Chapter 6
Linear Scalarization of Vector Variational Inequalities

This chapter deals with linear scalarization techniques for vector variational inequality problems and Minty vector variational inequality problems. Such concepts are important for deriving numerical algorithms for solving vector variational inequalities.

For each given $\ell \in \mathbb{N}$, we denote by $\mathbb{R}^\ell_+$ the non-negative orthant of $\mathbb{R}^\ell$, that is,

$$\mathbb{R}^\ell_+ = \left\{x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell : x_i \geq 0, \ \text{ for } i = 1, 2, \ldots, \ell\right\},$$

so that $\mathbb{R}^\ell_+$ has a nonempty interior with the topology induced in terms of convergence of vectors with respect to the Euclidean metric. That is,

$$\operatorname{int}(\mathbb{R}^\ell_+) = \left\{x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell : x_i > 0, \ \text{ for } i = 1, 2, \ldots, \ell\right\}.$$

We denote by $\mathbb{T}^\ell_+$ and $\operatorname{int}(\mathbb{T}^\ell_+)$ the simplex of $\mathbb{R}^\ell_+$ and its relative interior, respectively, that is,

$$\mathbb{T}^\ell_+ = \left\{x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell_+ : \|x\| = \sum_{i=1}^{\ell} x_i = 1\right\},$$

and

$$\operatorname{int}(\mathbb{T}^\ell_+) = \left\{x = (x_1, x_2, \ldots, x_\ell) \in \operatorname{int}(\mathbb{R}^\ell_+) : \|x\| = \sum_{i=1}^{\ell} x_i = 1\right\}.$$

e denotes the unit vector in $\mathbb{R}^\ell$, that is, $e = (1, 1, \ldots, 1)$.

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Let K be a nonempty convex subset of $\mathbb{R}^n$. For each $i = 1, 2, \dots, \ell$, let $T_i : K \to \mathbb{R}^n$ be a vector-valued function such that $T = (T_1, T_2, \dots, T_\ell) : K \to \mathbb{R}^{\ell \times n}$ is a matrix-valued function. For abbreviation, we put

$$\langle T(x), v\rangle_\ell := (\langle T_1(x), v\rangle, \dots, \langle T_\ell(x), v\rangle), \quad \text{for all } x \in K \text{ and all } v \in \mathbb{R}^n.$$

Let us define the vector variational inequality problem and weak vector variational inequality problem in these settings.

- *Vector Variational Inequality Problem* (in short, FVVIP): Find $\bar{x} \in K$ such that for all $y \in K$

$$\langle T(\bar{x}), y - \bar{x}\rangle_\ell := (\langle T_1(\bar{x}), y - \bar{x}\rangle, \dots, \langle T_\ell(\bar{x}), y - \bar{x}\rangle) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{6.1}$$

- *Weak Vector Variational Inequality Problem* (in short, FWVVIP): Find $\bar{x} \in K$ such that for all $y \in K$

$$\langle T(\bar{x}), y - \bar{x}\rangle_\ell := (\langle T_1(\bar{x}), y - \bar{x}\rangle, \dots, \langle T_\ell(\bar{x}), y - \bar{x}\rangle) \notin -\text{int}(\mathbb{R}^\ell_+). \tag{6.2}$$

We denote the solution set of FVVIP and FWVVIP by Sol(FVVIP) and Sol(FWVVIP), respectively.

Let $W = (W_1, W_2, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ be arbitrary. The *weighted variational inequality problem* (in short, WVIP) consists of finding $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, W_2, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$W \cdot \langle T(\bar{x}), y - \bar{x}\rangle_\ell := \sum_{i=1}^{\ell} W_i \langle T_i(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K. \tag{6.3}$$

The solution set of WVIP is denoted by Sol(WVIP).

If $W \in \mathbb{T}^\ell_+$, then the solution of WVIP is called *normalized*. The set of normalized solutions of WVIP is denoted by Sol(WVIP)$_n$.

