Chapter 7
Nonsmooth Vector Variational Inequalities

We have seen in Chap. 5 that if the objective function of a vector optimization problem is smooth (that is, differentiable), then its solution, namely, weak efficient solution, strong efficient solution, efficient solution, properly efficient solution, can be characterized by the corresponding vector variational inequality problems. If the objective function is not smooth but it has some kind of directional derivative, namely, (upper or lower) Dini directional derivative, Clarke directional derivative, Dini-Hadamard directional derivative, etc., then the vector variational inequality problems studied in Chap. 5 would not be useful, and therefore, we need to define different kinds of vector variational inequality problems by means of bifunctions, called nonsmooth vector variational inequality problems. In the formulation of nonsmooth vector variational inequality problems, we consider different kinds of directional derivatives as a bifunction. For a comprehensive study of different kinds of directional derivatives and nonsmooth (scalar) variational inequalities, we refer the recent book [2]. Some recent papers on this topic are [1, 5, 7–9].

In this chapter, we define different kinds of nonsmooth vector variational inequality problems by means of a bifunction. Several existence results for solutions of these nonsmooth vector variational inequality problems are studied. We give some relations among different kinds of solutions of nonsmooth vector optimization problems and nonsmooth vector variational inequality problems.

7.1 Formulations and Preliminary Results

Throughout the section, unless otherwise specified, we assume that K is a nonempty convex subset of $\mathbb{R}^n$ and $C = \mathbb{R}^\ell_+$. Let $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function such that for each fixed $x \in K$, $h(x; d)$ is positively homogeneous in d, that is, $h(x; \alpha d) = \alpha h(x; d)$ for all $\alpha > 0$.

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We consider the following nonsmooth vector variational inequality problems:

- *Strong h-Vector Variational Inequality Problem* (h-SVVIP): Find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) = \big(h_1(\bar{x}; y - \bar{x}), \ldots, h_\ell(\bar{x}; y - \bar{x})\big) \in C, \quad \text{for all } y \in K. \tag{7.1}$$

- *h-Vector Variational Inequality Problem* (h-VVIP): Find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) = \big(h_1(\bar{x}; y - \bar{x}), \ldots, h_\ell(\bar{x}; y - \bar{x})\big) \notin -C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.2}$$

- *Weak h-Vector Variational Inequality Problem* (h-WVVIP): Find $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) = \big(h_1(\bar{x}; y - \bar{x}), \ldots, h_\ell(\bar{x}; y - \bar{x})\big) \notin -\text{int}(C), \quad \text{for all } y \in K. \tag{7.3}$$

As we have seen in Chap. 5, the Minty vector variational inequality problems are closely related to the (Stampacchia) vector variational inequality problems, therefore, we also consider the following Minty nonsmooth vector variational inequality problems.

- *Minty Strong h-Vector Variational Inequality Problem* (h-MSVVIP): Find $\bar{x} \in K$ such that

$$h(y; \bar{x} - y) = \big(h_1(y; \bar{x} - y), \ldots, h_\ell(y; \bar{x} - y)\big) \in -C, \quad \text{for all } y \in K. \tag{7.4}$$

- *Minty h-Vector Variational Inequality Problem* (h-MVVIP): Find $\bar{x} \in K$ such that

$$h(y; \bar{x} - y) = \big(h_1(y; \bar{x} - y), \ldots, h_\ell(y; \bar{x} - y)\big) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.5}$$

- *Minty Weak h-Vector Variational Inequality Problem* (h-MWVVIP): Find $\bar{x} \in K$ such that

$$h(y; \bar{x} - y) = \big(h_1(y; \bar{x} - y), \ldots, h_\ell(y; \bar{x} - y)\big) \notin \text{int}(C), \quad \text{for all } y \in K. \tag{7.6}$$

The set of solutions of h-SVVIP, h-VVIP, h-WVVIP, h-MSVVIP, h-MVVIP and h-MWVVIP are denoted by Sol(h-SVVIP), Sol(h-VVIP), Sol(h-WVVIP), Sol(h-MSVVIP), Sol(h-MVVIP) and Sol(h-MWVVIP), respectively.

Let $f = (f_1, f_2, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be avector-valued function and

$$D^+ f(x; d) = \big(D^+ f_1(x; d), \ldots, D^+ f_\ell(x; d)\big),$$

where $D^+ f_i(x; d)$ denotes the upper Dini directional derivative of f_i at x in the direction d.

When $h(x; \cdot) = D^+ f(x; \cdot)$, then h-SVVIP, h-VVIP, h-WVVIP, h-MSVVIP, h-MVVIP and h-MWVVIP become the following nonsmooth vector variational inequality problems.

- (D^+-SVVIP): Find $\bar{x} \in K$ such that

$$D^+ f(\bar{x}; y - \bar{x}) \in C, \quad \text{for all } y \in K. \tag{7.7}$$

- (D^+-VVIP): Find $\bar{x} \in K$ such that

$$D^+ f(\bar{x}; y - \bar{x}) \notin -C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.8}$$

- (D^+-WVVIP): Find $\bar{x} \in K$ such that

$$D^+ f(\bar{x}; y - \bar{x}) \notin -\text{int}(C), \quad \text{for all } y \in K. \tag{7.9}$$

- (D^+-MSVVIP): Find $\bar{x} \in K$ such that

$$D^+ f(y; \bar{x} - y) \in -C, \quad \text{for all } y \in K. \tag{7.10}$$

- (D^+-MVVIP): Find $\bar{x} \in K$ such that

$$D^+ f(y; \bar{x} - y) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.11}$$

- (D^+-MWVVIP): Find $\bar{x} \in K$ such that

$$D^+ f(y; \bar{x} - y) \notin \text{int}(C), \quad \text{for all } y \in K. \tag{7.12}$$

Similarly, we can define D_+-SVVIP, D_+-VVIP, D_+-WVVIP, D_+-MSVVIP, D_+-MVVIP and D_+-MWVVIP by considering $D_+ f(x; \cdot)$ in place of $h(x; \cdot)$ in h-SVVIP, h-VVIP, h-WVVIP, h-MSVVIP, h-MVVIP and h-MWVVIP, respectively.

If we consider (upper or lower) Dini directional derivative as a bifunction $h(x; d)$, with x referring to a point in $\mathbb{R}^n$ and d referring to a direction from $\mathbb{R}^n$, then (7.1), (7.2), (7.3), (7.4), (7.5) and (7.6) are equivalent to (7.7), (7.8), (7.9), (7.10), (7.11) and (7.12), respectively. In general, if we treat any generalized directional derivative as a bifunction $h(x; d)$ with x referring to a point in $\mathbb{R}^n$ and d referring to a direction from $\mathbb{R}^n$, then the corresponding nonsmooth vector variational inequality problems can be defined in the same way.

Definition 7.1 A vector-valued bifunction $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be:

(a) *strongly C-pseudomonotone* if for all $x, y \in K$,

$$h(x; y - x) \in C \quad \text{implies} \quad h(y; x - y) \in -C;$$

(b) *C-pseudomonotone* if for all $x, y \in K$,

$$h(x; y - x) \notin -C \setminus \{\mathbf{0}\} \quad \text{implies} \quad h(y; x - y) \notin C \setminus \{\mathbf{0}\};$$

(c) *weakly C-pseudomonotone* if for all $x, y \in K$,

$$h(x; y-x) \notin -\mathrm{int}(C) \quad \text{implies} \quad h(y; x-y) \notin \mathrm{int}(C);$$

(d) *C-properly subodd* if

$$h(x; d_1) + h(x; d_2) + \cdots + h(x; d_m) \in C,$$

for every $d_i \in \mathbb{R}^n$ with $\sum_{i=1}^m d_i = \mathbf{0}$ and for all $x \in K$.

