Chapter 1
Preliminaries

This chapter deals with basic definitions from convex analysis and nonlinear analysis, such as convex sets and cones, convex functions and their properties, generalized derivatives, and continuity for set-valued maps. We also gather some known results from fixed point theory for set-valued maps, namely, Nadler's fixed point theorem, Fan-KKM lemma and its generalizations, Fan section lemma and its generalizations, Browder fixed point theorem and its generalizations, maximal element theorems and Kakutani fixed point theorem. A brief introduction of scalar variational inequalities, nonsmooth variational inequalities, generalized variational inequalities and equilibrium problems is given.

1.1 Convex Sets and Cones

Throughout the book, all vector spaces are assumed to be defined over the field of real numbers, and we adopt the following notations.

We denote by $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ the set of all real numbers, rational numbers and natural numbers, respectively. The interval $[0, \infty)$ is denoted by $\mathbb{R}_+$. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$. The zero element in a vector space will be denoted by $\mathbf{0}$. Let A be a nonempty set. We denote by 2^A (respectively, $\Pi(A)$) the family of all subsets (respectively, nonempty subsets) of A and by $\mathscr{F}(A)$ the family of all nonempty finite subsets of A. If A and B are nonempty subsets of a topological space X such that $B \subseteq A$, we denote by $\text{int}_A(B)$ (respectively, $\text{cl}_A(B)$) the interior (respectively, closure) of B in A. We also denote by $\text{int}(A)$, $\text{cl}(A)$ (or $\overline{A}$), and $\text{bd}(A)$ the interior of A in X, the closure of A in X, and the boundary of A, respectively. Also, we denote by A^c the complement of the set A. If X and Y are topological vector spaces, then we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear functions from X to Y.

Q.H. Ansari et al., *Vector Variational Inequalities and Vector Optimization*,
Vector Optimization, DOI 10.1007/978-3-319-63049-6_1

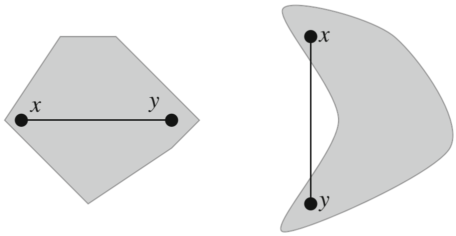


Fig. 1.1 Illustration of a convex set and of a nonconvex set, respectively

Definition 1.1 Let X be a vector space, and x and y be distinct points in X. The set $L = \{z : z = \lambda x + (1 - \lambda)y \text{ for all } \lambda \in \mathbb{R}\}$ is called the *line through x and y*.

The set $[x, y] = \{z : z = \lambda x + (1 - \lambda)y \text{ for } 0 \leq \lambda \leq 1\}$ is called a *line segment* with the endpoints x and y.

Definition 1.2 A subset W of a vector space X is said to be a *subspace* if for all $x, y \in W$ and $\lambda, \mu \in \mathbb{R}$, we have $\lambda x + \mu y \in W$.

Geometrically speaking, a subset W of X is a subspace of X if for all $x, y \in W$, the plane through the origin, x and y lies in W.

Definition 1.3 A subset M of a vector space X is said to be an *affine set* if for all $x, y \in M$ and $\lambda, \mu \in \mathbb{R}$ such that $\lambda + \mu = 1$ imply that $\lambda x + \mu y \in M$, that is, for all $x, y \in M$ and $\lambda \in \mathbb{R}$, we have $\lambda x + (1 - \lambda)y \in M$.

Geometrically speaking, a subset M of X is an affine set if it contains the whole line through any two of its points.

Definition 1.4 A subset K of a vector space X is said to be a *convex set* if for all $x, y \in K$ and $\lambda, \mu \geq 0$ such that $\lambda + \mu = 1$ imply that $\lambda x + \mu y \in K$, that is, for all $x, y \in K$ and $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in K$.

Geometrically speaking, a subset K of X is *convex* if it contains the whole line segment with endpoints through any two of its points (see Fig. 1.1).

Definition 1.5 A subset C of a vector space X is said to be a *cone* if for all $x \in C$ and $\lambda \geq 0$, we have $\lambda x \in C$.

A subset C of X is said to be a *convex cone* if it is convex and a cone; that is, for all $x, y \in K$ and $\lambda, \mu \geq 0$ imply that $\lambda x + \mu y \in C$ (see Fig. 1.2 and 1.3).

Remark 1.1 If C is a cone, then $\mathbf{0} \in C$. In the literature, it is mostly assumed that the cone has its apex at the origin. This is the reason why $\lambda \geq 0$ is chosen in the definition of a cone. However, some references define a set $C \subset X$ to be a cone if $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$. In this case, the apex of the "shifted" cone may not be at the origin, or $\mathbf{0}$ may not belong to C.

Remark 1.2 It is clear from the above definitions that every subspace is an affine set as well as a convex cone, and every affine set and every convex cone are convex. But the converse of these statements may not be true in general.

Evidently, the empty set, each singleton set $\{x\}$ and the whole space X are all both affine and convex. In $\mathbb{R}^n$, straight lines, circular discs, ellipses and interior of

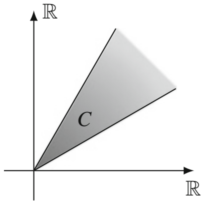


Fig. 1.2 A convex cone

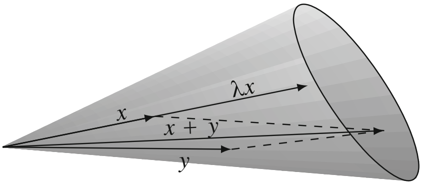


Fig. 1.3 A convex cone in $\mathbb{R}^3$

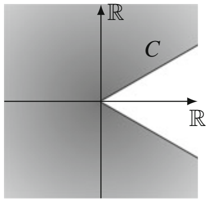


Fig. 1.4 A cone which is not convex

triangles are all convex. A ray, which has the form $\{x_0 + \lambda v : \lambda \geq 0\}$, where $v \neq \mathbf{0}$, is convex, but not affine.

Remark

(a) A cone C may or may not be convex (see Figs. 1.2 - 1.4).
(b) A cone C may be open, closed or neither open nor closed.
(c) A set C is a convex cone if it is both convex as well as a cone.
(d) If C_1 and C_2 are convex cones, then $C_1 \cap C_2$ and $C_1 + C_2$ are also convex cones.

Definition 1.6 Let X be a vector space. Given $x_1, x_2, \ldots, x_m \in X$, a vector $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$ is called

(a) a *linear combination* of $x_1, x_2, \ldots, x_m$ if $\lambda_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, m$;
(b) an *affine combination* of $x_1, x_2, \ldots, x_m$ if $\lambda_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$;
(c) a *convex combination* of $x_1, x_2, \ldots, x_m$ if $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$;
(d) a *cone combination* of $x_1, x_2, \ldots, x_m$ if $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, m$.

A set K is a subspace, affine, convex or a cone if it is closed under linear, affine, convex or cone combination, respectively, of points of K.

Theorem 1.1 *A subset K of a vector space X is convex (respectively, subspace, affine, convex cone) if and only if every convex (respectively, linear, affine, cone) combination of points of K belongs to the set K.*

Proof Since a set that contains all convex combinations of its points is obviously convex, we only consider K is convex and prove that it contains any convex combination of its points, that is, if K is convex and $x_i \in K$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$, then we have to show that $\sum_{i=1}^{m} \lambda_i x_i \in K$. We prove this by induction on the number m of points of K occurring in a convex combination. If $m = 1$, the assertion is simply $x_1 \in K$ implies $x_1 \in K$, evidently true. If $m = 2$, then $\lambda_1 x_1 + \lambda_2 x_2 \in K$ for $\lambda_i \geq 0$, $i = 1, 2$, $\sum_{i=1}^{2} \lambda_i = 1$, holds because K is convex. Now suppose that the result is true for m. Then (for $\lambda_{m+1} \neq 1$)

$$\begin{aligned}
\sum_{i=1}^{m+1} \lambda_i x_i &= \sum_{i=1}^{m} \lambda_i x_i + \lambda_{m+1} x_{m+1} \\
&= \sum_{i=1}^{m} (1 - \lambda_{m+1}) \frac{\lambda_i x_i}{1 - \lambda_{m+1}} + \lambda_{m+1} x_{m+1} \\
&= (1 - \lambda_{m+1}) \sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} x_i + \lambda_{m+1} x_{m+1} \\
&= (1 - \lambda_{m+1}) \sum_{i=1}^{m} \mu_i x_i + \lambda_{m+1} x_{m+1},
\end{aligned}$$

where $\mu_i = \dfrac{\lambda_i}{(1 - \lambda_{m+1})}$, $i = 1, 2, \ldots, m$. But then $\mu_i \geq 0$ for $i = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{m} \mu_i = \frac{\sum_{i=1}^{m} \lambda_i}{1 - \lambda_{m+1}} = \frac{1 - \lambda_{m+1}}{1 - \lambda_{m+1}} = 1,$$

so by the result for m, $y = \sum_{i=1}^{m} \mu_i x_i \in K$. Immediately, by convexity of K, we have

$$\sum_{i=1}^{m+1} \lambda_i x_i = (1 - \lambda_{m+1}) y + \lambda_{m+1} x_{m+1} \in K.$$

The proof for subspace, affine and convex cone cases follows exactly the same pattern. □

Remark 1.3

(a) The intersection of any number of convex sets (respectively, subspaces, affine sets, convex cones) is a convex set (respectively, subspace, affine set, convex cone).
(b) The union of any number of convex sets need not be convex.
(c) For $i \in \mathbb{N}$, let K_i be convex. If $K_i \subseteq K_{i+1}$, $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} K_i$ is convex.

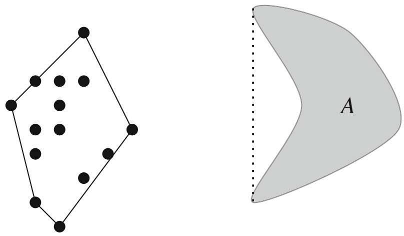


Fig. 1.5 Illustration of the convex hull of 14 points and a convex hull of a set A

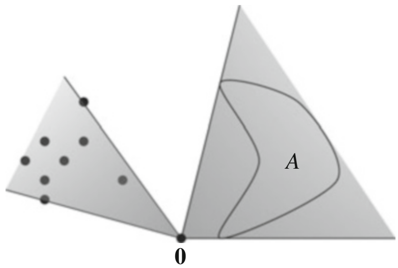


Fig. 1.6 Conic hull of 8 points and the cone generated by the set A

(d) If K_1 and K_2 are convex subsets of a vector space X and $\alpha \in \mathbb{R}$, then $K_1 + K_2 = \{x + y : x \in K_1,\ y \in K_2\}$ and $\alpha K_1 = \{\alpha x : x \in K_1\}$ are convex sets.

(e) A subset K of a vector space X is convex if and only if $(\lambda + \mu)K = \lambda K + \mu K$ for all $\lambda \geq 0$, $\mu \geq 0$.

Definition 1.7 Let A be a nonempty subset of a vector space X. The intersection of all convex sets (respectively, subspaces, affine sets) containing A is called a *convex hull* (respectively, *linear hull, affine hull*) of A, and it is denoted by $\text{co}(A)$ (respectively, $[A]$, $\text{aff}(A)$) (see Fig. 1.5). Similarly, the intersection of all convex cones containing A is called a *conic hull* of A, and it is denoted by $\text{cone}(A)$ (see Fig. 1.6).

By Remark 1.3 (a), the convex (respectively, affine, conic) hull is a convex set (respectively, affine set, convex cone). In fact, $\text{co}(A)$ (respectively, $\text{aff}(A)$, $\text{cone}(A)$) is the smallest convex set (respectively, affine set, convex cone) containing A.

The cone $\text{cone}(A)$ can also be written as

$$\text{cone}(A) = \{x \in X : x = \lambda y \text{ for some } \lambda \geq 0 \text{ and some } y \in A\}.$$

It is also called a *cone generated* by A (see Fig. 1.6).

Theorem 1.2 *Let A be a nonempty subset of a vector space X. Then $x \in \text{co}(A)$ if and only if there exist x_i in A and $\lambda_i \geq 0$, for $i = 1, 2, \ldots, m$, for some positive integer m, where $\sum_{i=1}^{m} \lambda_i = 1$ such that $x = \sum_{i=1}^{m} \lambda_i x_i$.*

Proof Since $\text{co}(A)$ is a convex set containing A, therefore, from Theorem 1.1, every convex combination of its points lies in it, that is, $x \in \text{co}(A)$.

Conversely, let $K(A)$ be the set of all convex combinations of elements of A. We claim that the set

$$K(A) = \left\{ \sum_{i=1}^{m} \lambda_i x_i : x_i \in A,\ \lambda_i \geq 0, i = 1, 2, \ldots, m,\ \sum_{i=1}^{m} \lambda_i = 1,\ m \geq 1 \right\}$$

is convex. Indeed, consider $y = \sum_{i=1}^{m} \lambda_i y_i$ and $z = \sum_{j=1}^{\ell} \mu_j z_j$ where $y_i \in A$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} \lambda_i = 1$ and $z_j \in A$, $\mu_j \geq 0, j = 1, 2, \ldots, \ell$, $\sum_{j=1}^{\ell} \mu_j = 1$, and let $0 \leq \lambda \leq 1$. Then

$$\lambda y + (1 - \lambda) z = \sum_{i=1}^{m} \lambda \lambda_i y_i + \sum_{j=1}^{\ell} (1 - \lambda) \mu_j z_j,$$

where $\lambda \lambda_i \geq 0, i = 1, 2, \ldots, m$, $(1 - \lambda)\mu_j \geq 0, j = 1, 2, \ldots, \ell$ and

$$\sum_{i=1}^{m} \lambda \lambda_i + \sum_{j=1}^{\ell} (1 - \lambda) \mu_j = \lambda \sum_{i=1}^{m} \lambda_i + (1 - \lambda) \sum_{j=1}^{\ell} \mu_j = \lambda + (1 - \lambda) = 1.$$

Also, the set $K(A)$ of convex combinations contains A (each x in A can be written as $x = 1 \cdot x$). By the definition of $\operatorname{co}(A)$ as the intersection of all convex supersets of A, we deduce that $\operatorname{co}(A)$ is contained in $K(A)$.

Thus the convex hull of A is the set of all (finite) convex combinations from within A. □

The above result also holds for an affine set and a convex cone.

Corollary 1.1

(a) *The set A is convex if and only if* $A = \operatorname{co}(A)$.
(b) *The set A is affine if and only if* $A = \operatorname{aff}(A)$.
(c) *The set A is a convex cone if and only if* $A = \operatorname{cone}(A)$.
(d) *The set A is a subspace if and only if* $A = [A]$.

Definition 1.8 The *relative interior* of a set C in a topological vector space X, denoted by $\operatorname{relint}(C)$, is defined as

$$\operatorname{relint}(C) = \{x \in C : N_\varepsilon(x) \cap \operatorname{aff}(C) \subseteq C \text{ for some } \varepsilon > 0\},$$

where $N_\varepsilon(x)$ denotes the neighborhood of x.

Remark 1.4

(a) We have $\operatorname{relint}(C) \subseteq \operatorname{aff}(C)$.
(b) $\operatorname{relint}(C) = \operatorname{aff}(C)$ if and only if $\operatorname{aff}(C) = X$.

Example 1.1

(a) Consider the set $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\}$. Then $\text{int}(C) = \emptyset$, but $\text{relint}(C) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, z = 0\}$.
(b) For the set $C = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$, we have $\text{int}(C) = \text{relint}(C) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$.

Definition 1.9 The *relative boundary* of a set C in a topological vector space X, denoted by $\text{relb}(C)$ or $\text{rb}(C)$, defined as

$$\text{relb}(C) = \text{cl}(C) \setminus \text{relint}(C).$$

Example 1.2 Consider a square in the (x_1, x_2)-plane in $\mathbb{R}^3$ defined as

$$C = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1,\ x_3 = 0\right\}.$$

Its affine hull is the (x_1, x_2)-plane, that is,

$$\text{aff}(C) = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\right\}.$$

The interior of C is empty, but the relative interior is

$$\text{relint}(C) = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : -1 < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\right\}.$$

Its boundary (in $\mathbb{R}^3$) is itself; its relative boundary is the wire-frame outline,

$$\text{relb}(C) = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \max\{|x_1|, |x_2|\} = 1,\ x_3 = 0\right\}.$$

Definition 1.10 A subset C of a topological vector space X is called *relatively open* if $\text{relint}(C) = C$.

Remark 1.5 If $C_1 \subseteq C_2$, then

(a) $\text{cl}(C_1) \subseteq \text{cl}(C_2)$ and
(b) $\text{int}(C_1) \subseteq \text{int}(C_2)$.

Note that property (b) does not hold for relative interior, that is, $\text{relint}(C_1) \subseteq \text{relint}(C_2)$ is not true in general. For example, if C_2 is a cube in $\mathbb{R}^3$ and C_1 is one of the faces of C_2. Then $\text{relint}(C_2)$ and $\text{relint}(C_1)$ are both nonempty but disjoint.

Remark 1.6 Let C be a subset of a topological vector space X.

(a) Every affine set is relatively open by definition and at the same time closed.
(b) $\text{cl}(C) \subset \text{cl}(\text{aff}(C)) = \text{aff}(C)$ for every $C \subseteq X$.
(c) Any line through two different points of $\text{cl}(C)$ lies entirely in $\text{aff}(C)$.

If $C \subseteq X$ is convex, then we have the following assertions:

(d) $\text{int}(C)$ and $\text{relint}(C)$ are convex.

(e) $\mathrm{cl}(C)$ is also convex.
(f) If $C \subseteq X$ is a convex set with nonempty interior, then $\mathrm{cl}(\mathrm{int}(C)) = \mathrm{cl}(C)$.
(g) If $C \subseteq X$ is a convex set with nonempty interior, then $\mathrm{int}(\mathrm{cl}(C)) = \mathrm{int}(C)$.
(h) $\mathrm{relint}(C) = \mathrm{relint}(\mathrm{cl}(C))$. Moreover, it holds $\mathrm{int}(C) = \mathrm{int}(\mathrm{cl}(C))$.
(i) $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{relint}(C))$ as well as $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{int}(C))$ if $\mathrm{int}(C) \neq \emptyset$.

Remark 1.7 ([127, Corollary 6.3.2]) If C is a convex set in $\mathbb{R}^n$, then every open set which meets $\mathrm{cl}(C)$ also meets $\mathrm{relint}(C)$.

Proposition 1.1 ([86]) *Let Y be a topological vector space with a cone C, $c_0 \in \mathrm{int}(C)$ and $V := \mathrm{int}(C) - c_0$. Then $Y = \{\lambda V : \lambda \geq 0\}$.*

Definition 1.11 A cone C in a vector space X is said to be

(a) *nontrivial* or *proper* if $C \neq \{\mathbf{0}\}$ and $C \neq X$;
(b) *reproducing* if $C - C = X$;
(c) *pointed* if for $x \in C$, $x \neq \mathbf{0}$, the negative $-x \notin C$, that is, $C \cap (-C) = \{\mathbf{0}\}$.

Definition 1.12 A cone C in a topological vector space X is said to be a

(a) *closed cone* if it is also closed;
(b) *solid cone* if it has nonempty interior.

Below we give some properties of a cone.

Remark 1.8

(a) If C is a cone, then the convex hull of C, $\mathrm{co}(C)$ is a convex cone.
(b) If C_1 and C_2 are convex cones, then $C_1 + C_2 = \mathrm{co}(C_1 \cup C_2)$.

Example 1.3 Let

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, 2, \ldots, n\}.$$

Then $\mathbb{R}^n_+$ is a proper, closed, pointed, reproducing convex cone in the vector space $\mathbb{R}^n$.

Example 1.4 Let $C[0, 1]$ be the vector space of all real-valued continuous linear functionals defined on the interval $[0, 1]$. Then

$$C_+[0, 1] = \{f \in C[0, 1] : f(t) \geq 0 \text{ for all } t \in [0, 1]\}$$

is a proper, reproducing, pointed, convex cone in $C[0, 1]$. Note that the set

$$C^+ = \{f \in C_+[0, 1] : f \text{ is nondecreasing}\}$$

is also a proper, pointed, convex cone in the space $C[0, 1]$ but it is not reproducing as $C^+ - C^+$ is the proper subspace of all functions with bounded variation of $C[0, 1]$.

Example 1.5 Let

$$C = \big\{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > 0, \text{ or} \\ x_1 = 0,\ x_2 > 0, \text{ or} \\ \dots \\ x_1 = \dots = x_{n-1} = 0,\ x_n > 0, \text{ or} \\ x = \mathbf{0}\big\},$$

where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$. Then C is a proper, closed, pointed, reproducing convex cone in the vector space $\mathbb{R}^n$.

Let C be a subset of a vector space X. We denote by $\ell(C) = C \cap (-C)$.

Definition 1.13 Let X be a topological vector space. A convex cone C in X is said to be

(a) *acute* if its closure $\mathrm{cl}(C)$ is pointed;
(b) *correct* if $\mathrm{cl}(C) + C \setminus \ell(C) \subseteq C$.

Example 1.6

(a) The nonnegative orthant $\mathbb{R}^n_+$ of all vectors of $\mathbb{R}^n$ with nonnegative coordinates is a convex, closed, acute and correct cone. The set $\{\mathbf{0}\}$ is also such a cone, but it is a trivial cone. The set composed of zero and of the vectors with the first coordinates being positive, is a pointed, correct cone, but it is not acute.
(b) Let

$$C = \big\{(x, y, z) \in \mathbb{R}^3 : x > 0,\ y > 0,\ z > 0\big\} \\ \cup \big\{(x, y, z) \in \mathbb{R}^3 : x \geq y \geq 0,\ z = 0\big\}.$$

Then C is a convex, acute cone but not correct.
(c) Let Ω be the vector space of all sequences $x = \{x_m\}$ of real numbers. Let

$$C = \{x = \{x_m\} \in \Omega : x_m \geq 0 \text{ for all } m\}.$$

Then C is a convex pointed cone. We cannot say whether it is correct or acute because no topology has been given on the space.

Proposition 1.2 *A cone C is correct if and only if* $\mathrm{cl}(C) + C \setminus \ell(C) \subseteq C \setminus \ell(C)$.

Proof If $\mathrm{cl}(C) + C \setminus \ell(C) \subseteq C \setminus \ell(C)$, then the cone is obviously correct because $C \setminus \ell(C) \subseteq C$.

Conversely, assume that C is a correct convex cone. Since $\ell(C)$ is a subspace and C is convex, for all $a, b \in C$, $a + b \in \ell(C)$ implies $a, b \in \ell(C)$. Therefore,

$$C \setminus \ell(C) + C \setminus \ell(C) = C \setminus \ell(C),$$

and

$$C + C \setminus \ell(C) \subseteq C \setminus \ell(C).$$

Thus,

$$\begin{aligned} \mathrm{cl}(C) + C \setminus \ell(C) &= \mathrm{cl}(C) + C \setminus \ell(C) + C \setminus \ell(C) \\ &\subseteq C + C \setminus \ell(C) \subseteq C \setminus \ell(C). \end{aligned}$$

This completes the proof. □

The cone $\mathbb{R}^n_+ \subset \mathbb{R}^n$ has the following interesting property: Consider the set

$$B = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}.$$

Then for every $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, there exist a unique $b \in B$ and $\lambda > 0$ such that $x = \lambda b$. Indeed, consider $\lambda = x_1 + x_2 + \cdots + x_n$ (> 0) and $b = \lambda^{-1} x$. In view of this property, we have the following definition.