The following lemmas show the relationship among Sol(FVVIP), Sol(FWVVIP) and Sol(WVIP).

Lemma 6.1 *For any given weight vector $W = (W_1, W_2, \dots, W_\ell) \in \text{int}(\mathbb{R}^\ell_+)$ (respectively, $W = (W_1, W_2, \dots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$), Sol(WVIP) $\subseteq$ Sol(FVVIP) (respectively, Sol(WVIP) $\subseteq$ Sol(FWVVIP)).*

Proof Let $\bar{x} \in$ Sol(WVIP) w. r. t. the weight vector $W \in \text{int}(\mathbb{R}^\ell_+)$ (respectively, $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$) but $\bar{x} \notin$ Sol(FVVIP) (respectively, $\bar{x} \notin$ Sol(FWVVIP)). Then, there would exist some $y \in K$ such that

$$\langle T(\bar{x}), y - \bar{x}\rangle_\ell = \angle T_1(\bar{x}, y - \bar{x}\rangle, \dots, \langle T_\ell(\bar{x}, y - \bar{x}\rangle \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K$$

$$\Big(\langle T(\bar{x}), y - \bar{x}\rangle_\ell = \angle T_1(\bar{x}, y - \bar{x}\rangle, \dots, \langle T_\ell(\bar{x}, y - \bar{x}\rangle \in -\text{int}(\mathbb{R}^\ell_+), \quad \text{for all } y \in K\Big).$$

Since $W \in \text{int}(\mathbb{R}^\ell_+)$ (respectively, $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$), we have

$$W \cdot \langle T(\bar{x}), \bar{x} - y \rangle_\ell = \sum_{i=1}^{\ell} W_i \langle T_i(\bar{x}), y - \bar{x} \rangle > 0,$$

that is,

$$W \cdot \langle T(\bar{x}), y - \bar{x} \rangle_\ell < 0,$$

which contradicts our assumption that $\bar{x} \in K$ is a solution of WVIP. Hence, $\bar{x} \in K$ is a solution of FVVIP (respectively, FWVVIP). □

Lemma 6.2 *If $\bar{x}$ is a solution of FWVVIP, then there exists a weight vector $W = (W_1, W_2, \ldots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that $\bar{x} \in$ Sol(WVIP) w. r. t. W.*

Proof Let $\bar{x} \in$ Sol(FWVVIP). Then,

$$\{\langle T(\bar{x}), y - \bar{x} \rangle_\ell : y \in K\} \cap \{-\text{int}(\mathbb{R}^\ell_+)\} = \emptyset.$$

So, by a separation theorem, there exists $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$\inf_{y \in K} W \cdot \langle T(\bar{x}), y - \bar{x} \rangle_\ell \geq \sup_{v \in -\text{int}(\mathbb{R}^\ell_+)} W \cdot v.$$

This implies that $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. Then, the right-hand side of the above inequality is 0, and therefore, $W \cdot \langle T(\bar{x}), y - \bar{x} \rangle_\ell \geq 0$ for all $y \in K$. Hence, $\bar{x} \in K$ is a solution of WVIP. □

By combining Lemma 6.1 and L:5.6.2, we have the following relations in terms of Cheng [2] and Lee et al. [4].

Remark 6.1

(a) $$\bigcup_{W \in \text{int}(\mathbb{R}^\ell_+)} \text{Sol(WVIP)} \subseteq \text{Sol(VVIP)} \subseteq \text{Sol(WVVIP)}$$
$$= \bigcup_{W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}} \text{Sol(WVIP)}.$$

(b) Since the solution set Sol(WVIP) of the WVIP w. r. t. the weight vector $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ is equal to the solution set of the WVIP w. r. t. the weight vector αW, for any $\alpha > 0$, the above inclusion can be rewritten as

$$\bigcup_{W \in \text{int}(\mathbb{T}^{\ell_i}_+)} \text{Sol(WVIP)} \subseteq \text{Sol(VVIP)} \subseteq \text{Sol(WVVIP)} = \bigcup_{W \in \mathbb{T}^{\ell_i}_+} \text{Sol(WVIP)}.$$

(c) Cheng [2] and Lee et al. [4] studied the nonemptyness, compactness, convexity and connectedness of the solution set Sol(WVIP).

