International Perspectives on the Teaching and Learning of Mathematical Modelling

Gloria Ann Stillman
Werner Blum
Gabriele Kaiser *Editors*

Mathematical Modelling and Applications

Crossing and Researching Boundaries in Mathematics Education

Springer

International Perspectives on the Teaching and Learning of Mathematical Modelling

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ISSN 2211-4920 ISSN 2211-4939 (electronic)
International Perspectives on the Teaching and Learning of Mathematical Modelling
ISBN 978-3-319-62967-4 ISBN 978-3-319-62968-1 (eBook)
https://doi.org/10.1007/978-3-319-62968-1

Library of Congress Control Number: 2017954330

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Series Preface

Applications and modelling and their learning and teaching in schools and universities have become a prominent topic in the last decades in view of the growing worldwide relevance of the usage of mathematics in science, technology and everyday life. There is consensus that modelling should play an important role in mathematics education, and the situation in schools and universities is slowly changing to include real-world aspects, frequently with modelling as real-world problem-solving, in several educational jurisdictions. Given the worldwide impending shortage of students who are interested in mathematics and science, it is essential to discuss accelerating possible changes of mathematics education in school and tertiary education towards the inclusion of real-world examples and the competencies to use mathematics to solve real-world problems.

This innovative book series established by Springer, *International Perspectives on the Teaching and Learning of Mathematical Modelling,* aims at promoting academic discussion on the teaching and learning of mathematical modelling at various educational levels all over the world. The series will publish books from different theoretical perspectives from around the world, dealing with teaching and learning of mathematical modelling in schooling and at tertiary level. This series will also enable the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), an International Commission on Mathematical Instruction-affiliated study group, to publish books arising from its biennial conference series. ICTMA is a unique worldwide group where not only mathematics educators dealing with education at school level are included but also applied mathematicians interested in teaching and learning modelling at tertiary level are represented as well. Four of these books published by Springer have already appeared.

The planned books will display the worldwide state of the art in this field, most recent educational research results and new theoretical developments and will be of interest for a wide audience. Themes dealt with in the books will be teaching and learning of mathematical modelling in schooling and at tertiary level including the usage of technology in modelling, psychological, social and cultural aspects of modelling and its teaching, modelling competencies, curricular aspects, modelling examples and courses, teacher education and teacher education courses. The book series aims to support the discussion on mathematical modelling and its teaching internationally and will promote the teaching and learning of mathematical modelling and research of this field all over the world in schools and universities.

The series is supported by an editorial board of internationally well-known scholars, who bring in their long experience in the field as well as their expertise to this series. The members of the editorial board are Maria Salett Biembengut (Brazil), Werner Blum (Germany), Helen Doerr (USA), Peter Galbraith (Australia), Toshikazu Ikeda (Japan), Mogens Niss (Denmark) and Jinxing Xie (China).

We hope this book series will inspire readers in the present and the future to promote the teaching and learning of mathematical modelling all over the world.

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The original version of this book was revised.
An update to this book can be found at https://doi.org/10.1007/978-3-319-62968-1_53

Chapter 1
Crossing Boundaries in Mathematical Modelling and Applications Educational Research and Practice

Gloria Ann Stillman, Werner Blum, and Gabriele Kaiser

Abstract This chapter gives an overview on the current state-of-the-art on the teaching and learning of mathematical modelling and applications and its contribution to educational research and practice which is reflected in the various contributions in this book. Several chapter authors use the opportunity to strengthen and build our research practices by reaching out to others in educational research, beyond the boundaries of our community, and those in fields other than education. By researchers recognising boundaries in applications and modelling research that limit our vision and what we are currently able to do, a more entrepreneurial view of research groups could lead to the brokerage of knowledge in multidisciplinary or multi-community teams to work on some of the more perplexing research questions that have faced our research community. Fluid social alliances in research groups that coalesce and then disperse could result in a much wider dissemination of knowledge both to, and from, our community in the future.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_1

1.1 Introduction

This volume includes a selection of chapters arising from presentations at the 17th International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA17) which took place in Nottingham, England, in 2015, addressing the theme of *Modelling perspectives: Looking within and across boundaries*. Mathematical modelling and applications have the potential to appeal to a wider audience than they currently do within the field of mathematics education research. Thus, the book is intended to provide a stimulus to consider new approaches drawing on related research in mathematics education and associated domains. Mathematical modelling and applications have inherent qualities fostering multidisciplinary work (Andresen and Petersen 2011) that is required for effective problem-solving in many areas (English and Gainsburg 2016). This book is therefore an opportunity to strengthen and build our research community by reaching out to others in educational research, beyond the boundaries of our community, and those in other fields.

In a research community that values authenticity (Galbraith 2013) and real-world aspects in mathematics education at all levels of education (Blum et al. 2007), there are boundaries coming from the subject researched that should not be crossed. There are, however, boundaries that are artificial and should be crossed. As FitzSimmons and Mitsui (2013) point out, "different communities of practice have different knowledge bases which may prevent the effective communication between them, so that [these] boundaries need to be crossed" (p. 99). A teacher in a classroom deals with multiple audiences (students, administrators, parents) and a different set of tasks (preparing learning activities, facilitating learning, reporting what is learnt in a prescribed manner) from those a researcher does collecting data for a research project or a mathematical modeller in industry, for example, collecting information about constraints for a client brief. The teacher, researcher and mathematician are from distinct communities of practice and social worlds with different viewpoints, but all could be involved in the resolution of the same scientific problem, for example, when modelling how a modeller gets at the nub of a problem so the essence of a real-world situation becomes tractable mathematically.

"Boundary recognition involves making difference which may have previously been implicit, explicit" (Garraway 2010, p. 220). Researchers having a disposition to recognise such boundaries could lead to entrepreneurial research groups brokering knowledge in multidisciplinary or multi-community teams working on some of the most intractable research questions that have faced our research community (e.g. how to facilitate development of problem formulation abilities). According to Star and Griesemer (1989), "consensus is not necessary for cooperation nor for the successful conducting of work" (p. 388). However, as objects and methods can have different meanings in different social worlds, tensions can be generated by trying to ensure integrity of information and reconciling meanings if cooperation is desired, or needed, to advance scientific knowledge. To manage such diversity of viewpoints and simultaneously cooperate, *boundary objects* are used (Star and Griesemer

1989) as a means of translation between worlds. Despite having different meanings (e.g. a table of values; see Psycharis and Potari this volume) in the different worlds (e.g. in the classroom versus in a fuel depot), the structure of the boundary object is common enough to be recognised in the different worlds and become a means of translation between worlds (Star and Griesemer 1989). Boundaries then convey Star's (2010) idea of being indicators of a shared space where different groups work on the resolution of a problem confronting different communities of practice. In this way, knowledge is propagated through fluid social alliances in research groups that coalesce and then disperse, seeding knowledge growth in several different social worlds or communities of practice rather than the knowledge being isolated within one community.

1.2 New Approaches from Crossing Boundaries in Research, Teaching and Practice

New approaches arise in research, teaching and modelling practice from crossing boundaries. In the first part of this book, the included chapters are indicative of the broad spectrum of innovation this can bring.

Even though mathematical modelling has been one of the competencies in the mathematics educational standards in Germany for more than 10 years and there have been several efforts to implement modelling and real-world applications in schooling (Greefrath and Vorhölter 2016), mathematical modelling still plays a minor role in German classrooms. The chapter by *Alfke*, a teacher researcher, is thus of paramount interest to advocates, in many parts of the world, of the inclusion of mathematical modelling in curriculum documents and its incorporation into everyday classroom practice. Alfke reports first results of a research study aimed at supporting students' mathematical modelling competency by the use of increasing learning aids in a self-regulated classroom learning environment, in keeping with the tenets of quality mathematics teaching (Blum 2008). The theoretical background of the work draws from the learning sciences, scaffolding and differentiated instruction (Pea 2004; Jain et al. 2006) and science educational research (Schmidt-Weigand and Di Fuccia 2014) where this type of increasing learning aid has been applied to fostering autonomous, cooperative learning and on encouraging problem-solving competence and specialised self-awareness. Increasing scaffolds are offered to students in the form of aid cards to support their working on a problem, thus freeing up the teacher for more targeted differentiation in the classroom. Thirty seventh-graders were video- and audio-recorded while working on complex modelling problems supported by increasing learning aids and a diagram of the modelling cycle, enhanced to indicate potential areas of difficulty or blockages to progress, as a metacognitive aid. First results point to the use of increasing learning aids for supporting modelling activities increasingly independent from the teacher.

Aymerich, *Gorgorió* and *Albarracín* undertake an interesting diagrammatic analysis of the solution paths of students engaged in mathematical and statistical modelling. They report on the solutions of 22 groups of Year 10 students to a model-eliciting activity, in the tradition of Lesh and Harel (2003), involving interpretation of data. Student groups were asked to see what could be ascertained about the structure of a company based on their mathematical/statistical analysis of data related to salary distribution within the company. In order to analyse student work and characterise models constructed, the authors developed a schematic visualisation tool for these models. The graphs displayed the variety of solutions and the complexity of output from the groups. A graph displayed the mathematical objects present in the solution at each of its nodes, as well as the concepts and procedures involved in the development of the model with segments connecting the concepts displaying relationships detected. This analysis tool distinguished significant differences between student responses. It complements other tools for analysing students' model creation process developed, for example, by Stillman (2002) who extended the response mapping technique of Collis and Watson (1991) to analyse contributors (in terms of cues, concepts and procedures) to lowering rates of success on applications and modelling tasks used in assessment. This tool has since been modified by Matsuzaki (2011) to incorporate display of the influences of prior knowledge, related to mathematics and to reality, on model construction.

Buchholtz broaches the subject of how teachers can promote mathematising by means of mathematical city walks, a kind of mathematics trail excursion crossing the boundary between inside and outside classroom practice. Tasks have to be carefully selected to match students' cognitive skills and to include basic ideas specific to mathematical topics. In the chapter, Buchholtz analyses an example of a task according to the basic ideas contained therein, potential difficulties and possible solutions. He suggests that such tasks in a mathematical city walk can be used diagnostically as well as to provide practice in applying known mathematics and motivating students to engage in real-world applications.

One approach to facilitating a stronger presence of modelling in the learning of mathematics is to look for openings in current curricula and teacher practice to make a platform to build on what is already being done, or has the potential to be done, with existing goals, levers and constraints (Burkhardt 2014). *Caron* and *Pineau* use this approach to work with mathematics and physics teachers, to transform a rich application problem into an engaging activity that allows students to experience some elements of the modelling process while learning or applying specific strategies, concepts and skills, useful for laying bare the underlying structure of the situation (Jensen et al. 2017). The development and first validation of the activity in a transitional mathematics class in a technical engineering school allowed Caron and Pineau to test the extent to which teachers and students could cross the boundaries between application and modelling and between mathematics and physics. The crossing into physics, however, proved to be more difficult to implement than expected.

Mathematical models impact all levels of society, and so mathematical modelling is being seen as an important topic in mathematics education. Representations of mathematical modelling processes are increasingly being used in curriculum documents on national (e.g. Common Core State Standards for Mathematics, Council of Chief State School Officers 2010) and transnational levels (e.g. in PISA, OECD 2013). In their chapter, *Doerr*, *Ärlebäck* and *Misfeldt* critically discuss the dominance of the single image of mathematical modelling that is shown by the *modelling cycle* and offer alternative representations to more fully capture multiple aspects of modelling in mathematics education. They suggest that a complex process such as mathematical modelling should be conveyed in policy and curriculum documents by multiple images. These could accommodate aspects such as student modelling activities that move beyond creating descriptive models validated by comparison to empirical data, to working with a full range of models including models with explanatory power, models with social and political implications and models using computational media.

The notion of productive modelling oriented noticing is proposed by *Galbraith*, *Stillman* and *Brown* within an anticipatory metacognition framework for the implementation of successful modelling. Productive modelling oriented noticing involves modellers noticing what is, or is not, important in order to generate strategies for responding to, or initiating, activities necessary for successful engagement in modelling. Galbraith et al. address the question: How does "noticing" feature as an enabler and a displayer of modelling ability? Based on student work at an extracurricular modelling event, they identify global and specific noticing of a strategic and explanatory nature, which provides evidence of anticipatory aspects of mental activity taking place during modelling, and illustrate a coding system for identifying and labelling components of productive modelling oriented noticing.

Crossing the boundaries between mathematical modelling and statistics education is the aim of the chapter by *Kawakami*. He views the learning process as combining distribution-related models: firstly, a model from the "modelling world" constructed in students' internal world corresponding to an image of a particular data distribution and, secondly, a model of the real distribution constructed in the students' external world and generated by the collection and visual arrangement of real data. These models are combined and reconstructed into a coherent whole to include statistical and contextual elements by comparing, contrasting and coordinating between these models. The chapter examines how primary students combined distribution-related models in experimentation that included conjecturing and validation. Although the reported findings are limited to a small sample, they provide empirical evidence that experimentation can foster students' model development of distribution. The trigger for combining students' models can be the phase of model validation in experimentation.

The boundary *Manouchehri* and *Lewis* propose to cross is that between the realities of the researcher and the researched, in this case beginning modellers at school. Students' interpretations and expectations of what counts as a precise and adequate method are key players in whether the modelling cycle is revisited or refined. Explanations offered by researchers for beginning modellers' reluctance to seek and produce refined models rarely account for epistemological elements that influence students' choices including the criteria they consider when validating their solutions.

Manouchehri and Lewis draw from data collected in three interrelated research projects about intermediate and high school students' mathematical practices to problematise two implementation issues for mathematical modelling in schools:

1. The gap between students' intuitions regarding variables they legitimately consider as prior constraints, based on their real-life experiences, and the conflict their choices create when asked to consider conventional mathematics to solve the same problems; and
2. The complexity of converging intuitive and analytical domains of students' work so as not to devalue their intuitions.

Mousoulides, *Nicolaidou* and *Evagorou* cross community boundaries to address an important issue, namely, designing teacher professional development learning communities to better understand, analyse and support teachers in the development of modelling tasks and the teaching of mathematical modelling. Their study examined the impact of a three-tiered professional learning community (students, teachers and parents, researchers and teacher educators) on teachers' knowledge and skills in designing and implementing inquiry-based modelling problems in their classrooms. Results confirmed that teachers improved their knowledge and pedagogical approaches to modelling changing from a focus on the minutia of day-to-day implementation difficulties to appreciating the substantive contribution of model-eliciting activities in developing students' mathematical constructs. Teachers gradually improved their self-confidence in teaching more complex modelling-based tasks and became more motivated in designing modelling activities. As well as participation in the learning community improving teachers' knowledge and pedagogical approaches, attitudes and self-confidence, it also increased the communication and collaboration between all the different groups in the professional learning community.

The chapter by *Palharini*, *Tortola* and *Almeida* crosses boundaries within mathematics itself which most would not anticipate to be a passage to new ideas for teaching modelling and proof. The authors describe a study that aims to investigate whether it is possible to consider the recurrence process inside a mathematical modelling activity as a mathematical proof. The study refers to the work of Ludwig Wittgenstein, on proof by recurrence. Palharini et al. base their arguments on the analysis of two mathematical modelling activities: one at tertiary level and one for basic education in Brazil. A qualitative approach and an interpretative analysis of Wittgenstein's writings are used to make inferences from written data and data collected through audio-recordings from students for the first activity but from suggested solutions in the textbook for the second activity. Their analysis indicates that mathematical modelling activities, in a sense, could lead to the need for mathematical proof, particularly proof by recurrence.

The chapter by *Perrenet*, *Zwaneveld*, *van Overveld* (†) and *Borghuis* is a meaningful contribution to the literature on engineering education in the work with models and modelling. The assessment of models in terms of criteria and purposes is described. The usefulness of purposes and criteria for models differentiating between understanding and misunderstanding is illuminated clearly in the chapter.

The study, in providing a model for assessing the usefulness of models in terms of purposes and criteria for the model, opens a new field of research for all modelling investigations not just engineering education.

Rosa and *Orey* focus on a combination of emic (local), etic (global) and dialogical (glocal) approaches to ethnomodelling, adding to their previous theorising with respect to modelling research (e.g. Rosa and Orey 2013). Ethnomodelling is the study of mathematical ideas and procedures developed, used, practised and present in diverse situations found in the daily lives of distinct cultural groups such as roofing contractors. Implementation of the dialogical perspective is emphasised by Rosa and Orey. In the dialogical approach to ethnomodelling research, the use of both approaches can deepen our understanding of important issues in scientific research and investigation. Many local mathematical practices have disappeared because of the intrusion or imposition of *foreign* (etic) knowledge value systems and technologies that emerged from the development of concepts promising short-term gains or solutions to problems faced by the members of these distinct cultural groups without considering emic knowledge that could solve these very same problems. The application of ethnomathematical techniques and the tools of modelling allow us to see a different reality and give us insight into the mathematics all of us perform in a holistic manner.

A recent innovation in upper secondary classrooms is the flipped classroom where there is a swapping of classroom and homework activities through the use of electronic technologies and, in some cases, an expansion of the curriculum (Bishop 2013). From an intensive survey of the literature, *Stillman* concludes that meta-analyses of findings from flipped classroom studies in mathematics classrooms to date need to be treated with caution, until the teachers and students involved develop mindsets that maximise and integrate the learning potential of both the out-of-classroom and in-classroom learning environments. However, rather than a flipped classroom approach being used as a means to cover the curriculum, it can be used to enrich the curriculum. Exploring the latter approach, Stillman examines the question of whether a flipped classroom approach to teaching could provide both vicarious experiences and fostering of critical thinking skills associated with modelling. A local secondary school implementation is used to illustrate how the approach could build meta-knowledge about mathematical modelling and facilitate associated critical thinking skills, such as anticipating and visualisation, to expand the learning experiences of secondary mathematics students.

As metacognitive competencies are an essential component of modelling competency, *Vorhölter* looks at finding a method or instrument to reliably measure metacognitive modelling competencies of larger groups of students. Techniques currently being used with smaller samples are considered too costly in terms of both time and money for use with large cohorts. Results are presented of a design-based process aimed at the development of a questionnaire for measuring metacognitive modelling competencies. Selected items of the questionnaire are presented and discussed. Preliminary results point to fostering of metacognitive competencies in modelling being possible by using special examples and metacognitive means.

1.3 Researching Boundaries in Mathematical Modelling Education

Researching boundaries in mathematical modelling education both in the sense of being at the periphery or edge of current practice and Star's (2010) notion of a shared space where different groups (e.g. archaeologists, history, technology and mathematics teachers and secondary students—see Sala et al. later) work on solving a shared problem are impetuses for deepening our research base and establishing research evidence for bringing new ideas into future classroom practice or to show where there are current gaps.

Alpers carries out a textbook analysis of items in statics in engineering texts to identify what part of the modelling cycle could be found in solved examples and students' solutions. His aim is to try to reveal where engineering students are given opportunities in their tertiary courses to develop the necessary competencies needed to model when problem-solving in their future careers. Two textbooks were analysed and two well-performing students solved a sample of 25 items that cover different areas of statics. The results and analysis are presented with examples from the textbooks, and the main conclusion is that not all aspects of the modelling cycle can be found in the solved examples or students' solutions. Among other difficulties, the problems examined were already clarified from the outset, and no assumptions or simplifications were needed, and hardly any considerations needed to be made to validate the real model. This points to a change of textbooks being decisive for students' learning.

There has been recent emphasis on educating future teachers in how to enhance their professional reflection so as to increase competency. Within the seminar for pre-service teachers, "Modelling Days", at Kassel University, a focus on reflection was promoted explicitly. *Borromeo Ferri* reports on an explorative study where the goal was to investigate pre-service teachers' levels of reflectivity after conducting modelling activities with high school students. Analysis of written reflections showed different levels of reflectivity, with high levels rarely being reached. In conducting the study, a new model of levels of reflectivity for teacher education and teacher training in mathematical modelling was developed by adapting the model of Hatton and Smith (1995).

The benefits of student engagement with real-world contexts seem to be well accepted by the mathematical modelling and applications community. Yet, concerns related to difficulties *necessarily* arising through engagement with the messy real world in mathematics classes (Sullivan et al. 2003) continue to be raised. *Brown* presents a qualitative analysis from a study of Year 9 students to illustrate how engagement with context offers opportunities to demonstrate and deepen genuine mathematical understanding of rate of change. Genuine collaboration and inter-thinking (Mercer and Howe 2012) when students *work as groups*, rather than just *being in groups working*, were found to facilitate the development of mathematical understanding, clearly enabled by the real-world context. The illustration and analytical tools from this study may act as a boundary device (Garraway 2010) that enables the passage of knowledge between those who write about the value (or otherwise) of using real-world contexts in the classroom.

An online questionnaire was conducted by *Cabassut* and *Ferrando* on an exploratory voluntary sample to gain insight into the difficulties encountered in the teaching of mathematical modelling in France and Spain. The timing of the data collection was prior (1 year) to the introduction of a new French curriculum for primary and lower secondary education in 2016, introducing modelling as a main mathematical activity and compulsory practical interdisciplinary projects. With respect to perceived difficulties, different conceptions were used to construct four clusters of respondents, from those having positive and confident conceptions to those who were negative and lacked confidence. In order to offer training and resources effectively in response to the expressed difficulties, the roles of country, age, gender or school level need to be clarified further in semi-structured interviews.

Research has shown that problem posing, as the "inverse activity" of problem-solving, can positively affect students' problem-solving skills (Silver 1994). The research by De Bock and colleagues to date has focussed on finding ways to improve beginner modellers' proficiency in moving from understanding a problem situation to a mathematical model. Extending on the problem-posing idea, *De Bock*, *Veracx* and *Van Dooren* look at the potential of a specific problem-posing variant, "inverse modelling", that is the selection of a real-world situation given a mathematical model, to progress modelling. They examined two groups of 11th grade students, one first receiving a modelling task and then an inverse-modelling task and the other receiving both tasks in reverse order. Results indicated that inverse modelling did not have an overall positive effect on modelling with accuracy scores for modelling significantly improving only after inverse modelling with affine functions with negative slope. However, the researchers question the ecological validity of their approach of multiple-choice items and suggest using a future research design in which students are invited to think freely about situations in their own environment that can be modelled with specific types of functions.

Durandt and *Jacobs* report preliminary results of the thinking and planning strategies of 38 South African Grade 10–12 mathematics pre-service teachers, who were exposed to mathematical modelling for the first time. Participants worked in eight groups on a textbook-based traffic flow model-eliciting activity. The open-ended nature of the task, handling intra-group dynamics, the construction of appropriate equations and the interpretation of findings were the most pressing challenges for the beginning modellers. Participants' attitudes towards modelling, attained via a post-questionnaire, were very positive, and they all appreciated the exposure to mathematics in a real-world setting. Findings with respect to the pre-service teachers' planning strategies, experiences and attitudes will contribute to a set of guidelines aimed at the integration of mathematical modelling into the pre-service education of future mathematics teachers at the university in question.

The notion of mathematical literacy (or numeracy) has gained momentum internationally recently through the influence of the Programme of International Student Assessment (PISA) (OECD 2013) and national concerns in some countries about the ability of their citizens to use mathematics effectively in personal, civic and work life. Accordingly, it might be expected that responses to these concerns should be reflected in relevant curriculum documents for compulsory schooling. *Frejd* and *Geiger* present a content analysis of a sample of 12 national curricula documents in relation to mathematical literacy and numeracy. The analysis showed that there was not a common definition of mathematical literacy across the analysed documents and that the idea of mathematical literacy is represented in a quite limited fashion.

Gallart, *Ferrando*, *García-Raffi*, *Albarracín* and *Gorgorió* present a tool for analysing the work of secondary students when they solve a type of Fermi problem. The authors note that previous research has related to representing the processes of modelling these Fermi problems (Ärlebäck and Bergsten 2010), so they concentrate on the products of Fermi problems instead. The tool is based on the characterisation of the concepts, procedures and languages used to construct models aligning with the characterisation of models by Lesh and Harel (2003). It was used with two groups of upper secondary students, each group from a different school working on two Fermi problems. The study shows the proposed analytical tool is useful to describe the models produced by students and to distinguish different aspects between the models produced by students with, and without, previous modelling experience.

Kreckler defines global modelling competence as the ability to undertake a full modelling process and to possess the meta-knowledge of the procedure. A 4-h teaching unit to foster global modelling competence in regular school lessons was developed. The unit was tested with 332 German secondary school students (tenth grade). The goal of this empirical study was to increase global modelling competence independent of influencing factors such as the topic for the teaching unit, student gender and mathematics assessment grades. The study resulted in a significant increase in global modelling competence independent of grade and topic, but no significant changes could be identified concerning motivation.

In previous work, Ärlebäck and Frejd (2013) have attempted to formulate a commognitive perspective (Sfard 2008) of mathematical modelling and models to provide a framework for analysing communication in modelling, taking into consideration social (i.e. group) and cognitive (i.e. individual) dimensions. *Park* uses a commognitive framework in combination with Stillman et al.'s (2007) blockages framework, in the study she reports, to examine pre-service secondary mathematics teachers' conception of mathematical modelling and content knowledge when engaging in mathematical modelling activities within a group. As the group comprised only three students, what these particular students' conceptions and content knowledge were is of less interest than the validity of the combination of analytical tools for the purposes proposed. This combination does appear to be fit-for-purpose but full evaluation awaits wider testing and empirical use.

Psycharis and *Potari* use activity theory (Engeström 2001) and the construct of boundary crossing (Engeström et al. 1995) to study the process of teachers' professional learning when they link mathematics teaching to the workplace through modelling. Transforming authentic workplace situations in the classroom for engaging students in mathematical modelling is complex. Learners need support to develop critically and mathematically informed models of complex realities, to construct and deconstruct real and mathematical models of complex situations, and to understand how the structure of models of workplace realities relates to the models of their mathematical counterparts. In particular, Psycharis and Potari analyse teachers' boundary crossings between two activity systems to advance their learning: mathematics teaching and a workplace, a fuel depot. Results indicate that collaborative task design and reflection helped teachers combine elements from the workplace into mathematics teaching. Different ways of linking reality and mathematics teaching were identified in the modelling process in which the teachers engaged.

Saeki, *Kaneko* and *Saito* used a multidisciplinary team including a general educational consultant and an art expert to investigate how pre-service teachers responded to a meta-question asking them to locate geometric figures hidden in the composition of four replicas of Renaissance paintings at an Art museum. To fulfil the *meta-question*, students presented a demonstration to an audience of museum visitors. The purpose of the study was (1) to investigate how teacher education students critiqued and meta-validated models connecting paintings and the mathematical world and (2) to analyse the students' decision-making. Students were given a physical device, a lens, to help them locate various figures said to be used by painters to compose elements of paintings and to verify the models they proposed. To locate and explain why the particular figure might have been used, students were expected to research extra-mathematical knowledge themselves. The pre-service teachers' critique and meta-validation of the models they constructed for their decision-making resulted from both mathematical considerations and extra-mathematical considerations. They benefitted from the input from different knowledge communities—the consultant on interdisciplinary integration and the art expert.

A multidisciplinary team was also the knowledge community involved in the study reported by *Sala*, *Font*, *Giménez* and *Barquero*. The aim was to promote inquiry and student modelling competencies and to investigate how interaction between multiple disciplines could enhance modelling and inquiry processes. A teaching sequence based on an archaeological context, the ruins of a Roman theatre discovered in Badalona (Catalonia), was designed and implemented with 12- to 14-year-old students in secondary school. The initial historical situation was presented by archaeologists from the local museum, and students visited the site with their teachers. Historical documents about Roman ruins also were made available so the students had access to domain knowledge from several domain experts. Mathematical modelling appeared as a central tool in the teaching and learning processes which are evaluated by the authors. A constant dialectic between mathematics and history was required to facilitate evolution of the modelling process. It also enabled validation of the modelling.

Van Reeth and *De Bock* present an interesting survey on the phenomenon of "over reliance on linearity", a topic the second author has been studying for some time. This time, the focus is the teaching and practice of economic modelling. There is a widespread use of linearity in economics education and practice which the authors illustrate by discussing the treatment, in major economic textbooks, of demand and supply functions and of the Phillips curve modelling the relationship between unemployment and inflation. This raises the mathematical points of when it is a good idea in any context to approximate a non-linear function with a linear function and how to decide the length of the interval in which it is acceptable. In this instance, it is important to establish where the boundary between mathematics and economics is determined and for what purpose (educational or practical). Typically, the phenomenon of overreliance on linearity is described only incidentally in economics education research studies which focus more on other concerns. The study by De Bock et al. (2014), in which tertiary students' over reliance on linearity in economic thinking was the main research focus, is discussed in some detail.

1.4 Pedagogical Issues for Teachers and Teacher Educators Using Mathematical Modelling and Applications

Given the interests of the ICTMA research community, it is no surprise that pedagogical issues for teachers and teacher educators using mathematical modelling and applications in their teaching are always areas of motivation and study for the research community. The first two chapters in this part of the book arose from two of the plenaries that were presented in Nottingham.

In their chapter, *Burkhardt* and *Swan* (†) set out their views, based on over 30-years of experience at the Shell Centre, on methods and challenges in teaching students the strategies and skills needed to model real problems using mathematics and approaches to helping education systems make this happen in classrooms. They conclude that substantial progress has been made on understanding the methods and challenges of teaching modelling and applications but much less with systems taking responsibility to make this happen in everyday classrooms. Examples from a sequence of modelling projects are used to illustrate design principles that have proved powerful for materials to support teaching and professional development in the area of modelling and applications. Barriers in school systems to the implementation of important improvements like modelling are discussed and how they might be tackled to bring systemic change.

Drawing on empirical and theoretical research studies, *Frejd* discusses similarities and differences between working with mathematical modelling in "school" and mathematical modelling as a "professional task" in the workplace. The extent of major differences between modelling work in educational and workplace contexts indicates that mathematical modelling in school will remain an unreachable goal in terms of coherence to professional practice according to Frejd. He suggests using innovative teaching methods, such as simulation, gaming and role playing, to bridge the gap between modelling as a professional activity and as a school activity. He then turns the spotlight on simulations of workplaces and role playing for the remainder of the chapter.

Guerrero-Ortiz and *Mena-Lorca* take the first steps in a Chilean university to help bridge the silos of expertise that so often hinder students' integration of knowledge across what appear to be compatible disciplines of mathematics and the sciences (English 2016). Their focus is on the separation at tertiary level between the teaching of modelling and models in mathematics and scientific disciplines. The differences result from what is considered a model and the goals of teaching modelling in the different disciplines. The science lecturers emphasised establishing relationships between mathematical and extra-mathematical knowledge so models and output could be understood and interpreted. For the mathematics lecturer, models were a tool to study mathematical characteristics of objects, where extra-mathematical knowledge was not considered. There was thus a tension between how tertiary students in this context were taught about models in mathematics and how they would apply them in the sciences. Lovric (2017) points out that tensions as identified here should not be viewed as barriers, but as opportunities to facilitate enriching the teaching of both mathematics and the sciences as modelling tasks are designed. This is the next step for Guerrero-Ortiz and Mena-Lorca.

The pedagogical potential of interpreting students' modelling from the perspective of interactive translations among plural worlds as opposed to mathematical modelling as involving transitions only between two fixed worlds—a real world and a mathematical world—is explored by *Ikeda* and *Stephens*. Experimental lessons for Japanese Year 10 students exemplify benefits of the plural interacting worlds perspective as (a) enabling teachers to direct attention to intermediate models that can help the building of further abstract models and (b) focusing teacher attention on meaningful contradictions supporting student verification, critique or modification of their original models.

Lamb, *Matsuzaki*, *Saeki* and *Kawakami* address the vexed question of appropriate teacher assistance for struggling modellers. They show, by use of the Dual Modelling Cycle Framework, how students who have difficulties finding a solution to one modelling task are introduced to a second similar simpler modelling task in a second modelling cycle so success there can lead to a transition back to the first modelling cycle where the modelling can be progressed. The authors therefore add to the corpus of work on the use of the Dual Modelling Cycle Framework (e.g. Matsuzaki and Saeki 2013) where the utility of the framework for different student groups in different educational contexts has been demonstrated.

The issue of educating practising teachers to implement mathematical modelling in their classroom in a long-term professional development programme brings challenges for both the teachers participating in the professional development and the teacher educators conducting professional development. In the study by *Manouchehri* of a professional development programme in the USA to prepare teachers to meet the expectation of the Common Core State Standards for Mathematics that they help school learners develop mathematical modelling skills (Council of Chief State School Officers 2010), teachers' level of comfort with mathematical modelling increased as time progressed. In contrast, teacher concerns persisted about the management of the short- and long-term demands of the curriculum, preparing students for skills-based standardised tests and guiding student discussions without infringing their autonomy. Concurrently, teacher educator challenges included managing teachers' diverse mathematical backgrounds and limiting direct instruction of mathematics.

Framing mathematical content (e.g. the velocity concept) that has arisen from the modelling of a phenomenon in its historical development is advocated by *Moeller* as a means to develop teacher competence as a teacher of modelling. A scheme to develop the velocity concept giving adequate attention to it being a historical modelling of a phenomenon is proposed for students, and it is suggested as being essential subject matter in pre-service teacher courses. Such preparation would allow future teachers to draw on the history of science so as to maximise educational possibilities of a historically developed modelling of a phenomenon such as velocity.

Teacher assistance for low-achieving students, students with learning difficulties and special needs, is the issue that concerns *Reilly* and *Scott-Wilson*, *Wessels*, and *Wessels* and *Swart*. The use of applications and mathematical modelling for students of all abilities has been approached in past ICTMA publications (e.g. Swan 1991), but these authors specifically focus on students with special learning needs. Inclusive task design so all students can access and show mastery of appropriate skills is the focus of the chapter by Reilly. A mathematical application task for Year 7 students is used to illustrate the design principles chosen to achieve this goal. The study shows task adaptation criteria allowed all students to work towards the overarching goal with students making individual progress. On the other hand, the efficacy of students with disabilities learning through mathematical modelling tasks, given that typically they learn at a slower rate and are already developmentally behind their peers, is explored by Scott-Wilson et al. Findings of Scott-Wilson's doctoral study included evidence of engagement and meaningful mathematical learning. In addition, extra benefits from the programme for students with disabilities were development of literacy, social skills practice during collaboration and social negotiations and support for development of potentially more robust thinking operations.

Stender, *Krosanke* and *Kaiser* focus on the issue of identifying appropriate teacher interventions for scaffolding when students are solving complex, realistic problems over an extended period of time so as to preserve student autonomy and independence in the direction and conduct of the modelling. This chapter supplements previous work by Kaiser and Stender (2013) in this area. Empirical work emphasises the importance of adequate diagnostic work preceding adaptive assistance. The intervention, "present status of work", is a powerful scaffolding measure at the beginning of every intervention in complex modelling processes, because it has the potential for a positive impact on the solution process in several aspects.

Vos and *Roorda* address the issue of student development of their treatment of real-world context over time through solving, or attempting to solve, contextualised tasks that integrate mathematical and real-world context aspects. Busse's framework of ideal types of students' treatment of context in real-world tasks (Busse 2011) was useful to analyse students' different approaches to tasks. The use of the framework in the analysis of data showed how students' preferences for particular approaches to context develop and that contexts and mathematics are not disjoint spheres but show complementarity, leading to integration for many students, but it takes time: at least 1 year.

Identification of aspects of creativity in pre-service teachers' mathematical modelling processes and products is the focus of *Helena Wessel*'s research report. The research confirmed that pre-service teachers' intuitive notions of what creativity is coincided with indicators for creativity (fluency, flexibility, novelty and usefulness) described in the literature. The pre-service teachers' solutions to the modelling task used in the study elicited flexibility as well as novel and useful models, but the students' fluency in terms of variety of approaches was restricted as the task was not mathematically challenging enough for the participants. This outcome indicates the importance of careful matching of tasks to a specific cohort as the potential for mathematical modelling to foster creativity can be task dependent.

Authenticity of students' experience of modelling in Dutch schools is the focus issue for *Zwaneveld*, *Perrenet*, *van Overveld* (†) and *Borghuis*. These authors examine the authenticity of textbook tasks as genuine mathematical modelling in terms of having a modelling purpose and students needing to perform characteristic modelling activities. Two Dutch mathematics textbooks for upper secondary schools, that were examined by the team of researchers, had hardly any genuine mathematical modelling tasks (based on these criteria), although modelling is explicitly mentioned in the formal curriculum. This points to the influence of textbooks being a critical consideration for student learning and implementation of the intended curriculum in the classroom.

1.5 Influences of Technologies on Modelling and Applications

There is a long tradition in the ICTMA book series from Berry et al. (1984) onwards of presenting examples (e.g. Clements 1986) and issues (e.g. Maass and Schlöglmann 1991) related to the use of technological resources in the teaching and learning of mathematical modelling and the influences these might have on teaching with modelling and applications. Given the nature of modelling and early computing power, simulations were often used and still are. Digital technologies in the form of software or small applets are becoming ubiquitous in society and in many educational contexts as a means of data collection, generation and analysis and for delivery of curriculum or as collaborative communicative tools.

Frejd and *Arlebäck* present a classroom activity using a simulation provided by a commercial game app for smartphones and tablets as the focus. Harnessing the potential of mobile technologies and apps and other new technologies in the digital landscape for mathematics education has been advocated for some time now as an opportunity to cater to the tastes of digital natives (e.g. LaPointe 2008; Kyriakides et al. 2016). An analysis focusing on the interplay between the designed intervention environment and students' work from two upper secondary classes identified affordances of the technological environment (Gibson 2015) to inform the re-design of the modelling activity. Results provide the basis for a discussion of the role of mobile technology and simulation apps within classroom teaching and learning of mathematical modelling and the future development of principles to design and implement modelling activities using game apps.

Greefrath and *Siller* consider theoretically, and by illustration, the added benefits of integration of digital tools in modelling and simulations, thus continuing the theme above. A qualitative, empirical study complements the theory by examining the activities of Year 10 students using GeoGebra to work on a modelling task requiring deterministic simulation. The students used the tools differently for researching, constructing, drawing, calculating, measuring, experimenting and visualising. Tools were employed primarily between the situation model, real model, mathematical model and the mathematical results and between the situation model and the mathematical model in the modelling cycle. They were not used while interpreting and validating the mathematical results. The study showed that the general modelling cycle used (Blum and Leiss 2007) was sufficient for describing use of digital tools while modelling. It appears that a special modelling cycle that represents technology as a separate area of activities (e.g. Daher and Shahbari 2015), or restricting its use between the mathematical model and the mathematical solution (e.g. Greefrath and Siller 2010), does not describe the modelling processes sufficiently.

The influence of mathematical modelling on South African engineering diploma students' visualisation when solving differential equations with a computer algebra system was investigated by *Kotze*, *Jacobs* and *Spangenberg*. Participants comprised 80 second year vocational engineering diploma students at a comprehensive university. Students' abilities to make contextual connections between different representations through a model-eliciting task were assessed using content analysis. By reversing the curricular approach, most participants constructed a meaningful differential equation that deepened understandings of the world in which they modelled. In the normal curriculum approach, students usually struggle to interpret numerical tables and computer graphs derived from symbolic differential equations and often leave interpretative questions unanswered. The modelling environment stimulated development of adequate schema through experimentation with paper-and-pen and computer algebra technologies.

How 14-year-old Israeli students participated in collaborative learning processes and developed skills in analysis of models was investigated by *Naftaliev*, while they worked on one modelling activity, a foot race, presented digitally as three different interactive diagrams. Three interactive diagram settings were designed as an animation of multi-process motion, but each differed in its pedagogical function. The interactive diagrams draw on purposely designed representations that can be used by modellers to illustrate real motion to give insight into the connections between the model and reality and important mathematical concepts. The students explored sets of characteristics of the mathematical models in the diagrams to analyse related phenomena presented as a real model and to develop meaning of the mathematical models regarding the phenomenon. Shared knowledge was developed when students engaged in a reflective activity concerning other group members' reasoning and instruments involved in the collaborative process. Analysis showed choosing and combining models from different interactive diagrams was based on personal choice to anchor the inquiry in the more familiar ones.

Ortega and *Puig* present modelling teaching material for a Year 11 class to model a real-life phenomenon: a basketball rebounding and falling. The teaching material uses electronic tablet apps to collect and process real data in the classroom. The research study was aimed at analysing which phases of the modelling cycle were influenced by the qualitative analysis of characteristics of the phenomenon, the associated families of functions and the students' prior knowledge of these. After analysing a classroom implementation, the authors confirmed previous findings by Puig and Monzó (2013) that a qualitative analysis of the phenomenon and the families of functions and the students' prior knowledge about these functions are key elements in managing and controlling the modelling process, especially when choosing the model and interpreting results in terms of the real phenomenon.

Digital technologies as the means of delivery or as cognitive collaborative communicative tools can enable expansion across geographical boundaries and overcome lack of provision locally as Borba and Gadanidis (2008) have pointed out in their survey paper. In this vein, *Orey* and *Rosa* describe how a group of mixed-ability tertiary students used long-distance education technologies to develop mathematical models in relation to their experience with nationwide protests in Brazil related to a rise in bus fares. Mathematical modelling as a teaching tool focused on the development of a critical and reflective efficacy engaging students in a contextualised teaching-learning process that allowed them to become involved in the construction of solutions of social significance, aiming at democratising mathematics through the development of the modelling process in virtual learning environments.

1.6 Assessment of Mathematical Modelling in Schools and Universities

Modes of assessment in mathematical modelling fall into two categories: (1) a holistic approach, where the modeller works (individually or in groups) on a complete modelling problem, and (2) an atomistic approach, where the modeller is asked to demonstrate separate competencies (e.g. making assumptions) needed for only part of the modelling cycle (Blomhoej and Jensen 2003). The assessment of modelling competencies has been a strong area of focus since the 1990s in an effort to develop reliable and valid modes of assessment, balancing what can be known from holistic and atomistic modes (Kaiser and Brand 2015).

Biccard and *Wessels* demonstrate how the six instructional design principles for model-eliciting activities (Lesh et al. 2000) can be reworded to serve as principles for assessing modelling abilities of students working in groups. The six principles form a framework for a holistic evaluation of group modelling. A design research study investigated the modelling competencies of a group of Year 7 students, holistically.

The group as a whole was assessed, not individual students. It was found that the six reworded principles allowed evaluation of significant aspects of model-eliciting activities such as model construction, reality integration, quality of documentation, self-evaluation, development of prototypes for thinking and generalisation.

In contrast, *Djepaxhija*, *Vos* and *Fuglestad* take what they consider a holistic/atomistic approach using a multiple-choice test, similar to the one developed by Haines et al. (2000) at the tertiary level, but designed for pairs of Year 9 students and only to assess mathematising competencies. Three PISA-released items were used to develop six tasks. This chapter is a report of the validity study of these tasks where the task format has the specific feature of centring around a holistic modelling problem while asking for a separate mathematising competency related to that problem. The students were well able to distinguish between the holistic modelling problem and the atomistic part of the task. They were able to handle the format, and their actions in response to the tasks were the intended mathematising activities.

1.7 Applicability at Different Levels of Schooling

Researching or demonstrating the applicability of applications and modelling as an educational experience, or a means of approaching curriculum content through well documented implementation examples, is important to provide continued support for adoption of applications and modelling in schooling.

