \{\mathbf{0}\};$$

(c) *generalized C_x-monotone$_+$* on K if for every $y \in K$ and for all $\xi \in T(y)$, there exists $\zeta \in T(x)$ such that

$$\langle \zeta - \xi, x - y \rangle \notin -C(x) \setminus \{\mathbf{0}\};$$

(d) *generalized C_x-pseudomonotone* on K if for every $y \in K$ and for all $\zeta \in T(x)$ and $\xi \in T(y)$, we have

$$\langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\} \quad \text{implies} \quad \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\};$$

(e) *generalized C_x-pseudomonotone$^+$* on K if for every $y \in K$ and for all $\zeta \in T(x)$, we have

$$\langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\} \text{ implies } \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\},$$

for some $\xi \in T(y)$;

(f) *generalized C_x-pseudomonotone$_+$* on K if for every $y \in K$, we have

$$\begin{aligned} &\text{for some } \zeta \in T(x), \quad \langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\} \\ &\text{implies } \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}, \quad \text{for all } \xi \in T(y). \end{aligned}$$

When we replace $C(x) \setminus \{\mathbf{0}\}$ by $\text{int}(C(x))$ in the above definitions, then T is called *weakly generalized C_x-monotone*, *weakly generalized C_x-monotone$^+$*, *weakly generalized C_x-monotone$_+$*, *weakly generalized C_x-pseudomonotone*, *weakly generalized C_x-pseudomonotone$^+$*, and *weakly generalized C_x-pseudomonotone$_+$*, respectively.

The following example shows that the weakly generalized C_x-pseudomonotonicity does not imply weakly generalized C_x-monotonicity.

Example 8.4 Let $X = Y = \mathbb{R}$, $C(x) = [0, \infty)$ for all $x \in X$, and let $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as $T(x) =]-\infty, x]$ for all $x \in \mathbb{R}$. Then it is easy to see that T is weakly generalized C_x-pseudomonotone but not weakly generalized C_x-monotone.

From the above definition, we have the following diagram (Fig. 8.2).

The implications in the following lemma follow from the definition of different kinds of monotonicities, and therefore, we omit the proof.

Lemma 8.5 *Let K be a nonempty subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values. Then*

(a) $\text{Sol(GSVVIP)}_w^d \subseteq \text{Sol(MGSVVIP)}_w^d$ *if T is strongly generalized C_x-pseudomonotone$^+$;*
(b) $\text{Sol(GSVVIP)}_w^d \subseteq \text{Sol(MGSVVIP)}_g^d$ *if T is strongly generalized C_x-pseudomonotone$_+$;*
(c) $\text{Sol(GVVIP)}_w^d \subseteq \text{Sol(MGVVIP)}_w^d$ *if T is generalized C_x-pseudomonotone$^+$;*
(d) $\text{Sol(GVVIP)}_w^d \subseteq \text{Sol(MGVVIP)}_g^d$ *if T is generalized C_x-pseudomonotone$_+$;*

$$\begin{tikzcd}
\mathrm{GM} \arrow[r] \arrow[d] \arrow[rrr, bend left] & \mathrm{GPM} \arrow[r] \arrow[d] & \mathrm{GPM}^{+} \arrow[d] & \mathrm{GM}^{+} \arrow[l] \arrow[d] \\
\mathrm{GM}_{+} \arrow[rrr, bend right] & \mathrm{GPM}_{+} & \mathrm{GPM}^{+}_{+} & \mathrm{GM}^{+}_{+}
\end{tikzcd}$$

Fig. 8.2 Relations among different kinds of generalized C_x-monotonicity. GM and GPM stand for generalized C_x-monotonicity and generalized C_x-pseudomonotonicity, respectively

(e) $\mathrm{Sol(GWVVIP)}_w^d \subseteq \mathrm{Sol(MGWVVIP)}_w^d$ *if T is weakly generalized C_x-pseudomonotone$^+$;*

(f) $\mathrm{Sol(GWVVIP)}_w^d \subseteq \mathrm{Sol(MGWVVIP)}_g^d$ *if T is weakly generalized C_x-pseudomonotone$_+$.*

Next we give the first result on the existence of a solution of $\mathrm{(GWVVIP)}_w$.

Theorem 8.1 *Let X and Y be Banach spaces and K be a nonempty compact convex subset of X. Let $C : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a proper, closed and convex (not necessarily pointed) cone with $\mathrm{int}(C(x)) \neq \emptyset$; and let $W : K \to 2^Y$ be defined by $W(x) = Y \setminus \{-\mathrm{int}(C(x))\}$, such that the graph $\mathcal{G}(W)$ of W is closed in $X \times Y$. Let $x \in K$ be arbitrary and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is weakly generalized C_x-pseudomonotone$_+$ and weakly generalized C_x-upper sign continuous$^+$ on K. Then there exists a solution of $\mathrm{(GWVVIP)}_w$.*

Proof Define set-valued maps $S_w^W, M_g^W : K \to 2^K$ by

$$S_w^W(y) = \{x \in K : \exists \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x\rangle \notin -\mathrm{int}(C(x))\},$$

and

$$M_g^W(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x\rangle \notin -\mathrm{int}(C(x))\},$$

for all $y \in K$. Then by Proposition 8.3, S_w^W is a KKM map on K. By generalized C_x-pseudomonotonicity$_+$ of T, $S_w^W(y) \subseteq M_g^W(y)$ for all $y \in K$. Since S_w^W is a KKM map, so is M_g^W. Also,

$$\bigcap_{y\in K} S_w^W(y) \subseteq \bigcap_{y\in K} M_g^W(y).$$

by Lemma 8.4 (i),

$$\bigcap_{y\in K} M_g^W(y) \subseteq \bigcap_{y\in K} S_w^W(y),$$

and thus,

$$\bigcap_{y\in K} S_w^W(y) = \bigcap_{y\in K} M_g^W(y).$$

By Proposition 8.2 (b) and the assumption that the graph $\mathcal{G}(W)$ of W is closed, $M_g^W(y)$ is closed for all $y \in K$. Since K is compact, so is $M_g^W(y)$ for all $y \in K$. By Fan-KKM Lemma 1.14, we have

$$\bigcap_{y\in K} S_w^W(y) = \bigcap_{y\in K} M_g^W(y) \neq \emptyset.$$

Hence, there exists $\bar{x} \in K$ such that for all $y \in K$, there exists $\bar{\zeta} \in T(\bar{x})$ satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle \notin -\operatorname{int}(C(\bar{x})).$$

The proof of theorem is complete. □

Remark 8.6 We note that the assumptions of Theorem 8.1 imply that, in case of an infinite-dimensional space Y, the cone $C(x)$ cannot be pointed for each $x \in K$. Indeed, the assumptions imply that $Y \setminus \{-\operatorname{int}(C(x))\}$ is closed for each $x \in K$; hence $\operatorname{int}(C(x))$ is open. Since Y is infinite-dimensional, $\operatorname{int}(C(x))$ contains a whole straight line. That is, there exist $y, z \in Y$ such that $y + tz$, $y - tz \in \operatorname{int}(C(x))$ for all $t \in \mathbb{R}$. By convexity, $\mathbf{0} \in C(x)$ which gives $(1/t)y + z$, $(1/t)y + z \in C(x)$ for all $t > 1$. Since $C(x)$ is closed, $z \in C(x)$ and $-z \in C(x)$. Consequently, $C(x)$ cannot be pointed.

Analogously to Theorem 8.1, we have the following existence result for a solution of $(\text{GVVIP})_w$.

Theorem 8.2 *Let X, Y, K, C and W be the same as in Theorem 8.1. Let $x \in K$ be arbitrary and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is generalized C_x-pseudomonotone$_+$ and generalized C_x-upper sign continuous$^+$ on K such that the set $M_g^W(y) = \{x \in K : \forall \xi \in T(y)$ satisfying $\langle \xi, y - x \rangle \notin -\operatorname{int}(C(x))\}$ is closed for all $y \in K$. Then there exists a solution of* $(\text{GVVIP})_w$.

Remark 8.7 Theorem 8.1 and 8.2 also hold when K is nonempty weakly compact convex subset of a Banach space X.

Since S_w^S is not a KKM map, the argument similar to Theorem 8.2 cannot be used for proving the existence of a solution of $(\text{GSVVIP})_w$. Therefore, we define the following concept of pseudomonotonicity.

Definition 8.6 Let $x \in K$ be an arbitrary element. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be *generalized C_x-pseudomonotone** on K if for every $y \in K$ and for all $\zeta \in T(x)$ and $\xi \in T(y)$, we have

$$\langle \zeta, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\} \quad \text{implies} \quad \langle \xi, y - x \rangle \in C(x).$$

We use the above definition of pseudomonotonicity and establish the following existence result for a solution of $(\text{GSVVIP})_w$.

Theorem 8.3 *Let X, Y, K and C be the same as in Theorem 8.1. In addition, we assume that the graph of C is closed. Let $x \in K$ be arbitrary and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is generalized C_x-pseudomonotone* and strongly generalized upper sign continuous$^+$ on K. Then there exists a solution of* $(\text{GSVVIP})_w$.