As we have seen in Chap. 5, the Minty vector variational inequalities are useful to establish the existence of a solution for (Stampacchia) vector variational inequalities and also have their own importance while dealing with vector optimization problems. Therefore, we consider the following Minty weighted variational inequality problem.

Let $W = (W_1, W_2, \ldots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ be arbitrary. The *Minty weighted variational inequality problem* (in short, MWVIP) consists in finding $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, W_2, \ldots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ such that

$$W \cdot \langle T(y), y - \bar{x} \rangle_\ell := \sum_{i=1}^{\ell} W_i \langle T_i(y), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{6.4}$$

The solution set of MWVIP is denoted by Sol(MWVIP).

It can be easily seen that the solution set Sol(MWVIP) is convex for every $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.

The following lemmas show that the relationship among Sol(FVVIP), Sol(FWVVIP) and Sol(WVIP).

Lemma 6.3 *For any given weight vector* $W = (W_1, W_2, \ldots, W_\ell) \in \text{int}(\mathbb{R}^\ell_+)$ *(respectively,* $W = (W_1, W_2, \ldots, W_\ell) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$*), Sol(MWVIP)* $\subseteq$ *Sol(FMVVIP) (respectively, Sol(MWVIP)* $\subseteq$ *Sol(FMWVVIP)).*

Proof Let $\bar{x} \in$ Sol(MWVIP) w. r. t. the weight vector $W \in \text{int}(\mathbb{R}^\ell_+)$ (respectively, $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$) but $\bar{x} \notin$ Sol(FVVIP) (respectively, $\bar{x} \notin$ Sol(FWVVIP)). Then, there would exist some $y \in K$ such that

$$\langle T(\bar{x}), y - \bar{x} \rangle_\ell \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K$$

$$\left(\langle T(\bar{x}), y - \bar{x} \rangle_\ell \in -\text{int}(\mathbb{R}^\ell_+), \quad \text{for all } y \in K \right).$$

Since $W \in \text{int}(\mathbb{R}^\ell_+)$ (respectively, $W \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$), we have

$$W \cdot \langle T(\bar{x}), \bar{x} - y \rangle_\ell = \sum_{i=1}^{\ell} W_i \langle T_i(\bar{x}), y - \bar{x} \rangle > 0,$$

that is,

$$W \cdot \langle T(\bar{x}), y - \bar{x} \rangle_\ell < 0,$$

which contradicts our assumption that $\bar{x} \in K$ is a solution of WVIP. Hence, $\bar{x} \in K$ is a solution of FVVIP (respectively, FWVVIP). □

In general, we have

$$\bigcup_{W\in\operatorname{int}(\mathbb{R}^{\ell}_{+})} \mathrm{Sol(WMVIP)} \subseteq \mathrm{Sol(MVVIP)} \subseteq \mathrm{Sol(MWVVIP)}$$

$$= \bigcup_{W\in\mathbb{R}^{\ell}_{+}\setminus\{\mathbf{0}\}} \mathrm{Sol(MWVIP)}.$$

Definition 6.1 Let $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+}\setminus\{\mathbf{0}\}$ be a weight vector. A matrix-valued function $T = (T_1, T_2, \dots T_\ell) : K \to \mathbb{R}^{\ell\times n}$ is said to be

(a) *weighted monotone w. r. t. the weight vector* W if for all $x, y \in K$, we have

$$W \cdot \langle T(x) - T(y), x - y\rangle_\ell \geq 0,$$

and *weighted strictly monotone w. r. t. the weight vector* W if the inequality is strict for all $x \neq y$;