Definition 1.14 Let X be a vector space and C be a proper cone in X. A nonempty subset $B \subset C$ is called a *base* for C if each nonzero element $x \in C$ has a unique representation of the form $x = \lambda b$ for some $\lambda > 0$ and some $b \in B$ (Figs. 1.7 and 1.8).

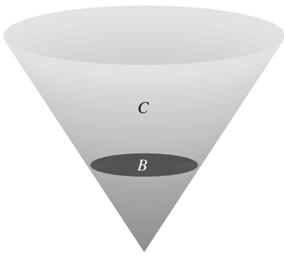


Fig. 1.7 B is a base for the cone C

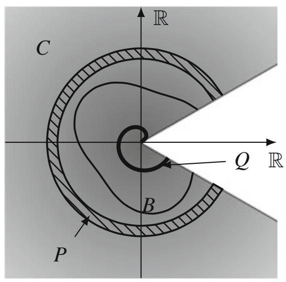


Fig. 1.8 B is base for C, but Q and P are not a base for C

Remark 1.9 Note that if B is a convex base of a proper convex cone C, then $\mathbf{0} \notin B$. Indeed, suppose that $\mathbf{0} \in B$. Since B is convex, for every element $b \in B$, the convex combination of $\mathbf{0}$ and b also belongs to B. Then we also have $b = 2 \cdot \frac{1}{2}b \in B$, contradicting the uniqueness of the representation of $b \in C \setminus \{\mathbf{0}\}$.

Theorem 1.3 *Let C be a proper convex cone in a vector space X and $B \subset X$ be a convex set. Then the following assertions are equivalent:*

(a) *B is a base for C;*
(b) *$C = \mathbb{R}_+B$ and $\mathbf{0} \notin \mathrm{aff}(B)$;*
(c) *There exists a linear functional $\phi : X \to \mathbb{R}$ such that $\phi(x) > 0$ for every $x \in C \setminus \{\mathbf{0}\}$ and $B = \{x \in C : \phi(x) = 1\}$.*

Proof (a) $\Rightarrow$ (b) Let B be a base for C. Then by Definition 1.14, $C = \mathbb{R}_+B$. Because B is convex, $\mathrm{aff}(B) = \{\mu b + (1 - \mu)b' : b, b' \in B,\ \mu \in \mathbb{R}\}$. Assume that $\mathbf{0} \in \mathrm{aff}(B)$, then $\mathbf{0} = \mu b + (a - \mu)b'$ for some $b, b' \in B$ and $\mu \in \mathbb{R}$. Since $\mathbf{0} \notin B$, $\mu \notin [0, 1]$. Thus, there exists some $\mu_0 > 1$, $b_0, b_0' \in B$ such that $\mu_0 b_0 = (\mu_0 - 1)b_0' \in C$, in contradiction to the definition of the base. Therefore, $\mathbf{0} \notin \mathrm{aff}(B)$.

(b) $\Rightarrow$ (c) Assume that $C = \mathbb{R}_+B$ and $\mathbf{0} \notin \mathrm{aff}(B)$. Consider $b_0 \in B$ and $X_0 := \mathrm{aff}(B) - b_0$. Then X_0 is a linear subspace of X and $b_0 \notin X_0$. Let $L_0 \subset X_0$ be a base of X_0. Then $L_0 \cup \{b_0\}$ is linearly independent, so, we can complete $L_0 \cup \{b_0\}$ to a base L of X. There exists a unique linear function $\phi : X \to \mathbb{R}$ such that $\phi(x) = 0$ for all $x \in L \setminus \{b_0\}$ and $\phi(b_0) = 1$. Since $\mathrm{aff}(B) = b_0 + X_0$, it holds $\phi(x) = 1$ for all $x \in \mathrm{aff}(B)$, thus, $B \subset \{x \in C : \phi(x) = 1\}$. Conversely, let $x \in C$ be such that $\phi(x) = 1$. Then $x = tb$ for some $t > 0$ and $b \in B$. It follows that $1 = \phi(x) = t\phi(b) = t$, thus, $x \in B$.

(c) $\Rightarrow$ (a) Assume that $\phi : X \to \mathbb{R}$ is linear, $\varphi(x) > 0$ for every $x \in C \setminus \{\mathbf{0}\}$, and $B = \{x \in C : \phi(x) = 1\}$. Consider $x \in C \setminus \{\mathbf{0}\}$ and take $t := \phi(x) > 0$ and $b := t^{-1}x$. Then $x = tb$. Since $b \in C$ and $\phi(b) = 1$, we have $b \in B$. Suppose that $x = t'b'$ for some $t' > 0$ and $b' \in B$. Then $t = \phi(x) = t'\phi(b') = t'$, whence $b = b'$. So, every nonzero element x of C has a unique representation tb with $t > 0$ and $b \in B$. This means that B is a base of C. □

Lemma 1.1 *Each proper convex cone with a convex base in a vector space is pointed.*

Proof Let C be a proper convex cone with a convex base B. Take any $x \in C \cap (-C)$ and assume that $x \neq \mathbf{0}$. Then there are $b_1, b_2 \in B$ and $\lambda_1, \lambda_2 > 0$ with $x = \lambda_1 b_1 = -\lambda_2 b_2$. Since B is convex, we have

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} b_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} b_2 = \mathbf{0} \in B,$$

a contradiction to Remark 1.9. □

Example 1.7 The cone $C = \{x : x = \lambda \cdot (1, 2), \lambda \geq 0\} \cup \{x : x = \lambda \cdot (2, 1), \lambda \geq 0\}$ is pointed, proper and has a base $B = \{(1, 2), (2, 1)\}$, but C is not convex.

Remark 1.10 If B is a base of a cone C, then $\text{cone}(B) = C$. If $\mathbf{0} \in \text{cor}(C)$, the core of C, for a nonempty subset C of a vector space X, then $\text{cone}(C) = X$.

The following result can be found in Jameson [82, p. 80] and known as Jameson lemma.

Proposition 1.3 (Jameson Lemma) *Let X be a Hausdorff topological vector space with its zero vector being denoted by $\mathbf{0}$. Then a cone $C \subset X$ with a closed convex bounded base B is closed and pointed.*

Proof We show that C is closed. Let $\{c_\alpha\} \subseteq C$ be a net converging to c. Since B is a base, there exist a net $\{b_\alpha\} \subseteq B$ and a net $\{t_\alpha\}$ of positive numbers such that $c_\alpha = t_\alpha b_\alpha$. We claim that t_α is bounded. Suppose, contrary, that $\lim_\alpha t_\alpha = \infty$. Then the net $\left\{b_\alpha = \frac{c_\alpha}{t_\alpha}\right\}$ converges to $\mathbf{0}$ as X is Hausdorff. Since B is closed, $\mathbf{0} = \lim_\alpha b_\alpha \in B$ which contradicts to the fact that B does not contain the zero element. So, we may assume that $\{t_\alpha\}$ converges to some $t_0 \geq 0$. If $t_0 = 0$, then by the boundedness of B, $\lim_\alpha t_\alpha b_\alpha = \mathbf{0}$. Hence, $c = \mathbf{0}$ and, of course, $c = \mathbf{0} \in C$. If $t_0 > 0$, we may assume that $t_\alpha > \varepsilon$ for all α and some positive ε. Now, $b_\alpha = \frac{c_\alpha}{t_\alpha}$ converges to $\frac{c}{t_0}$ and again by the closedness of B, $\frac{c}{t_0} \in B$. Hence, $c \in C$ and so C is closed. The pointedness of C can be easily seen. □

Definition 1.15 Let Y be a topological vector space with its topological dual Y^*, and C be a convex cone in Y. The *dual cone* C^* of C is defined as

$$C^* = \{\xi \in Y^* : \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\},$$

where $\langle \xi, y \rangle$ denotes the evaluation of ξ at y. The *strict dual cone* C^*_+ of C is defined as

$$C^*_+ = \{\xi \in Y^* : \langle \xi, y \rangle > 0 \text{ for all } y \in C\}.$$

The *quasi-interior* of C^* is defined as

$$C^\# := \{\xi \in Y^* : \langle \xi, y \rangle > 0 \text{ for all } y \in C \setminus \{\mathbf{0}\}\}.$$

If C is empty, then C^* interprets as the whole space Y^*.

For example, in $\mathbb{R}^2$ the dual of a convex cone C consists of all vectors making a non-acute angle with all vectors of the cone C (see Fig. 1.9). For an example of a dual cone in $\mathbb{R}^3$, see Fig. 1.10.

The following proposition can be proved easily by using the definition. Therefore, we omit the proof.

Proposition 1.4 *Let Y be a topological vector space with its topological dual Y^*. Let C, C_1 and C_2 be convex cones in Y.*

(a) *The dual cone C^* is a closed convex cone.*
(b) *The strict dual cone C^* is a convex cone.*

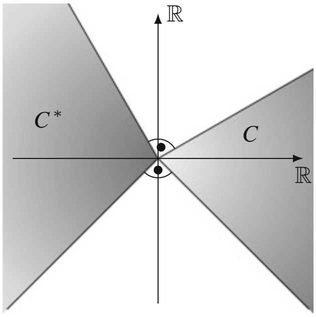


Fig. 1.9 A dual cone in $\mathbb{R}^2$

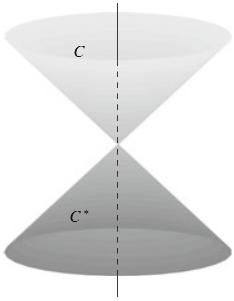


Fig. 1.10 A dual cone in $\mathbb{R}^3$

(c) $C^* = (\mathrm{cl}(C))^*$.
(d) *If* $C_1 \subset C_2$, *then* $C_2^* \subset C_1^*$ *and* $C_{2+}^* \subset C_{1+}^*$.
(e) $(C^*)^* = C^{**} = \mathrm{cl}(C)$.
(f) $(C_1 + C_2)^* = C_1^* \cap C_2^* = (C_1 \cap C_2)^*$.
(g) $(C_1 \cap C_2)^* \supset C_1^* + C_2^* = \mathrm{co}\,(C_1 \cup C_2)^*$.
(h) *If* C_1 *and* C_2 *are closed convex cones with nonempty intersection, then*

$$(C_1 \cap C_2)^* = \mathrm{cl}\left(C_1^* + C_2^*\right) = \mathrm{cl}\left(\mathrm{co}\,(C_1 \cup C_2)^*\right).$$

We now define the recession cone and asymptotic cone and discuss their properties.

Definition 1.16 Let C be a nonempty subset of a vector space Y. A vector $d \in Y$ is said to be a *direction of recession* if for any $x \in C$, the ray $\{x + \lambda d : \lambda \geq 0\}$ (starting from x and going indefinitely along d) lies in C (or never crosses the relative boundary of C).

Definition 1.17 Let C be a nonempty subset of a vector space Y. The set of all directions of recession is called *recession cone* and it is denoted by C_∞ (see Fig. 1.11). That is, for any $x \in C$,

$$C_\infty = \{d \in Y : x + \lambda d \in C \text{ for all } \lambda \geq 0\}.$$

Below we collect some properties of a recession cone.

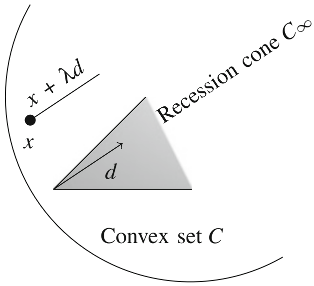


Fig. 1.11 A recession cone (see [21])

Remark 1.11

(a) C_∞ depends only on the behavior of C at infinity. In fact, $x + \lambda d \in C$ implies $x + \alpha d \in C$ for all $\alpha \in [0, \lambda]$. Thus, C_∞ is just the set of all directions from which one can go straight from x to infinity, while staying in C.
(b) If C is closed and convex, then for all $x \in C$, we have

$$C_\infty = \bigcap_{\lambda > 0} \frac{C - x}{\lambda}.$$

(c) C_∞ does not depend on $x \in C$.

Definition 1.18 Let Y be a topological vector space. A recession cone of a nonempty closed convex set $C \subset Y$ is called *asymptotic cone*.

In other words, if C is a nonempty closed convex subset of Y, then the asymptotic cone of C is defined as

$$C_\infty = \left\{ d \in Y : \exists\, \lambda_m \to +\infty,\ \exists\, x_m \in C \text{ with } \lim_{m\to\infty} \frac{x_m}{\lambda_m} = d \right\}.$$

If Y is a reflexive Banach space and C is a weakly closed convex set C in Y, then the asymptotic cone C_∞ of C is defined as

$$C_\infty = \{x \in X : \exists\, \lambda_m \downarrow 0 \text{ and } \exists\, x_m \in C \text{ such that } \lambda_m x_m \rightharpoonup x\},$$

where "$\rightharpoonup$" means convergence in the weak topology.

We set $\emptyset_\infty = \emptyset$.

Example 1.8

(a) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \le x_2\right\}$, then C is unbounded and $C_\infty = C$.
(b) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < x_2\right\}$, then C is unbounded and $C_\infty = \text{cl}(C)$.
(c) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^k \le x_2,\ k > 1\right\}$, then C is unbounded and

$$C_\infty = \{(0, x_2) : x_2 \ge 0\}.$$

(d) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \geq \frac{1}{x_1}\right\}$, then C is unbounded and

$$C_\infty = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\right\}.$$

(e) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1^2\right\}$, then C is unbounded and

$$C_\infty = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0,\ x_2 \geq 0\right\}.$$

(f) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\right\}$, then C is bounded and

$$C_\infty = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = 0\right\} = \left\{(0, 0)\right\}.$$

(g) If $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\right\} \cup \{(0, 0)\}$, then C is unbounded and $C_\infty = C$.

(h) The recession cone of a nonempty affine set M is the subspace L parallel to M.

Theorem 1.4 *Let Y be a topological vector space and C be a nonempty closed convex subset of Y.*

(a) *The recession cone C_∞ is a closed convex cone containing the origin, that is, $C_\infty = \{d : C + d \subset C\}$.*

(b) *Furthermore, let $(Y, \|\cdot\|)$ be a normed vector space. Then $d \in C_\infty$ if and only if there exists a vector $x \in C$ such that $x + \lambda d \in C$ for all $\lambda \geq 0$, that is,*

$$C_\infty = \{d : \text{there exists } x \in C,\ x + \lambda d \in C \text{ for all } \lambda \geq 0\}.$$

(c) *If $(Y, \|\cdot\|)$ is a normed vector space, then C is bounded if and only if $C_\infty = \{\mathbf{0}\}$.*

Proof

(a) Let $d \in C_\infty$, then $x + d \in C$ for any $x \in C$, that is, $C + d \subset C$.

On the other hand, if $C + d \subset C$, then

$$C + 2d = (C + d) + d \subset C + d \subset C,$$

and so forth, implying $x + md \in C$ for any $x \in C$ and for any positive integer m. The line segments joining the points $x, x + d, x + 2d, \ldots$, are then all contained in C by convexity, so that $x + \lambda d \in C$ for every $\lambda \geq 0$. Thus, $d \in C_\infty$. Since positive scalar multiplication does not change directions, C_∞ is truly a cone.

It remains to show that C_∞ is convex. Let $d_1, d_2 \in C_\infty$ and $0 \leq \lambda \leq 1$, then we have

$$\begin{aligned}(1 - \lambda)d_1 + \lambda d_2 + C &= (1 - \lambda)(d_1 + C) + \lambda(d_2 + C)\\ &\subset (1 - \lambda)C + \lambda C = C.\end{aligned}$$

Hence, $(1 - \lambda)d_1 + \lambda d_2 \in C_\infty$.

(b) If $d \in C_\infty$, then $x + \lambda d \in C$ for all $\lambda \geq 0$ for all $x \in C$ by the definition of C_∞.

Conversely, let $d \neq \mathbf{0}$ be such that there exists a vector $x \in C$ such that $x + \lambda d \in C$ for all $\lambda \geq 0$. We fix $\bar{x} \in C$ and $\lambda > 0$, and we show that $\bar{x} + \lambda d \in C$. It is sufficient to show that $\bar{x} + d \in C$, that is, to assume that $\lambda = 1$, since the general case where $\lambda > 0$ can be reduced to the case where $\lambda = 1$ by replacing d with d/λ.

Let $z_m = x + md$ for $m = 1, 2, \ldots$ and note that $z_m \in C$ for all m, since $x \in C$ and $d \in C_\infty$. If $\bar{x} = z_m$ for some m, then $\bar{x} + d = x + (m+1)d$, which belongs to C and we are done. We thus assume that $\bar{x} \neq z_m$ for all m, and we define

$$d_m = \frac{(z_m - \bar{x})}{\|z_m - \bar{x}\|}\|d\|, \quad m = 1, 2, \ldots.$$

so that $\bar{x} + d_m$ lies on the line that starts at $\bar{x}$ and passes through z_m. We have

$$\frac{d_m}{\|d\|} = \frac{\|z_m - x\|}{\|z_m - \bar{x}\|}\frac{z_m - x}{\|z_m - x\|} + \frac{x - \bar{x}}{\|z_m - \bar{x}\|} = \frac{\|z_m - x\|}{\|z_m - \bar{x}\|}\frac{d}{\|d\|} + \frac{x - \bar{x}}{\|z_m - \bar{x}\|}.$$

Since $\{z_m\}$ is an unbounded sequence,

$$\frac{\|z_m - x\|}{\|z_m - \bar{x}\|} \to 1, \quad \frac{x - \bar{x}}{\|z_m - \bar{x}\|} \to 0,$$

so by combining the preceding relations, we have $d_m \to d$. The vector $\bar{x} + d_m$ lies between $\bar{x}$ and z_m in the segment connecting $\bar{x}$ and z_m for all m such that $\|z_m - \bar{x}\| \geq \|d\|$, so by the convexity of C, we have $\bar{x} + d_m \in C$ for all sufficiently large m. Since $\bar{x} + dm \to \bar{x} + d$ and C is closed, it follows that $\bar{x} + d \in C$.

(c) If C is bounded, then it is clear that C_∞ can not contain any nonzero direction.

Conversely, let $\{x_m\} \subseteq C$ be such that $\|x_m\| \to +\infty$ (we assume $x_m \neq \mathbf{0}$). The sequence $\left\{d_m : \dfrac{x_m}{\|x_m\|}\right\}$ is bounded, so we can extract a convergent subsequence, namely, $\{d_k\}$ such that $\lim\limits_{k \to \infty} d_k = d$ with $k \in \mathbb{K} \subseteq \mathbb{N}$ ($\|d\| = 1$). Now, given $x \in C$ and $\lambda > 0$, take k so large that $\|x_k\| \geq \lambda$. Then we see that

$$x + \lambda d = \lim_{k \to \infty}\left[\left(1 - \frac{\lambda}{\|x_k\|}\right)x + \frac{\lambda}{\|x_k\|}x_k\right]$$

is in the closed convex set C and hence $d \in C_\infty$. □

Below we present some properties of recession cones and asymptotic cones.

Remark 1.12 Let X be a topological vector space.

(a) For a nonempty convex set $C \subseteq X$, we have $(\mathrm{cl}(C))_\infty = (\mathrm{relint}(C))_\infty$, where $\mathrm{relint}(C)$ denotes the relative interior of C; Furthermore, for any $x \in \mathrm{relint}(C)$, one has $d \in (\mathrm{cl}(C))_\infty$ if and only if $x + \lambda d \in \mathrm{relint}(C)$ for all $\lambda > 0$.

(b) Moreover, for a nonempty convex set $C \subseteq X$, it holds $(C + x)_\infty = C_\infty$ for all $x \in X$.
(c) For two nonempty closed convex sets $\tilde{C}, \hat{C} \subset Y$, $\tilde{C} \subseteq \hat{C}$ implies $\tilde{C}_\infty \subseteq \hat{C}_\infty$.
(d) Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be any family of nonempty sets in X, then

$$\Big(\bigcap_{\alpha \in \Lambda} C_\alpha\Big)_\infty \subset \bigcap_{\alpha \in \Lambda} (C_\alpha)_\infty.$$

If, in addition, $\bigcap_{\alpha \in \Lambda} C_\alpha \neq \emptyset$ and each set C_α is closed and convex, then we obtain an equality in the previous inclusion.

Moreover, If $C_1 \subseteq X_1, C_2 \subseteq X_2, \ldots, C_m \subseteq X_m$ are closed convex sets, where X_i, $i = 1, 2, \ldots, m$ are topological vector spaces, then

$$(C_1 \times C_2 \times \cdots \times C_m)_\infty = (C_1)_\infty \times (C_2)_\infty \times \cdots \times (C_m)_\infty.$$

We present the definition of a contingent cone and its properties.

Definition 1.19 Let C be a nonempty subset of a normed space X.

(a) Let $\bar{x} \in \mathrm{cl}(C)$ be given. An element $u \in X$ is said to be a *tangent* to C at $\bar{x}$ if there exist a sequence $\{x_m\}$ of elements $x_m \in C$ and a sequence $\{\lambda_m\}$ of positive real numbers λ_m such that $\lim\limits_{m\to\infty} x_m \to \bar{x}$ and $\lim\limits_{m\to\infty} \lambda_m(x_m - \bar{x}) = u$.
(b) The set $T(C, \bar{x})$ of all tangents to C at $\bar{x}$ is called the *contingent cone* (or the *Bouligand tangent cone*) to C at $\bar{x}$.

In other words, a contingent cone $T(C, \bar{x})$ to C at $\bar{x}$ is defined as

$$T(C,\bar{x}) = \big\{u \in X : \exists\, \{x_m\} \subset C \text{ and } \{\lambda_m\} \subset (0, \infty) \\ \text{such that } x_m \to \bar{x} \text{ and } \lambda_m(x_m - \bar{x}) \to u\big\}.$$

Figure 1.12 visualizes two contingent cones.

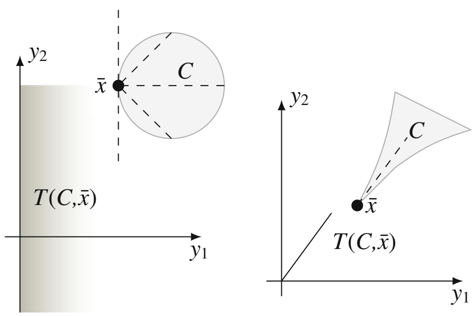


Fig. 1.12 Tangent to C at $\bar{x}$ and contingent cone $T(C, \bar{x})$

It is easy to see that the above definition of contingent cone can be written as

$$T(C,\bar{x}) = \Big\{ u \in X : \exists \{x_m\} \subset C \text{ and } \{\lambda_m\} \text{ such that } x_m \to \bar{x},\ \lambda_m \to 0^+ \text{ and } \frac{x_m - \bar{x}}{\lambda_m} \to u \Big\}.$$

If $\bar{x} \in \text{int}(C)$, then $T(C,\bar{x})$ is clearly the whole space. That is why we considered $\bar{x} \in \text{cl}(C)$.

If $u_m = \dfrac{x_m - \bar{x}}{\lambda_m}$ $(\to u)$, that is, $x_m = \bar{x} + \lambda_m u_m$ $(\in C)$, then we have $d \in T(C,\bar{x})$ if and only if there exist sequences $\{u_m\} \to u$ and $\{\lambda_m\} \to 0^+$ such that $\bar{x} + \lambda_m u_m \in C$ for all $m \in \mathbb{N}$.