Proof Define set-valued maps $S_w, M_g^S : K \to 2^K$ by

$$S_w(y) = \{x \in K : \exists \zeta \in T(x) \text{ satisfying } \langle \zeta, y - x\rangle \notin -C(x) \setminus \{\mathbf{0}\}\},$$

and

$$M_g^S(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x\rangle \in C(x)\},$$

for all $y \in K$. Then by Proposition 8.3, S_w is a KKM map on K. By generalized C_x-pseudomonotonicity* of T, $S_w(y) \subseteq M_g^S(y)$ for all $y \in K$. Since S_w is a KKM map, so is M_g^M. Also,

$$\bigcap_{y\in K} S_w(y) \subseteq \bigcap_{y\in K} M_g^S(y).$$

By using strongly generalized C_x-upper sign continuity$^+$ of T and Lemma 8.4 (b), we have

$$\begin{aligned}\bigcap_{y\in K} M_g^S(y) &= \text{Sol(MGSVVIP)}_g^d \subseteq \text{Sol(GSVVIP)}_s^d \\ &\subseteq \text{Sol(GVVIP)}_s^d \subseteq \text{Sol(GVVIP)}_w^d \\ &= \bigcap_{y\in K} S_w(y),\end{aligned}$$

and thus,

$$\bigcap_{y\in K} S_w(y) = \bigcap_{y\in K} M_g^S(y).$$

Since the graph $\mathcal{G}(C)$ of C is closed and K is compact, we have that $M_g^S(y)$ is compact for all $y \in K$. By Fan-KKM Lemma 1.14, we have

$$\bigcap_{y\in K} S_w(y) = \bigcap_{y\in K} M_g^S(y) \neq \emptyset.$$

Hence, there exists $\bar{x} \in K$ such that for all $y \in K$, there exists $\zeta \in T(\bar{x})$ satisfying

$$\langle \zeta, y - \bar{x} \rangle \notin -C(x) \setminus \{\mathbf{0}\}.$$

This completes the proof. $\square$

To give the existence results for solutions of $(\text{GWVVIP})_w$ defined on a closed (not necessarily bounded) convex subset K of a Banach space X, we need the following coercivity conditions.

Definition 8.7 The set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *weakly generalized υ-coercive* on K if there exist a compact subset B of X and $\tilde{y} \in B \cap K$ such that for every $\zeta \in T(x)$,

$$\langle \zeta, \tilde{y} - x \rangle \in -\operatorname{int}(C(x)), \quad \text{for all } x \in K \setminus B. \tag{8.16}$$

(b) *generalized υ-coercive* on K if there exist a compact subset B of X and $\tilde{y} \in B \cap K$ such that for every $\zeta \in T(x)$,

$$\langle \zeta, \tilde{y} - x \rangle \in -C(x) \setminus \{\mathbf{0}\}, \quad \text{for all } x \in K \setminus B. \tag{8.17}$$

Theorem 8.4 *Let X, Y, C, W and $\mathcal{G}(W)$ be the same as in Theorem 8.1, and K be a nonempty closed convex subset of X. Let $x \in K$ be an arbitrary element and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is weakly generalized C_x-pseudomonotone$_+$, weakly generalized C_x-upper sign continuous$^+$ and weakly generalized υ-coercive on K and it has nonempty values. Then $(\text{GWVVIP})_w$ has a solution.*

Proof Let S_w^W and M_g^W be the set-valued maps defined as in the proof of Theorem 8.1. Choose a compact subset B of X and $\tilde{y} \in B \cap K$ such that for every $\zeta \in T(x)$, (8.16) holds.

We claim that the closure $\operatorname{cl}\big(S_w^W(\tilde{y})\big)$ of $S_w^W(\tilde{y})$ is a compact subset of K. If $S_w^W(\tilde{y}) \not\subseteq B$, then there exists $x \in S_w^W(\tilde{y})$ such that $x \in K \setminus B$. It follows that, for some $\zeta \in T(x)$,

$$\langle \zeta, \tilde{y} - x \rangle \notin -\operatorname{int}(C(x)),$$

which contradicts (8.16). Therefore, we have $S_w^W(\tilde{y}) \subseteq B$; hence, $\operatorname{cl}\big(S_w^W(\tilde{y})\big)$ is a compact subset of K.

As in the proof of Theorem 8.1, by Fan-KKM Lemma 1.14, we have

$$\bigcap_{y \in K} \operatorname{cl}\big(S_w^W(\tilde{y})\big) \neq \emptyset.$$

Again, as in the proof of Theorem 8.1, $M_g^W(y)$ is closed for all $y \in K$. By weakly generalized C_x-pseudomonotonicity$_+$ of T, $S_w^W(y) \subseteq M_g^W(y)$ for all $y \in K$. Therefore,

$$\operatorname{cl}\big(S_w^W(\tilde{y})\big) \subseteq M_g^W(y), \quad \text{for all } y \in K.$$

Consequently,

$$\bigcap_{y \in K} M_g^W(y) \neq \emptyset.$$

Furthermore, as in the proof of Theorem 8.1, we have

$$\bigcap_{y \in K} S_w^W(y) = \bigcap_{y \in K} M_g^W(y) \neq \emptyset.$$

Hence, $(\mathrm{GWVVIP})_w$ has a solution. □

Analogous to Theorem 8.4, we can prove the following existence result for a solution of $(\mathrm{GVVIP})_w$.

Theorem 8.5 *Let X, Y, C, W and $\mathcal{G}(W)$ be the same as in Theorem 8.2, and K be a nonempty closed convex subset of X. Let $x \in K$ be an arbitrary element and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty valued, generalized C_x-pseudomonotone$_+$, generalized C_x-upper sign continuous$^+$ and generalized υ-coercive on K such that the set*

$$M_g(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$$

is closed for all $y \in K$. Then $(\mathrm{GVVIP})_w$ has a solution.

Definition 8.8 The set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *weakly generalized d-coercive* on K if there exist a point $\tilde{y}$ and a number $d > 0$ such that for every $\zeta \in T(x)$,

$$\langle \zeta, \tilde{y} - x \rangle \in -\operatorname{int}(C(x)), \text{ if } x \in K \text{ and } \|\tilde{y} - x\| > d;$$

(b) *generalized d-coercive* on K if there exist a point $\tilde{y}$ and a number $d > 0$ such that for every $\zeta \in T(x)$,

$$\langle \zeta, \tilde{y} - x \rangle \in -C(x) \setminus \{\mathbf{0}\}, \text{ if } x \in K \text{ and } \|\tilde{y} - x\| > d.$$

Now we present an existence theorem for a solution of problem $(\mathrm{GWVVIP})_w$ under weakly generalized C_x-pseudomonotonicity$_+$ assumption.

Theorem 8.6 *Let X, Y, C, W and $\mathcal{G}(W)$ be the same as in Theorem 8.1. Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a weakly generalized C_x-pseudomonotone$_+$, weakly generalized C_x-upper sign continuous$^+$ on K with nonempty compact values. Suppose that at least one of the following assumptions holds:*

(i) *K is weakly compact.*
(ii) *X is reflexive, K is closed, and T is generalized d-coercive on K.*

Then $(\mathrm{GWVVIP})_w$ has a solution.

Proof Let S_w^W be the set-valued map defined as in the proof of Theorem 8.1. Define a set-valued map M_w^W by

$$M_w^W(y) = \{x \in K : \exists \xi \in T(y) \text{ satisfying } \langle \xi y - x\rangle \notin -\operatorname{int}(C(x))\},$$

for all $y \in K$. In order to prove the theorem under assumptions (i) it suffices to follow the proof of Theorem 8.1.

As in the proof of Theorem 8.1, S_w^W is a KKM map. By weakly generalized C_x-pseudomonotone$_+$, $S_w^W(y) \subseteq M_w^W(y)$ for all $y \in K$, and so M_w^W is a KKM-map. As in the proof of Proposition 8.2 (d), we can easily show that $M_w^W(y)$ is weakly closed for all $y \in K$.

Let us now consider the case (ii). Let B_r denote the closed ball (under the norm) of X with center at origin and radius r. If $K \cap B_r \neq \emptyset$, part (i) guarantees the existence of a solution x_r for the following problem, denoted by $(\text{GWVVIP})_w^r$:

$$\begin{aligned} &\text{find } x_r \in K \cap B_r \text{ such that for all } y \in K \cap B_r,\\ &\text{there exists } \zeta_r \in T(x_r) \text{ satisfying } \langle \zeta_r, y - x_r\rangle \notin -\operatorname{int}(C(x_r)). \end{aligned}$$

We observe that $\{x_r : r > 0\}$ must be bounded. Otherwise, we can choose r large enough so that $r \geq \|\tilde{y}\|$ and $d < \|\tilde{y} - x_r\|$, where $\tilde{y}$ satisfies the weakly generalized d-coercivity of T. It follows that, for every $\zeta_r \in T(x_r)$,

$$\langle \zeta_r, y_0 - x_r\rangle \in -\operatorname{int}(C(x_r)),$$

that is, x_r is not a solution of problem $(\text{GWVVIP})_w^r$, a contradiction. Therefore, there exist r such that $\|x_r\| < r$. Choose for any $x \in K$. Then we can choose $\varepsilon > 0$ small enough such that $x_r + \varepsilon(x - x_r) \in K \cap B_r$. If we suppose that for every $\zeta_r \in T(x_r)$,

$$\langle \zeta_r, x - x_r\rangle \in -\operatorname{int}(C(x_r)),$$

then

$$\langle \zeta_r, x_r + \varepsilon(x - x_r) - x_r\rangle = \varepsilon\langle \zeta_r, x - x_r\rangle \in -\operatorname{int}(C(x_r)),$$

that is, x_r is not a solution of $(\text{GWVVIP})_w^r$. Thus, x_r is a solution of $(\text{GWVVIP})_w$. $\square$

Analogous to Theorem 8.6, we have the following existence result for a solution of $(\text{GVVIP})_w$.