Grafenhofer and *Siller* add to the literature on interdisciplinary (e.g. Brinkmann and Brinkmann 2007) and multidisciplinary approaches in modelling (e.g. Andresen and Petersen 2011) when they discuss the use of an interesting modelling problem about setting up a network system for refuelling hydrogen cars in Germany. Groups of students from three secondary schools attempted to solve the problem over three to four modelling days. The activities of four different student groups were analysed focussing on extra-mathematical knowledge used when the students were working in an interdisciplinary context (mathematics, physics, chemistry). The outcome was that students spent much of their time (over 1 day) understanding the underlying phenomena. The students' interdisciplinary processes and preparations by teachers had no detectable influence on the modelling conducted. Students tended to find their own way by using their real-life extra-mathematical knowledge and experiences in other subjects such as geography and economics rather than the ones targeted. This investigation suggests that teachers have to pay attention to how questions are posed and student prior experience at integrating knowledge in the targeted disciplines in interdisciplinary approaches to modelling if they want to influence the direction of student modelling and to reduce time spent by student groups merely understanding the context.

A research project conducted by *Spooner* in New Zealand demonstrates that it is possible for authentic mathematical modelling, based on the characteristics and behaviours of a professional modelling team, to be carried out at secondary school level. From the author's previous opportunity to work as a member of a professional modelling team, an authentic mathematical modelling experience for secondary school students was developed and researched. Classroom activities were created and trialled with a group of 16- and 17-year-old students. The activities used took the form of messy real-world situations. No manipulation of the situation was carried out to make the model more accessible. The structure of the activity was in the prompts given by the teacher to direct students through the stages of the process. Data collected from the learning activities showed all aspects of the mathematical modelling process were taught to, and learnt by, the group with the exception of refining the model.

1.8 Conclusion

As the most obvious output from the ICTMA research community is this series of volumes produced biennially, it is important that the lines of connection between the studies and theorising in the present volume to previous work and that of others researching similar and related ideas outside the community be established and demonstrated. In a similar fashion, many chapters in this volume have demonstrated the deliberate crossing by researchers of boundaries at the periphery of what has been researched previously or the inclusion of members of different communities into multidisciplinary research teams to bring a different lens on what is being researched. The identification of boundary objects or boundary devices in some of the work in this volume could lead to more productivity within research teams operating or formed on this basis.

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Part I
New Approaches in Research, Teaching and Practice from Crossing Boundaries

Chapter 2
Mathematical Modelling with Increasing Learning Aids: A Video Study

Deike S. Alfke

Abstract This study aims at supporting learners' competency of mathematical modelling in ordinary mathematics lessons by using *increasing learning aids* in a self-regulated learning environment. The study intends to evaluate the feasibility of the approach by carrying out a case study. Thirty seventh-graders were video- and audio-recorded while working on complex modelling problems supported by increasing learning aids and a diagram of the modelling cycle enhanced to indicate potential areas of difficulty or blockages to progress as a metacognitive aid. First results point out that the usage of increasing learning aids for solving mathematical modelling problems supports modelling activities. In this chapter, an overview on the modelling tasks is presented, with one task presented in detail. General results and results for one specific group will be reported.

Keywords Increasing learning aids • Aid cards • Teacher's support • Video study • Self-regulated learning • Mathematics education • Metacognitive prompts • Modelling cycle

2.1 Introduction

For mathematical modelling, the competencies as well as the use of modelling activities in mathematics education, is a nationally and internationally highly discussed topic of didactics of mathematics (Kaiser et al. 2011, 2015; Kaiser and Sriraman 2006; Maaß 2006). The competency of mathematical modelling is one of the six central competencies in the German mathematics educational standards already having been implemented for more than 10 years. Thus, this competency has to be acquired by students during their school career. According to Blum (2011), research shows that students display various difficulties when solving modelling

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_2

problems. Furthermore, an imbalance exists between the amount of the teacher's support and the students' ability to work independently (Blum 2015). The qualitative empirical study reported in this chapter addresses this imbalance. It aims at supporting the competency of mathematical modelling in mathematics education using increasing learning aids in a self-regulated learning environment and intends to evaluate the efficiency of the approach.

2.2 Theoretical Background and Research Focus

In this section, the theoretical framework of the project is presented. It includes the historical background and the framework of increasing learning aids. At the end of the section, the research focus is given.

Increasing learning aids are a form of scaffolding (Van de Pol 2010) and primarily follow the taxonomy of assistance by Zech (2002) based on the principle of minimal aid by Aebli (1985). The approach of increasing learning aids, with reference to task-oriented issues, has been developed within natural science education by Joseph Leisen (1999).

On the one hand, increasing learning aids in natural science education aim at fostering autonomous, cooperative learning of students, and on the other hand, on encouraging problem-solving competence and specialised self-awareness related to the field (Wodzinski et al. 2006). By providing these increasing scaffolds, the format of the problems offers students different aids for their working process in the form of aid cards. Every *aid card* comprises two parts, a prompting part suggesting general-strategic learning behaviour and a solution part giving single solution steps. As the form of scaffolding is these pre-created cards with hints or questions, in addition to aiding learning, the form of help is such that students can access if, and when, needed; that is, they are not dependent on the teacher being available to offer support at the time they need it. This way, students learn to control their learning process and to assume responsibility for it. Thus, the principle of adaptive instruction by Weinert (1996) takes effect. The increasing learning aids are an example of internal differentiation. They offer an approach to respond to heterogeneity. While students usually work in pairs, the teacher is given space for individual advice and support if unexpected difficulties occur. The aid cards contain general-strategic as well as content-specific learning aids and have the following functions: focusing, paraphrasing, visualising, elaboration of sub-aims, activation of background knowledge and verification. All in all, these increasing learning aids have a rather closed character.

So far, this kind of increasing support has been primarily applied in natural science education for task-oriented problem-solving exercises (Schmidt-Weigand et al. 2012), but there is nearly no existing research about increasing learning aids in mathematics education and their efficiency in self-regulated learning environments. This empirical feasibility study aims to close this gap by adapting the approach of increasing learning aids from natural science education to the specific requirements of mathematical modelling. The intention is to investigate whether students can be supported in solving complex modelling problems by increasing learning aids and how students use increasing learning aids when solving mathematical modelling problems. Following Kaiser et al. (2013) and Maaß (2006), complex modelling problems are based on a real complex problem for which students have to find the mathematics themselves in order to solve the problem.

2.3 Methods

In this section, the methods used are described. The project follows a qualitative design in which the theory of cognitive apprenticeship (Collins et al. 1989) serves as the basis for the designed learning environment.

The teaching unit was implemented between November 2014 and April 2015 in two seventh-grade classes with 59 students (37 female and 22 male), aged 12 and 13 years old, at a German grammar school in Hamburg. Not all students took part in the research project. The teaching unit comprises six lessons of 90 min each, which were held during regular mathematics lessons. The first phase of the unit lasted two lessons. It started with an introduction to mathematical modelling in which the teacher solved a modelling problem for the students and externalised her thinking processes which she usually carries out internally. Based on this activity, the students acquired meta-knowledge about the modelling cycle and the provided aid topics by going through a worked example where an ideal group of students solves a modelling problem and uses the increasing learning aids. Thus, the students in class learnt how to use the prompts, the aid cards and when most likely to ask for the teacher's assistance. During the following phase, the students worked on four complex modelling problems in groups of three for about 60–70 min during four lessons. Scaffolding was provided during these lessons by the increasing learning aids. For the students, it was supposed to be a group decision when, and to what extent, they sought assistance, but it was also possible for individuals to do so. As the students worked more independently, the support of the learning aids was expected to fade, as they consulted different and particularly fewer learning aids (including the aid cards) and less teacher support. The teacher mentored the students throughout the entire working process adapting her assistance to the students' needs (van de Pol 2010).

The focus of the research lies on the handling of the learning aids by the students and whether the aid cards and the teacher's support were effective, especially concerning increasing support in mathematical modelling. The study focused on an external as well as an internal perspective. The external perspective was mainly surveyed by video-recording and audio-recording in order to analyse classroom activities in more detail. Due to organisational reasons, five groups of three students were video- and audiotaped in each class during their working process; thus there were 30 students in ten groups in total who took part in the study. The students in these ten groups worked together for the whole unit and were determined and chosen by the teacher researcher, who taught both classes in mathematics during the entire school year. The study included groups of the same and of different gender as well as heterogeneous and more or less homogenous groups. Besides the video- and audio-recordings, data for this analysis include the students' posters of the solved modelling problems and their scripts written during modelling. To evaluate the internal perspective of the students, post-task interviews were conducted. Only the students of four out of the five groups in each class could be interviewed due to organisational reasons. In order to take student mathematical modelling competencies and their changes into account, the teaching unit was framed by a pre- and a post-modelling test.

2.4 Concept of Increasing Learning Aids

In this section, the increasing learning aids, as developed for mathematical modelling, are introduced with their components. Increasing learning aids in mathematical modelling are a form of scaffolding to assist students in their learning process.

The study uses the modelling cycle by Kaiser and Stender (2013) enhanced by the metacognitive prompts as shown in Fig. 2.1. Initially, the *modelling cycle with metacognitive prompts*, A–H, is used in order to focus students on their difficulty. These prompts are supplied for all stages and transitions of the modelling cycle and are given in the form of questions. The prompts with letters B, C, D, H and I indicate topics for problems in the real world, whereas the letters E, F and G indicate problems within mathematics. As lack of progress can occur at any time, prompt A is not attached to any stage or transition as are the other prompts. Therefore, there are nine

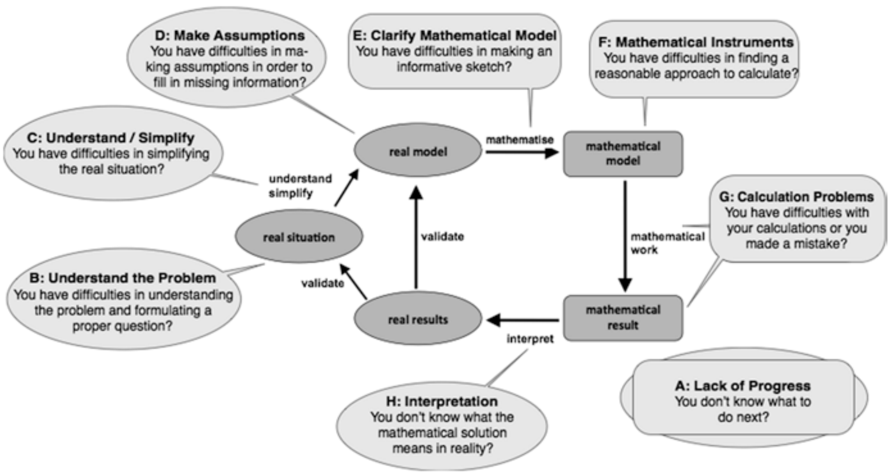

Fig. 2.1 Modelling cycle (Kaiser and Stender 2013) with topics of aid cards

different aid topics. For every aid topic, a *card box* has been developed. Every card box contains *general-strategic*, *content-related* and *content aids*, which increase the offered support step by step and build on each other. The students use the aid cards in the given order according to the needed degree of assistance. The more cards they use, the more intense the help becomes. The last card of every card box recommends asking the *teacher for assistance* as a final aid.

A problem is identified by using the modelling cycle with the aid topics, in order to be able to consult one of the nine card boxes with learning aids given for every identified problem. So, for example, if a group has difficulties in understanding or simplifying the real situation, the modelling cycle gives as corresponding heading and question next to the transition 'understand/simplify'. Therefore, the students know that they should consult card box C for assistance. Not only are the aid cards strongly related to the specific modelling examples dealt with in class; but also they offer more general support related to modelling strategies such as the usage of the modelling cycle as a metacognitive orientation, affective support and finally references to problems previously dealt with. The students, as novice modellers, were introduced into mathematical modelling, the modelling cycle and the card boxes used in this study at the beginning of the teaching unit.

The implemented kind of teacher's assistance follows the Zech taxonomy (2002) based on the principle of minimal aid, that is, support is predominantly given if the teacher is asked for advice. The teacher first gives general-strategic help, for example, when approaching the group table, she asks the students to present the state of their work, and later she might assist the group in finding the right card box to use in case the students cannot identify their problem by themselves. Thus, the *increase of the learning aids* is formed by the successively growing level of support within one topic for help. This applies to all the specific card boxes and the teacher's support.

The aids were developed by the researcher and teacher jointly with the research group of the supervisor of the study. During a development phase possible areas of difficulty, of blockages and of progress were anticipated, thus one for almost every phase and for every transition of the modelling cycle. In order to evaluate the efficiency of the approach a pilot study was conducted, the aid topics (and also the aid cards mentioned below) were revised based on the reactions of the students and mathematics educators from the already mentioned research group.

2.5 Tasks

In this section, the topics of the tasks used in this study are given. Following this, one of the modelling problems is presented in detail. The students were given four complex modelling problems. The problems dealt with realistic topics in the students' field of experience: the reduced lifetime of smokers (recently worked on in biology), salvage of a boulder from the river Elbe in Hamburg (on the shore of the

Fig. 2.2 Modelling task *The Wild Mouse*. THE WILD MOUSE XXL has only been on the funfair in Hamburg since summer 2013. They speed downwards on the tracks from 30 m high with up to 60 km/h. But don't worry: the average speed is 16.2 km/h and therefore a lot slower. On a track length of 585 m, you will ride down three steep slopes with water fountains in the valleys which will give you a little refreshment. As every year, you stroll along the streets on the summer fair with your father. And you have already wanted to try out the even faster new WILD MOUSE last year. But last year, the queue was extremely long, and your dad did not want to wait that long. He put you off until next year. And that is now! As you reach THE WILD MOUSE you start complaining: 'Oh no! The queue is at least a hundred meters long. Look at them: The people are waiting way down at the 100-metre-sign'. 'The fair will close in about half an hour. We won't make that in time', your father replies. 'Oh no! Please, let's try it! Look, there are five cars on the track at the same time and four cars are waiting over there. And look how fast they are!' *Will it be worth waiting?*

river where students live), the dimensions of a clog and the time needed to queue for a ride at the funfair in Hamburg. The level of mathematics required in solving these tasks is related to content from previous years of schooling; hence, any difficulties could not be attributed to lack of relevant mathematical knowledge. One of the modelling problems, for which first results will be given below, is presented in more detail: *The Wild Mouse* at the funfair in Hamburg. Almost every child who visits this funfair knows *The Wild Mouse*, and so the students know *The Wild Mouse* or have even ridden it. The task is shown in Fig. 2.2.

A roller coaster with five cars can be seen in the large picture. Therefore, five wild mice are driving on the tracks at the same time. There is also a sign that can be seen through a magnifying glass. The sign tells the customer that the ride is still 100 m away. In the small picture, there are four cars in the transition zone. Three to four people are sitting in each car. The students have to find out whether the child and his father will make it onto a ride with *The Wild Mouse* before the funfair closes.

2.6 First Results

At first, an overview of the use of the increasing learning aids in one class using the modelling problem, *The Wild Mouse*, is given. Then, as an example, the intense work of one learning group working on *The Wild Mouse* and using increasing learning aids will be presented.

2.6.1 General Results

An overview of the application of the use of increasing learning aids in one class using the modelling problem, *The Wild Mouse*, is shown in Fig. 2.3. In the first row, the existing aid cards for this modelling problem are displayed. The 24 general-strategic aids, which are the same cards for every modelling problem, are shaded in light grey. The 15 content-related and the content aids, which are different for every modelling problem, are shaded in dark grey. The *last* card of each box is indicated with an *L*. As mentioned above, the last card recommends asking the teacher for advice as a last aid. However, the *teacher* was also asked for advice on other occasions. Sometimes cards for alternative aids for one problem are given, for example, if students have difficulties in simplifying the real situation, they choose card box C. Part C1 gives the general-strategic aid to look again at the text and the illustration. It supports by stating two questions: 'Where can you find really important information? (Part C2A)' and 'How can you use this information? (Part C2B)'. If the students cannot answer these questions, they can use the indicated cards, in this case C2A and C2B. The use of the aid cards for each of the five videotaped groups is shown in the rows underneath. The legend in the second table explains the letters

aid – group	A 1	A 2	A L	B 1	B 2	B 3	B L	C 1	C 2 A	C 2 B	C L	D 1	D 2	D 3	D L	E 1	E 2	E 3	E L	F 1	F2	F 3	F L	G 1	G 2	G 3	G 4	G L	H 1	H 2	H L	I1	I2	I3 A	I3 B	I4 A	I4 B	I5	IL
1																s_5				o_1	s_2		t_4	u_6	u_7			t_3	s_8		t_9								t_{10}
2									o_3	s_4	t_5	o_1	o_2			o_6	u_7	s_8										t_9											t_{10}
3																				o_1	o_2 u_6	s_4	t_5				u_3												t_7
4							t_1	o_2	s_3												o_6 o_{10} s_{12}		t_{11}	o_4	o_5 o_7	u_9	u_8												t_{13} t_{14}
5												o_1	o_2	s_3							s_5				u_4														

Legend	
general-strategic aid	(light grey)
contend-related/content aid	(dark grey)
successfully used card without a following card for the specific problem	s
successfully used card with a following card for the specific problem	o
unsuccessfully used card	u
teacher was asked for assistance	t
order of cards used	x_i

Fig. 2.3 Application of increasing learning aids

used, *s*, *o*, *u* and *t*. The subscript indicates the order in which the cards were used by the groups. As stated above, the decision to use a learning aid was supposed to be made by the group; but it was also possible to decide this individually.

The chart provides evidence that all the groups used aid cards. Every group used at least one aid card successfully in their work. Some aid card topics were used more successfully than others. In the process of understanding the problem and making assumptions, for example, the use of aid cards of topic C was more successful than the use of aid cards of topic G that offers support in calculating problems. Some aid card topics A, B and I were not used in this modelling problem. The cards of topics B and I were mostly used in the first two modelling problems, whereas this was the last modelling problem. However, the following points are also evident: The students did not always ask the teacher for advice in the case when an aid card did not help enough. In addition, they did not always fetch the next aid card of a topic in the case of insufficient help by the current card. Observation showed students did not always use increasing learning aids if they could not progress.

2.6.2 Results of an Individual Group

In this section, an excerpt from the transcript of group 4 will give insight into the working process of one group. Some results of the working process will be presented afterwards. The three girls were simplifying the situation and were making assumptions. The girls had read the task silently, and each one had marked passages of the text. They had framed the problem of the task together and had collected important information from the text. After about 7 min of independent work and a 5-min discussion about the task problem and the important information given, they decided to use an aid card. Their problem is connected to the relevance of the three steep drops in the track.

S1: But we have to include the three steep drops in our calculation.
S3: No, I think the steep drops with fountains means the water splashes out of the sides.
S2: Yes, but steep drops/
S3: Because the water splashes out of the sides. (She gestures with her hands up.) Steep drops. The section where the drop is. You know?
S1: Yes, but/
S3: Everything is included she said.
S2: Shall I get an aid card?
S3: Yes, no, we need to find out what we have. (S1 and S3 look at the modelling cycle with the topics of the aid cards.) We need, we have, (She reads.) 'You've got difficulties making assumptions. Simplify the situation. You've got difficulties simplifying the real world's situation'.
S1: Yes.
S2: Yes, I'd better fetch support cards C1 and 2. (She gets up and goes to fetch an aid card.)

S1: Yes, 1 and 2.
S2: C, isn't it?
S1: Yes.
S3: Yes. (She writes.) The highest point of the tracks. Aid card/
S2: Well, the first card says: 'Look at the text and the pictures again. Where can you find particularly important information? How can you use the given information?'
S3: We've got that.
S2: (She returns with two aid cards, sits down and reads.) 'Card 2A: The cars, important information: The cars have an average speed of 16.2 km/h. The tracks are 585 m long. The queue is 100 m long'.
S3: Oh, where does it say that?
S2: (She reads.) 'The fairground closes in about half an hour. There are five cars on the track at the same time and four cars in the transition area'.
S3: Oh! I've got an idea!
S2: (She reads.) 'Four persons can sit in each car. Additional information: The cars of the Wild Mouse can go as fast as 60 km/h'.
S3: That's a good card. Let's keep this one.
S2: Shall I take the other one back?
S3: I would say yes.

It is evident that the girls made a conscious decision to use the increasing learning aids in the form of an aid card. The girls identified their problem by using the modelling cycle and the possible problems connected to it. One girl took the first two cards of aid card topic C deliberately. Another girl read card 1 of the topic aloud. The girls reflected on the general-strategic help briefly, and they continued to part 2A immediately. Part 2A was also read aloud. The comments of the girls show that they grasped the given information, analysed and used it. This aid card solved their problem. The two cards of aid topic C, which they had taken, simplified the situation, and it was exactly this information which helped the girls to make their final assumptions. These assumptions helped to make a real model and developed a mathematical model of the situation in order to finally calculate a realistic number of people waiting in the queue.

2.7 Final Remarks

First results of the study show that it is helpful to use increasing learning aids for mathematical modelling problems due to the following reasons: Every aid topic was used successfully at least in one of the four modelling problems. Aid cards helped to overcome students' difficulties and barriers and sometimes even made the teacher's support superfluous. This fostered the independent work of the students. Furthermore, it makes sense to use increasing learning aids, because they helped the students to progress in solving the modelling problems at several steps of their working process.

Following these first results, a further, and particularly more detailed, examination of the collected material will be undertaken. The research questions, whether students can be supported in solving complex modelling problems by increasing learning aids and how students approach increasing learning aids when solving mathematical modelling problems, will be answered by means of qualitative social research (Kuckartz 2014). Therefore, incidents playing a decisive role will be transcribed and evaluated. The collected material and the modelling tests will be taken into consideration as well. It is also intended to provide material with increasing learning aids that can be used in mathematics education. The learning aids idea already requires expansion; therefore, exercises will be developed connecting complex modelling problems with textbook or test exercises. In addition, the concept is meant to be implemented in various school grades.

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Chapter 3
Modelling with Statistical Data: Characterisation of Student Models

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Abstract This chapter reports on the solutions of 22 groups of Year 10 students (15–16 years old) to a model-eliciting activity involving interpretation of data, namely, lists of salaries from five companies. Students were asked to see what could be ascertained about the structure of the company based on their mathematical or statistical analysis of the data. The students had no previous modelling experience but some understanding of statistics. Solutions based on the concepts and the processes involved in the models are represented in a graph. This analysis tool allowed distinguishing of significant differences between students' responses. Results show a wide range of concepts and mathematical procedures were used. The activity promoted mathematical modelling and could be the first of a didactic sequence aimed at working on data distribution and dispersion.

Keywords Model-eliciting activities • Secondary education • Statistics • Visualisation tool

3.1 Introduction

This chapter presents an exploratory study with the aim of determining whether students of Obligatory Secondary Education (15–16 years old) would create mathematical models as defined by Lesh and Harel (2003) during the execution of an activity in the field of statistics. The data provided to the students involved salary distributions from different companies, which gave a realistic context to the task. The students were then asked to describe the type of company these distributions corresponded to. This activity combines mathematical and statistical modelling processes. Modelling is considered as both a learning means and aim and statistics as a tool for understanding the world. In addition, statistics invites the students to

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_3

combine different calculus and data processing techniques to develop their reasoning and explanation potential, allowing them to extract conclusions applicable to everyday life.

The type of analysis we suggest is based on the definition of mathematical model offered by Lesh and Harel (2003) and is centred on determining which concepts and procedures make up the students' models. In order to analyse the students' work and characterise the models they obtained, we have developed a visualisation tool for these models in the form of diagrams. Specifically, the diagrams allow us to easily visualise the wide variety of proposals collected and the complexity of some of the working teams' output.

3.2 Theoretical Framework: Mathematical Modelling and Statistics

Although many definitions of models are available in the literature, we use that given by Lesh and Harel (2003) from the field of mathematics education. These authors consider that models are conceptual systems used to construct, describe or explain other systems. Models include both (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions or patterns and (b) the accompanying procedures for generating useful constructions, manipulations or predictions. Therefore, mathematical modelling can be understood as the creation of complex structures or systems (that have interrelated integrating elements) created on the basis of interaction cycles. These systems allow for the understanding of reality by its simplification. Models can be exported to different real-life situations, and they can be generalised and validated. This modelling cycle requires feedback from reality and applicability, as well as effectiveness in solving the situation (Blum and Leiss 2007).

Modelling presents two educational branches: firstly, as a vehicle to convey certain mathematical content and secondly, as a content in itself, in order to encourage and motivate students to face mathematical problems related to the everyday world (Julie and Mudaly 2007). In the latter case, mathematical modelling not only entails having knowledge of mathematics but also mastering the mathematisation process and identifying real-life situations that can be mathematised. This learning process is geared towards applying mathematics to different situations with real-life contexts and adopting new learning outcomes as consolidated knowledge (Siller and Kuntze 2011).

A field in which the task of connecting to reality is most evident is statistics, in addition to measure activities. For instance, the NCTM standards recommend emphasising statistics and probability as essential facts to promote reasoning and the integration of citizens that participate in today's society (NCTM 2000, p. 4). However, despite curricular recommendations, statistics is usually taught in a technical manner in Spain. As argued by Batanero (2000):

> The new primary and secondary education curricula include general recommendations on the teaching of statistics. However, in practice, few teachers teach this topic and in other cases it is touched upon very briefly, or in an excessively formal manner. (p. 6)

Statistics didactics have not received the same attention as other mathematical contents, but the notion of statistical culture or literacy is starting to be introduced as a necessity for citizens of our society. This statistical culture has two main components: (a) people's ability to interpret and critically evaluate statistical information, which they may encounter in diverse contexts, and (b) their ability to discuss or communicate their reactions to such statistical information, such as their understanding of the meaning of the information, their opinions or their concerns (Gal 2002). From this perspective, and following other authors' recommendations (e.g. Doerr and Lesh 2011; Kazak 2009), we understand that the use of statistical activities in the classroom may be helpful in modelling tasks. This is due to statistics being inseparable from its context, and it is possible to offer the students projects and activities which have significant contexts.

3.3 Methodology

3.3.1 Design of the Activity

In this chapter, we will study the output of Year 10 students (15–16 years old) in a classroom activity designed to promote modelling using statistical data. Our study is exploratory and is based on a qualitative analysis. We worked with 72 students (three natural groups, called G1, G2 and G3) from a school in a town close to Barcelona at the beginning of the course and without any previous modelling experience and with basic statistical notions. The students were mainly grouped into teams of three, since we believe teamwork allows for the construction of mathematical models in collaboration with one another. The first author guided the activity in the classroom.

The activity proposed had to be complex enough for the students to create models that were also complex enough, following the guidelines of model-eliciting activities (Lesh 2010). We highlight the following considerations introduced to the design of the activity:

- Model-eliciting activities were not to be developed to be instructional 'treatments' whose worth depends on their use inducing significant learning gains in students.
- Average-ability students often make significant conceptual adaptations related to powerful mathematical concepts during 60–90 min model-eliciting activities.
- Students whose model development work is most impressive in the preceding situations often do not have outstanding academic records on problem sets of the type emphasised in traditional textbooks and tests.

We chose to adapt an activity from the project NRICH, giving two different distributions and asking to determine what type of distribution had been used. We contextualised the different distributions following an idea by Levitt and Dubner (2005) that suggested determining the type of company according to its distribution of salaries. The problem statement of the activity used is the following:

Salaries and Companies
In the following table, we find a list of salaries from five different companies:

(a) Analyse the data so that you can obtain an idea of the structure of each company. Write out your conclusions and the methods you used.
(b) This analysis should provide you with criteria to classify the different companies. Classify them and argue why each of them fulfils the criteria established.

The data collected is made up of 22 audio-recordings of the work developed by each team in the classroom, as well as the final reports given by each of the 22 working teams.

3.3.2 Data Analysis

The analysis developed is based on determining the mathematical objects, such as concepts and procedures developed by the students in the class activity, as well as the relationships they establish during the activity with one another while trying to find the salary structure of each company. After that we constructed a graph for each of the solutions, displaying the mathematical objects present in the resolution at each node, as well as the concepts and procedures involved in the development of the model. In addition, the segments connecting the concepts display the relationships detected during the activity. Table 3.1 shows the mathematical objects most

Table 3.1 Most used mathematical objects by the students during the activity

Symbol	Mathematical objects	Frequency
N	Number of elements in the distribution (used)	16
$\sum x_i$	Sum of values in the distributions (salaries)	5
$\bar{x}$	Average of each distribution	14
$\bar{X}$	Average of averages	2
m	Minimum of the distribution	12
M	Maximum of the distribution	16
I	Intervals	5
$x_i - \bar{x}$	Difference between each element and the distribution average	3
Dx_i	Difference between each consecutive element	2

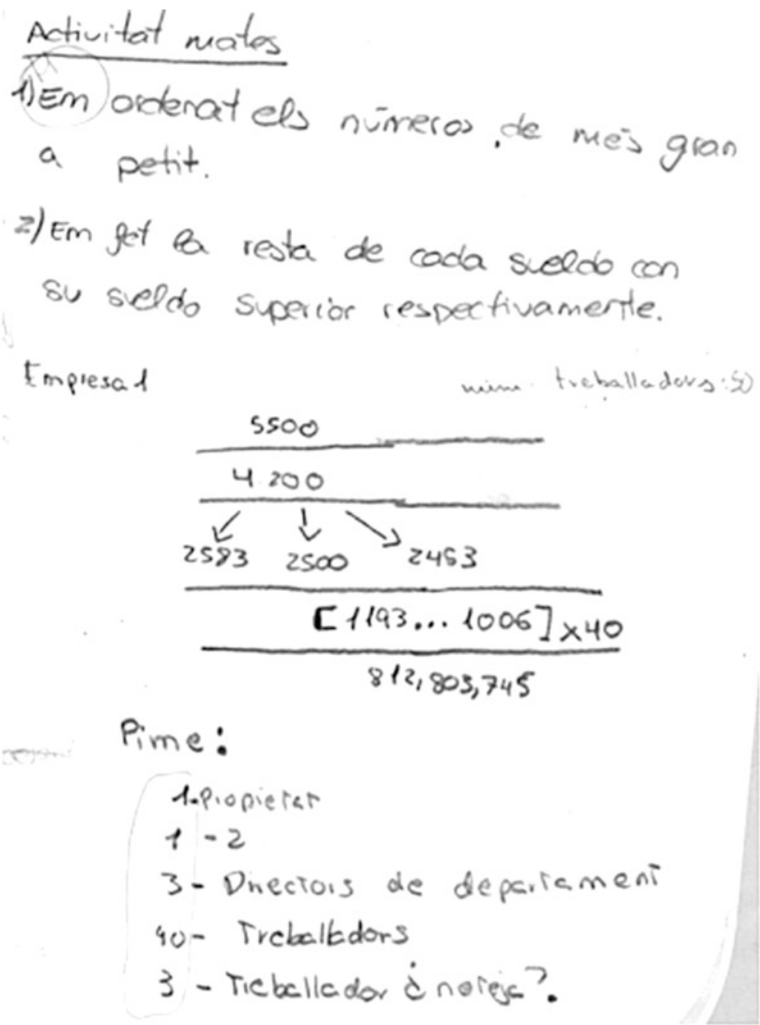

Fig. 3.1 A sketch of the work of group G2_g07

used by the students in this activity (due to space constraints). This list is a result in itself as it shows a wide range of concepts and procedures that support the models generated by the students.

Figure 3.1 shows one of the diagrams from the final output of a team who separated the company's data into different parts. In the text, the students wrote: "*1) We have ordered the numbers from large to small, 2) We have subtracted each salary from the previous salary*." The solution obtained by these students gives effective structural information, since they classified the data into different levels, defined by

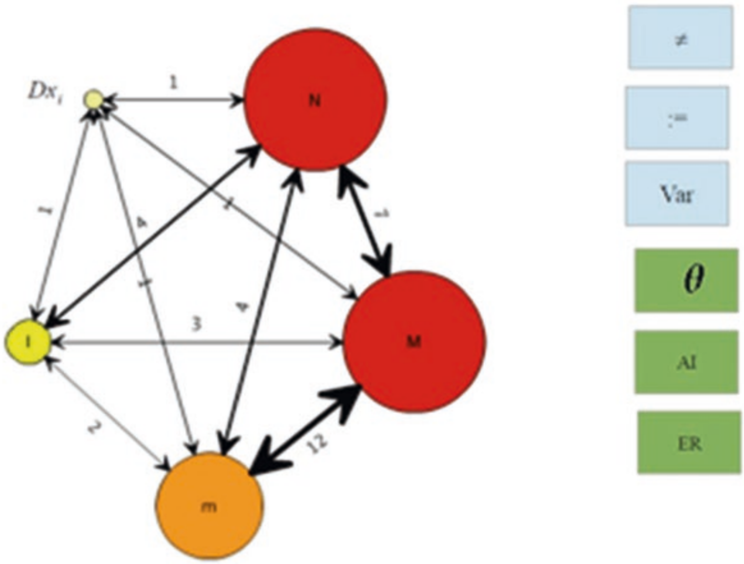

Fig. 3.2 Graph elaborated on the basis of the resolution of group G2_g07

intervals (I). Specifically, they arranged the salaries in decreasing order and grouped them into intervals. However, their intervals were not chosen intuitively; they calculated consecutive differences (Dx_i) between salaries and introduced a new interval of data every time they identified a significant jump from one salary to the next. In the lower part of Fig. 3.1, we can observe the structure proposed for the company with a first owner, a second owner, three department directors, 40 employees and three low-skilled workers.

For each solution, we have generated a graph including the mathematical objects used in the developed model and their connections during the solving process. The radius of each circle is proportional to the number of times the object is referred to (also the colour intensity), and the number in each edge indicates the number of times that connected objects are related during problem solving. In the right part of each graph, we place the concepts (in blue) and procedures (in green) that emerge during the discussion that are proposed but not used in the final model developed by students.

The graph generated for G2_g07 group is shown in Fig. 3.2, where it is also evident that the students have used other concepts such as the number of people in the company (N) and the maximum (M) and minimum salaries. Among other procedures proposed, but not used, we highlight the processes of ordering from minor to major (θ), of comparing with other real and known companies (ER) and the discussion of external variables that may lead to such a distribution (Var). This is how we developed the analysis of the students' solutions and products and created the diagrams that describe each of the generated models when attempting to explain the structure of companies. We display the results of our analysis in Sect. 3.4.

3.4 Results

3.4.1 Relevant Concepts Detected and Complexity

Some of the models proposed by the students are based on concepts that they developed for the activity. An example of this is the work of group G03_g01 (Fig. 3.3), who based their model on calculating the average of the differences between each data item and the total group average. The latter concept clearly includes the main ideas that characterise standard deviation.

On the other hand, the complexity of the models produced by the students can be measured by the number of mathematical objects used to develop their solutions. Table 3.2 shows that four of the productions analysed are based on two or fewer objects, nine of the groups used three objects and nine groups used four or more objects. These results show that this activity allows the students to independently develop concepts relevant to statistics while elaborating high-level mathematical models with a high conceptual complexity.

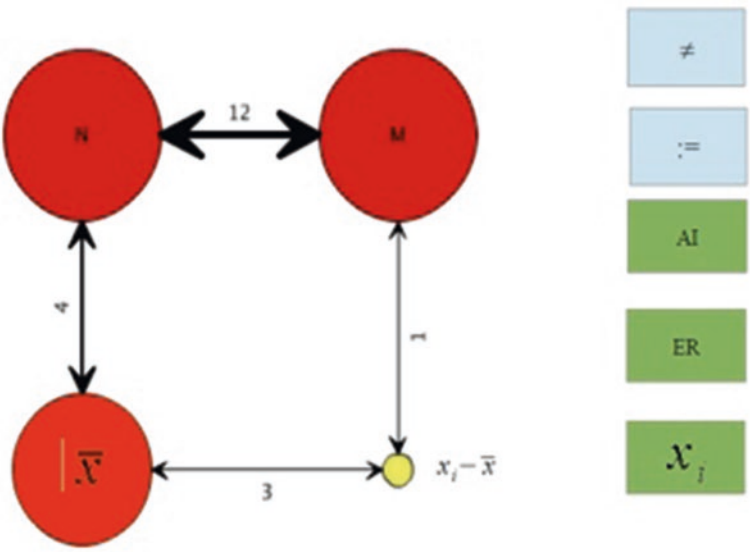

Fig. 3.3 Diagram of the model produced by group G03_g01

Table 3.2 Number of mathematical objects used by each group

1	2	3	4	5	6 or more
G1_g03	G2_g06	G1_g01	G1_g05	G2_g07	G1_g04
G3_g08	G3_g09	G1_g02	G3_g01	G3_g04	G1_g07
		G1_g06	G3_g05		G3_g03
		G2_g01	G3_g07		
		G2_g02			
		G2_g03			
		G2_g04			
		G3_g02			
		G3_g03			

3.4.2 Variety of Resulting Models

Figure 3.4 gathers six of the diagrams resulting from the analysis of the students' productions during the activity, showing the structural variety of the proposed models. We can see that some groups work on the basis of different concepts and procedures but do not relate them. Others establish relationships between all the concepts used, while others base their solution on a small number of concepts and procedures which they modify. The latter case yields diagrams in which some mathematical objects are only related to one other of the objects used.

3.4.3 Responding to a Different Perspective

The proposed activity contains two questions geared to encourage students to approach the main activity – to classify companies according to their salary distribution – from different perspectives. However, we observed that the students did not directly respond to questions (a) and (b). Specifically, in 18 out of 22 products analysed, the students focussed on determining the type of company that corresponded to each distribution. This can be observed in the diagrams, since they only include one mathematical model to respond to both activities. In previous research, we have found situations in which students change the objectives of the activity to approach them to what they consider most natural for the corresponding context, as explained in Albarracín and Gorgorió (2013).

3.5 Discussion

Using the visualisation proposed for the students' models in the form of diagrams that derive from the definition provided by Lesh and Harel (2003), we have searched for patterns in these models. On an initial review of the data, the models proposed

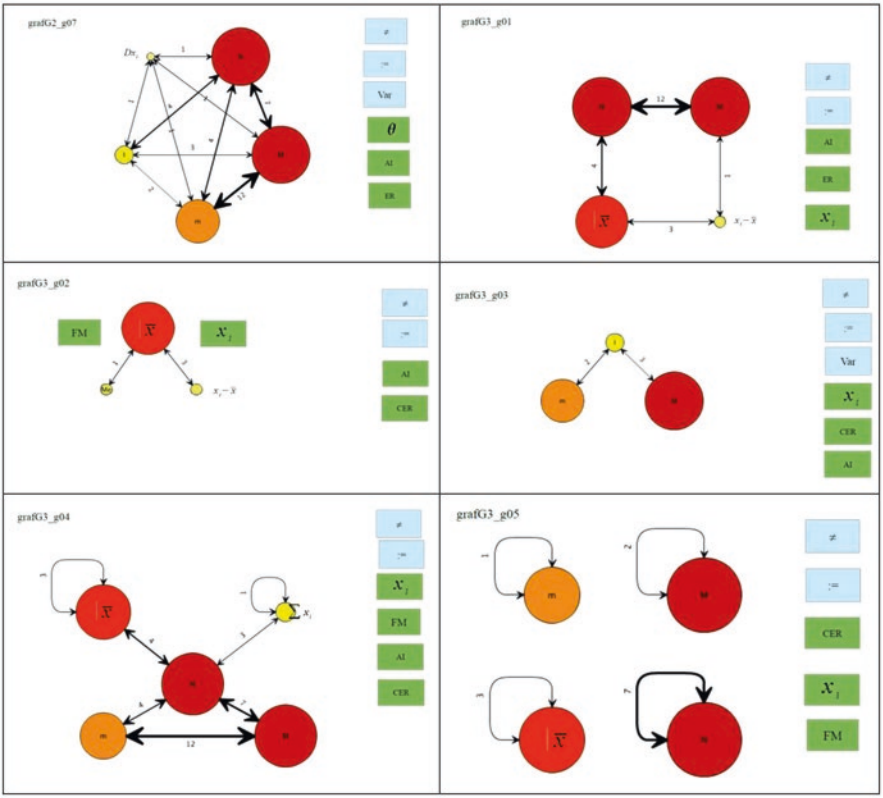

Fig. 3.4 A sample of six diagrams obtained from the analysis

by the students may not seem very different in structure. However, using the aforementioned visualisation, we have been able to distinguish significant differences between them. We were especially able to detect differences in relations between the participating mathematical objects, the concepts that appeared and the procedures used by the students. These graphs gave us a perspective of the complexity of the models developed by the students.

It is noted that this activity has promoted the appearance of important mathematical ideas that the students had not worked on previously, such as the concept of normality, present in many discussions but without taking shape in the last models. Another important concept that appears – that is also difficult to deal with in depth – is that of data dispersion. It was observed that students used different types of diagrams to represent the latter and some teams even elaborated their own methods to determine it. The developed analysis has allowed us to visualise the models generated by the students with a high level of detail, and this complements other tools for analysing students' model-creating process, such as that proposed by Matsuzaki (2011).

On the other hand, the model-eliciting activity allows for different modifications that will lead to working on different concepts, only changing the type of data offered. There are several ways of modifying this activity to obtain data that will allow us to promote modelling processes more effectively. An option would be to increase the amount of data in order to avoid modellers searching for data elements one by one. Alternatively, there could be more data available, and the students' task would be to choose which items are necessary to complement their model in order to identify the structure of the company and its categorisation (hours worked, the company sector, etc.). Providing companies with different numbers of workers could be an obstacle, since on the one hand the modelling should be as realistic as possible; however, the number of workers is, to many, already a criterion strong enough to make a classification. The data could be entered in the problem formulation such as the number of workers and the same average salary to force students to search for other criteria. It would also be interesting to use activities that present equivalent mathematical procedures but include new contexts to promote the use of different solution strategies, as studied in Albarracín and Gorgorió (2014).

3.6 Conclusions

Based on the results of our study and following the definition of Lesh and Harel (2003), students create models with different levels of complexity to give answers to the questions posed to them in the activity. Some of the models studied can be applied to real-life situations similar to that which was proposed, where the same mathematical concepts and procedures may intervene and questions arise of a similar modelling level of difficulty. However, this study does not guarantee that the students will be able to solve these alternative activities. In our study, only two groups used a single mathematical object to generate the model on which they based their solution. Other groups used a larger number of concepts to construct their models, but we understand that the key indicator is the number of related concepts that allow for establishing a model. It is necessary to carry out new studies in this direction.

We observe how the activity proposed created the need to quantify the difference or similarity of salaries. It can therefore be a good activity to introduce all the content related to dispersion and the different ways of measuring it. During the resolution process, students are informally discussing a large number of statistical concepts (such as the normality of a distribution). From this point of view, this activity could be the first of a didactic sequence aimed at working on data distribution and dispersion.

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Chapter 4
How Teachers Can Promote Mathematising by Means of Mathematical City Walks

Nils Buchholtz

Abstract By using mathematical city walks, teachers can promote competences in mathematising. In this out-of-school activity, the independent setting up of mathematical models is practised based on meaningful reality-based tasks. It is crucial that the tasks are appropriately selected for the cognitive skills of the students and include basic ideas specific to mathematical topics. The chapter analyses an example of a task according to basic ideas contained, potential difficulties and possible solutions. Based on the reconstructed basic ideas, teachers can also use the tasks of a mathematical city walk diagnostically. For this purpose, students can be interviewed in a diagnostic interview about their solution approaches.