(b) *weighted pseudomonotone w. r. t. the weight vector* W if for all $x, y \in K$, we have

$$W \cdot \langle T(x), y - x\rangle_\ell \geq 0 \quad\Rightarrow\quad W \cdot \langle T(y), y - x\rangle_\ell \geq 0,$$

and *weighted strictly pseudomonotone w. r. t. the weight vector* W if the second inequality is strict for all $x \neq y$;

(c) *weighted maximal pseudomonotone w. r. t. the weight vector* W if it is weighted pesudomonotone and for all $x, y \in K$, we have

$$W \cdot \langle T(z), z - x\rangle_\ell \leq 0 \;\; \forall z \in \;]x, y] \quad\Rightarrow\quad W \cdot \langle T(x), y - x\rangle \geq 0, \tag{6.5}$$

and *weighted maximal strictly pseudomonotone w. r. t. the weight vector* W if it is weighted strictly pseudomonotone and (6.5) holds.

It can easily seen that if each T_i is monotone, then T is weighted monotone w. r. t. the any weight vector $W \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$.

Definition 6.2 Let $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$ be a weight vector. A matrix-valued function $T = (T_1, T_2, \dots, T_\ell) : K \to \mathbb{R}^{\ell\times n}$ is said to be *weighted hemicontinuous w. r. t. the weight vector* W if for all $x, y \in K$ and $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \sum_{i=1}^{\ell} W_i \cdot \langle T_i(x + \lambda(y - x)), y - x\rangle$ is continuous.

If each T_i is continuous, then T is continuous, and hence, T is hemicontinuous.

Proposition 6.1 *Let $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$ be a weight vector and $T = (T_1, T_2, \dots, T_\ell) : K \to \mathbb{R}^{\ell\times n}$ be weighted hemicontinuous and weighted pseudomonotone w. r. t. the weight vector $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$. Then, T is weighted maximal pseudomonotone w. r. t. the same weight vector $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$.*

Proof Assume that for all $x, y \in K$,

$$W \cdot \langle T(z), z - x \rangle_\ell \geq 0, \quad \text{for all } z \in \,]x, y].$$

Then,

$$W \cdot \langle T(x + \lambda(y - x)), y - (x + \lambda(y - x)) \rangle_\ell \geq 0, \quad \text{for all } \lambda \in \,]0, 1]$$

which implies that

$$W \cdot \langle T(x + \lambda(y - x)), y - x \rangle_\ell \geq 0, \quad \text{for all } \lambda \in \,]0, 1].$$

By the weighted hemicontinuity of T, we have

$$W \cdot \langle T(y), y - x \rangle_\ell \geq 0.$$

Hence, T is weighted maximal pseudomonotone w. r. t. the weight vector W. □

The following lemma can be viewed as a generalization of the Minty lemma (see, [3]).

Lemma 6.4 *Let* $W = (W_1, W_2, \ldots, W_n) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ *be a weight vector and* $T = (T_1, T_2, \ldots, T_\ell) : K \to \mathbb{R}^{\ell \times n}$ *be weighted maximal pseudomonotone w. r. t. the weight vector* $W = (W_1, W_2, \ldots, W_n) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$. *Then, Sol(WVIP) = Sol(MWVIP).*

Proof It is obvious that Sol(WVIP) $\subseteq$ Sol(MWVIP) by the weighted pseudomonotonicity of T.

Let $\bar{x} \in$ Sol(MWVIP), then

$$W \cdot \langle T(y), y - \bar{x} \rangle_\ell \geq 0, \quad \text{for all } y \in K.$$

Since K is convex, we have $]\bar{x}, y] \subseteq K$, and therefore,

$$W \cdot \langle T(z), z - \bar{x} \rangle_\ell \geq 0, \quad \text{for all } z \in \,]\bar{x}, y].$$

By the weighted maximal pseudomonotonicity of T, we have

$$W \cdot \langle T(\bar{x}), y - \bar{x} \rangle_\ell \geq 0, \quad \text{for all } y \in K.$$

This shows that $\bar{x} \in$ Sol(WVIP), and hence, Sol(WVIP) = Sol(MWVIP). □

Remark 6.2 In view of Proposition 6.1 and Lemma 6.4, we have that if T is weighted hemicontinuous and weighted pseudomonotone w. r. t. the same weight vector $W = (W_1, W_2, \ldots, W_n) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$, then Sol(WVIP) = Sol(MWVIP).