Theorem 8.7 *Let X, Y, C, W and $\mathcal{G}(W)$ be the same as in Theorem 8.2. Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be nonempty valued, generalized C_x-pseudomonotone$_+$ and generalized C_x-upper sign continuous$^+$ on K such that the set*

$$S_g^M(y) = \{x \in K : \forall \xi \in T(y) \textit{ satisfying } \langle \xi, y - x\rangle \notin -C(x) \setminus \{\mathbf{0}\}\}$$

is weakly closed for all $y \in K$. Suppose that at least one of the following assumptions holds:

(i) *K is weakly compact.*
(ii) *X is reflexive, K is closed, and T is generalized d-coercive on K.*

Then $(\mathrm{GVVIP})_w$ *has a solution.*

In order to derive the existence results for solution of $(\mathrm{GWVVIP})_w$ and $(\mathrm{GWVVIP})_s$ by the way of solving an appropriate Stampacchia generalized (scalar) variational inequality problem (in short, GVIP), we use the following scalarization technique.

Let $s \in Y^*$ and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values. We define a set-valued map $T_s : K \to 2^{X^*}$ by

$$\langle T_s(x), y\rangle = \langle s, T(x), y\rangle, \quad \text{for all } x \in K \text{ and } y \in X.$$

Also, set

$$H(s) = \{y \in Y : \langle s, y\rangle \geq 0\}.$$

Then for all $s \in Y^*$, $H(s)$ is a closed convex cone in Y.

Recall that a set-valued map $Q : X \to 2^{X^*}$ is said to be *generalized pseudomonotone* on X if for every pair of points $x, y \in X$ and for all $u \in Q(x)$, $v \in Q(y)$, we have

$$\langle u, y-x\rangle \geq 0 \quad \text{implies} \quad \langle v, y-x\rangle \geq 0.$$

Also, a set-valued map $Q : X \to 2^{X^*}$ is said to be *generalized pseudomonotone*$^+$ on X if for every pair of points $x, y \in X$ and for all $u \in Q(x)$, we have

$$\langle u, y-x\rangle \geq 0 \quad \text{implies} \quad \langle v, y-x\rangle \geq 0, \text{ for some } v \in Q(y).$$

Obviously, every generalized pseudomonotone set-valued map is generalized pseudomonotone$^+$.

Proposition 8.4 *Let X and Y be Banach spaces and K be a nonempty closed convex subset of X. Suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is strongly generalized $H(s)$-pseudomonotone (respectively, strongly generalized $H(s)$-pseudomonotone$^+$) for some $s \in Y^* \setminus \{\mathbf{0}\}$. Then the mapping T_s is generalized pseudomonotone (respectively, generalized pseudomonotone$^+$) on K.*

Proof For any $x, y \in K$, let

$$\langle \zeta_s, y-x\rangle \geq 0, \quad \text{for all } \zeta_s \in T_s(x). \tag{8.18}$$

Then $\langle s, \langle \zeta, y-x\rangle\rangle \geq 0$ for all $\zeta \in T(x)$. Therefore, $\langle \zeta, y-x\rangle \in H(s)$ for all $\zeta \in T(x)$. If T is strongly generalized $H(s)$-pseudomonotone, then we must have

$\langle \xi, y-x\rangle \in H(s)$ for all $\xi \in T(y)$, and thus $\langle s, \langle \xi, y-x\rangle\rangle \geq 0$ for all $\xi \in T(y)$. Hence, for all $\xi_s \in T_s(y)$,

$$\langle \xi_s, y-x\rangle \geq 0, \tag{8.19}$$

that is, T_s is generalized pseudomonotone on K. Analogously, if T is strongly generalized $H(s)$-pseudomonotone$^+$, (8.18) implies (8.19) for some $\xi_s \in T_s(y)$ and T_s is generalized pseudomonotone$^+$ on K. □

Theorem 8.8 *Let X and Y be Banach spaces and K be a nonempty compact convex subset of X. Let $C : K \to 2^Y$ be defined as in Theorem 8.1 such that $C_+^* \setminus \{\mathbf{0}\} \neq \emptyset$. Let $x \in K$ be arbitrary and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is weakly generalized C_x-upper sign continuous and weakly generalized $H(s)$-pseudomonotone on K for some $s \in C_+^*$ where $H(s) \neq Y$, and has nonempty values. Then the following statements hold.*

(a) *There exists a solution of* $(\mathrm{GWVVIP})_w$.
(b) *If for each $x \in K$, the set $T(x)$ is convex and weakly compact in $\mathcal{L}(X,Y)$, then there exists a solution of* $(\mathrm{GWVVIP})_s$.

Proof

(a) Since $H(s) \neq Y$, we note that $\operatorname{int} H(s) = s^{-1}((0,\infty))$. To see this, consider the following argument. It is clear that $s^{-1}((0,\infty)) \subset \operatorname{int}(H(s))$.

Conversely, let $y \in \operatorname{int}(H(s))$. Then there exists $r > 0$ such that $B_r(y) \subset H(s)$, where $B_r(y)$ denotes the ball with center at y and radius r. Hence, $\langle s, y + rz\rangle \geq 0$ for all $\|z\| < 1$. If $\langle s, y\rangle = 0$, then from the above inequality we conclude that $\langle s, w\rangle \geq 0$ for all $w \in Y$ or $Y \subset H(s)$ which is a contradiction. Therefore, $\langle s.w\rangle > 0$ and $y \in s^{-1}((0,\infty))$. Consequently, $\operatorname{int}(H(s)) = s^{-1}((0,\infty))$.

As $s \in C_+^* \setminus \{\mathbf{0}\}$, the mapping T_s is generalized pseudomonotone on K due to Proposition 8.4. Beside, since T is weakly generalized C_x-upper sign continuous, so is T_s. Now, in the special case where $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$ for all $x \in K$. Theorem 8.1 guarantees the existence of a solution $\bar{x} \in K$ of $(\mathrm{GVIP})_w^s$, that is, for all $y \in K$, there exists $\zeta_s \in T_s(\bar{x})$ satisfying

$$\langle \zeta_s, y-\bar{x}\rangle \geq 0. \tag{8.20}$$

Consequently, for every $y \in K$, there exists $\bar{\zeta} \in T(\bar{x})$ such that

$$\langle s, \langle \bar{\zeta}, y-\bar{x}\rangle\rangle \geq 0,$$

hence, $\langle \bar{\zeta}, y-\bar{x}\rangle \notin -\operatorname{int}(H(s))$. Since $s \in C_+^*$, $-\operatorname{int}(H(s)) \supseteq -\operatorname{int}(C_+) \supseteq -\operatorname{int}(C(\bar{x}))$, so that

$$\langle \bar{\zeta}, y-\bar{x}\rangle \notin -\operatorname{int}(C(\bar{x})).$$

Therefore, $\bar{x}$ is a solution of $(\mathrm{GWVVIP})_w$.

(b) Let, in addition, the set $T(\bar{x})$ be convex and compact. Then $T_s(\bar{x})$ is obviously convex in X^*. We show that $T_s(\bar{x})$ is also compact.

Let $\{z_\alpha\}$ be a net in $T_s(\bar{x})$. Then there exists a net $\{\zeta_\alpha\}$ in $T(\bar{x})$ such that

$$\langle z_\alpha, x\rangle = \langle s, \langle \zeta_\alpha, x\rangle\rangle, \quad \text{for all } x \in X.$$

Since $T(\bar{x})$ is compact, there exists a subnet of $\{\zeta_\alpha\}$ which is converging to some $\zeta \in T(\bar{x})$. Without loss of generality, we suppose that ζ_α converges to ζ. Fix any $x \in X$. Then we can define

$$\langle l, u\rangle = \langle s, \langle u, x\rangle\rangle, \quad \text{for all } u \in \mathcal{L}(X, Y),$$

hence, $l \in \mathcal{L}(X, Y)^*$. Therefore, there exists $\bar{z} \in X^*$ such that

$$\lim_\alpha \langle z_\alpha, x\rangle = \lim_\alpha \langle l, \zeta_\alpha\rangle = \langle l, \zeta\rangle = \langle s, \langle \zeta, x\rangle\rangle = \langle \bar{z}, x\rangle,$$

that is, $\bar{z} \in T_s(\bar{x})$. Thus, $T_s(\bar{x})$ is compact set in X^*.

By (8.20) and the well known minimax theorem [4], we have

$$\max_{\zeta_s \in T_s(\bar{x})} \min_{y \in K} \langle \zeta_s, y - \bar{x}\rangle = \min_{y \in K} \max_{\zeta_s \in T_s(\bar{x})} \langle \zeta_s, y - \bar{x}\rangle \geq 0.$$

Hence, there exists $\zeta_s \in T_s(\bar{x})$ such that

$$\langle \zeta_s, y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K,$$

that is, there exists $\zeta \in T(\bar{x})$ such that

$$\langle s, \langle \zeta, y - \bar{x}\rangle\rangle \geq 0, \quad \text{for all } y \in K.$$

Analogously, it follows that

$$\langle \zeta, y - \bar{x}\rangle \notin -\operatorname{int}(C(\bar{x})), \quad \text{for all } y \in K.$$

Therefore, $\bar{x}$ is a strong solution of $(\text{GWVVIP})_w$. □

In order to solve $(\text{GWVVIP})_w$ with an unbounded domain, we need the following coercivity conditions. We first note that

$$C_+^* = \{l \in Y^* : \langle l, y\rangle \geq 0 \text{ for all } y \in C_+\},$$

and

$$\operatorname{int}(C_+^*) = \{l \in Y^* : \langle l, y\rangle > 0 \text{ for all } y \in C_+\},$$

where $C_+ = \operatorname{co}(\{C(x) : x \in K\})$.