Keywords Mathematising • Tasks • Basic ideas • Math trails • Percentage • Diagnostic dimension

4.1 Introduction

Students are often faced with specific challenges and difficulties when solving real-world application tasks in mathematics education as the fundamental step between the reality context of a task and its mathematical solution – the so-called mathematising – requires skills in putting up adequate mathematical models and interpreting the outcomes from using a model in relation to the problem situation (Blum 2007; Galbraith et al. 2007; Turner et al. 2015). The mediation of these central skills is one of the tasks of teachers when dealing with modelling activities, but its instructional implementation can run the risk to be limited to the use of deficient embedded word problems following the strategy: "*Ignore the context, just extract all data from the text and calculate something according to a familiar schema*" (Blum 2015, p. 79).

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
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In the worst case, tasks can even neglect an autonomous mathematisation (e.g. if mathematical models are already pre-made and only their fit in practical situations has to be evaluated). To avoid this, teaching mathematising requires, however, not only the ability to develop suitable learning environments and to select appropriate tasks but also a specific diagnosis to assess the skills of the students in this area properly (Borromeo Ferri and Blum 2010).

This chapter presents possibilities for the promotion of mathematising specifically by out-of-school activities and their respective requirements for teachers. For this purpose, the chapter is based on the development of meaningful reality-based tasks of a mathematical city walk for percentage calculations in Hamburg – a special type of a math trail (Shoaf et al. 2004). One focus of the chapter is the didactical considerations for the development, selection and use of appropriate reality-based tasks, if such teaching projects are planned in mathematics lessons. A second focus is the diagnostic dimension of such non-formal learning opportunities. The chapter illustrates exemplarily, by an analysis of a student solution, how teachers can use the respective tasks to identify errors and misconceptions of students in order to develop appropriate individual support measures.

4.2 Theoretical Embedding

For the field of mathematical modelling, already different approaches exist to describe the relevant skills of teachers (e.g. Blum 2015; Doerr 2007; Kaiser et al. 2010). Borromeo Ferri and Blum (2010) describe a four-dimensional model of teacher competences for teaching mathematical modelling, which includes (1) a theoretical dimension and (2) a task dimension, that comprises skills that teachers must have in order to assess the potential of modelling tasks (e.g. in terms of multiple solutions or the cognitive challenge of the content). The model also includes (3) a curricular dimension about knowledge for planning and implementation of modelling in the classroom and (4) a diagnostic dimension of teacher competences that refers to the assessment of student achievements in the field of modelling. This fourth dimension includes, for example, not only an awareness of the various substeps in the modelling cycle but also the assessment and evaluation of student solutions in terms of their technical adequacy, conformity and accuracy. In addition, knowledge about learning disabilities and anticipation of possible errors related to modelling belongs to this dimension. When teaching mathematising, especially knowledge in the area of tasks (see Sects. 4.3 and 4.4) and the diagnostic dimension (see Sect. 4.5) is required.

The transformation of reality-related contexts into mathematical structures, concepts or models – the so-called “mathematising” (Freudenthal 1983) – occurs not only in complex modelling processes but also in much more elementary tasks such as simple measuring. In educational standards and international assessment studies like PISA, mathematising is accordingly seen as a central aspect of mathematical modelling; however, even simple translation processes between an extra-mathematical situation and mathematical models are recognised (see Turner et al.

2015). When students mathematise, they have to enable content-specific images that are related to the context in which the mathematisation takes place.

These content-specific images are described within the German mathematics education discussion as so-called *Grundvorstellungen* or basic ideas (vom Hofe 1998; Kleine et al. 2005) or internationally as concept images (Tall and Vinner 1981). Basic ideas can be understood both as distinct individual patterns for the acquiring of a mathematical concept and ideas for operating with mathematical content. Basic ideas build on known facts or contexts and support the development of visual representations for the internalisation of a mathematical concept and so enable learners to apply concepts to reality by recognising mathematical structures in real contexts (vom Hofe 1998). So, especially for mathematising, basic ideas play a decisive role.

To exemplify and specify the subject-specific context of mathematisation, three basic ideas of percentage calculation are described (see Hafner 2011):

1. *The ratio- or of-hundred idea*: The proportion $p\%$ is transferred to a (fictional) situation with 100 units. Here, the percent sign is linguistically interpreted as "… of hundred", that is, of 100 units, p units are referred to.
2. *The hundredth- or operator idea*: The specified $p\%$ is seen as a hundredth fraction $p/100$ and multiplicative operator. For a fixed percentage, each percentage is calculated by multiplying the variable reference with the constant hundredth fraction.
3. *The unit- or quasi-cardinal idea*: By an assignment of the size range of the reference variable to a notional size range of percentages, one can calculate percentages. Percentages are in this case treated as an independent entity with the unit %. The whole corresponds to 100%, equivalent to one hundredth being 1 %. In particular, $p\%$ of the whole corresponds to p times the hundredth part of the whole.

Teachers need "to accurately gauge the difficulties and cognitive demands of the tasks and the prior knowledge of their students" (Brunner et al. 2013, p. 230). For this, basic ideas have a specific diagnostic use in mathematics education: When it comes to the learning of mathematical content or processes like mathematising, basic ideas can be used from a normative point of view for a didactic description of mathematical tasks. They describe in this case what students should ideally learn based on a task. So when selecting or developing appropriate tasks, teachers can consider the extent to which a cognitive challenge is ensured by the integration of different basic ideas and to which extent students are able to set up mathematical models (i.e. the task dimension of teacher competence). On the other hand, basic ideas can also be used in a descriptive way for a description of individual images and strategies of the students. Interpretively, they describe what students actually imagine when working on tasks and how these ideas influence their solution of the tasks. By studying the solution strategies of students, teachers can gain insight into the formation of specific basic ideas by individual students and thus diagnose mathematising competences (i.e. diagnostic dimension of teacher competence) in order to provide individual learning support (cf. Brunner et al. 2013).

4.3 The Mathematical City Walk

Mathematical city walks are a special type of math trails. Shoaf, Pollak and Schneider describe them as walks to discover mathematics:

> A math trail can be almost anywhere – a neighborhood, a business district or shopping mall, a park, a zoo, a library, even a government building. The math trail map or guide points to places where walkers formulate, discuss, and solve interesting mathematical problems. (Schoaf et al. 2004, p. 6)

They have been known since the 1980s and exist as an out-of-school leisure activity for families and persons interested in mathematics (Blane and Clarke 1984). The range of mathematical content in math trails can extend from primary to secondary education and so accordingly the complexity and difficulty of the selected tasks. A central idea of the math trails is that students solve mathematical tasks and problems on specific objects in the city or in the surroundings by estimation or measurement of realistic sizes (see Ludwig and Jesberg 2015). However, the mathematics required for the tasks of math trails often varies greatly. But in this special form of mathematical modelling, in particular, methods of simplifying and mathematising play a central role that represent cognitive hurdles for many students. It therefore seems advisable to promote mathematising via mathematical city walks exclusively in only one content-specific area, which should have played a central role in the classroom before, and to adapt the tasks to the cognitive abilities of the students (also with regard to a manageable processing time).

The mathematical city walk for percentage calculation in Hamburg has been tested with a class of seventh-graders of a Hamburg Gymnasium. The class consisted of 25 pupils, 9 of them girls and 16 boys. The class did not have much experience with modelling tasks, but percentages had recently been taught in class. The city walk consists of four tasks with respective subtasks, comprising concepts and basic ideas of the percentage calculation (see Fig. 4.1 for an example). The tasks are designed to ensure that the students have to carry out concrete measurements and identify required quantities on the objects autonomously in groups of threes. They process the mathematisation by a meaningful assignment of determined variables into a mathematical model. At each individual station of the trail, the groups are allowed a processing time of 20 min (so the walk takes about 90 min in total, which is two teaching lessons). Each group is equipped with a tape measure and a folder in which the tasks are collected. All solutions to the tasks must be recorded on a group worksheet and submitted for a later consideration of the learning process.

The process of mathematising within the tasks takes place in comparatively small steps and the developed mathematical models are of low complexity. On the one hand, this is done to support independent mathematising piece by piece (the students are grade 7), and, on the other hand, this keeps the processing time of the tasks generally quite short. For an overview, the following important criteria for tasks which teachers should consider when planning a mathematical city walk are summarised.

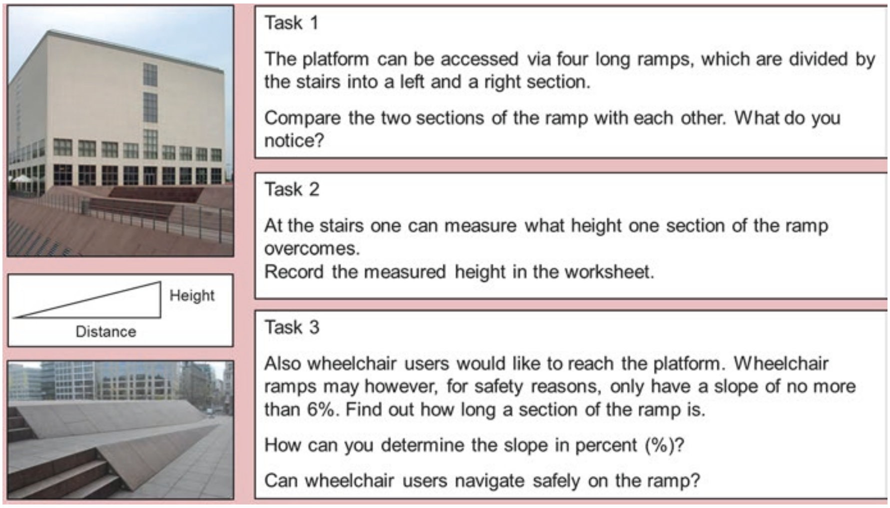

Fig. 4.1 Task *Art Gallery*

The tasks of a mathematical city walk:

- Relate to content previously treated in class,
- Take into account different basic ideas of the topic,
- Encourage students to mathematise independently,
- Are characterised by a sufficient degree of openness (from the number of possible approaches for determining a particular solution up to the number of possible solutions),
- Are related to the associated objects and should not be solvable without them,
- Can be solved by determining sizes on the spot,
- Have a realistic problem orientation,
- Have differentiating features, such as a stepped task format,
- Promote collaborative work,
- Should not exceed the processing time of 20 min,
- Should be accessible within 10-min walk.

In the following example, the task, *Art Gallery*, as well as its requirements with respect to mathematising, will be analysed. Attention is paid to the basic ideas contained in the task and the anticipation of learning difficulties or potential errors. Such task analyses are important for teachers in the area of knowledge of tasks, referring to the second dimension of the framework of Borromeo Ferri and Blum (2010).

4.4 Task Analysis

In the task, *Art Gallery*, the students have to measure the sizes of the ramps on the platform of the Hamburg Art Gallery so that the slope of the ramps can be determined in percent (see Fig. 4.1).

First, the students have to estimate, measure or count floor tiles or step lengths to make a comparison between the lower and upper sections of the ramps on the platform, which have the same dimensions. To determine the slope of the ramp in the third task, the students are to determine in the second task the height difference that is overcome by the ramps. This perpendicular distance (75 cm) can be best measured by means of the stairs; however, the students might also come up with the idea to take the sloping side of the stairs (104 cm) as a measure to look for the height of the ramp, in particular if the meaning of the term "perpendicular" is still unclear (cf. Fig. 4.2). The third task involves identifying the length of the ramp and finding a method for calculating the slope, the actual mathematisation. Since the exact distance (1152.25 cm) is not directly measurable, we are content with determining the horizontal distance of the ramp from the ground tiles (11.5 m = 1150 cm). The difference between the two sizes is small and can be revisited in grade 8 during the treatment of the Pythagoras theorem. Also an interpretation of the result with regard to the problem is demanded. One possible approach for a mathematisation can be seen in setting up a relational equation (75 cm/1150 cm equals the sought percentage/100 %) or using the formula to calculate the percentage ($p = (75 \times 100)/1150$). Both approaches are mainly based on the ratio- or of-hundred idea (see Sect. 4.2), in which the difference in height must be understood as a certain hundredth part of the horizontal distance. If the sizes were determined correctly, the result is a slope of about 6.5%; in the case of erroneously using 104 cm, the result is a slope of about 9%. With regard to the problem for wheelchairs, it can be stated that driving onto the ramps would not be safe in either case, although in the first safe driving on the

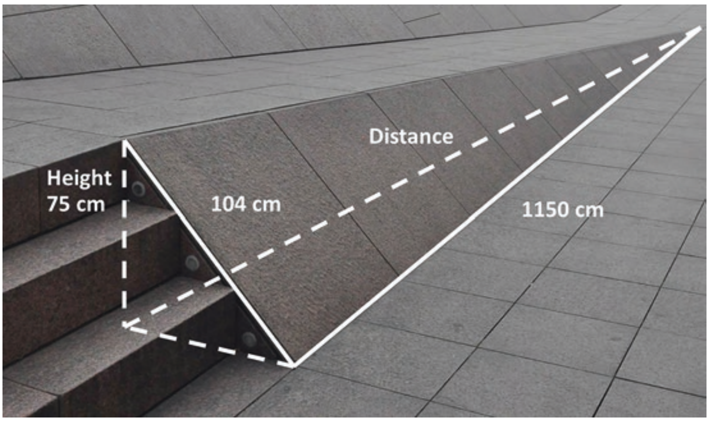

Fig. 4.2 Relevant measures of the ramps of the task, *Art Gallery*

ramps would certainly still be possible. Possible learning difficulties for an adequate mathematisation might occur by the difficult assignment of determined variables into a mathematical model by the already mentioned incorrect determination of the height and when it comes to dealing with the various units m and cm. Another possible source of error is if the students already interpret the ratio of height and the horizontal distance (0.065) as a percentage (0.065%).

4.5 Diagnostic Use of Tasks

Looking at the diagnostic dimension of teacher competences for teaching mathematical modelling (see Sect. 4.2), teachers can obtain valuable information about the students' competences in mathematising from an analysis of the solutions of the students and a comparison between the basic ideas intended by the tasks and the solutions of students (see Hafner and vom Hofe 2008). For this purpose, solutions can be examined according to possible misconceptions in order to derive appropriate supporting measures. In the following, one example of a students' solution of the task, *Art Gallery*, is analysed in this way.

4.5.1 Sarah's Solution

In order to find a reasonable solution, Sarah mathematised by making a sketch of the situation in which she entered the measured dimensions for the variables (Fig. 4.3). Unfortunately, the quantities (height and horizontal distance) are interchanged in her solution and furthermore measured in different units (m and cm), which results in a calculation error. The notation of 77.5 ÷ 11.5 indicates – despite the mathematical errors – that Sarah wanted to set up a ratio equation by comparing two determined distances to each other. Looking closer at the mathematisation, it is noticeable that she obviously was uncertain about which determined size had to be divided by which. Together with the other students of her group, Sarah has finally agreed on the proper mathematical model but has operated with incorrect units. The result 6.73 for the percentage of the slope is coincidently correct in this case because the quotient of height and horizontal distance (0.0673) needs to be multiplied by 100 to specify the percentage of the slope (6.73%). Sarah's answer to the task however is related to the problem and interprets the mathematical result correctly. This re-translation of mathematical results with regard to a real solution actually shows the progress of Sarah in the acquisition of mathematisation competence.

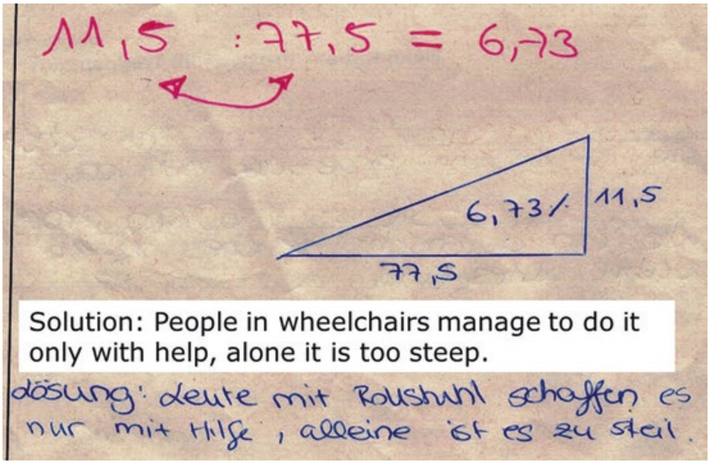

Fig. 4.3 Sarah's solution of the task, *Art Gallery*

4.5.2 Diagnostic Interview

Findings from the New Zealand Numeracy Development Projects recommend the use of diagnostic interviews, in which the students are asked about their solutions (cf. Ministry of Education 2008; see also Hafner and vom Hofe 2008). With these kinds of interviews, at least partially, insights into the development of competences in the field of mathematisation can be gained. However, a considerable number of such interviews based on diagnostic findings can be conducted for individual cases only. A list of diagnostic questions that teachers might use follows:

- Can you describe the task in other words?
- What can you find out from this task, anyway?
- Can you explain your steps to solve the problem?
- Are there also other solutions?
- Which variables play a role in the calculation?
- How did you determine these sizes? In what unit are they given?
- How are the variables brought together in your calculation?
- Can you describe your calculation in words?

The diagnostic interview with Sarah and the other students of their group revealed that Sarah had difficulties with mathematical quantities. Only with a strong focus on her sketch and a lot of help could she identify and explain her calculation approach:

T: OK, why is this supposed to be calculated that way?
S: Because 77.5 is this lower surface [sic], and 11.5 is this area [sic]. 11.5 divided by 77.5 would make no sense, we tried that. [...] That is why we then exchanged the values. [...]
T: So why did you divide the sizes at all?

S: I always memorised, that if you have to calculate percentages, you have to divide the sizes.

T: That's right. But do you remember why you have to do that?

S: *(points to the base of the triangle)*
This is the basic value, I think, because that's down here. [...] This is the percentage *(points to the height)*. And that I have to divide.

T: Aha. What do I have to divide by what? Percentage by basic value or basic value by a percentage?

S: Percentage by basic value! For example, if one has 12% slope, that is 12 m divided by 100 m.

The interview revealed that Sarah principally has an understanding of the concept of percentage as a ratio, as she refers to her prior content knowledge and has the ability to assign the sizes in the sketch to her calculation. However, there are still uncertainties with the mathematical terminology. Sarah's individual image of percentage seems to be a ratio, but from her comments, it becomes clear that this conceptual understanding could be memorised without understanding the background for the assignment of sizes and their dimension. Sarah's competences in the field of mathematisation must therefore be promoted conceptually, for example, by explaining other solutions of the students. In particular, further exercises for the comprehension of the concept of percentage calculation can also be useful here.

4.6 Conclusion

The results of the mathematical city walk revealed difficulties the students had with the tasks, but they are encouraging from the diagnostic point of view. The diagnostic interview with Sarah was useful and revealed some of her misconceptions, which can be subsequently worked on. Obviously, it is very difficult for students in grade 7 to deal with reality-based tasks when mathematical approaches need to be set up autonomously; an experience that is also shared by Blum (2007). The tasks of the mathematical city walk therefore can provide the students at least a learning opportunity for mathematising. Teachers should seek to integrate this central mathematical activity in mathematics lessons more often and need to select tasks and look at solutions thoughtfully. By using a mathematical city walk, it is possible to create incentives for autonomous mathematising based on real-world problems in a delimited thematic context.

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Chapter 5
L'Hospital's Weight Problem: Testing the Boundaries Between Mathematics and Physics and Between Application and Modelling

France Caron and Kathleen Pineau

Abstract Although this classical optimisation problem can be considered an application, some of the difficulties it brings to students have to do with modelling. We show how an activity, designed from this problem and trialled in a transitional mathematics course of a technical engineering school in Montreal, allows students to experience, within the goals and practical boundaries of the course, some elements of the modelling process and to develop skills useful for that purpose. As such, it can serve as inspiration for gradually introducing modelling considerations in content-driven mathematics courses that do not traditionally allot time for exploring open situations. The crossing into physics, despite the strong potential envisioned at the design stage, proved to be more difficult to implement.

Keywords Modelling process • Modelling paradigms • Application • Interdisciplinarity • Mathematisation • Scaffolding • Function composition • Interpretation • Validation

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_5

5.1 Introduction

Modelling has been a topic of growing interest in mathematics education research, since before it was even mentioned explicitly in the mathematics curriculum (Lingefjärd 2007). Although some national school curricula or specific university courses or programmes now clearly refer to modelling as an objective or as an approach to the learning of mathematics, there is still substantial effort required to make it a reality in most mathematics classes, as "the majority of the teachers have yet to experience the role of a modeller and hence have difficulty acknowledging the potentials of the use of modelling tasks in their classrooms" (Ng 2013, p. 339). This is even more of a challenge in courses that are essentially content driven and for which the learning objectives and time constraints appear to prevent students from entering any rich modelling activity, let alone approaching an open-ended situation. In these courses, introducing problems where the concepts that have just been taught can be applied directly seems to be the best a teacher can do in order to connect mathematics with some of their uses in the 'real world'.

With the teaching of functions, this interest in short application problems, combined with the presence of technology, tends to reduce modelling to curve-fitting exercises in many school curricula. If such an approach can help select among possible function candidates for a given relationship within a given domain, it also runs the risk of implicitly promoting "subversion of reality by choices available on the menus of calculators" (Galbraith 2007, p.81). More generally, restricting modelling to the *empirical* paradigm may well miss on the essential aspects of a given reality, its underlying structure with the key relations and interactions at play. Although such a paradigm can assist in making predictions, it does little to deepen understanding of a situation; for that, one has to turn to the *theoretical* paradigm (Maull and Berry 2001), by studying the underlying processes and by identifying appropriate laws and principles, which may come from another discipline and from which a model can be built and later validated with data. Maull and Berry noted that students tend to approach a problem with the empirical paradigm, even when a theoretical approach would be accessible and more appropriate.

To allow for a better representation and a stronger presence of modelling in the learning of mathematics, one can either promote major curricular changes and hope that teachers will embrace and implement these changes as soon as they are made effective – the 'big bang' approach, as labelled by Burkhardt (2014) – or look for openings in today's curriculum and teachers' practices to find ways in which to build gradually on what is already done or could be done with existing goals, levers and constraints. The activity presented here fits this second approach: by working with teachers of mathematics and physics, a rich application problem was transformed into an engaging activity that allows students to experience some elements of the modelling process (Blum et al. 2002) while learning or applying specific strategies, concepts and skills, useful for getting to the underlying structure of the situation. The collaborative effort that took place could be associated to a design research approach, where there was a clear intention of understanding the teachers' needs and institutional constraints. The development and first validation of the activity within a class of transitional mathematics in a technical engineering school allowed us to test to which extent teachers and students could cross, in such context, the boundary between application and modelling and the one between mathematics and physics.

5.2 Origin of the Activity

The activity was designed as part of a larger project initiated in 2007 by three Montreal school boards (Caron and Savard 2012). Faced with a lack of learning activities for a new applied stream of secondary school mathematics in Québec, pedagogical counsellors turned to École de technologie supérieure (ÉTS) to collaborate on the design of hands-on activities for the concepts and skills to be learned.

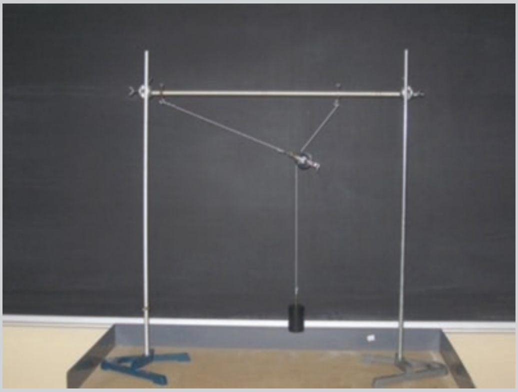
ÉTS is a technical engineering school whose programmes are offered in the form of cooperative education and are tailored to persons holding a technical college diploma in engineering technology. The school relies on senior lecturers in the Department of General Studies to administer its common core courses and ensure, through transitional courses, the meeting of requirements by all students, who come from diverse scientific backgrounds. The close proximity of lecturers in mathematics, physics, chemistry and computer science within the department was key in developing hands-on learning activities.

The interdisciplinary and inter-level collaboration developed further with the progressive entry of college (cegep) teachers. From the initial goal of motivating students for mathematics and sciences, the project added to its objectives the support of secondary-tertiary transition. This allowed some of the activities, such as *L'Hospital's Weight Problem*, to also be used in cegep courses and in the transitional mathematics course offered at ÉTS.

5.3 L'Hospital's Weight Problem

This problem was initially presented in 1691 by Johann Bernoulli to the Marquis de L'Hospital as an introduction to differential calculus.

L'Hospital's Weight Problem
A string is tied to a pulley and attached on the left of a horizontal bar. Another string, attached to the right end of the bar, passes through the pulley and ends with a weight. Knowing the lengths of the strings and the distance between the two end points of these strings on the horizontal bar, the problem is to predict the equilibrium position of the weight.



The potential of this problem for use in a contemporary calculus class was explored by Van Maanen (1991) and, later with the use of a symbolic calculator, by Drijvers (1996, 1999). Van Maanen's classroom experiment was conducted using one physical apparatus for the class and fixed values for the parameters of the problem (length of the strings and distance between the end points). Drijvers returned to the problem (1996) to show the rich learning activities one can approach with a CAS calculator. By observing how students struggled in solving the generalised form of the problem, he attributed the obstacle they faced to their difficulties in discerning between variables and parameters (Drijvers 1999).

Although the problem itself can be considered an *application*, in the sense of Stillman (2008), both these experiments suggest that modelling is at the core of some of the difficulties experienced by the students and that some form of scaffolding should be considered to allow students to benefit from the rich learning potential of the problem.

5.4 Developing Strategies and Skills for Modelling

In keeping with the learning objectives of the applied stream for secondary schools, the problem was transformed into a hands-on activity where all students could interact with their own apparatus and use measuring tools to assist them in mathematising the system.

As it developed, the transitional mathematics course at ÉTS became the field where the activity would be first experimented, near the end of the semester, in two sections of about 40 students, grouped in teams of two or three. Within this context, the learning goals for the activity were to consolidate knowledge and the use of trigonometry, functions and the recently encountered derivative while developing skills for constructing and validating equations to model a system, playing with variables, parameters and composition of functions. The learning activity comprises the following phases:

1. Students become familiar with the components of the apparatus through exploration and assess the dependencies between the different lengths and angles.
2. Using geometry and trigonometry, they construct equations that encode the relationships between the different lengths and angles.
3. They test the validity of their mathematical model by measuring the various lengths on the apparatus.
4. Using their validated formulas, they predict the equilibrium position of the weight for a new parameterisation of the apparatus, by looking for the maximum on a graph of a function and by using differential calculus.
5. They check how well they predicted the weight's position by setting the apparatus to the given parameters and measuring.
6. They refer back to their initial exploration and assess the extent to which the equations reflect the dependencies that they had anticipated.

An additional phase was initially included, where the same problem was to be approached with the vector analysis of forces, as is typically done in physics. But it was tried only with the first group and then dropped, due to lack of time, heterogeneity in students' prior knowledge and the perceived distance with the content covered in the course.

Although empirical data is used and the goal seems to be for students to predict, the modelling performed, even without vector analysis, does not reduce to a curve-fitting exercise. Rather than being confined to the *empirical paradigm* (Maull and Berry 2001), the model aimed at by the activity embraces the *theoretical paradigm*, as students must analyse the situation and call upon known concepts and properties to build their model, understand the situation and explain their results.

The interaction with the physical apparatus offers major benefits with respect to modelling. First, it provides a learning environment that enables students to explore and anticipate the behaviour of the system, to formulate their model and to interpret and validate the formulas they build. Second, with the possibility of changing both the length of the two strings and the distance between their end points, it lends itself

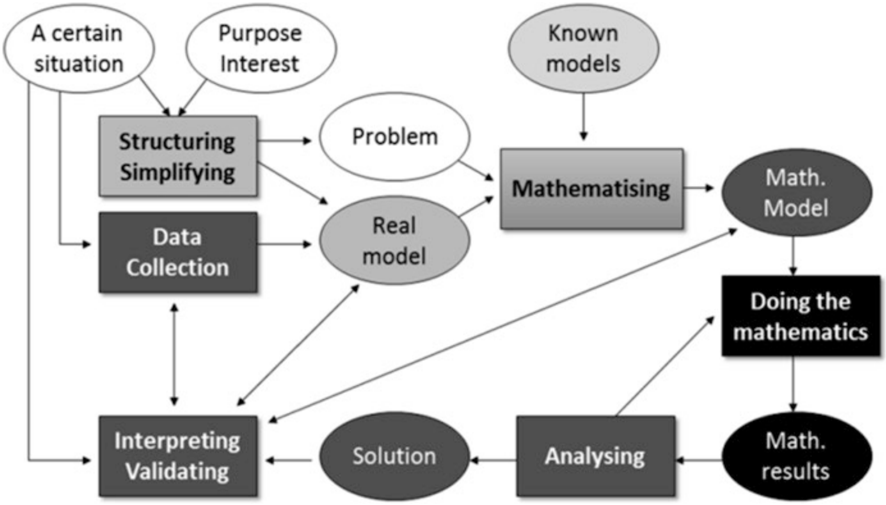


Fig. 5.1 Phases of the modelling process addressed by the activity

to testing various configurations. These benefits could be realised with a simulator, but the physical apparatus offers additional advantages. It confronts the differences between the results produced by a mathematical model and the measurements on the "real thing". In addition, with the effort required to change the lengths involved (by undoing the screws, adjusting the length and redoing the screws), as opposed to simply moving the pulley and observing the resulting position of the weight, it helps distinguish between parameters and variables.

Built on our own representation of the modelling process described in Blum et al. (2002), Fig. 5.1 illustrates in three shades of grey the phases that are experienced by students during the activity. A lighter shade is used for a phase where students are less autonomous. For instance, although they are free to explore the situation, they are guided in structuring it with questions such as "*Does the equilibrium position of the pulley change when you change: a) the length of the string that holds the pulley? b) the length of the string that holds the weight? c) the distance between the fixed end-points of the two strings*?" Yet, they get to experience, within a qualitative version of the empirical paradigm, a systematic analysis of the distinct role of the parameters, a strategy that they could reinvest in a future situation. Similarly, as will be shown in Sect. 5.5, the mathematisation part is also assisted with strong scaffolding. This is not only to prevent students from getting lost in the amount of variables and equations, but it also aims at having them learn how simple relations can be combined to build a rather elaborate model, a necessary skill to enter the theoretical paradigm for modelling a complex situation and gain a deeper understanding of the interactions at play.

5.5 Integrating Scaffolding

In providing scaffolding for building the model, we did not try to comply with Aebli's "principle of minimal support" (cited in Kaiser and Stender 2013). As the teacher of mathematics expressed genuine concern that a greater degree of modelling freedom could lead students away from the concepts and skills aimed at by the activity or prevent them from completing the task within the allotted time, we rather looked to provide support that would allow most teams, while retaining some responsibility in the modelling process, to evolve on their own in the direction that served the mathematical concepts at the core of the lesson, to assess the value of their progress and to correct their work if required. The support provided by the workbook, the apparatus and the teammates was accompanied with teacher interventions, on an as-per-needed basis.

A partially defined model with a schematic representation of the situation was given for students to complete. The intention was to have students experience how a relatively complex model can be built from combining simple relations, through arithmetic operations and variable substitution (or function composition). Figure 5.2 shows an example of the model completed by a team, with a rather creative (if not rigorous) use of references.

We also provoked, explicitly in the workbook, regular confrontations between models and apparatus. In particular, the activity has students frequently move back and forth, through mathematisation and interpretation, between their real model (as expressed in their qualitative description of the system) and their mathematical model.

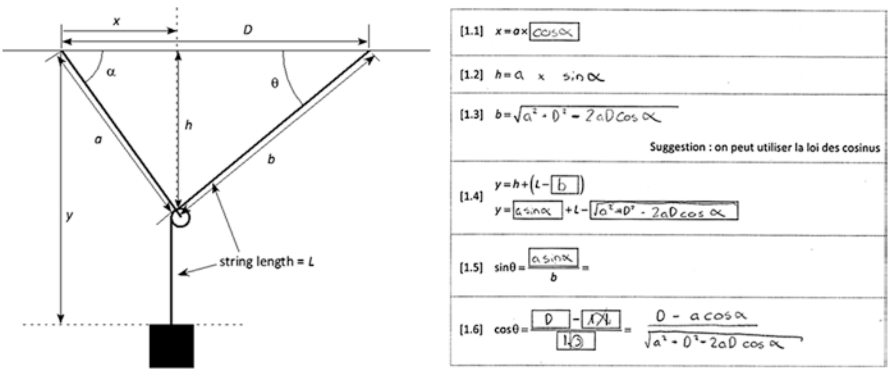


Fig. 5.2 Completing the model

5.6 Some Experimental Results

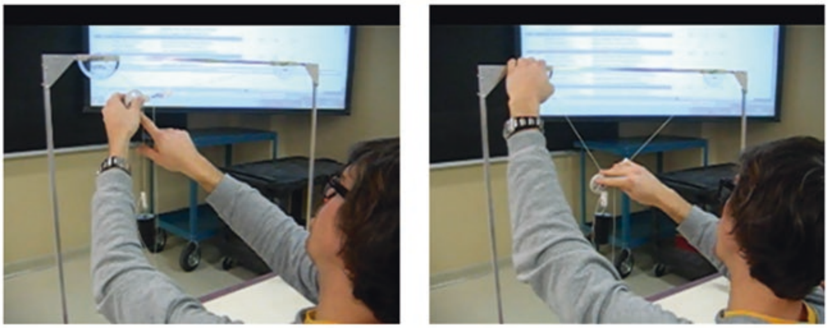
In the last 4 years, the activity has been used in the transitional mathematics course at ÉTS and, more recently, adapted versions in cegep and secondary classes. We report on the initial experiment at ÉTS. Photos and videos were taken in two classes of about 40 students and added to the qualitative data from the completed workbooks. A questionnaire was also distributed to collect information on students' perception of the activity.

Of the students, 77% declared having enjoyed the activity "a lot", while 23% said they enjoyed it "a little". Their appreciation had to do with the "concrete" character of the situation, the discovery of a "real-life application of calculus", a "different approach than the one seen in physics" and the "progressive learning" provided by the scaffolding in the workbook, with a "possibility to review and understand, to validate and self-correct". The derivative, the trigonometric ratios and the relationships between variables, parameters and functions were the main topics that the students felt they understood better after the activity.

Most teams correctly assessed in qualitative terms the relationships between the different variables and parameters. However, a few teams reviewed their initial appreciation of these relationships after having built and worked with their mathematical model.

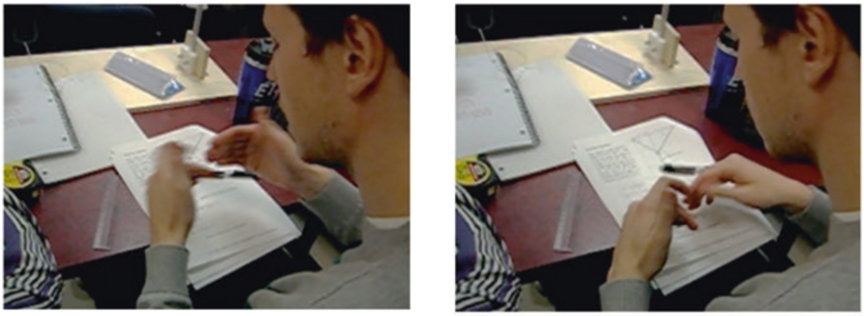
We were struck by how often, throughout the activity, students took the initiative of returning to the apparatus in order to get a better appreciation of the dynamics involved (Fig. 5.3), and this, in addition to the planted provocations.

We expected the construction of the equations would be greatly facilitated by the schematic representation of the system and the organisation of the partially defined model. Yet, this apparently simple exercise clearly marked a moment of revelation



"Check, check… If the angle changes, then obviously, the pulley has changed position."

Fig. 5.3 Exploring the apparatus by playing with the pulley



"You take your h here, you put it there. You take what's in your b, and you put it there. Actually, it's just… You go down. You look at your base formula, you can't use the b, what you do, you unpack it, you can't use your h, you unpack it with what you've found in 1.1. […] The order doesn't make a difference, actually."

Fig. 5.4 Substituting variables in composing functions

for some students, as they worked their way through the different substitutions (Fig. 5.4). In answer to the appreciation questionnaire, one student stated that he had learned with the activity how to "play with and construct complex and universal formulas".

In trying to have them maintain a critical perspective on the modelling, students were asked to explain where the differences between the values predicted by the model and the measurements came from. It is interesting to note that all of them pointed the finger at the data collection (measurements, instruments), ignoring the fact that the model assumes many ideal conditions that cannot be met: a weightless pulley, no friction, etc. This is not surprising, as the structuration and the mathematisation had been done essentially from a geometric viewpoint.

5.7 Crossing Disciplinary Boundaries

At the border of secondary and post-secondary instruction, *L'Hospital Weight Activity* also sits at the boundary of mathematics and physics. The contribution of a physicist to the design brought forward the value of crossing into physics, and the vector analysis of forces, as this perspective added to the explanatory value of the modelling activity. Reconciling the equations generated by this approach with those that come from calculus also offers moments of intense algebraic manipulations.

The physics approach extended the duration of the activity beyond what could be afforded in the transitional mathematics course at ÉTS. In addition, as vectors are not part of the content covered in the course and not all students can be expected to have prior knowledge of them, the integration of the analysis of forces was not perceived as sustainable.

With secondary schools, the connection of the problem with physics clearly worked against its use, as physics can rarely be assumed to be a course shared by all students of a given class. The removal of the physics portion eventually led some secondary teachers to try the activity, in its pre-calculus version, and use it for some of their groups.

In its first implementation in the science programme at one of the cegeps, the interdisciplinary learning potential of the problem was tackled by the physics and the mathematics teachers who decided to divide the modelling activity into sections to be handled in their respective courses. As the laboratory portion of the activity typically falls under the responsibility of the physics teacher, the apparatus was not used in the mathematics classes where it would have served a clear purpose in anticipating relationships and validating equations. In trying to have the activity serve as an application in both disciplines of concepts that had just been taught, a delay occurred that hindered the synergy that could have otherwise developed. There is now a will to establish more and stronger connections between the two disciplines in future implementations of the activity.

5.8 Conclusion

In summary, the collaboration of teachers of mathematics with a colleague of physics led to the design of a mathematics learning activity, usable in today's classes, that not only supports the learning of specific math concepts but also opens to the development of modelling skills. Despite the richness of the interdisciplinary collaboration at the design stage and the promising avenues it opens, the crossing into physics at the classroom level appears to depend on the concurrent contribution of a physics class taken by all students.

Scaffolding was key to enabling the progress of students in their modelling of the situation, along the axes of interest for the course and within its practical boundaries. One could envision such a scaffolding to fade out, if the students were exposed to more activities of that nature. Even if the activity did not have the openness typically associated with a modelling task, its outcome confirmed our intuition that the first-time experience of building an elaborate mathematical model by combining simpler relations and the constant movement between a situation and its model lay important foundations for the development of modelling skills. Recognising the value of such steps could help cross the boundary between research and practice in mathematics education.

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Chapter 6
Representations of Modelling in Mathematics Education

Helen M. Doerr, Jonas B. Ärlebäck, and Morten Misfeldt

Abstract Mathematical models have a substantial impact at all levels of society, and hence mathematical modelling stands as an important topic in mathematics education. Mathematical modelling has a particular pedagogical/didactical discourse as modelling continues to garner attention in educational research. Diagrammatic representations of mathematical modelling processes are increasingly being used in curriculum documents on national and transnational levels. In this chapter, we critically discuss one of the most frequently used representations of modelling processes in the literature, namely, that of the *modelling cycle*, and offer alternative representations to more fully capture multiple aspects of modelling in mathematics education.

Keywords Modelling cycle • Modelling competences • Technology • Social-critical education • Mathematical modelling • Prescriptive models

6.1 Introduction

Both in society more broadly and in the workplace in particular, mathematical models are used to control processes, to design products, to monitor and influence economic systems, to enhance human agency, and to structure and understand the natural world. Given the widespread use and impact of mathematical models (Niss

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_6

2015), it is not surprising to find mathematical modelling competencies as an educational goal in various curriculum standards documents on national and transnational levels. Prominent examples are the PISA 2012 framework and the recently adopted Common Core State Standards in Mathematics (CCSSM) in the United States (Council of Chief State School Officers [CCSSO] 2010). The 2012 PISA framework defines mathematical literacy as "an individual's capacity to *formulate*, *employ*, and *interpret* mathematics in a variety of contexts. It includes *reasoning* mathematically and *using* mathematical concepts, procedures, facts and tools to *describe*, *explain,* and *predict* phenomena" (OECD 2013, p. 25, italics added). In CCSSM, modelling with mathematics is one of eight standards for mathematical practices that teachers should seek to develop in their students at all grade levels, K-12. Modelling is described in terms of what students are able to do:

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society and the workplace. …[They] are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyse those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. (CCSSO 2010, p. 7)

Both the PISA (OECD 2013) and the CCSSM (CCSSO 2010) standards documents include representations of mathematical modelling that are intended to convey to stakeholders and practitioners the key elements involved in learning to do mathematical modelling and in learning about mathematical models and their role in society. As images of modelling, these representations necessarily convey some important aspects of modelling, but as with all images and representations, other important aspects of modelling are pushed into the background or left out in some way. Hence, our concern with the dominance of particular images of modelling is with the influence that dominant images will have as modelling is taken up by writers of curriculum materials, by textbook authors, by teachers, by teacher educators and others involved in professional development and by developers of large-scale and high-stakes assessments. One of the most frequently used representations of mathematical modelling in curricular documents and in the research literature is that of the *modelling cycle*. Our goal in this chapter is to critically examine the question of what important aspects of modelling are pushed to the background or omitted by widely used representations of the modelling cycle.

6.2 The Modelling Cycle

We begin our analysis of the cyclic representations of modelling with the PISA framework (OECD 2013), followed by the CCSSM (CCSSO 2010) and then the research literature. The PISA document situates modelling in real-world contexts,

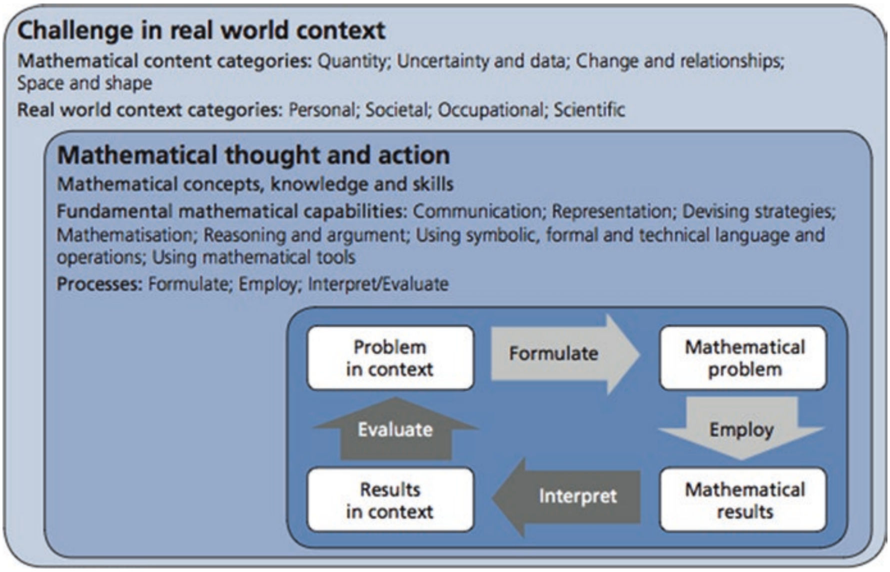

Fig. 6.1 Representation of modelling in the 2012 PISA framework (OECD 2013, p. 26)

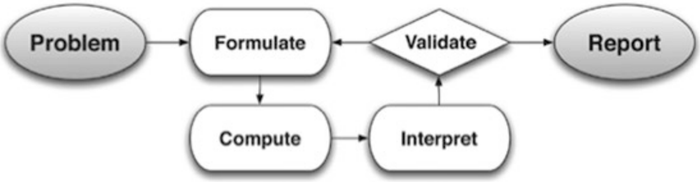

Fig. 6.2 Representation of modelling in the Common Core (CCSSO 2010, p. 72)

noting that this includes four contexts: personal, societal, occupational and scientific. Mathematical concepts, knowledge and skill are drawn upon in order to engage in the four processes of *formulating* the model, *employing* mathematical skills to obtain mathematical results, *interpreting* those results in context and *evaluating* the goodness of the solution (Fig. 6.1).