If $m = 2$, the definition of proper suboddness reduces to the definition of suboddness.

Example 7.1 The function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, with $h(x, d) = (x, -x-d)$ is strongly $\mathbb{R}^2_+$-pseudomonotone, $\mathbb{R}^2_+$-pseudomonotone and weakly $\mathbb{R}^2_+$-pseudomonotone, but h is not $\mathbb{R}^2_+$-properly subodd.

Definition 7.2 A vector-valued bifunction $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *C-upper sign continuous* (respectively, *strongly C-upper sign continuous* and *weakly C-upper sign continuous*) if for all $x, y \in K$ and $\lambda \in]0, 1[$,

$$h(x + \lambda(y-x); x-y) \notin C \setminus \{\mathbf{0}\} \quad \text{implies} \quad h(x; x-y) \notin C \setminus \{\mathbf{0}\}$$

$$\big(\text{respectively,} \quad h(x + \lambda(y-x); x-y) \in -C \quad \text{implies} \quad h(x; x-y) \in -C$$

$$\text{and} \quad h(x + \lambda(y-x); x-y) \notin \mathrm{int}(C) \quad \text{implies} \quad h(x; x-y) \notin \mathrm{int}(C)\big).$$

Definition 7.3 A vector-valued bifunction $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *v-hemicontinuous* if for each fixed $d \in \mathbb{R}^n$ and for all $x, y \in K$,

$$\lim_{\lambda \to 0^+} h(x + \lambda(y-x)); d) = h(x; d).$$

It can be easily seen that if each component h_i, $i = 1, 2, \ldots, \ell$, of h is hemicontinuous, that is,

$$\lim_{\lambda \to 0^+} h_i(x + \lambda(y-x)); d) = h_i(x; d),$$

then h is v-hemicontinuous.

Remark 7.1 If h is v-hemicontinuous, then it is strongly C-upper sign continuous and weakly C-upper sign continuous as C and $\mathbb{R}^\ell \setminus \{\mathrm{int}(C)\}$ are closed sets.

Example 7.2 The function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, which is defined by $h(x; d) = \big(|x| \cdot x^2 \cdot d, \exp(x) \cdot d\big)$, is strongly $\mathbb{R}^2_+$-upper sign continuous.

The following result provides the relationship between nonsmooth vector variational inequality problems and Minty nonsmooth vector variational inequality problems.

Lemma 7.1 *Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-pseudomonotone (respectively, strongly C-pseudomonotone and weakly C-pseudomonotone) and C-upper sign continuous (respectively, strong C-upper sign continuous and weakly C-upper sign continuous) such that for each fixed $x \in K$, $h(x;\cdot)$ is C-properly subodd and positively homogeneous. Then $\bar{x} \in K$ is a solution of* h-VVIP *(respectively,* h-SVVIP *and* h-WVVIP*) if and only if it is a solution of* h-MVVIP *(respectively,* h-MSVVIP *and* h- MWVVIP*)*.

Proof The C-pseudomonotonicity of h implies that every solution of h-VVIP is a solution of h-MVVIP.

Conversely, let $\bar{x} \in K$ be a solution of h-MVVIP. Then

$$h(y;\bar{x}-y) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.13}$$

Since K is convex, we have $y_\lambda := \bar{x}+\lambda(y-\bar{x}) \in K$ for all $\lambda \in]0,1[$, therefore, (7.13) becomes

$$h(y_\lambda;\bar{x}-y_\lambda) \notin C \setminus \{\mathbf{0}\}.$$

Since $\bar{x}-y_\lambda = \lambda(\bar{x}-y)$ and $h(x;\cdot)$ is positively homogeneous, we have

$$h(y_\lambda;\bar{x}-y) \notin C \setminus \{\mathbf{0}\}.$$

Thus, the C-upper sign continuity and the C-proper suboddness of h imply that $\bar{x} \in K$ is a solution of h-VVIP.

Similarly, we can prove Sol(h-SVVIP) = Sol(h-MSVVIP) and Sol(h-WVVIP) = Sol(h-MWVVIP). □

Example 7.3 Let $X = \mathbb{R}$, $K = [0,1]$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$. Consider the function $h(x;d) = \left(x^2 d, |x|d\right)$. Note that h is strongly $\mathbb{R}^2_+$-pseudomonotone, strongly $\mathbb{R}^2_+$-upper sign continuous, $\mathbb{R}^2_+$-properly subodd and positive homogeneous in the second variable. The element $\bar{x} = 0$ is the only solution of the strong h-vector variational inequality problem h-SVVIP as well as the only solution of the Minty strong h-vector variational inequality problem h-MSVVIP.

In general, Sol(h-SVVIP) $\neq$ Sol(h-MSVVIP), Sol(h-VVIP) $\neq$ Sol(h-MVVIP) and Sol(h-WVVIP)) $\neq$ Sol(h-MWVVIP).

To overcome this deficiency, we define the following perturbed h-vector variational inequality problems.

- *ε-Perturbed Strong h-Vector Variational Inequality Problem* (ε-h-PSVVIP): Find $\bar{x} \in K$ for which there exists $\bar{\varepsilon} \in]0,1[$ such that

$$h(\bar{x}+\varepsilon(y-\bar{x});y-\bar{x}) \in -C, \quad \text{for all } y \in K \text{ and all } \varepsilon \in]0,\bar{\varepsilon}[. \tag{7.14}$$

- *ε-Perturbed h-Vector Variational Inequality Problem* (ε-h-PVVIP): Find $\bar{x} \in K$ for which there exists $\bar{\varepsilon} \in]0, 1[$ such that

$$h(\bar{x} + \varepsilon(y - \bar{x}); y - \bar{x}) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K \text{ and all } \varepsilon \in]0, \bar{\varepsilon}[. \tag{7.15}$$

- *ε-Perturbed Weak h-Vector Variational Inequality Problem* (ε-h-PWVVIP): Find $\bar{x} \in K$ for which there exists $\bar{\varepsilon} \in]0, 1[$ such that

$$h(\bar{x} + \varepsilon(y - \bar{x}); y - \bar{x}) \notin \text{int}(C), \quad \text{for all } y \in K \text{ and all } \varepsilon \in]0, \bar{\varepsilon}[. \tag{7.16}$$

Proposition 7.1 *Let $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-pseudomonotone (respectively, strongly C-pseudomonotone and weakly C-pseudomonotone) and C-properly subodd such that it is positively homogeneous in the second argument. Then $\bar{x} \in K$ is a solution of ε-h-*PVVIP *(respectively, ε-h-*PSVVIP *and ε-h-*PWVVIP*) if and only if it is a solution of h-*MVVIP *(respectively, h-*MSVVIP *and h-* MWVVIP*).*

Proof Let $\bar{x}$ be a solution of h-MVVIP. Then

$$h(y; \bar{x} - y) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{7.17}$$

Since K is convex, we have

$$x_\varepsilon := \bar{x} + \varepsilon(z - \bar{x}) \in K, \quad \text{for all } z \in K \text{ and all } \varepsilon \in [0, 1].$$