It is equivalent to saying that $u \in T(C,\bar{x})$ if and only if there exist sequences $\{u_m\} \to u$ and $\{\lambda_m\} \subset \mathbb{R}_+$ such that

$$\bar{x} + \lambda_m u_m \in C, \quad \text{for all } m \in \mathbb{N} \quad \text{and} \quad \{\lambda_m x_m\} \to 0.$$

Remark 1.13

(a) A contingent cone to a set C at a point $\bar{x} \in \text{cl}(C)$ describes a local approximation of the set $C - \{\bar{x}\}$. This concept is very helpful for the investigation of optimality conditions.
(b) From the definition of $T(C,\bar{x})$, we see that $\bar{x}$ belongs to the closure of the set C. It is evident that the contingent cone is really a cone.

Lemma 1.2 *Let C and D be nonempty subsets of a normed space X.*

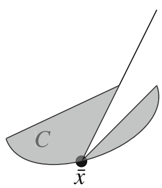
(a) *If $\bar{x} \in \text{cl}(C) \subset \text{cl}(D)$, then $T(C,\bar{x}) \subset T(D,\bar{x})$.*
(b) *If $\bar{x} \in \text{cl}(C \cap D)$, then $T(C \cap D,\bar{x}) \subset T(C,\bar{x}) \cap T(D,\bar{x})$.*

Definition 1.20 Let C be a subset of a vector space X is called *starshaped* at $\bar{x} \in C$ if for all $x \in C$ and for every $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)\bar{x} \in C.$$

An example for a starshaped set $C \subset \mathbb{R}^2$ is given in Fig. 1.13.

Fig. 1.13 A starshaped set C



Theorem 1.5 *Let C be a nonempty subset of a normed space X. If C is starshaped at some $\bar{x} \in C$, then* $\text{cone}\,(C - \{\bar{x}\}) \subset T(C, \bar{x})$.

Proof Take any $x \in C$. Then we have

$$x_m = \bar{x} + \frac{1}{m}(x - \bar{x}) = \frac{1}{m}x + \left(1 - \frac{1}{m}\right)\bar{x} \in C, \quad \text{for all } m \in \mathbb{N}.$$

Hence, we get $\bar{x} = \lim_{m\to\infty} x_m$ and $x - \bar{x} = \lim_{m\to\infty} m(x_m - \bar{x})$. But this implies that $x - \bar{x} \in T(C, \bar{x})$ and therefore, we obtain $C - \{\bar{x}\} \subset T(C, \bar{x})$.

Since $T(C, \bar{x})$ is a cone, it follows further that $\text{cone}\,(C - \{\bar{x}\}) \subset T(C, \bar{x})$. □

Theorem 1.6 *Let C be a nonempty subset of a normed space X. For every $\bar{x} \in \text{cl}(C)$, we have $T(C, \bar{x}) \subset \text{cl}\,(\text{cone}(C - \{\bar{x}\}))$.*

Proof Take an arbitrary tangent u to C at $\bar{x}$. Then there exist a sequence $\{x_m\}$ of elements in X and a sequence $\{\lambda_m\}$ of positive real numbers such that

$$\bar{x} = \lim_{m\to\infty} x_m \quad \text{and} \quad u = \lim_{m\to\infty} \lambda_m(x_m - \bar{x}).$$

The last equality implies $u \in \text{cl}\,(\text{cone}(C - \{\bar{x}\}))$. □

Theorem 1.7 *Let C be a nonempty subset of a normed space X. The contingent cone $T(C, \bar{x})$ is closed for every $\bar{x} \in \text{cl}(C)$.*

Proof Let $\{u_m\}$ be an arbitrary sequence in $T(C, \bar{x})$ with $\lim_{m\to\infty} u_m = u \in X$. For every tangent u_m, there exist a sequence $\{x_{m_i}\}$ of elements in C and a sequence $\{\lambda_{m_i}\}$ of positive real numbers such that

$$\bar{x} = \lim_{i\to\infty} x_{m_i} \quad \text{and} \quad u_m = \lim_{i\to\infty} \lambda_{m_i}(x_{m_i} - \bar{x}).$$

Consequently, for every $m \in \mathbb{N}$, there exists an $i(m) \in \mathbb{N}$ such that

$$\|x_{m_i} - \bar{x}\| \le \frac{1}{m}, \quad \text{for all } i \ge i(m),$$

and

$$\|\lambda_{m_i}(x_{m_i} - \bar{x}) - u_m\| \le \frac{1}{m}, \quad \text{for all } i \ge i(m).$$

If we define $y_m = x_{m_{i(m)}} \in C$ and $\mu_m = \lambda_{m_{i(m)}} > 0$ for all $m \in \mathbb{N}$, then we get $\bar{x} = \lim_{m\to\infty} y_m$ and

$$\begin{aligned}\|\mu_m(y_m - \bar{x}) - u\| &\le \|\mu_m(y_m - \bar{x}) - u_m\| + \|u_m - u\| \\ &\le \frac{1}{m} + \|u_m - u\|, \quad \text{for all } m \in \mathbb{N}.\end{aligned}$$

This implies that $u = \lim_{m\to\infty} \mu_m(y_m - \bar{x})$. Hence, $u \in T(C, \bar{x})$ and so $T(C, \bar{x})$ is closed. □

Corollary 1.2 *Let C be a nonempty subset of a normed space X. If C is starshaped at some $\bar{x} \in C$, then $T(C, \bar{x}) = \mathrm{cl}\,(\mathrm{cone}(C - \{\bar{x}\}))$.*

Theorem 1.8 *Let C be a nonempty convex subset of a normed space X. The contingent cone $T(C, \bar{x})$ is convex for every $\bar{x} \in \mathrm{cl}(C)$.*

Proof Since C is convex, $C - \{\bar{x}\}$ and $\mathrm{cone}(C - \{\bar{x}\})$ are convex as well. Since the closure of a convex set is convex, we have $\mathrm{cl}\,(\mathrm{cone}(C - \{\bar{x}\}))$ is also convex. Finally, from above corollary, we have $T(C, \bar{x}) = \mathrm{cl}\,(\mathrm{cone}(C - \{\bar{x}\}))$. □

1.2 Convex Functions and Their Properties

Definition 1.21 Let X be a vector space. A function $f : X \to \mathbb{R}$ is said to be

(a) *positively homogeneous* if for all $x \in X$ and all $r > 0$, we have $f(rx) = rf(x)$;
(b) *subodd* if for all $x \in X \setminus \{\mathbf{0}\}$, we have $f(x) \geq -f(-x)$.

Example 1.9

(a) Every linear function is positively homogeneous.
(b) The function $f(x) = |x|$ is positively homogeneous.
(c) Every norm is positively homogeneous.
(d) The function $f(x) = \begin{cases} x, & \text{if } x > 0, \\ -\frac{1}{2}x, & \text{if } x \leq 0 \end{cases}$ is positively homogeneous.
(e) $f(x) = x^2$ is subodd.

Definition 1.22 Let K be a subspace of a vector space X. A function $f : K \to \mathbb{R}$ is said to be *linear* if for all $x, y \in K$ and all $\lambda, \mu \in \mathbb{R}$,

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y). \tag{1.1}$$

Definition 1.23 Let K be a nonempty affine subset of a vector space X. A function $f : K \to \mathbb{R}$ is said to be *affine* if (1.1) holds for all $x, y \in K$ and all $\lambda, \mu \in \mathbb{R}$ such that $\lambda + \mu = 1$.

In other words, f is affine if and only if

$$f\,(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y), \tag{1.2}$$

for all $x, y \in K$ and all $\lambda \in \mathbb{R}$.

Definition 1.24 Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is said to be *convex* if for all $x, y \in K$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$, we have

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y). \tag{1.3}$$

In other words, f is convex if and only if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \tag{1.4}$$

for all $x, y \in K$ and all $\lambda \in [0, 1]$.

The functional f is said to be *strictly convex* if inequality (1.4) is strict for all $x \neq y$.

A function f is said to be *concave* if $-f$ is convex.

Example 1.10

(a) Let $K = X = \mathbb{R}$ and $f(x) = x^2$ for all $x \in K$. Then f is a convex function.
(b) Let $K = [0, \pi]$ and $f(x) = \sin x$ for all $x \in K$. Then f is a convex function.
(c) Let $K = X = \mathbb{R}$ and $f(x) = |x|$ for all $x \in K$. Then f is a convex function. In fact, the functions in (a) and (b) are strictly convex but the function in (c) is not.
(d) The functions $f(x) = \ln|x|$ for $x > 0$, and $g(x) = +\sqrt{1 - x^2}$ for $x \in [-1, 1]$ are concave.

Remark 1.14 An affine function is both convex and concave.

Definition 1.25 Let K be a nonempty subset of a vector space X and $f : K \to \mathbb{R}$ be a function. The set

$$\text{epi}(f) = \{(x, \alpha) \in K \times \mathbb{R} : f(x) \leq \alpha\}$$

is called *epigraph* of f.

Theorem 1.9 *Let K be a nonempty convex subset of a vector space X and $f : K \to \mathbb{R}$ be a function. Then f is convex if and only if its epigraph is a convex set.*

Proof Let f be a convex function. Then for any (x, α) and $(y, \beta) \in \text{epi}(f)$, we have $f(x) \leq \alpha$ and $f(y) \leq \beta$. Also, for all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \leq \lambda\alpha + (1 - \lambda)\beta.$$

Thus,

$$((\lambda x + (1 - \lambda)y), \lambda\alpha + (1 - \lambda)\beta) = \lambda(x, \alpha) + (1 - \lambda)(y, \beta) \in \text{epi}(f).$$

Hence, epi(f) is convex.

Conversely, let epi(f) be a convex set, and $(x, f(x)) \in \text{epi}(f)$ and $(y, f(y)) \in \text{epi}(f)$. Then for all $x, y \in K$ and all $\lambda \in [0, 1]$, we have

$$\lambda\,(x, f(x)) + (1 - \lambda)\,(y, f(y)) \in \text{epi}(f).$$

This implies that

$$(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \mathrm{epi}(f)$$

and thus,

$$f(\lambda x + (1-\lambda)y) \leq f(x) + (1-\lambda)f(y).$$

Hence, f is convex. □

Theorem 1.10 *Let K be a nonempty convex subset of a vector space X and $f : K \to \mathbb{R}$ be a convex function. Then the lower level set $L_\alpha = \{x \in K : f(x) \leq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$.*

Proof Let $x, y \in L_\alpha$. Then $(x, \alpha) \in \mathrm{epi}(f)$ and $(y, \alpha) \in \mathrm{epi}(f)$. Therefore, for all $\lambda \in [0, 1]$,

$$\lambda(x, \alpha) + (1-\lambda)(y, \alpha) \in \mathrm{epi}(f),$$

equivalently,

$$(\lambda x + (1-\lambda)y, \lambda\alpha + (1-\lambda)\alpha) \in \mathrm{epi}(f)$$

and thus

$$f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\alpha = \alpha.$$

Hence, $\lambda x + (1-\lambda)y \in L_\alpha$ and so L_α is convex. □

Remark 1.15 The converse of above theorem may not hold. For example, the function $f(x) = x^3$ defined on $\mathbb{R}$ is not convex but its lower level set $L_\alpha = \{x \in \mathbb{R} : x \leq \alpha^{1/3}\}$ is convex for every $\alpha \in \mathbb{R}$.

Theorem 1.11 *Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is convex if and only if for all $x_1, x_2, \ldots, x_m \in K$ and $\lambda_i \in [0, 1]$, $i = 1, 2, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$,*

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i). \tag{1.5}$$

The inequality (1.5) is called *Jensen's inequality*.

Proof Suppose that the Jensen's inequality (1.5) holds. Then trivially, f is convex.

Conversely, we assume that the function f is convex. Then we show that the Jensen's inequality (1.5) holds. We prove it by induction on m. For $m = 1$ and $m = 2$, the inequality (1.5) trivially holds. Assume that the inequality (1.5) holds for m. We shall prove the result for $m + 1$. If $\lambda_{m+1} = 1$, the result holds because

$\lambda_i = 0$, for $i = 1, 2, \ldots, m$ and the result is true for $m = 1$. If $\lambda_{m+1} \neq 1$, we have

$$\begin{aligned} f\left(\sum_{i=1}^{m+1} \lambda_i x_i\right) &= f\left(\sum_{i=1}^{m} \lambda_i x_i + \lambda_{m+1} x_{m+1}\right) \\ &= f\left(\sum_{i=1}^{m} (1 - \lambda_{m+1}) \frac{\lambda_i x_i}{1 - \lambda_{m+1}} + \lambda_{m+1} x_{m+1}\right) \\ &= f\left((1 - \lambda_{m+1}) \sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} x_i + \lambda_{m+1} x_{m+1}\right) \\ &= f\left((1 - \lambda_{m+1}) \sum_{i=1}^{m} \mu_i x_i + \lambda_{m+1} x_{m+1}\right) \\ &\leq (1 - \lambda_{m+1}) f\left(\sum_{i=1}^{m} \mu_i x_i\right) + \lambda_{m+1} f(x_{m+1}) \\ &\leq (1 - \lambda_{m+1}) \sum_{i=1}^{m} \mu_i f(x_i) + \lambda_{m+1} f(x_{m+1}), \end{aligned}$$

where $\mu_i = \dfrac{\lambda_i}{(1 - \lambda_{m+1})}$, $i = 1, 2, \ldots, m$ with $\mu_i \geq 0$ for $i = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{m} \mu_i = \frac{\sum_{i=1}^{m} \lambda_i}{1 - \lambda_{m+1}} = \frac{1 - \lambda_{m+1}}{1 - \lambda_{m+1}} = 1.$$

This completes the proof. □

The following theorems provide some properties of convex functions. The proof of these theorems is quite trivial, and hence, omitted.

Theorem 1.12 *Let K be a nonempty convex subset of a vector space X.*

(a) *If $f_1, f_2 : K \to \mathbb{R}$ are two convex functions, then $f_1 + f_2$ is a convex function on K.*
(b) *If $f : K \to \mathbb{R}$ is a convex function and $\alpha \geq 0$, then αf is a convex function on K.*
(c) *For each $i = 1, 2, \ldots, m$, if $f_i : K \to \mathbb{R}$ is a convex function and $\alpha_i \geq 0$, then $\sum_{i=1}^{m} \alpha_i f_i$ is a convex function. Further, if at least one of the functions f_i is strictly convex with the corresponding $\alpha_i > 0$, then $\sum_{i=1}^{m} \alpha_i f_i$ is strictly convex on K.*

Theorem 1.13 *Let K be a nonempty convex subset of a vector space X. For each $i = 1, 2, \ldots, m$, if $f_i : K \to \mathbb{R}$ is a convex function, then* $\max\{f_1, f_2, \ldots, f_m\}$ *is also a convex function on K.*

Next we provide characterizations of a differentiable convex function.

Theorem 1.14 ([10, 110]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then*

(a) *f is convex if and only if for all $x, y \in K$,*

$$\langle \nabla f(x), y - x \rangle \leq f(y) - f(x). \tag{1.6}$$

(b) *f is strictly convex if and only if the inequality (1.6) is strict for $x \neq y$.*

Proof

(a) If f is a convex function, then for all $\lambda \in [0, 1]$

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

For $\lambda > 0$, we have

$$\frac{f((1-\lambda)x + \lambda y) - f(x)}{\lambda} \leq f(y) - f(x),$$

which on taking limit $\lambda \to 0^+$ leads to (1.6) as f is a differentiable function.

Conversely, let $\lambda \in [0, 1]$ and $u, v \in K$. On taking $x = (1-\lambda)u + \lambda v$ and $y = u$ in (1.6), we have

$$\lambda \langle \nabla f((1-\lambda)u + \lambda v), u - v \rangle \leq f(u) - f((1-\lambda)u + \lambda v). \tag{1.7}$$

Similarly, on taking $x = (1-\lambda)u + \lambda v$ and $y = v$ in (1.6), we have

$$-(1-\lambda)\langle \nabla f((1-\lambda)u + \lambda v), u - v \rangle \leq f(v) - f((1-\lambda)u + \lambda v). \tag{1.8}$$

Multiplying inequality (1.7) by $(1-\lambda)$ and inequality (1.8) by λ, and then adding the resultants, we obtain

$$f((1-\lambda)u + \lambda v) \leq (1-\lambda)f(u) + \lambda f(v).$$

(b) Suppose that f is strictly convex and $x, y \in K$ be such that $x \neq y$. Since f is convex, the inequality (1.6) holds. We need to show the inequality is strict. Suppose on the contrary that

$$\langle \nabla f(x), y - x \rangle = f(y) - f(x).$$

Then for $\lambda \in]0, 1[$, we have

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y) = f(x) + \lambda \langle \nabla f(x), y - x \rangle.$$

Let $z = (1 - \lambda)x + \lambda y$, then $z \in K$ and the above inequality can be written as

$$f(z) < f(x) + \langle \nabla f(x), z - x \rangle,$$

which contradicts the inequality (1.6). Proof of the converse part follows as given for the convex case. ◻

Theorem 1.15 ([63, 110]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if for all $x, y \in K$,*

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0.$$

Proof Let f be a differentiable convex function. Then by Theorem 1.14 (a), we have

$$\langle \nabla f(x), y - x \rangle \leq f(y) - f(x), \quad \text{for all } x, y \in K.$$

By interchanging the roles of x and y, we have

$$\langle \nabla f(y), x - y \rangle \leq f(x) - f(y), \quad \text{for all } x, y \in K.$$

On adding the above inequalities we get the conclusion.

Conversely, by mean value theorem, for all $x, y \in K$, there exists $z = (1-\lambda)x + \lambda y$ for some $\lambda \in]0, 1[$ such that

$$\begin{aligned} f(y) - f(x) &= \langle \nabla f(z), y - x \rangle = (1/\lambda)\langle \nabla f(z), z - x \rangle \\ &\geq (1/\lambda)\langle \nabla f(x), z - x \rangle = \langle \nabla f(x), y - x \rangle, \end{aligned}$$

where the above inequality is obtained on using the given hypothesis. Hence, by Theorem 1.14 (a), f is a convex function. ◻

The following example illustrates the above theorem.

Example 1.11 The function $f(x) = x_1^2 + x_2^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$, is a convex function on $\mathbb{R}^2$ and $\nabla f(x) = 2(x_1, x_2)$. For $x, y \in \mathbb{R}^2$,

$$\begin{aligned} \langle \nabla f(y) - \nabla f(x), y - x \rangle &= \langle 2(y_1 - x_1, y_2 - x_2), (y_1 - x_1, y_2 - x_2) \rangle \\ &= 2(y_1 - x_1)^2 + 2(y_2 - x_2)^2 \geq 0. \end{aligned}$$

Definition 1.26 Let K be a nonempty subset of a normed space X and $x \in K$ be a given point. A function $f : K \to \mathbb{R}$ is said to be *locally Lipschitz around* x if for some $k > 0$

$$|f(y) - f(z)| \leq k\|y - z\|, \quad \text{for all } y, z \in N(x) \cap K, \tag{1.9}$$

where $N(x)$ is a neighborhood of x. The constant k is called *Lipschitz constant* and it varies as the point x varies.

The function f is said to be *Lipschitz continuous* on K if the inequality (1.9) holds for all $y, z \in K$.

A continuously differentiable function always satisfies the Lipschitz condition (1.9). However, a locally Lipschitz function at a given point need not be differentiable at that point. For example, the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x|$, satisfies the Lipschitz condition on $\mathbb{R}$. But f is not differentiable at 0.

The class of Lipschitz continuous functions is quite large. It is invariant under usual operations of sum, product and quotient.

It is clear that every Lipschitz continuous function is continuous. Also, every convex function is not only continuous but also locally Lipschitz in the interior of its domain.

Theorem 1.16 ((See [6, Theorem 1.14])) *Let K be a nonempty convex subset of $\mathbb{R}^n$, $f : K \to \mathbb{R}$ be a convex function and x be an interior point of K. Then f is locally Lipschitz at x.*

As we have seen, the convex functions cannot be characterized by lower level sets. However, if the function is convex then lower level sets are convex but the converse is not true. Now we define a class of such functions, called quasiconvex functions, which are characterized by convexity of their level sets.

Definition 1.27 Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is said to be

(a) *quasiconvex* if for all $x, y \in K$ and all $\lambda \in]0, 1[$,

$$f(x + \lambda(y - x)) \leq \max\{f(x), f(y)\};$$

(b) *strictly quasiconvex* if for all $x, y \in K$, $x \neq y$ and all $\lambda \in]0, 1[$,

$$f(x + \lambda(y - x)) < \max\{f(x), f(y)\};$$

(c) *semistrictly quasiconvex* if for all $x, y \in K$ with $f(x) \neq f(y)$,

$$f(x + \lambda(y - x)) < f(x), \quad \text{for all } \lambda \in]0, 1[.$$

A function $f : K \to \mathbb{R}$ is said to be (*strictly, semistrictly*) *quasiconcave* if $-f$ is (strictly, semistrictly) quasiconvex.

Note that in Definition 1.27 (b), the premise excludes the case $f(x) = f(y)$. Therefore, the formulation of the definition of semistrictly quasiconvexity differs from (a). Also, note that a strictly quasiconvex function was referred to as a strongly quasiconvex function in [17] and a semistrictly quasiconvex function was referred to as a strictly quasiconvex function in [17, 19, 110].

Example 1.12

(a) Every convex function is quasiconvex.
(b) The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \sqrt{|x|}$, is quasiconvex, but not convex.
(c) Every strictly convex function is semistrictly quasiconvex.
(d) The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x$, is semistrictly quasiconvex, but f is not strictly convex.

Obviously, every (strictly) convex function is (strictly) quasiconvex but the converse is not necessarily true. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a quasiconvex function but not a convex function. Also, a convex function is semistrictly quasiconvex but the converse may not be true. Again, we see that $f(x) = x^3$ is semistrictly quasiconvex but not convex. We note that the strict quasiconvexity is not a generalization of convexity as a constant function is convex but not strictly quasiconvex. Obviously, a strictly quasiconvex function is quasiconvex but the converse is not true, For example, the greatest integer function $f(x) = [x]$ is quasiconvex but not strictly quasiconvex on $\mathbb{R}$.

We now give the characterization of a quasiconvex function in terms of convexity of its lower level sets.

Theorem 1.17 *Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is quasiconvex if and only if the lower level sets $L(f, \alpha)$ are convex for all $\alpha \in \mathbb{R}$.*

Proof Let f be a quasiconvex function and for $\alpha \in \mathbb{R}$, let $x, y \in L(f, \alpha)$. Then $f(x) \leq \alpha$ and $f(y) \leq \alpha$. Since f is a quasiconvex function, for all $\lambda \in [0, 1]$, we have

$$f((1 - \lambda)x + \lambda y) \leq \max\{f(x), f(y)\} \leq \alpha,$$

that is, $(1 - \lambda)x + \lambda y \in L(f, \alpha)$ for all $\lambda \in [0, 1]$. Hence, $L(f, \alpha)$ is convex.