However, Charitha et al. [1] proved that Sol(WVIP) = Sol(MWVIP) if each T_i, $i = 1, 2, \ldots, \ell$ is continuous and monotone.

We now have some existence results for solutions of WVIP.

Theorem 6.1 *Let $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$ be a weight vector and K be a nonempty convex subset of $\mathbb{R}^n$. Let $T = (T_1, T_2, \dots, T_\ell) : K \to \mathbb{R}^{\ell \times n}$ be weighted maximal pseudomonotone w. r. t. $W = (W_1, W_2, \dots, W_n) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$. Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for each $x \in K \setminus D$, $W \cdot \langle T(x), y - x \rangle_\ell < 0$. Then, there exists a solution $\bar{x} \in K$ of WVIP.*

Proof For each $x \in K$, define set-valued maps $F, G : K \to 2^K$ by

$$F(x) = \{y \in K : W \cdot \langle T(y), y - x \rangle_\ell < 0\}$$

and

$$G(x) = \{y \in K : W \cdot \langle T(x), y - x \rangle_\ell < 0\}.$$

Then, it is clear that for each $x \in K$, $G(x)$ is convex. By weighted pseudomonotonicity of T, we have $F(x) \subseteq G(x)$ for all $x \in K$.

For each $y \in K$, the complement of $F^{-1}(y)$ in K is

$$[F^{-1}(y)]^c = \{x \in K : W \cdot \langle T(y), y - x \rangle_\ell \geq 0\}$$

is closed in K, and hence, $F^{-1}(y)$ is open in K. Therefore, $F^{-1}(y)$ is compactly open.

Assume that for all $x \in K$, $F(x)$ is nonempty. Then all the conditions of Theorem 1.36 are satisfied, and therefore, there exists $\hat{x} \in K$ such that $\hat{x} \in G(\hat{x})$. It follows that

$$0 = W \cdot \langle T(\tilde{x}), \tilde{x} - \tilde{x} \rangle < 0,$$

a contradiction. Hence, there exists $\bar{x} \in K$ such that $F(\bar{x}) = \emptyset$. This implies that for all $y \in K$,

$$W \cdot \langle T(y), y - \bar{x} \rangle_\ell \geq 0,$$

that is, there exists $\bar{x} \in K$ w. r. t. the weight vector $W = (W_1, W_2, \dots, W_\ell) \in \mathbb{R}^{\ell}_{+} \setminus \{\mathbf{0}\}$ such that

$$W \cdot \langle T(y), y - \bar{x} \rangle_\ell \leq 0, \quad \text{for all } y \in K.$$

By Lemma 6.4, $\bar{x} \in K$ is a solution of WVIP. □

Remark 6.3 In view of Remark 6.2, the assumption that T is weighted maximal monotone in Theorem 6.1 can be replaced by weighted hemicontinuous and weighted pseudomonotone w. r. t. W.

Remark 6.4 In Theorem 6.1, if T is weighted maximal strictly pseudomonotone w. r. t. W, then solution of WVIP is unique.

Indeed, assume that there exist two solutions x' and x'' of WVIP. Then, we have

$$W \cdot \langle T(x''), x' - x'' \rangle_\ell \geq 0.$$

By the weighted strictly pseudomonotonicity of T, we have

$$W \cdot \langle T(x'), x' - x'' - \rangle_\ell > 0, \quad \text{i.e.} \quad W \cdot \langle T(x'), x'' - x' \rangle_\ell < 0,$$

that is, x' is not a solution of WVIP, a contradiction.

Now we present the definition of weighted B-pseudomonotonicity.