Taking $y = x_\varepsilon$ with $\bar{\varepsilon} = 1$ and $\varepsilon \in]0, \bar{\varepsilon}[$ in (7.17), we have

$$h(x_\varepsilon; \bar{x} - x_\varepsilon) \notin C \setminus \{\mathbf{0}\}.$$

Since $\bar{x} - x_\varepsilon = \varepsilon(\bar{x} - z)$ and $h(x; .\cdot)$ is positively homogeneous, we have

$$h(x_\varepsilon; \bar{x} - z) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } z \in K \text{ and all } \varepsilon \in]0, \bar{\varepsilon}[\,. \tag{7.18}$$

Since $(z - \bar{x}) + (\bar{x} - z) = \mathbf{0}$ and h is C-properly subodd, we have

$$h(x_\varepsilon; z - \bar{x}) + h(x_\varepsilon; \bar{x} - z) \in C. \tag{7.19}$$

Combining (7.18) and (7.19), we obtain

$$h(x_\varepsilon; z - \bar{x}) \notin C \setminus \{\mathbf{0}\}, \quad \text{for all } z \in K \text{ and all } \varepsilon \in]0, \bar{\varepsilon}[.$$

Therefore, $\bar{x} \in K$ is a solution of ε-h-PVVIP.

Conversely, suppose that $\bar{x} \in K$ is a solution of ε-h-PVVIP, but not a solution of h-MVVIP. Then there exists z $\in K$ such that

$$h(z; \bar{x} - z) \in C \setminus \{\mathbf{0}\}.$$

Since K is convex, we have

$$x_\varepsilon := \bar{x} + \varepsilon(z - \bar{x}) \in K, \quad \text{for all } \varepsilon \in [0, 1].$$

Since $x_\varepsilon - z = (1 - \varepsilon)(\bar{x} - z)$ and $h(x; \cdot)$ is positively homogeneous, we have

$$h(z; \bar{x} - z) = \frac{1}{1 - \varepsilon} h(z; x_\varepsilon - z) \in C \setminus \{\mathbf{0}\}, \quad \text{for all } \varepsilon \in]0, 1[;$$

thus,

$$h(z; x_\varepsilon - z) \in C \setminus \{\mathbf{0}\}, \quad \text{for all } \varepsilon \in]0, 1[.$$

By C-pseudomonotonicity of h, we obtain

$$h(x_\varepsilon; z - x_\varepsilon) \in -C \setminus \{\mathbf{0}\}, \quad \text{for all } \varepsilon \in]0, 1[.$$

Since $z - x_\varepsilon = (1 - \varepsilon)(z - \bar{x})$ and $h(x; \cdot)$ is positively homogeneous, we have

$$h(x_\varepsilon; z - \bar{x}) \in C \setminus \{\mathbf{0}\}, \quad \text{for all } \varepsilon \in]0, 1[,$$

which contradicts our supposition that $\bar{x}$ is a solution of ε-h-PVVIP.

The rest of the part can be proved in a similar way. □

7.2 Existence Results for Solutions of Nonsmooth Vector Variational Inequalities

We first present an existence result for a solution of h-VVIP without using any kind of monotonicity.

Theorem 7.1 *Let K be a nonempty compact convex subset of $\mathbb{R}^n$. Let $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be a vector-valued function such $h(x; \mathbf{0}) = \mathbf{0}$ and $h(x; \cdot)$ is positively homogeneous for each fixed $x \in K$, and the set $\{x \in K : h(x; y - x) \in -C \setminus \{\mathbf{0}\}\}$ is open in K for every fixed $y \in K$. Then h-*VVIP *has a solution $\bar{x} \in K$.*

Proof Suppose that h-VVIP has no solution. Then for every $\bar{x} \in K$, there exists $y \in K$ such that

$$h(\bar{x}; y - \bar{x}) \in -C \setminus \{\mathbf{0}\}. \tag{7.20}$$

For every $y \in K$, define the set N_y by

$$N_y = \{x \in K : h(x; y - x) \in -C \setminus \{\mathbf{0}\}\}. \tag{7.21}$$

By assumption, the set N_y is open in K for each $y \in K$. Therefore, from (7.20), $\{N_y : y \in K\}$ is an open cover of K. Since K is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_k\}$ of K such that

$$K = \bigcup_{i=1}^{k} N_{y_i}.$$

Thus, there exists a continuous partition of unity $\{\beta_1, \beta_2, \ldots, \beta_k\}$ subordinated to $\{N_{y_1}, N_{y_2}, \ldots, N_{y_k}\}$ such that for all $x \in K$,

- $\beta_j(x) \geq 0, j = 1, 2, \ldots, k$
- $\sum_{j=1}^{k} \beta_j(x) = 1$
- $\beta_j(x) = 0$ whenever $x \notin N_{y_j}$, and $\beta_j(x) > 0$ whenever $x \in N_{y_j}$

Let $p : K \to \mathbb{R}^n$ be defined by

$$p(x) = \sum_{j=1}^{k} \beta_j(x) y_j, \quad \text{for all } x \in K.$$

Since each β_i is continuous, we have p is continuous. Let $\Delta = \text{co}(\{y_1, y_2, \ldots, y_k\}) \subset K$. Then Δ is a simplex of the finite dimensional space and p maps Δ into itself. By Brouwer's Fixed Point Theorem 1.39, there exists $\hat{x} \in \Delta$ such that $p(\hat{x}) = \hat{x}$.

Define $q : K \to \mathbb{R}^\ell$ by

$$q(x) = h(x; x - p(x)) = \sum_{j=1}^{k} \beta_j h(x; x - y_j), \quad \text{for all } x \in K. \tag{7.22}$$

For any given $x \in K$, let $J = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}$. Obviously, J is nonempty. It follows from (7.21) and (7.22) that

$$q(x) = \sum_{j \in J} \beta_j(x) h(x; y_j - x) \in -C \setminus \{\mathbf{0}\}, \quad \text{for all } x \in K.$$

Since $\hat{x} \in \Delta \subset K$ is a fixed of p, from (7.21), we have

$$q(\hat{x}) = h(\hat{x}; \hat{x} - \hat{x}) = \mathbf{0} \in -C \setminus \{\mathbf{0}\},$$

a contradiction. Hence, h-VVIP has a solution $\bar{x} \in K$. □

The following result provides the existence of a solution of h-MWVVIP and h-WVVIP in the setting of compact convex set but under weakly C-pseudomonotonicity.

Theorem 7.2 *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be a positively homogeneous in the second argument, C-properly subodd and weakly C-pseudomonotone vector-valued function such that for all $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for each fixed $x \in K$, $h_i(x; \cdot)$ is continuous. Then* h-MWVVIP *has a solution $\bar{x} \in K$.*

Furthermore, if h is weakly C-upper sign continuous, then $\bar{x} \in K$ is a solution of h-WVVIP.