Conversely, let $x, y \in K$ and $\bar{\alpha} = \max\{f(x), f(y)\}$. Then $x, y \in L(f, \bar{\alpha})$, and by convexity of $L(f, \bar{\alpha})$, we have $(1 - \lambda)x + \lambda y \in L(f, \bar{\alpha})$ for all $\lambda \in [0, 1]$. Thus for all $\lambda \in [0, 1]$,

$$f((1 - \lambda)x + \lambda y) \leq \bar{\alpha} = \max\{f(x), f(y)\}.$$

This completes the proof. □

Next result gives the characterization of a quasiconvex function in terms of its gradient.

Theorem 1.18 ([6, 10, 110]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then f is quasiconvex if and only if for all $x, y \in K$,*

$$f(y) \leq f(x) \quad \Rightarrow \quad \langle \nabla f(x), y - x \rangle \leq 0. \tag{1.10}$$

It can be easily seen that if $f_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, m$, is a quasiconvex function on a nonempty convex subset K of a vector space X, then $\max\{f_1, f_2, \ldots, f_m\}$ is also a quasiconvex function on K.

Definition 1.28 Let X be a topological space. A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *lower semicontinuous* (respectively, *upper semicontinuous*) at a point $x \in X$ if for every $\varepsilon > 0$, there exists a neighborhood U of x such that $f(y) \leq f(x) + \varepsilon$ (respectively, $f(y) \geq f(x) - \varepsilon$) for all $y \in U$ when $f(x) > -\infty$, and $f(y) \to -\infty$ as $y \to x$ when $f(x) = -\infty$ (respectively, $f(x) < +\infty$, and $f(y) \to +\infty$ as $y \to x$ when $f(x) = +\infty$).

A function f is lower semicontinuous (respectively, upper semicontinuous) on X if it is lower semicontinuous (respectively, upper semicontinuous) at every point of X.

If X is a metric space then it can be expressed as

$$\limsup_{y \to x} f(y) \leq f(x) \quad \text{(respectively, } \liminf_{y \to x} f(y) \geq f(x)\text{)}.$$

For non-metric spaces, an equivalent definition using nets may be stated.

It can be easily seen that a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous (respectively, upper semicontinuous) on X if and only if the set $\{x \in X : f(x) \leq r\}$ (respectively, $\{x \in X : f(x) \geq r\}$) is closed for all $r \in \mathbb{R}$.

Theorem 1.19 *Let X be a topological space and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Then f is lower semicontinuous if and only if* $\text{epi}(f) := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ *is closed.*

The following theorem gives a sufficient condition for a semistrictly quasiconvex function to be quasiconvex.

Theorem 1.20 *Every lower semicontinuous semistrictly quasiconvex function on a convex set is quasiconvex.*

Proof Let f be a semistrictly quasiconvex function defined on a convex subset K of a vector space X. Then for all $x, y \in K$, $f(x) \neq f(y)$ and $\lambda \in]0, 1[$, we have

$$f((1 - \lambda)x + \lambda y) < \max\{f(x), f(y)\}.$$

It remains to show that if $f(x) = f(y)$ and $\lambda \in]0, 1[$, then

$$f((1 - \lambda)x + \lambda y) \leq \max\{f(x), f(y)\}.$$

Assume contrary that $f(z) > f(x)$ for some $z \in]x, y[$. Then $z \in \Omega := \{z \in]x, y[: f(z) > f(x)\}$. Since f is a lower semicontinuous function, the set Ω is open. Therefore, there exists $z_0 \in]x, z[$ such that $z_0 \in \Omega$. Since $z, z_0 \in \Omega$, by semistrict quasiconvexity of f, we have

$$f(x) < f(z) \quad \Rightarrow \quad f(z_0) < f(z),$$

and

$$f(y) < f(z_0) \quad \Rightarrow \quad f(z) < f(z_0),$$

which is a contradiction. □

Definition 1.29 Let K be a nonempty open subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is said to be

(a) *pseudoconvex* if for all $x, y \in K$,

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \Rightarrow \quad f(y) \geq f(x);$$

(b) *strictly pseudoconvex* if for all $x, y \in K$, $x \neq y$,

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \Rightarrow \quad f(y) > f(x).$$

A function f is (strictly) pseudoconcave if $-f$ is (strictly) pseudoconvex.

Clearly, every differentiable convex function is pseudoconvex, but the converse is not true. For example, the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + x^3$, is pseudoconvex, but not convex.

Theorem 1.21 ([6, 19]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty, open and convex set and $f : K \to \mathbb{R}$ be a differentiable and pseudoconvex function. Then f is both semistrictly quasiconvex and quasiconvex.*

The converse of above theorem does not hold. For example, the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3$, is quasiconvex, but not pseudoconvex because for $x = 0$ and $y = -1$, $\langle \nabla f(x), y - x \rangle = 0$ and $f(y) < f(x)$.

Definition 1.30 Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is said to be *pseudolinear* if it is both pseudoconvex and pseudoconcave.

Some of the examples of pseudolinear function defined on $\mathbb{R}$ are e^x, $x + x^3$ and $\tan^{-1} x$.

We present certain characterizations of a pseudolinear function given by Chew and Choo [44].

Theorem 1.22 *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then the following statements are equivalent:*

(a) *f is a pseudolinear function;*
(b) *For any $x, y \in K$, $\langle \nabla f(x), y - x \rangle = 0$ if and only if $f(x) = f(y)$;*
(c) *There exists a real-valued function p defined on $K \times K$ such that for any $x, y \in K$,*

$$p(x, y) > 0 \quad \textit{and} \quad f(y) = f(x) + p(x, y)\langle \nabla f(x), y - x \rangle.$$

The function p obtained in the above theorem is called the *proportional function* of f.

Theorem 1.23 ([6, Thoerem 1.39]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a continuously differentiable function. Then f is pseudolinear if and only if for any $x, y \in K$,*

$$\langle \nabla f(x), y-x \rangle = 0 \quad \Rightarrow \quad f(x) = f((1-\lambda)x + \lambda y), \quad \textit{for all } \lambda \in [0,1]. \tag{1.11}$$

Now we give a brief introduction of the concept of monotonicity and give some characterizations of convex and generalized convex functions in terms of monotonicity of their gradient function.

Definition 1.31 Let K be a nonempty subset of $\mathbb{R}^n$. A map $F : K \to \mathbb{R}^n$ is said to be

(a) *monotone* if for all $x, y \in K$, $x \neq y$, we have

$$\langle F(y) - F(x), y - x \rangle \geq 0;$$

(b) *strictly monotone* if for all $x, y \in K$, $x \neq y$, we have

$$\langle F(y) - F(x), y - x \rangle > 0;$$

(c) *strongly monotone* with modulus σ if there exists a real number $\sigma > 0$ such that for all $x, y \in K$, $x \neq y$, we have

$$\langle F(y) - F(x), y - x \rangle \geq \sigma \|y - x\|^2.$$

It is clear that a strictly monotone map is monotone but the converse is not true. For example, the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x_1, x_2) = (2x_1, 0)$, is monotone but not strictly monotone as the definition fails at $x = (0, 1)$, $y = (0, 2)$.

Also, every strongly monotone map is strictly monotone but the converse is not true. For instance, the map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 1 + x^2, & \text{if } x \geq 0, \\ 1 - x^2, & \text{if } x < 0, \end{cases}$$

is strictly monotone but it is not strongly monotone. We observe that if we restrict the domain of the function F defined above to $[1, \infty)$, then it is strongly monotone with modulus $\sigma = 2$.

In view of Theorem 1.15, we have the following result.

Theorem 1.24 *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is*

(a) *convex if and only if its gradient ∇f is monotone;*
(b) *strictly convex if and only if its gradient ∇f is strictly monotone.*

Analogous to Theorem 1.24, we have the following theorem.

Theorem 1.25 ([6, Theorem 4.2]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is strongly convex with modulus $\rho > 0$ if and only if its gradient ∇f is strongly monotone with modulus $\sigma = 2\rho$.*

Next we define generalized monotone maps and relate generalized convexity with generalized monotonicity of its gradient function. Karamardian [88] introduced the concept of pseudomonotone maps whereas the notions of strict pseudomonotonicity and quasimonotonicity were introduced by Hassouni [73] and independently by Karamardian and Schaible [89].

Definition 1.32 Let K be a nonempty subset of $\mathbb{R}^n$. A map $F : K \to \mathbb{R}^n$ is said to be

(a) *quasimonotone* if for all $x, y \in K$, $x \neq y$, we have

$$\langle F(x), y - x\rangle > 0 \quad \Rightarrow \quad \langle F(y), y - x\rangle \geq 0;$$

(b) *pseudomonotone* if for all $x, y \in K$, $x \neq y$, we have

$$\langle F(x), y - x\rangle \geq 0 \quad \Rightarrow \quad \langle F(y), y - x\rangle \geq 0;$$

(c) *strictly pseudomonotone* if for all $x, y \in K$, $x \neq y$, we have

$$\langle F(x), y - x\rangle \geq 0 \quad \Rightarrow \quad \langle F(y), y - x\rangle > 0.$$

It is clear that a (strictly) monotone map is (strictly) pseudomonotone but the converse is not true. For example, the map $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = xe^{-x^2}$ is pseudomonotone but the definition of monotonicity fails at $x = 1$, $y = 2$.

A strictly pseudomonotone map is pseudomonotone but the converse is not true. For instance, the map $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = \max\{x, 0\}$ is pseudomonotone but it is not strictly pseudomonotone.

Also, every pseudomonotone map is quasimonotone but the converse is not true as the map $F(x) = x^2$ is quasimonotone on $\mathbb{R}$ but it is not pseudomonotone on $\mathbb{R}$.

The following result gives a characterization of quasiconvex functions.

Theorem 1.26 ([6, Theorem 4.3]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is quasiconvex if and only if its gradient ∇f is quasimonotone.*

As expected we have a similar characterization for (strict) pseudoconvexity of a function in terms of the (strict) pseudomonotonicity of the gradient map.

Theorem 1.27 ([6, Theorem 4.4]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is pseudoconvex (respectively, strictly pseudoconvex) if and only if its gradient ∇f is pseudomonotone (respectively, strictly pseudomonotone).*

Proof Assume that f is pseudoconvex but ∇f is not pseudomonotone. Then there exist $x, y \in K$, $x \neq y$, such that

$$\langle \nabla f(x), y - x\rangle \geq 0 \quad \text{and} \quad \langle \nabla f(y), x - y\rangle > 0.$$

Since f is pseudoconvex, the first inequality leads to $f(y) \geq f(x)$, and the second one leads to $f(x) > f(y)$ as every pseudoconvex function is quasiconvex. We thus arrive at a contradiction as the two conclusions are contradictory to each other.

Conversely, assume on the contrary that ∇f is pseudomonotone but f is not pseudoconvex. Then there exist $x, y \in K$ such that

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \text{and} \quad f(y) < f(x).$$

By the mean value theorem, there exists $z = (1 - \lambda)x + \lambda y$ for some $\lambda \in]0, 1[$ such that

$$f(y) - f(x) = \langle \nabla f(z), y - x \rangle = (1/\lambda)\langle \nabla f(z), z - x \rangle.$$

Since $f(y) < f(x)$, it follows that $\langle \nabla f(z), z - x \rangle < 0$. Now by pseudomonotonicity of ∇f, we have

$$\langle \nabla f(x), z - x \rangle < 0, \text{ that is, } \langle \nabla f(x), y - x \rangle < 0,$$

which leads to a contradiction. □

The following concepts of strict and semistrict quasimonotonicity were introduced by Blum and Oettli [28].

Definition 1.33 Let K be a nonempty convex subset of $\mathbb{R}^n$. A map $F : K \to \mathbb{R}^n$ is said to be

(a) *strictly quasimonotone* if F is quasimonotone and for all $x, y \in K$, $x \neq y$, there exists $z \in]x, y[$ such that $\langle F(z), y - x \rangle \neq 0$;
(b) *semistrictly quasimonotone* if F is quasimonotone and for $x, y \in K$, $x \neq y$,

$$\langle F(x), y - x \rangle > 0 \Rightarrow \text{ there exists } z \in](x + y)/2, y[\quad \text{such that } \langle F(z), y - x \rangle > 0.$$

Obviously, a pseudomonotone map is semistrictly quasimonotone, and a strictly pseudomonotone map is strictly quasimonotone.

The following diagram gives the relationship among different classes of monotone maps defined above.

Strong monotonicity				
⇓				
Strict monotonicity	⇒	Monotonicity		
⇓		⇓		
Strict pseudomonotonicity	⇒	Pseudomonotonicity	⇒	Semistrict Quasimonotonicity
⇓		⇓		
Strict quasimonotonicity	⇒	Quasimonotonicity		

Theorem 1.28 *Let K be a nonempty convex subset of $\mathbb{R}^n$. If $F : K \to \mathbb{R}^n$ is strictly quasimonotone, then it is semistrictly quasimonotone.*

Proof If $\langle F(x), y - x\rangle > 0$ for all $x, y \in K$, $x \neq y$, then

$$\langle F(x), z - x\rangle > 0, \quad \text{for all } z \in]x, y[.$$

Since F is quasimonotone, we have $\langle F(z), z - x\rangle \geq 0$ which implies that

$$\langle F(z), y - x\rangle \geq 0, \quad \text{for all } z \in]x, y[.$$

Since F is strictly quasimonotone, there exists $\hat{z} \in](x + y)/2, y[$ such that $\langle F(\hat{z}), y - x\rangle \neq 0$. Thus, we have $\langle F(\hat{z}), y - x\rangle > 0$, that is, F is semistrictly quasimonotone. □

We now link strict quasiconvexity of a function with strict quasimonotonicity of its gradient.

Theorem 1.29 ([6, Theorem 4.7]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is strictly quasiconvex if and only if its gradient ∇f is strictly quasimonotone.*

The following theorem relates semistrict quasiconvexity of a function with semistrict quasimonotonicity of its gradient.

Theorem 1.30 ([6, Theorem 4.8]) *Let K be a nonempty open convex subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is semistrictly quasiconvex if and only if its gradient ∇f is semistrictly quasimonotone.*

1.3 Generalized Derivatives

In order to deal with the optimality conditions for optimization problems of functions whose ordinary derivative does not exist but they have some kind of generalized derivatives, we give the concept of some generalized derivatives.

Definition 1.34 Let X and Y be locally convex topological vector spaces, K be a nonempty convex subset of X, and $f : X \to Y$ be a given mapping.

(a) If for some $x \in K$ and some $d \in X$, the limit

$$\left\langle f'(x), d\right\rangle := \lim_{t \to 0} \frac{1}{t}\left[f(x + td) - f(x)\right]$$

exists, then $\langle f'(x), d\rangle$ is called the *directional derivative* of f at x in the direction d. If this limit exists for all $d \in X$, then f is called directionally differentiable at x.

(b) If for some $x \in K$ and all $d \in X$, the limit

$$\langle Df(x), d\rangle := \lim_{t \to 0} \frac{1}{t}\left[f(x + td) - f(x)\right]$$

exists and $Df(x)$ is a continuous linear map from X to Y, then $Df(x)$ is called the *Gâteaux derivative* of f at x, and f is called Gâteaux differentiable at x. If f is Gâteaux differentiable at every $x \in K$, then we say that f is Gâteaux differentiable on K.

Example 1.13 It is well known that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x|$, is not Gâteaux differentiable, but it is directionally differentiable at 0.

Definition 1.35 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, K be a nonempty open subset of X, and $f : X \to Y$ be a mapping. Let $x \in K$ be given. If there is a continuous linear map $f'(x) : X \to Y$ with

$$\lim_{\|h\|_X \to 0} \frac{\|f(x+h) - f(x) - \langle f'(x), h\rangle\|_Y}{\|h\|_X} = 0,$$

then $f'(x)$ is called the *Fréchet derivative* of f at x and f is called *Fréchet differentiable* at x.

Lemma 1.3 ([81, Lemma 2.17]) *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces,* K *be a nonempty open subset in* X*, and* $f : X \to Y$ *be a mapping. If the Fréchet derivative of* f *at* $x \in K$ *exists, then the Gâteaux derivative of* f *at* x *exists and both are equal.*

Definition 1.36 Let $K \subseteq \mathbb{R}^n$ be an open convex set and $f : \mathbb{R}^n \supseteq K \to \mathbb{R}$ be a real-valued function. The *upper* and *lower Dini directional derivatives of* f *at* $x \in K$ in the direction $d \in \mathbb{R}^n$ are defined as

$$D^+f(x; d) = \limsup_{t \downarrow 0} \frac{f(x+td) - f(x)}{t},$$

and

$$D_+f(x; d) = \liminf_{t \downarrow 0} \frac{f(x+td) - f(x)}{t},$$

respectively.

Remark 1.16 It is easy to see that $D_+f(x; d) \leq D^+f(x; d)$. If the function f is convex, then the upper and lower Dini directional derivatives are equal to the directional derivative.

Definition 1.37 Let K be a nonempty convex subset of $\mathbb{R}^n$. The function $f : K \to \mathbb{R}$ is called *radially upper (lower) semicontinuous* (also known as *upper (lower) hemicontinuous* on K) if for every pair of distinct points $x, y \in K$, the function f is upper (lower) semicontinuous along the line segment $[x, y]$.

Theorem 1.31 (Diewert Mean Value Theorem) [50] *Let* K *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $f : K \to \mathbb{R}$ *be radially upper semicontinuous on* K*. Then for any* *pair x, y of distinct points of K, there exists $w \in [x, y[$ such that*

$$f(y) - f(x) \geq D^+ f(w; y-x),$$

where $[x, y[$ denotes the line segment joining x and y including the endpoint x. In other words, there exists $\lambda \in [0, 1[$ such that

$$f(y) - f(x) \geq D^+ f(w; y-x), \quad \text{where } w = x + \lambda(y-x).$$

If f is radially lower semicontinuous on K, then for any pair x, y of distinct points of K, there exists $v \in [x, y[$ such that

$$f(y) - f(x) \leq D_+ f(v; y-x),$$

Definition 1.38 [97] Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $q : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. A function $f : K \to \mathbb{R}$ is said to be

(a) *q-quasiconvex* if for all $x, y \in K$,

$$f(x) \leq f(y) \quad \Rightarrow \quad q(y; x-y) \leq 0;$$

(b) *q-quasiconcave* if $-g$ is q-quasiconvex;
(c) *q-pseudoconvex* if for all $x, y \in K$, $x \neq y$,

$$f(x) < f(y) \quad \Rightarrow \quad q(y; x-y) < 0;$$

(d) *strictly q-pseudoconvex* if for all $x, y \in K$, $x \neq y$,

$$f(x) \leq f(y) \quad \Rightarrow \quad q(y; x-y) < 0;$$

(e) *q-pseudoconcave* if $-f$ is q-pseudoconvex;
(f) *q-pseudolinear* if it is both q-pseudoconvex as well as q-pseudoconcave.

If $q(x; d) = D^+ f(x; d)$ ($q(x; d) = D_+ f(x; d)$), then the above definitions are called D^+-quasiconvex, D^+-quasiconcave, D^+-pseudoconvex, strictly D^+-pseudoconvex, D^+-pseudoconcave, and D^+-pseudolinear (D_+-quasiconvex, D_+-quasiconcave, D_+-pseudoconvex, strictly D_+-pseudoconvex, D_+-pseudoconcave, and D_+-pseudolinear), respectively.

Remark 1.17 It is clear that strict q-pseudoconvexity implies q-pseudoconvexity and also q-quasiconvexity. But, as pointed out in [97], neither q-quasiconvexity implies q-pseudoconvexity nor the reverse implication holds.

Example 1.14 Let $K = [-1, 1]$ and

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \frac{1}{2}x, & \text{if } x < 0. \end{cases}$$

Then f is D^+-pseudolinear over K.

Remark 1.18 Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

(a) If f is D^+-pseudoconvex over K, then it is pseudoconvex over K in the sense of Diewert [50], that is, for all $x, y \in K$, $f(x) < f(y)$ implies $D_+f(y; x-y) < 0$.
(b) If f is D^+-pseudoconvex (D_+-pseudoconcave) over K and lower semicontinuous (upper semicontinuous), then it is quasiconvex (quasiconcave) over K (see Corollary 15 in [50]).
(c) If f is quasiconvex over K, then for all $x, y \in K$,

$$f(x) \le f(y) \quad \Rightarrow \quad D^+f(y; x-y) \le 0.$$

(d) If f is quasiconcave over K, then for all $x, y \in K$,

$$f(x) \ge f(y) \quad \Rightarrow \quad D^+f(y; x-y) \ge 0.$$

(e) Any linear fractional function whose denominator is positive over K is D^+-pseudolinear.

Lemma 1.4 ([50]) *Let $K \subseteq \mathbb{R}^n$ be nonempty set and $f : K \to \mathbb{R}$ be upper semicontinuous and D^+-pseudoconvex, that is, for all $x, y \in K$, $f(x) < f(y) \Rightarrow D^+f(y; x-y) < 0$. Then f is quasiconvex and semistrictly quasiconvex.*

Lemma 1.5 ([97]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $f : K \to \mathbb{R}$ be a function, and $p, q : K \times \mathbb{R}^n \to \mathbb{R}$ be bifunctions such that for all $x \in K$ and all $d \in \mathbb{R}^n$, $p(x; d) \le q(x; d)$. Then q-quasiconvexity, q-pseudoconvexity, and strict q-pseudoconvexity imply p-quasiconvexity, p-pseudoconvexity, and strict p-pseudoconvexity, respectively.*

Proof Let f be q-quasiconvex. Then we have for all $x, y \in K$,

$$f(x) \le f(y) \quad \Rightarrow \quad q(y; x-y) \le 0.$$

Because $p(y; x-y) \le q(y; x-y)$ for all $x, y \in K$, the implication

$$f(x) \le f(y) \quad \Rightarrow \quad p(y; x-y) \le 0$$

also holds, and thus, f is p-quasiconvex. The remaining assertions can be proven in a similar way. □

Definition 1.39 [68, 97] Let $K \subseteq \mathbb{R}^n$ be a nonempty set. A bifunction $q : K \times \mathbb{R}^n \to \mathbb{R}$ is said to be *pseudomonotone* if for every pair of distinct points $x, y \in K$, we have

$$q(x; y-x) \ge 0 \quad \Rightarrow \quad q(y; x-y) \le 0. \tag{1.12}$$

Remark 1.19 The above implication (1.12) is equivalent to the following implication:

$$q(y; x-y) > 0 \quad \Rightarrow \quad q(x; y-x) < 0. \tag{1.13}$$

Lemma 1.6 *A bifunction $q : K \times \mathbb{R}^n \to \mathbb{R}$ is pseudomonotone if and only if for every pair of distinct points $x, y \in K \subseteq \mathbb{R}^n$, we have*

$$q(x; y - x) > 0 \quad \Rightarrow \quad q(y; x - y) < 0. \tag{1.14}$$

Proof The implication (1.14) is equivalent to the following implication:

$$q(y; x - y) \geq 0 \quad \Rightarrow \quad q(x; y - x) \leq 0.$$

Interchanging x and y, we get (1.12). □

Lemma 1.7 ([129]) *Let $f : K \to \mathbb{R}$ be radially upper semicontinuous on $K \subseteq \mathbb{R}^n$ and $q : K \times \mathbb{R}^n \to \mathbb{R}$ be subodd and positively homogeneous in the second argument such that for all $x \in K$, $q(x; \cdot) \leq D^+ f(x; \cdot)$. Then*

(a) *f is quasiconvex over K if and only if it is q-quasiconvex;*
(b) *f is q-pseudoconvex if and only if q is pseudomonotone.*

Definition 1.40 Let K be a nonempty subset of a Banach space, $f : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$. The *Clarke directional derivative* of f at $x \in K$ in the direction of a vector $v \in K$, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

Clearly, for all $x, v \in K$, we have $D^+ f(x; v) \leq f^\circ(x; v)$.