Definition 8.9 Let X and Y be Banach spaces and K be a nonempty closed convex subset of X. Let $C : K \to 2^Y$ be a set-valued map such that $C_+^* \setminus \{\mathbf{0}\} \neq \emptyset$. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *generalized* v*-coercive* if there exist $x_0 \in K$ and $s \in C_+^* \setminus \{\mathbf{0}\}$ such that

$$\inf_{\zeta \in T_s(x)} \frac{\langle \zeta, x - x_o \rangle}{\|x - x_0\|} \to \infty, \quad \text{as } x \in K,\ \|x\| \to \infty.$$

(b) *weakly generalized* v*-coercive* if there exist $y \in K$ and $s \in C_+^* \setminus \{\mathbf{0}\}$ such that

$$\inf_{\zeta \in T_s(x)} \langle \zeta, x - y \rangle \to \infty, \quad \text{as } x \in K,\ \|x\| \to \infty.$$

It is clear that if T is generalized v-coercive, then it is weakly generalized v-coercive.

Under the assumption of the weak generalized v-coercivity of T, we have the following existence theorem for solutions of $(\text{GWVVIP})_w$ and $(\text{GWVVIP})_s$.

Theorem 8.9 *Let X, Y and C be the same as in Theorem 8.8 and, in addition, X be reflexive. Let K be a nonempty convex closed subset of X. Suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is weakly generalized $H(s)$-upper sign continuous, weakly generalized $H(s)$-pseudomonotone, and weakly generalized v-coercive with respect to an $s \in C_+^* \setminus \{\mathbf{0}\}$ on K, where $H(s) \neq Y$, and has nonempty values. Then the following statements hold.*

(a) *There exists a solution of* $(\text{GWVVIP})_w$.
(b) *If, for each $x \in K$, the set $T(x)$ is convex and weakly compact in $\mathcal{L}(X,Y)$, there exists a solution of* $(\text{GWVVIP})_s$.

Proof If, for the given $s \in C_+^* \setminus \{\mathbf{0}\}$, there exists $\bar{x} \in K$ which is a solution of $(\text{GVIP})_w$, that is, for all $y \in K$, there exists $\zeta_s \in T_s(\bar{x})$ satisfying

$$\langle \zeta_s, y - \bar{x} \rangle \geq 0.$$

Then as in the proof of Theorem 8.8, assertions (a) and (b) are true. So, for the proof of this theorem, it is sufficient to prove that there exists a solution of $(\text{GVIP})_w$.

Let B_r denote the closed ball (under the norm) of X with center at origin and radius r. In the special case where $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$ for all $x \in K \cap B_r$, Proposition 8.4 and Theorem 8.1 with Remark 8.7 guarantee the existence of a solution x_r for the following problem, denoted by $(\text{GVIP})_w^r$:

$$\begin{aligned}
&\text{Find } x_r \in K \cap B_r \text{ such that for all } y \in K \cap B_r,\\
&\text{there exists } \zeta_s \in T_s(\bar{x}) \text{ satisfying } \langle \zeta_s, y - \bar{x} \rangle \geq 0,
\end{aligned}$$

if $K \cap B_r \neq \emptyset$. Choose $r \geq \|x_0\|$, where x_0 satisfies the weak generalized υ-coercivity of T. Then for some $\zeta_s' \in T_s(\bar{x})$, we have

$$\langle \zeta_s', y - \bar{x} \rangle \geq 0. \tag{8.21}$$

We observe that $\{x_r : r > 0\}$ must be bounded. Otherwise, we can choose r large enough so that the weak generalized υ-coercivity of T yields

$$\langle \zeta_s, x_0 - \bar{x} \rangle < 0, \quad \text{for all } \zeta_s \in T_s(\bar{x}),$$

which contradicts (8.21). Therefore, there exists r such that $\|x_0\| < r$. Now, for each $x \in K$, we can choose $\varepsilon > 0$ small enough such that $x_r + \varepsilon(x - x_r) \in K \cap B_r$. Then

$$\langle \zeta_s, x_r + \varepsilon(x - x_r) - \bar{x} \rangle \geq 0, \quad \text{for some } \zeta_s \in T_s(\bar{x}).$$

Dividing by ε on both sides of the above inequality, we obtain

$$\langle \zeta_s, x - x_r \rangle \geq 0, \quad \text{for all } x \in K,$$

which shows that x_r is s solution of $(\text{GVIP})_w^s$ and the result follows. □

We now obtain similar results in the case of weak generalized $H(s)$-pseudomonotonicity.

Theorem 8.10 *Let X, Y and C be the same as in Theorem 8.8. Let K be a nonempty convex subset of X and $T : K \to 2^{\mathcal{L}(X,Y)}$ be a weakly generalized $H(s)$-upper sign continuous, weakly generalized $H(s)$-pseudomonotone mapping with nonempty compact values on K with respect to $s \in C_+^* \setminus \{\mathbf{0}\}$ where $H(s) \neq Y$. Suppose that at least one of the following conditions hold:*

(i) *K is weakly compact.*
(ii) *K is closed, T is weakly υ-coercive on K with respect to the same $s \in C_+^* \setminus \{\mathbf{0}\}$, and X is reflexive.*

Then the following statements hold.

(a) *There exists a solution of* $(\text{GWVVIP})_w$.
(b) *If, for each $x \in K$, the set $T(x)$ is convex, there exists a solution of* $(\text{GWVVIP})_s$.

Proof We first note that, in case (i), the existence of a solution to the $(\text{GVIP})_w$ defined in (8.20) is guaranteed by Theorem 8.6 (a). In addition, under assumptions of (ii), the set $T_s(x)$ is also convex and sequential compact. Therefore, in order to prove this theorem it suffices to follow the proofs of Theorems 8.8 and 8.9 with the corresponding modifications, respectively. □

Remark 8.8 Let X and Y be Banach spaces and K be a closed convex pointed cone in X. Let $C : K \to 2^Y$ be such that for all $x \in K$, $C(x)$ is a closed convex pointed cone with $\text{int}(C(x)) \neq \emptyset$. Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map with nonempty values. The *generalized vector complementarity problem* (in short, GVCP) is to find

$(\bar{x}, \bar{\zeta}) \in K \times T(\bar{x})$ such that

$$\langle \bar{\zeta}, \bar{x} \rangle \notin \operatorname{int}(C(\bar{x})) \text{ and } \langle \bar{\zeta}, y \rangle \notin -\operatorname{int}(C(\bar{x})), \quad \text{for all } y \in K.$$

It can be shown that if $(\text{GWVVIP})_s$ has a solution, then (GVCP) has a solution. Then by using Theorems 8.9 and 8.10, we can derive existence results for solutions of (GVCP). For further details, we refer [5].

Definition 8.10 Let $x \in K$ be an arbitrary element. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *generalized C_x-quasimonotone* on K if for every $y \in K$ and for all $\zeta \in T(x)$ and all $\xi \in T(y)$, we have

$$\langle \zeta, y - x \rangle \notin -C(x) \quad \text{implies} \quad \langle \xi, y - x \rangle \notin -\operatorname{int}(C(x));$$

(b) *generalized C_x-quasimonotone$^+$* on K if for every $y \in K$ and for all $\zeta \in T(x)$, we have

$$\langle \zeta, y - x \rangle \notin -C(x) \text{ implies } \langle \xi, y - x \rangle \notin -\operatorname{int}(C(x)), \text{ for some } \xi \in T(y).$$

Daniilidis and Hadjisavvas [2] established some existence results for a solution of $(\text{GWVVIP})_w$ under generalized C_x-quasimonotonicity or generalized C_x-quasimonotonicity$^+$.

Now we establish some existence results for solutions of $(\text{GSVVIP})_s$, $(\text{GVVIP})_s$ and $(\text{GWVVIP})_s$.

Definition 8.11 Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be a set-valued map. A single-valued map $f : K \to \mathcal{L}(X, Y)$ is said to be a *selection* of T if for all $x \in K$, $f(x) \in T(x)$. It is called *continuous selection* if, in addition, f is continuous

Lemma 8.6 *If u is a selection of T, then every solution of* SVVIP (5.1)*,* VVIP (5.2) *and* WVVIP (5.3) *(all these defined by means of f) is a solution of* $(\text{GSVVIP})_s$*,* $(\text{GVVIP})_s$ *and* $(\text{GWVVIP})_s$*, respectively.*

Proof Assume that $\bar{x} \in K$ is a solution of SVVIP (5.1), that is,

$$\langle f(\bar{x}), y - \bar{x} \rangle \in C(x), \quad \text{for all } y \in K.$$

Let $\bar{\zeta} = f(\bar{x})$. Then, $\bar{\zeta} \in T(\bar{x})$ such that

$$\langle \bar{\zeta}, y - \bar{x} \rangle \in C(x), \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is a solution of $(\text{GSVVIP})_s$.