Proof For all $y \in K$, we define two set-valued maps $S, M : K \to 2^K$ by

$$S(y) = \{x \in K : h(x; y - x) \notin -\text{int}(C)\}$$

and

$$M(y) = \{x \in K : h(y; x - y) \notin \text{int}(C)\}.$$

We show that S is a KKM map. Let $\hat{x} \in \text{co}\left(\{y_1, y_2 \ldots, y_p\}\right)$, then $\hat{x} = \sum_{k=1}^{p} \lambda_k y_k$ with $\lambda_k \geq 0$ and $\sum_{k=1}^{p} \lambda_k = 1$. If $\hat{x} \notin \bigcup_{k=1}^{p} S(y_k)$, then

$$h(\hat{x}; y_k - \hat{x}) \in -\text{int}(C), \quad \text{for all } k = 1, 2, \ldots, p.$$

Since $-C$ is a convex cone and $\lambda_k \geq 0$ with $\sum_{k=1}^{p} \lambda_k = 1$, we have

$$\sum_{k=1}^{p} \lambda_k h(\hat{x}; y_k - \hat{x}) \in -\text{int}(C). \tag{7.23}$$

Since

$$\sum_{k=1}^{p} \lambda_k (y_k - \hat{x}) = \sum_{k=1}^{p} \lambda_k y_k - \sum_{k=1}^{p} \lambda_k \hat{x} = \hat{x} - \hat{x} = \mathbf{0},$$

by C-proper suboddness of h, we have

$$\sum_{k=1}^{p} h(\hat{x}; \lambda_k (y_k - \hat{x})) \in C.$$

By positive homogenuity of h, we have

$$\sum_{k=1}^{p} \lambda_k h(\hat{x}; y_k - \hat{x}) \in C,$$

which contradicts (7.23). Therefore, $\text{co}\big(\{y_1, y_2, \ldots, y_p\}\big) \subseteq \bigcup_{k=1}^{p} S(y_k)$. Hence, S is a KKM map.

The weak C-pseudomonotonicity of h implies that $S(y) \subseteq M(y)$ for all $y \in K$; hence, M is a KKM map.

We claim that $M(y)$ is a closed set in K for all $y \in K$. Indeed, let $\{x_m\}$ be a sequence in $M(y)$ which converges to $x \in K$. Then

$$h(y; x_m - y) \notin \text{int}(C), \quad \text{that is,} \quad h(y; x_m - y) \in \mathbb{R}^\ell \setminus \{\text{int}(C)\}.$$

Since $\mathbb{R}^\ell \setminus \{\text{int}(C)\}$ is a closed set and each $h_i(x; \cdot)$ is continuous, we have $h(y; x - y) \notin \text{int}(C)$, and hence, $x \in M(y)$. Thus, $M(y)$ is closed in K.

Further, since K is compact, it follows that $M(y)$ is compact for all $y \in K$. Then, by Fan-KKM Lemma 1.14,

$$\bigcap_{y \in K} M(y) \neq \emptyset,$$

that is, there exists $\bar{x} \in K$ such that

$$h(y; \bar{x} - y) \notin \text{int}(C), \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is a solution of h-MVVIP.

By Lemma 7.1, $\bar{x} \in K$ is a solution of h-WVVIP. □

Definition 7.4 A vector-valued function $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ is said to be *C-pseudomonotone$_+$* if for all $x, y \in K$,

$$h(x; y - x) \notin -C \setminus \{\mathbf{0}\} \quad \text{implies} \quad h(y; x - y) \in -C.$$

Clearly, C-pseudomonotonicity$_+$ is stronger than C-pseudomonotonicity.

Example 7.4 Let $X = \mathbb{R}$, $K = [-2, 2]$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$. The function $h : K \times \mathbb{R} \to \mathbb{R}^2$, defined by $h(x; d) = \big(x \cdot d, x^2 \cdot d\big)$, is $\mathbb{R}^2_+$-pseudomonotone, but not $\mathbb{R}^2_+$-pseudomonotone$_+$.

We have the following existence result for a solution of h-SVVIP under C-pseudomonotonicity$_+$ assumption.

Theorem 7.3 *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be a positively homogeneous in the second argument, C-properly subodd and C-pseudomonotone$_+$ vector-valued function such that for all $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for each fixed $x \in K$, $h_i(x; \cdot)$ is continuous. Furthermore, if h is strongly C-upper sign continuous, then h-*VVIP *has a solution.*

Proof For all $y \in K$, we define set-valued maps $S, M : K \to 2^K$ by

$$S(y) = \{x \in K : h(x; y - x) \notin -C \setminus \{\mathbf{0}\}\}$$

and

$$M(y) = \{x \in K : h(y; x - y) \in -C\}.$$

By C-pseudomonotonicity$_+$, Sol(h-VVIP) $\subseteq$ Sol(h-MSVVIP). From Lemma 7.1, Sol(h-MSVVIP) $\subseteq$ Sol(h-SVVIP) $\subseteq$ Sol(h-VVIP). Thus, Sol(h-VVIP) $\subseteq$ Sol(h-MSVVIP) $\subseteq$ Sol(h-VVIP), and hence, Sol(h-VVIP) $=$ Sol(h-MSVVIP), that is,

$$\bigcap_{y \in K} S(y) = \bigcap_{y \in K} M(y).$$

We prove that S is a KKM map. Let $\hat{x} \in \text{co}\left(\{y_1, y_2, \ldots, y_p\}\right)$, then $\hat{x} = \sum_{k=1}^{p} \lambda_k y_k$ with $\lambda_k \geq 0$ and $\sum_{k=1}^{p} \lambda_k = 1$. If $\hat{x} \notin \bigcup_{k=1}^{p} S(y_k)$, then

$$h(\hat{x}; y_k - \hat{x}) \in -C \setminus \{\mathbf{0}\}, \quad \text{for all } k = 1, 2, \ldots, p.$$

Since $-C$ is a convex cone and $\lambda_k \geq 0$ with $\sum_{k=1}^{p} \lambda_k = 1$, we have

$$\sum_{k=1}^{p} \lambda_k h(\hat{x}; y_k - \hat{x}) \in -C \setminus \{\mathbf{0}\}. \tag{7.24}$$

Since

$$\sum_{k=1}^{p} \lambda_k (y_k - \hat{x}) = \sum_{k=1}^{p} \lambda_k y_k - \sum_{k=1}^{p} \lambda_k \hat{x} = \hat{x} - \hat{x} = \mathbf{0},$$

by C-proper suboddness of h, we have

$$\sum_{k=1}^{p} h(\hat{x}; \lambda_k (y_k - \hat{x})) \in C.$$

The positive homogenuity of h in the second argument implies that

$$\sum_{k=1}^{p} \lambda_k h(\hat{x}; y_k - \hat{x}) \in C,$$

which contradicts (7.24). Therefore, $\text{co}\left(\{y_1, y_2, \ldots, y_p\}\right) \subseteq \bigcup_{k=1}^{p} S(y_k)$. Hence, S is a KKM map.

By C-pseudomonotonicity$_+$ of h implies that $S(y) \subseteq M(y)$ for all $y \in K$; hence, M is a KKM map. Since $-C$ is closed and each $h_i(x; \cdot)$ is continuous, it can be easily seen that $M(y)$ is closed subset of a compact set K, and hence, compact. Therefore, by Fan-KKM Lemma 1.14,

$$\bigcap_{y\in K} S(y) = \bigcap_{y\in K} M(y) \neq \emptyset,$$

that is, there exists a solution of h-VVIP. □

Definition 7.5 Let K be a nonempty convex subset of $\mathbb{R}^n$. A vector-valued function $h = (h_1, h_2, \dots, h_\ell) : K \to \mathbb{R}^\ell$ is said to be

(a) *strongly proper C-quasimonotone*$_*$ if for every finite set $\{y_1, y_2, \dots, y_p\}$ in K and $x \in \text{co}\left(\{y_1, y_2, \dots, y_p\}\right)$, there exists $i \in \{1, 2, \dots, p\}$ such that $h(x; y_i - x) \in C$;
(b) *proper C-quasimonotone*$_*$ if for every finite set $\{y_1, y_2, \dots, y_p\}$ in K and $x \in \text{co}\left(\{y_1, y_2, \dots, y_p\}\right)$, there exists $i \in \{1, 2, \dots, p\}$ such that $h(x; y_i - x) \notin -C \setminus \{\mathbf{0}\}$;
(c) *weakly proper C-quasimonotone*$_*$ if for every finite set $\{y_1, y_2, \dots, y_p\}$ in K and $x \in \text{co}\left(\{y_1, y_2, \dots, y_p\}\right)$, there exists $i \in \{1, 2, \dots, p\}$ such that $h(x; y_i - x) \notin -\text{int}(C)$.