Definition 1.41 Let K be a nonempty subset of a Banach space with its dual space X^*, $f : K \to \mathbb{R}$ be locally Lipschitz at a given point $x \in K$. The *Clarke generalized subdifferential* of f at $x \in K$, denoted by $\partial^c f(x)$, is defined by

$$\partial^c f(x) = \{\xi \in X^* : f^\circ(x; v) \geq \langle \xi, v \rangle \text{ for all } v \in K\}.$$

Remark 1.20 It follows from the definition that for every $v \in K$,

$$f^\circ(x; v) = \max\{\langle \xi, v \rangle : \xi \in \partial^c f(x)\}.$$

If f is convex, then the Clarke generalized subdifferential coincides with the subdifferential of f in the sense of convex analysis [127].

Proposition 1.5 ([48, Proposition 2.1.1]) *Let K be a nonempty subset of a normed space X and $f : K \to \mathbb{R}$ be a locally Lipschitz at a point $x \in K$.*

(a) *The function $v \mapsto f^\circ(x; v)$ is finite, positively homogeneous, and subadditive, and satisfies $|f^\circ(x; v)| \leq k||v||$.*
(b) *$f^\circ(x; v)$ is upper semicontinuous as a function of $(x; v)$ and, satisfies the Lipschitz condition as a function of v alone.*
(c) $f^\circ(x; -v) = (-f)^\circ(x; v)$.

Lemma 1.8 ([48]) *Let K be a nonempty subset of a Banach space and $f : K \to \mathbb{R}$ be locally Lipschitz. Then the set-valued map $\partial^c f$ is upper semicontinuous (see, Sect. 1.4 for upper semicontinuity of a set-valued map).*

Theorem 1.32 (Lebourg's Mean Value Theorem) [48] *Let x and y be points in a Banach space X, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point u in $]x, y[$ such that*

$$f(y) - f(x) \in \langle \partial^c f(u), y - x \rangle.$$

Since when f is convex, the Clarke subdifferential coincides with the subdifferential of f in the sense of convex analysis, Theorem 1.32 also holds for subdifferential in the sense of convex analysis.

1.4 Tools from Nonlinear Analysis

In this section, we recall some concepts and results from nonlinear analysis which will be used in the sequel.

1.4.1 Continuity for Set-Valued Maps

Definition 1.42 ([15, 20]) Let X and Y be topological spaces. A set-valued map $T : X \to 2^Y$ is said to be

(a) *upper semicontinuous at* $x_0 \in X$ if for any open set V in Y containing $T(x_0)$, there exists an open neighborhood U of x_0 in X such that $T(x) \subseteq V$ for all $x \in U$;
(b) *lower semicontinuous at* $x_0 \in X$ if for any open set V in Y such that $V \cap T(x_0) \neq \emptyset$, there exists an open neighborhood U of x_0 in X such that $T(x) \cap V \neq \emptyset$ for all $x \in U$;
(c) *upper semicontinuous* (respectively, *lower semicontinuous*) on X if it upper semicontinuous (respectively, lower semicontinuous) at every point $x \in X$;
(d) *continuous* on X if it is upper semicontinuous as well as lower semicontinuous on X;
(e) *compact* if there exists a compact subset $\mathcal{K} \subseteq Y$ such that $T(X) \subseteq \mathcal{K}$;
(f) *closed* if its graph $\mathcal{G}(T) := \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$.

Remark 1.21 If $T(x)$ is a singleton in a neighborhood of x, then the upper semicontinuous and the lower semicontinuous of T at x are equivalent.

Example 1.15 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and consider the set-valued mapping $T : X \to 2^Y$ given by

$$T(x) := [(1 - x, x), (1, 1)],$$

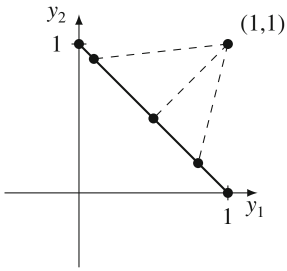


Fig. 1.14 An illustration of the set-valued mapping T defined in Example 1.15

where $[(a, b), (c, d)]$ is the line segment between (a, b) and (c, d) (see Fig. 1.14). Then T is upper and lower semicontinuous, and therefore continuous. If the set-valued map is changed slightly to

$$T_1(x) := \begin{cases} [(1-x, x), (1,1)], & \text{if } x \in [0,1], \\ \emptyset, & \text{else,} \end{cases}$$

then T_1 is upper semicontinuous, but T_1 is not lower semicontinuous. If we choose

$$T_2(x) := \begin{cases} [(1-x, x), (1,1)], & \text{if } x \in]0,1[, \\ \emptyset, & \text{else,} \end{cases}$$

then T_2 is lower semicontinuous, but not upper semicontinuous, and therefore not continuous.

Several other examples of upper semicontinuous and lower semicontinuous set-valued maps can be found in [3, 20].

Lemma 1.9 ([15, 20]) *Let X and Y be topological spaces. A set-valued map $T : X \to 2^Y$ is lower semicontinuous at $x \in X$ if and only if for any net $\{x_\alpha\} \subset X$, $x_\alpha \to x$ and for any $y \in T(x)$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 1.10 *Let X be a topological space, Y be a topological vector space and $T : X \to 2^Y$ be a set-valued map such that $T(x)$ is nonempty and compact for all $x \in X$. Then T is upper semicontinuous at $x \in X$ if and only if for any nets $\{x_\mu\} \subset X$ with $x_\mu \to x$ and $\{y_\mu\} \subset Y$ with $y_\mu \in T(x_\mu)$, there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that $y_\nu \to y$ for some $y \in T(x)$.*

Proof Let T be upper semicontinuous at $x \in X$. Assume that $\{x_\mu\} \subseteq X$ with $x_\mu \to x$ and $\{y_\mu\} \in Y$ with $y_\mu \in T(x_\mu)$. Let $\mathcal{V} \in \mathfrak{V}$, where $\mathfrak{V}$ stands for a basis of neighborhoods of $\mathbf{0}$. Then $\bigcup_{u \in T(x)} \{u + \mathcal{V}\}$ is an open covering of $T(x)$. Since $T(x)$ is compact, there exists a finite subset $\{u_1, u_2, \ldots, u_m\} \subseteq T(x)$ such that $\bigcup_{i=1}^{m} \{u_i + \mathcal{V}\} \supseteq T(x)$. Since T is upper semicontinuous at $x \in X$, there exists a neighborhood $\mathcal{U}$ of x such that

$$T(x') \subseteq \bigcup_{i=1}^{n} \{u_i + \mathcal{V}\}, \quad \text{for all } x' \in \mathcal{U}.$$

Since $x_\mu \to x$, there exists μ' such that $\{x_\mu\} \subset \mathcal{U}$ for all $\mu \geq \mu'$. Hence for each $\mu \geq \mu'$, $y_\mu \in u_i + \mathcal{V}$ for some $u_i \in \{u_1, u_2, \ldots, u_m\}$. Therefore, there exist a subnet $\{y_\nu\} \subseteq \{y_\mu\}$ and $u_i \in \{u_1, u_2, \ldots, u_m\}$ such that $\{y_\nu\} \subset u_i + \mathcal{V}$. Let this $u_i =: u^{\mathcal{V}}$. Corresponding to $\mathcal{V} \in \mathfrak{V}$, that is, for each $\mathcal{V} \in \mathfrak{V}$, there exist a subnet $\{y_\nu\} \subset \{y_\mu\}$ and $u^{\mathcal{V}} \in T(x)$ such that

$$\{y_\nu\} \subseteq u^{\mathcal{V}} + \mathcal{V}. \tag{1.15}$$

Let $\mathcal{V}_2 \geq \mathcal{V}_1$ for each $\mathcal{V}_1, \mathcal{V}_2 \in \mathfrak{V}$, if $\mathcal{V}_2 \subset \mathcal{V}_1$. Then $\mathfrak{V}$ is a directed set and $\{u^{\mathcal{V}}\}$ is a net of $T(x)$. Since $T(x)$ is compact, there exist $\mathfrak{V}' \subset \mathfrak{V}$ and $y \in T(x)$ such that $u^{\mathcal{V}'} \to y$, where $\mathcal{V}' \in \mathfrak{V}'$. Let V be a neighborhood of y. Since Y is a topological vector space, there exists $\bar{\mathcal{V}} \in \mathfrak{V}$ such that $\bar{\mathcal{V}} + \bar{\mathcal{V}} \in V - y$. Since $u^{\mathcal{V}'} \to y$, there exists $\mathcal{V}'' \in \mathfrak{V}'$ such that

$$u^{\mathcal{V}'} \in y + \bar{\mathcal{V}}, \quad \text{for all } \mathcal{V}' \geq \mathcal{V}''.$$

Hence for any $\Upsilon \in \mathfrak{V}'$ with $\Upsilon \geq \bar{\mathcal{V}} \cap \mathcal{V}''$,

$$\{u^{\Upsilon}\} + \mathcal{V}' \subset (y + \bar{\mathcal{V}}) + \bar{\mathcal{V}} \subset V.$$

Therefore by (1.15), there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that $\{y_\nu\} \subset V$. Thus there exists a subnet $\{y_\nu\} \subset \{y_\mu\}$ such that $y_\nu \to y$ for some $y \in T(x)$.

Suppose that T is not upper semicontinuous at $x \in X$. Then there exists an open set $\mathcal{V}$ containing $T(x)$ such that for any neighborhood $\mathcal{U}_\mu$ of x, there is a point $x_\mu \in \mathcal{U}_\mu$ with $T(x_\mu) \cap \mathcal{V}^c \neq \emptyset$. Hence, there exist $\{x_\nu\} \subset X$ converging to x and $y_\nu \in T(x_\nu) \cap \mathcal{V}^c$. Since $y_\nu \notin T(x)$ for all ν, $\{y_\nu\}$ does not have subnet converging to some point of $T(x)$. □

The following results provide the characterization of upper semicontinuity and lower semicontinuity, respectively.

Proposition 1.6 *Let X and Y be topological spaces and $T : X \to 2^Y$ be a set-valued map such that $T(x)$ is compact for each $x \in X$. Then T is upper semicontinuous if and only if for each open subset G of Y, the set*

$$T_+^{-1}(G) = \{x \in X : T(x) \subseteq G\}$$

is open.

Proposition 1.7 *Let X and Y be topological spaces. A set-valued map $T : X \to 2^Y$ is lower semicontinuous if and only if $T^{-1}(G) = \{x \in X : T(x) \cap G \neq \emptyset\}$ is open for every open subset G of Y.*

Proposition 1.8 ([20, p.112, Theorem 6]) *Let X and Y be topological spaces and $T : X \to 2^Y$ be a set-valued map. If T is upper semicontinuous, then T is closed.*

Proposition 1.9 ([20, p.112, Theorem 7]) *Let X and Y be topological spaces and $T_1, T_2 : X \to 2^Y$ be set-valued maps. If T_2 is upper semicontinuous, then the mapping $T = T_1 \cap T_2$ is upper semicontinuous.*

The following proposition shows that under the compactness assumption on the space Y, a set-valued map is closed if and only if it is upper semicontinuous. Notice that this proposition cannot be applied to Example 1.15, as $\mathbb{R}^2$ is not compact.

Proposition 1.10 ([20, p.112, Corollary]) *Let X and Y be topological spaces such that Y is compact and $T : X \to 2^Y$ be a set-valued map. Then T is closed if and only if it is upper semicontinuous.*

Proof Assume that T is closed. Let $\widetilde{T}$ be a set-valued map such that $\widetilde{T}(x) = Y$ for each $x \in X$. Then, by Proposition 1.9, $T = T \cap \widetilde{T}$ is upper semicontinuous, because $\widetilde{T}$ is upper semicontinuous.

The reverse implication follows from Proposition 1.8. □

Lemma 1.11 ([20, Theorem 3]) *Let X and Y be topological spaces. If $T : X \to 2^Y$ is upper semicontinuous on X and D is a compact subset of X, then $T(D)$ is compact.*

Example 1.16 The function T_1, defined in Example 1.15, is upper semicontinuous, and for $D_1 = [0, 1]$, $T_1(D_1)$ is compact. However, for the compact set $D_2 = \left[\frac{1}{4}, \frac{1}{2}\right]$, the set $T_2(D_2)$, where T_2 is also defined in Example 1.15, is compact, but T_2 is not upper semicontinuous.

The following lemma has been applied to the study of game theory (see [141]).

Lemma 1.12 ([141]) *Let X and Y be Hausdorff topological vector spaces such that Y is compact. Let $f : X \times Y \to \mathbb{R}$ be a lower semicontinuous function and for each fixed $y \in Y$, the function $x \mapsto f(x, y)$ be upper semicontinuous on X. Then the function $\Psi : X \to \mathbb{R}$ defined by*

$$\Psi(x) = \min_{y \in Y} f(x, y), \quad \text{for all } x \in X$$

is continuous on X.

Let X be a metric space with metric d. We use the following notations:

$$2^X_q = \text{ set of all nonempty and compact subsets of } X;$$

$$2^X_{cl} = \text{ set of all nonempty, closed and bounded subsets of } X;$$

For any nonempty subset M and N of X and for any $x \in M$, we define the distance from x to N by

$$d(x, N) = \inf_{y \in N} d(x, y).$$

We define the number $d(M, N)$ as

$$d(M.N) = \sup_{x \in M} d(x, N) = \sup_{x \in M} \inf_{y \in N} d(x, y).$$

The *Hausdorff metric* $\mathscr{H}(M,N)$ on 2^X_{cl} is defined as

$$\mathscr{H}(M,N) = \max\{d(M,N), d(N,M)\}, \quad \text{for all } M,N \in 2^X_{cl}.$$

Then $\mathscr{H}$ is metric on 2^X_{cl}. If (X,d) is complete metric space with metric d, then $(2^X_{cl}, \mathscr{H})$ is a complete metric space.

Lemma 1.13 (Nadler's Theorem) [115] *Let (X,d) be a metric space and $\mathscr{H}$ be a Hausdorff metric on 2^X_{cl}. If M and N are compact sets in X, then for each $x \in M$, there exists $y \in N$ such that*

$$d(x,y) \leq \mathscr{H}(M,N).$$

Now we define the continuity of a set-valued map in terms of ε and δ.

Definition 1.43 Let (X,d) and (Y,ρ) be metric spaces. A set-valued map $T : X \to 2^Y_q$ is said to be *$\mathscr{H}$-continuous* on X if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,y \in X$

$$\mathscr{H}(T(x), T(y)) < \varepsilon \quad \text{whenever} \quad d(x,y) < \delta.$$

Remark 1.22 The notions of continuity in the sense of Definitions 1.42 and 1.43 are equivalent if T is compact-valued.

Definition 1.44 ($\mathcal{H}$-Hemicontinuity) [145] Let K be a nonempty convex subset of a normed space X and Y be a normed vector space. A nonempty compact-valued map $T : K \to 2^{\mathcal{L}(X,Y)}$ is said to be *$\mathscr{H}$-hemicontinuous* if for any $x,y \in K$, the mapping $\alpha \mapsto \mathscr{H}(T(x+\alpha(y-x)), T(x))$ is continuous at 0^+, where $\mathscr{H}$ is the Hausdorff metric defined on $2^{\mathcal{L}(X,Y)}_{cl}$.

Definition 1.45 (u-Hemicontinuity) Let X and Y be topological vector spaces. A set-valued map $T : X \supseteq K \to 2^Y \setminus \{\emptyset\}$ is said to be *u-hemicontinuous* if for any $x,y \in K$ and $\alpha \in [0,1]$, the set-valued map $\alpha \mapsto T(\alpha x + (1-\alpha)y)$ is upper semicontinuous at 0^+.

1.4.2 Fixed Point Theory for Set-Valued Maps

In 1929, Knaster, Kuratowski and Mazurkiewicz [96] formulated the so-called KKM principle in the finite dimensional Euclidean space. Later, in 1961, it has been generalized to infinite dimensional Hausdorff topological vector spaces by Ky Fan [59]. Fan also established an elementary but very basic geometric lemma for set-valued maps which is called Fan's geometric lemma. In 1968, Browder gave a fixed point version of Fan's geometric lemma and this result is known as Browder fixed point theorem. Since then there have been numerous generalizations of Browder fixed point theorem their applications to coincidence and fixed point theory, minimax inequalities, variational inequalities, convex analysis, game theory, mathematical economics, social sciences, and so on.

It is well known that the famous Browder fixed point theorem [33] is equivalent to a maximal element theorem (see [138]). Such kind of maximal element theorems are useful to establish the existence of solutions of vector variational inequalities, vector equilibrium problems and their generalizations.

In this section, we recall some basic definitions from nonlinear analysis and present Fan-KKM lemma and its generalizations and some famous fixed point theorems for set-valued maps, namely, Nadler's fixed point theorem, Browder fixed point theorem and its generalizations, Kakutani fixed point theorem, etc.

Definition 1.46 Let X be a metric space and $T : X \to 2^X$ be a set-valued map with nonempty values. A point $x \in X$ is said to be a *fixed point* of T if $x \in T(x)$.

Definition 1.47 Let (X, d) and (Y, ρ) be metric spaces and $\mathscr{H}$ be a Hausdorff metric on 2^Y_{cl}. A set-valued map $T : X \to 2^Y_{cl}$ is said to be *set-valued Lipschitz map* if there exists a constant $\alpha > 0$ such that

$$\mathscr{H}(T(x), T(y)) \leq \alpha d(x, y), \quad \text{for all } x, y \in X.$$

The constant α is called a *Lipschitz constant* for T. If $\alpha < 1$, then T is called a set-valued contraction map. If $\alpha = 1$, then T is called *nonexpansive*.

In 1969, Nadler [115] extended the well-known Banach contraction principle for set-valued maps and established the following fixed point theorem.

Theorem 1.33 (Nadler's Fixed Point Theorem) [115] *Let (X, d) be a complete metric space. If $T : X \to 2^X_{cl}$ is a set-valued contraction map, then T has a fixed point.*

Definition 1.48 Let X be a topological vector space and K be a nonempty subset of X. A set-valued map $T : K \to 2^X$ is said to be a *KKM-map* if

$$\text{co}\,(\{x_1, x_2, \ldots, x_m\}) \subseteq \bigcup_{i=1}^{m} T(x_i)$$

for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of X.

Obviously, if T is a KKM-map, then $x \in T(x)$ for every $x \in K$.

Example 1.17 Let $X = K = \mathbb{R}$ and the set-valued map $T : X \to 2^X$ be defined by $T(x) = [0, x]$, where $[0, x]$ is the line segment between 0 and x. Then T is a KKM-map.

Lemma 1.14 (Fan-KKM Lemma) [59] *Let X be a Hausdorff topological vector space and K a nonempty subset of X. Let $T : K \to 2^X$ be a KKM-map such that $T(x)$ is a closed subset of X for all $x \in K$ and compact for at least one $x \in K$. Then $\bigcap_{x \in K} T(x) \neq \emptyset$.*

Chang and Zhang [41] introduced the following concept of generalized KKM mapping.

Definition 1.49 Let K be a nonempty subset of a Hausdorff topological vector space X. A set-valued map $T : K \to 2^X$ is called a *generalized KKM map* if for any finite set $\{x_1, x_2, \dots, x_m\} \subset K$, there is a finite subset $\{y_1, y_2, \dots, y_m\} \subset X$ such that for any subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \{y_1, y_2, \dots, y_m\}$, $1 \le k \le m$, we have

$$\operatorname{co}(\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}) \subset \bigcup_{j=1}^{k} T(x_{i_j}).$$

Clearly, if $T : K \to 2^X$ is a KKM map, then it is generalized KKM map. Indeed, for any finite set $\{x_1, x_2, \dots, x_m\} \subset K$, taking $y_i = x_i$, $i = 1, 2, \dots, m$, then since T is a KKM map, we have

$$\operatorname{co}(\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}) \subset \bigcup_{j=1}^{k} T(x_{i_j}).$$

However, if T is a generalized KKM map, then it may not be a KKM map.

Example 1.18 [41] Let $X = \mathbb{R}$, $K = [-2, 2]$ and $T : K \to 2^X$ be defined by

$$T(x) = \left[-\left(1 + \frac{x^2}{5}\right), 1 + \frac{x^2}{5}\right], \quad \text{for all } x \in K.$$

Then $\bigcup_{x\in K} T(x) = \left[-\frac{9}{5}, \frac{9}{5}\right]$ and $x \notin T(x)$ for all $x \in [-2, -9/5) \cup (9/5, 1]$. It follows that T is not a KKM map. Next we prove that T is a generalized KKM map. If for any finite subset $\{x_1, x_2, \dots, x_m\}$ of K, we take $\{y_1, y_2, \dots, y_m\} \subset [-1, 1]$, then for any finite subset $\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\} \subset \{y_1, y_2, \dots, y_m\}$, we have

$$\operatorname{co}(\{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}) \subset [-1, 1] = \bigcap_{x\in K} T(x) \subset \bigcup_{j=1}^{k} T(x_{i_j}),$$

that is, T is a generalized KKM map.

The following lemma is proved in [41] where convexity on K is assumed. However, Ansari et al. [7] pointed out that this lemma is true without convexity assumption on K.

Lemma 1.15 *Let K be a nonempty subset of a Hausdorff topological vector space X. If $T : K \to 2^X$ is a set-valued map and for each $x \in K$, the set $T(x)$ is finitely closed (i.e., for every finite-dimensional subspace L in X, $T(x) \cap L$ is closed in the Euclidean topology in L). Then the family of sets $\{T(x) : x \in K\}$ has the finite intersection property if and only if $T : K \to 2^X$ is a generalized KKM mapping.*

Definition 1.50 [135] Let X and Y be topological spaces. A set-valued map $T : X \to 2^Y$ is said to be *transfer open-valued* (respectively, *transfer closed-valued*) if for every $x \in X$, $y \in T(x)$ (respectively, $y \notin T(x)$), there exists a point $z \in X$ such that $y \in \mathrm{int}(T(z))$ (respectively, $y \notin \mathrm{cl}(T(z))$).

It is easy to see that a open-valued (respectively, closed-valued) set-valued map is a transfer open-valued (respectively, transfer closed-valued) set-valued map. But the converse is not true.

Lemma 1.16 ([42]) *Let X be a nonempty set, Y be a topological space and $T : X \to 2^Y$ be a set-valued map.*

(a) *T is transfer closed-valued if and only if $\bigcap_{x\in X} T(x) = \bigcap_{x\in X} \mathrm{cl}(T(x))$.*
(b) *T is transfer open-valued if and only if $\bigcup_{x\in X} T(x) = \bigcup_{x\in X} \mathrm{int}(T(x))$.*
(c) *X is a topological space, $T(x)$ is nonempty for each $x \in X$ and T^{-1} is transfer open-valued, then $X = \bigcup_{y\in Y} \mathrm{int}(T^{-1}(y))$.*

Ansari et al. [7] established the following generalized form of Fan-KKM lemma.