Similarly, we can prove the other cases. □

Lemma 8.7 *Let $f : K \to \mathcal{L}(X,Y)$ be a selection of $T : K \to 2^{\mathcal{L}(X,Y)}$ and $x \in K$ be an arbitrary element. If T is (respectively, strongly and weakly) generalized C_x-pseudomonotone, then f is (respectively, strongly and weakly) C_x-pseudomonotone.*

Theorem 8.11 *Let X and Y be Banach spaces and K be a nonempty compact convex subset of X. Let $C : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a proper closed convex (not necessarily pointed) cone with $\text{int}(C(x)) \neq \emptyset$; and let $W : K \to 2^Y$ be defined by $W(x) = Y \setminus \{-\text{int}(C(x))\}$, such that the graph $\mathcal{G}(W)$ of W is closed in $X \times Y$. For arbitrary $x \in K$, suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty valued, weakly generalized C_x-pseudomonotone$_+$ and has continuous selection f on K. Then there exists a solution of* (GWVVIP)$_s$.

Proof By the hypothesis, there is a continuous function $f : K \to \mathcal{L}(X,Y)$ such that $f(x) \in T(x)$ for all $x \in K$. From Lemma 8.7, f is weakly C_x-pseudomonotone. Then all the conditions of Theorem 5.2 are satisfied. Hence, there exists a solution of the following WVVIP: Find $\bar{x} \in K$ such that

$$\langle f(\bar{x}), y - \bar{x} \rangle \notin -\text{int}(C(\bar{x})), \quad \text{for all } y \in K.$$

By Lemma 8.6, $\bar{x}$ is a solution of (GWVVIP)$_s$.

Similarly, by using Lemmas 8.6 and 8.7, and Theorem 5.3, we can establish the following result.

Theorem 8.12 *Let X and Y be Banach spaces and K be a nonempty compact convex subset of X. Let $C : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $C(x)$ is a proper closed convex (not necessarily pointed) cone with $\text{int}(C(x)) \neq \emptyset$; and let $W : K \to 2^Y$ be defined by $W(x) = Y \setminus \{-\text{int}(C(x))\}$, such that the graph $\mathcal{G}(W)$ of W is closed in $X \times Y$. Let $x \in K$ be arbitrary and suppose that $T : K \to 2^{\mathcal{L}(X,Y)}$ is nonempty valued, generalized C_x-pseudomonotone$_+$ and has continuous selection f on K such that the set*

$$M_g^W(y) = \{x \in K : \forall \xi \in T(y) \text{ satisfying } \langle \xi, y - x \rangle \notin -\text{int}(C(x))\}$$

is closed for all $y \in K$. Then there exists a solution of (GVVIP)$_s$.

Remark 8.9 If K is compact and $T : K \to 2^{\mathcal{L}(X,Y)}$ is continuous, then T has a continuous selection, see, for example [3].

8.3 Existence Results Without Monotonicity

Let X and Y be two Banach spaces, $K \subset X$ be a nonempty, closed and convex set, and $C \subset Y$ be a closed, convex and pointed cone with $\text{int}(C) \neq \emptyset$.

Recall that a mapping $g : X \to Y$ is said to be *completely continuous* if the weak convergence of x_n to x in X implies the strong convergence of $g(x_n)$ to $g(x)$ in Y.

Definition 8.12 Let K be a nonempty, closed and convex subset of a Banach space X and Y be a Banach space ordered by a closed, convex and pointed cone C with $\text{int}(C) \neq \emptyset$. A set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be

(a) *completely semicontinuous* if for each $y \in K$,

$$\{x \in K : \langle \zeta, y - x \rangle \in -\text{int}(C) \text{ for all } \zeta \in T(x)\}$$

is open in K with respect to the weak topology of X;

(b) *strongly semicontinuous* if for each $y \in K$,

$$\{x \in K : \langle \zeta, y - x \rangle \in -\text{int}(C) \text{ for all } \zeta \in T(x)\}$$

is open in K with respect to the norm topology of X.

Remark 8.10

(a) Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X and Y be a Banach space ordered by a closed, convex and pointed cone C with $\text{int}(C) \neq \emptyset$. Let $T : K \to \mathcal{L}(X, Y)$ be completely continuous. Then T is completely semicontinuous.
(b) Let K be a nonempty, compact and convex subset of a Banach space X and Y be a Banach space ordered by a closed, convex and pointed cone C with $\text{int}(C) \neq \emptyset$. Let $T : K \to \mathcal{L}(X, Y)$ be continuous. Then T is strongly semicontinuous.
(c) When $X = \mathbb{R}^n$, complete continuity is equivalent to continuity, and complete semicontinuity is equivalent to strong semicontinuity.

Next we state and prove the existence result for a solution of $(\text{GWVVIP})_s$ with $C(x)$ is a fixed pointed solid closed convex cone in Y.

Theorem 8.13 *Let K be a nonempty, bounded closed and convex subset of a reflexive Banach space X and Y be a Banach space ordered by a proper closed convex and pointed cone C with $\text{int}(C) \neq \emptyset$. Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be a completely semicontinuous set-valued map with nonempty values. Then there exists a solution of $(\text{GWVVIP})_s$ for a fixed pointed solid closed convex cone C in Y, that is, there exist $\bar{x} \in K$ and $\zeta \in T(\bar{x})$ such that*

$$\langle \zeta, y - \bar{x} \rangle \notin -\text{int}(C), \quad \textit{for all } y \in K.$$

Proof Suppose that the conclusion is not true. Then for each $\hat{x} \in K$, there exists $y \in K$ such that

$$\langle \hat{\zeta}, y - \hat{x} \rangle \in -\text{int}(C), \quad \text{for all } \hat{\zeta} \in T(\hat{x}). \tag{8.22}$$

For every $y \in K$, define the set N_y as

$$N_y = \{x \in K : \langle \zeta, y - x \rangle \in -\text{int}(C) \text{ for all } \zeta \in T(x)\}.$$

Since T is completely semicontinuous, the set N_y is open in K with respect to the weak topology of X for every $y \in K$.

We assert that $\{N_y : y \in K\}$ is an open cover of K with respect to the weak topology of X. Indeed, first it is easy to see that

$$\bigcup_{y \in K} N_y \subseteq K.$$

Second, for each $\hat{x} \in K$, by (8.22) there exists $y \in K$ such that $\hat{x} \in N_y$. Hence $\hat{x} \in \bigcup_{y \in K} N_y$. This shows that $K \subseteq \bigcup_{y \in K} N_y$. Consequently,

$$K = \bigcup_{y \in K} N_y.$$

So, the assertion is valid.

The weak compactness of K implies that there exists a finite set of elements $\{y_1, y_2, \ldots, y_m\} \subseteq K$ such that $K = \bigcup_{i=1}^{m} N_{y_i}$. Hence there exists a continuous (with respect to the weak topology of X) partition of unity $\{\beta_1, \beta_2, \ldots, \beta_m\}$ subordinated to $\left\{N_{y_1}, N_{y_2}, \ldots, N_{y_m}\right\}$ such that $\beta_j(x) \geq 0$ for all $x \in K, j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} \beta_j(x) = 1$ for all $x \in K$, and

$$\beta_j(x) \begin{cases} = 0, & \text{whenever } x \notin N_{y_j}, \\ > 0, & \text{whenever} x \in N_{y_j}. \end{cases}$$

Let $p : K \to X$ be defined by

$$p(x) = \sum_{j=1}^{m} \beta_j(x) y_j, \quad \text{for all } x \in K. \tag{8.23}$$

Since β_i is continuous with respect to the weak topology of X for each i, p is continuous with respect to the weak topology of X. Let $\Delta := \mathrm{co}(\{y_1, y_2, \ldots, y_m\}) \subseteq K$. Then Δ is a simplex of a finite dimensional space and p maps Δ into itself. By Brouwer's Fixed Point Theorem 1.39, there exists $\tilde{x} \in \Delta$ such that $p(\tilde{x}) = \tilde{x}$. For any given $x \in K$, let

$$k(x) = \left\{j : x \in N_{y_j}\right\} = \left\{j : \beta_j(x) > 0\right\}.$$

Obviously, $k(x) \neq \emptyset$.

Since $\tilde{x} \in \Delta \subseteq K$ is a fixed point of p, we have $p(\tilde{x}) = \sum_{j=1}^{m} \beta_j(\tilde{x})y_j$ and hence by the definition of N_y, we derive for each $\tilde{\zeta} \in T(\tilde{x})$

$$\begin{aligned}
\mathbf{0} &= \langle \tilde{\zeta}, \tilde{x} - \tilde{x} \rangle \\
&= \langle \tilde{\zeta}, \tilde{x} - p(\tilde{x}) \rangle \\
&= \left\langle \tilde{\zeta}, \tilde{x} - \sum_{j=1}^{m} \beta_j(\tilde{x})y_j \right\rangle \\
&= \sum_{j \in k(x_0)} \beta_j(x_0) \langle \tilde{\zeta}, \tilde{x} - y_j \rangle \in \operatorname{int}(C)
\end{aligned}$$

which leads to a contradiction. Therefore, there exist $\bar{x} \in K$ and $\zeta \in T(\bar{x})$ such that

$$\langle \zeta, y - \bar{x} \rangle \notin -\operatorname{int}(C), \quad \text{for all } y \in K.$$

This completes the proof. □

The proof of the following result can be easily derived on the lines of the proof of Theorem 8.13.

Theorem 8.14 *Let K be a nonempty, compact and convex subset of a Banach space X and Y be a Banach space ordered by a proper closed convex and pointed cone C with $\operatorname{int}(C) \neq \emptyset$. Let $T : K \to 2^{\mathcal{L}(X,Y)}$ be strongly semicontinuous with nonempty values. Then there exist $\bar{x} \in K$ and $\zeta \in T(\bar{x})$ such that*

$$\langle \zeta, y - \bar{x} \rangle \notin -\operatorname{int}(C), \quad \textit{for all } y \in K.$$

Now we establish an existence theorem for a solution of $(\text{GWVVIP})_g$ under lower semicontinuity assumption on the underlying set-valued map T.