Example 7.5 Let $X = \mathbb{R}$, $K = [0, 1]$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$. The function $h : K \times \mathbb{R} \to \mathbb{R}^2$, defined by $h(x; d) = \left(x, \sqrt{x} \cdot |d|\right)$, is strongly proper C-quasimonotone$_*$.

Theorem 7.4 *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $h = (h_1, h_2, \dots, h_\ell) : K \to \mathbb{R}^\ell$ be a C-properly subodd and strongly proper C-quasimonotone$_*$ vector-valued function such that for all $i \in \mathscr{I} = \{1, 2, \dots, \ell\}$ and for each fixed $x \in K$, $h_i(x; \cdot)$ is continuous and $h(x; \mathbf{0}) = \mathbf{0}$ for all $x \in K$. Then* h-SVVIP *has a solution $\bar{x} \in K$.*

Proof For all $y \in K$, we define a set-valued map $S : K \to 2^K$ by

$$S(y) = \{x \in K : h(x; y - x) \in C\}.$$

Since $h(x; y-y) = h(x; \mathbf{0}) = \mathbf{0} \in C$ for each $y \in K$, $y \in S(y)$, and thus, $S(y) \neq \emptyset$. We show that S is a KKM map. Let $\hat{x} \in \text{co}\left(\{y_1, y_2, \dots, y_p\}\right)$ such that $\hat{x} \notin \bigcup_{k=1}^{p} S(y_k)$. This implies that

$$h(\hat{x}; y_k - \hat{x}) \notin C, \quad \text{for all } k = 1, 2, \dots, p.$$

This contradicts the strong proper C-quasimonotonicity$_*$ of h. Hence, S is a KKM map.

Since $h(x; \cdot)$ is continuous and C is closed, it can be easily seen that $S(y)$ is a closed subset of a compact set K, and hence, $S(y)$ is compact for all $y \in K$. Then, by Fan-KKM Lemma 1.14,

$$\bigcap_{y\in K} S(y) \neq \emptyset,$$

that is, there exists $\bar{x} \in K$ such that

$$h(\bar{x}; y - \bar{x}) \in C, \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is a solution of h-SVVIP. □

Similarly, we can prove the following results.

Theorem 7.5 *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be a C-properly subodd and weakly proper C-quasimonotone$_*$ vector-valued function such that for all $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for each fixed $x \in K$, $h_i(x; \cdot)$ is continuous and $h(x; \mathbf{0}) = \mathbf{0}$ for all $x \in K$. Then h-*WVVIP *has a solution $\bar{x} \in K$.*

Theorem 7.6 *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $h = (h_1, h_2, \ldots, h_\ell) : K \to \mathbb{R}^\ell$ be a C-properly subodd and proper C-quasimonotone$_*$ vector-valued function such that for all $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for each fixed $x \in K$, the set $\{x \in K : h(x; y - x) \notin -C \setminus \{\mathbf{0}\}\}$ is closed and $h(x; \mathbf{0}) = \mathbf{0}$ for all $x \in K$. Then h-*VVIP *has a solution $\bar{x} \in K$.*

7.3 Nonsmooth Vector Variational Inequalities and Nonsmooth Vector Optimization

The optimization problem may have a nonsmooth objective function. Therefore, Crespi et al. [3, 4] introduced Minty variational inequality for scalar-valued functions defined by means of lower Dini directional derivative. More recently, the same authors extended their formulation to the vector case in [5]. They have also established the relations between a Minty Vector Variational Inequality problem (in short, MVVIP) and solutions of VOP (both ideal and weak efficient but not efficient) solution. Crespi et al. [5] used the scalarization method to obtain their results. The similar VVIP is also considered by Lalitha and Mehta [8] and proved some existence results. They also provided some relationships between the solutions of VOP and this kind of VVIP.

In this section, we propose some relations between vector optimization and vector variational inequalities when the objective functions are not necessarily smooth.

Throughout this section, unless otherwise specified, we assume that K is nonempty convex subset of $\mathbb{R}^n$, $C = \mathbb{R}^\ell_+$ and $h = (h_1, h_2, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is a vector-valued function.

Definition 7.6 A vector-valued function $f = (f_1, f_2, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ is said to be

(a) *C-h-convex* if for all $x, y \in K$,

$$f(y) - f(x) - h(x; y - x) \in C;$$

(b) *strictly C-h-convex* if for all $x, y \in K$, $x \neq y$,

$$f(y) - f(x) - h(x; y - x) \in \text{int}(C);$$

(c) *strongly C-h-pseudoconvex* if for all $x, y \in K$,

$$f(y) - f(x) \notin C \quad \text{implies} \quad h(x; y - x) \notin C,$$

equivalently,

$$h(x; y - x) \in C \quad \text{implies} \quad f(y) - f(x) \in C;$$

(d) *C-h-pseudoconvex* if for all $x, y \in K$,

$$f(y) - f(x) \in -C \setminus \{\mathbf{0}\} \quad \text{implies} \quad h(x; y - x) \in -C \setminus \{\mathbf{0}\};$$

equivalently,

$$h(x; y - x) \notin C \setminus \{\mathbf{0}\} \quad \text{implies} \quad f(y) - f(x) \notin -C \setminus \{\mathbf{0}\};$$

(e) *weakly C-h-pseudoconvex* if for all $x, y \in K$,

$$f(y) - f(x) \in -\text{int}(C) \quad \text{implies} \quad h(x; y - x) \in -\text{int}(C),$$

equivalently,

$$h(x; y - x) \notin -\text{int}(C) \quad \text{implies} \quad f(y) - f(x) \notin -\text{int}(C).$$

Obviously, strictly C-h-convexity implies C-h-convexity, and C-h-convexity implies C-h-pseudoconvexity.

If $h(x; d) = D^+f(x; d)$ (respectively, $D_+f(x; d)$) upper (respectively, lower) Dini directional derivative of a function f at x in the direction d, then C-h-convexity is called C-D^+-convexity (respectively, C-D_+-convexity), and so on.

Example 7.6 Let $X = \mathbb{R}$, $K = [0, 1]$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$. Let the function $h : K \times \mathbb{R} \to \mathbb{R}^2$, be given as $h(x; d) = \left(-x^2, -|x| - |d|\right)$. Furthermore, let $f : K \to \mathbb{R}^2$ be defined by $f = \left(x^2, |x|\right)$. Then f is C-h-convex, but not strictly C-h-convex. Moreover, f is strongly C-h-pseudoconvex and C-h-pseudoconvex, but f is not weakly C-h-pseudoconvex.

The following result provides the relation among the weakly efficient solution of VOP and the solutions of h-WVVIP and (D^+-WVVIP).