Theorem 1.34 *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^X$ be a transfer closed-valued map such that $\mathrm{cl}(T(x_0))$ is compact for at least one $x_0 \in K$, and let $\mathrm{cl}\, T : K \to 2^X$ be a generalized KKM map. Then $\bigcap_{x\in K} T(x) \neq \emptyset$.*

Proof Since $\mathrm{cl}\, T : K \to 2^X$ is defined by $(\mathrm{cl}\, T)(x) = \mathrm{cl}(T(x))$ for all $x \in K$, we have that $\mathrm{cl}\, T$ is a generalized KKM map with closed values. By Lemma 1.15, the family of sets $\{T(x) : x \in K\}$ has the finite intersection property. Since $\mathrm{cl}(T(x_0))$ is compact, we have $\bigcap_{x\in K} \mathrm{cl}(T(x)) \neq \emptyset$. Since T is transfer closed-valued,

$$\bigcap_{x\in K} T(x) = \bigcap_{x\in X} \mathrm{cl}(T(x)) \neq \emptyset.$$

This completes the proof. □

The following section lemma, due to Xiang and Debnath [137], is a generalization of Fan section lemma [61] which can be derived by using Fan-KKM Lemma 1.14.

Lemma 1.17 (Fan Section Lemma) *Let K be a nonempty subset of a Hausdorff topological vector space X. Let A be a subset of $K \times K$ such that the following conditions hold.*

(i) *$(x, x) \in A$ for all $x \in K$;*
(ii) *For all $y \in K$, the set $A_y = \{x \in K : (x, y) \in A\}$ is closed in K;*
(iii) *For all $x \in K$, the set $A_x = \{y \in K : (x, y) \notin A\}$ is convex or empty;*
(iv) *For a nonempty compact convex subset $D \subset K$ with each $x \in K$, there exists $y \in D$ such that $(x, y) \notin A$.*

Then there exists $\bar{x} \in K$ such that $\{\bar{x}\} \times K \subset A$.

The following lemma is a generalization of Lemma 1.14.

Lemma 1.18 ([46]) *Let K be a nonempty convex subset of a topological vector space X. Let $T : K \to 2^K$ be a KKM-map such that the following conditions hold.*

(i) $\mathrm{cl}_K(T(\tilde{x}))$ *is compact for some* $\tilde{x} \in K$*;*
(ii) *For each* $A \in \mathscr{F}(K)$ *with* $\tilde{x} \in A$ *and each* $x \in \mathrm{co}(A)$*,* $T(y) \cap \mathrm{co}(A)$ *is closed in* $\mathrm{co}(A)$.
(iii) *For each* $A \in \mathscr{F}(K)$ *with* $\tilde{x} \in A$*,*

$$\left(\mathrm{cl}_K\left(\bigcap_{x\in\mathrm{co}(A)} T(x)\right)\right) \cap \mathrm{co}(A) = \left(\bigcap_{x\in\mathrm{co}(A)} T(x)\right) \cap \mathrm{co}(A).$$

Then $\bigcap_{x\in K} T(x) \neq \emptyset$.

Definition 1.51 Let X be a topological space and Y be a nonempty set. A set-valued map $T : X \to 2^Y$ is said to have *open lower section* if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for every $y \in Y$.

Lemma 1.19 ([136]) *Let X be a topological space and Y be a convex subset of a topological vector space. Let $S, T : X \to 2^Y$ be set-valued maps with open lower sections. Then*

(a) *the set-valued map* $H : X \to 2^Y$*, defined by* $H(x) = \mathrm{co}(S(x))$ *for all* $x \in X$*, has open lower sections;*
(b) *the set-valued map* $J : X \to 2^Y$*, defined by* $J(x) = S(x) \cap T(x)$ *for all* $x \in X$*, has open lower sections.*

The following fixed-point theorem has been proven by Browder [33].

Lemma 1.20 (Browder Fixed Point Theorem) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Suppose that $T : K \to 2^K$ is a set-valued map with nonempty convex values and has open lower sections. Then T has a fixed point.*

We present a Browder type fixed point theorem for set-valued maps under noncompact setting.

Theorem 1.35 ([9]) *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $S, T : K \to 2^K$ be set-valued maps such that the following conditions hold.*

(i) *For all* $x \in K$*,* $\mathrm{co}(S(x)) \subseteq T(x)$ *and* $S(x) \neq \emptyset$*;*
(ii) $K = \bigcup\{\mathrm{int}_K(S^{-1}(x)) : x \in K\}$*;*
(iii) *If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that* $x \in \mathrm{int}_K(S^{-1}(\tilde{y}))$.

Then there exists $\bar{x} \in K$ *such that* $\bar{x} \in T(\bar{x})$.

If S has open lower sections, then condition (ii) in Theorem 1.35 holds, and hence, we have the following result.

Corollary 1.3 [9] *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $S, T : K \to 2^K$ be set-valued maps such that the following conditions hold.*

(i) *For all $x \in K$, $\mathrm{co}(S(x)) \subseteq T(x)$ and $S(x) \neq \emptyset$;*
(ii) *The set $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is open;*
(iii) *If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that $x \in S^{-1}(\tilde{y})$.*

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.

Chowdhury and Tan [47] establish the following version of Browder type fixed point theorem for non-Hausdorff spaces.

Theorem 1.36 *Let K be a nonempty convex subset of a topological vector space Y and $S, T : K \to 2^K$ be set-valued maps. Assume that the following conditions hold:*

(a) *For all $x \in K$, $S(x) \subseteq T(x)$.*
(b) *For all $x \in K$, $T(x)$ is convex and $S(x)$ is nonempty.*
(c) *For all $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is compactly open.*
(d) *There exists a nonempty closed compact (not necessarily convex) subset D of K and an element $\tilde{y} \in D$ such that $K \setminus D \subset T^{-1}(\tilde{y})$.*

Then there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

The following maximal element theorem for a set-valued map is equivalent to Corollary 1.3.

Theorem 1.37 ([49, 105]) *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $S, T : K \to 2^K$ be set-valued maps satisfying the following conditions:*

(i) *For all $x \in K$, $\mathrm{co}(S(x)) \subseteq T(x)$;*
(ii) *For all $x \in K$, $x \notin T(x)$;*
(iii) *For all $y \in K$, $S^{-1}(y) = \{x \in K : y \in S(x)\}$ is open in K;*
(iv) *There exist a nonempty compact subset D of K and a nonempty compact convex subset B of K such that for all $x \in K \setminus D$, $S(x) \cap B \neq \emptyset$.*

Then there exists $\bar{x} \in K$ such that $S(\bar{x}) = \emptyset$.

Definition 1.52 (Φ-Condensing Map) [125, 126] Let X be a Hausdorff topological vector space, L be a lattice with a minimal element, and let $\Phi : 2^X \to L$ be a measure of noncompactness on X and $D \subseteq X$. A set-valued map $T : D \to 2^X$ is called *Φ-condensing* if $M \subseteq D$ with $\Phi(T(M)) \geq \Phi(M)$ implies that M is precompact.

Remark 1.23 Note that every set-valued map defined on a compact set is necessarily Φ-condensing. If X is locally convex, then a compact set-valued map (that is, $T(D)$ is precompact) is Φ-condensing for any measure of noncompactness Φ. Obviously, if $T : D \to 2^X$ is Φ-condensing and if $S : D \to 2^X$ satisfies $S(x) \subseteq T(x)$ for all $x \in D$, then S is also Φ-condensing.

Remark 1.24 If K is a nonempty, closed and convex subset of a locally convex Hausdorff topological vector space X, then condition (iii) of Theorem 1.35 and condition (iv) of Theorem 1.37 can be replaced by the following condition:

(iv)′ The set-valued map $S : K \to 2^K$ is Φ-condensing,

see Corollary 4 in [43].

Theorem 1.38 (Kakutani Fixed Point Theorem) [85] *Let K be a nonempty compact convex subset of a locally convex topological vector space X and Y be a topological vector space. Let $T : K \to 2^Y$ be a set-valued map such that for each $x \in K$, $T(x)$ is nonempty, compact and convex. Then T has a fixed point, that is, there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.*

The Kakutani fixed point theorem is a set-valued version of the following Brouwer fixed point theorem.

Theorem 1.39 (Brouwer's Fixed Point Theorem) *Let K be a nonempty, compact and convex subset of a finite dimensional space $\mathbb{R}^n$ and $f : K \to K$ be a continuous map. Then there exists $x \in K$ such that $f(x) = x$.*

1.5 Variational Inequalities

Theory of variational inequalities is one of the powerful tools of current mathematical technology, introduced separately by G. Fichera and G. Stampacchia in early sixties. The ideas and techniques of variational inequalities are being applied in various fields of mathematics, engineering, management and social sciences including fluid flow through porous media, contact problems in elasticity, optimal control, nonlinear optimization, transportation and economic equilibria, etc. During the last three decades, variational inequalities are used as tools to solve optimization problems; See for example [2, 6, 7, 11–16, 18, 30–33, 45, 55, 56, 64, 67, 69, 94, 98, 116, 121, 124, 132, 134, 139, 140, 142] and the references therein. In this section, we give a brief introduction to the theory of variational inequalities.

Let X be a topological vector space with its topological dual X^*, and K be a nonempty convex subset of X. The value of $l \in X^*$ at x is denoted by $\langle l, x\rangle$. Let $F : K \to X^*$ be a mapping. The *variational inequality problem* (in short, VIP) is to find $\bar{x} \in K$ such that

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K. \tag{1.16}$$

The inequality (1.16) is called *variational inequality*.

Roughly speaking, the variational inequality (1.16) states that the vector $F(\bar{x})$ must be at a non-obtuse angle with all the feasible vectors emanating from $\bar{x}$. In other words, the vector $\bar{x}$ is a solution of VIP if and only if $F(\bar{x})$ forms a non-obtuse angle with every vector of the form $y - \bar{x}$ for all $y \in K$.

First let us consider an application of varianal inequalities in Partial Differential Equations.

Example 1.19 (Inverse Problems in Partial Differential Equations) There is a large number of examples in applied sciences that can be modeled by means of partial differential equations (PDEs). The corresponding PDEs often involve certain unknown variable parameters when a measurement of a solution of the PDE is available. This leads to so-called *inverse problems*. The direct problem, on the other hand, is to solve the PDE. As an example, let us consider the following elliptic boundary value problem (BVP)

$$-\nabla \cdot (q\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.17}$$

where Ω denotes a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\partial\Omega$ is its boundary. Problems of the form (1.17) have been studied in great detail in the literature due to their wide real-world applications. For example, $u = u(x)$ may be the steady-state temperature at a fixed point x of a body. Then q would represent a variable thermal conductivity coefficient and f would constitute the external heat source. When the problem (1.17) needs to be solved, one can choose from a large number of concepts proposed in the literature. Most approaches either regard problem (1.17) as a hyperbolic PDE in q or pose an optimization problem whose solution is an estimate of q. There exist two approaches involving the reformulation of (1.17) as an optimization problem: The problem (1.17) can either be formulated as an unconstrained optimization problem, or it can be handled as a constrained optimization problem which involves the PDE in as a constraint. Since the solution of equations corresponds to minimization problems and therefore to variational inequalities as optimality conditions, the results to be presented in this chapter are directly applicable to (1.17). For further applications and in-depth analysis of inverse problems, we refer to [76].

The simplest example of a variational inequality problem is the problem of solving a system of nonlinear equations.

Proposition 1.11 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping. A vector $\bar{x} \in \mathbb{R}^n$ is a solution of* VIP *if and only if $F(\bar{x}) = \mathbf{0}$.*

Proof Let $F(\bar{x}) = \mathbf{0}$. Then, obviously, inequality (1.16) holds with equality.

Conversely, suppose that $\bar{x}$ satisfies the inequality (1.16). Then, by taking $y = \bar{x} - F(\bar{x})$ in (1.16), we get

$$\langle F(\bar{x}), \bar{x} - F(\bar{x}) - \bar{x}\rangle = \langle F(\bar{x}), -F(\bar{x})\rangle \geq 0,$$

that is, $-\|F(\bar{x})\|^2 \geq 0$, which implies that $F(\bar{x}) = \mathbf{0}$. □

If $F(x)$ is the gradient of a differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$, then the VIP provides the necessary and sufficient condition for a solution of an optimization problem.

Proposition 1.12 *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. If $\bar{x}$ is a solution of the following optimization problem:*

$$\textit{minimize } f(x), \quad \textit{subject to } x \in K, \tag{1.18}$$

then $\bar{x}$ is a solution of VIP *with $F \equiv \nabla f$.*

Proof For any $y \in K$, define a function $\varphi : [0, 1] \to \mathbb{R}$ by

$$\varphi(\lambda) = f(\bar{x} + \lambda(y - \bar{x})), \quad \text{for all } \lambda \in [0, 1].$$

Since $\varphi(\lambda)$ attains its minimum at $\lambda = 0$, therefore, $\varphi'(0) \geq 0$, that is,

$$\langle \nabla f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.19}$$

Hence, $\bar{x}$ is a solution of VIP with $F \equiv \nabla f$. □

Proposition 1.13 *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a pseudoconvex function. If $\bar{x}$ is a solution of* VIP *with $F(\bar{x}) = \nabla f(\bar{x})$, then it is a solution of the optimization problem* (1.18)*.*

Proof Suppose that $\bar{x}$ is a solution of VIP, but not an optimal solution of the optimization problem (1.18). Then there exists a vector $y \in K$ such that $f(y) < f(\bar{x})$. By pseudoconvexity of f, we have $\langle \nabla f(\bar{x}), y - \bar{x} \rangle < 0$, which is a contradiction to the fact that $\bar{x}$ is a solution of VIP. □

Let K be a closed convex cone in a topological vector space X and $F : K \to X^*$ be a mapping. The *nonlinear complementarity problem* (NCP) is to find a vector $\bar{x} \in K$ such that

$$F(\bar{x}) \in K^* \quad \text{and} \quad \langle F(\bar{x}), \bar{x} \rangle = 0, \tag{1.20}$$

where K^* is the dual cone of K.

For further details and applications of complementarity problems, we refer to [56, 64, 74, 75, 87–89, 131] and the references therein.

The next result provides the equivalence between a nonlinear complementarity problem and a variational inequality problem.

Proposition 1.14 *If K is a closed convex pointed cone in a topological vector space X, then* VIP *and* NCP *have precisely the same solution sets.*

Proof Let $\bar{x} \in K$ be a solution of VIP. Then

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.21}$$

In particular, taking $y = x + \bar{x}$ in the above inequality, we get

$$\langle F(\bar{x}), x\rangle \geq 0, \quad \text{for all } x \in K,$$

which implies that $F(\bar{x}) \in K^*$.

By substituting $y = 2\bar{x}$ in inequality (1.21), we obtain

$$\langle F(\bar{x}), \bar{x}\rangle \geq 0, \tag{1.22}$$

and again taking $y = \mathbf{0}$ in inequality (1.21), we get

$$\langle F(\bar{x}), -\bar{x}\rangle \geq 0. \tag{1.23}$$

Inequalities (1.22) and (1.23) together imply that $\langle F(\bar{x}), \bar{x}\rangle = 0$. Hence, $\bar{x}$ is a solution of NCP.

Conversely, suppose that $\bar{x} \in K$ is a solution of NCP, then we have

$$\langle F(\bar{x}), \bar{x}\rangle = 0 \text{ and } \langle F(\bar{x}), y\rangle \geq 0, \quad \text{for all } y \in K.$$

Thus,

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of VIP. □

Let K be a nonempty subset of a normed space X and $T : K \to K$ be a mapping. The *fixed point problem* (FPP) is to find $\bar{x} \in K$ such that

$$T(\bar{x}) = \bar{x}. \tag{1.24}$$

Now we give a relationship between a VIP and a FPP.

Proposition 1.15 *Let K be a nonempty subset of a normed space X and $T : K \to K$ be a mapping. If the mapping $F : K \to K$ is defined by*

$$F(x) = x - T(x), \tag{1.25}$$

then VIP (1.16) *coincide with* FPP (1.24).

Proof Let $\bar{x} \in K$ be a fixed point of the problem (1.24). Then, $F(\bar{x}) = \mathbf{0}$, and thus, $\bar{x}$ solves (1.16).

Conversely, suppose that $\bar{x}$ solves (1.16) with $F(\bar{x}) = \bar{x} - T(\bar{x})$. Then $T(\bar{x}) \in K$ and letting $y = T(\bar{x})$ in (1.16) gives $-\|\bar{x} - T(\bar{x})\|^2 \geq 0$, that is, $\bar{x} = T(\bar{x})$. □

A problem closely related to the VIP is the following problem, known as *Minty variational inequality problem* (MVIP): Find $\bar{x} \in K$ such that

$$\langle F(y), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K. \tag{1.26}$$

The inequality (1.26) is known as *Minty variational inequality* (MVI). Minty [113] gave a complete characterization of the solutions of VIP in terms of the solutions of MVIP. Since the origin of VIP, most of the existence results for a solution of a VIP are established by showing the equivalence between VIP and MVIP.

To distinguish between a variational inequality and Minty variational inequality, we sometimes write Stampacchia variational inequality (SVI) instead of a variational inequality.

Contrary to the Stampacchia variational inequality problem (SVIP), Minty variational inequality problem (MVIP) is a sufficient optimality condition for the optimization problem (1.18) which becomes necessary if the objective function f is pseudoconvex and differentiable.

Theorem 1.40 (Giannessi 1998) *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. The following statements hold:*

(a) *If $\bar{x} \in K$ is a solution of* MVIP *with $F \equiv \nabla f$, then $\bar{x}$ is a solution of optimization problem* (1.18).
(b) *If f is pseudoconvex and $\bar{x} \in K$ is a solution of the optimization problem* (1.18)*, then it is a solution of* MVIP *with $F \equiv \nabla f$.*

Proof

(a) Let $y \in K$ be arbitrary. Consider the function $\varphi(\lambda) = f(\bar{x} + \lambda(y - \bar{x}))$ for all $\lambda \in [0, 1]$. Since $\varphi'(\lambda) = \langle \nabla f(\bar{x} + \lambda(y - \bar{x})), y - \bar{x} \rangle$ and $\bar{x}$ is a solution of MVIP with $F \equiv \nabla f$, it follows that

$$\varphi'(\lambda) = \langle \nabla f(\bar{x} + \lambda(y - \bar{x})), y - \bar{x} \rangle \geq 0, \quad \text{for all } \lambda \in [0, 1].$$

This implies that φ is a nondecreasing function on $[0, 1]$, and therefore,

$$f(y) = \varphi(1) \geq \varphi(0) = f(\bar{x}).$$

Thus, $\bar{x}$ is a solution of the optimization problem (1.18).
(b) Let $\bar{x}$ be an optimal solution of the optimization problem (1.18). Then for all $y \in K$, $f(\bar{x}) \leq f(y)$. Since f is a pseudoconvex differentiable function, by Theorem 1.21, f is quasiconvex. Then by Theorem 1.18, we have

$$\langle \nabla f(y), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K.$$

Thus, $\bar{x}$ is a solution of MVIP. □

Definition 1.53 Let K be a nonempty convex subset of topological vector space X. A mapping $F : K \to X^*$ is said to be

(a) *lower hemicontinuous* or *radially lower semicontinuous* if for any fixed $x, y \in K$, the function $\lambda \mapsto F(x + \lambda(y - x))$ defined on $[0, 1]$ is lower semicontinuous;
(b) *upper hemicontinuous* or *radially upper semicontinuous* if for any fixed $x, y \in K$, the function $\lambda \mapsto F(x + \lambda(y - x))$ defined on $[0, 1]$ is upper semicontinuous;

(c) *hemicontinuous* or *radially semicontinuous* if for any fixed $x, y \in K$, the mapping $\lambda \mapsto F(x + \lambda(y - x))$ defined on $[0, 1]$ is continuous, that is, if F is continuous along the line segments in K.

The following Minty lemma is an important tool in the theory of variational inequalities when the mapping is monotone and the domain is convex.

Lemma 1.21 (Minty Lemma) *Let K be a nonempty subset of a topological vector space X and $F : K \to X^*$ be a mapping. The following assertions hold.*

(a) *If K is convex and F is hemicontinuous, then every solution of* MVIP *is a solution of* VIP.
(b) *If F is pseudomonotone, then every solution of* VIP *is a solution of* MVIP.

Proof

(a) Let $\bar{x} \in K$ be a solution of MVIP. Then for any $y \in K$ and $\lambda \in]0, 1]$, $z = \bar{x} + \lambda(y - \bar{x}) \in K$, and hence,

$$\langle F(z), z - \bar{x}\rangle \geq 0, \quad \text{for all } \lambda \in]0, 1],$$

which implies that

$$\langle F(y + \lambda(\bar{x} - y)), y - \bar{x}\rangle \geq 0, \quad \text{for all } \lambda \in]0, 1].$$

By the hemicontinuity of F, we have

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of VIP.
(b) Obvious, by pseudomonotonicity of F. □

It can be easily seen that if K is a nonempty closed convex subset of X and $F : K \to X^*$ be hemicontinuous and pseudomonotone, then the solution set of VIP is closed and convex. Moreover, if the F is strictly monotone, then the solution of VIP is unique, provided it exists. Finally, we present a result on the existence of a solution of VIP (1.16).

Theorem 1.41 ([131, Theorem 3.1]) *Let X be a reflexive Banach space, K be a nonempty bounded closed convex subset of X and $T : K \to X^*$ be a mapping. Suppose that T is pseudomonotone and hemicontinuous. Then there exists a solution $x \in K$ of* VIP (1.16)*. Furthermore, if in addition T is strictly pseudomonotone, then the solution is unique.*

1.5.1 Nonsmooth Variational Inequalities

Motivated by the optimality conditions in terms of the generalized directional derivatives, we associate an optimization problem with the variational inequality problem defined by means of a bifunction h.

Let K be a nonempty subset of $\mathbb{R}^n$ and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a bifunction. The *variational inequality problem in terms of a bifunction h* is defined as follows:

$$\text{Find } \bar{x} \in K \text{ such that } h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K. \qquad (\text{VIP})_h$$

When $h(x; y - x) = \langle F(x), y - x \rangle$, where $F : \mathbb{R}^n \to \mathbb{R}^n$, then $(\text{VIP})_h$ reduces to the VIP studied in the previous section.

As we have seen in the previous section that the Minty variational inequality problem is closely related to VIP, and also provides a necessary and sufficient optimality condition for a differentiable optimization problem under convexity or pseudoconvexity assumption. Therefore, the study of Minty variational inequality defined by means of a bifunction h is also very important in the theory of nonsmooth variational inequalities. The *Minty variational inequality problem in terms of a bifunction h* is defined as follows:

$$\text{Find } \bar{x} \in K \text{ such that } h(y; \bar{x} - y) \leq 0, \quad \text{for all } y \in K. \qquad (\text{MVIP})_h$$

To prove the equivalence between $(\text{VIP})_h$ and $(\text{MVIP})_h$, we introduce the following concept of upper sign continuity.

Definition 1.54 Let K be a nonempty convex subset of $\mathbb{R}^n$. A bifunction $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is said to be *upper sign continuous* if for all $x, y \in K$ and $\lambda \in]0, 1[$,

$$h(x + \lambda(y - x); x - y) \leq 0 \quad \text{implies} \quad h(x; y - x) \geq 0.$$

This notion of upper sign continuity for a bifunction extends the concept of upper sign continuity introduced in [70].

Clearly, every subodd radially upper semicontinuous bifunction is upper sign continuous.

The following lemma is a generalization of Minty Lemma 1.21.