Theorem 8.15 *Let X and Y be Hausdorff topological vector spaces, K be a nonempty convex subset of X and the set-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ be lower semicontinuous such that the set*

$$A_x := \{y \in K : \langle \zeta, y - x \rangle \in -\operatorname{int}(C(x)) \textit{ for all } \zeta \in T(x)\}$$

is convex for all $x \in K$. Let the set-valued map $W : K \to 2^Y$, defined by $W(x) = Y \setminus \{-\operatorname{int}(C(x))\}$ for all $x \in K$, be closed. Assume that for a nonempty compact convex set $D \subset K$ with each $x \in D \setminus K$, there exists $y \in D$ such that for any $\zeta \in T(x)$, $\langle \zeta, y - x \rangle \in -\operatorname{int}(C(x))$. Then $(\text{GWVVIP})_g$ has a solution.

Proof Let

$$A = \{(x, y) \in K \times K : \langle \zeta, y - x \rangle \notin -\operatorname{int}(C(x)) \text{ for all } \zeta \in T(x)\}.$$

Clearly, $(x,x) \in A$ for all $x \in K$. For each fixed $y \in K$, let

$$\begin{aligned} A_y &:= \{x \in K : (x,y) \in A\} \\ &= \{x \in K : \langle \zeta, y-x\rangle \notin -\operatorname{int}(C(x)) \text{ for all } \zeta \in T(x)\}. \end{aligned}$$

Then by Proposition 8.2 (f), A_y is closed. By hypothesis, for each fixed $y \in K$, the set $A_x := \{y \in K : (x,y) \notin A\}$ is convex.

By Lemma 1.17, there exists $\bar{x} \in K$ such that $\{\bar{x}\} \times K \subset A$, that is, $\bar{x} \in K$ such that $\langle \xi, \bar{x}-y\rangle \notin -\operatorname{int}(C(\bar{x}))$, for all $\xi \in T(\bar{x})$ and $y \in K$. □

8.4 Generalized Vector Variational Inequalities and Optimality Conditions for Vector Optimization Problems

Throughout this section, unless otherwise specified, we assume that K is a nonempty convex subset of $\mathbb{R}^n$ and $f = (f_1, f_2, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. The subdifferential of a convex function f_i is denoted by ∂f_i.

Corresponding to K and ∂f_i, the (Stampacchia) generalized vector variational inequality problems and Minty generalized vector variational inequality problems are defined as follows:

$(\mathrm{GVVIP})^\ell_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$,

$$\langle \bar{\zeta}, y - \bar{x}\rangle_\ell := \big(\langle \bar{\zeta}_1, y-\bar{x}\rangle, \ldots, \langle \bar{\zeta}_\ell, y-\bar{x}\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{8.24}$$

$(\mathrm{GVVIP})^\ell_s$: Find $\bar{x} \in K$ such that there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, such that for all $y \in K$

$$\langle \bar{\zeta}, y - \bar{x}\rangle_\ell := \big(\langle \bar{\zeta}_1, y-\bar{x}\rangle, \ldots, \langle \bar{\zeta}_\ell, y-\bar{x}\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{8.25}$$

$(\mathrm{GVVIP})^\ell_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, satisfying

$$\langle \bar{\zeta}, y - \bar{x}\rangle_\ell := \big(\langle \bar{\zeta}_1, y-\bar{x}\rangle, \ldots, \langle \bar{\zeta}_\ell, y-\bar{x}\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{8.26}$$

$(\mathrm{MGVVIP})^\ell_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$,

$$\langle \xi, y - \bar{x}\rangle_\ell := \big(\langle \xi_1, y-\bar{x}\rangle, \ldots, \langle \xi_\ell, y-\bar{x}\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{8.27}$$

(MGVVIP)$^\ell_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exist $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$,

$$\langle \xi, y - \bar{x} \rangle_\ell := \big(\langle \xi_1, y - \bar{x} \rangle, \ldots, \langle \xi_\ell, y - \bar{x} \rangle \big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{8.28}$$

(GWVVIP)$^\ell_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$,

$$\langle \bar{\zeta}, y - \bar{x} \rangle_\ell := \big(\langle \bar{\zeta}_1, y - \bar{x} \rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x} \rangle \big) \notin -\mathrm{int}\left(\mathbb{R}^\ell_+ \right). \tag{8.29}$$

(GWVVIP)$^\ell_s$: Find $\bar{x} \in K$ such that there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, such that for all $y \in K$

$$\langle \bar{\zeta}, y - \bar{x} \rangle_\ell := \big(\langle \bar{\zeta}_1, y - \bar{x} \rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x} \rangle \big) \notin -\mathrm{int}\left(\mathbb{R}^\ell_+ \right). \tag{8.30}$$

(GWVVIP)$^\ell_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, satisfying

$$\langle \bar{\zeta}, y - \bar{x} \rangle_\ell := \big(\langle \bar{\zeta}_1, y - \bar{x} \rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x} \rangle \big) \notin -\mathrm{int}\left(\mathbb{R}^\ell_+ \right). \tag{8.31}$$

(MGWVVIP)$^\ell_g$: Find $\bar{x} \in K$ such that for all $y \in K$ and all $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$,

$$\langle \xi, y - \bar{x} \rangle_\ell := \big(\langle \xi_1, y - \bar{x} \rangle, \ldots, \langle \xi_\ell, y - \bar{x} \rangle \big) \notin -\mathrm{int}\left(\mathbb{R}^\ell_+ \right). \tag{8.32}$$

(MGWVVIP)$^\ell_w$: Find $\bar{x} \in K$ such that for all $y \in K$, there exist $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, such that

$$\langle \xi, y - \bar{x} \rangle_\ell := \big(\langle \xi_1, y - \bar{x} \rangle, \ldots, \langle \xi_\ell, y - \bar{x} \rangle \big) \notin -\mathrm{int}\left(\mathbb{R}^\ell_+ \right). \tag{8.33}$$

We denote the solution sets of the above mentioned problems (GVVIP)$^\ell_g$, (GVVIP)$^\ell_s$, (GVVIP)$^\ell_w$, (MGVVIP)$^\ell_g$, (MGVVIP)$^\ell_w$, (GWVVIP)$^\ell_g$, (GWVVIP)$^\ell_s$, (GWVVIP)$^\ell_w$, (MGWVVIP)$^\ell_g$ and (MGWVVIP)$^\ell_w$ by Sol(GVVIP)$^\ell_g$, Sol(GVVIP)$^\ell_s$, Sol(GVVIP)$^\ell_w$, Sol(MGVVIP)$^\ell_g$, Sol(MGVVIP)$^\ell_w$, Sol(GWVVIP)$^\ell_g$, Sol(GWVVIP)$^\ell_s$, Sol(GWVVIP)$^\ell_w$, Sol(MGWVVIP)$^\ell_g$ and Sol(MGWVVIP)$^\ell_w$, respectively.

As in Remark 8.1, we have

(a) Sol(GVVIP)$^\ell_g \subseteq$ Sol(GVVIP)$^\ell_s \subseteq$ Sol(GVVIP)$^\ell_w$;
(b) Sol(GWVVIP)$^\ell_g \subseteq$ Sol(GWVVIP)$^\ell_s \subseteq$ Sol(GWVVIP)$^\ell_w$;
(c) Sol(GVVIP)$^\ell_g \subseteq$ Sol(GWVVIP)$^\ell_g$;
(d) Sol(GVVIP)$^\ell_s \subseteq$ Sol(GWVVIP)$^\ell_s$;

(e) $\text{Sol(GVVIP)}_w^{\ell} \subseteq \text{Sol(GWVVIP)}_w^{\ell}$;
(f) $\text{Sol(MGVVIP)}_g^{\ell} \subseteq \text{Sol(MGWVVIP)}_g^{\ell}$;
(g) $\text{Sol(MGVVIP)}_w^{\ell} \subseteq \text{Sol(MGWVVIP)}_w^{\ell}$.

The following example shows that $\text{Sol(GVVIP)}_w^{\ell} \subseteq \text{Sol(GVVIP)}_s^{\ell}$ may not be true.

Example 8.5 [7] Let $K = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0,\ -\sqrt{-x_1} \leq x_2 \leq 0\right\}$ and

$$f_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2} + x_2, \quad \text{for all } (x_1, x_2) \in K,$$
$$f_2(x_1, x_2) = x_2, \quad \text{for all } (x_1, x_2) \in K.$$

If $(x_1, x_2) = (0, 0)$, then

$$\begin{aligned}\partial f_1(x_1, x_2) &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \zeta_1^2 + \zeta_2^2 \leq 1\} + \{(0, 1)\} \\ &= \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 : \zeta_1^2 + (\zeta_2 - 1)^2 \leq 1\}.\end{aligned}$$

If $(x_1, x_2) \neq (0, 0)$, then

$$\partial f_1(x_1, x_2) = \left\{\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} + 1\right)\right\}.$$

It can be easily checked that for all $(\zeta_1, \zeta_2) \in \partial f_1(0, 0)$, there exists $(x_1, x_2) \in K$ such that

$$(\zeta_1 x_1 + \zeta_2 x_2, x_2) \in -\mathbb{R}_+^2 \setminus \{\mathbf{0}\},$$

and that for all $(x_1, x_2) \in K$, there exists $(\xi_1, \xi_2) \in \partial f_1(0, 0)$ such that

$$(\xi_1 x_1 + \xi_2 x_2, x_2) \notin -\mathbb{R}_+^2 \setminus \{\mathbf{0}\}.$$

Hence, $(0, 0) \in \text{Sol(GVVIP)}_w^{\ell}$, but $(0, 0) \notin \text{Sol(GVVIP)}_s^{\ell}$.