Theorem 7.7 *Let $f : K \to \mathbb{R}^\ell$ be a vector-valued function. Then the following statements hold.*

(a) *Every strongly efficient solution of* VOP *is a solution of* D^+-SVVIP (7.7).

(b) *If f is strongly C-h-pseudoconvex, then every solution of h-*SVVIP *is a strongly efficient solution of* VOP.

Proof

(a) Let $\bar{x}$ be a strongly efficient solution of VOP. Then

$$f(y) - f(\bar{x}) \in C, \quad \text{for all } y \in K.$$

Since K is a convex set, we have $\bar{x} + \lambda(y - \bar{x}) \in K$ for all $\lambda \in [0, 1]$; thus,

$$\frac{f(\bar{x} + \lambda(y - \bar{x})) - f(\bar{x})}{\lambda} \in C, \quad \text{for all } \lambda \in]0, 1[.$$

Taking the limit sup as $\lambda \downarrow 0$, we obtain

$$D^+ f(\bar{x}; y - \bar{x}) = \limsup_{\lambda \downarrow 0} \frac{f(\bar{x} + \lambda(y - \bar{x})) - f(\bar{x})}{\lambda} \in C, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of D^+-SVVIP (7.7).

(b) Assume that $\bar{x} \in K$ is a solution of D^+-SVVIP (7.7) but not a strongly efficient solution of VOP. Then there exists $y \in K$ such that

$$f(y) - f(\bar{x}) \notin C.$$

Since f is strongly C-h-pseudoconvex, we have

$$h(\bar{x}; y - \bar{x}) \notin C,$$

contradicting our assumption that $\bar{x}$ is a solution of h-SVVIP. □

Since $\mathbb{R}^\ell \setminus \{- \operatorname{int}(C)\}$ is a closed convex cone, in a similar way, we can prove the following result.

Theorem 7.8 *Let $f : K \to \mathbb{R}^\ell$ be a vector-valued function. Then the following statements hold.*

(a) *Every weakly efficient solution of* VOP *is a solution of D^+-*WVVIP (7.9).

(b) *If f is weakly C-h-pseudoconvex, then every solution of h-*WVVIP *is a weakly efficient solution of* VOP.

Theorem 7.9 *Let $f : K \to \mathbb{R}^\ell$ be a C-h-pseudoconvex vector-valued function. Then every solution of h-VVIP is an efficient of VOP.*

Proof It lies on the lines of the proof of Theorem 7.7 (b), therefore, we omit it. □

Theorem 7.10 *Let* $f : K \to \mathbb{R}^\ell$ *be a vector-valued function such that* $(-f)$ *is strictly* C-h-*convex, that is,*

$$-(f(y) - f(x)) - h(\bar{x}; y - \bar{x}) \in \text{int}(C), \quad \textit{for all } x, y \in K. \tag{7.25}$$

Then every weakly efficient solution of VOP *is a solution of* h-VVIP.

Proof Assume that $\bar{x}$ is a weakly efficient solution of VOP but not a solution of h-VVIP. Then there exists $y \in K$ such that

$$h(\bar{x}; y - \bar{x}) \in C \setminus \{\mathbf{0}\}. \tag{7.26}$$

Combining (7.26) and (7.25), we obtain

$$f(y) - f(\bar{x}) \in -\text{int}(C),$$

a contradiction to our assumption that $\bar{x}$ is a weakly efficient solution of VOP. Hence, $\bar{x}$ is a solution of h-VVIP. □

Theorem 7.11 *If* $f : K \to \mathbb{R}^\ell$ *is strictly* C-D^+-*convex function, then every weakly efficient solution of* VOP *is an efficient solution of* VOP.

Proof Assume that $\bar{x}$ is a weak efficient solution of VOP, but not an efficient solution of VOP. Then there exists $y \in K$ such that

$$f(y) - f(\bar{x}) \in -C \setminus \{\mathbf{0}\}. \tag{7.27}$$

Since f is strictly C-D^+-convex, we have

$$D^+f(\bar{x}; y - \bar{x}) - f(y) + f(\bar{x}) \in -\text{int}(C). \tag{7.28}$$

By combining (7.27) and (7.28), we obtain

$$D^+f(\bar{x}; y - \bar{x}) \in -\text{int}(C).$$

Thus, $\bar{x}$ is not a solution of D^+-WVVIP (7.9). By using Theorem 7.8 (a), we see that $\bar{x}$ is not a weak efficient solution of VOP, a contradiction to our assumption. □

Theorem 7.12 *For each* $i = 1, 2, \ldots, \ell$, *let* $f_i : K \to \mathbb{R}$ *be* D^+-*pseudoconvex and lower semicontinuous. If* $\bar{x} \in K$ *is a solution of* D^+-VVIP, *then it is an efficient solution of* VOP.

Proof Suppose that $\bar{x} \in K$ is not an efficient solution of VOP. Then there exists $\hat{y} \in K$ such that $f_i(\hat{y}) < f_i(\bar{x})$ for some i and $f_j(\hat{y}) \leq f_j(\bar{x})$ for all $j \neq i$. By D^+-pseudoconvexity of f_i, $D^+f_i(\bar{x}; \hat{y} - \bar{x}) < 0$. By Remark 1.18 (b), f_j is quasiconvex for all $j \neq i$, and hence, $D^+f_j(\bar{x}; \hat{y} - \bar{x}) \leq 0$ for all $j \neq i$. Thus, $\bar{x} \in K$ is not a solution of D^+-VVIP. □

The following result, due to Ansari and Lee [1], provides the relationship between a solution of an h-MVVIP and an efficient solution of VOP. It can be treated as a nonsmooth version of Theorem 5.28.

Theorem 7.13 *For each $i \in \mathscr{I}$, let $f_i : K \to \mathbb{R}$ be upper semicontinuous and D^+-pseudoconvex. For each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for all $x \in K$, let $h_i(x; \cdot)$ be positively homogeneous and subodd such that $h_i(x; \cdot) \leq D^+ f_i(x; \cdot)$. Then $\bar{x} \in K$ is a solution of h-*MVVIP *if and only if it is an efficient solution of* VOP.

Proof Let $\bar{x} \in K$ be a solution of h-MVVIP but not an efficient solution of VOP. Then there exists $z \in K$ such that

$$f(\bar{x}) - f(z) \in C \setminus \{\mathbf{0}\}. \tag{7.29}$$

Set $z(\lambda) := \lambda \bar{x} + (1 - \lambda) z$ for all $\lambda \in [0, 1]$. Since K is convex, $z(\lambda) \in K$ for all $\lambda \in [0, 1]$. Also since each f_i is D^+-pseudoconvex, it follows from Lemma 1.4 that f_i is quasiconvex and semistrictly quasiconvex. By using quasiconvexity, semistrictly quasiconvexity and (7.29), we get

$$f_i(\bar{x}) - f_i(z(\lambda)) \in C \setminus \{\mathbf{0}\}, \quad \text{for all } \lambda \in]0, 1[.$$

That is,

$$f_i(\bar{x}) \geq f_i(z(\lambda)), \quad \text{for all } \lambda \in]0, 1[\text{ and all } i = 1, 2, \ldots, \ell, \tag{7.30}$$

with strict inequality holds in (7.30) for some k such that $1 \leq k \leq \ell$.