In the CCSSM (CCSSO 2010), modelling is both a standard of mathematical practices at all grade levels and a content standard in high school (grades 9 through 12). As with the PISA framework (OECD 2013), modelling is about analysing empirical situations: "Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modelled using mathematical and statistical methods" (CCSSO 2010, p. 72), as shown in Fig. 6.2. The vision of modelling includes both descriptive models (such as graphs of observations) and analytic models that seek to explain phenomena. Computational technology (such as graphing utilities, spreadsheets, computer algebra systems, dynamic geometry software) plays a role in "varying assumptions, exploring consequences, and comparing

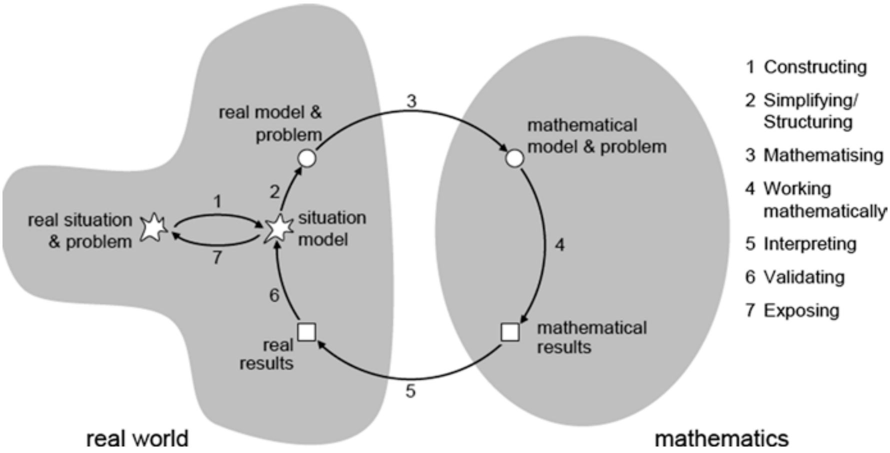

Fig. 6.3 The modelling cycle depicted by Blum and Leiß (2007, p. 225)

predictions with data" (CCSSO 2010, p. 72). The CCSSM elaborates each of these modelling processes, including a clarification that "compute" does not mean to "calculate" per se, but rather means to analyse, to perform operations on relationships between variables and to draw conclusions.

In the research literature on modelling, there are several variants of the modelling cycle, such as the widely cited image of Blum and Leiß (2007) shown in Fig. 6.3. A similar image has been developed by Blomhøj and Jensen (2007), where modelling competency is defined as "someone's insightful readiness to carry through all parts of a mathematical modelling process in a certain context" (p. 48). All representations of modelling have their strengths and weaknesses, a point also made by Blum (2015). There are some striking similarities among many of these cyclic representations, even when the specific words chosen to describe the subprocesses of modelling differ. All of these representations capture some sense that a mathematical model is a simplified version of some aspect of the real world that is formalized in mathematics for the purpose of solving a problem situation in the real world.

Given the recent manifestations and importance of these representations of modelling in curriculum standards documents for policy-makers, curriculum developers, teachers and researchers, we put forward four important aspects of mathematical modelling that are *not* well captured by the images we have shown: the non-linearity of modelling, the role of multiple models and pre-existing models within modelling activity, the social and critical aspects of modelling and the role of computational media in modelling.

6.3 The Non-linearity of the Modelling Process

These widely used representations of mathematical modelling processes share the same problem: they provide a useful analytical abstraction of the processes involved when engaged in the creative thinking when an individual (or a group of individuals) maps a real problem situation onto some subset of mathematics for some particular purpose. However, all the individual differences that occur when students engage in doing mathematical tasks make the transition from an abstract analytical representation of modelling to a more normative tool for planning teaching and learning of modelling at best problematic. Teaching approaches that would guide students through predetermined boxes would be inadequate for embracing the multitude of learning pathways that are known to occur in the classroom (Borromeo Ferri 2007; Lesh and Doerr 2012). In her work, for example, Borromeo Ferri illustrates both the non-linearity (in terms of following steps or sub-competencies shown in the modelling cycles) and the differences between two pupils in their individual modelling routes or pathways, as shown in Fig. 6.4.

Just as importantly, when digital technologies are introduced into modelling tasks, the non-linearity of students' actual modelling pathways becomes more dynamic and stochastic. As illustrated by Lesh and Doerr (2012), students' actual modelling activity does not move in a linear path through the boxes and subprocesses of the modelling cycle. As students work, they "bounce around" as they attend to different aspects of the problem situation (sometimes re-defining the problem), their mathematical work (revising the relationships between objects), the data and their representations (selecting new objects to represent) and their interpretations of their outcomes in terms of perceived criteria (Doerr and Pratt 2008). We suggest that an image of moving between "nodes" or multiple paths in a network (as shown in Fig. 6.5) might offer teachers and researchers new ways of thinking about both teaching and researching mathematical modelling.

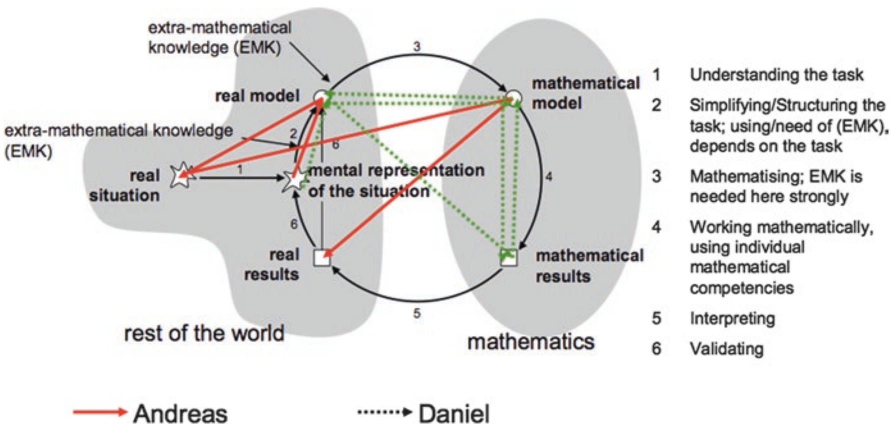

Fig. 6.4 Individual modelling pathways (Borromeo Ferri 2007, p. 2087)

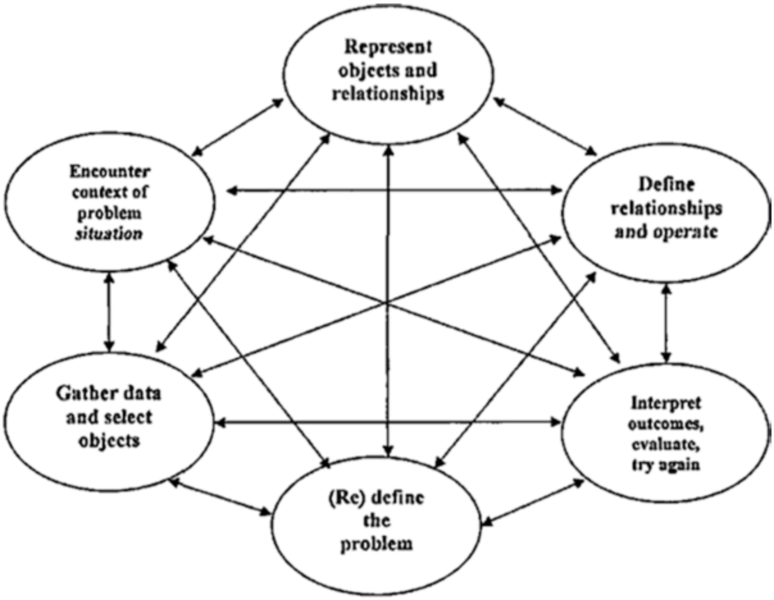

Fig. 6.5 The nodes of the modelling process (Doerr and Pratt 2008, p. 264)

6.4 The Role of Multiple Models or Pre-existing Models

As noted earlier, models serve many purposes in society and the workplace. Models sometimes serve descriptive purposes, where the modeller wants to describe or predict the behaviour of some real phenomena. Both the PISA framework (OECD 2013) and the CCSSM (CCSSO 2010) point to the role of graphs in describing physical phenomena. However, as Hestenes (2010) and others have pointed out, models often need to serve explanatory purposes. To accomplish this, the modeller may need to draw on multiple models within the modelling process or on other pre-existing models, whose structure may need to be explored and understood. Consider, for example, the well-known problem of modelling light intensity as a function of distance from a light source. The graph of this relationship can readily be found to follow an inverse square relationship, but this leaves an important question unanswered: why is this an inverse square relationship? A graph is *descriptive* but not *explanatory*. To understand why light behaves in this way, another model is needed, namely, the geometry of the sphere (see Ärlebäck and Doerr 2015). Most representations of the modelling cycle do not include how these two models (one descriptive and the other explanatory) are brought together in the modelling process.

6.5 The Social and Critical Aspects of Modelling

We know from the work of Barbosa (2006), Niss (2015) and many others that models are projected back into the world. Recent years have provided us with numerous examples in governance and finance, as well as in science and engineering. For example, macroeconomic models of the development of state finances and welfare increasingly control political decision-making. New public management structures that encourage people to deliver more work and output on certain measurable parameters can be seen as the result of underlying models on how to increase worker productivity. In finance, the complexity of the models that govern the stock exchange (Johansen and Sørensen 2014), and the large losses that occurred as a consequence of these models, places new kinds of responsibilities on the mathematicians and financial analysts for the major economic losses that occurred during the dramatic events in the financial crisis. Our claim here is simple: models have a huge impact on our world; but the social and critical aspects of the role of models in such areas as governance, management and finance are not captured by the modelling cycle. As Barbosa (2006) noted, “mathematical models are not neutral descriptions about an independent reality” (p. 294). Barbosa described the kinds of critical mathematical modelling activity that occurs when pupils investigate a real social problem as “quite removed from the characterization of modelling as involving diagrammatic representations” (p. 294). Rosa and Orey (2015) have recently put forward a representation (Fig. 6.6) that captures some of the dynamic and humanized aspects of

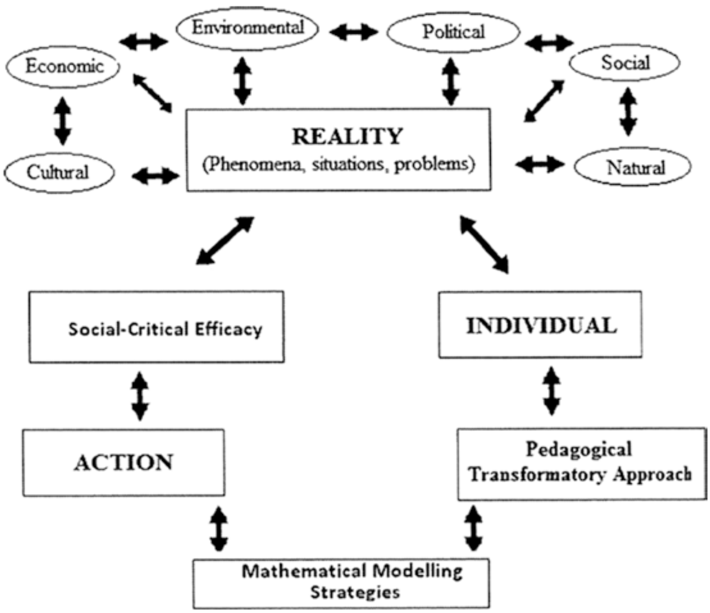

Fig. 6.6 Representation of socio-critical modelling (Rosa and Orey 2015, p. 394)

modelling, capturing the role of the individual modeller, the transformative nature of the pedagogy involved and the orientation towards action as models are projected back into a social context.

6.6 The Role of Digital Technologies

Some attempts to characterize the role of computational media have aimed at augmenting the modelling cycle (Greefrath et al. 2011). For instance, the representation shown in Fig. 6.7 depicts the "computer model" as distinctly separated from the mathematical model and suggests a sense in which technology becomes a medium for helping in the process of moving from mathematical problem (model) to mathematical results.

The interplay between the world and the mathematics that are shown in the modelling cycle (Figs. 6.1, 6.2 and 6.3) might have described the mathematical modelling done in an era when many crucial insights were gained from the interplay between mathematical analysis and real-world experiments. However, advances in computational media have changed this situation because a new kind of "experimental" work is now done through computational models of various phenomena. Moreover, these computational models often involve mathematics (particularly in the case of stochastic phenomena) that is simply not possible with the closed form solutions suggested by the image of the modelling cycle. Validation of such computational models is often far more complex than a mapping back to the problem situation would suggest. With computational media we often have several types of models involved in much modelling work. Indeed, we have only to look at the role of mathematical modelling in biology to see the role that computational experiments play. One representation that captures this interplay between physical phenomena (or empirical data), simulation (or computational data) and analysis (or explanatory theory) is shown in Fig. 6.8.

Research has shown that modern mathematical software can be a powerful tool in supporting a multitude of mathematical work processes and can act as a tool

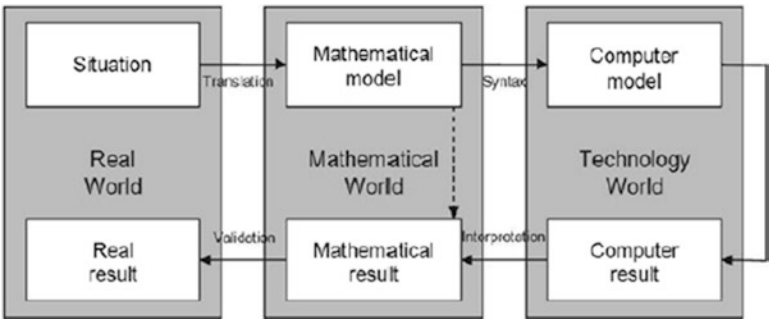

Fig. 6.7 Modelling cycle augmented with technology (Greefrath et al. 2011, p. 316)

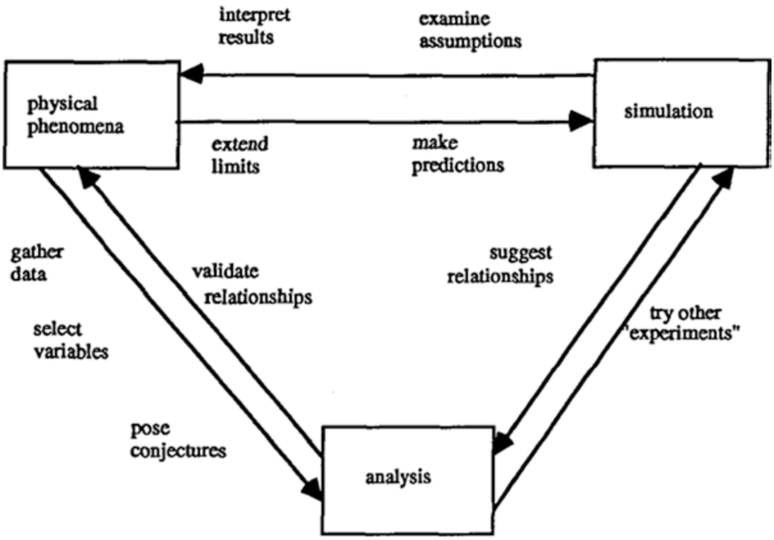

Fig. 6.8 The interplay of phenomena, simulation and analysis (Doerr 1997, p. 269)

towards enhancing the mathematical abilities of their users (Guin et al. 2005; Laborde 2005). But computational media have also been described as a new "universe" for mathematical activity, in the sense of a new type of mathematical reality. This has been articulated as computational media offering mathematical "microworlds" for students to tinker with in order to develop their mathematical curiosity and start mathematical investigations. Hence, modern computational media allow for new mathematical venues to be investigated and also allow professional mathematicians to investigate types of mathematical realities that previously were inaccessible (Borwein and Devlin 2009). Furthermore, the computational speed of computers allows mathematical models to project their results back into the world in real time, hence shaping the real world. In other words, computational media both empower the mathematical processes involved in modelling activities by providing new "worlds" to explore and potentially shape the world we try to model. These different roles can be summed up in a representation focusing on the roles of computational media in modelling activities rather than the modelling process as such, if we think of them as overlapping spheres of influence, as shown in Fig. 6.9.

6.7 Conclusion: The Necessity of Multiple Representations

The issue addressed in this chapter is the dominance of the one single image of mathematical modelling that is shown by the *modelling cycle* in international and national curriculum documents such as PISA (OECD 2013) and the Common Core Standards (CCSSO 2010). As noted earlier, any one representation of modelling has its strengths

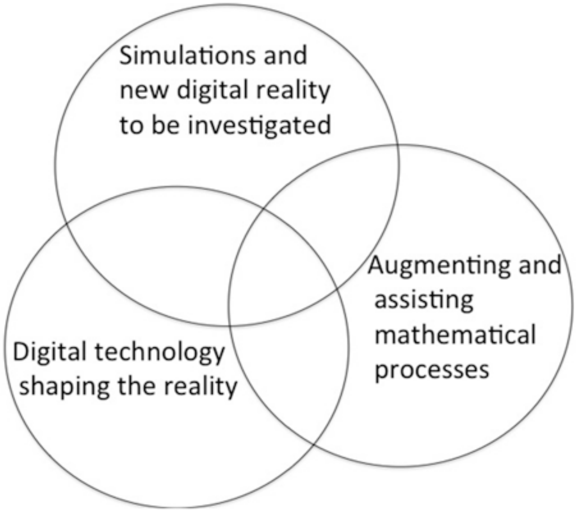

Fig. 6.9 Spheres of digital technology influence on the modelling process

and weaknesses, and hence we argue that we need multiple representations and images to capture and convey the richness of modelling for mathematics education for policy-makers, curriculum developers and teachers. Curriculum materials that would guide students through predetermined steps in a modelling cycle would be inadequate for conveying to teachers the non-linearity of the multiple learning pathways that would occur in a classroom. Similarly, modelling activities for students need to move beyond creating descriptive models that can be validated by comparison to empirical data to working with a full range of models including those with explanatory power, those with social and political implications and those using computational media. Representations of these aspects of modelling imply modelling tasks that explore and bring to bear existing models, that are socially relevant and engage students in action as the models are projected back into the world and that open up new realms of mathematical venues. Our recommendation is not that we should improve or revise the modelling cycle to encompass these important aspects of modelling. Rather, we suggest that a complex process such as mathematical modelling should be conveyed in policy and curriculum documents by multiple images that accommodate the aspects addressed in this chapter and through future research.

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Chapter 7
The Primacy of 'Noticing': A Key to Successful Modelling

Peter Galbraith, Gloria Ann Stillman, and Jill P. Brown

Abstract The notion of productive modelling-oriented noticing (pMON) is proposed within an anticipatory metacognition framework for the implementation of successful modelling. pMON involves modellers noticing what is important (or not) in order to generate strategies for responding to, or initiating, activities necessary for successful engagement in modelling. In this chapter, we address the question: How does 'noticing' feature as an enabler and a displayer of modelling ability? From student work at an extra-curricular event, we identify global and specific noticing of a strategic and explanatory nature, which evidences anticipatory aspects of mental activity taking place during modelling, and illustrate a coding system for identifying and labelling components of pMON.

Keywords Noticing • Anticipatory metacognition • Productive modelling-oriented noticing • Global noticing • Specific noticing

7.1 Background

The term 'noticing' in education (e.g. Choy 2016; Jacobs et al. 2010; Santagata 2011; Star and Strickland 2008) has commonly been applied to how teachers identify, interpret and act upon classroom events to enhance student learning. Star and Strickland (2008) assert that learning from teaching depends on teachers' ability to

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_7

'notice', by identifying what is (and therefore what is not) noteworthy and important**.** Previously, Sternberg and Davidson (1983) noted that gifted professionals such as doctors and lawyers had an ability to sift through information, selecting the relevant from the irrelevant, as a basis for subsequent actions. Jacobs et al. (2010) go further in stating that 'noticing' is an essential competence needed within *any* profession.

Additionally, in terms of the attributes of a community of practice, Wenger and Wenger (2015) point out that skilled 'noticing' should be nurtured in novices, as well as being exhibited by masters – in our case within the field of mathematical modelling. In developing insights to enhance teaching, a central aspect is to identify how students perform when faced with unfamiliar modelling situations. This provides data on associated competencies (or lack of them) displayed in extended modelling settings involving the world outside the classroom. In focusing on student noticing and related decision-making within modelling activity, our approach is different from the way that 'noticing' research has been typically conducted, where the emphasis has been on teacher behaviour across a range of topics in classroom settings. Of course, observations and evaluation of student activity are designed to contribute to the developing corpus of knowledge used to inform modelling pedagogy. We commence by locating 'noticing' within metacognitive activity.

7.1.1 Anticipatory Metacognition

Metacognition is of central concern in the research and practice of mathematical modelling with experts and novices acknowledging the importance of reflection on actions when addressing a real-world problem (Lambert et al. 1989). The focus of such reflection can be checking the accuracy of mathematics, evaluating a solution against contextual implications or examining decisions made at some intermediate stage of the modelling process (Stillman 2011), that is, reflection on the mathematics employed and the modelling undertaken. These metacognitive abilities remain crucial especially for beginners. A new development arising from our previous work is the construct of anticipatory metacognition. *Anticipatory metacognition* shifts the emphasis towards reflection that points *forward* to actions not yet undertaken, that is, recognising (noticing) possibilities of what 'might be'. These reflections can emerge as a consequence of prior progress (or lack of it), so they might not all be identifiable at the outset of modelling. Included is an aspect not provided for when a modelling problem is presented to, rather than chosen by, a group of students – the 'noticing' involved in recognising a situation has modelling potential and the identification of an associated and relevant mathematical question to pursue. Anticipatory metacognition encompasses three distinct dimensions (Fig. 7.1): meta-metacognition (see Stillman 2011), implemented anticipation (see Niss 2010; Stillman and Brown 2014) and modelling-oriented noticing (Galbraith 2015). We focus on the last of these here.

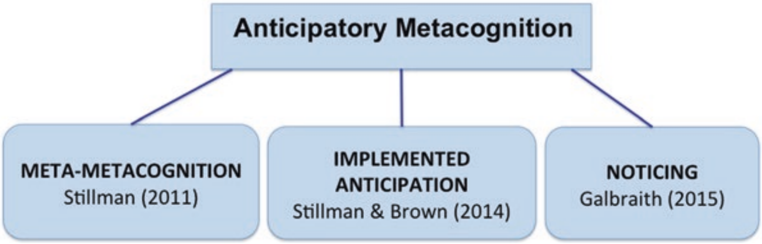


Fig. 7.1 Dimensions of anticipatory metacognition

7.1.2 Modelling-Oriented 'Noticing'

In a broad sense, modelling-oriented noticing involves 'noticing' how mathematicians as well as educators act when operating within the field of modelling, from both mathematical and pedagogical points of view (Galbraith 2015). The former 'noticing' provides a means for studying aspects central to modelling such as problem recognition and posing as well as in the conduct of modelling proper. Novices, as learners of modelling, also engage in these activities, for discerned 'noticing' is needed to select, develop and document modelling products appropriately. The pedagogical viewpoint (the one closest to teacher noticing literature, e.g. Santagata 2011) involves a mentor role: observing how students operate when working on modelling problems and focusing on identifying, interpreting and acting on emerging events as deemed necessary. In the present case, the emphasis is on the first of these – monitoring and observing student decision-making during modelling activity in a collaborative team context. The characteristics identified, when distilled, contribute to the goal of enhancing the quality of mentor activity.

However, successful carrying out of tasks requires more than mere 'noticing', in the same way that successful metacognitive monitoring of previous work requires appropriate follow-up. Discernment of the relevance of what is noticed is also essential. In a different context, Choy (2016) combined Santagata's notion of noticing (2011) and Sternberg and Davidson's (1983) processes of insight to characterise what he calls *productive mathematical noticing* that takes such relevance into account – applied within a conventional classroom. Developing a parallel but different structure for the modelling context, we define *productive Modelling-Oriented Noticing* (pMON) as the process of modellers noticing in a productive manner what is important and what is not in order to generate strategies for responding to, or initiating, activities necessary for successful engagement while carrying out a modelling activity.

The context of our work involves the development of skills of mathematical modelling as real-world problem-solving. This is a necessary ability for individuals to possess, in order to apply their mathematical knowledge in personal, work-related and civic situations – as specifically declared a curriculum goal in several countries (e.g. ACARA 2016). The educational challenge contains both a task and a person dimension, as there is a difference between helping students to solve individual modelling problems and the greater challenge of nurturing the development of an effective problem-solver over time – one who has developed and honed this pMON quality. An effective modeller is someone who can successfully engage with the modelling cycle, identifying cues and evidence that are present before and that emerge during an activity, and can act upon these appropriately (Stillman 2004). Their relevance is noticed, that is, metacognitive knowledge is generated with respect to task and strategy characteristics and cognitive goals. In this sense, enhanced 'noticing' is an attribute of emerging proficiency, and the identification and documentation of its characteristics are important for both research and practice. In this chapter, while continuing to recognise and value the role of 'noticing' in general teacher activity, we focus, as foreshadowed above, on the 'noticing' attributes of students. The question we address is: *How does 'noticing' feature as an enabler and a displayer of modelling ability?*

7.2 Extracurricular Modelling Event

The setting for our observations has been a 2-day modelling challenge sponsored annually by AB Paterson College, Queensland, Australia. Typically, in the upper secondary level (years 10–11) of the event, groups of 20 students mentored by a modelling expert were divided into five teams of four students drawn from a mix of schools located in south-east Queensland and Singapore. The authors of this chapter were group mentors at this event. The teams were assigned so members came from different schools.

During an introductory 2-h session, students were introduced by their mentors to the cyclic modelling process in part through its application to a real-life problem selected by the mentor. After student teams had worked on the problem, it was addressed interactively with the group as a whole with contributions from the students. The student teams then used the time until early afternoon on day 2 (approx. 9 h) to identify, address and report on a modelling problem through a written poster and an oral presentation. The task of identifying the context and the specific problem(s) on which to work was the teams' own, as was the choice to do further work in their own time.

7.3 Data Collection

For the introductory modelling task, the scripts of teams were digitally photographed as was the board work of the team that presented their solution to the entire group. The work of teams was photographed and videotaped at various intervals over the 2 days as they explored possibilities for a situation to model, collected data once a problem was found, constructed posters to display their modelling solution and presented their modelling to their group and mentor. Students gave their posters, PowerPoint presentations and rough working to their mentors for further analysis. The poster summarising the modelling outcome for each team, as well as the team presentation to the whole group, provided structural data about the substantive modelling. Additionally, students progressively completed open questionnaire items about their approach to aspects of their chosen task, as they reached different stages. The content of the written products, and observation of student activity, provided evidence of the approach, structure and detail of ‘noticing’ as displayed and recorded by the students during respective phases of the modelling process.

For the detailed analysis that follows, two teams of four, Gold and Silver, were selected from the 2012 Challenge. Each team was mixed gender with two year 10 and two year 11 students. Students were from Australia (3) and Singapore (5). Experience of similar activities varied from frequently, to occasionally, to never having done such activities in mathematics classes.

7.4 Analysis

In order to generate initial coding categories for pMON, the PowerPoint presentation and poster from a team of four students from the 2009 Challenge investigating inundation of the Gold Coast was subjected to macro and micro qualitative analysis (see Galbraith 2011 for a full description of this modelling project). Figure 7.2 shows the categories and definitions that resulted from this process. In reflecting on the instances of ‘noticing’ embedded in the data, sub-families were identified which we label, respectively, as (1) *Global Noticing* (*GN*) and (2) *Specific Noticing* (*SN*).

Global Noticing is the noticing needed for alertness in respect of the total modelling process, for example, how the process is activated and sustained throughout the different modelling phases. It involves awareness of information (present or identified as necessary) and what is needed to proceed productively towards an outcome.

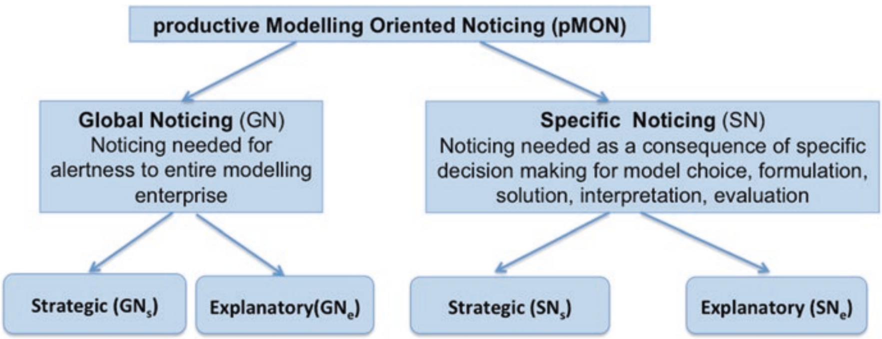


Fig. 7.2 Categorisation with definitions of subcategories of productive modelling-oriented noticing

Specific Noticing is the noticing that a modeller is called upon to do as a consequence of specific decision-making in the choice of a particular mathematical model, its formulation, solution, interpretation and evaluation. Both global noticing (*GN*) and specific noticing (*SN*) can be expressed as *strategic noticing (s)* or *explanatory noticing (e)* coded as GN_s, GN_e, SN_s and SN_e.

This coding scheme will now be used in a detailed analysis of data from the 2012 Challenge. Firstly, student contributions that took place during the whole group introduction to modelling facilitated by mentors will be discussed, illustrated and interpreted. This task has been chosen for present purposes mainly on the basis that students were able to reach a team solution quickly – so the nature of modelling was understood in the same way by all participants. Secondly, we discuss and illustrate aspects of 'noticing' that characterised student work in different modelling phases using data from the work of the two teams mentioned above. In this part of the challenge, teams chose their problem or situation to model according to their own interests and anticipated collective ability to solve in the given time frame. The description and analysis are restricted, because of space considerations, to examining alertness to taking control of the modelling process at a macro level and noticing related to problem finding and problem posing and decision-making related to formulation at the micro level.

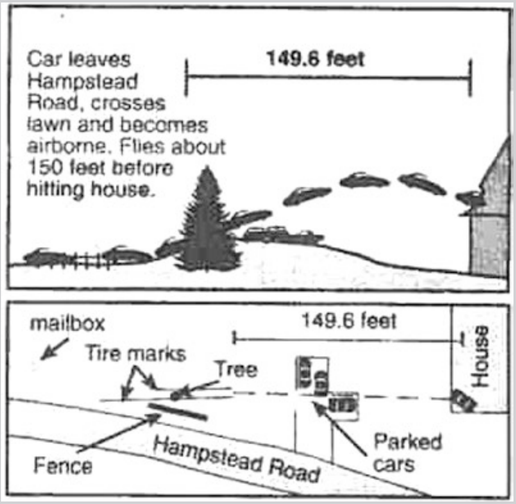
7.4.1 Productive Noticing During Introductory Modelling Task

For the 2012 Challenge, mentors used as the introductory modelling task, *Aerial Car*, based on a newspaper report by Shamroth Reiff (2000). (Note: The top diagram is artistic not realistic as is common in newspapers, but there were photographs of the scene provided to students as well.)

Aerial Car

According to the *New Hampshire Sunday News* May 7, 2000, two sleeping residents of East Derry had the most fortunate escape on Easter Sunday 2000. A car came crashing through the roof of their upstairs bedroom and landed on their bedroom dresser. The car was being driven along Hampstead Road at 3:35 am by a young woman when the car left the road and careered through a neighbouring property before ascending a slope of grade 20% then becoming airborne. The car flew over three parked cars travelling a horizontal distance of 48.62 m before crashing through the roof. The point of impact on the roof was about 30 cm lower than the point of take-off as the house was on lower ground.

(continued)



(Used with permission)

When using this real situation as the basis for modelling, the modeller must first pose a problem in a way that can be mathematised. Those with a penchant for exactness in mathematics might ask: What speed was she travelling? From a real-life modelling perspective, a better question is: 'Was she speeding?' Having separately identified *projectile motion* as an appropriate model, most teams began without any correction for air resistance (for such a large object), with most choosing to work out the speed of her car when it became airborne. After solution development, one team was asked to appoint a spokesperson to present their solution to the entire group. He clearly articulated *strategic* and *explanatory noticing* and inferences the group had made in deciding on their global approach to the task. He started by pointing out that his team had assumed that the driver had not accelerated once her car had left the road as that would affect their conclusion (GN_e). He also sketched his group's interpretation of the situation, explaining they intended to use projectile motion which in 'theory' modelled the trajectory of the car in the air by a parabola. However, this was not the 'real' trajectory, which would mean the car travelling with their calculated launch speed fell short (SN_e). He added the 'real' trajectory for this speed (Fig. 7.3a) explaining that if they found the speed for their simple model was above or near the speed limit, there would be no need to refine their approach to modelling the car's actual path as they would be already able to answer the question. She would be deemed to be speeding (SN_s). The teams were showing evidence of pMON as they had been able to sift through all the information given (other newspaper detail and photographs), to discern what was relevant, to relate it to previous experiences of using projectile motion and to combine this metacognitive

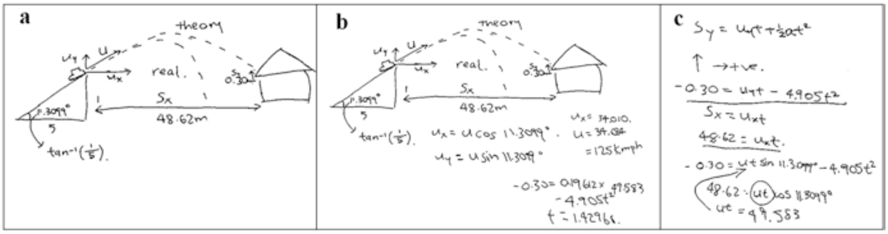


Fig. 7.3 Modelling the *Aerial Car* scenario

knowledge so as to anticipate a 'sense of direction' (Treilibs 1979) for their modelling (GN_s). This was then followed through in implementation (Fig. 7.3b, c). This sense of direction was serviced by reflection on where activities were leading in terms of potential contributions to an appropriate solution (anticipatory metacognitive activity).

7.4.2 *Macro-analysis of pMON in Self-Chosen Modelling Projects*

Following the introductory activity described above, students were provided with scaffolding devices in the form of a diagram summarising the modelling process and a guide to making a report. However, one team, Silver, produced their own set of questions to guide their modelling poster design. The teams then independently began problem finding and problem posing following a team's brainstorming of possible situations of interest to them.

Gold team was interested in the notion of 'wealthiness' put forward by a male year 11 Australian student who was considering economic modelling as a future career. Their initial ideas for a situation to model included changes over time in (a) wealth in the percentage of a population and (b) different areas of the world and (c) the purchasing power of a population that could be considered wealthy. This discussion led to initial questions: How wealthy can a specific person or country become? What are the most ideal conditions to heighten wealth? An Internet search produced a paper by Sala-i-Martin and Mohapatra (2002/2003) on the distribution of income in the G20 countries. The team finally settled on studying the income distribution in Australia and India with the guiding question: *How wealthy can you get?* All team members thus had noticed enough about fiscal issues in the region where they lived to feel confident in choosing this phenomenon to model.

Choosing a situation for modelling requires group-initiated proactive reflections about potential situations, such that the group becomes convinced that the situation chosen can generate a problem that can be modelled mathematically in the time frame, given the expertise of its members. More is needed than suggesting possible situations that are of interest. The task ahead must be anticipated, and considered globally, as evidenced in the Silver team's brainstorming and eventual choice.

Initially, Silver team was interested in how different caffeine-containing foods affect mental capacity and appeared set to conduct an empirical experiment. One of the foods they looked at was chocolate. In their Internet searching, they were surprised to find BBC News articles (e.g. Pritchard 2012) dealing with an occasional note in a medical journal (Messerli 2012) about chocolate consumption, cognitive function and Nobel laureates. They therefore decided to focus their investigation on the modelling of chocolate consumption. Their initial thinking was recorded as:

> When people in a country get wealthier, their purchasing power increases, and they are more willing and able to buy chocolate products whenever they want them. A higher chocolate consumption level may result in extra calories-intake, leading to more people being obese. (Notes – Silver team, November 22)

In terms of *Global Noticing* to select a phenomenon they could potentially model, the team needed to anticipate that they knew enough about the mathematical structure of the intended model, at an appropriate level to achieve their objective. This reflection required judgments about access to mathematical tools (including digital tools) that would allow them to model the chosen problem. This is evident in notes (in extract following) that the team recorded about their intentions to investigate what they later termed a myth: ‘as people become wealthier they would become more obese’ (poster, Day 2). They decided to investigate this at the level of a country’s average chocolate consumption and obesity proportion. The codes in the extract identify evidence of *Global Noticing* as the team demonstrated alertness of how the modelling process would be activated and sustained throughout different modelling phases. They also showed awareness of information (identified as necessary) and what was needed to proceed productively towards an outcome.

> *What variables are there?*
> The gross domestic product (GDP) per capita; Chocolate consumption rate; Obesity proportion.
>
> *What data do we need?*
> GDP per capita of selected countries. The proportion of obese people in the selected countries. The amount of chocolate consumed per country for a particular year.
>
> *What mathematics do we need?*
> We use tables and graphs to represent the data we find. Visually from the graph we find the best fit trend line and get the equation of it. We then use the data to generate a regression based on the line and produce an equation to model the data. Using the equation, we can predict the amount of chocolate consumption, then compare it with more data to test and refine our model. (GN_s)
>
> *Do we know how to solve this mathematical model?*
> Yes. To find if there is a relationship between the GDP per capita and the amount of chocolate consumed, we substitute the GDP per capita into the equation we found from the graph (Chocolate consumption against the GDP per capita) and compare the result in the actual data. We repeat this with the graph of the proportion of obese people in a country against the chocolate consumption to find the relationship between these two variables. (GN_s)
>
> *What does this output mean mathematically? In the real context?*
> If the calculated results are close to the actual data of the countries, it would prove that there is indeed a relationship between the amount of chocolate consumed, the GDP per capita and also the proportion of obese people in a country. (GN_e) (Notes – Silver team 22 November)

7.4.3 Micro-analysis of pMON in Team Projects

At the micro level, we look for instances of noticing related to problem finding and problem posing and decision-making related to formulation locally. The rationales (see Fig. 7.4) of the cases chosen for elaboration indicate that the students have identified an issue they deem significant. The modelling intentions (Fig. 7.4) are relevant to satisfy interest (Gold team) and curiosity (Silver team). From a mathematical perspective, both teams added a second criterion for their choice of question(s) to pose: The mathematics involved needed to be complex (Gold team), and the question had to be 'able to be modelled using the mathematical modelling method' (Silver team) (Questionnaire data).

The initial assumptions (Fig. 7.5) of both teams involved only *strategic Specific Noticing* (SN_s). They contain anticipatory elements that include simplifications basic to model creation (e.g. third assumption for Silver). In implementing proposed formulations, pragmatic decisions were made when limits of technological tools, or insufficient data, were realised. For the Gold team, this meant an adjustment to their mathematical model, providing an example of *explanatory Specific Noticing* (SN_e).

> Contextually, there is a tendency for incomes in a country to be distributed in a manner such that most of the population belongs to the middle income group with only the minority being at each end. This is reflected in a lognormal distribution curve that is skewed to the right, as opposed to a standard [normal] distribution curve, which is symmetrical. However, due to the limits of calculator technology, it was unable to directly model the data from the lognormal density function. Instead, since the data fits a lognormal density function it can fit a standard density function if the 'disposable income' values are 'logarithmised'. (Poster, Gold Team)

a) In all areas of the world, each country is unique and differs in wealthiness [sic]. In each region there is a certain amount of income that each part of the population earns. Australia and India were chosen as the two countries whose wealth would be analysed and compared. The data that were collected to be modelled are based primarily on income distribution over time (equalised disposable income versus percentage of population) to determine income equality in the countries and estimate their future wealth.	**b)** People have the mindset that as people get wealthier they would consume more chocolate, which would in turn cause obesity. So, as people become wealthier they would become more obese. However, we wanted to test whether this myth is true.

Fig. 7.4 Rationale for choice of modelling topic by (**a**) Gold team and (**b**) Silver team

Gold Team:	*Silver Team:*
1. Socio-economic status does not change over the time domain. 2. Political status of the countries we are collecting data on remains the same (e.g., market/demand economies/mass production, poverty, human rights).	1. The amount of exercise done remains constant 2.The intake of other food besides chocolate remains the same. 3. The price of chocolate does not change as people's income changes. 4. People's age group is not taken into account. 5. The type of chocolate consumed remains constant. 6. The countries selected are representative of global rate. 7. People's preference for eating chocolate remains the same over the years.

Fig. 7.5 Initial assumptions made by Gold team and Silver team

The Silver team, on the other hand, found there was insufficient data for their model construction, but decided to continue with their proposed model, compromised by data drawn from different years, rather than 1 year as had been originally intended.

7.5 Conclusion

In considering the data emerging from student activity in the *modelling challenge*, it is clear that goal-directed ‘noticing’ preceded decision-making and production. We refer again to the anticipatory aspects of mental activity that take place during modelling. It is clear that both global and specific noticing focus heavily not only on current or recent activity (so that checking of work remains important) but also on future aspects perceived to be related to the task being undertaken. Anticipatory metacognitive activity both directs the global implementation of the modelling and impacts on how a specific formulation and mathematisation remain relevant (or not) for the task assigned to it.

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Chapter 8
Combining Models Related to Data Distribution Through Productive Experimentation

Takashi Kawakami

Abstract This chapter illustrates how students combined distribution-related models using a case study involving 31 year 5 students (10–11-year-olds) in a *Paper Helicopter Eperimentation*. Through the experimental activities that included conjecturing and validation, the students grasped relevant statistical and/or contextual elements from models of real distribution constructed in their external world and contrasted and coordinated these elements with their distribution-related models constructed in their internal world (modelling world). The approach of combining distribution-related models through experimentation establishes an alternative way of utilising ideas of models and modelling in mathematics education in statistics instruction.