Moreover, $\text{Sol(GVVIP)}_s^{\ell} = \left\{(x, -\sqrt{-x}) : x < 0\right\}$ and $\text{Sol(GVVIP)}_w^{\ell} = \left\{(x, -\sqrt{-x}) : x \leq 0\right\}$.

Proposition 8.5 *For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be convex. Then* $\text{Sol(GVVIP)}_w^{\ell} \subseteq \text{Sol(MGVVIP)}_g^{\ell} \subseteq \text{Sol(MGVVIP)}_w^{\ell}$.

Proof Let $\bar{x} \in K$ be a solution of (GVVIP)_w^{ℓ}. Then for all $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, such that

$$\left(\langle \bar{\zeta}_1, y - \bar{x}\rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x}\rangle\right) \notin -\mathbb{R}_+^{\ell} \setminus \{\mathbf{0}\}. \tag{8.34}$$

Since each f_i is convex, ∂f_i, $i \in \mathscr{I}$, is monotone, and therefore, we have

$$\langle \xi_i - \bar{\zeta}_i, y - \bar{x} \rangle \geq 0, \quad \text{for all } \xi_i \in \partial f_i(y) \text{ and for each } i \in \mathscr{I}. \tag{8.35}$$

From (8.34) and (8.35), it follows that for all $y \in K$ and all $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I}$,

$$\left(\langle \xi_1, y - \bar{x} \rangle, \ldots, \langle \xi_\ell, y - \bar{x} \rangle\right) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

Thus, $\bar{x} \in K$ is a solution of $(\text{MGVVIP})^\ell_g$. □

The converse of the above proposition may not be true, that is, $\text{Sol}(\text{MGVVIP})^\ell_g \not\subseteq \text{Sol}(\text{GVVIP})^\ell_w$.

Example 8.6 Let $K =]-\infty, 0]$ and $f_1(x) = x, f_2(x) = x^2$. Since $(x, 0) \in -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ for all $x \in]-\infty, 0[$, we have $0 \notin \text{Sol}(\text{GVVIP})^\ell_w$.

But, since $(x, 2x^2) \notin -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}$, we have $0 \in \text{Sol}(\text{MGVVIP})^\ell_g$. Moreover, we can easily verify that $\text{Sol}(\text{GVVIP})^\ell_w =]-\infty, 0[$ and $\text{Sol}(\text{MGVVIP})^\ell_g =]-\infty, 0]$.

The following result provides the relationship between the solutions of $(\text{MGWVVIP})^\ell_g$ and $(\text{GWVVIP})^\ell_g$.

Theorem 8.16 *For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be convex. Then $\bar{x} \in K$ is a solution $(\text{GWVVIP})^\ell_w$ if and only if it is a solution of $(\text{MGWVVIP})^\ell_w$.*

Proof Let $\bar{x} \in K$ be a solution of $(\text{GWVVIP})^\ell_w$. Then for any $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, such that

$$\left(\langle \bar{\zeta}_1, y - \bar{x} \rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x} \rangle\right) \notin -\text{int}\left(\mathbb{R}^\ell_+\right). \tag{8.36}$$

Since each f_i is convex, ∂f_i $(i \in \mathscr{I})$ is monotone, and therefore, we have

$$\langle \xi_i - \bar{\zeta}_i, y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K,\ \xi_i \in \partial f_i(y) \text{ and for each } i \in \mathscr{I}. \tag{8.37}$$

From (8.36) and (8.36), it follows that for any $y \in K$ and any $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I}$,

$$\left(\langle \xi_1, y - \bar{x} \rangle, \ldots, \langle \xi_\ell, y - \bar{x} \rangle\right) \notin -\text{int}\left(\mathbb{R}^\ell_+\right).$$

Thus, $\bar{x} \in K$ is a solution of $(\text{MGWVVIP})^\ell_g$. Since $\text{Sol}(\text{MGWVVIP})^\ell_g \subseteq \text{Sol}(\text{MGWVVIP})^\ell_w$, $\bar{x} \in K$ is a solution of $\text{Sol}(\text{MGWVVIP})^\ell_w$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{MGWVVIP})^\ell_w$. Consider any $y \in K$ and any sequence $\{\alpha_m\} \searrow 0$ with $\alpha_m \in]0, 1]$. Since K is convex, $y_m := \bar{x} + \alpha_m(y - \bar{x}) \in K$. Since $\bar{x} \in K$ is a solution of $(\text{MGWVVIP})^\ell_w$, there exist $\xi_i^m \in \partial f_i(y_m)$, $i \in \mathscr{I}$, such that

$$\left(\langle \xi_1^m, y_m - \bar{x})\rangle, \ldots, \langle \xi_\ell^m, \eta(y_m, \bar{x})\rangle\right) \notin -\text{int}\left(\mathbb{R}^\ell_+\right).$$

Since each f_i is convex and so it is locally Lipschitz (see Theorem 1.16), and hence, there exists $k > 0$ such that for sufficiently large m and for all $i \in \mathscr{I}$, $\left\|\xi_i^m\right\| \leq k$. So, we can assume that the sequence $\left\{\xi_i^m\right\}$ converges to $\bar{\zeta}_i$ for each $i \in \mathscr{I}$. Since the set-valued map $y \mapsto \partial f_i(y)$ is closed (see Lemma 1.8), $\xi_i^m \in \partial f_i(y_m)$ and $y_m \to \bar{x}$ as $m \to \infty$, we have $\bar{\zeta}_i \in \partial f_i(\bar{x})$ for each $i \in \mathscr{I}$. Therefore, for any $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i \in \mathscr{I}$, such that

$$\left(\langle\bar{\zeta}_1, y-\bar{x}\rangle, \ldots, \langle\bar{\zeta}_\ell, y-\bar{x}\rangle\right) \notin -\operatorname{int}\left(\mathbb{R}_+^\ell\right).$$

Hence, $\bar{x} \in K$ is a solution of (GWVVIP)$_w^\ell$. □

Next theorem provides the necessary and sufficient conditions for an efficient solution of VOP.

Theorem 8.17 ([6]) *For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be convex. Then $\bar{x} \in K$ is an efficient solution of VOP if and only if it is a solution of* (MGVVIP)$_w^\ell$.

Proof Let $\bar{x} \in K$ be a solution of (MGVVIP)$_w^\ell$ but not an efficient solution of VOP. Then there exists $z \in K$ such that

$$\left(f_1(z) - f_1(\bar{x}), \ldots, f_\ell(z) - f_\ell(\bar{x})\right) \in -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}. \tag{8.38}$$

Set $z(\lambda) := \lambda z + (1-\lambda)\bar{x}$ for all $\lambda \in [0,1]$. Since K is convex, $z(\lambda) \in K$ for all $\lambda \in [0,1]$. Since each f_i is convex, we have

$$f_i(z(\lambda)) = f_i(\lambda z + (1-\lambda)\bar{x}) \leq \lambda f_i(z) + (1-\lambda) f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell,$$

that is,

$$f_i(\bar{x} + \lambda(z-\bar{x})) - f_i(\bar{x}) \leq \lambda[f_i(z) - f_i(\bar{x})],$$

for all $\lambda \in [0,1]$ and for each $i = 1, 2, \ldots, \ell$. In particular, for $\lambda \in]0,1[$, we have

$$\frac{f_i(z(\lambda)) - f_i(\bar{x})}{\lambda} \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell. \tag{8.39}$$

By Lebourg's Mean Value Theorem 1.32, there exist $\lambda_i \in]0,1[$ and $\xi_i \in \partial f_i(z(\lambda_i))$ such that

$$\langle\xi_i, z-\bar{x}\rangle = f_i(z(\lambda)) - f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell. \tag{8.40}$$

By combining (8.39)–(8.40), we obtain

$$\langle\xi_i, z-\bar{x}\rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell. \tag{8.41}$$

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are all equal. Then it follows from (8.38) and (8.41) that $\bar{x}$ is not a solution of $(\text{MGVVIP})_w^\ell$. This contradicts to the fact the $\bar{x}$ is a solution of $(\text{MGVVIP})_w^\ell$.