By Diewert Mean-Value Theorem 1.31, there exists $\alpha_i \in]0, 1[$ such that

$$f_i(z(\lambda)) - f_i(\bar{x}) \geq D^+ f_i\left(z(\alpha_i); z(\lambda) - \bar{x}\right), \quad \text{for all } \lambda \in]0, 1[\text{ and all } i \in \mathscr{I}. \tag{7.31}$$

Combining inequalities (7.30) and (7.31), we obtain

$$D^+ f_i(z(\alpha_i); z(\lambda) - \bar{x}) \leq 0, \quad \text{for all } \lambda \in]0, 1[\text{ and all } i = 1, 2, \ldots, \ell,$$

with strict inequality holds for some k such that $1 \leq k \leq \ell$. Since, for each fixed $x \in K$, $h_i(x; \cdot) \leq D^+ f_i(x; \cdot)$, we have

$$h_i(z(\alpha_i); z(\lambda) - \bar{x}) \leq 0, \quad \text{for all } \lambda \in]0, 1[\text{ and all } i = 1, 2, \ldots, \ell,$$

where strict inequality holds for some k such that $1 \leq k \leq \ell$. By using the positive homogeneity of h_i in the second argument, we get

$$h_i(z(\alpha_i); z(\lambda) - \bar{x}) = h_i(z(\alpha_i); \lambda \bar{x} + (1 - \lambda) z - \bar{x}) = (1 - \lambda) h_i(z(\alpha_i); z - \bar{x})$$

and so,

$$h_i(z(\alpha_i); z - \bar{x}) \leq 0, \quad \text{for all } i = 1, 2, \ldots, \ell, \tag{7.32}$$

where strict inequality holds for some k such that $1 \leq k \leq \ell$. By the suboddness of h_i in the second argument, we have

$$h_i(z(\alpha_i); \bar{x} - z) \geq 0, \quad \text{for all } i = 1, 2, \ldots, \ell, \tag{7.33}$$

where strict inequality holds for some k such that $1 \leq k \leq \ell$.

Suppose that $\alpha_1, \ldots, \alpha_\ell$ are all equal. Then by (7.33), the positive homogeneity of h_i in the second argument, and the fact that

$$\bar{x} - z(\alpha_i) = (1 - \alpha_i)(\bar{x} - z),$$

we have

$$h_i(z(\alpha_i); \bar{x} - z(\alpha_i)) \geq 0, \quad \text{for all } i = 1, 2, \ldots, \ell,$$

where strict inequality holds for some i, that is,

$$\big(h_1(z(\alpha_1); \bar{x} - z(\alpha_1)), \ldots, h_\ell(z(\alpha_\ell); \bar{x} - z(\alpha_\ell))\big) \in C \setminus \{\mathbf{0}\},$$

which contradicts to our assumption that $\bar{x}$ is a solution of (h-MVVIP).

Consider the case when $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are not equal. Let $\alpha_1 \neq \alpha_2$.

If $\alpha_1 < \alpha_2$, then by the positive homogeneity and the suboddness of $h_i(x; \cdot)$, we get

$$h_1(z(\alpha_1); z(\alpha_2) - z(\alpha_1)) = (\alpha_2 - \alpha_1) h_1(z(\alpha_1); \bar{x} - z),$$

and by using (7.33), we obtain

$$h_1(z(\alpha_1); z(\alpha_2) - z(\alpha_1)) = (\alpha_2 - \alpha_1) h_1(z(\alpha_1); \bar{x} - z) \geq 0, \tag{7.34}$$

where strict inequality holds for $k = 1$.

Since each f_i is D^+-pseudoconvex and $h_i(x; \cdot) \leq D^+ f_i(x; \cdot)$, by Lemma 1.5, f_i is h_i-pseudoconvex; further, by Lemma 1.7 (b), h_i is pseudomonotone. Therefore, we have

$$h_1(z(\alpha_2); z(\alpha_2) - z(\alpha_1)) \geq 0, \tag{7.35}$$

where strict inequality holds for $k = 1$ by virtue of Lemma 1.6. The positive homogeneity of $h_i(x; \cdot)$ implies that

$$(\alpha_2 - \alpha_1) h_1(z(\alpha_2); \bar{x} - z) \geq 0,$$

where strict inequality holds for $k = 1$. Since $\alpha_2 - \alpha_1 > 0$, we have

$$h_1(z(\alpha_2); \bar{x} - z) \geq 0,$$

with strict inequality for $k = 1$.

If $\alpha_1 > \alpha_2$, then by the positive homogeneity and the suboddness of $h_i(x; \cdot)$, we get

$$h_2(z(\alpha_2); z(\alpha_1) - z(\alpha_2)) = (\alpha_1 - \alpha_2) h_2(z(\alpha_2); \bar{x} - z),$$

and by using (7.33), we obtain

$$h_2(z(\alpha_2); z(\alpha_1) - z(\alpha_2)) = (\alpha_1 - \alpha_2) h_2(z(\alpha_2); \bar{x} - z) \geq 0, \tag{7.36}$$

with strict inequality for $k = 2$.

As above, each h_i is pseudomonotone; therefore,

$$h_2(z(\alpha_1); z(\alpha_1) - z(\alpha_2)) \geq 0, \tag{7.37}$$

with strict inequality for $k = 2$ by virtue of Lemma 1.6 Again as above, by using the positive homogeneity of $h_i(x; \cdot)$, we get

$$h_2(z(\alpha_1); \bar{x} - z) \geq 0,$$

with strict inequality for $k = 2$.

For the case $\alpha_1 \neq \alpha_2$, let $\bar{\alpha} = \max\{\alpha_1, \alpha_2\}$. Then we have

$$h_i(z(\bar{\alpha}); \bar{x} - z) \geq 0, \quad \text{for all } i = 1, 2.$$

By continuing this process, we can find $\alpha^* \in]0, 1[$ such that

$$h_i(z(\alpha^*); \bar{x} - z) \geq 0, \quad \text{for all } i = 1, 2, \ldots, \ell,$$

with strict inequality for some k such that $1 \leq k \leq \ell$. By multiplying the above inequality by $1 - \alpha^*$, we obtain

$$h_i(z(\alpha^*); \bar{x} - z(\alpha^*)) \geq 0, \quad \text{for all } i = 1, 2, \ldots, \ell,$$

with strict inequality for some k such that $1 \leq k \leq \ell$. Thus,

$$\big(h_1(z(\alpha^*); \bar{x} - z(\alpha^*)), \ldots, h_\ell(z(\alpha^*); \bar{x} - z(\alpha^*))\big) \in C \setminus \{\mathbf{0}\},$$

which contradicts our supposition that $\bar{x}$ is a solution of h-MVVIP.

Conversely, suppose that $\bar{x} \in K$ is an efficient solution of VOP, but not a solution of h-MVVIP. Then there exists $z \in K$ such that

$$h(z; \bar{x} - z) \in C \setminus \{\mathbf{0}\},$$

that is,

$$h_i(z; \bar{x} - z) \geq 0, \quad \text{for all } i \in \mathscr{I},$$

with strict inequality holds for some i. Since $h_i(z; \ ) \leq D^+ f_i(z; \cdot)$ for all $i \in \mathscr{I}$,

$$D^+ f_i(z; \bar{x} - z) \geq 0, \quad \text{for all } i \in \mathscr{I},$$

with strict inequality holds for some i. Since each f_i is D^+-pseudoconvex, we have

$$f_i(\bar{x}) \geq f_i(z), \quad \text{for all } i \in \mathscr{I}.$$

Let $j \in \mathscr{I}$ be such that $D^+ f_j(z; \bar{x} - z) > 0$. Since f_j is upper semicontinuous and D^+-pseudoconvex, it follows from Lemma 1.4 that f_j is quasiconvex; hence, it follows from Theorem 4 in [6] that $f_j(\bar{x}) > f_j(z)$. Thus, $f(z) - f(\bar{x}) \in -C \setminus \{\mathbf{0}\}$; hence, $\bar{x}$ is not an efficient solution of VOP. This contradiction proves our result. □

The following result gives the relation between a solution of h-VVIP and a properly efficient solution (in the sense of Henig) of VOP.