Keywords Distribution • Experimentation • Combining distribution-related models • Statistics • Context • Statistical reasoning • Paper helicopter experimentation • Primary school students

8.1 Introduction

Recent research by the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) and elsewhere has increasingly emphasised the need to cross the boundaries between mathematical modelling and statistics education (e.g. Engel and Kuntze 2011; Makar and Confrey 2007). Data-based modelling has been highlighted and demonstrated as a powerful vehicle for developing primary to tertiary students' fundamental statistical ideas, such as data,

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_8

variability, and distribution (e.g. Engel and Kuntze 2011; English 2012; Lehrer and Schauble 2004). Limited research exists, however, on how students can create and develop models and in statistics instruction use the modelling approach, especially at the primary level (English 2012; Kawakami 2015). Distribution is at the heart of statistics and is a fundamental component of statistical reasoning. The necessity of fostering young students' informal views of distribution arises because of the difficulties of developing an aggregate view of distribution (Garfield and Ben-Zvi 2008). This chapter proposes strategies that foster primary school students' modelling activities on distribution through autonomous experimentation.

8.2 Theoretical Framework

8.2.1 Using Model and Modelling Ideas in Teaching and Learning Statistics

Several researchers have highlighted differences in the nuances and emphases of models and modelling between mathematics education and statistics education (Engel and Kuntze 2011; Garfield and Ben-Zvi 2008). In mathematics education, models *that have to be built*, for example, to solve real-world tasks using mathematics and to construct new mathematical knowledge, are emphasised (Ikeda and Stephens 2015). Although the objective of building models and the nuance of models depend on the standpoint taken (e.g. pragmatic or scientific/humanistic), the process of mathematising real-world situations is common. In contrast, in statistics education, models *that have been built*, for example, to measure and explain variation, are often emphasised (Garfield and Ben-Zvi 2008). Engel and Kuntze (2011) highlighted that "a core concept in statistical modelling is the signal-noise metaphor" (p. 400). Modelling in statistics instruction (usually at secondary level) typically involves activities of statistical model selection (e.g. regression models, time-series models, normal distribution) and calibration (model-to-data fitting). In the whole process, "consideration of variation" is crucial (Wild and Pfannkuch 1999, p. 226).

This study employs the ideas of models and modelling in mathematics education in primary statistics education. Specifically, the researcher intended to explore students' development of statistical ideas and representations as models, through a modelling process (Kawakami 2015). Students are encouraged to create, verify, modify, and apply their own models in solving tasks with real data and to handle variation, such as data modelling (English 2012; Lehrer and Schauble 2004).

8.2.2 Combining Distribution-Related Models

In statistics, data variation is organised "under the key notion of *distribution*" (Lehrer and Schauble 2004, p. 638). Distribution is a complex and multifaceted entity. Bakker and Gravemeijer (2004) demonstrated the various aspects of distribution, such as *centre*, *spread*, *density*, and *skewness*. Prodromou and Pratt (2006) emphasised the balance between two distribution perspectives: *data-centric* and *modelling* perspectives. A data-centric perspective means "distribution is seen as a collection of data results" (p. 70). In contrast, in a modelling perspective, "data distributions are seen as variations from the ideal model" (p. 71) (e.g. normal or binomial theoretical distributions). In addition, statistical investigation requires developing models with statistical and real contextual elements (Wild and Pfannkuch 1999). These studies support encouragement of students to create their own models of distribution in exploring real-world contexts and to develop models of probability distribution as the final outcome.

In research on modelling instruction, specifically from a cognitive perspective, the iterative nature of modelling connects and overlaps models (Ärlebäck and Doerr 2015; Lesh and Doerr 2000). This study adapts the modelling epistemology and views the learning process as combining distribution-related models (see Fig. 8.1). In the process, two models are emphasised. First, the "modelling world" is constructed in students' internal world and is corresponded to an image of a particular data distribution. Second, a model of the real distribution is constructed in the students' external world (i.e. the real world) and is generated by the collection and arrangement of real data. These models are combined and reconstructed into a coherent whole to include statistical and contextual elements by comparing, contrasting, and coordinating between these models. Ideally, combined models form aggregate views of data by selecting and connecting the relevant elements of the distribution and associating these elements with the real context behind the data.

8.2.3 Adopting Experimental Activities

Several studies have suggested that the use of experimentation in modelling that includes conjecturing and validation offers rich opportunities for students to develop their models and ideas (e.g. Carreira and Baioa 2011; Halverscheid 2008).

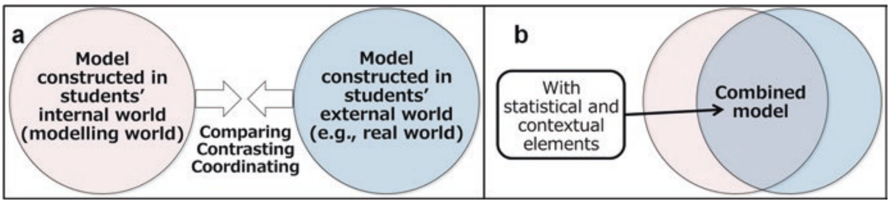

Fig. 8.1 (**a**) Conceptualisation of combining models (**b**) Conceptualisation of combined models

Halverscheid (2008) asserted that the experimentation becomes the real-world context. In their study, Carreira and Baioa (2011) observed that "to investigate through experimentation reflects on mental actions and on past and subsequent learning of mathematical ideas and becomes a way to develop understanding of mathematical models" (p. 214). Hence, experimentation fosters the process of combining distribution-related models (Fig. 8.1). This study applies experimentation as a pedagogical approach to teaching statistics through modelling and elaborates the four-step experimentation process. In the process, students (1) create and express distribution-related models as images, (2) conduct experiments, (3) compare and contrast distribution-related models as images and as real results, and (4) conjecture once again about modifying and improving their own distribution-related models.

8.3 The Study

This chapter addresses the following research question: How do students combine distribution-related models in experimentation that included conjecturing and validation? To investigate this question, student work in a series of statistics lessons with experimental activities (Kawakami 2013) from a modelling perspective is considered. The 10–11-year-old participants comprised 31 year 5 students (21 males and 10 females). The lessons took place in a private primary school in Tokyo. Although the students had learnt about mathematical average, they were inexperienced with statistical enquiry and the histogram before the lessons. In the lessons, they learnt about the mean as a representative value, the range, and using dot plots.

8.3.1 Design

8.3.1.1 The Paper Helicopter Experimentation (PHE)

The *Paper Helicopter Experimentation* (*PHE*) (Fig. 8.2), though simple, is a "statistically rich" activity that originated from the field of quality engineering (Box 1992). It focuses on measuring the flight times of paper helicopters dropped from a certain height. The PHE has been adopted in primary and secondary statistics education, and it is in the Japanese year 7 mathematics textbook (12–13-year-olds) for the introduction of using a representative value and histogram. Ainley et al. (2000) adopted the PHE activity to examine students' understanding of the utility of graphs, where year 3 students (8–9-year-olds) changed and examined the blade length parameters of the helicopter using a scatter graph to induce the longest flight time. There are similarities here with *The Paper Airplane Problem* (see Lesh 2003). In contrast, this study focuses on the following opportunities through the PHE: to (a) form and validate conjectures about flight-time distribution and (b) explore changing conditions of the experiment.

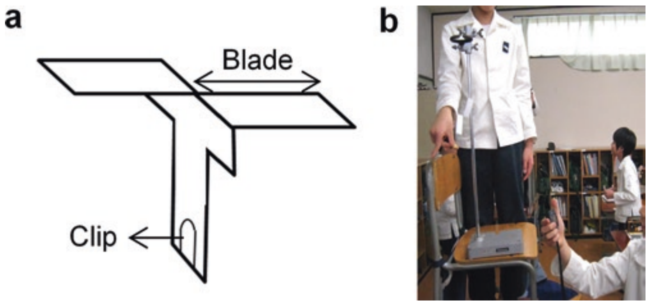

Fig. 8.2 (**a**) The paper helicopter (**b**) Student dropping a helicopter and student measuring its flight time with a stopwatch

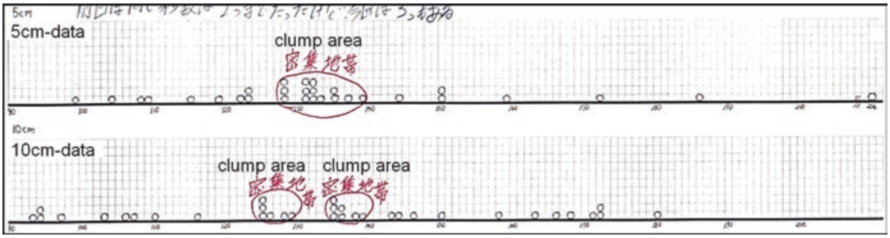

Fig. 8.3 Sample of flight-time data of 5 and 10-cm bladed helicopters in the second experiment (5-cm blades, $n = 30$; 10-cm blades, $n = 34$)

8.3.1.2 Tasks and Outline of Lessons

Three tasks were presented in a series of five 45-min lessons. The main task was *Task 1*: “Compare the helicopter with 5-cm blades and 10-cm blades. Which do you think will have a longer flight time?” For this task, the students performed the first experiment, collected two data sets of flight times, and arranged them in dot plots (taking around 90 min). Data were converted into 0.01 bps for easier data management. Although the students drew a conclusion by calculating the mean of the flight times, they noted “We cannot judge which paper helicopter will have a longer flight time based only on these results” due to many variations in the first experiment’s results. Therefore, the teacher set up *Task 2*: “What can you do to reduce the variation of the data?” In *Task 2*, the students used the four-step experimental approach to the activities (for around 100 min). Firstly, the students conjectured what would influence the data variation and drew sketches of a distribution of either helicopter’s flight time. Secondly, they conducted the second experiment and arranged two data sets of flight time in dot plots (Fig. 8.3). Here, they focused on the concentration of data and called it the “clump area”. Thirdly, they verified their initial conjectures (source and result) by comparing with the real distribution. Fourthly, based on the validation, they conjectured possible variation sources in the data of the second experiment and made conjectures and drew sketches of the distribution for future refined experiments. Finally, the teacher set up *Task 3*: “Refer to the results of *Task 2* and answer *Task 1*”.

Table 8.1 Types of students' conjectures ($N = 31$)

Model	Conjecture type	Example of students' conjectures	Total conjectures (n)	
			Initial	Final
A	No elements of distribution and context	Enumerating dots/no description of graph	5	1
B	Separated elements of distribution and/or context	Describing thin distribution, although describing "the spread is large"	17	11
C	Connected elements of distribution	Describing a unimodal distribution in which the mean is peak	7	16
D	Connected elements of distribution and context	Linking the shape of distribution and the source of variation	2	3

8.3.2 Data Collection and Analysis

The data collection comprised video-recordings, lesson artefacts, and field notes. The artefacts for analysis included students' sketches and descriptions of a hypothetical distribution. The analysis tracked their initial and final conjectured models in *Task 2* and examined the elements of distribution (e.g. centre, spread, density, shape) and/or the context used and the connections of these elements to illustrate how the students' original models were maintained or modified after experimentation.

Students' models of distribution in each conjecture were categorised into four types including elements of distribution (see Table 8.1). Furthermore, the analysis summarised changes in students' models from initial conjecture to the final one. Finally, the work of two students showing successful change type were selected to illustrate in more depth how students combined distribution-related models through experimentation.

8.4 Results

This section illustrates the findings for the research question about students' model combination, describing the results of the whole group and illustrating with two cases.

8.4.1 Changes in Students' Models Through Experimentation

Table 8.1 shows the results of 31 students' models in initial and final conjectures. Overall, 19 students could connect the elements of distribution in the final conjecture, whereas only nine students did so in the initial conjecture. For *Model B* in each

Table 8.2 Change types of students' conjectures ($N = 31$)

Type	Model change	Frequency
1	Model A → Model A	1
2	Model B → Model B	8
3	Model C → Model C	6
4	Model D → Model D	2
5	Model A → Model B	4
6	Model B → Model C	9
7	Model B → Model D	1

case, although almost all students used informal language to describe the distribution (e.g. "variation", "cluster", "clump", and "spread out"), they did not reflect them explicitly in the features of distribution (e.g. shape). Concerning *Model C* in initial conjectures, five students linked the centre and the density of the distribution (e.g. drawing a unimodal distribution in which the mean is the peak), but only two students connected the shape of distribution and the mean. However, for *Model C,* in the final conjectures, nine students linked explicitly the shape of the distribution and other features (e.g. mean, frequency, and range). A decrease in occurrence of *Model A* and *Model B* and an increase in *Model C* indicate that the exploration and its expression of features in empirical distribution (e.g. Fig. 8.3) facilitated students' multifaceted views of distribution. For *Model D*, in each conjecture, all students explained the distribution features linking measurement error, such as the accurate use of a stopwatch.

Table 8.2 shows how 31 students changed their models related to distribution through the experimental activities. Seven types of changes were distinguished, and from these, 17 students maintained their models, whereas 14 changed theirs. However, because few *types 2* and *3* included additional distribution features, we are uncertain if these students could develop their model; therefore, they may require further consideration. In *types 6* and *7*, all students, ($n = 10$), could focus on the distribution shape in the final conjecture. However, only one student connected the features and context more. It appears that for *types 6* and *7*, students could successfully combine distribution-related models through experimentation. The next section focuses on two cases to analyse the trigger of combining distribution-related models and focusing on the context, leading to rational views of distribution.

8.4.2 Case Studies

8.4.2.1 Type 6: Yuri's Case

In *Task 2*, Yuri initially conjectured the following sources of variation: "difference in the length of dropping", "accurate use of the stopwatch", and "angle of blades". However, she did not relate explicitly possible sources of variation to the dots in the dot plots. Her initial conjecture of flight-time distribution of 10-cm bladed

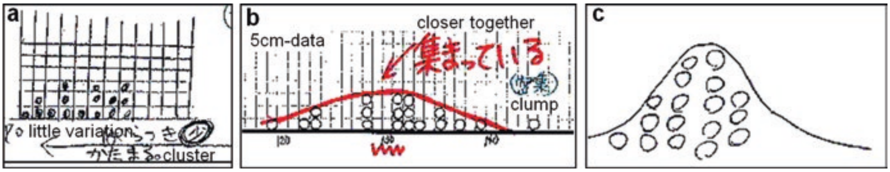

Fig. 8.4 (**a**) Yuri's Model B in the initial conjecture of distribution (**b**) Yuri's model of real distribution from the second experiment (**c**) Yuri's Model C in the final conjecture of distribution

helicopter in *Model B* had some bumps (Fig. 8.4a). She informally used the terms "variation" and "cluster" and explained about her conjecture as follows: "I think the difference between values will decrease a little (i.e., the dots will cluster in an area)". She started to reason about spread and density but could not connect these elements explicitly. In her initial conjecture and real distribution comparison, she focused on the distribution features of a 5-cm bladed helicopter's flight time, where the data were closer together around the value "130 (1.3 seconds)". She, then, modelled the distribution by using the notion of distribution shape and characterising it as "closer together" and "clump" (Fig. 8.4b). She idealised the dots in her mind and verified her initial model in terms of density with "It was an expected result. We could make an exact measure, since the use of the stopwatch was accurate". Thus, she connected shape with density and became aware of the context. In validation, she conjectured "individuals who measure using stopwatches" as the source of data variation in the second experiment and the flight-time distribution of a 10-cm bladed helicopter in *Model C* (Fig. 8.4c). She connected shape and density by applying an informal notion of a distribution shape (i.e. "I think the dot plot will be *just* mountain-shaped") and charted mountain-shaped plots. However, she did not identify her model's context.

8.4.2.2 Type 7: Ayu's Case

In *Task 2*, Ayu was initially aware of the context behind the data, since she related possible sources of data variation concerning flight data (e.g. "timing of stopwatches", "make of helicopter", and "angle of blades") to the values in the dot plots. Her initial conjecture of distribution for *Model B* had some bumps (Fig. 8.5a). She added the following supplemental explanation to the distribution sketch: "Although all values will not be equal, as people experiment by themselves, the data will come close to some values" and "although only at most three values were equal in the last experiment, about four values will be equal next time". She began reasoning about centre, spread, and context but could not connect these elements explicitly. In her initial conjecture and real distribution comparison, she explored the distribution features and modelled the shape of the distribution (Fig. 8.5b). Although she also considered the difference between her image model and real distribution with, "I thought the dots would clump more, the dots clumped only a little. I think the timing error was large". Thus, she connected shape, density, and context. Based on the

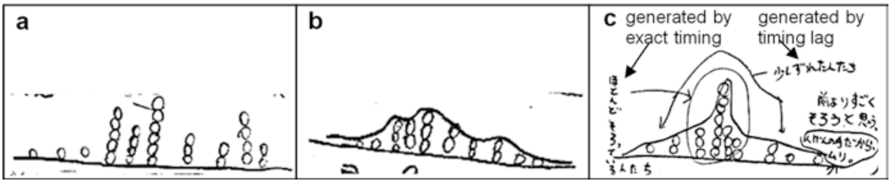

Fig. 8.5 (**a**) Ayu's Model B in the initial conjecture of distribution (**b**) Ayu's model of real distribution from the second experiment (**c**) Ayu's Model D in the final conjecture of distribution

validation, she considered "the use of stopwatches" as the variation source in the second experiment and conjectured the flight-time distribution of a 5-cm bladed helicopter again in *Model D* (Fig. 8.5c). She related the variation source (i.e. timing) to distribution and explained her following conjecture: "I think the dots will clump around a value [more] than the last experiment. All values will not be equal, as people experiment by themselves". She connected statistical elements and context as well as centre and shape.

8.5 Discussion and Conclusion

This chapter has addressed some aspects of students' development of distribution-related models based on rational distribution features using the PHE. Half of the students clearly developed their views of distribution, and the highest number of students took more rational views of distribution (see Table 8.2). Although 12 students did not relate with distribution elements in the final conjecture, the number not doing this decreased from in the initial conjecture (see Table 8.1). Although the reported findings are limited to a small sample, they provide empirical evidence that experimentation can foster students' model development of distribution. Nevertheless, nine students maintained *Model A* or *Model B* throughout the experimentation (see Table 8.2). This result confirms the difficulty of developing aggregate views of distribution (Garfield and Ben-Zvi 2008), indicating more research is needed into developing students' statistical reasoning about distribution.

Two case studies provided empirical evidence that isolated statistical and/or contextual elements in the students' distribution-related models formed by initial conjectures were able to be coordinated in their models by re-conjecturing after trial and validation. In Yuri's case, in the validation, she newly adopted a relevant statistical element (shape) from the distribution-related model in her external world into one in her internal world, resulting in coordinated statistical elements in her final model (Fig. 8.4c). However, Yuri could not construct explicitly the combined model with statistical and contextual elements because of her inexperience in correlating the variation sources to the graph, in contrast to Ayu's case. On the other hand, Ayu created a distribution-related model that had contextual and statistical elements in her internal world at the initial conjecture, but these were separated (Fig. 8.5a). The PHE's simple and observable activities also encouraged her to become aware of data variation and its source. In the validation, Ayu abstracted the real distribution shape and idealised the whole (Fig. 8.5b). Then, she newly adopted a relevant statistical element (shape) from the distribution-related model in her external world into one in her internal world, thus coordinating statistical and contextual elements (Fig. 8.5c).

The trigger for combining students' models can be the phase of model validation in experimentation. Both students recognised real distribution as a *model* of the experimental phenomenon (population distribution) by comparing their conjectures with the real distribution and resolving their conflict. Their real distribution models in the external world drew out analogies about the relationship between the distributions elements, which was lacking in their internal world models. There appeared to be the nature of models "as entities for comparing and contrasting and for drawing out analogies" in mathematics education (Ikeda and Stephens 2015, p. 354). In addition, these distribution generalisations could have been influenced productively by their prior knowledge (Stillman 2000). Authentic context in experimentation might reinforce their prior knowledge that flight-time distribution grows into nearly normal distribution through sophisticated experiments. These episodes strengthen the experimentation effects in the development of mathematical ideas indicated in previous literature (Carreira and Baioa 2011; Halverscheid 2008), demonstrating that these extend to the development of statistical ideas. In statistics education, conjecturing and validation about distributions are useful to guide students to look at aggregate features of distributions (e.g. Bakker and Gravemeijer 2004); therefore, this chapter suggests the possibility of a modelling approach combining students' distribution-related models through experimentation.

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Chapter 9
Reconciling Intuitions and Conventional Knowledge: The Challenge of Teaching and Learning Mathematical Modelling

Azita Manouchehri and Stephen T. Lewis

Abstract In this chapter we present illustrative examples from school learners' modelling efforts to highlight how knowledge of extra-mathematical contexts can influence students' mathematical practices during various phases in the mathematical modelling cycle. We propose that future research may need to focus on how to constructively utilize students' intuitions drawn from their personal and cultural backgrounds to advance their modelling cognition.

Keywords Epistemology • Intuition • Modelling • Validation • Reflection • Cognition

9.1 Introduction

The goal of improving mathematical modelling skills among school learners has been a major global source of scholarly efforts in mathematics education (Cai et al. 2014). There is agreement that successful modelling hinges upon the ability to produce increasingly more precise and generalized solutions through iterative reflection and validation actions (Blum 1991). As such, an individual's interpretation and expectation of what counts as a precise and adequate method becomes a key player in whether the modelling cycle is revisited or refined. While past research has considered the type of learner deficiencies that interfere with precise modelling (e.g. background knowledge, reading skills, errors), explanations offered for learners' reluctance to seek and produce refined models rarely account for epistemological elements that influence their choices including the criteria they consider when validating the solutions they produce. We argue that unpacking these issues is central to

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_9

advancement of the field and merits careful consideration. In establishing this point, we will draw from data collected over the course of 4 years and through three interrelated research projects on intermediate and high school learners' mathematical practices to problematize two issues associated with implementation of mathematical modelling in schools: (1) the chasm between learners' intuitions regarding variables they legitimately consider as prior constraints, based on their real-life experiences, and the conflict their choices create when confronted with the demand that conventional mathematics be considered in solving the same problems and (2) complexity of converging intuitive and analytical domains of children's work in a manner that learners' intuitions are not dismissed. It is our contention to argue that difficulties associated with "reflecting" and "validating" processes might be due to learners' extra-mathematical knowledge gained through personal and cultural practices in which their lives are grounded.

9.2 Background Literature

Blum and Leiβ (2007) provide an overview of the mathematical modelling cycle, where a problem situation has to first be understood by the modeller and a *situation model* is constructed with support from the data presented in the problem. Through simplifying and structuring, a *real model*, or a simplistic representation of the situation with more precise details (p. 226), is established and mathematized for analytical purposes, and through the use of mathematical tools, a solution is generated, and the results are matched against the real-world situation allowing for validation and refinement of this model (p. 227). Findings of some studies highlight that learners do not volunteer, without specific intervention, reflecting, validating and generalizing attempts (Galbraith and Stillman 2006; Verschaffel and De Corte 1997). Others have noted learners' inability to consider real-life knowledge as a barrier to their successful modelling practices. For instance, in a study of elementary school learners' work on application problems, Verschaffel and De Corte (1997) indicated that students tended to neglect real-world knowledge and other appropriate considerations when solving modelling problems. They associate erroneous decisions students make about modelling contexts with this gap in their knowledge (p. 589). Other challenges in producing mathematical models have been tied to learners' tendency to focus on isolated parts of a problem instead of viewing it globally (Mousoulides et al. 2010) and their inability to connect mathematical concepts to real-world situations (Kehle and Lester 2003).

While the literature provides insight into the competencies learners need to develop to be successful in solving modelling problems, it is less clear how modelling behaviours might be nurtured in ways that students' voices are taken into account when extending their mathematization processes. Assuming the absence of knowledge of the real world on the part of the learners might be less than accurate when judging children's understandings or how they mathematize what they assume as authentic questions regarding real-life events. Attempts at altering children's modelling practices without maintaining a responsive stance towards their priorities and intuitions might fail to nurture modelling as an intellectualized disposition that potentially anchors their decision-making in real-life and instead may be reduced to a set of procedures that students will learn to mimic to satisfy what they perceive to be mandatory academic expectations for success.

9.3 Context and Background

Our program of research combines and augments data across three different projects. Collectively, all three projects aim to examine mathematical practices of school learners as they solve different types of problems from various mathematical areas with the intent to gain an understanding of their mathematical choices and ways those understandings interact with instruction. Through our first project, we collected a large data bank of school learners' work through written responses they provided to selected problems (project A). Approximately 1,000 students from 28 different school districts in two major states in the USA were sampled in this study (Tague 2015). The second project documents, using individual and group interviews, sources that seemingly contributed to students' written responses (project B) (Manouchehri and Zhang 2015; Zhang and Manouchehri 2016). The third project involves the use of teaching experiments to explore ways that learners' practices, as identified in previous cycles of study, might be altered in the presence of particular scaffolding techniques, interventions and semiotic tools (project C) (Manouchehri et al. 2014). Both projects B and C rely on work with approximately 120 children enrolled in grades 8–12. The project activities consist of individual or teamed interviews with students along with short- and long-term teaching experiments that attempt to access their interpretations of, and responses to, tasks and ways that they shift, or refuse to shift, their approaches in the presence of designed interventions.

The problems used in these studies, collectively, elicit learners' modelling and modelling with mathematics processes. Two questions guide our data collection and analysis across the three studies: (1) What factors influence student choices of mathematical tools when confronted with a variety of problems ranging from application to modelling contexts? (2) How do learners rationalize their choices?

A major dimension of our analysis concerns understanding the role of learners' intuitions in how they tackle and move through the modelling cycle. This consideration is significant as Dixon and Moore (1996) characterize *intuitive understanding* of a problem as a representation that is distinct from the representation of the formal solution procedure for solving a problem. In this sense, intuition is not a special mechanism, but a form of reasoning guided by people's interactions with the environment. As such, intuition is a product of prior experience and reason where hypotheses are examined by performing probabilistic judgements. It is sensible to imagine that these intuitive understandings can influence individuals' understanding of, and interactions with, the modelling process.

In coding the data, we seek occasions in which the use of intuitive understanding and reasoning becomes paramount and directly shapes the learners' mathematical work manifested in how they interpret the problem, mathematize the context, solve and validate their responses. Results highlight that how children perceive, interpret and then mathematize towards solving a problem is closely linked to their primary and secondary intuitions (Fischbein 1993). While primary intuitions are grounded in life experiences, secondary intuitions are developed through schooling and the result of repeated exposure to particular practices, some of which are in sharp contrast with the type of thinking demanded in the modelling process (Manouchehri and Zhang 2015). In revealing the problematic nature of this issue, in the next section, we will provide illustrative examples to point out the relevance of children's thinking in the ways that they draw on extra-mathematical knowledge when responding to tasks. We will show how learners go about authenticating tasks using their experiences, which often counter conventional interpretations of valid responses. Our goal in this paper is not to report on findings of any one of the three projects, as a unit, but to outline what we have learned through 4 years of research using multiple cycles of investigation about key features that have challenged our thinking regarding ways to nurture modelling skills among school learners.

9.4 Modelling and Intuitions

9.4.1 Risk Analysis

Fischbein (1993) articulated that, "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (p. 14). In turn, *intuitive acceptance* shapes whether one accepts a certain solution or interpretation without explicit or detailed justification. Accordingly, these interpretations and intuitions directly influence the children's mathematizing, validating and refining efforts. The following section provides an illustrative example of this point.

Risk Analysis Problem
Suppose you have $1000 and wish to invest the entire amount on two business proposals you were given…What factors do you consider in order to decide how much to invest in each business venture? What questions do you ask to decide what to do?

The conversation depicted below occurred between the first author and two ninth-grade students, Jasmine and Tonya around this problem during a teamed interview in project B. The goal of the interview was to document variables the learners considered and the type of tools they used in responding to a task that required risk analysis. The dialogue depicts the robustness of children's convictions when defining and justifying their choices. Furthermore, it problematizes the task of altering children's personal values.

Jasmine: Do I know them? (She means the investing outlets.)
Interviewer: Does it make any difference to you?
Tonya: Sure, if they are friends, then I will give the money to the one I like best. If family then they get my money?
Interviewer: But what if there are risks involved? What if one of them has a higher risk of you losing your investment?
Tonya: Same difference (Jasmine nods in agreement).
Interview: So, suppose these people are neither friends nor family members. Assume that you are investing your money determined to make money. What then?
Tonya: How do we know these are legitimate businesses? How do we know they are not gonna take the money and run?
Interviewer: (pause) Good point, good point… so, how do you go about determining whether these are legitimate ventures? What data do you look for?
Jasmine: Ask around and find out from people we trust what they know about them.
Interviewer: What else? What specific information would you ask for? Suppose they are both legitimate…what then?
Tonya: How long they have been in business? Who their clients are, stuff like that.
Interviewer: What if they have been in business the same amount of time? Or, what if they are both new business proposals?
Jasmine: Best not to invest in new business…I personally go for the older, established one…that is, if I had the money to invest (Tonya nods in agreement). First of all, in real life, you always must know the person; otherwise there is not much you can do if you lose money. If you know them, then you can go and make arrangements to at least get your money back later.

Notice that in structuring her questions, the interviewer attempted to gauge the learners' thinking on factors that would be appropriate to consider so as to provide them with an analytical structure they could use in sequencing their analysis. We draw attention to the fact that students remained strongly tied to their views about the desire to invest with someone that they knew and trusted. Although aspects of the problem could be classified as inauthentic (Vos 2011) such as not actually having $1000 to invest, students still clung heavily to their framing notions and considered variables in a very personal, realistic and valid way. They continued to defend their choices about how they would invest *their* money. As such when attempting to promote mathematization, the interviewer's suggestions were rejected.

Table 9.1 Responses and explanations to the wage problem

Number of responses	Final response	Explanation
75	Not enough information	We don't know how many hours
83	$300	You earn $1,200 a month
62	$750	You can work 100 h a week and earn $750
58	It depends on the number of hours worked	
48	There is no difference; they are the same	Working 40 h a week gives you the same amount

9.4.2 The Wage Problem

In examining mathematical modelling and application tasks, students draw on extra-mathematical knowledge and personal life experiences. These domains of knowledge authenticate (Niss 1992; Vos 2011) tasks and shape how students enter and navigate the modelling process. Niss (1992) characterizes an authentic extra-mathematical situation as one that is embedded in practices or subject areas outside of mathematics that deals with problems or issues recognized as being genuine to those working within it. Vos (2011) expands the notion of authenticity by including the stakeholders' relationship to that context and considers aspects of modelling tasks that are constructed for educational purposes to be inauthentic, since students may have little to no stake in determining a correct or viable answer in the contextual domain of the problem (p. 721). The general concern deriving from this principle is how to manage tasks if they are indeed authenticated by students based on their relative intuitions about the context at large. The following example offers an illustration of this challenge.

The Wage Problem
Suppose you have two job options: One pays you $7.50 an hour and the other one gives a fixed amount of $300 a week. Which option would you take?

We administered this question to approximately 500 students enrolled in grades 5 through 8 in 23 different schools across two large states in the USA (project A). In asking the question, we had intended to evaluate students' conditional reasoning and decision-making under hypothesis. The central mathematical idea was for students to recognize "hours of work per week" as the primary variable and then to examine conditions under which either option may be considered better. Table 9.1 indicates the breakdown of student responses for this task.

Interviews with students in project B revealed that those who had selected $300 a week as the optimal option had assumed hourly pay to mean having to work less than 40 h a week. They also argued that the expectation to work more than 40 h a week would not hold since it would obligate the employer to provide health insurance to them which many businesses tried to avoid.

Those who had claimed there was not enough information to answer the question reasoned that knowledge about distance from work to home, and whether they would be given the option to set their own work hours (night shifts; hours per day), could have made a job option more appealing even if the pay were less. Some students argued that since transportation was key in whether they could make it to work, then the primary consideration for them was the location of the workplace, for example, whether a bus route ran between their home and place of employment. The individuals who had identified $7.50 an hour argued that since they did not wish to work for more than 40 h a week, and they could potentially be asked to do so with the first option ($300 a week), this motivated their selection of the hourly pay option.

These responses showcase that students considered the problem as real, and as such a barrier that was faced was in structuring a real model from the situation model that they had constructed (Blum and Leiβ 2007). They recognized the need to consider factors such as health insurance, distance to work and transportation, among others. These responses support that students are capable of thinking deeply about real-world contexts, making assumptions about those contexts and applying their experiences to answer questions posed about them. Even though their final answers may be considered as incorrect, the learners did, in fact, transform the problem into a more realistic version than was originally presented and solved it accordingly.

9.4.3 The Gasoline Problem

Greer et al. (2007) found that students distinguish between mathematical problems and real-world contexts based on how the tasks are presented. For example, in solving application word problems, students in their study exhibited the tendency to disregard answers they deemed problematic to the real-world situation. The authors argued that students may not exhibit mathematical behaviours if they do not deem the situation to be mathematical in its nature. The authors' points are profound, and the following illustration serves to expand on their notion.

The Gasoline Problem
You just won a "gasoline for life" prize. Should you take the option of a lump sum of $250,000 instead? (NCTM 2011)

The Gasoline Problem has been proposed as a recommended activity to be used in high school classrooms by a nationally recognized professional network. The authors had predicted that in solving this problem, students would rely on their own personal circumstances to make assumptions regarding the problem. To funnel student thinking, the authors also outlined suggestions for how teachers could gauge students' responses around issues of interest including considering the geographic location of their residence and whether they would have access to public transportation, whether they had to endure a daily long commute or if they were to take a long trip over the summer. The authors claimed that these questions could help student focus on determining a model suitable for answering the question.

During one teaching experiment episode in project C, we posed this question to a group of high school students with the intent to see the extent in which their responses matched those anticipated by the curriculum developers and also the utility of suggestions for shaping their work. The range of responses students provided deviated from the anticipated range and are illustrated below:

- $250,000 "cause I can buy the things I need right now or like pay for my tuition and dorm. For all 4 years in advance".
- $250,000 "cause I can start investing it now and will be making more money that way".
- Who knows where I will be 10 years? I take the cash.
- $250,000 now. I won't get to drive till I am 16 and even then I probably won't have money to buy a car – So even assuming that I will get one say in 7 years, I am almost sure that by then, we will have solar cars and no need for fuel.

In considering students' responses, it is obvious that the means by which students prioritized and organized variables was not compatible with those considered to be central to the problem developers. Corresponding to the modelling cycle, this highlights that organizing variables within a context and establishing a real model are heavily influenced by the particular views students hold about the situation under study. Note that in their responses, learners had relied on their intuitions about future outcomes and their experiences. Due to these issues, they were reluctant to further engage in considering factors that the curricular authors had proposed to be pursued. Indeed, even in the presence of the instructor's persistence that the students must consider alternative approaches, they refused to do so as they viewed them unnecessary. Despite our efforts at problematizing the task, the students assumed it was resolved. What if questions we posted failed to receive attention or establish serious intellectual engagement on their part?

We note here that the students' responses are both valid and sophisticated. For example, the students considered the issues of the time value of money, which is a distinguishing characteristic of lump sums versus annuity relative to finance, where individuals attempt to value a future cash flow based on current dollars. Thus, their perspectives are not only drawn from their realities and experiences but also reflect sophisticated mathematical ideas that could be pursued.

9.5 Discussion

Mathematics is often characterized as a discipline that is culture-free, in which statements have unambiguous meanings, and, due to its preference for specific structured reasoning and symbolism, has also been described as a universal language (Devlin 2000). It is assumed that problems have common interpretations, language used in describing a phenomenon is shared, and individuals study and interpret the subject similarly. There is, however, mounting evidence that the knowledge that students bring with them to school are powerful influences on how they interpret and view problems or go about solving them (Mousoulides et al. 2010). We support these claims and further highlight the intense connections among analytical, relational and intuitive reasoning used in mathematical work. While students' perspectives rely on relational understanding, they are *real* in the sense that they reference tangible needs. As such, an important issue facing the mathematics education community is conceptualizing how instruction may address these intuitions and assist students in acquiring analytical competencies needed in modelling processes.

Schwarzkopf (2007) argued that in order for learners to be successful in solving modelling problems, teachers need to find ways to help their students balance the visible real-world into mathematical forms. This implies that modelling in its formal sense may not be inherent, but may need to be structured in a way that promotes engagement in the process. Ultimately, this means that activities need to be developed for instructional purposes that consider the cognitive processes that students need so as to re-engage them in tasks when their determined solutions may not be adequate in their prediction or utility. We further stress that success of the attempts at improving mathematical modelling skills among students is closely linked to bridging the gap between students' "realities," "intuitions" and the desired constraints in mathematical modelling processes. Learners' intuitions and experiences impact how they assess accuracy of the responses they obtain and thus influence whether they revisit their so-called "incorrect" answers. The process of reflection is ignited by detecting an anomaly of some sort in the solution or in the model. If the gap between what is real in mathematics and what is real in real-life is such that even unreasonable answers are perceived as legitimate, then from the learners' perspective there is no need for reflection. From an epistemological standpoint in association with mathematical modelling, the successfulness of re-engaging in the problem is linked to bridging these gaps in realities.

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Chapter 10
A Modelling Perspective in Designing Teacher Professional Learning Communities

Nicholas Mousoulides, Marilena Nicolaidou, and Maria Evagorou

Abstract The chapter addresses a professional learning community development approach, namely, a multi-tiered research design which involves a modelling approach to learning. The chapter describes how a *models and modelling perspective* was employed in developing a teacher learning community and how it was used to better understand, analyse, and support the nature and development of teacher knowledge. Results have revealed that teachers improved their knowledge and pedagogical approaches to modelling. Changes in their attitudes, self-confidence, and motivation, and in their collaboration with other teachers, were also evident. We conclude that such an approach, although very demanding both in skills and competences, might serve in establishing and supporting a teacher learning community.

Keywords Teacher professional development • Learning community • Model-eliciting activity • Multi-tiered research design • Technological tools • Attitudes and beliefs

10.1 Introduction and Theoretical Framework

The purpose of this chapter is to address how a models and modelling perspective (English and Mousoulides 2015; Lesh and Doerr 2003) was used in the development of a teacher professional learning community. By adopting a multi-tiered research design, involving a models and modelling approach to learning, we aimed to understand the nature and development of mathematics teacher knowledge, and what it means for a teacher to develop mathematics content, pedagogy, and an understanding of how students develop their mathematical ideas when working with inquiry-based modelling problems.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_10

Increasingly, researchers are realizing that it is crucial to work collaboratively with teachers and their students in the reality of their own classrooms, in an attempt to better inform practice (Ball 1996; Robutti et al. 2016). Such collaborative paradigms include design studies, which are process oriented, and theory driven, such as the multi-tiered teaching experiments, and professional development approaches that involve content-based collaborative inquiry (Robutti et al. 2016). In the latter, the professional learning communities that are developed could better support teachers' shift and professional development to reform-oriented approaches, through a focus on their students' developments and understandings (Lave and Wenger 1991).

The goal in a learning community is to advance the collective knowledge and in that way to support the growth of individual teacher knowledge and skills (Robutti et al. 2016). A successful and productive learning community must have four distinct characteristics, namely:

(a) Diversity of expertise and experiences among its members, who are valued for their contributions and given support to develop
(b) A shared vision of continually advancing the collective (and individual) knowledge and skills
(c) An emphasis on the development of participants' metacognitive abilities, reflective thinking, and the notion of learning how to learn
(d) Mechanisms for sharing what is learned (Lave and Wenger 1991)

The learning community developed in our study was organized around a three-tiered research paradigm (see Fig. 10.1) that mainly addresses the development of teachers. The research paradigm also addresses the development of researchers, parents, and students, in an attempt to examine how the collaborative environment developed for both classroom practitioners and researchers served in generating meaningful change within modelling learning contexts (Lesh and Kelly 2000). Parents were engaged in the professional learning community, as a larger study (within which the present study is situated) aimed to investigate the ways engaging parents in such settings could further facilitate the introduction of inquiry-based approaches in the teaching and learning of mathematics and science. Specifically, the collaboration aimed at designing and implementing teaching and learning

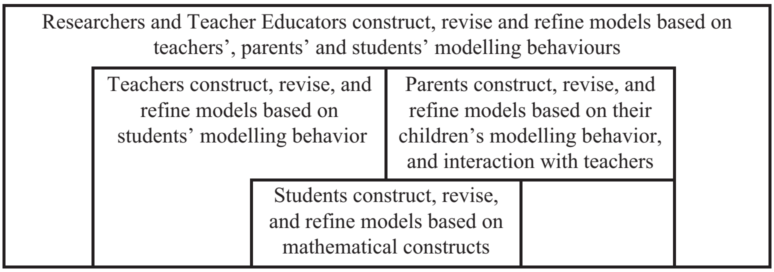

Fig. 10.1 Tiers of the learning community development

experiences (involving the construction and application of models) that maximize learning at each level. The collaboration also focused on the documentation and analysis of learning, together with reflection on learning.

As students are engaged in complex modelling situations (e.g. providing mathematically based arguments to validate a statement (young people involved in car accidents during weekends are usually drunk) or investigating the reasons a person is gaining weight, although he/she is doing sports on a daily basis) that repeatedly challenge them to reveal, test, refine, and revise important aspects of mathematical constructs (e.g. various concepts from statistics and algebra), teachers are focused on their own thought revealing problems that focus their attention on their students' modelling behaviour (Cai et al. 2014). Further, as the teachers revise and refine their models, this in turn affects the students' models and vice versa. At the same time, researchers (and parents) are focused on the nature of teachers' and students' developing knowledge and abilities which in turn are constantly affecting each other (English and Mousoulides 2015; Mousoulides 2013).

10.2 The Present Study

10.2.1 Purpose

The purpose of the study was to examine the impact of a multi-tiered professional learning community on teachers' knowledge and skills in designing and implementing inquiry-based modelling problems in their classrooms. We hypothesized that a collaborative modelling-based professional learning community would have a positive impact on teachers' mathematical knowledge, attitudes, and motivation, and their pedagogical approaches in teaching modelling problems.

10.2.2 Participants and Procedures

Four mathematics teachers (three females and one male) teaching in fifth- and sixth-grade classrooms in one urban-situated school agreed to participating in the study presented in this chapter. All participants held master's degrees in mathematics education. Other participants of the learning community developed were the students, the parents (around 40% of parents actively participated in the learning community), two mathematics inspectors who served as teacher trainers (together with the researchers), and the members of the research team in the MASCIL project. The project focuses its actions on designing and carrying out activities to support teachers in implementing inquiry-based teaching as well as connecting mathematics and science education to the world of work. Due to the emphasis of the chapter and page limitations, only the results related to the teachers are presented here.

The overview of the design of the learning community is presented in Fig. 10.2. In a period of 3 months, the teachers (and parents) participated in five 3-h workshops on mathematical modelling, inquiry-based learning (IBL), and world of work (WoW) and appropriate pedagogical approaches on mathematical modelling and problem-solving.

Teachers and parents were invited to participate in a collaborative design approach to develop and implement a number of model-eliciting activities (see Lesh and Doerr 2003) in their classrooms. Two thematic areas emerged, namely, *car accidents and road safety* and *health and exercise*. Over the course of the next 2 months, teachers worked collaboratively with researchers, parents, and teacher educators to develop lesson plans and learning activities for the two thematic areas and to implement the activities in their classrooms. Participants met weekly, communicated via email and via a blog, to develop five 80-min lesson plans for each thematic area. Further, two interactive applets were also designed to support and facilitate students' work in the two thematic areas. A screenshot of the applet designed for the 'car accidents and road safety' activity is presented in Fig. 10.3.