Consider the case when $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are not equal. Let $\lambda_1 \neq \lambda_2$. Then from (8.41), we have

$$\langle \xi_1, z - \bar{x} \rangle \leq f_1(z) - f_1(\bar{x}) \tag{8.42}$$

and

$$\langle \xi_2, z - \bar{x} \rangle \leq f_2(z) - f_2(\bar{x}). \tag{8.43}$$

Since f_i and f_2 are convex, ∂f_1 and ∂f_2 are monotone, that is,

$$\langle \xi_1 - \xi_2^*, z(\lambda_1) - z(\lambda_2) \rangle \geq 0, \quad \text{for all } \xi_2^* \in \partial f_1(z(\lambda_2)), \tag{8.44}$$

and

$$\langle \xi_1^* - \xi_2, z(\lambda_1) - z(\lambda_2) \rangle \geq 0, \quad \text{for all } \xi_1^* \in \partial f_2(z(\lambda_1)). \tag{8.45}$$

If $\lambda_1 > \lambda_2$, then by (8.44), we obtain

$$0 \leq \langle \xi_1 - \xi_2^*, z(\lambda_1) - z(\lambda_2) \rangle = (\lambda_1 - \lambda_2) \langle \xi_1 - \xi_2^*, z - \bar{x} \rangle,$$

and so,

$$\langle \xi_1 - \xi_2^*, z - \bar{x}) \rangle \geq 0 \quad \Leftrightarrow \quad \langle \xi_1, z - \bar{x} \rangle \geq \langle \xi_2^*, z - \bar{x} \rangle.$$

From (8.42), we have

$$\langle \xi_2^*, z - \bar{x} \rangle \leq f_1(z) - f_1(\bar{x}), \quad \text{for all } \xi_2^* \in \partial f_1(z(\lambda_2)).$$

If $\lambda_1 < \lambda_2$, then by (8.45), we have

$$0 \leq \langle \xi_1^* - \xi_2, z(\lambda_1) - z(\lambda_2) \rangle = (\lambda_1 - \lambda_2) \langle \xi_1^* - \xi_2, z - \bar{x} \rangle,$$

and so,

$$\langle \xi_1^* - \xi_2, z - \bar{x} \rangle \leq 0 \quad \Leftrightarrow \quad \langle \xi_1^*, z - \bar{x} \rangle \leq \langle \xi_2, z - \bar{x} \rangle.$$

From (8.43), we obtain

$$\langle \xi_1^*, z - \bar{x} \rangle \leq f_2(z) - f_2(\bar{x}), \quad \text{for all } \xi_1^* \in \partial f_2(z(\lambda_1)).$$

Therefore, for the case $\lambda_1 \neq \lambda_2$, let $\bar{\lambda} = \min\{\lambda_1, \lambda_2\}$. Then, we can find $\bar{\xi}_i \in \partial f_i(z(\bar{\lambda}))$ such that

$$\langle \bar{\xi}_i, z - \bar{x} \rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for all } i = 1, 2.$$

By continuing this process, we can find $\lambda^* \in]0, 1[$ and $\xi_i^* \in \partial f_i(z(\lambda^*))$ such that $\lambda^* = \min\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ and

$$\langle \xi_i^*, z - \bar{x} \rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell. \tag{8.46}$$

From (8.38) and (8.46), we have $\xi_i^* \in \partial f_i(z(\lambda^*))$, $i = 1, 2, \ldots, \ell$, and

$$\left(\langle \xi_1^*, z - \bar{x} \rangle, \ldots, \langle \xi_\ell^*, z - \bar{x} \rangle\right) \in -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}.$$

By multiplying above inclusion by $-\lambda^*$, we obtain

$$\left(\langle \xi_1^*, z(\lambda^*) - \bar{x} \rangle, \ldots, \langle \xi_\ell^*, z(\lambda^*) - \bar{x} \rangle\right) \in -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}.$$

which contradicts to our supposition that $\bar{x}$ is a solution of $(\text{MGVVIP})_w^\ell$.

Conversely, suppose that $\bar{x} \in K$ is an efficient solution of VOP. Then we have

$$\left(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\right) \notin -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{8.47}$$

Since each f_i is convex, we deduce that

$$\langle \xi_i, \bar{x} - y \rangle \leq f_i(\bar{x}) - f_i(y), \quad \text{for all } y \in K,\ \xi_i \in \partial f_i(y) \text{ and } i \in \mathscr{I}.$$

Also, we obtain

$$\langle \xi_i, y - \bar{x} \rangle \geq f_i(y) - f_i(\bar{x}), \quad \text{for all } y \in K,\ \xi_i \in \partial f_i(y) \text{ and } i \in \mathscr{I}. \tag{8.48}$$

From (8.47) and (8.48), it follows that $\bar{x}$ is a solution of $(\text{MGVVIP})_w^\ell$. □

Theorem 8.17 is extended for Dini subdifferential by Al-Homidan and Ansari [1].

Theorem 8.18 *[6] For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be convex. If $\bar{x} \in K$ is a solution* $(\text{GVVIP})_w^\ell$, *then it is an efficient solution of* VOP *and hence a solution of* $(\text{MGVVIP})_w^\ell$.

Proof Since $\bar{x} \in X$ is a solution of $(\text{GVVIP})_w^\ell$, for any $y \in K$, there exist $\bar{\zeta}_i \in \partial f_i(\bar{x})$, $i = 1, 2, \ldots, \ell$, such that

$$\left(\langle \bar{\zeta}_1, y - \bar{x} \rangle, \ldots, \langle \bar{\zeta}_\ell, y - \bar{x} \rangle\right) \notin -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}. \tag{8.49}$$

Since each f_i is convex, we have

$$\langle \bar{\zeta}_i, y - \bar{x} \rangle \leq f_i(y) - f_i(\bar{x}) \quad \text{for any } y \in K \text{ and all } i \in \mathscr{I}. \tag{8.50}$$

By combining (8.49) and (8.50), we obtain

$$\big(f_1(y)-f_1(\bar{x}),\ldots,f_\ell(y)-f_\ell(\bar{x})\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is an efficient solution of VOP. □

From Theorem 8.18, we see that $(\text{GVVIP})^\ell_w$ is a sufficient optimality condition for an efficient solution of VOP. However, it is not, in general, a necessary optimality condition for an efficient solution of VOP.

Example 8.7 Let $K = [-1, 0]$ and $f(x) = (x, x^2)$. Consider the following differentiable convex vector optimization problem:

$$\text{minimize } f(x), \quad \text{subject to } x \in K, \tag{VOP}$$

Then $\bar{x} = 0$ is an efficient solution of VOP and $\bar{x} = 0$ is a solution of the following (MVVIP): Find $\bar{x} \in K$ such that for all $y \in K$,

$$\big(\langle \nabla f_1(y), y-\bar{x}\rangle, \langle \nabla f_2(y), y-\bar{x}\rangle\big) = \big(y-\bar{x}, 2y(y-\bar{x})\big) \notin -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}.$$

However, $\bar{x} = 0$ is not a solution of the following (VVIP): Find $\bar{x} \in K$ such that for all $y \in K$,

$$\big(\langle \nabla f_1(\bar{x}), y-\bar{x}\rangle, \langle \nabla f_2(\bar{x}), y-\bar{x}\rangle\big) = \big(y-\bar{x}, 2\bar{x}(y-\bar{x})\big) \notin -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}.$$

The following result presents the equivalence between the solution of $(\text{GWVVIP})^\ell_w$ and a weakly efficient solution of VOP.

Theorem 8.19 *For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be convex. If $\bar{x} \in K$ is a weakly efficient solution of* VOP *if and only if it is a solution of* $(\text{GWVVIP})^\ell_w$.

Proof Suppose that $\bar{x}$ is a solution of $(\text{GWVVIP})^\ell_w$ but not a weakly efficient solution of VOP. Then there exists $y \in K$ such that

$$\big(f_1(y)-f_1(\bar{x}),\ldots,f_\ell(y)-f_\ell(\bar{x})\big) \in -\text{int}\left(\mathbb{R}^\ell_+\right). \tag{8.51}$$

Since each f_i, $i \in \mathscr{I}$, is convex, we have

$$\langle \zeta_i, y-\bar{x}\rangle \le f_i(y)-f_i(\bar{x}), \quad \text{for all } \zeta_i \in \partial f_i(\bar{x}). \tag{8.52}$$

Combining (8.51) and (8.52), we obtain

$$\big(\langle \zeta_1, y-\bar{x}\rangle, \ldots, \langle \zeta_\ell, y-\bar{x}\rangle\big) \in -\text{int}\left(\mathbb{R}^\ell_+\right), \quad \text{for all } \zeta_i \in \partial f_i(\bar{x})$$

which contradicts to our supposition that $\bar{x}$ is a solution of $(\text{GWVVIP})^\ell_w$.

Conversely, assume that $\bar{x} \in K$ is a weakly efficient solution of VOP but not a solution of $(\text{GWVVIP})^\ell_w$. Then by Theorem 8.16, $\bar{x}$ is not a solution of

$(\text{MGWVVIP})_w^\ell$. Thus, there exist $y \in K$ and $\xi_i \in \partial f_i(y)$, $i \in \mathscr{I}$, such that

$$\left(\langle \xi_1, y - \bar{x}\rangle, \ldots, \langle \xi_\ell, y, \bar{x}\rangle\right) \in -\text{int}\left(\mathbb{R}_+^\ell\right). \tag{8.53}$$

By convexity of f_i, $i \in \mathscr{I}$, we have

$$0 > \langle \xi_i, y - \bar{x}\rangle \geq f_i(y) - f_i(\bar{x}). \tag{8.54}$$

From (8.53) and (8.54), we then have

$$\left(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\right) \in -\text{int}\left(\mathbb{R}_+^\ell\right).$$

which contradicts to our assumption that $\bar{x}$ is a weakly efficient solution of VOP. □

The following example shows that the weakly efficient solution of VOP may not be a solution of $(\text{GWVVIP})_g^\ell$.

Example 8.8 ([7]) Let $K =]-\infty, 0]$ and

$$f_1(x) = x, \quad f_2(x) = \begin{cases} x^2, & x < 0 \\ x, & x \geq 0. \end{cases}$$

Then $\text{sol}(\text{GWVVIP})_g^\ell =]-\infty, 0[$, but the set of weakly efficient solution of VOP is $]-\infty, 0]$.

The relations between a properly efficient solution in the sense of Geoffrion and a solution of $(\text{GVVIP})_w^\ell$ is studied in [6].

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