Lemma 1.22 *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a pseudomonotone and upper sign continuous bifunction such that h is positively homogeneous in the second argument. Then $\bar{x} \in K$ is a solution of* $(\text{VIP})_h$ *if and only if it is a solution of* $(\text{MVIP})_h$.

Proof The pseudomonotonicity of h implies that every solution of $(\text{VIP})_h$ is a solution of $(\text{MVIP})_h$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{MVIP})_h$. Then

$$h(y; \bar{x} - y) \leq 0, \quad \text{for all } y \in K. \qquad (1.27)$$

Since K is convex, we have $y_\lambda = \bar{x} + \lambda(y - \bar{x}) \in K$ for all $\lambda \in]0, 1[$. Therefore, inequality (1.27) becomes

$$h(y_\lambda; \bar{x} - y_\lambda) \leq 0.$$

As $\bar{x} - y_\lambda = \lambda(\bar{x} - y)$ and h is positively homogeneous in the second argument, we have

$$h(y_\lambda; \bar{x} - y) \leq 0.$$

Thus, the upper sign continuity of h implies that $\bar{x} \in K$ is a solution of $(\text{VIP})_h$. □

Let us recall the optimization problem:

$$\text{minimize } f(x), \quad \text{subject to } x \in K, \tag{P}$$

where K is a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ is a function.

In the subsequent theorems, we relate the solutions of the problem (P) and $(\text{VIP})_h$.

Theorem 1.42 *Let K be a nonempty convex subset of $\mathbb{R}^n$, $f : K \to \mathbb{R}$ be a function and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a bifunction. If f is h-convex and $\bar{x} \in K$ is a solution of $(\text{VIP})_h$, then $\bar{x}$ solves the problem (P).*

Proof By h-convexity of f, we have

$$f(y) - f(\bar{x}) \geq h(\bar{x}; y - \bar{x}), \quad \text{for all } y \in K.$$

Since $\bar{x}$ is a solution of $(\text{VIP})_h$, we have

$$h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K.$$

The last two inequalities together imply that

$$f(y) - f(\bar{x}) \geq 0, \quad \text{for all } y \in K,$$

that is, $\bar{x}$ is a solution of problem (P). □

The h-convexity assumption in the above theorem can be weakened to h-pseudoconvexity.

For the converse of Theorem 1.42 to hold, we do not require the function f to be h-convex. However, we assume that the function f and the bifunction h satisfy the following condition:

$$\forall\, x \in K,\ d \in \mathbb{R}^n :\ D_+f(x; d) \leq h(x; d). \tag{1.28}$$

Theorem 1.43 *Let K be a nonempty convex subset of $\mathbb{R}^n$, $f : K \to \mathbb{R}$ be a function and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfy the condition (1.28). If $\bar{x}$ is an optimal solution of the problem (P), then $\bar{x} \in K$ is a solution of $(\text{VIP})_h$.*

Proof Since K is convex and $\bar{x}$ is an optimal solution of problem (P), for any $y \in K$, we have

$$f(\bar{x}) \leq f(\bar{x} + \lambda(y - \bar{x})), \quad \text{for all } \lambda \in]0, 1].$$

This implies that

$$\frac{f(\bar{x} + \lambda(y - \bar{x})) - f(\bar{x})}{\lambda} \geq 0, \quad \text{for all } \lambda \in]0, 1].$$

Taking lim inf as $\lambda \to 0^+$, we obtain

$$D_+ f(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K,$$

which on using (1.28) implies that

$$h(\bar{x}; y - \bar{x}) \geq 0, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of $(\text{VIP})_h$. □

Thus, it is possible to identify the solutions of the optimization problem (P) with those of the $(\text{VIP})_h$ provided the objective function is h-convex or h-pseudoconvex.

Theorem 1.44 *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a function such that*

$$h(x; y - x) > f(y) - f(x), \quad \textit{for all } x, y \in K \textit{ and } x \neq y. \tag{1.29}$$

Then every solution of the problem (P) *is a solution of* $(\text{VIP})_h$.

Proof Assume that $\bar{x}$ is a solution of the problem (P) but not a solution of $(\text{VIP})_h$. Then there exists $y \in K$ such that

$$h(\bar{x}; y - \bar{x}) < 0. \tag{1.30}$$

From (1.29), we reach to a contradiction to our assumption that $\bar{x}$ is a solution of the problem (P). Hence, $\bar{x}$ is a solution of $(\text{VIP})_h$. □

Next we establish that a solution of the Minty variational inequality problem $(\text{MVIP})_h$ is an optimal solution of the problem (P) under specific assumptions.

Theorem 1.45 *Let K be a nonempty convex subset of $\mathbb{R}^n$, $f : K \to \mathbb{R}$ be a radially lower semicontinuous function and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfy condition* (1.28) *and be positively homogeneous in the second argument. If $\bar{x} \in K$ is a solution of* $(\text{MVIP})_h$*, then it is a solution of the problem* (P).

Proof Let $\bar{x} \in K$ be a solution of $(\text{MVIP})_h$. Then

$$h(y; \bar{x} - y) \leq 0, \quad \text{for all } y \in K. \tag{1.31}$$

Let $y \in K$, $y \neq \bar{x}$ be arbitrary. Since f is radially lower semicontinuous, by Theorem 1.31, there exists $\theta \in [0, 1[$ such that for $w = y + \theta(\bar{x} - y)$, we have

$$D_+ f(w; \bar{x} - y) \geq f(\bar{x}) - f(y). \tag{1.32}$$

As $\theta < 1$, by the positive homogeneity of h in the second argument, we have from relation (1.32) and the condition (1.28) that

$$(1-\theta)^{-1} h(w; \bar{x} - w) \geq f(\bar{x}) - f(y).$$

From (1.31), we have

$$0 \geq h(w; \bar{x} - w) \geq (1-\theta)(f(\bar{x}) - f(y)),$$

and as $\theta < 1$, it follows that $f(\bar{x}) - f(y) \leq 0$. Since $y \in K$ was arbitrary, it follows that $\bar{x}$ is a solution of problem (P). □

As in the differentiable case, the problem $(\text{MVIP})_h$ is a necessary optimality condition under the assumption of the convexity (or pseudoconvexity) of f.

Theorem 1.46 *Let K be a nonempty convex subset of $\mathbb{R}^n$, $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a bifunction and $f : K \to \mathbb{R}$ be a h-convex function. If $\bar{x} \in K$ is solution of problem* (P)*, then it solves* $(\text{MVIP})_h$.

Proof Since f is h-convex, we have

$$f(\bar{x}) - f(y) - h(y; \bar{x} - y) \geq 0, \quad \text{for all } y \in K.$$

Since $\bar{x}$ is a solution of problem (P), we obtain

$$0 \geq f(\bar{x}) - f(y) \geq h(y; \bar{x} - y), \quad \text{for all } y \in K,$$

thus $\bar{x}$ solves $(\text{MVIP})_h$. □

In the following theorem, we relax the h-convexity assumption but we add some other assumptions.

Theorem 1.47 *Let K be a nonempty convex subset of $\mathbb{R}^n$, $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfy the condition*

$$\forall\, x \in K,\ d \in \mathbb{R}^n : \ h(x; d) \leq D^+ f(x; d) \tag{1.33}$$

and be positively homogeneous and subodd in the second argument and $f : K \to \mathbb{R}$ be a h-pseudoconvex function. If $\bar{x} \in K$ is solution of problem (P)*, then it solves* $(\text{MVIP})_h$.

Proof Since $\bar{x}$ is a solution of problem (P), we have

$$f(\bar{x}) \leq f(y), \quad \text{for all } y \in K.$$

By Lemma 1.7(a), f is h-quasiconvex and hence,

$$h(y; \bar{x} - y) \leq 0, \quad \text{for all } y \in K,$$

thus $\bar{x}$ solves $(\text{MVIP})_h$. □

We close this section by giving the following existence result for a solution of $(\text{VIP})_h$.

Theorem 1.48 ([6, Theorem 6.8]) *Let K be a nonempty compact convex subset of $\mathbb{R}^n$ and $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a pseudomonotone bifunction such that h is proper subodd in the second argument and the function $x \mapsto h(y; x - y)$ is lower semicontinuous. Then $(\text{MVIP})_h$ has a solution $\bar{x} \in K$. Furthermore, if h is upper sign continuous and positively homogeneous as well as subodd in the second argument, then $\bar{x} \in K$ is a solution of $(\text{VIP})_h$.*

For a thorough study on nonsmooth variational inequalities, we refer to [6].

1.5.2 Generalized Variational Inequalities

Let X be a topological vector space with its dual X^*, K be a nonempty subset of X, $F : K \to 2^{X^*}$ be a set-valued map with nonempty values. The *generalized variational inequality problem* (GVIP) is to find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that

$$\langle \bar{u}, y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.34}$$

An element $\bar{x} \in K$ is said to be a *strong solution* of GVIP if there exists $\bar{u} \in F(\bar{x})$ such that the inequality (1.34) holds.

The weak form of the GVIP is the problem of finding $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{u} \in F(\bar{x})$ satisfies

$$\langle \bar{u}, y - \bar{x} \rangle \geq 0. \tag{1.35}$$

It is called a *weak generalized variational inequality problem* (WGVIP). An element $\bar{x} \in K$ is said to be a *weak solution* of GVIP if for each $y \in K$, there exists $\bar{u} \in F(\bar{x})$ such that the inequality (1.35) holds. It should be noted that $\bar{u}$ in WGVIP depends on y. Of course, if F is a single-valued map, then both the problems mentioned above reduce to the variational inequality problem (1.16).

Clearly, every strong solution of GVIP is a weak solution. However, the converse is not true in general, see, for example, Example 8.1 in [6].

For the next result, we need the following theorem.

Theorem 1.49 (Kneser Minimax Theorem) [95] *Let K be a nonempty convex subset of a vector space X and D be a nonempty compact convex subset of a topological vector space Y. Suppose that $f : K \times D \to \mathbb{R}$ is lower semicontinuous* *and convex in the second argument and concave in the first argument. Then*

$$\min_{y \in D} \sup_{x \in K} f(x, y) = \sup_{x \in K} \min_{y \in D} f(x, y).$$

The following lemma says that every weak solution of GVIP is a strong solution if the set-valued map F is nonempty, compact and convex valued.

Lemma 1.23 *Let K be a nonempty convex subset of X and $F : K \to 2^{X^*}$ be a set-valued map such that for each $x \in K$, $F(x)$ is nonempty, compact and convex. Then every weak solution of* GVIP *is a strong solution.*

Proof Let $\bar{x} \in K$ be a weak solution of GVIP. Then for each $y \in K$, there exists $\bar{u} \in F(\bar{x})$ such that

$$\langle \bar{u}, \bar{x} - y \rangle \leq 0,$$

that is,

$$\inf_{u \in F(\bar{x})} \langle u, \bar{x} - y \rangle \leq 0, \quad \text{for all } y \in K.$$

Define a functional $f : K \times F(\bar{x}) \to \mathbb{R}$ by

$$f(y, u) := \langle u, \bar{x} - y \rangle.$$

Then for each $y \in K$, the real-valued functional $u \mapsto f(y, u)$ is lower semicontinuous and convex, and for each $u \in F(\bar{x})$, the functional $y \mapsto f(y, u)$ is concave. Since $F(\bar{x})$ is compact and convex, by Theorem 1.49, we have

$$\inf_{u \in F(\bar{x})} \sup_{y \in K} \langle u, \bar{x} - y \rangle = \sup_{y \in K} \inf_{u \in F(\bar{x})} \langle u, \bar{x} - y \rangle \leq 0. \tag{1.36}$$

Since $F(\bar{x})$ is compact, there exists $\bar{u} \in F(\bar{x})$ such that

$$\sup_{y \in K} \langle \bar{u}, \bar{x} - y \rangle \leq 0,$$

and hence

$$\langle \bar{u}, y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K.$$

□

If $K = X$, then clearly, WGVIP reduces to the following *set-valued inclusion problem* : Find $\bar{x} \in X$ such that

$$\mathbf{0} \in F(\bar{x}). \tag{1.37}$$

We consider the generalized complementarity problem which is one of the most important problems from operations research. For details on complementarity problems and their generalizations, we refer [2, 56, 64, 74, 75, 87–89, 130, 131, 135] and the references therein.

Let K be a convex cone in X with its dual cone $K^* = \{u \in X^* : \langle u, x \rangle \geq 0$ for all $x \in K\}$. The *generalized complementarity problem* (GCP) is to find $\bar{x} \in K$ and $\bar{u} \in F(\bar{x})$ such that

$$\bar{u} \in K^* \quad \text{and} \quad \langle \bar{u}, \bar{x} \rangle = 0. \tag{1.38}$$

Proposition 1.16 ([6, Proposition 8.1]) $(\bar{x}, \bar{u})$ *is a solution of* GVIP *if and only if it is a solution of* GCP.

Let K be a nonempty subset of a normed space X and $T : K \to 2^K$ be a set-valued map with nonempty values. The *set-valued fixed point problem* (in short, SVFPP) associated with T is to find $\bar{x} \in K$ such that

$$\bar{x} \in T(\bar{x}). \tag{1.39}$$

The point $\bar{x} \in K$ is called a *fixed point* of T if the relation (1.39) holds. This problem can be converted into a generalized variational inequality formulation as shown below in the set-valued version of Proposition 1.15.

Proposition 1.17 ([6, Proposition 8.2]) *Let K be a nonempty subset of a normed space X and $T : K \to 2^K$ be a set-valued map with nonempty values. If the set-valued map $F : K \to 2^X$ is defined by*

$$F(x) = x - T(x), \tag{1.40}$$

then an element $\bar{x} \in K$ is a strong solution of GVIP (1.34) *if and only if it is a fixed point of T.*

Let K be a nonempty convex subset of a Banach space X and $f : K \to \mathbb{R}$ be a function. Consider the following optimization problem:

$$\text{minimize } f(x), \quad \text{subject to } x \in K. \tag{1.41}$$

The following result shows that the GVIP with $F(x) = \partial f(x)$, the subdifferential of a convex function f, is a necessary and sufficient optimality condition for the optimization problem (1.41).

Proposition 1.18 *Let K be a nonempty convex subset of a Banach space X and $f : K \to \mathbb{R}$ be a convex function. If $\bar{x} \in K$ is a solution of the minimization problem (1.41), then it is a strong solution of* GVIP *with $F(x) = \partial f(x)$ for all $x \in K$. Conversely, if $(\bar{x}, \bar{u})$ is a solution of* GVIP *with $\bar{u} \in \partial f(\bar{x})$, then $\bar{x}$ solves the optimization problem (1.41).*

Proof Let $\bar{x} \in K$ be a solution of the minimization problem (1.41). Then, $f(\bar{x}) \leq f(y)$ for all $y \in K$. By the definition of subdifferential of a convex function, $\mathbf{0} \in \partial f(\bar{x})$. Hence, $(\bar{x}, \mathbf{0})$ is a solution of GVIP, that is, $\bar{x}$ is a strong solution of GVIP with $F(x) = \partial f(x)$ for all $x \in K$.

Conversely, assume that $(\bar{x}, \bar{u})$ is a solution of GVIP with $F(x) = \partial f(x)$ for all $x \in K$. Then $\bar{u} \in \partial f(\bar{x})$ and

$$\langle \bar{u}, y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.42}$$

Since $\bar{u} \in \partial f(\bar{x})$, we have

$$\langle \bar{u}, y - \bar{x} \rangle \leq f(y) - f(\bar{x}), \quad \text{for all } y \in X. \tag{1.43}$$

By combining inequalities (1.42) and (1.43), we obtain

$$f(\bar{x}) \leq f(y), \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of the minimization problem (1.41). □

It can be easily seen that if $\bar{x}$ is a weak solution of GVIP, even then it is a solution of the minimization problem (1.41).

Theorem 1.50 ([130]) *Let K be a nonempty compact convex subset of $\mathbb{R}^n$ and $F : K \to 2^{\mathbb{R}^n}$ be an upper semicontinuous set-valued map such that for each $x \in K$, $F(x)$ is nonempty, compact and convex. Then there exists a solution $(\bar{x}, \bar{u})$ of* GVIP.

If K is not necessarily bounded, then we have the following result.

Theorem 1.51 ([6, Theorem 8.2]) *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $F : K \to 2^{\mathbb{R}^n}$ be an upper semicontinuous set-valued map such that for each $x \in K$, $F(x)$ is nonempty, compact and convex. If there exist an element $\tilde{y} \in K$ and a constant $r > \|\tilde{y}\|$ such that*

$$\max_{u \in F(x)} \langle u, \tilde{y} - x \rangle \leq 0, \tag{1.44}$$

for all $x \in K$ with $\|x\| = r$, then there exists a solution $(\bar{x}, \bar{u})$ of GVIP.

Theorems 1.50 and 1.51 also hold in the setting of Banach spaces. Some existence results for a solution of GVIP under the assumption that the underlying set K is convex but neither bounded nor closed, are derived in [62].

The following problem is the set-valued version of the Minty variational inequality problem, known as *generalized Minty variational inequality problem* (in short, GMVIP): Find $\bar{x} \in K$ such that for all $y \in K$ and all $v \in F(y)$, we have

$$\langle v, y - \bar{x} \rangle \geq 0. \tag{1.45}$$

A weak form of the generalized Minty variational inequality problem is the following problem which is called *weak generalized Minty variational inequality problem* (WGMVIP): Find $\bar{x} \in K$ such that for all $y \in K$, there exists $v \in F(y)$ satisfying the inequality (1.45).

A solution of WGMVIP is called a *weak solution* of GMVIP. It is clear that every solution of GMVIP is a weak solution of GMVIP.

The following result provides a necessary and sufficient condition for a solution of the minimization problem (1.41).

Proposition 1.19 *Let K be a nonempty convex subset of a Banach space X and $f : K \to \mathbb{R}$ be a convex function. Then $\bar{x} \in K$ is a solution of the minimization problem* (1.41) *if and only if it is a solution of* GMVIP (1.45) *with $F(x) = \partial f(x)$.*

Proof Let $\bar{x} \in K$ be a solution of GMVIP (1.45) but not a solution of minimization problem (1.41). Then there exists $z \in K$ such that

$$f(z) < f(\bar{x}). \tag{1.46}$$

By Theorem 1.32, there exist $\lambda \in]0, 1[$ and $v \in \partial f(z(\lambda))$, where $z(\lambda) = \lambda z + (1 - \lambda)\bar{x}$, such that

$$\langle v, z - \bar{x} \rangle = f(z) - f(\bar{x}). \tag{1.47}$$

By combining (1.46) and (1.47), we obtain

$$\langle v, z - \bar{x} \rangle < 0.$$

Since $\lambda(z - \bar{x}) = z(\lambda) - \bar{x}$, we have

$$\langle v, z(\lambda) - \bar{x} \rangle < 0,$$

a contradiction to our supposition that $\bar{x}$ is a solution of GMVIP (1.45).

Conversely, suppose that $\bar{x} \in K$ is a solution of the minimization problem (1.41). Then we have

$$f(y) - f(\bar{x}) \geq 0, \quad \text{for all } y \in K. \tag{1.48}$$

Since f is convex, we deduce that

$$\langle v, y - \bar{x} \rangle \geq f(y) - f(\bar{x}), \quad \text{for all } y \in K \text{ and all } y \in \partial f(y). \tag{1.49}$$

From inequalities (1.48) and (1.49), it follows that $\bar{x}$ is a solution of GMVIP (1.45). □

Definition 1.55 Let K be a nonempty convex subset of a topological vector space X. A set-valued map $F : K \to 2^X$ is said to be *generalized hemicontinuous* if for any $x, y \in K$ and for all $\lambda \in [0, 1]$, the set-valued map

$$\lambda \mapsto \langle F(x + \lambda(y - x)), y - x \rangle = \bigcup_{w \in F(x+\lambda(y-x))} \langle w, y - x \rangle$$

is upper semicontinuous at **0**.

Definition 1.56 Let K be a nonempty convex subset of a topological vector space X. A set-valued map $F : K \to 2^{X^*}$ is said to be *generalized pseudomonotone* if for every pair of distinct points $x, y \in K$ and for any $u \in F(x)$ and $v \in F(y)$, we have

$$\langle u, y - x\rangle \geq 0 \quad \Rightarrow \quad \langle v, y - x\rangle \geq 0.$$

F is called *generalized weakly pseudomonotone* if for every pair of distinct points $x, y \in K$ and for any $u \in F(x)$, we have

$$\langle u, y - x\rangle \geq 0 \quad \Rightarrow \quad \langle v, y - x\rangle \geq 0 \text{ for some } v \in F(y).$$

Now we present some existence results for solutions of GVIP under different kinds of generalized monotonicities.

The following result which was established by Konnov and Yao [103], is a set-valued version of the Minty lemma.

Lemma 1.24 (Generalized Linearization Lemma) [6, Lemma 8.2] *Let K be a nonempty convex subset of a topological vector space X and $F : K \to 2^{X^*}$ be a set-valued map with nonempty values. The following assertions hold.*

(a) *If F is generalized hemicontinuous, then every solution of* WGMVIP *is a solution of* WGVIP.
(b) *If F is generalized pseudomonotone, then every solution of* WGVIP *is a solution of* GMVIP.
(c) *If F is generalized weakly pseudomonotone, then every solution of* WGVIP *is a solution of* WGMVIP.

Theorem 1.52 *Let K be a nonempty compact convex subset of a Banach space X and $F : K \to 2^{X^*}$ be a generalized pseudomonotone and generalized hemicontinuous set-valued map such that for each $x \in K$, $F(x)$ is nonempty. Then there exists a solution $\bar{x} \in K$ of* WGVIP. *If, in addition, the set $F(\bar{x})$ is also compact and convex, then $\bar{x} \in K$ is a strong solution of* GVIP.

Proof For each $y \in K$, define two set-valued maps $S, T : K \to 2^K$ by

$$S(y) = \{x \in K : \exists u \in F(x),\ \langle u, y - x\rangle \geq 0\},$$

and

$$T(y) = \{x \in K : \forall v \in F(y),\ \langle v, y - x\rangle \geq 0\},$$

respectively. We divide the proof into five steps.

(i) We claim that S is a KKM map, that is, the convex hull $\mathrm{co}(\{y_1, y_2, \ldots, y_m\})$ of every finite subset $\{y_1, y_2, \ldots, y_m\}$ of K is contained in the corresponding union $\bigcup_{i=1}^{m} S(y_i)$.

Let $\hat{x} \in \text{co}(\{y_1, y_2, \ldots, y_m\})$. Then

$$\hat{x} = \sum_{i=1}^{m} \lambda_i y_i, \quad \text{for some } \lambda_i \geq 0 \text{ with } \sum_{i=1}^{m} \lambda_i = 1.$$

If $\hat{x} \notin \bigcup_{i=1}^{m} S(y_i)$, then for all $w \in F(\hat{x})$,

$$\langle w, y_i - \hat{x} \rangle < 0, \quad \text{for all } i = 1, 2, \ldots, m.$$

For all $w \in F(\hat{x})$, it follows that

$$\begin{aligned} 0 &= \langle w, \hat{x} - \hat{x} \rangle \\ &= \left\langle w, \sum_{i=1}^{m} \lambda_i y_i - \sum_{i=1}^{m} \lambda_i \hat{x} \right\rangle \\ &= \left\langle w, \sum_{i=1}^{m} \lambda_i (y_i - \hat{x}) \right\rangle \\ &= \sum_{i=1}^{m} \lambda_i \langle w, y_i - \hat{x} \rangle < 0, \end{aligned}$$

which is a contradiction. Therefore, we must have

$$\text{co}(\{y_1, y_2, \ldots, y_m\}) \subseteq \bigcup_{i=1}^{m} S(y_i),$$

and hence, S is a KKM map.