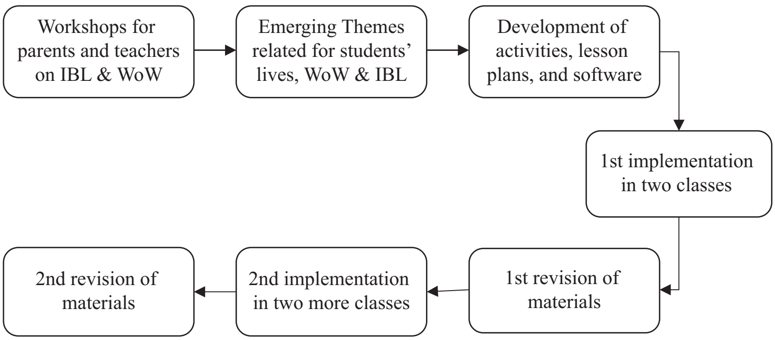

Fig. 10.2 Stages of the learning community design

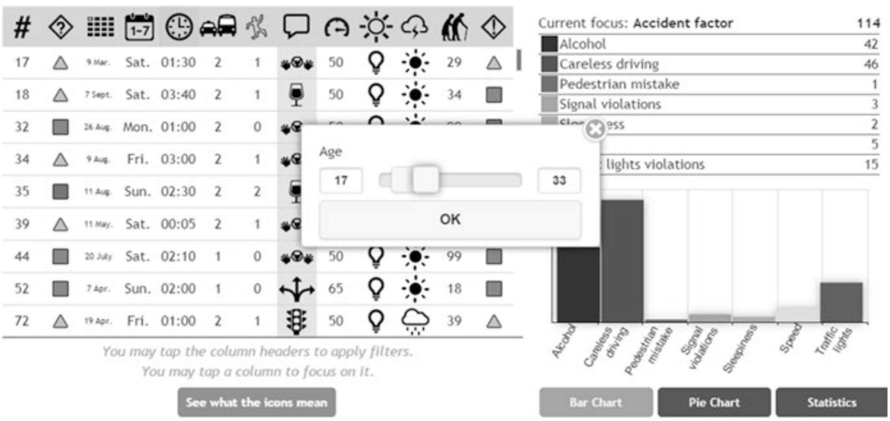

Fig. 10.3 Screenshot of the *Car Accidents and Road Safety* applet

The implementation of the first thematic area activities took place in two classes in the school. Other teachers, teacher trainers, some parents, and the research team observed each lesson, debriefed and analysed teacher approaches and student work immediately following each observed lesson, and reflected on their understandings throughout the process. Following each lesson implementation, an individual interview with the teacher that implemented the lesson was carried out by the research team. Revised activities were then tried out in other classrooms, followed by interviews (see above) and final modifications in the activities. Following a similar track, the second activity was also delivered in all participating classes. The study presented here is informed by data from the 2014 to 2015 school calendar year.

10.2.3 Data Sources and Analysis

Extensive field notes from lesson observations and group discussions were collected. All discussions through the planning stages, observations, debriefing meetings, lesson implementations, and lesson revisions were videotaped and transcribed. Data were triangulated with individual and group interviews and written reflections from each participating teacher.

A grounded theory approach to qualitative data collection and analysis was adopted (Corbin and Strauss 2008). Data analysis involved applying the constant comparative method. As Corbin and Strauss (2008) proposed, three approaches were used for data coding: (a) open coding, for examining, comparing, conceptualizing, and categorizing the data; (b) axial coding, for making connections between the categories revealed from open coding; and (c) selective coding, for selecting the core category and relating it to other categories and for validating the relationships between categories.

Three themes were revealed through coding and categorizing patterns in teachers' discourse and teaching practice. The first theme, teacher knowledge and pedagogical approaches to modelling, emerged as teachers gradually adopted and used more appropriate teaching methods and sought to assist students in improving their models and solutions. The second theme, teacher attitudes, self-confidence, and motivation, emerged as an outcome of participants' responses in interviews and in their reflections. The third theme, communication and collaboration, emerged both in group and individual interviews, as well as in the collaborative lesson plan design, and the use of email and the blog.

10.3 Results

The results are presented with regard to the three themes that emerged, namely, (a) teacher knowledge and pedagogical approaches to modelling; (b) teacher attitudes, self-confidence, and motivation towards teaching modelling; and (c) communication and collaboration.

10.3.1 Teacher Knowledge and Pedagogical Approaches to Modelling

Throughout the design, implementation, and modification of the activities, participants (researchers, teachers, and inspectors) identified a number of changes in teacher knowledge and pedagogical approaches. Results revealed that teachers synthesized their own and each other's prior knowledge, experiences, and resources in teaching and learning modelling activities. Teachers engaged in recursive interactions between their shared and prior experiences in teaching mathematics, using an inquiry-based approach. For instance, teachers' discourse (during lesson planning and reflective meetings that took place after lesson implementations) gradually improved, both in terms of the mathematical knowledge used and discussed and the appropriate pedagogical approaches used in the implementations. In these observations, discussions, and interviews, the following sub-themes emerged: (i) types and quality of questions, (ii) time allocated for student work and (iii) better feedback to student teams.

10.3.1.1 Types and Quality of Questions and Time Allocated for Student Work

Gradually teachers moved from addressing more closed (and answer oriented) questions to students to more open and inquiry-oriented ones that required conceptual understanding. This was evident not only in the observations that took place but also during the discussions. Throughout the implementation, all four teachers managed to provide more time for their students' individual and group work, devote less time for guidelines and instructions, and better facilitated orchestrated discussions between student teams, rather than addressing questions to individual students. The following exchange also reflects that shift:

Mary: I am always concerned with the questions I ask. I try to engage all, not just few, students, which is not easy.
Nina: Well, your questions [today] were appropriate; demanding and challenging. I believe they helped your students.
Mary: Well, I tried to do so. I was also stressed about time.
Harry: Yes, but I believe you provided enough time for each question, and your questions were clever […] they helped them [students] to progress through the task.

10.3.1.2 Better Feedback to Student Teams

Provided feedback significantly improved, as teachers focused on noticing important elements and aspects of students' work and addressed crucial suggestions for the improvement of student models. The following exchange shows how one teacher changed the way he provided feedback to his students:

Harry: It is now [working with modelling problems] more difficult to reply [provide feedback] to your students.
Researcher: Why it is more difficult?
Harry: There are not simple answers …I cannot just say ‘Ok’ and move one or say ‘Think again’. I have to listen carefully to what students say, and try to find aspects of their work that can be improved. This is difficult.
Researcher: Do you feel that you have improved your feedback to students?
Harry: Absolutely. Do you not agree? [laughs] I actually listen to them now […] it is not just crosstalk. Also, students learnt that they have to discuss with me, not just give me a number.

10.3.2 Teacher Attitudes, Confidence, and Motivation Towards Teaching Modelling

The second theme is related to the teacher affective domain and specifically to teachers’ (1) confidence to teach modelling activities, (2) motivation to integrate modelling in their day-to-day teaching, and (3) gradually more positive attitudes and task value for modelling tasks. Throughout the study, in various instances teachers reported that modelling tasks are good vehicles to teach and learn mathematics. During the group interviews and discussions, a tendency for participants to integrate modelling tasks in their day-to-day teaching practice was revealed. An increase in their level of confidence was also evident. The following remark expresses participants’ shared views about modelling and their willingness to teach more modelling-oriented in their lessons: ‘Is teaching modelling demanding? A lot! But I feel I can handle it now, and it is what kids need […] work with real problems with their peers’. Further, it was a shared understanding among participants that their engagement in the learning community was beneficial, and gradually they could see themselves becoming more independent in working with modelling tasks. The following exchange (from an interview at the end of the study) shows how two teachers changed the way they felt about modelling throughout the course of the study:

Researcher: So, will you continue using modelling tasks after the end of the project?
Anne: You can bet! [laughs] We spent so much time to learn to work like this, so I am not going to leave it! Honestly, I like modelling. It is not easy but it is rewarding. The students also enjoy it a lot.
Researcher: What do you mean by ‘not easy’?
Anne: You have to prepare a lot. Many different and diverse questions might appear. I remember at the beginning I was even scared, especially when I had visitors (researchers, parents) in classroom. It is totally different now.

Mary: I also feel the same way. I keep facing difficulties and sometimes I am not sure how to help my students overcoming a constraint, but I will continue working with modelling problems. I am more confident, but not 100%!

Participants sought an equilibrium in their teaching, by using modelling activities through adopting one (or more) inquiry-based instructional method(s). Teachers also expressed positive attitudes and task value for modelling tasks. Both in individual and group interviews, teachers appeared motivated to more frequently use modelling in their lessons. Quite often, all teachers referred to the great benefits (for students) from using modelling tasks in their lessons, albeit mentioning the various systemic and other constraints (e.g. time-consuming, or modelling is not included in the assessment). The following exchange shows the willingness of one teacher to keep working with modelling: ‘It is amazing how students work. They like the problems and I like watching them be so engaged in mathematics. […] You have to promise [points to one researcher] that you will include our school in any similar future projects on modelling!’

10.3.3 Communication and Collaboration

One of the core ideas when designing a professional learning community is the development of a culture of communication and collaboration between the participants. Our results revealed that there was (1) a constant and productive collaboration between teachers and (2) a fruitful communication of teachers with researchers and teacher trainers. Collaboration and communication between teachers improved both in quantity and in quality. Teachers collaboratively prepared, tested, and revised the lesson plans and communicated to each other on a daily basis, during and after school time. Further, there was a clear shift from discussing ‘simple’ issues (e.g. time devoted in each task) to sharing challenging questions, to designing extensions to tasks, and to co-teaching modelling activities. Similarly, the topics and questions teachers addressed to researchers shifted towards more insightful and reflective ones. Teachers’ questions moved from more ‘procedural’ (e.g. organize students’ groups, how to use the applet) to more ‘conceptual’ (e.g. develop a rubric for assessing student work, alternative teaching methods, classroom management during student presentations).

There is also evidence that participants, and especially teachers, recognized that it was crucial to work together to achieve their collective purpose of learning. The classroom observations, the group interviews, the collaborative designs, and the interactions with many others (e.g. parents, inspectors, and researchers) assisted teachers in reflecting on their collaboration. The following exchange (from a group interview at the end of the study) shows some aspects:

Nina: You cannot imagine [refers to the two researchers] how much time we spent every afternoon and night discussing with Anne.

Researcher: Is that a good thing?

Nina: Not for our families! [laughs]
Researcher: For you?
Nina: Much more than good! I feel it is usual to talk to Anne and Mary on a daily basis and work together. You know, this is not the case for our school [..] not only our school, but in schools in general. I like it, really!
[…]
Anne: It is not only that we work together, which is nice of course. Every day I have questions, important questions I want to discuss with others.

10.4 Discussion

In the study presented in this chapter, participating teachers were engaged in the design and modification of their modelling learning context through their selection and design of collaborative modelling activities, co-planned lessons, observations, and collaborative analysis of their students' models. This teacher-driven and modelling-based professional learning community offered unique opportunities for participants to collaborate, synthesize, and integrate appropriate pedagogies and teaching methods in the teaching and learning of mathematical modelling and problem-solving. Gradually teachers lessened their focus on difficulties in implementing modelling in their day-to-day mathematics teaching. As they investigated how various approaches to teaching complex problems engaged their students, and assisted them in building better and more refined models, their confidence in teaching complex problem-solving rose, and they appreciated the contribution of model-eliciting activities in developing students' mathematical constructs (English and Mousoulides 2015).

While the majority of teachers were very confident in their ability to teach mathematics (all had a master's degree in mathematics education), they were significantly less confident in teaching modelling activities and more complex inquiry-based problems – a trend that clearly pointed to the need to focus training and collaboration on building skills, expanding resources, and enhancing teachers' sense of efficacy and confidence and, therefore, motivation to work with modelling activities. Results revealed that teachers gradually improved their self-confidence in teaching more complex and modelling-based tasks, and they became more motivated in designing modelling activities, a very demanding and difficult task (Cai et al. 2014; English and Mousoulides 2015).

The learning community served as a structure that promoted a more collaborative culture. Teachers' willingness to collaborate did not stop at the classroom door, but teachers joined forces with researchers and parents to fruitfully collaborate in better implementing the modelling activities and to analyse students' models and developments. We can claim that the powerful collaboration that was established characterizes the professional learning community, and it was a systematic process in which teachers worked together (and with researchers and parents) to analyse and improve their classroom practice (Lave and Wenger 1991). This process, in turn, is expected to head to higher levels of student developments in modelling and problem-solving.

10.5 Conclusion

The notion of learning communities, including teachers and other significant groups, like parents, inspectors, and researchers, will grow as we try to address the professional needs of teachers, the demanding nature of teaching and learning in and through modelling, and the communication and collaboration with people from diverse backgrounds and views, and share what one learns with others. The example presented here showed that teachers' participation in the learning community contributed in improving teachers' knowledge and pedagogical approaches, resulted in more positive attitudes and increased self-confidence, and increased the communication and collaboration between them. Clearly, more research on teachers' development as they construct knowledge and skills for teaching modelling and inquiry-based learning is needed in order to illustrate ways that professional learning communities can meet the learning needs of teachers so that teachers can meet the learning needs of their students.

Acknowledgements This chapter is based on the work within the project MASCIL – Mathematics and Science for Life (www.mascil-project.eu). MASCIL has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement no. 320693. This chapter reflects only the authors' views, and the European Union is not liable for any use that may be made of the information contained herein.

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Chapter 11
Mathematical Modelling and Proof by Recurrence: An Analysis from a Wittgensteinian Perspective

Bárbara Nilvada Palharini, Emerson Tortola, and Lourdes Maria Werle de Almeida

Abstract This chapter describes a study that aims to investigate the question: Is it possible to consider the recurrence process inside a mathematical modelling activity as a mathematical proof? The study is based in the writings of Ludwig Wittgenstein, in particular, on the subject of proof by recurrence. We based our arguments on the analysis of two mathematical modelling activities. A qualitative approach and an interpretative analysis of Wittgenstein's writings were used to infer points from written data and data collected through audio-recordings. Our analysis indicated that mathematical modelling activities, in a sense, may lead to the need for mathematical proof, particularly proof by recurrence.

Keywords Mathematical education • Mathematical modelling • Mathematical proof • Modelling as a vehicle • Modelling as content • Proof by recurrence • Proof in modelling activities • Wittgebsteinain perspective

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_11

11.1 Introduction

Discussions regarding mathematical modelling as a way to deal with mathematics are important and necessary to strengthen mathematical modelling in the context of mathematics education, since the development of modelling activities encourages the exercise of specific procedures and promotes the study of various mathematical subjects.

Blum et al. (2002) discussed the research goals of the ICMI study for applications and modelling in mathematics education and identified several mathematical modelling issues and challenges. Even now some of these points need our attention, for example:

> What parts of mathematics, if any, are less likely to be represented in applications and modelling? What parts of applications and modelling, if any, are less likely to be represented in mathematics? What is the meaning and role of abstraction, formalization and generalization in applications and modelling? What is the meaning and role of proof and proving in applications and modelling? Are there common features of proving and modelling? (p.159)

According to Almeida (2014), reflections on mathematical modelling and philosophy 'may possibly clarify something in regards to the lack of unanimity about the modelling activity or the almost inherent complexity of the modelling process, about the place of mathematics in modelling activities' (p. 100). It is at this last aspect that this chapter is directed, that is, about the place of mathematics in modelling activities; in particular, the aim of this research is to answer the question: Is it possible to consider the recurrence process inside a mathematical modelling activity as a mathematical proof?

11.2 Mathematical Modelling: A Way to Deal with Mathematics

Around the world mathematical modelling is designed from different perspectives, as indicated by Kaiser and Sriraman (2006), which indicate different contexts and different interpretations of what is named mathematical modelling. According to Galbraith (2012), these interpretations are presented in two different settings, namely, as an object of study and as a way of teaching and learning mathematics. These two sets of mathematical modelling approaches are classified in accordance with the objective that we take to address it, respectively, 'modelling as content' and 'modelling as a vehicle' (Galbraith 2012).

By considering modelling as content, Galbraith (2012) refers to an activity that 'sets out to enable students to use their mathematical knowledge to solve real problems, and to continue to develop this ability over time' (p. 13). In this case, the focus is on modelling itself and how to learn to do it and how to use the procedures of modelling and to use mathematics to solve the problems which the modeller faces. On the other hand, mathematical modelling may be used as a vehicle, where 'some parts of a modelling process, or aspects related to modelling, are used to enhance the learning of mathematical concepts that form part of the curricular mathematics included in syllabuses' (p. 13).

An idea that can be associated with mathematical modelling is to analyse and interpret, through mathematics, situations that are around us (Lingefjärd 2007). In recent years, this idea has been implemented inside classrooms to contribute to mathematics teaching and learning. One of the teacher's main objectives is to confront students with problem situations and, while they are seeking solutions, to promote mathematical discussions regarding mathematical or modelling content. To develop modelling activities, it is important to identify which mathematical content is necessary to solve the problem. These choices depend, among other factors, on the characteristics stated in the problem, on the abilities of the modeller and on the modelling tools at his/her disposal Almeida (2014). However, in different activities some mathematical characteristics are recurrent, such as the mathematical recurrence (Matheus and Reed 2007). It is used in the works of Yanagimoto (2003) to estimate the number of bluegills in Lake Biwa, in Japan, Bassanezi (2004) to study the population dynamics of Nile Tilapia and Almeida et al. (2012a) to determine the amount of mercury released for the environment, among others.

We focus our discussion on the use of recurrence in mathematical modelling activities, and we propose this examination from a Wittgensteinian perspective, mainly considering that in mathematics education, particularly in modelling, different uses of mathematics can be seen as different language games as stated by Wittgenstein (1996). Wittgenstein was particularly interested in the philosophies of mathematics, breaking with an absolutist view of mathematics, as defended by the philosophical currents of logicism, intuitionism and formalism (Gerrard 1991; Putnam and Conant 1997; Wright 1980), since such a view sets aside specificities of mathematical uses in different political, social and cultural contexts. The philosophical examination of the language uses, in particular of mathematical language, its rules, proofs and propositions, appears in the articulations between the Wittgensteinian perspective and mathematics education (Duarte and Taschetto 2014; Gottschalk 2014). Specifically, in modelling, some researchers have signalled the use of different models in different language games (Almeida 2014; Almeida et al. 2012b) and in mathematics teaching and learning (Souza and Barbosa 2014).

11.2.1 Wittgenstein's Perspective About Proof by Recurrence

Ludwig Wittgenstein was an Austrian philosopher with great importance in the linguistic turn of the twentieth century. Books such as *Philosophical Grammar* (Wittgenstein 1974), *Philosophical Remarks* (Wittgenstein 1975) and *Philosophical Investigations* (Wittgenstein 2012) discuss the construction of a new perspective for language, as opposed to a merely representative function.

Part of Wittgenstein's writings is dedicated to mathematics and, in particular, the role of mathematical proofs. According to Wright (1980), 'no question receives more attention in *RFM* than that of the nature of the distinction between calculation,

First Principle of Induction Let $p(n)$ be a propositional function whose universe is the integer set greater than or equal to a given integer a. Suppose we can prove that $p(a)$ is true; if $r \geq a$ and $p(r)$ is true, then $p(r+1)$ is also true. Then, $p(n)$ is true for all $n \geq a$.
Second Principle of Induction Let $p(n)$ be a propositional function whose universe is the integer set greater than or equal to a given integer a. Suppose that we can prove $p(a)$ is true; if $r > a$ and $p(k)$ is true, and for all k such that $a \leq k < r$, then $p(r)$ is also true. Then, $p(n)$ is true for all $n \geq a$.

Fig. 11.1 Principle of finite induction by Domingues and Iezzi (2003, pp. 31–32)

or proof in general, and experiment'[1] (p. 318). For Wittgenstein (1996), 'the proof... is a *single* pattern, at one end of which are written certain sentences and at the other end a sentence (which we call the "proved proposition")' (p. 48). Regarding the proposition and its meaning, Wittgenstein (2003) notes that a proposition is all that can be true or false, in a particular language system: 'It is only in a language that something is a proposition. Understanding a proposition is to understand a language' (p. 97).

The proposition and its proof are related in a logical and unique way. In many situations, what we have in mathematics are not proofs but a hypothesis that must be verified and validated. However, only a proof establishes connections between the hypothesis and its validity. The importance of proof is in 'showing us' things as they really are. 'A proof ought to show not merely that this is how it is, but this is how it has to be' (Wittgenstein 1996, p. 149). In particular, Wittgenstein presents reflections regarding proof by recurrence and *touches on* the validity of this process as a procedure related with mathematical proof.

> As we say the proof by recurrence shows that algebraic equations are valid for all cardinal numbers; for now, it doesn't matter if the expression is a good or bad choice; what matters is that you have the same meaning clearly defined in all cases.
>
> And is it not clear that proof by recurrence actually show the same for all "proved" equations?
>
> And does not mean that between the proof by recurrence and the proposition, that it proves, there is always the same (internal) relationship?
>
> Anyway, it is clear that there must be such proof by recurrence or, rather, an interactive one. (A proof communicates the insight that "it is the way it should be for every number"). (Wittgenstein 2003, p. 325)

The proof by recurrence procedure may be associated with the axiomatic method of the mathematician Giuseppe Peano (1858–1932) and his infinite sets, as well as the principle of finite induction (Fig. 11.1). Thus, the proof by recurrence may be considered as a mathematical proof, the iteration, that shows one term depending on another, generates a chain of propositions culminating with 'and so on'. Wittgenstein raises the validity of 'and so on': How can we infer from two or three cases an infinite class of validity? 'The connection with the most finite domains is entirely clear. In a finite domain, it would certainly be a proof that $f(x)$ is valid for all values of x, and that is why we say in the arithmetic case that $f(x)$ is valid for all numbers' (Wittgenstein 2003, p. 326).

[1] *RFM* refers to the work *Remarks on the Foundations of Mathematics*.

According to Wittgenstein (2003), ‘the only time that it is unwise to call something “proof” is when the grammar of the word “proof” does not conform with the grammar of the object under consideration’ (p. 333). It is in the grammar of the considered object that it is possible to validate the proof, where we should consider the mathematical proposition and the propositions associated with it. In this way, proof by recurrence may represent the proved proposition via an internal relationship, and the connections between the hypothesis and the concept, and between the proposition and its foundations. In Wittgenstein’s words, ‘We can always write a proof by recurrence as a limited series with “and so on” without it loses none of its rigor’ (Wittgenstein 2003, p. 344). Considering Wittgenstein’s philosophy on proof by recurrence, we analyse mathematical modelling activities.

11.3 Methodological Procedures

To investigate if mathematical modelling activities may allow procedures related to the proof by recurrence, we analysed two situations from a bigger collection: the activity *Concentration of the Contraceptive Pill in the Body* (see more details in Palharini et al. 2015), developed during a mathematical modelling course by three Brazilian students who were in the second year of a mathematics degree, and *It Is Time to Turn Off the Lights* activity from Almeida et al. (2012a), a Brazilian book that addresses modelling in the classroom.

Concentration of the Contraceptive Pill in the Body
Composition (coated tablet):

Each coated tablet contains 0.100 mg levonorgestrel and 0.020 mg ethinylestradiol.

After single-dose administration, maximum blood concentration of levonorgestrel is achieved within 1–2.5 h, and ‘steady state’ is reached after 19 days of continuous use. After a single dose, maximum ethinylestradiol concentration in the serum is reached within 1–2 h and ‘steady state’ after 6 days of continuous use. The plasma elimination half-life of levonorgestrel with ethinylestradiol is from 8 to 13 h.

Hypothesis:

(1) The contraceptive half-life is 12 h; (2) the pill is ingested each 24 h; and (3) adequate contraceptive use is to administer equal doses, every day, for 21 consecutive days.

Concentration of the Contraceptive Pill in the Body presents information regarding modelling activity developed in the classroom by a group of three students, named as A, B and C, from a total of four groups. From such modelling activity, to support our analysis, we used data collection including audio-recording of the classes, written data and solution of students' modelling activity.

It Is Time to Turn Off the Lights

Metallic mercury contained in the lamp, in its gaseous form, is toxic for humans and for the environment. When lamps are thrown in landfills, it contaminates the soil, water resources, fauna and local flora, reaching the food chain. If the lamps were handled or disposed of incorrectly, they may break and release mercury vapour – around 20 mg per lamp.

Considering this problem and the way the lamps out of use are discarded in a public institution, a group of students, motivated by a story from a local newspaper, decided to investigate this theme in mathematics class.

According to a newspaper article, around 3000 fluorescent lamps were out of use in a deposit, with the risk of breakdown and release of mercury into the environment. In addition, around 420 lamps would be forwarded for this deposit every month.

To investigate the problem, consider that 3000 lamps broken and the other 420 sent monthly also suffer damage and release mercury into the environment.

Hypothesis:

(1) Each fluorescent lamp has 20 mg of mercury; (2) there are, initially, 3,000 lamps out of use in the deposit. (3) 420 new lamps are deposited in the same deposit monthly; (4) all lamps are broken as soon as they arrive in the deposit, releasing mercury; and (5) the mercury half-life is approximately 2 months.

The analysis was conducted by three researchers from the set of data collected through empirical research, mainly considering records and audio of the students during modelling activity and the suggestions for the modelling activity development in the book, *Mathematical Modelling in Basic Education* (Almeida et al. 2012a). These methodological procedures were considered to support an interpretative analysis based in the writings of Ludwig Wittgenstein.

11.4 An Analysis of the Recurrence Process in Mathematical Modelling

From the empirical data shown in Fig. 11.2, students tried assuming a hypothesis and tried to formulate a mathematical model using the initial concentration of salts in the pill, $Q(0) = 0.12$, where Q is the amount, in milligrammes, of the salt

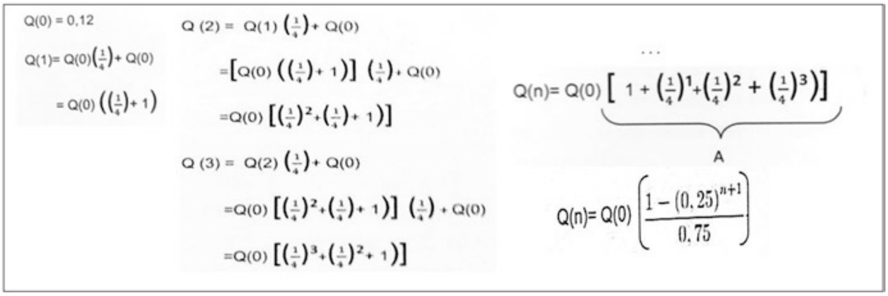

Fig. 11.2 Students' records

concentration in the body. It is only in the language regarding the half-life of the drug that $Q(0)$, $Q(1)$, $Q(2)$...$Q(n)$ are considered propositions (Fig. 11.2). According to Wittgenstein (2003, p. 97), 'to understand a proposition is to understand a language. A proposition is a sign on a system of signs'.

By formulating the model, for the concentration in the body of a dose of the pill $Q(n) = Q(0)(1/2)^n$, or for n doses $Q(n) = Q(0)(1 - 0, 25^{n+1}/0, 75)$, considering a dose each 24 h, students used mathematics as a guide that indicates propositions. This guide was indicated when the students argue that:

Student A: For the model to work it is necessary to take the medicine at the right time every day, if the person takes it after or before the time, the model does not work.

The construction of a mathematical model that considers n doses each 24 h is stated by student B as:

Student B: Now I will do the modelling to find a model for n pills taken from the $Q(0) = 0.12$ mg. We will do a recurrence modelling to find our model. …In $Q(2)$ all this here will suffer the decay in the organism... . If we do $Q(3)$ we will have ¼ of $Q(2) + Q(0)$ ... . We can generalise, who will be the $Q(n)$, will be $Q(0)$ times that sum here, which is a finite GP [Geometric Progression] right?

When they are trying to generalise, from a simple case to a more general case, the recurrence procedure is used. This procedure was used to obtain a rule that directs its use and shows how the situation must be (Wittgenstein 1996).

With the obtained rule students may calculate the salt concentration at any time. However, to obtain this rule, another mathematical tool is necessary, to consider the sum of a finite geometric progression as C points out:

Student C: We try to put it as an infinite sum of a GP and never validated, so we arrived at the sum of a finite geometric progression ... . In the other way it would never work, because it is not infinite, there is a period that we are taking this pill; it is for 21 days.

0	$Q(0) = 60000$
1	$Q(1) = \left(\frac{1}{2}\right)^{\frac{1}{2}}.Q(0) + 8400$
2	$Q(2) = \left(\frac{1}{2}\right)^{\frac{1}{2}}.Q(1) + 8400 = \left(\frac{1}{2}\right)^{\frac{1}{2}}.\left[\left(\frac{1}{2}\right)^{\frac{1}{2}}.Q(0) + 8400\right] + 8400 = Q(0).\frac{1}{2} + 8400.\left(\left(\frac{1}{2}\right)^{\frac{1}{2}} + 1\right)$
…	…
n	$Q(n) = \left(\frac{1}{2}\right)^{\frac{n}{2}}\left[Q(0) + 8400\left(\frac{\sqrt{2}(1-\sqrt{2})}{1-\sqrt{2}}\right)\right]$

Fig. 11.3 Recursive process to formulate a model adapted from Almeida et al. (2012a)

Thus, within these considerations, the students of the group generalised a mathematical model using a recurrence process and communicated it to the other students.

The second modelling activity analysed in this chapter is from Almeida et al. (2012a). From this source, we have the information: each fluorescent lamp has 20 mg of mercury; with 3000 initial lamps out of use, 60,000 mg of mercury is in the environment; and 420 lamps are broken monthly, this is, 8400 mg more of mercury. Assuming this hypothesis, one of the ways to estimate the mercury concentration in the environment, at any time, is stated in Fig. 11.3.

Both modelling activities (*Concentration of the Contraceptive Pill in the Body* and *It Is Time to Turn Off the Lights*) allow a mathematical analysis that may lead the modeller to the use of a recurrence process. Considering our question: 'Is it possible to consider the recurrence process inside a mathematical modelling activity as a mathematical proof?', we may note that the students' recurrence process development is not mathematically as sophisticated as the induction process (Fig. 11.1). Students A, B and C start from a situation where $Q(0)$ is valid, given the nature of what they comprehend of the real data studied and the hypothesis made, by the students, considering the real data. They surmise the iteration that generates the propositions $Q(1)$, $Q(2)$, …, to $Q(n)$.

Wittgenstein (2003) argues about the validity of iteration showing one term depending on another, what generates the chain of propositions, that is, the validity of the 'and so on', the inaccessibility of making conjectures unto infinity. However, when we are engaged in a mathematical modelling activity, we have to consider a finite domain. For example, the use of the contraceptive pill is made in a finite number of days, for 21 days uninterrupted. Even when we thought about the lamps that every month are breaking in the deposit, we know that sooner or later, this will end. Indeed, this mathematical modelling practice corroborates with Wittgenstein (2003) when he says that the connection with most finite domains is entirely clear.

During the iteration between one and another proposition, it is necessary to consider some mathematical artifices to simplify the expression for n doses of the drug or the amount of mercury at any time. A system of rules is required that allows the modeller to define, and redefine, the initial proposition from a set of hypotheses to a final proposition, what demonstrates an internal relationship with the hypothesis adopted. Even when the modeller redefines the proposition, we may say that he/she is not falling into the trap of making a serious reduction and/or simplification.

> In this calculus, the "proof" has a fixed meaning. If now I call the inductive calculus a proof, it is not a proof that spares me to check if the steps in the chain of equations were made in accordance with these particular rules (or paradigms). If they are gone, I say the last equation of the chain is proved or that the chain of equations is correct. (Wittgenstein 2003, p. 337)

Wittgenstein (1996) indicates on the basis of the proposition:

> The proof belongs to the background of the proposition. To the system in which the proposition has an effect. ... Every empirical proposition may serve as a rule if it is fixed, like a machine part, made immovable, so that now the whole representation turns around it and it becomes part of the coordinate system, independent of the facts. (p. 437)

When we discussed about proofs in mathematical modelling activities, doubts arise regarding their validity. This may occur due to the origin of the initial proposition, an empirical proposition and mathematical model; such is often related with an experiment. However, regarding mathematical proof, an initial proposition may be an empirical one. Thus, in mathematical modelling activity, the modeller does not worry with the validity of the proof.

In Wittgenstein's writings (1996, 2012) regarding proof by recurrence, one of the main obstacles is having to conjecture unto infinity, this is what he named 'and so on', because inside the different situations, with which we find ourselves, we do not deal with infinite domains. The empirical propositions are about facts, whereas infinite domains exist only in our language, through grammatical propositions[2], as in the case of mathematical language. According to Lingefjärd (2007), in mathematical modelling, we often need to analyse and interpret, through mathematics, situations that are around us; most of these situations have finite domains, and thus we do not fall into 'the trap' of conjecture unto infinity. To deal with this kind of conjecture, it is necessary to appeal to grammatical propositions, defined by Wittgenstein (1996) as those used in mathematical language.

11.5 Discussion and Implications for Teaching and Learning

Concerned with the use of mathematics in mathematical modelling, we proposed to investigate the question 'Is it possible to consider the recurrence process inside a mathematical modelling activity as a mathematical proof?' We based our reflections in Wittgenstein's writings regarding mathematical proof, in particular proof by recurrence. This perspective helps us to support the analysis of a mathematical modelling activity performed by students and a mathematical modelling activity suggested by modelling literature. We conclude that students, engaged in

[2] Propositions resulting from conventions crystallised into linguistic expressions are defined by Wittgenstein (1996, 2012) as grammatical propositions.

mathematical modelling activities, used mathematical processes that correspond to proof by recurrence, as indicated in mathematical language. On the one hand, we note that the use of this procedure was made without the students' knowledge that they were doing a mathematical proof. However, students used a recurrence process closely connected with the principle of finite induction. From this activity, teachers could suggest different modelling activities related to other infinite processes to teach, introduce or remind students regarding the sum of an infinite series and about mathematical propositions that make a series convergent or divergent depending on characteristics of the series and its limit value, using, again, aspects from formal mathematics.

By reflecting on modelling as content and as vehicle, as denoted by Galbraith (2012), we now focus on the contributions of mathematical modelling and proof by recurrence for teaching and learning mathematics. According to the author, modelling as content should propose empowering students to use their mathematical knowledge to solve real problems and to further develop this capability. The fact that the use of proof by recurrence, in mathematical modelling, can take place without the knowledge that the modeller is using a formal mathematical process for justification of propositions leads us to consider the second genre for modelling as stated by Galbraith (2012), that is, modelling as vehicle. In this context, parts of the modelling process are used to enhance learning mathematical concepts or processes that are part of the course included in mathematical programmes. In particular, regarding proof by recurrence, Wittgenstein's writings (1996) help us to understand the dynamic of this kind of proof and what we have to consider to validate a proof, mainly, the nature of the propositions used. Generally, proofs in mathematical modelling depend on the conditions stated in the problem-situation addressed, the empirical propositions assumed and the abilities of the modellers, among other factors that make possible the use of mathematical language in the solution of mathematical proofs.

From this study, we saw in the philosophy, particularly in the philosophies of mathematics – that consider the foundations of mathematics and the specificities of different contexts – as Wittgenstein philosophy, a way to investigate proof in modelling activities, how students use mathematical language in mathematical proofs when day-to-day situations are the beginning of the activity and the transition, made by them, between propositions that are valid within mathematics and its interpretation in the real world.

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Chapter 12
Quality Criteria for Mathematical Models in Relation to Models' Purposes: Their Usefulness in Engineering Education

Jacob Perrenet, Bert Zwaneveld, Kees van Overveld, and Tijn Borghuis

Abstract A taxonomy of eight quality criteria for mathematical models was developed for the common basic modelling course in the innovated BSc curriculum of Eindhoven University of Technology. First year engineering students of all disciplines reflected on their group modelling projects, indicating how their models could be improved, using the criteria. The students were also asked to indicate the purpose(s) of their models from a list of 16 purposes. This study explores the usefulness of the purposes and criteria, defined as relevance combined with understandability. *Optimisation* proved to be the most relevant purpose, followed by *analysis*, *prediction (what)*, and *verification. Specialisation*, *genericity*, *scalability*, *distinctiveness,* and *convincingness* criteria proved useful; but *audience*, *impact*, and *surprise* did not.

Keywords Engineering • Explorative research • Group projects • Model purposes • Model quality criteria

12.1 Introduction

For many years, the Eindhoven University of Technology (TU/e) offered about a dozen engineering Bachelor programmes (e.g. in biomedical engineering, software science, applied physics, and applied mathematics). Only in applied mathematics did mathematical modelling have a prominent position (Perrenet and Adan 2011).

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_12

In 2012, the totally innovated Bachelor college started, comprising all engineering Bachelor programmes as majors. The students were offered much more freedom to fill their programmes, but a set of basic courses, including modelling, became mandatory for all.

The aim of the modelling course is learning how to convert a non-mathematical problem into a form which can be tackled using mathematical tools, without losing sight of the original question. Starting with a generic introduction, the course offered four variants for specific disciplinary groups: dynamical systems (wherein time is the most important factor), data modelling (with acquired data as a starting point), process modelling (concerning systems with distinct states), and modelling from scratch (not directly related to a specific discipline). The education period takes about 10 weeks with lectures, take-home assignments, intermediate tests, and a group project (with five students per group).

We will focus on the generic introduction of 7 weeks, developed by the late Kees van Overveld, in consultation with Tijn Borghuis, and with representatives of all TU/e disciplines to make the content useful for all students. Much was constructed especially for this course. In this explorative study, we will zoom in on the topic of modelling criteria in relation to modelling purposes. We explore the usefulness of those criteria for the engineering students as apparent from their reflection on their modelling activities.

12.2 Theory

In ICTMA publications, respectively, general mathematical education literature, much has been published about the structure of the mathematical modelling *process* and its quality, that is, the steps to be performed and their order. Examples are Blum and Leiß (2006), Girnat and Eichler (2011), Blomhøj and Hoff Kjeldsen (2006), and Borromeo Ferri (2006). This structure is often represented by a modelling cycle (see Perrenet and Zwaneveld 2012). Less attention has been given to the quality of the successful *result*: a working model, that is, a model and its interpretation, with validation and verification. Only a few lists of quality criteria for a working model can be found in literature. An example is Meyer's list (1984): accuracy (is the model's output correct?), descriptive realism (is the model based on assumptions which are correct?), precision (are the model's predictions *definite* numbers, functions, geometrical figures? or are those *a range of* numbers, etc.?), robustness (is the model immune to errors in input data?), generality (is the model applicable to a wide range of applications), and fruitfulness (are the conclusions useful, inspiring, or pointing the way to another good model?). A second example is Agoshkov's list (2002): adequacy (the extent of qualitative or quantitative agreement between the model and the modelled system concerning its properties), sufficient simplicity (balanced between giving reasonably accurate results that fulfil the stated purpose and economy in terms of costs), completeness (yielding the best possibility of obtaining desired outcomes), productivity (ease of measurement of input data), robustness (stability with respect to errors in input data), and clearness (direct, clear, substantial sense of a model's components).

The *purpose* of a model has not received much attention in modelling education research. Van Overveld and Borghuis (2013) developed a *taxonomy* of quality criteria that relates to the lists mentioned before but takes into account the *purpose* of a model as well. In 'one model performs better than another', 'better' should include 'with respect to the models' *purpose*'. They distinguish the following purposes (with typical questions added):

– *Explanation*: Why …? How come …?
– *Prediction 1 (when)*: When …?
– *Prediction 2 (what)*: What …? What if …?
– *Compression*: Can these data be summarised in less data or in formulas?
– *Inspiration*: Maybe X could be tried…? Maybe Y could be true …?
– *Communication*: How to inform a specific audience?
– *Unification*: How to capture the similarity of phenomena from different domains?
– *Abstraction*: How to capture the essence of a phenomenon leaving out details?
– *Analysis*: What are the properties of the system under study?
– *Verification*: Is it true that … (statement about the modelled system)? (+ give argument)
– *Exploration*: What are the options …? In what ways can we connect A to B?
– *Decision*: Which of these is the best option (a closed set of alternatives)?
– *Optimisation*: What is the best value for these parameters or dimensions?
– *Specification*: What external properties should some artefact have? What should it do?
– *Realisation*: What internal properties should some artefact have? How should it do it?
– *Steering and control*: What (real time, online) interventions should this system do?

Three independent dimensions of the modelling process – beginning (definition stage) or end (conclusion stage), inside (model, modelled system) or outside (stakeholders, context), and qualitative or quantitative perspective – produce an eight-criteria taxonomy (Table 12.1).

Van Overveld and Borghuis stress that a complete taxonomy requires properties where all possible values can be enumerated. They develop operationalisations for all criteria. We will illustrate that here with two examples, *impact* and *convincingness*. *Impact* needs four quantities to express it as a number: r_1, the profit or income in the present situation, without the model outcome; r_2, the profit or income with the model outcome in place; c_1, the cost of ownership in the current situation; and c_2, the cost of ownership with the model outcome in place. For all quantities, the same timescale is taken (e.g. lifetime or yearly amounts). The quantity $\rho = ((r_2 - r_1) - (c_2 - c_1))/(|r_2 - r_1| - |c_2 - c_1|)$ is a number between −1 and 1. Positive values mean a beneficial contribution; negative values mean that the impact is adverse. The absolute value $|\rho|$ indicates the size of the impact.

Table 12.1 Taxonomy of quality criteria

Dimension			Criterion	
Stage	Inside or outside	Perspective	Description	Name
Define	Stakeholders and context	Qualitative	To which extent is the approach capable to handle various types of modelled systems or purposes?	Genericity
Define	Stakeholders and context	Quantitative	To which extent can some characteristic dimensions of the problem increase, where the model still functions?	Scalability
Define	Model and modelled system	Qualitative	To which extent does the model/model outcome require specialised knowledge on behalf of the problem owner?	Specialisation
Define	Model and modelled system	Quantitative	What size of intended audience does the model address?	Audience
Conclude	Stakeholders and context	Qualitative	How plausible are the assumptions of the model?	Convincingness
Conclude	Stakeholders and context	Quantitative	How similar can alternatives be in order for the model to allow distinction between these alternatives?	Distinctiveness
Conclude	Model and modelled system	Qualitative	What is the extent to which the model outcome may bring unforeseen new ideas?	Surprise
Conclude	Model and modelled system	Quantitative	What is the extent to which the model outcome can affect the stakeholders?	Impact

Convincingness hinges on plausibility of assumptions. This can be related to an ordinal scale, from high to low, as follows:

5. Assumptions are logically deducible from other non-problematic assumptions.
4. There is a law or theory in a well-accepted discipline (physics, economics, etc.), such that the current assumption is an instance of that law or theory.
3. There is a plausible formal model system to which the current system may be compared.
2. There is an empirical model supporting the assumption and a similarity argument.
1. The model behaviour is consistent with intuition.

For details, see Van Overveld and Borghuis (2013). Van Overveld and Borghuis conclude with combining the purposes and the criteria. The relevance of the various criteria is related to the purpose(s) of a model (see Table 12.2). For example, if the purpose is *communication*, the criterion *audience* (how large can the intended audience be?) is very relevant; if the purpose is *inspiration*, it is not.

Our explorative research questions concerning purposes and criteria are the following:

- To what extent are the criteria useful for students, that is, relevant as well as understood?
- To what extent are the purposes useful for students, that is, relevant as well as understood?