Theorem 7.14 *If $\bar{x} \in K$ is a properly efficient solution (in the sense of Henig) of* VOP*, then it is a solution of* D^+-VVIP.

Proof Since $\bar{x} \in K$ is a properly efficient solution (in the sense of Henig) of VOP, there is convex cone D in $\mathbb{R}^\ell$ such that $C \setminus \{\mathbf{0}\} \subseteq \text{int}(D)$, and

$$f(y) - f(\bar{x}) \notin -D \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K.$$

Since $-\text{int}(D) \subseteq -D \setminus \{\mathbf{0}\}$, we have

$$f(y) - f(\bar{x}) \notin -\text{int}(D), \quad \text{for all } y \in K.$$

Since K is a convex set, we have $\bar{x} + \lambda(y - \bar{x}) \in K$ for all $\lambda \in [0, 1]$; thus,

$$\frac{f(\bar{x} + \lambda(y - \bar{x})) - f(\bar{x})}{\lambda} \in \mathbb{R}^\ell \setminus \{-\text{int}(D)\}, \quad \text{for all } \lambda \in]0, 1[.$$

Since $\mathbb{R}^\ell \setminus \{-\text{int}(D)\}$ is a closed convex cone, by taking the limit sup as $\lambda \downarrow 0$, we obtain

$$D^+ f(\bar{x}; y - \bar{x}) = \limsup_{\lambda \downarrow 0} \frac{f(\bar{x} + \lambda(y - \bar{x})) - f(\bar{x})}{\lambda} \in \mathbb{R}^\ell \setminus \{-\text{int}(D)\}, \quad \text{for all } y \in K.$$

Therefore,

$$\langle D^{+}f(\bar{x}), y-\bar{x}\rangle \notin -\operatorname{int}(D), \quad \text{for all } y \in K.$$

Since $C \setminus \{\mathbf{0}\} \subseteq \operatorname{int}(D)$ for all $x \in K$, we obtain the result. □

Theorem 7.15 ([10]) *If $\bar{x} \in K$ is a properly efficient solution (in the sense of Geoffrion) of* VOP *and for each $i = 1, 2, \ldots, \ell$, $D^{+}f_i(\bar{x}; \cdot)$ is finite on $K - \bar{x}$, then $\bar{x} \in K$ is a solution of D^{+}*-VVIP.

Proof Let $\bar{x} \in K$ be a properly efficient solution (in the sense of Geoffrion) of VOP. Suppose on contrary that there exists $d \in \mathbb{R}^n$ such that $d \in K - \bar{x}$, $D^{+}f_i(\bar{x}; d) < 0$ and $D^{+}f_j(\bar{x}; d) \leq 0$, $j \neq i$. We choose $v \in K$ such that $d = v - \bar{x}$. Since K is convex, we can choose a sequence $\{t_n\}$ of positive real numbers such that $t_n \downarrow 0$, $\bar{x} + t_n(v - \bar{x}) \in K$ for all n and

$$D^{+}f_i(\bar{x}; d) = \lim_{n\to\infty} \frac{f_i(\bar{x} + t_n(v-\bar{x})) - f_i(\bar{x})}{t_n}.$$

Since $D^{+}f_i(\bar{x}; d) < 0$,

$$\lim_{n\to\infty} \frac{f_i(\bar{x} + t_n(v-\bar{x})) - f_i(\bar{x})}{t_n} < 0,$$

and hence, there exists a natural number N such that for all $n \geq N$,

$$\frac{1}{t_n}\left[f_i(\bar{x} + t_n(v-\bar{x})) - f_i(\bar{x})\right] < 0,$$

that is, $f_i(\bar{x} + t_n(v - \bar{x})) < f_i(\bar{x})$. Since $\bar{x}$ is an efficient solution VOP, choosing a subsequence of the sequence $\{\bar{x} + t_n(v - \bar{x})\}$, if necessary, we may assume that

$$I := \{j : f_j(\bar{x} + t_n(v-\bar{x})) > f_j(\bar{x})\}$$

is constant for all $n \geq N$. So, for all $j \in I$, we have

$$\limsup_{n\to\infty} \frac{f_j(\bar{x} + t_n(v-\bar{x})) - f_j(\bar{x})}{t_n} \geq 0.$$

Since $D^{+}f_j(\bar{x}; d) \leq 0$ for all $j \in I$, we have

$$\limsup_{n\to\infty} \frac{f_j(\bar{x} + t_n(v-\bar{x})) - f_j(\bar{x})}{t_n} = 0,$$

for all $j \in I$. So, choosing a subsequence of $\{t_n\}$, if necessary, we may assume that

$$\lim_{n\to\infty} \frac{f_j(\bar{x} + t_n(v - \bar{x})) - f_j(\bar{x})}{t_n} = 0,$$

for all $j \in I$. So, for all $j \in I$, we have

$$\frac{f_i(\bar{x}) - f_i(\bar{x} + t_n(v - \bar{x}))}{f_j(\bar{x} + t_n(v - \bar{x})) - f_j(\bar{x})} = \frac{\frac{1}{t_n}[f_i(\bar{x}) - f_i(\bar{x} + t_n(v - \bar{x}))]}{\frac{1}{t_n}[f_j(\bar{x} + t_n(v - \bar{x})) - f_j(\bar{x})]} \to +\infty \text{ as } n \to \infty,$$

which contradicts to the proper efficiency (in the sense of Geoffrion) of $\bar{x}$. ☐

Remark 7.2 Since proper efficiency in the sense of Benson and in the sense of Geoffrion are equivalent when $C = \mathbb{R}^\ell_+$, Theorem 7.15 also holds for proper efficiency in the sense of Benson.

Corollary 7.1 ([11, Theorem 4]) *For each $i = 1, 2, \ldots, \ell$, let $f_i : K \to \mathbb{R}$ be convex and differentiable. If $\bar{x} \in K$ is a properly efficient solution (in the sense of Benson) of* VOP*, then it is a solution of* ∇-VVIP.

The following example shows that the Corollary 7.1 cannot be extended to efficient solutions of VOP even though each f_i is convex.

Example 7.7 Let $K = [-1, 0]$ and $f(x) = (x, x^2)$. Then $\bar{x} = 0$ is an efficient solution of VOP, but it is not a solution of the following ∇-VVIP: Find $\bar{x} \in K$ such that for all $y \in K$,

$$(\langle \nabla f_1(\bar{x}), y - \bar{x}\rangle, \langle \nabla f_2(\bar{x}), y - \bar{x}\rangle) = (y - \bar{x}, 2\bar{x}(y - \bar{x})) \notin -\mathbb{R}^2_+ \setminus \{\mathbf{0}\}.$$

We notice that $\bar{x} = 0$ is not a properly efficient solution (in the sense of Benson) of VOP.

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