(ii) We show that $S(y) \subseteq T(y)$ for all $y \in K$, and hence T is a KKM map.

By generalized pseudomonotonicity of F, we have that $S(y) \subseteq T(y)$ for all $y \in K$. Since S is a KKM map, so is T.

(iii) We assert that $\bigcap_{y \in K} S(y) = \bigcap_{y \in K} T(y)$.

From step (ii), we have

$$\bigcap_{y \in K} S(y) \subseteq \bigcap_{y \in K} T(y),$$

and from Lemma 1.24, we have

$$\bigcap_{y \in K} S(y) \supseteq \bigcap_{y \in K} T(y).$$

Therefore, the conclusion follows.

(iv) We prove that for each $y \in K$, $T(y)$ is a closed subset of K.

For any fixed $y \in K$, let $\{x_m\}$ be a sequence in $T(y)$ such that $x_m \to \tilde{x} \in K$. Since $x_m \in T(y)$, for all $v \in F(y)$, we have $\langle v, y - x_m \rangle \geq 0$ for all m. As $\langle v, y - x_m \rangle$ converges to $\langle v, y - \tilde{x} \rangle$, therefore $\langle v, y - \tilde{x} \rangle \geq 0$, and hence, $\tilde{x} \in T(y)$. Consequently, $T(y)$ is closed.

(v) Finally, we show that the WGVIP is solvable.

From step (iv), $T(y)$ is a closed subset of the compact set K, and hence, it is compact. By step (ii) and Lemma 1.14, we have $\bigcap_{y \in K} T(y) \neq \emptyset$. Consequently, by step (iii), we also have $\bigcap_{y \in K} S(y) \neq \emptyset$. Hence, there exists $\bar{x} \in K$ such that

$$\forall y \in K,\ \exists \bar{u} \in F(\bar{x}) \ : \ \langle \bar{u}, y - \bar{x} \rangle \geq 0. \tag{1.50}$$

Thus, $\bar{x}$ is a solution of WGVIP.

If, in addition, the set $F(\bar{x})$ is also compact and convex, then by Lemma 1.23, $\bar{x} \in K$ is a strong solution of GVIP. □

1.6 Equilibrium Problems

Investigations of equilibrium states of a system play a central role in such diverse fields as economics, mechanics, biology and social sciences. There are many general mathematical problems which were suggested for modeling and studying various kinds of equilibria. Many researchers were / are considering these problems in order to obtain existence and uniqueness results and to propose solution methods. The Ky Fan [59–61] type inequality is one of such problems, which plays an important role in the theory of nonlinear analysis and optimization. It was W. Oettli who coined the name "Equilibrium Problem" to the Ky Fan type inequality, perhaps, because it is equivalent to find the equilibrium point of an optimization problem under certain conditions. The mathematical formulation of an equilibrium problem (in short, EP) is to find an element $\bar{x}$ of a set K such that

$$f(\bar{x}, y) \geq 0, \quad \text{for all } y \in K, \tag{1.51}$$

where $f : K \times K \to \mathbb{R}$ is a bifunction such that $f(x, x) \geq 0$ for all $x \in K$. It seems the most general problem and includes other equilibrium type ones such as optimization problem, saddle point problem, fixed point problem, complementarity problems, variational inequality problems, Nash equilibrium problem, etc. In this general form, EP was first considered by H. Nikaido and K. Isoda [119] as an auxiliary problem to establish existence results for Nash equilibrium points in non-cooperative games [117, 118]. This transformation allows one to extend various iterative methods, which were proposed for saddle point problems, for the case of EP. In the theory of EPs, the key contribution was made by Ky Fan [59–61], whose new existence results contained the original technique which became a basis for most further existence theorems in topological spaces. The work of Ky Fan perhaps motivated by the min-max problems appearing in economic equilibrium. Within the context of calculus of variations, motivated mainly by the work of Stampacchia [132], there arises the work of Brézis, Niremberg and Stampacchia [31] establishing a more general result than that in [61]. After the work of Blum and Oettli [29], it emerged as a new direction of research in nonlinear analysis, optimization, optimal control, game theory, mathematical economics, etc.

Example 1.20

(a) **Minimization Problem.** Let K be a nonempty set and $\varphi : K \to \mathbb{R}$ be a real-valued function. The *minimization problem* (in short, MP) is to find $\bar{x} \in K$ such that

$$\varphi(\bar{x}) \le \varphi(y), \quad \text{for all } y \in K. \tag{1.52}$$

If we set $f(x, y) = \varphi(y) - \varphi(x)$ for all $x, y \in K$, then MP is equivalent to EP.

(b) **Saddle Point Problem.** Let K_1 and K_2 be nonempty sets and $\ell : K_1 \times K_2 \to \mathbb{R}$ be a real-valued bifunction. The *saddle point problem* (in short, SPP) is to find $(\bar{x}_1, \bar{x}_2) \in K_1 \times K_2$ such that

$$\ell(\bar{x}_1, y_2) \le \ell(y_1, \bar{x}_2), \quad \text{for all } (y_1, y_2) \in K_1 \times K_2. \tag{1.53}$$

Set $K := K_1 \times K_2$ and define $f : K \times K \to \mathbb{R}$ by

$$f((x_1, x_2), (y_1, y_2)) = \ell(y_1, x_2) - \ell(x_1, y_2) \tag{1.54}$$

for all $(x_1, x_2), (y_1, y_2) \in K_1 \times K_2$. Then SPP coincides with EP.

(c) **Nash Equilibrium Problem.** Let $I = \{1, 2, \ldots, m\}$ be the set of players. For each player $i \in I$, let K_i be the strategy set of the ith player. Let $K = \prod_{i=1}^{m} K_i$. For every player $i \in I$, let $\varphi_i : K \to \mathbb{R}$ be the loss function of the ith player, depending on the strategies of all players. For $x = (x_1, x_2, \ldots, x_m) \in K$, we define $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. Then $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m) \in K$ is called a *Nash equilibrium point* if for all $i \in I$,

$$\varphi_i(\bar{x}) \le \varphi_i(\bar{x}^i, y_i), \quad \text{for all } y_i \in K_i. \tag{1.55}$$

This means that no player can reduce his loss by varying his strategy alone. We now define

$$f(x, y) = \sum_{i=1}^{m} \left(\varphi_i(x^i, y_i) - \varphi_i(x) \right).$$

For such f, EP coincides with *Nash equilibrium problem* (in short, NEP) of finding $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m) \in K$ such that (1.55) holds. Indeed, if (1.55) holds for all $i \in I$, obviously (1.51) is fulfilled. If, for some $i \in I$, we choose $y \in K$

such that $y^i = \bar{x}^i$, then

$$f(\bar{x}, y) = \varphi_i\left(\bar{x}^i, y_i\right) - \varphi_i(\bar{x}).$$

Thus EP implies NEP.

(d) **Fixed Point Problem.** Let X be an inner product space whose inner product is denoted by $\langle ., . \rangle$. Let K be a nonempty subset of X and $\varphi : K \to K$ be a given mapping. The *fixed point problem* (in short, FPP) is to find $\bar{x} \in K$ such that $\varphi(\bar{x}) = \bar{x}$.

Setting $f(x, y) = \langle x - \varphi(x), y - x \rangle$. Then $\bar{x}$ is a solution of FPP if and only if it is a solution of EP.

Indeed, FPP implies EP is obvious. If EP is satisfied, then by choosing $y = \varphi(\bar{x})$, we obtain

$$0 \leq f(\bar{x}, \varphi(\bar{x})) = -\|\bar{x} - \varphi(\bar{x})\|^2.$$

(e) **Variational Inequality Problem.** Let X, X^*, K and F be the same as in the formulation of variational inequality problem defined by (1.16). We set $f(x, y) = \langle F(x), y - x \rangle$ for all $x, y \in K$. Then VIP is equivalent to EP.

Let K and h be the same as defined in the formulation of nonsmooth variational inequality problem $(\text{VIP})_h$. If we define $f(x, y) = h(x; y - x)$, then $(\text{VIP})_h$ is equivalent to EP.

For further details on different special cases of EP, we refer to [3, 29, 65, 66, 80, 91–93] and the references therein.

Most of the results on the existence of solutions for equilibrium problems are derived in the setting of topological vector spaces either by using Browder type or Kakutani type fixed point theorems or by using Fan-KKM type theorems. Blum, Oettli and Théra [29, 120] have studied the existence of solutions of equilibrium problems in the setting of complete metric spaces inspired by the well-known Ekeland's variational principle [53, 54]. They extended Ekelend's variational principle for bifunctions and established several equivalent formulations, namely, Takahashi's minimization theorem [133] and Caristi-Kirk's fixed point theorem [34]. After the work of Blum, Oettli and Théra, several people have started working in this direction and established existence results for solutions of equilibrium problems in different settings or under different assumptions, see, for example, [1, 3–5, 26, 90, 106, 122, 123] and the references therein.

For solution methods for equilibrium problems, we refer to [51, 79, 99–101, 104, 109, 144] and the references therein.

Let X be a topological vector space, K be a nonempty convex subset of X and $f : K \times K \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in K$. A problem closely related to EP is the following problem, called *dual equilibrium problem* (in short, DEP) or *Minty equilibrium problem* (in short, MEP): find $\bar{x} \in K$ such that

$$f(y, \bar{x}) \leq 0, \quad \text{for all } y \in K. \tag{1.56}$$

Konnov and Schaible [102] defined the duality for equilibrium problems by using the rule that the dual of the dual is the primal, and used dual equilibrium problem. They proposed various duals of EP. The duality of equilibrium problems is also studied by Martínez-Legaz and Sosa [111] and Bigi et al. [27] but by using different approaches. However, Mastroeni [112] studied gap functions for equilibrium problems which convert an equilibrium problem to an optimization problem.

When $f(x, y) = g(x, y) + h(x, y)$ for all $x, y \in K$ with $g, h : K \times K \to \mathbb{R}$ such that $g(x, x) \geq 0$ and $h(x, x) = 0$ for all $x \in K$, then EP reduces to find $\bar{x} \in K$ such that

$$g(\bar{x}, y) + h(\bar{x}, y) \geq 0, \quad \text{for all } y \in K. \tag{1.57}$$

It was first proposed by Blum and Oettli [29] and further studied by Chadli et al. [38, 39], Kalmoun [83] and Chadli et al. [40] with applications to eigenvalue problems, hemivariational inequalities and anti-periodic solutions for nonlinear evolution equations, see also [35, 37] and the references therein.

Let $l : K \times K \to \mathbb{R}$ be a function. The *implicit variational problem* (for short, IVP) is to find $\bar{x} \in K$ such that

$$l(\bar{x}, \bar{x}) + g(\bar{x}, \bar{x}) \geq l(\bar{x}, y) + g(\bar{x}, y), \quad \text{for all } y \in K. \tag{1.58}$$

It is considered and studied by Mosco [114] and it contains EP (1.51) and (1.57) as special cases. It also includes variational and quasi-variational inequalities [18], fixed point problem and saddle point problem, Nash equilibrium problem of non-cooperative games as special cases. The existence of solutions of IVP was studied by Mosco [114], while Dolcetta and Matzeu [52] discussed its duality and applications.

Let $F, G : K \to \mathcal{L}(X, Y)$ be nonlinear operators. Set

$$l(x, y) = \langle F(x), y - x \rangle \quad \text{and} \quad g(x, y) = \langle G(x), y - x \rangle, \quad \text{for all } x, y \in K.$$

Then IVP reduces to the problem of finding $\bar{x} \in K$ such that

$$\langle F(\bar{x}) + G(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K. \tag{1.59}$$

It is known as *strongly nonlinear variational inequality problem* (in short, SNVIP).

Now we present some basic results on the existence of solutions for EP (1.51).

Theorem 1.53 *Let K be a nonempty convex subset of a Hausdorff topological vector space X and $f : K \times K \to \mathbb{R}$ be a bifunction vanishing on the diagonal, i.e. $f(x, x) = 0$ for all $x \in K$ such that the following conditions hold.*

(i) *f is quasiconvex in the second variable;*
(ii) *$\liminf_{x \to x^*} f(x, y) \leq f(x^*, y)$ for all $y \in K$ whenever $x \to x^* \in K$;*
(iii) *There exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$, there exists $\tilde{y} \in B$ such that $f(x, \tilde{y}) < 0$.*

Then EP (1.51) *has a solution in K.*

Proof For each $y \in K$, define

$$P(y) = \{x \in K : f(x, y) \geq 0\}.$$

Then the solution set of EP (1.51) is $\mathbb{S} = \bigcap_{y \in K} P(y)$. By condition (ii), for each $y \in K$, $P(y)$ is closed.

Now we prove that the solution set $\mathbb{S}$ is nonempty. Assume contrary that $\mathbb{S} = \emptyset$. Then for each $x \in K$, the set

$$S(x) := \{y \in K : x \notin P(y)\} = \{y \in K : f(x, y) < 0\rangle\} \neq \emptyset.$$

By quasiconvexity of f in the second variable, we have that $S(x)$ is convex for each $x \in K$. Thus, $S : K \to 2^K$ defines a set-valued map such that for each $x \in K$, $S(x)$ is nonempty and convex. Now for each $y \in K$, the set

$$\begin{aligned} S^{-1}(y) &= \{x \in K : y \in S(x)\} = \{x \in K : f(x, y) < 0\} \\ &= \{x \in K : f(x, y) \geq 0\}^c = [P(y)]^c \end{aligned}$$

is open in K. Then the set-valued map $S : K \to 2^K$ satisfies all the conditions of Corollary 1.3 (with $S = T$), and therefore, there exists a point $\hat{x} \in K$ such that $\hat{x} \in S(\hat{x})$, that is, $0 = f(\hat{x}, \hat{x}) < 0$, which is a contradiction. Hence the solution set $\mathbb{S}$ of EP (1.51) is nonempty. □

Allen [2] also proved a similar result with different coercivity condition (iii) but by using Fan-KKM Lemma. If K is compact, then the condition (iii) in the above theorem is satisfied. Therefore, if K is compact and f is upper semicontinuous in the first argument, then Theorem 1.53 reduces to the well-known Ky Fan theorem [61].

The following result is a slight generalization of a particular form of Theorem 10 in [37].

Theorem 1.54 *Let X be a Hausdorff topological vector space, K be a closed convex subset of X and $f : K \times K \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in K$. Suppose that*

(i) *for each finite subset E of K, $\min\limits_{x \in \mathrm{co}(E)} \max\limits_{y \in E} f(x, y) \geq 0$;*
(ii) *for each fixed $y \in K$, the function $x \mapsto f(x, y)$ is upper semicontinuous;*
(iii) *there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$, there exists $\tilde{y} \in B$ such that $f(x, \tilde{y}) < 0$.*

Then EP (1.51) *has a solution.*

Now we present a theorem which will be used in the sequel.

Theorem 1.55 ([29, Lemma 1]) *Let X be a Hausdorff topological vector space, K be a nonempty compact convex subset of X, D be a convex subset of X and $f : K \times D \to \mathbb{R}$ be concave and upper semicontinuous in the first argument, and convex* *in the second argument. Assume that*

$$\max_{x\in K} f(x, y) \geq 0, \quad \textit{for all } y \in D.$$

Then there exists $\bar{x} \in K$ *such that* $f(\bar{x}, y) \geq 0$ *for all* $y \in D$.

Proof Assume contrary that the conclusion does not hold. Then for every $x \in K$, there exist $y \in D$ and $\varepsilon > 0$ such that $f(x, y) < -\varepsilon$. Therefore, the open sets

$$S_\varepsilon(y) := \{x \in K : f(x, y) < -\varepsilon\}, \quad \text{for } y \in D,\ \varepsilon > 0,$$

cover the compact set K. Hence there exists a finite subcover $\{S_{\varepsilon_i}(y_i)\}_{i=1}^m$ of K. Let $\varepsilon := \min_{1\leq i\leq m} \varepsilon_i$. Then from the fact that $K \subseteq \bigcup_{i=1}^m S_{\varepsilon_i}(y_i)$, we have

$$\min_{1\leq i\leq m} f(x, y_i) \leq -\varepsilon, \quad \text{for all } x \in K.$$

Since the function $x \mapsto f(x, y_i)$ is concave, it follows from [127, Theorem 21.1] that there exist real numbers $\mu_i \geq 0$ for $i = 1, 2, \ldots, m$ with $\sum_{i=1}^m \mu_i = 1$ such that $\sum_{i=1}^m f(x, y_i) \leq -\varepsilon$ for all $x \in K$. The convexity of $y \mapsto f(x, y)$ implies with $\tilde{y} := \sum_{i=1}^m \mu_i y_i \in D$ that $f(x, \tilde{y}) \leq -\varepsilon$ for all $x \in K$. Hence $\max_{x\in K} f(x, \tilde{y}) < 0$, a contradiction of our hypothesis. □

Definition 1.57 ([146]) Let K be a nonempty convex subset of a topological vector space X. A bifunction $f : K \times K \to \mathbb{R}$ is said to be *diagonally quasiconvex* in y if for any finite set $\{y_1, y_2, \ldots, y_m\} \subset K$ and any $x_0 \in \operatorname{co}(\{y_1, y_2, \ldots, y_m\})$, we have

$$f(x_0, x_0) \leq \max_{1\leq i\leq m} f(x_0, y_i).$$

f is said to be *diagonally quasiconcave* in y if $-f$ is diagonally quasiconvex in y.

A bifunction $f : K\times K \to \mathbb{R}$ is said to be γ*-diagonally quasiconvex* in y for some $\gamma \in \mathbb{R}$ if for any finite set $\{y_1, y_2, \ldots, y_m\} \subset K$ and any $x_0 \in \operatorname{co}(\{y_1, y_2, \ldots, y_m\})$, we have

$$\gamma \leq \max_{1\leq i\leq m} f(x_0, y_i).$$

f is said to be γ*-diagonally quasiconcave* in y for some $\gamma \in \mathbb{R}$ if $-f$ is $-\gamma$-diagonally quasiconvex in y.

Definition 1.58 ([41]) Let K be a nonempty convex subset of a topological vector space X. A bifunction $f : K \times K \to \mathbb{R}$ is said to be γ*-generalized diagonally quasiconvex* in y if for any finite set $\{y_1, y_2, \ldots, y_m\} \subset K$, there is a finite set $\{x_1, x_2, \ldots, x_m\} \subset K$ such that for any set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subset \{x_1, x_2, \ldots, x_m\}$ and any $x_0 \in \text{co}(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})$, we have

$$\gamma \leq \max_{1 \leq j \leq k} f(x_0, y_{i_j}).$$

f is said to be *γ-generalized diagonally quasiconcave* in y if $-f$ is $-\gamma$-generalized diagonally quasiconvex in y.

Chang and Zhang [41] gave the relation between generalized KKM maps and γ-generalized diagonally quasiconvexity (quasiconcavity).

Proposition 1.20 *Let K be a nonempty convex subset of a topological vector space X, $f : K \times K \to \mathbb{R}$ be a bifunction and $\gamma \in \mathbb{R}$. Then the following statements are equivalent:*

(a) *The set-valued map $T : K \to 2^K$ defined by*

$$T(y) = \{x \in K : f(x, y) \leq \gamma\} \text{ (respectively, } T(y) = \{x \in K : f(x, y) \geq \gamma\})$$

is a generalized KKM map.

(b) *$f(x, y)$ is γ-generalized diagonally quasiconcave (respectively, γ-generalized diagonally quasiconvex) in y.*

Tian [136] introduced the following definition of γ-transfer lower semicontinuous functions.

Definition 1.59 Let X and Y be topological spaces. A bifunction $f : X \times Y \to \mathbb{R}$ is said to be *γ-transfer lower semicontinuous* (respectively, *γ-transfer lower semicontinuous*) function in the first argument for some $\gamma \in \mathbb{R}$ if for all $x \in X$ and $y \in Y$ with $f(x, y) > \gamma$ (respectively, $f(x, y) < \gamma$), there exist a point $z \in Y$ and a neighborhood $N(x)$ of x such that $f(u, z) > \gamma$ (respectively, $f(u, z) < \gamma$) for all $u \in N(x)$.

The bifunction f is said to be to *γ-transfer lower semicontinuous (respectively,* γ-transfer lower semicontinuous*) in the first argument if it is γ-transfer lower semicontinuous (respectively, γ-transfer lower semicontinuous) for every $\gamma \in \mathbb{R}$.*

Ansari et al. [7] established the following minimax inequality theorem.

Theorem 1.56 *Let K be a nonempty closed convex subset of a Hausdorff topological vector space X and $f, g : K \times K \to \mathbb{R}$ be bifunctions such that the following conditions hold.*

(i) *For any fixed $y \in K$, the function $x \mapsto f(x, y)$ is 0-transfer upper semicontinuous.*
(ii) *For any fixed $x \in K$, the function $y \mapsto g(x, y)$ is 0-generalized diagonally quasiconvex.*
(iii) *$f(x, y) \geq g(x, y)$ for all $(x, y) \in K \times K$.*
(iv) *The set $\{x \in K : f(x, y_0) \geq \gamma\}$ is precompact (that is, its closure is compact) for at least one $y_0 \in K$.*

Then there exists a solution $\bar{x} \in K$ of EP (1.51).

Proof Define set-valued maps $S, T : K \to 2^K$ by

$$S(y) = \{x \in K : f(x, y) \geq \gamma\} \quad \text{and} \quad T(y) = \{x \in K : g(x, y) \geq \gamma\},$$

for all $y \in K$. Condition (i) implies that S is a transfer closed-valued map. Indeed, if $x \notin S(y)$, then $f(x, y) < 0$. Since $f(x, y)$ is 0-transfer lower semicontinuous in x, there is a $z \in K$ and a neighborhood $N(x)$ of x such that $f(u, z) < 0$ for all $u \in N(x)$. Then $S(z) \subset K \setminus N(x)$. Hence, $x \in \text{cl}\ (S(z))$. Thus, S is transfer closed-valued.

From condition (ii) and Proposition 1.20, T is a generalized KKM map. From (iii), we have that $T(y) \subset S(y)$ for all $y \in K$, and hence S is also a generalized KKM map. So, cl S is also a KKM map. Condition (iv) implies that $S(y_0)$ is precompact. Hence, cl $S(y_0)$ is compact. By Theorem 1.34,

$$\bigcap_{y \in K} S(y) \neq \emptyset.$$

As a result, there exists $\bar{x} \in K$ such that $f(\bar{x}, y) \geq \gamma$ for all $y \in K$. □

Remark 1.25

(a) If for every fixed $y \in K$, the function $x \mapsto f(x, y)$ is upper semicontinuous in x, then condition (i) of Theorem 1.56 is satisfied immediately.
(b) The following condition implies condition (iv) in Theorem 1.56.
(iv)′ There exist a compact subset D of K and $y_0 \in K$ such that for all $x \in K \setminus D, f(x, y_0) < 0$.

For further details, existence results and applications of equilibrium problems, we refer [1, 3–5, 7, 8, 22–29, 31, 35–41, 46, 47, 49, 51, 52, 57, 58, 65, 66, 71, 72, 77, 78, 80, 83, 84, 90–93, 99–102, 104, 107–109, 111, 112, 120, 128, 135, 142, 143] and the references therein.

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