12.3 Methods

12.3.1 Participants and Materials

Our experimental subjects are 212 groups of first and second year engineering students of various disciplines. The tasks used are embedded in regular education. In the beginning of their project reports, the students had to state their modelling purpose(s). At the end, they had to reflect as groups on their modelling project structured by the following task.

Criteria Reflection Task

1. Necessity for improvement: In the lectures a set of criteria was presented to compare models. From the perspective of the model's purpose, on which of those criteria ought the model be improved, according to your opinion and why?
2. Possibility for improvement: For which of those eight criteria do you have ideas about how to actually improve your model? Describe these ideas briefly.

Table 12.2 Relevance of criteria related to modelling purposes

	Criterion							
Purpose	Genericity	Scalability	Specialisation	Audience	Convincingness	Distinctiveness	Surprise	Impact
Explanation	Medium relevance	No relevance	Medium relevance	Medium relevance	Medium relevance	Medium relevance	Medium relevance	Medium relevance
Prediction1	No relevance	Medium relevance	No relevance	Medium relevance	High relevance	High relevance	No relevance	High relevance
Prediction2	No relevance	Medium relevance	No relevance	Medium relevance	High relevance	High relevance	Medium relevance	High relevance
Compression	Medium relevance	High relevance	No relevance	High relevance	No relevance	High relevance	No relevance	Medium relevance
Inspiration	No relevance	No relevance	Medium relevance	No relevance	No relevance	No relevance	High relevance	No relevance
Communication	High relevance	High relevance	High relevance	High relevance	High relevance	High relevance	Medium relevance	High relevance
Unification	High relevance	No relevance	No relevance	No relevance	High relevance	Medium relevance	High relevance	No relevance
Abstraction	No relevance	No relevance	No relevance	No relevance	High relevance	High relevance	High relevance	No relevance
Analysis	High relevance	High relevance	High relevance	No relevance	High relevance	High relevance	Medium relevance	Medium relevance
Verification	Medium relevance	High relevance	No relevance	No relevance	High relevance	High relevance	No relevance	High relevance
Exploration	High relevance	No relevance	Medium relevance	No relevance	No relevance	Medium relevance	High relevance	No relevance
Decision	Medium relevance	Medium relevance	Medium relevance	No relevance	High relevance	High relevance	No relevance	High relevance
Optimisation	High relevance	High relevance	No relevance	No relevance	High relevance	Medium relevance	No relevance	High relevance
Specification	High relevance	Medium relevance	No relevance	No relevance	No relevance	High relevance	No relevance	Medium relevance
Realisation	High relevance	Medium relevance	No relevance	No relevance	No relevance	High relevance	No relevance	High relevance
Steering and control	No relevance	Medium relevance	Medium relevance	Medium relevance	Medium relevance	Medium relevance	No relevance	Medium relevance

12.3.2 Variables for Analysis

Usefulness is a two-sided concept for criteria as well as for purposes. Firstly, a purpose (criterion) should be relevant for the modelling problems the students meet. Secondly, the students should understand the concept. To account for both sides, we define the *usefulness of a criterion* (*purpose*) as the frequency of that criterion (purpose) mentioned with understanding by the students minus the frequency mentioned with misunderstanding. In both definitions, frequency is the number out of all reports (212). A relatively large positive score signifies usefulness; a low or even negative score signifies non-usefulness (because of low relevance and/or low understanding). As an indicator, we use that a purpose or criterion is mentioned in relation to the context of the problem; from the description of the relation in the report text, understanding or misunderstanding is deduced; mixed understanding-misunderstanding is considered as misunderstanding.

12.4 Results

12.4.1 Example Problem with Scoring of Student Group Answers

We present an example of a modelling assignment, completed with a scored example of student group answers to the purpose and criteria questions. Notation is as follows: '------' is used for omitted text, 'CAPITALISED' for use with understanding, and '<u>underlined</u>' for use with misunderstanding.

Assignment: Dynamic Modelling Project, Virus Infection

When a virus enters a human body, it may replicate fast at first; our immune system will react only after a threshold has been passed. If the body cannot cope with the virus growth, we need to administer an antiviral drug, in due course and in adequate amounts. Here is room for choices to be made. The possibilities also depend on the patient. Construct a model that indicates the results of a treatment as chosen by the specialist or general practitioner. Show how a responsible decision can best be made. Determine responsibility and what is best. Keep in mind that a model suitable for an adult may be inadequate for a child or infant.

Example of student group answers to assignment questions with scoring embedded:

Purposes: The purpose of our model is to ANALYSE the disease. By analysing how the virus behaves there can be control of the outcomes by using medication. To OPTIMISE and CONTROL the amount of medication prescribed is also one of the purposes. The amount of medication can be controlled by following the advice our model offers. Optimising the amountofmedicationleadstocuringtheinfectedandnotspendingtoomuchmoneyonmedication.------

Possibilities for improvement: Our model could be improved for the categorisations: GENERICITY, scalability, ------. Our model can be improved in genericity because our model is now only for "normal" people. If more human aspects are taken into account, the model can be used for special cases as well. Our model does for instant [sic] not take in account that some people might be allergic to the medicine. ------ The scalability can be improved because we have no limit on the amount of medication. This is not realistic. There must be a state in which the person gets an overdose. We did not take that into account, so our scalability is too large for the model to be realistic in all situations. ------

It should be noted that we did not include the reference to a criterion's operationalisation (Sect. 12.2) as a requirement nor as a proof for understanding. We observed that students almost never used these operationalisations.

12.4.2 Validity and Reliability

Did we measure what we intended to measure? Firstly, it should be noted that we only analysed limited sections of the project reports, that is, those sections that were expected to contain the answers to the specific questions related to purpose and criteria according to the prescribed report format. If a question was not answered where it should be, the immediate context was scanned, but not the whole report. A small chance remains that the answers or other signs of specific understanding or misunderstanding were present elsewhere. Secondly, as our indicators for usefulness are compound variables, scores sometimes might be ambiguous. However, if necessary, this will be solved by a division into understanding and misunderstanding.

Did we measure well what we measured? Scoring was mainly done by the first author. While the scoring of purposes posed few problems, the scoring of criteria was harder. In about 20% of the cases, the main rater had doubts; in those cases, discussion followed with the other authors until consensus was reached. Hence, drawing conclusions should be handled with care.

12.4.3 Results for Purposes

Figure 12.1 shows for all purposes the frequency of use (out of 212) with understanding and, below the axis, with misunderstanding. The data for quasi-use (other purposes) and non-use (no purposes) is included. The purpose of *optimisation* is clearly useful, followed by *analysis*, *prediction what*,[1] and *verification*. All other

[1] 'Prediction' was used often without further specification, but as the second variant of prediction did not play any role, we did not see this as a sign of misunderstanding.

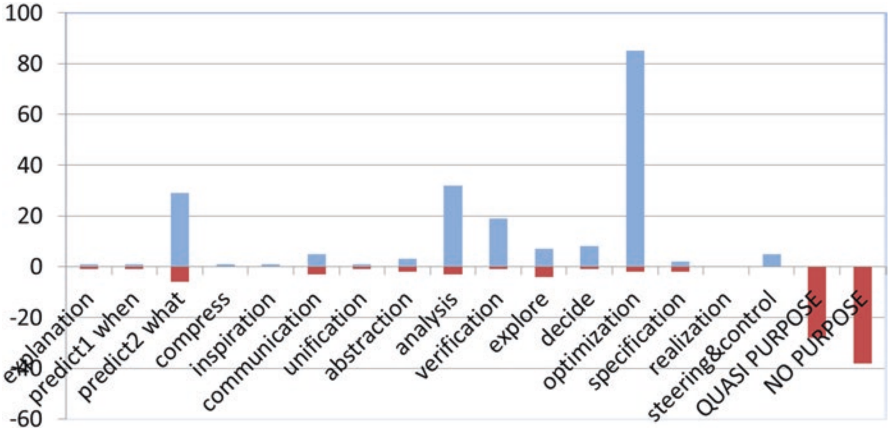

Fig. 12.1 Usefulness of purposes, split-up into understanding and misunderstanding

purposes were barely useful. Generally, when used, purposes were used with understanding: relevance is nearly equal to usefulness (total bar length is nearly equal to bar length above the axis). Also, the frequencies of cases where no purposes were mentioned or where other (quasi-)purposes were mentioned were substantial. Quasi-purpose examples are 'determine', 'calculate', and 'investigate'.

12.4.4 Results for Criteria

Figure 12.2 shows for all criteria the frequency of use (out of 212) with understanding and, below the axis, with misunderstanding. The data for no criteria and quasi-criteria are included. The criterion of *convincingness* is clearly useful, followed by *genericity*, *distinctiveness*, *specialisation*, and *scalability*. *Surprise* appears to be barely useful and *audience* and *impact* appear to be not even useful. Compared to the purposes results, there is much more misunderstanding. *Audience* was often misunderstood as being equal to *specialisation* and *impact* as meaning 'influence'. *Scalability* in general is useful, but it showed some confusion with *genericity* or was mistaken for 'measurable on a scale'; *distinctiveness* was mistaken for 'correctness' sometimes or 'like in reality'. The frequency of cases where no criteria were mentioned is substantial. No other (quasi-) criteria were mentioned.

12.5 Conclusions and Discussion

We will summarise our results:

– Only 4 of the 16 purposes presented are useful: especially *optimisation* and also *analysis*, *prediction 2* (*what*), and *verification.*

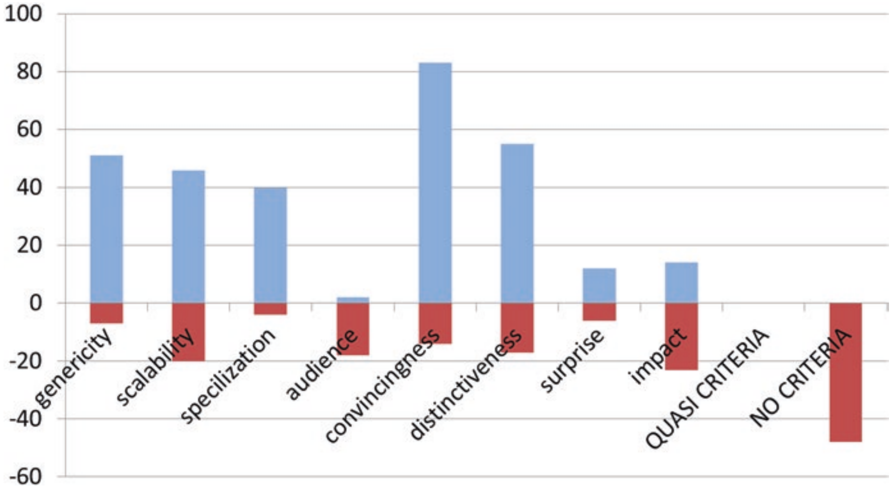

Fig. 12.2 Usefulness of criteria, split-up in understanding and misunderstanding

- When purposes are mentioned, it is with understanding most of the time.
- With a substantial frequency, other purposes than the presented ones are mentioned, such as determine, calculate, and investigate.
- With a substantial frequency, no purpose is mentioned at all.
- Five of the eight criteria are useful: especially *convincingness* and also *genericity*, *distinctiveness*, *specialisation*, and *scalability*. The other three criteria appear to be barely useful (i.e. *surprise*) or even not useful (*audience* and *impact*).
- When criteria are mentioned, it is sometimes with misunderstanding, such as *audience* as equal to *specialisation*, *scalability* as equal to *genericity*, *impact* as meaning influence, *scalability* as meaning measurable on a scale, and *distinctiveness* as meaning correctness or meaning like in reality.
- No other criteria than the presented ones are mentioned.
- With a substantial frequency, no criterion is mentioned at all.

The results should not be generalised without further research. Generalisation to higher *technical* education appears justified. In higher (non-technical) science and mathematics education, modelling is less prominent, and when present the spectrum of purposes and criteria would be different. The purpose of *explanation*, for example, would be more important. To what extent these purposes and criteria might be used at secondary level is another question. The chapter by Zwaneveld et al. (in press) explores this for the purposes. The result that there is more misunderstanding of criteria than of purposes might be explained by the fact that purposes are asked for in the beginning of the report, and criteria at the end, so possibly without tutor feedback. Also, the criteria were new for staff. Misunderstanding of criteria mainly falls into two categories: (1) blurring the quantitative and the qualitative aspect (e.g. *specialisation* and *audience*) and (2) blurring with familiar concepts that have similar names (e.g. *impact* and *influence*).

As there were no specific instructions to staff members of the various departments creating the assignments, one might suppose that *optimisation*, and the other useful purposes first came to mind when constructing an assignment. Especially concerning *optimisation* and also *analysis*, one might suppose they are in the kernel of engineering.

As the relevance of criteria is related to the purpose concerned, one might predict from Table 12.1 (Sect. 12.3) that the relevance of *audience* and *surprise* would be low and the relevance of *genericity*, *scalability*, *convincingness*, and *distinctiveness* would be high, as they are. However, it would not predict high relevance for *specialisation* and low relevance for *impact*. Maybe Table 12.1 (from Van Overveld and Borghuis 2013) can be improved, for example, one could defend that *specialisation* is relevant for *optimisation* after all (to facilitate discussion with the client) and that *audience* and *specialisation* are relevant for *specification*. In our exploration only frequencies of use of purposes and criteria were counted. It might be interesting to analyse the data further by constructing an empirical table like Table 12.1 containing the related use of purposes and criteria by the students.

Our final conclusion is that the quality criteria for models related to their purpose (Van Overveld and Borghuis 2013) are interesting but for optimal use in higher education some fine-tuning is necessary still; as for the students, some terms are still ambiguous and some criteria overlap. The relations between criteria and purposes require further research.

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Chapter 13
Ethnomodelling as the Mathematization of Cultural Practices

Milton Rosa and Daniel Clark Orey

Abstract In this chapter, we share how we have come to use a combination of emic (local), etic (global) and dialogical (glocal) approaches in our work in ethnomodelling. The acquisition of both *emic* and *etic* knowledge presents us with an alternative goal for the implementation of ethnomodelling research. Emic knowledge is essential for creating an intuitive understanding of mathematical ideas, procedures and practices developed by members of distinct cultural groups. Etic knowledge is essential for cross-cultural comparisons, based on the components of ethnology. The implementation of a dialogical perspective is a third approach for ethnomodelling research that uses both emic and etic knowledge traditions through processes of dialogue and interaction. Finally, ethnomodelling is defined as the study of mathematical phenomena within a culture because it is a social construct and is culturally bound.

Keywords Etic • Emic • Dialogical • Ethnomodelling • Ethnomathematics • Mathematization

13.1 Introduction

When investigating forms of knowledge possessed by the members of distinct cultural groups (emic),[1] we are able to find unique mathematical ideas, procedures and practices that are considered different forms of mathematics. This information can

[1] The concepts of emic and etic were introduced by the linguist Pike (1954) who drew upon an analogy with two linguistic terms: (a) phon***emic***, which are the sounds people use in a particular language, and (b) phon***etic*** that relates to general aspects and the actual vocal sounds produced in language.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_13

be used to both express and explore the relationships between culture and mathematics. Any outsider's (etic) understanding of cultural traits[2] is based on the many particular interactions and interpretations that emphasize inessential features of cultural groups as well as the misinterpretation of unique and culturally mathematical forms of knowledge. The challenge that arises from this understanding is how culturally bound mathematical ideas are better understood without letting the culture of the investigators and educators interfere with the culture of the members of the cultural group under study. This is not easy and may only happen when the members of cultural groups under study share the same interpretation of their culture (emic) as opposed to an outsider's interpretation (etic). On the other hand, an insider's (emic) view of cultural traits is based on factors such as cultural and linguistic backgrounds, the historical-social context and moral values that combine to influence mathematical ideas, procedures and practices developed by the people of their own culture and context.

Over time, different cultural groups shared, developed and evolved different ways of doing mathematics in order to understand and comprehend their own cultural, social, political, economic and natural environments (Rosa and Orey 2010). Each cultural group has developed often unique and distinct ways to *mathematize* their own realities (D'Ambrosio 2001). In this context, mathematization is the process in which members of distinct cultural groups come up with different mathematical tools that help them organize, analyse, solve and model specific problems located in the context of their own real-life contexts (Rosa and Orey 2012). These tools allow them to identify and describe specific mathematical ideas, procedures or practices by schematizing, formulating and visualizing problems in different ways, discovering relations and regularities and translating real-world phenomena to academic mathematics through the process of mathematization. As increasingly diverse elements engage with each other, it is important to search for alternative methodological approaches in order to record mathematical ideas, procedures and practices that occur in different cultural contexts. One alternative methodological approach to this is *ethnomodelling*, which is considered as the practical application of ethnomathematics (Rosa and Orey 2010). This need for culturally bound forms of mathematical modelling is deeply rooted in the theory of ethnomathematics.

13.2 Ethnomodelling

Ethnomodelling is the study of mathematical ideas and procedures developed, used, practiced and presented in diverse situations found in the daily life of the members of distinct cultural groups. This allows those engaged in this process to study

[2] Cultural traits are systems of knowledge that consist of patterns, traditions, meanings, beliefs, values, actions, experiences, attitudes, hierarchies, religion, notions of time, norms, roles, spatial relations, concepts of the universe, artefacts, mentifacts, sociofacts and symbols acquired by a group of people, which are diffused and shared from generation to generation (D'Ambrosio 2001).

mathematics as a system regarding their own contextual reality in which there is an equal effort to create an understanding of all components of these systems as well as the interrelationship among them. It is a tool towards the development of pedagogical actions found in an ethnomathematics program (Rosa and Orey 2012). It is necessary to reveal a diversity of sophisticated mathematical practices developed in distinct cultural contexts, which allows students to work with authentic situations and real-life problems such as geometric principles in craftwork, architectural concepts and practices in the activities and artefacts of many local cultures. Many of these notions relate to numerically based relations found in measuring, calculating, gaming, divining, navigating and astronomy and modelling (Eglash et al. 2006).

In this process, the term *translation* is used to describe the process of modelling local cultural systems (*emic*) which may have global Western-academic representations (*etic*) (Rosa and Orey 2010). An effective use of ethnomathematics also applies modelling in order to establish relations between local conceptual frameworks (emic) and the mathematics embedded in relation to local designs. More often than not, local designs have been analysed, interpreted and valued from a Western view (etic). One example of this practice might include the applications found in the symmetry and classifications in crystallography to indigenous textile patterns. In some cases, the translation of mathematical procedures and practices into the language of Western or Academic mathematics is as direct and simple as found in counting systems and calendars. However, there are cases in which mathematical ideas and procedures are embedded in processes such as in iterations found in beadwork or in Eulerian paths in sand drawings. It is this act of translation that is best referred to as ethnomodelling in that the mathematical knowledge can be seen as arising from emic (local) rather than etic (global) origins (Eglash et al. 2006).

Ethnomodelling takes into consideration diverse processes found in the development of local forms of mathematical knowledge, which include the unique aspects as well as patterns of creativity and invention. Thus, it has become impossible to imprison the development of mathematical knowledge in only one form of reality because the members of distinct cultural groups interact in local and globalized contexts, which can provide different representations of real-world phenomena (Rosa and Orey 2012). In this regard, mathematics is no longer conceived as a universal language. From our perspective, it may be a language, but one that has a variety of regional dialects, accents and diverse forms of vocabulary, unique to the culture it is used in and reflecting the problems that it came from. The dynamic processes found in the production of a diversity of mathematical ideas, procedures and practices operate in the register of interpretative singularities that regard possibilities for a symbolic construction of knowledge in different cultural groups (Rosa and Orey 2010).

Emic constructs are the accounts, descriptions and analyses expressed in terms of the conceptual schemes and categories regarded as meaningful and appropriate by the members of the cultural group under study. Etic constructs are the accounts, descriptions and analyses of mathematical ideas, procedures and practices expressed in terms of conceptual schemes and categories regarded as meaningful by the community of scientific observers and investigators (Lett 1996). Thus, the issue is whether it is necessary to understand cultural specificity against the background of universal and generic theories and methods (etic knowledge) or whether this behaviour can only be understood within its cultural context and therefore requires culturally specific theories and concepts (emic knowledge). Thus, any theories and methods seem to be susceptible to cultural differences and to demand a cultural contextualization.

13.3 Mathematical Phenomena and Their Ethnomodels

Many investigators and educators have made extensive use of mathematical procedures ranging from statistical methods for the interpretation of patterns in behaviour to mathematical representations in the processes of local conceptual and logical systems such as in workplaces. For example, Duarte (2004) investigated the uniqueness of mathematical knowledge produced by workers (emic) through the study of mathematical ideas and practices that they develop in home construction sites. In this study, there was a reflection on the mathematical knowledge possessed by the members of this working class in order to legitimate and validate their knowledge and determine the pedagogical and curricular implications that are inferred in the process of production of this knowledge. In this context, mathematical modelling is considered a pedagogical tool and by others as a way to understand anthropological and archaeological perspectives of mathematics. Others still have decried the use of the mathematical and, in particular, statistical and quantitative modelling as fundamentally in opposition to a humanistic approach to understanding human behaviour and the knowledge that takes into account the contingency and historical *embeddedness* which in turn decries universality. Traditional mathematical modelling practices have not fully taken into account widespread implications of diverse aspects of human social behaviour.

These social and cultural components are extremely critical and emphasize the "unity of culture, viewing culture as a coherent whole, a bundle of [mathematical] practices and values" (Pollak and Watkins 1993, p. 490) that often appears incompatible with the rationality and the elaboration of traditional mathematical modelling processes. This approach relates to the socio-critical perspective of modelling in which pedagogical goals are rooted in the critical and reflective understanding of the surrounding world (Sriraman and Kaiser 2006). However, in the context of diverse mathematical forms of knowledge, what is meant by *social* and *cultural components* varies widely and ranges from viewing mathematical practices as learned and transmitted to and from members of diverse groups to the mathematical practices viewed as abstract symbolic systems with a deep internal history and logic that provides a symbolic system to its mathematical structure (D'Ambrosio 2001). Mathematical knowledge developed by members of distinct cultural groups often consists of abstract symbol systems and is the consequence of social, historical and cultural events that people have developed, accumulated, diffused and learned through history.

Cognitive aspects needed in this framework become primary decision-making processes by which members either accept or reject an ethnomodel as part of their own repertoire of mathematical knowledge. The conjunction of these two scenarios appears to be adequate in relation to the depth needed to encompass a full range of cultural mathematical phenomena. There appears to be a level of cognition that we all share, to varying degrees, with the members of our own and other cultural groups. This level includes cognitive models that we elaborate on a non-conscious level, which serve to provide an internal organization of mathematical phenomena in order to provide the basis upon which diverse mathematical practices take place. However, according to Eglash et al. (2006), this representation arises through the formulation of abstract and conceptual structures that form a sense of organization for external phenomena we encounter. Cultural constructs provide us with representations for systems taken from reality.

The implications for this form of modelling are that these models engage cultural constructs and are symbolic systems organized by an internal logic of the members of cultural group themselves. Models built without a first-hand sense for the world being modelled should be viewed with suspicion. Thus, investigators and educators, if not blinded by their own cultural backgrounds, are influenced by the paradigm in which they are immersed, which includes all prior history, theory and ideology they have absorbed. If they are aware of this, they might develop an informed sense of distinction that makes a difference from the point of view of mathematical knowledge of the work being modelled. They will in the end be better able to explain to outsiders (etic, glocal) what matters to insiders (emic, local). Moreover, local users of mathematics communicate directly their uses and worldviews without the help of outsiders.

13.3.1 Ethnomodel of the Mathematization of the Gable

Informants from a roofing contractor cultural group can easily describe the practices acquired for the construction of a roof gable, which is the most commonly used type of pitched roof construction. After choosing the type of tile such as red roofing tiles or shingles to begin the construction of the roof, it is necessary that roofing contractors calculate the slopes (pitches) of the beams that form the triangles in the gable. In general, the "roof is constituted by the composition of inclined planes. The simplest roof is the one that has only two inclined planes. It is called the gable roof" (Moreira and Pardal 2012, p. 41). Gabled roofs often possess a ridge near or at the centre and slope in two directions. It is simple and common in design and economical to construct and can be used on any type of structure and in any type of climate.

Roofing contractors use triangles because they are stable and rigid and have immobility. Hence, the main objective of the roof is to provide protection from season change because they must be strong enough to withstand high winds and shed moisture and often snow and ice quickly. Roof slope and rigidness are for shedding

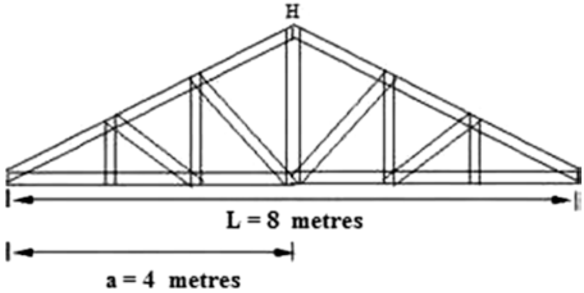

Fig. 13.1 Scheme of a gable used in Brazilian roof constructions

water and any excess weight provided by snow and ice and bearing any extra additional weight. For example, in the case of many roof styles and the amount of rain in Brazil, foremen calculate the slope (pitch) of the roof by applying a ratio between the height and the length of the gable, which is expressed as a percentage. For example, the percentage of the slope (trim) for the roof to the tiles is at least 30 % so that rainwater (snow and ice are not a problem in Brazil) will drain quickly. Figure 13.1 shows the scheme of a gable used in Brazilian roof constructions.

According to this procedure, for each metre (100 cm) that runs horizontally, there is a vertical rise of 30 cm. Thus, if the length of the gable is $L = 8$ m, roofing contractors mentally perform the percentage calculation by using $a = 4$ m, which is half of that measure. Then, they multiply this result by the percentage of the slope of the roof. For example, 30 % of 4 m corresponds to the height of 1.20 m. This procedure represents an emic (local) view of this mathematical practice. Conversely, framers have used the Pythagorean theorem (etic, global) to cut roof rafters, and roofers apply formulas to determine the amount of roofing material needed. For educational and pedagogical purposes, mathematics educators perceive the construction of gable roofs as the placement of two right triangles together.

An emic observation of this mathematical practice sought to understand it in the context of how Brazilians build gabled roofs from the perspective of internal dynamics and relations as influenced within the culture of roofers. On the other hand, an etic perspective provides cross-cultural contrasts and comparative perspectives by using aspects of academic mathematics that translate this practice to create a new understanding of investigators and educators from different cultural backgrounds. It is also important to understand the dialogical (glocal) relationship between these two approaches because the informal calculation (emic knowledge) of the height (trim, flow) of the gable does not preclude the use of the *Pythagorean theorem* (etic knowledge) by these professionals. Thus, the members of this specific cultural group strive to compare, interpret and explain the mathematical knowledge they observe and are experiencing. According to Rosa and Orey (2010), in order for ethnomodelling to be successful, it is necessary to value and link distinct forms of mathematical knowledge by applying a dialogical approach in this process.

13.4 Dialogical (Glocal) Approach in Ethnomodelling Research

In any new analogies regarding ethnomodelling, it may be possible to state that emic perspectives are concerned with the differences that make mathematical practices unique from the *insider's* viewpoint. We argue that emic ethnomodels are grounded in what matters in the mathematical world of those being modelled. Conversely, etic ethnomodels are built on data gleaned from an outsider's view. Etic ethnomodels represent how modellers think the world works through systems taken from reality, while emic ethnomodels represent how people living in such a context think these systems work in their own reality. It is important to emphasize how etic approaches play an important role in ethnomodelling research, yet at the same time, emic approaches should be taken into consideration in this process. However, the focus of this analysis is emic if the mathematical ideas, procedures and practices are unique to a subset of cultures rooted in diverse ways in which etic activities are carried out in specific cultural settings. According to Pike (1954), while emic and etic perspectives are often thought of as creating conflicting dichotomies, they were originally conceptualized as two complementary viewpoints.

In this regard, rather than posing a dilemma, the use of both approaches actually deepens understanding of important issues in scientific research and investigations. A suggestion for dealing with this dilemma is to use a combined emic-etic approach, rather than simply applying emic or etic dimensions to study or examine mathematical procedures and practices employed by members of distinct cultural groups. This requires investigators to attain the emic knowledge developed by members of cultural groups under study, which encourages them to put aside any perceived or unperceived cultural biases so that they may be able to become familiar with the cultural differences that are relevant to the members of these groups (Berry 1999). For example, the objective of the study conducted by Bortoli and Marchi (2013) with 34 students in the second school year, in a high school, in Caxias do Sul, in the state of Rio Grande do Sul, Brazil, was to investigate trigonometric knowledge applied in right triangles and its connection to mathematical knowledge used in civil construction. Different classroom activities were planned in order to encourage students to research, explore and interpret trigonometric knowledge by interviewing professionals who work in civil construction.

Thus, one of the nine groups of students decided to work with trigonometric relations involved in the construction of the roof gable in a house from foremen's viewpoints. By conducting interviews with foremen, students in this group found out about why the roof has different angulation (trim) according to the materials used in the roofing as well as aesthetics the customers want in their houses. According to the information obtained by the foremen, students in this group determined that the trim of the Roman roof tile (one of the different types of roof tiles in Brazil) is 40 %, which means that, for each metre (100 cm) that runs horizontally, there is a vertical rise of 40 cm. Thus, they applied the method used by the foremen to determine the height of the roof. For example, if the length of a house is 10 m, then they divide the

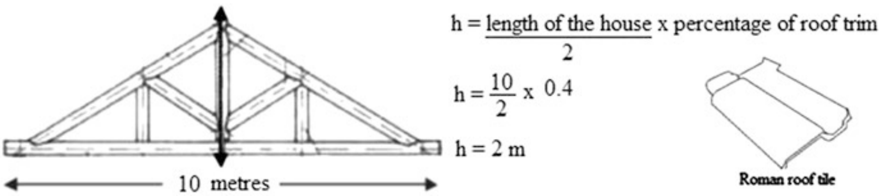

Fig. 13.2 Foremen's procedure used by the students

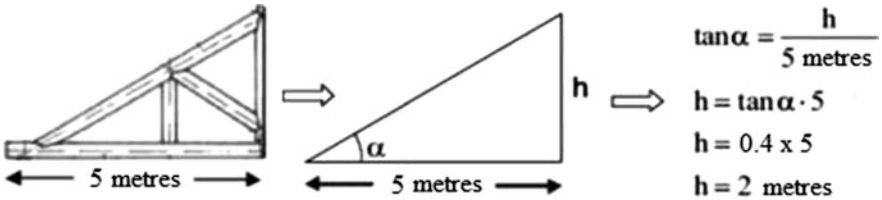

Fig. 13.3 Academic procedure used by the students

length by 2 and multiplied the result by the percentage of the trim of the Roman roof tile. Figure 13.2 shows the foremen's procedure used by the students. Subsequently, students in this group perceived the relation between mathematical knowledge of the foremen with the academic method to determine the height of the roof by applying trigonometric knowledge associated with the definition of tangent (Fig. 13.3).

The results of the study conducted by Bortoli and Marchi (2013) showed that students in this group were able to understand the connection between the local knowledge (emic) and the academic knowledge (etic) through a dialogical interaction. For example, one of the students in this group stated: "I was able to perceive the relation between the mathematical knowledge used by these professionals with mathematics applied in the schools". Usually, in ethnomodelling research, an emic analysis focuses on a single cultural perspective and employs both descriptive and qualitative methods to study a mathematical idea, procedure or practice of interest. Its focus becomes the study within a cultural context in which investigators examine internal logic found in the cultural system itself. In this regard, meaning is gained relative to the context and, therefore, not easily transferable to other contextual settings.

In contrast, an etic analysis is comparative and examines cultural practices by using standardized methods. The etic approach tries to identify lawful relationships and causal explanations valid across different cultures. Thus, if investigators and educators wish to make statements about universal or etic aspects of mathematical knowledge, these statements need to be phrased in abstract ways (Rosa and Orey 2010). While traditional concepts of emic and etic aspects are important approaches for understanding, and comprehending, cultural influences on ethnomodelling, we propose a dialogical view in this process (Rosa and Orey 2010), which makes use of "acts of translation between emic and etic perspectives" (Eglash et al. 2006, p. 347). In the ethnomodelling process, cultural specificity may be better understood with the background of communality and the universality of theories and methods and vice versa. It is important to analyse the insights that have been acquired through subjective and culturally contextualized methods. The rationale behind an emic-etic dilemma is that mathematical phenomena can only be fully understood within the cultural context they were developed.

13.5 Final Considerations

Many local mathematical practices have disappeared because of the intrusion or imposition of *foreign* (etic) knowledge value systems and technologies through the process of colonization and global capitalism. This *foreign* knowledge emerged from the development of concepts that promised short-term gains or solutions to problems faced by the members of distinct cultural groups without considering emic knowledge that solves these very same problems. The tragedy of the impending disappearance of local knowledge is equally obvious when a diversity of skills, technologies, cultural artefacts, problem-solving strategies and techniques and expertise are lost to all of us before being archived, understood and/or saved.

Defined in this manner, the usefulness of both emic and etic distinctions is evident in ethnomodelling research. Investigators and educators have been acculturated to some particular cultural worldview; we all therefore need a means for distinguishing between answers we derive as acculturated members of *my* group and the answers we derive as observers of *our* group. Culture is a blueprint that specifies a plan of action by utilizing the research provided by both approaches, which helps to gain a more complete understanding of the member of the cultural group under study.

On the other hand, mathematical ideas, procedures and practices used outside of school may be considered a modelling process rather than a mere set of techniques to manipulate numbers and procedures. The application of ethnomathematical techniques and the tools of modelling allow us to see a different reality and give us insight into the mathematics we all perform in a holistic manner. Ethnomodels are cultural artefacts and pedagogical tools used to facilitate the understanding and comprehension of systems taken from the reality of members of distinct cultural groups (Rosa and Orey 2010). Thus, the pedagogical approach that connects the cultural aspects of mathematics with its academic aspects is ethnomodelling, a process of translation and elaboration of problems and questions taken from systems that are part of any given cultural group.

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Chapter 14
Enabling Anticipation Through Visualisation in Mathematising Real-World Problems in a Flipped Classroom

Gloria Ann Stillman

Abstract Meta-analyses of findings from flipped classroom studies in mathematics classrooms to date need to be treated with caution until the teachers and students involved develop mindsets that maximise and integrate the learning potential of both the out-of-classroom and in-classroom learning environments. Rather than a flipped classroom being used as a means to get through the curriculum, it can be used to enrich the curriculum. The question addressed is: Could a flipped classroom provide both vicarious experiences and fostering of critical thinking skills associated with modelling? A local secondary school implementation is used to illustrate how the approach could build meta-knowledge about mathematical modelling and facilitate associated critical thinking skills such as anticipating and visualisation to expand the learning experiences of secondary mathematics students.

Keywords Flipped classroom • Critical thinking skills • Anticipating • Visualisation • Mathematisation

14.1 Introduction

The provision of vicarious experiences of real-world situations in upper secondary school to enrich students' understanding of the world is often considered a luxury a teacher in a classroom must forego in order to cover the core of the curriculum in the limited time available (Stillman 2007). According to the National Council of Teachers of Mathematics *Principles to Action*, on the contrary, there should be evidence of connections being made between "mathematics and the real world" (2014, p. 4). In conjunction with this, a second purpose for the teacher is to enable

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_14

opportunities for students to develop ways of thinking commensurate with modelling situations in their social and physical worlds.

A recent innovation in upper secondary classrooms is the flipped classroom where there is a swapping of classroom and homework activities through the use of electronic technologies and, in some cases, an expansion of the curriculum (Bishop 2013). In the example to be examined in this chapter, the teachers used video technology to present skill work and use of mathematical techniques, sometimes embedded in real-world scenarios, for students to view before class as assigned homework. In-class time was spent on providing teacher support as students worked on problem-solving activities.

The question arises: Could a flipped classroom provide the ideal situation for exacerbating the perceived restrictions on modelling or applications in the upper secondary school through judicious use of video clips furnishing both vicarious experiences (e.g. construction of ramps for wheelchairs on a lecture hall to meet building code standards) and the intentional fostering of the critical thinking skills associated with modelling? As an example of the latter, the teacher might plan to include in the video deliberate exposing of actions to assist students in mathematising a real-world problem situation such as explicitly demonstrating and anticipating the effects of confounding variables in the situation in finding a "sense of direction" (Treilibs 1979) in mathematising to model the situation adequately (e.g. required length of the ramp covering a car park entrance). This entails the teacher anticipate potential situational pitfalls needing consideration and from these select those that have the most dramatic effect to capitalise on use of the visual medium (Mayer 2005) to enhance critical thinking. This aligns with Brown's conclusion from her doctoral work that teachers "need to better understand the cognitive role of visualisation in modelling", noting that students in her upper secondary study were reluctant visualisers (2015, p. 440). It seems self-evident that the medium of video should be used to promote students' appreciation of how "visualisation can support their mathematisation" in modelling (Brown 2015, p. 440).

14.2 The Flipped Classroom

14.2.1 What Is It?

Baker (2000) and Lage and colleagues (e.g. Lage et al. 2000) were the first to publish on the flipped or inverted classroom (as it is also called). "Inverting the classroom means that events that have traditionally taken place inside the classroom now take place outside the classroom and vice versa" (Lage et al. 2000, p. 32). After surveying existing research on flipped classrooms, Bishop (2013) defined it as "an educational technique that consists of two parts: interactive group learning activities inside the classroom, and direct computer-based individual instruction outside the classroom" (p. 6). The interactive classroom activities require human interaction, whereas the explicit instruction methods can be automated through technology. Bishop (2013) sees this as enabling "a unique combination of learning theories once thought to be incompatible—active, problem-based learning activities founded upon constructivist schema and instructional lectures derived from direct instruction methods founded upon behaviourist principles" (p. iii). However, it is not merely a matter of assigning explicit instruction methods automated by technology to homework activities, whilst the classroom centres on student-centred interactive learning experiences, but rather there is *an opportunity to expand the curriculum* (Bishop 2013). Furthermore, "emphasis on students becoming the agents of their own learning" enables a "shift from teacher-driven instruction to student-centred learning" (Hamdan et al. 2013, p. 4) as is necessary to bring modelling into the classroom.

This role, as a potential enabler of teacher change (i.e. one of Burkhardt's change levers) that frees up time and works to "enable people like themselves to achieve these goals in their own classrooms" (Burkhardt 2006, p. 192), is what is attractive about the flipped classroom in the context of a curriculum innovation (mathematical modelling) perceived as being too time consuming (Stillman 2007). Also it is suggested that benefits of the flipped classroom are actually benefits of active learning which was not occurring in these classrooms previously, and this is being investigated by Yong et al. (2015). Thus, the major gains might be in allowing students to be involved in more collaborative learning environments and learning experiences aligned with such learning environments such as modelling that provide contexts for the development of critical thinking skills. In environments already involving a large amount of active learning, it might be more sensible to be measuring student performance and affect to ensure there is no loss of advantage through use of the flipped classroom model which frees up even more classroom time for collaborative activity, rather than to be measuring performance expecting gains.

14.2.2 Research in Mathematics Classrooms on Flipped Classroom

Research to date on the use of flipped classrooms in high school and senior secondary schooling is not extensive as the initial uptake and reports of practice have mainly been in tertiary education (see Bishop 2013). Despite reported widespread teacher interest in the Flipped Learning Ning, hosted by the University of Northern Colorado (Hamdan et al. 2013), and other such electronic communities, there are only a small, but growing, number of examples of published studies of flipped mathematics classrooms in schools.

14.2.2.1 Student Performance

Bormann (2014) identified the opportunity for students to be involved in a learning environment that can lead to higher achievement as a major affordance of the flipped classroom as a learning model. Fulton (2012) reported how Byron Public High School in Minnesota adopted the flipped classroom model in mathematics in 2009 when faced with low student numbers passing state mathematics testing, but by 2011 nearly 75% of students passed the state test, and by 2012, 86.6% of senior students had completed four or more credits in mathematics. Gains in performance were also evident at Niagara Falls High School (Western New York Regional Information Centre 2013) in algebra II and trigonometry classes on mastery tests. On the other hand, when Clark (2013) implemented the flipped classroom model in year 9 algebra 1 classes for 7 weeks at a public high school, student scores on end of unit tests did not differ significantly from those in a traditional lecture style class taking the same tests. Similarly, there was no significant difference when Saunders (2014) investigated the effects of the flipped classroom learning model on student achievement in two year 11 mathematics classes.

14.2.2.2 Student Engagement

The opportunity for students to be involved in a more engaging learning environment is another major affordance of the flipped classroom learning model (Bormann 2014). A learning environment is more engaging if it deepens students' interaction with the content both physically and cognitively (Butt 2014). This was not always supported in the studies surveyed, but there was support in secondary mathematics classroom studies. At Byron Public High School, by 2011 teachers were reporting increased engagement in mathematics classes (Fulton 2012). Students in Clark's study (2013) reported improved engagement in the flipped classroom classes "due to the interesting and meaningful activities completed throughout the study" (p. 91). Factors that were reported positively affecting engagement for the majority of participants included instruction quality, particularly strategic use of real-world activities and technology to enhance this, promotion of strong peer connections through collaboration and communication, classroom organisation and management promoting collaborative learning through projects and hands-on activities.

14.2.2.3 Fostering Students' Critical Thinking Skills

Bormann (2014) identified the opportunity "to provide more meaningful activities that put in place the critical thinking skills related to their content areas" (p. 13) as a benefit and major affordance of the flipped classroom that prepared students for future learning or work. However, Saunders' study (2014) in two secondary mathematics classes showed no significant gain in student critical thinking skills; but the thinking skills associated with modelling and application were not those investigated. Modelling activities were used in testing the effects of the flipped classroom model in undergraduate engineering in a numerical methods course (Bishop 2013). Modelling activities were used with the students undertaking the flipped classroom instruction because Bishop felt they would counteract shortcomings in other interactive student-centred learning methods such as a negative effect on objective knowledge outcomes but positive influences on skills. Examination scores of students involved in the flipped classroom model and the teacher-centred lecture programme were not significantly different; however, on the course evaluation, the comparison group rated their progress on “learning to apply course material (to improve thinking, problem solving and decisions)” more highly than students in the flipped classroom group (p. 65). The difference was statistically significant. There was no statistical difference between ratings of progress towards “learning to analyse and critically evaluate ideas, arguments, and points of view” (p. 65). (See Sect. 14.2.3 for possible reasons for these outcomes.)

14.2.2.4 Teacher Practice

Teacher practice in the studies surveyed varied widely, and, to a certain extent, this variation accounts for the mixed results as there is really a failure to implement the model fully in some studies.

In the Clark (2013) case study of 42 year 9 students in algebra I classes (7 weeks), students were expected to prepare for class by watching videos, listening to podcasts, reading articles, viewing presentations and contemplating questions outside class. All resources were original teacher products uploaded to the class learning management system. Completion of notes was taken as an indicator of evidence of being adequately prepared for class. Inside the classroom students participated in hands-on activities, real-world applications and sometimes completed independent practice.

On the other hand, Muir (2016) reports a case study involving two experienced Australian senior secondary mathematics teachers in Tasmania. One teacher had been flipping his classrooms for 3 years, whilst this was the first year for the other teacher. Both created their own skills-based video tutorials available on each school’s learning management system. Students preferred the teacher-prepared materials over other online tutorials, and teachers indicated in an interview that they thought it was important to make their own. Video tutorials were completed out of class, whilst in-class time was used for individual work from a textbook and a mastery test before proceeding to the next topic. Whole class teaching was minimised. The approach emphasised students developing the capacity to tackle standard exercises, and both teachers focussed on mastering content. Clearly, these implementations have not focussed on active learning where students develop critical thinking skills with challenging learning experiences such as modelling tasks. Instead, class time was optimised to cover the prescribed curriculum and prepare for assessment using textbook tasks.