Definition 6.3 Let $W = (W_1, W_2, \ldots, W_n) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ be a weight vector. A matrix-valued function $T = (T_1, T_2, \ldots, T_\ell) : K \to \mathbb{R}^{\ell \times n}$ is said to be *weighted B-pseudomonotone w. r. t. the weight vector W* if for each $x \in K$ and every sequence $\{x_m\}_{m\in\mathbb{N}}$ in K converging to x with

$$\limsup_{m\to\infty} W \cdot \langle T(x_m), x - x_m \rangle_\ell \geq 0,$$

we have

$$\limsup_{m\to\infty} W \cdot \langle T(x_m), y - x_m \rangle \leq W \cdot \langle T(x), y - x \rangle, \quad \text{for all } y \in K.$$

Theorem 6.2 *Let $W = (W_1, W_2, \ldots, W_n) \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ be a weight vector and K be a nonempty convex subset of $\mathbb{R}^n$. Let $T = (T_1, T_2, \ldots, T_\ell) : K \to \mathbb{R}^{\ell \times n}$ be weighted B-pseudomonotone w. r. t. W such that for each $A \in \mathscr{F}(K)$, $x \mapsto W \cdot \langle T(x), y - x \rangle_\ell$ is lower semicontinuous on* co$A$. *Assume that there exist a nonempty, closed and compact subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $W \cdot \langle T(x), \tilde{y} - x \rangle_\ell < 0$. Then, there exists a solution $\bar{x} \in K$ of WVIP.*

Proof For each $x \in K$, let $G : K \to 2^K$ be defined by

$$G(x) = \{y \in K : W \cdot \langle T(x), y - x \rangle_\ell < 0\}.$$

Then, for all $x \in K$, $G(x)$ is convex. Let $A \in \mathscr{F}(K)$, then for all $y \in \text{co}A$,

$$[G^{-1}(y)]^c \cap \text{co}A = \{x \in \text{co}A : W \cdot \langle T(x), y - x \rangle_\ell \geq 0\}$$

is closed in co$A$ by lower semicontinuity of the map $x \mapsto W \cdot \langle T(x), y - x \rangle_\ell$ on co$A$. Hence $G^{-1}(y) \bigcap \text{co}A$ is open in co$A$.

Suppose that $x, y \in \text{co}A$ and $\{x_m\}_{m\in\mathbb{N}}$ is a sequence in K converging to x such that

$$W \cdot \langle T(x_m), (\alpha x + (1 - \alpha)y) - x_m \rangle_\ell \geq 0, \quad \text{for all} \quad m \in \mathbb{N} \text{ and all } \alpha \in [0, 1].$$

For $\alpha = 0$, we have

$$W \cdot \langle T(x_m), x - x_m \rangle_\ell \geq 0, \quad \text{for all } m \in \mathbb{N},$$

and therefore,

$$\limsup_{m \to \infty} W \cdot \langle T(x_m), x - x_m \rangle_\ell \geq 0.$$

By the weighted B-pseudomonotonicity of T, we have

$$\limsup_{m \to \infty} W \cdot \langle T(x_m), y - x_m \rangle_\ell \leq W \cdot \langle T(x), y - x \rangle_\ell. \tag{6.6}$$

For $\alpha = 1$, we have

$$W \cdot \langle T(x_m), y - x_m \rangle \geq 0, \quad \text{for all } m \in \mathbb{N},$$

and therefore,

$$\limsup_{m \to \infty} W \cdot \langle T(x_m), y - x_m \rangle_\ell \geq 0. \tag{6.7}$$

From (6.6) and (6.7), we get

$$W \cdot \langle T(x), y - x \rangle_\ell \geq 0,$$

and thus $y \notin G(x)$.

Assume that for all $x \in K$, $G(x)$ is nonempty. Then all the conditions of Theorem 1.36 are satisfied. The rest of the proof follows the lines of the proof of Theorem 6.1. □

Some existence results for solutions of WVIP have been studied in [4] under strong monotonicity and Lipschitz continuity of each T_i and in [1] under continuity and monotonicity each T_i.

References

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