14.2.3 Flipped Mathematics Classrooms with Real-World Applications and Modelling

Two flipped classroom studies to date have involved modelling activities, an introductory differential equations course at a liberal arts college (Yong et al. 2015) and a numerical methods course in an undergraduate engineering course (Bishop 2013).

At the secondary school level, there were several reports of in-class time being spent solving real-world applied mathematical problems (e.g. Clark 2013), however.

Modelling activities were incorporated into the numerical methods course for students undertaking the flipped classroom model of instruction in the study by Bishop (2013) in an endeavour to increase the educational effectiveness of the student-centred approach for the flipped classroom group. Examination performance was similar for the two groups, but the flipped classroom group's results were significantly lower on homework quizzes. The flipped classroom students were unhappy to be in an experimental group that had an increased workload, so they neglected homework that they perceived to contribute little to final results. Students' opinions of the modelling activities were mixed but more negative than positive. Bishop (2013) recommended changes to his implementation such as earlier introduction of the modelling activities, tighter integration between these and other assigned work and several shorter modelling activities instead of a few long ones. The other tertiary study (Yong et al. 2015) used part of the extra class time in the flipped classroom section of a differential equations course to devote to getting a start on modelling tasks such as setting up a profitable, sustainable, fishery management strategy. Video lectures were assigned as homework to the flipped classroom section but were also available to the control section who attended lectures in class and completed all homework out of class. All tasks were the same. There was no significant difference in achievement on a content test or attitude test between the two sections, but results for the second year of the project when students in the flipped classroom worked on more-ambitious modelling tasks in-class are yet to be released. Attitudes to real-world tasks were positive.

As was typical of the studies in secondary school, the Clark study (2013) has no detail of the real-world applications students used during class time when participating in the flipped classroom model of instruction. The students perceived that they were already using real-life applications in their classes when the study began as, on a pre-survey, 28 (out of 42) agreed or strongly agreed their "learning activities focused on real life applications and improved [their] learning" (p. 70). In the post-survey, this had risen to 35 (out of 42). From the qualitative data, Clark concluded that activities involving real-world applications supported increased student engagement and performance.

14.3 Leveraging a Flipped Classroom Implementation for Modelling and Applications

14.3.1 The Flipped Classroom Implementation

The implementation that is the subject of this analysis occurred in a large metropolitan school in Victoria, Australia. The quotations come from emails or a 3-h conversation with the two mathematics teachers involved and two university academics, one of whom is the chapter author. The two teachers, Ned and Joe (pseudonyms), are highly experienced mathematics teachers who felt they were "time poor" in their year 11 and 12 mathematics classes with "very little time to apply and reflect". Ned had experimented previously with year 9 students when "supportable devices" in the form of phones were first available.

As the instructional model for the year 11 implementation, which began in 2015, was meant to facilitate student ownership of their learning, it was structured as follows:

> Joe: so they view the videos or view material *before* [emphasis] the class and then they come into the classroom and the idea is that their homework is to watch videos rather than doing problems so they can spend all the class time doing problems with our assistance is the idea, is the model.

The teachers made instructional videos involving narration of self-explanations (Chi et al. 1994) of the concepts and processes in real-time solving of worked-out examples (Paas and Van Merriënboer 1994) covering linear, quadratic and higher-order polynomial functions with "a couple of little ones that were more application, practical". More applications were planned for later as "they're the ones that take all the time to do". As there is a lot of content in year 11 mathematics, the intention was:

> Ned: We thought it might be a way of splitting up the content to get the kids to almost teach themselves. We're there now to take that skill set that they have got and to apply that to more in-depth problems but we have had a few hurdles on the way.

Teething problems with their flipped classroom approach were identified with the researchers as they reflected on the implementation of the modules for the topics mentioned above. These included changing student mindsets; adjusting for individual pacing, ideal chunk size for videos and number of examples per chunk; ownership or access to a supportable device at home; and production time for more practical application videos to complement the instructional skill/process videos.

As both teachers wanted "to get more value for what we're doing", the researchers suggested that the videos would be an ideal vehicle for promotion of applications and modelling particularly in association with preparation for, or in conjunction with, school-based assessment and a renewed emphasis, as from the next year, on modelling in the assessment of the Victorian senior secondary curriculum for 2016–2020 (VCAA 2015):

> Joe: Applications and modelling, that could be something we could focus on a little bit to get our kids more involved in the maths than they are doing at the moment; because at the moment, our videos have really been instructional and not, ah modelling...

The researchers suggested extra value could be added to the videos for applications or modelling in their flipped classroom model to enrich the curriculum rather than merely covering the curriculum. The videos could be used firstly as a means to engage students in mathematics classes by exposing them to various real-world current problems such as cleanup of a chemical spill (Yong et al. 2015) where they come to realise they can use their mathematics to understand and address such problems and, secondly, for enhancing critical thinking skills related to modelling successfully. Both involve teacher anticipation.

The first involves selection of a variety of situations (e.g. trajectory of a dig in volleyball) which the teachers themselves know are student interests. Once a suitable context is selected, instructional videos for homework could be assigned for particular mathematical techniques to ensure that the students have the mathematical skill set that could arise in handling the situation as there are particular elements of the situation they need to be able to mathematise and model. At the same time, they should not be held up prematurely in reaching a solution by lack of technical knowledge or unfamiliarity with how to handle particular mathematical dilemmas that might arise in the solution of any models. In addition, pedestrian solutions to the situation could result if the teacher did not convey that students who want to be innovative in their handling of the situation should do so. So, the practical videos can be used as a means to broaden student understanding of context in a way that text and talking cannot (e.g. the changing in speed and motion of an athlete over a 100 m sprint).

Secondly, promotion of anticipating in students as they model the situation can be facilitated by teachers through their own use of anticipation of ways to tackle the situation, as well as potentialities and constraints in the situation. In the past, secondary mathematics students have been criticised for not being able to keep track of all constraints in a real-world situation and were seen as lacking flexibility in dealing with them (Masingila et al. 1996). By raising some of the potentials and constraints of the situations, students could become aware that real situations are not as neat and tidy as what they encounter in textbook tasks. The key would be for the teacher to be able to showcase several ideas in the video without explicitly telegraphing privileging of a particular sense of direction.

Combining both of these elements, practical videos dedicated to the development of meta-knowledge about modelling in its own right as content could be used to show how a particular sense of direction was gained by the teacher in their own modelling of a real-world situation. The pedagogical intention of such a video would be the exposing of the teacher's critical thinking during modelling so students view how others model.

14.3.2 An Example: The Ramp Video

One of the practical video examples from this flipped classroom implementation concerned the design of access ramps to buildings. Ned's intention in making the video was "to pose a problem that would be confronted often in building and design and show how mathematics is an important part of the problem solving process". The mathematical concept applied was gradient, and the problem-solving aspect was "thinking about utilising space effectively". Students were given two homework videos on the gradient topic: an instructional video teaching the skills involved

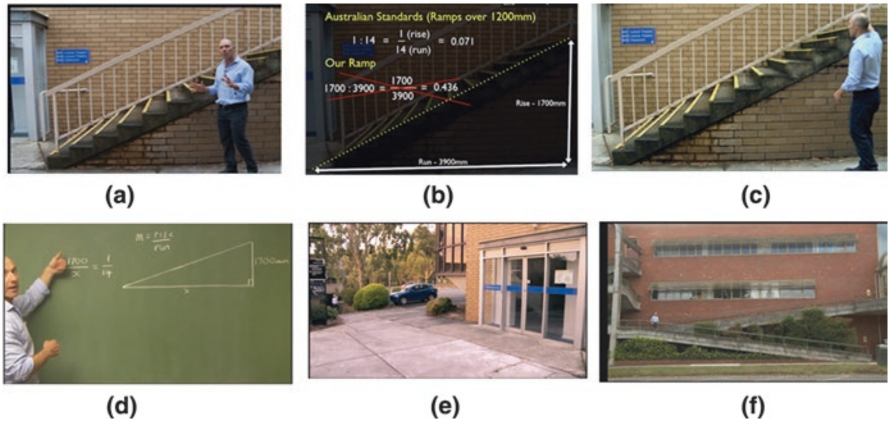

Fig. 14.1 Screenshots (**a**)–(**f**) from ramp video dealing with gradient for flipped classroom (Used with permission of photographer)

in finding the gradient and one related to the necessity for parallel ramps in a confined space to meet building regulations (see Fig. 14.1).

From a modelling perspective, the ramp video triggers questions that viewers can pose such as: Can parallel ramps really be set up in this space? If not, what other solutions would apply? If the intention is to stimulate anticipation when mathematising by students in later real-world problem solving, then in the particular context the modeller might ask: If I use this model will it do the job? Has it been used before? How? Where might I find out? By self-questioning such as this, through recalling images from this prior vicarious experience with the ramp video, the modeller might develop a sense of direction for the modelling applied in a different situation. For the task to involve modelling, it is not only seeing if conversion of existing stairs to a ramp complies (Fig. 14.1a, b) or calculating the length of a ramp that does comply (Fig. 14.1d) but also bringing in the constraints of the situation such as heights that cannot be varied (Fig. 14.1c), access points and available width (Fig. 14.1e) so that a ramp can realistically be constructed for community members unable to walk the stairs. This could mean thinking laterally such as using parallel ramps (Fig. 14.1f) if space permits.

This well-selected example involves several natural decision points where progress and interim mathematical results could be checked such as feasibility being subjected to a sense of the actual length and its relationship to access points of other infrastructure as modelled by the teacher in the video (Fig. 14.1e). By mentally working forward following their anticipated path, students can project feedback about adequacy of particular decisions back to previous decision points before acting on them or to revise those decisions if acted upon.

14.4 A Final Word

Despite mixed results on effectiveness in secondary mathematics classrooms, on balance, the flipped classroom model appears to have potential that is worthy of further development and research as a means to leverage more time in the classroom for engaging in *richer experiences* that give more than lip service to mathematical modelling activity (Yong et al. 2015). Meta-analyses of the findings from the studies to date need to be treated with care until the teachers and students engaged in the flipped classroom have developed mindsets that maximise and integrate the learning potential of both the out-of-classroom and in-classroom learning environments (Bishop 2013; Mayer 2009). Building meta-knowledge about mathematical modelling and facilitating critical thinking skills such as anticipating (Niss 2010; Stillman and Brown 2014) and visualisation in modelling (Brown 2015) seem ideal candidates for expanding the learning experiences of secondary mathematics students whose teachers have felt the continual pressure of lack of time to cover anything other than developing mathematical concepts and techniques. The current interest in this model is an opportunity not to be lost.

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Chapter 15
Measuring Metacognitive Modelling Competencies

Katrin Vorhölter

Abstract Following the discussion about modelling competency as well as respective research results, metacognitive competencies are considered to be an essential component of modelling competency. Until now, there is no method or instrument to reliably measure metacognitive modelling competencies of larger groups of students. In this chapter, different methods for measuring metacognitive modelling competencies are discussed. In addition, results of a design-based process aiming for the development of a questionnaire for measuring metacognitive modelling competencies as well as selected items of the questionnaire are presented.

Keywords Metacognition • Metacognitive strategies • Modelling competencies

15.1 Introduction

Metacognitive competencies have already, for a long time, been of major interest in general education and educational psychology. In recent years, the issue of metacognitive competencies and their promotion has become even more and more important in teaching, especially in mathematics teaching. In the last decade, the topic metacognition and its role in modelling processes has gained significant importance. Within the international community on mathematical modelling and due to the work of Maaß (2006) and Stillman (2011) in the last decade, the topic metacognition and its role in modelling processes has gained significant importance. Maaß (2006) defines *metacognitive competencies* as a sub-competence of modelling competencies; Stillman (2011) focuses on metacognitive barriers in the modelling process and the question of how to overcome them. Nevertheless, until now it could not be clarified how metacognitive competencies can be described

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_15

theoretically, particularly with regard to the question which domain-specific metacognitive competencies are important for the students' modelling process and how these metacognitive competencies can be measured. In this chapter, first steps to developing an instrument for measuring students' metacognitive modelling competencies are presented.

15.2 Metacognition in Modelling Processes

Working on modelling problems autonomously and successfully is challenging for students all over the world. The difficulties can be explained by the complexity of the problems that requires competencies on different levels. Referring to Blum (2015, pp. 77–78), "*modelling competency* in a comprehensive sense means the ability to construct and to use or apply mathematical models by carrying out appropriate steps as well as to analyse or to compare given models". In this sense, modelling competencies not only comprise the sub-competencies referring to single phases of the modelling process, but overall modelling competencies are needed in addition, such as cognitive skills and competencies that allow one to work on a modelling problem successfully and in a goal-oriented way (i.e. competency to structure a problem, to use heuristics and to work together in one group) (Kaiser 2007). According to Maaß (2006) and Blum (2011), metacognitive competencies are an essential facet of modelling competence as well. Thus, in the following a definition of metacognition in modelling processes and empirical results concerning the relevance of metacognition in modelling processes are given.

15.2.1 Definition of Metacognition

The concept of *metacognition* was introduced in the 1970s by John Flavell (1979) and Ann Brown (1978). Over the years, metacognition has become a fuzzy concept. Schneider and Artelt (2010, p. 149) define metacognition as

> people's knowledge of their own information-processing skills, as well as knowledge about the nature of cognitive tasks, and of strategies for coping with such tasks. Moreover, it also includes executive skills related to monitoring and self-regulation of one's own cognitive activities.

In this definition, metacognition is separated into metacognitive knowledge and metacognitive skills (often called metacognitive strategies): The former refers to declarative meta-knowledge that is taken as explicit knowledge or as knowledge to be made explicit. It is subdivided into knowledge of the characteristics of tasks, knowledge of appropriate strategies and knowledge of persons' own skills and competencies as well as those of other persons involved. Metacognitive skills consist of planning, monitoring and regulating the work as well as evaluating the whole process (e.g. Schneider and Artelt 2010; Veenman et al. 2006). As Veenman (2005) points out, the use of metacognitive knowledge depends on different motivational, cognitive and depositional aspects. Therefore, these aspects need to be taken into account when analysing metacognitive knowledge, although they may not easily be evaluated empirically. Another influencing factor for the usage of metacognition is the difficulty of the task: only tasks estimated on an intermediate difficulty level provoke the usage of metacognitive strategies (Hasselhorn 1992).

Referring to the distinction described above, metacognitive modelling competencies can also be divided into metacognitive knowledge and metacognitive strategies. However, until now it has not been researched how metacognitive competencies can be described theoretically.

15.2.2 Relevance of Metacognition in Modelling Processes

Looking at empirical findings concerning metacognition and mathematical learning in general and the role of metacognition in problem-solving processes, the role of metacognition is stated ambiguously. In their overview of theoretical and empirical work on metacognition in mathematics education from the previous four decades, Schneider and Artelt (2010) emphasised the importance of metacognition in mathematics education. They not only summarised the results of different studies that gave evidence of the positive correlation between metacognition and mathematical performance; but also they presented findings from intervention studies that succeeded in fostering students' metacognition and mathematical performance. In contrast, Lesh and Zawojewski (2007), in giving an overview about research on metacognition in problem-solving processes, questioned whether performance improvement was due to metacognition or to the students learning mathematics concepts better or differently. In addition, they gave examples when metacognition (or teachers' request for using metacognition) can be obstructive rather than helpful.

However, according to Blum (2011, p. 22), "there are many indications that meta-cognitive activities are not only helpful but even necessary for the development of modelling competency". For example, the relevance of metacognition in modelling processes is emphasised by the respective studies of Stillman et al. (2007) (for an overview about the current state of the art, see Stillman 2011). Especially the complete lack of (or only a very low level of) meta-knowledge about the modelling process can result in considerable problems when dealing with modelling tasks. Problems occur as well in the transitions between the various stages of the modelling process as in dissolving cognitive blockages while performing modelling tasks (Maaß 2006; Stillman 2011). To overcome such difficulties, the modelling cycle can be used as a metacognitive tool (Blum 2011, 2015). In contrast, Schukaljow and Leiss (2011), for example, did not find any significant correlation between cognitive and metacognitive self-reported strategies (in general or task orientated) on the one hand and mathematical modelling competence on the other hand.

Empirical research (Cohors-Fresenborg et al. 2010) further shows that, in particular, procedural aspects of metacognition have a significant influence on learning success; it is therefore proposed to focus on the promotion of procedural metacognition instead of declarative meta-knowledge. Especially planning of the solution process is essential for performing complex tasks successfully as Schoenfeld (1992) and Verschaffel (1999) point out. Mevarech and Kramarski (1997) indicate that reciprocal asking and answering of metacognitive questions by students while working on a complex task can improve mathematical performance as well as metacognitive competencies at the same time. This finding is confirmed by the conclusion of Goos (1998): collaborative interactions deliver metacognitive benefits. Adaptive support by the teacher is indispensable for bringing students onto a meta-level. Hence, strategic interventions are most adequate (Blum 2011; Kaiser and Stender 2013). Not only metacognitive strategies referring to planning, monitoring and regulating the modelling process are of great importance for solving modelling problems; but also Blum (2015) points out that reflecting on one's own activities is crucial for transferring knowledge and skills from one task to another.

15.3 Measuring Metacognitive Modelling Competencies

15.3.1 Methods for Measuring Metacognitive Competencies

In order to measure procedural metacognitive modelling competencies, there exist two possibilities: On the one hand, online methods like thinking aloud, observations, eye movement or log file registration enable process diagnostics concurrent with task performance. Thus, a deeper look into metacognitive behaviour of students without disturbing and influencing the subject too much is possible. But these methods cost a lot of time and money. Therefore, they can only be used for small samples (Veenman 2011). Especially the method of thinking aloud is often used for measuring metacognitive activities. Thinking-aloud protocols are considered to be fairly reliable, because thinking or doing, respectively, and verbalising are happening almost simultaneously. Furthermore, the pure verbalisation of metacognitive activities does not include any interpretations by the students. However, methods like thinking aloud and observation only lead to reliable results if students are able and motivated to verbalise all their thinking: neither activities and behaviours that are automatised and therefore do not occupy space in the working memory nor thoughts during phases of single work can be measured (Schellings et al. 2013). It is in the nature of online methods that data measured with the help of such instruments are bound to a given task.

On the other hand, offline methods like prospective or retrospective interviews or questionnaires can be used for measuring. In these cases, the results rely on the students' self-reports. This method bears the risk that strategies may be used unconsciously or their use may be forgotten by the students. Furthermore, the item formulation may remind the students of the usefulness of certain strategies. Consequently, they will answer according to their metacognitive knowledge and not on the basis of their behaviour. In contrast to observations and thinking-aloud protocols, processes which were not verbalised for different reasons can be measured with the help of questionnaires or interviews (Schellings et al. 2013; Veenman 2011).

As questionnaires are less labour intensive, they are often used for measuring metacognitive activities. Over the years, the validity of online and offline methods has been compared in several studies, many of these comparing thinking aloud to questionnaires (Schellings et al. 2013). Usually, the correlation is not very high, and therefore self-reports are qualified as less valid; the students' ability of reporting their applied strategies is doubted. However, Schellings et al. (2013) provide two different explanations concerning the low correlation between thinking aloud and questionnaires: The first assumption is that the compared measuring methods aim at different learning strategies. The second assumption refers to the fact that normally thinking aloud is task bound, whereas questionnaires often measure general learning strategies. Therefore, they developed a three-point-frequency questionnaire based directly on a taxonomy for coding thinking-aloud protocols. Twenty ninth-graders were asked to study a text, thinking aloud simultaneously. After studying the text, they were given the questionnaire. The overall correlation between the questionnaire and the thinking-aloud protocols was higher than in other studies (Schellings et al. 2013).

Thus, the development of questionnaires seems to be promising if you want to develop an instrument for evaluating the effectiveness of a learning environment for promoting students' metacognitive modelling competencies. In order to measure applied strategies, the students should be asked to fill out the questionnaire just after working on a modelling task.

15.3.2 Results of Studies Aiming at Development of a Questionnaire for Measuring Metacognitive Modelling Competencies

In order to develop items for measuring metacognitive strategies for modelling, different studies have been conducted. The first studies were aimed at reconstructing metacognitive skills that are important for solving modelling tasks. This was done in two different ways.

Firstly, videotapes of the working processes of several groups of students were analysed by coding metacognitive knowledge and strategies that could be reconstructed by the students' verbal expressions or their behaviour. In doing so, qualitative content analysis, according to Mayring (2014), was used. Thus the elaborated coding guideline gave an overview on the strategies that could be observed. Concerning metacognitive skills, the following strategies were observed:

- Competencies for orienting and planning the solution process
  - P1: Subdivide the solution process into several steps.
  - P2: Allocate parts of work to different persons.
  - P3: Structure the solution process according to the time available.
  - P4: Choose useful solution strategies.
- Competencies for monitoring and, if necessary, regulating the working process
  - M1: Identify different kinds of red-flag situations.
  - M2: Notice incomprehension.
  - M3: Keep track of the time available.
  - M4: Check the work habits.
  - M5: Reconsider solution strategies.
- Competencies for evaluating the modelling process in order to improve it
  - E1: Evaluate the strategies used.
  - E2: Reflect on the working habit.
  - E3: Validate the solution (cf. Schroeder 2014).

By analysing the videotapes, it became obvious that some students used metacognitive skills but did not express them explicitly; so after some time, they expressed the results of the use of special metacognitive strategies in different ways. Unfortunately, we were not able to figure out when exactly these strategies have been used.

As mentioned above, retrospective observations can ignore the usage of metacognitive strategies. Therefore, based on the coding guideline as well as based on the conceptualisation of existing metacognitive questionnaires for other domains (like Lingel et al. 2014; Rakozky and Klieme 2005), 27 items divided into the subprocesses of planning, monitoring, regulating and evaluating have been developed and tested. According to the fact that metacognitive strategies are only used when they are helpful (i.e. the task is not too easy and the students are motivated to use them; see Sect. 15.2.1), students are asked as well to judge their motivation and the difficulty of the task on a four-point scale. The items were given to 66 students of grade nine from five different classes.

For testing the questionnaire, the students were introduced to a modelling cycle (see Kaiser and Stender 2013) and then they worked in groups on a modelling task. The working process was videotaped. After working on the task, the students were asked to fill in the questionnaire. While filling in the questionnaire, they were allowed to speak to each other and discuss the items. Furthermore, four pairs of students were asked to explain their answers to the items during an interview. Moreover four experts rated the students' metacognitive behaviour with the help of the questionnaire as well as the videotapes.

Frequency distributions and item difficulties of the students' self-reports as well as of the experts' ratings were calculated. The results vary widely (for further information, see Janetzko 2014). Correlations between self-reports and expert ratings were low. With the help of the interviews, some reasons for low correlations were reconstructed:

- Those students with low metacognitive skills aligned their answers with the students who have higher metacognitive skills. Assumingly, this will not occur if students would not be allowed to discuss their answers with each other.
- Students claim that they monitor their working process but did not verbalise anything. This is a well-known problem (see Sect. 15.3.1) that can hardly be solved. But it becomes obvious that students' self-reports are of great importance, because pure observations of metacognitive skills cannot adequately capture those skills.
- Some formulations were simply not understood by the students. Especially the terminology of the modelling process was not familiar to them. So students have to become acquainted with the terminology beforehand.
- The difference between some items was not recognised by the students. So these items have to be combined or the difference has to be made more explicit.
- Sometimes the students did not know how to answer because some strategies they had only used on their own and did not share with the group. Others they had used only because a group member suggested doing so. So the items must clearly differentiate between the use of strategies in the whole group and strategies that were used for the monitoring and regulation of one's own behaviour.

Consequently, the questionnaire has been reworked in the outlined way paying special attention to the item formulation. Items with a very high average size as well as those that were similar were reformulated and made more explicit. Especially it has been differentiated between single strategy use and strategies used in the whole group (e.g. see Sect. 15.3.3). Furthermore, the introduction of the modelling cycle as a metacognitive tool to the students has been reviewed.

15.3.3 Items for Measuring Metacognitive Strategies

For measuring metacognitive strategies, the reviewed questionnaire consists of 39 five-point Likert items, divided into four parts. Contrary to the division of metacognitive strategies into the processes of *orientating/planning*, *monitoring/regulating* and *evaluating*, the items have been aligned in the order of their appearance during an ideal modelling process. Thereby, students were guided to recapitulate their working process. Beneath the three phases of *at the beginning*, *during* and *after working on the task*, the students are asked to judge their motivation to work on the task and the task difficulty at the end of the questionnaire.

The phase before the working process is measured by six items. All these items are primarily related to the first step of the modelling cycle, which contains developing a real model by understanding and simplifying the problem. Most items relate to metacognitive strategies for orienting and planning. The items refer to reading and understanding the task, capturing needed information as well as possible interim goals and agreeing on a common approach. Depending on the results mentioned

Table 15.1 Selected items of the questionnaire

Item no.	Item description	Relation to the coding guideline
1.1	At the beginning of the working process, *I captured important information out of the task*	
1.2	*At the beginning of the working process, we tried to get aware of possible steps*	P1
2.1	*I normally knew what was missing to get a solution*	M2
2.2	*We allocated work*	P2
2.3	*If we made no progress, we tried to find where exactly our problem is*	M1
2.4	*If our (interim) result seemed strange, I checked our assumptions*	M1
3.1	*When we had a solution, I was wondering if there is a better solution*	E3
3.2	*When we had a solution, we were wondering what we can do better next time*	E2

above, the items contain single processes (Table 15.1, item 1.1) and group processes (Table 15.1, item 1.2). Strategies that should be used on one's own as well as shared in the group were mentioned twice, one time as a single and one time as a group process. Having the goal in mind of developing a task-bound questionnaire that is applicable to different modelling tasks, some items were generalised.

The second part of the questionnaire refers to the phase of working on the problem (from developing a real model to validating the real results) and therefore merely relates to metacognitive strategies for monitoring and regulating the working process. As regulation can only occur when monitoring has been applied, these processes were combined in one section. This section can be divided into items that aim at processes which happen without the occurrence of a problem (13 items) and those metacognitive strategies that are helpful (or restraining) if a problem occurs (12 items). The first 13 items not only refer to strategies for monitoring and regulating (Table 15.1, item 2.1) but also to those for planning the working process (Table 15.1, item 2.2). Altogether students have to give information about their own behaviour as well as about the cooperation in the group during the time of working on the modelling problem. Filter questions were subsequently used for items concerning the occurrence of problems. After these filter questions, possible questions on regulating strategies are posed (Table 15.1, item 2.3 and item 2.4). In this part of the questionnaire as well as in the first part, formulations were tested several times, and formulations were divided into single and group processes.

The third part of the questionnaire consisting of seven items refers to the phases after working on the modelling problem. Primarily the items of this group relate to metacognitive strategies of evaluating the whole process. Using these strategies is – similar to validating results of modelling problems – often forgotten or there is not enough time to do it. As pointed out in Sect. 15.2.2, it is very important to learn through reflection and to overcome some kinds of behaviour that are restricting the quality of a learning process. These items aim at assessing if students reflected on the solution (Table 15.1, item 3.1) and if they had drawn any conclusions for the next working process (Table 15.1, item 3.2). Students evaluate only self-acting, if they came to a solution and had enough time left. In order to measure only the evaluation done during working time, another filter question is posed in the questionnaire.

15.4 Conclusions

Although different studies have already pointed out the importance of metacognitive modelling competencies for solving modelling problems successfully, research about metacognitive modelling competencies is still at its beginning. With regard to the evaluation of learning environments to promote metacognitive modelling competencies, it is especially necessary to develop instruments for measuring those competencies. Concerning metacognitive skills, there are existing different methods of measuring that have different advantages and disadvantages.

In order to develop a task-bound questionnaire for measuring metacognitive strategies that is applicable to different modelling tasks, two studies have been carried out. The results presented above clearly indicate that a questionnaire seems to be a possible instrument for measuring metacognitive modelling competencies. However, other aspects have to be taken into account. This includes not only the item formulation but also the particular circumstances under which the students are asked to fill in the questionnaires. In order to reconsider the reliability and validity of the revised questionnaire presented in extracts above, the questionnaire has to be tested once more.

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Part II
Researching Boundaries in Mathematical Modelling Education

Chapter 16
The Mathematical Modelling Competencies Required for Solving Engineering Statics Assignments

Burkhard Alpers

Abstract Engineers set up and work with mathematical models when solving engineering problems. For doing this successfully, mathematical modelling competencies are essential. Yet, it is open where in the engineering curriculum these competencies are acquired. In this chapter I investigate whether, and how, mathematical modelling competencies are addressed when students work on statics tasks appearing in one of the fundamental classes in many engineering study courses. As a theoretical framework for capturing modelling competencies, all assignments in two widely used statics textbooks were analysed relearning opportunities for modelling. It turned out that important subprocesses like understanding the situation and making assumptions and simplifications, interpretation or validation of results are not at all or only scarcely addressed.

Keywords Modelling competencies • Statics assignments • Engineering • Modelling cycle • Mathematical competence • Engineering curriculum

16.1 Introduction

Since engineers set up and work with mathematical models describing certain situations of interest, it seems to be quite obvious that mathematical modelling competencies should play a major role in engineering education. Therefore, these competencies have been included in the Curriculum Document of the Mathematics Working Group of the European Society for Engineering Education (Alpers et al.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_16

2013). It is open, however, where in engineering study courses the competencies should be acquired. In application subjects like statics or control theory, important modelling quantities are introduced (like force, torque, stress), and ways to set up models and to compute interesting quantities in these models are presented such that students at least "experience" mathematical modelling. But only by actively working on assignments can students acquire competence. It is the goal of this chapter to investigate for one fundamental application subject, statics, whether usual assignments are adequate for this purpose.

As a theoretical framework for conceptualising modelling competencies, I basically use the so-called modelling cycle (Blum and Leiß 2007) which is described in the next section. I chose two widespread textbooks on engineering statics that contain assignments for which sample solutions are available. For each competency identifiable in the modelling cycle, I analysed whether and how it is addressed when working on the assignments. In order to check empirically whether the solutions in the books match with "good" student solutions, I had two very successful students work on selected tasks and explain their thinking processes in detail. I describe the results of the document analysis and the students' work in Sect. 4. In Sect. 5 I discuss the results, relate them to the research literature and outline potential educational consequences.

16.2 Theoretical Framework

Kaiser and Brand (2015) provide an overview of different conceptualisations of modelling competencies during the last 30 years. Often, a more elaborated specification of these competencies has been based on the so-called modelling cycle which is a well-accepted idealisation of the modelling process. Figure 16.1 depicts an example of such a cycle as it was set up by Blum and Leiß (2007) (for similar cycles, see the overview in Frejd 2014). In the first step, a person working on solving a realistic problem tries to understand the problem situation and where the

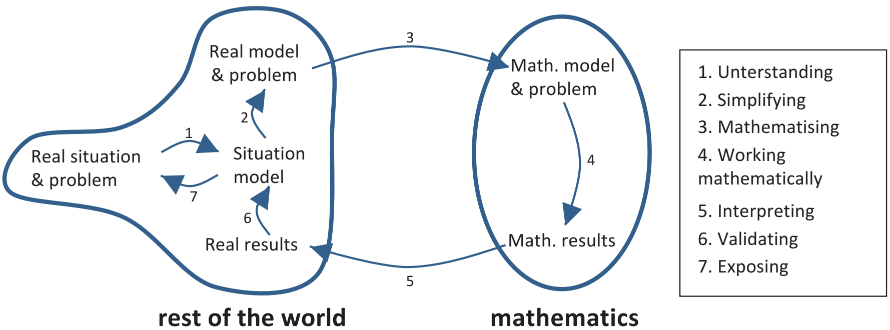


Fig. 16.1 Modelling cycle after Blum and Leiß (2007) and Borromeo Ferri and Blum (2010)

problem really is (1). This leads to a "situation model". Then, the situation is structured and simplified (2) to capture only presumably essential features which results in a "real model" where the problem is formulated. Then, in a process of mathematisation (3), the real model (and problem) is turned into a mathematical model (and problem), and the problem is solved by working mathematically (4). The results are interpreted in the real model (5) leading to "real results". These are then investigated regarding their sense-making in the situation model, that is, they are validated (6). In activity (7) (not explained in Blum and Leiß 2007 but in Borromeo Ferri and Blum 2010), the results are exposed to those who are interested, that is, they are documented and presented.

Maaß (2006) added so-called metacognitive modelling competencies comprising an awareness of the overall process and the position and meaning of the subprocesses when doing modelling. In Kaiser and Brand (2015), this is also termed "overall modelling competency" ("ability to carry out complete modelling tasks, metacognitive abilities, mainly monitoring the modelling process", p. 141). For investigating whether and how a learning activity addresses the acquisition of modelling competencies, it is therefore well grounded in the research literature to search for instances of the seven subprocesses mentioned above as well as reflections on the overall process.

16.3 Method of Investigation

I performed a content analysis as outlined in Robson and McCartan (2016, pp. 349–359), starting from the research question: Are mathematical modelling competencies addressed in statics tasks? As a sample, I chose two widely used statics textbooks, one from Germany (Gross et al. 2013; corresponding book of assignments: Hauger et al. 2012) and the German edition of a US textbook by Hibbeler (2012). Both textbooks are available in the 12th print edition from which I conclude that they are heavily used by lecturers. The book by Hauger et al. (2012) contains 83 tasks with sample solutions, the book by Hibbeler 1091 (!) tasks with sample solutions on the book's companion website. Many of Hibbeler's tasks are very small computational tasks, and there are also several quite similar tasks for training. The units of analysis are the single steps in the solutions. For analysing the content, I used as categories the subprocesses of the modelling cycle as described in the previous section. I investigated the sample solutions twice, first for understanding the procedure and secondly to find occurrences of subprocesses of the modelling cycle. I was also open for detecting other categories not covered by those subprocesses. I did not have another researcher check the way I mapped the solution steps to subprocesses of the modelling cycle since the latter have been used in many studies without any problems reported regarding the coding.

Moreover, I let two very successful students prepare and explain solutions to 25 selected assignments, 12 from Hauger et al. (2012) and 13 from Hibbeler (2012) in order to check whether successful students work on the tasks in the way the book authors assume them to do. I compared their solutions with the sample solutions and performed interviews for clarification. The 25 tasks cover all areas in statics and all kinds of occurring mathematical models, as far as they have been dealt with in the statics education of the students. Moreover, they address interesting aspects discovered in the content analysis and include a design task taken from the book by Hibbeler.

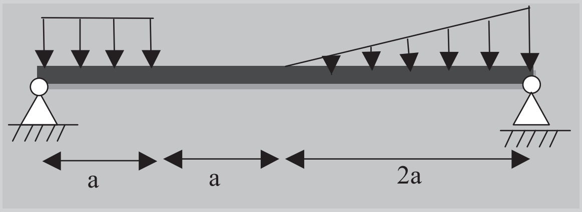
16.4 Results

In the sequel, I describe whether and how the competencies are addressed in the assignments. If I found an interesting deviation of the students' solutions from the sample solutions of the author, I will point this out. Since there were no tasks addressing the reflection of the whole modelling process, I omit the "overall modelling competency".

16.4.1 Constructing a Situation Model

In the set of assignments provided by Hauger et al. (2012), there is no real situation addressed that needs clarification. The tasks "live" in their own world of mechanics. The *Beam Task* is a typical example where the drawing already uses the "graphical language" of mechanics with special symbols for bearings and loads.

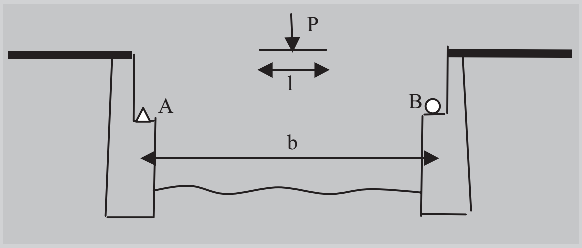
Beam Task (After Hauger et al. 2012, p. 27)



A beam is supported by a fixed and a movable bearing and loaded as shown in the figure. Determine the internal shear force and bending moment function.

In the textbook by Hibbeler (2012), often relations to underlying real situations are given. There is, for example, a photo of a pile of planks carried by two metal beams representing a constant line load (p. 185), and in an example (p. 189), a heap of gravel is shown that is modelled as a linearly decreasing line load. This gives readers an impression where such load functions might occur. In the assignments to be done by the students, however, there is no requirement for understanding a real situation and problem. The problems are already clearly stated, and often a drawing showing the essential properties of the situation is provided.

Bridge Task (After Hibbeler 2012, p. 367, Shortened)



Construct a truss for a bridge consisting of joints and pins that can carry the given load. Costs for pins and joints are given as well as restrictions on forces in pins. Make it cheap!

It is fair to state that in the textbook by Hibbeler (2012), there are a few more open tasks of a constructive nature (design tasks) like the *Bridge Task*. In design tasks like this, there are many possible solutions. Students are assumed to start with an initial design, check the restrictions on forces and improve the design iteratively using their knowledge on how the design variables influence the forces and the costs. Although this task offers good learning opportunities regarding mechanical concepts (distribution of force along connected pins) and resembles real design, the situation and problem are again clearly stated.

16.4.2 Constructing a Real Model

Constructing a real model means to simplify and to structure a problem. But again, in both textbooks the real models are already given in the assignments with simplifications already made as can be seen in the *Beam Task* where a body is already idealised as a beam and the loads are simplified as being linear or linearly decreasing. Simplifications are provided either directly (by statements like "weight, radius, etc. is neglectable") or indirectly via code words like "smooth surface" (i.e. friction can be neglected), "homogeneous" (i.e. the centre of volume can be considered instead of the centre of mass) or "thin" (i.e. the surface can be considered instead of the volume). The authors expect students to translate such words. Yet, the two students working on a selection of tasks which also contained such code words ignored them completely. When questioned, they stated that they expect everything important to be mentioned explicitly in the task with at least a symbol given (if not a concrete value). Unless the latter is the case, quantities are of no relevance. This expectation is probably based on their experience with such tasks.

The real models that are given in the assignments often use a kind of "graphical language", as is the case in the *Beam Task*. There are conventions for objects like beams, pins, joints or bearings and for loads as well (point loads, line loads, etc.) in two- or three-dimensional space. In order to understand the real models, students must be able to "read" and understand this graphical language. This situation is similar to the one Biehler et al. (2015) found when investigating the foundations of electrical engineering.

16.4.3 Setting Up Free-Body Diagrams

The analysis of assignments revealed the overwhelming role of creating so-called free-body diagrams (FBDs) in the solution process. Therefore, I suggest in Fig. 16.2 to refine the general modelling cycle for usage in statics assignments by inserting another stage "free-body diagrams" and a subprocess "subdividing and translating" which will be explained below.

In this step a part of the mechanical configuration is "cut-free", and all forces or torques that are applied are added to the drawing as well as all important geometric information. For doing this, the graphical language of statics has to be translated into forces and torques. In the *Framed Drum Task*, the upper left drum can be cut-free (as shown in the right-hand side of the drawing which is **not** part of the task description but placed there for space reasons). The reason for cutting free parts is that one can then set up equations of equilibrium for such parts and use these to compute unknown quantities (mostly forces or torques) from known ones. Therefore, when choosing a part to be cut-free, students already have in mind how they can use the resulting system of equations. There is no clear split between the "rest of the world" and the mathematical domain here.

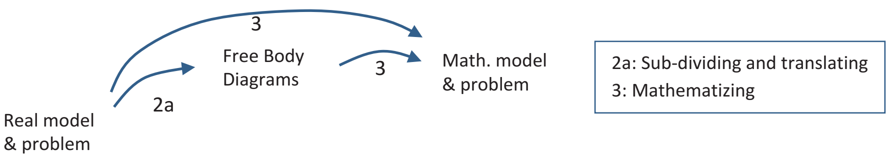
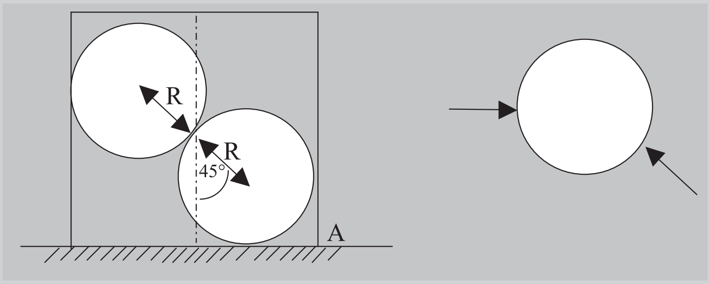


Fig. 16.2 Refinement of the modelling cycle

Framed Drum Task (After Hauger et al. 2012, p. 24)



Two cylindrical drums (weight m) are fixed by a u-shaped frame (weight M). Determine all contact forces. What is the meaning of the ratio M/m regarding the contact force in A?

A similar situation can be found in assignments involving infinitely small sections, for example, when computing the mass of an inhomogeneous object (with variable density). It is preferable to choose only one dimension infinitely small because the subsequent integration then requires only one-dimensional integration. So again, an anticipation of the solution process is at least very helpful when choosing a body to cut-free.

16.4.4 Setting Up a Mathematical Model

The process of setting up mathematical models is called "mathematising", and it is often a challenging task (cf. the examples in Niss 2010). In statics, the important modelling quantities like force and torque and the main principles for setting up relations (like equilibrium) are already treated and exemplified in the textbook. So, the remaining task consists of using the quantities and applying the principles correctly and effectively. This mainly includes:

- Setting up equations (scalar/vectorial ones), as in the *Framed Drum Task* above. In vectorial models, unit vectors are set up in assumed directions of forces or torques.
- Setting up functions in order to determine internal forces and moments as in the *Beam Task* above where one sets up a complete line load function first.
- Setting up inequalities in static friction models.
- Setting up formulae or functions for virtual work or potential energy.
- In infinitely small FBDs, determine the property of interest and build the integral.

16.4.5 Solving the Mathematical Problem

Many mathematical problems turned out to be predominantly of a geometric nature where students have to recall and use the whole "repertoire" of geometric properties, theorems and operations (sine rule, cosine rule, Pythagoras, scalar and cross product, etc.), and choose among a huge set of possible equations to find a subset which enables them to compute unknown geometric quantities from known ones. Further mathematical problem solving procedures recognised in the analysis are:

- Computing definite integrals in models with infinitely small objects,
- Performing integration to get from line load to shear force and bending moment.
- Using "classical calculus" to compute minima of energy functions,
- Decomposing bodies into components (centre of area, volume, mass),
- Solving sometimes larger systems of equations by smartly going through them.

In the German "tradition", the assignments rather use symbols than concrete values, whereas in the US "tradition", the provision of values seems to be usual. If symbols are used, then a proficiency with symbolic manipulations is required.

16.4.6 Interpreting the Results in the Real Model

There were only two types of assignments in both books where students were explicitly asked for interpretation as part of a "standard routine": In truss analysis it was required to state whether a member is under tension or pressure; in equilibrium analysis using potential energy functions, students had to distinguish between stable and unstable equilibrium. In both cases the sign of numerical results had to be interpreted properly. The *Framed Drum Task* was the only example for a required interpretation which was not part of a "standard routine". The result has a symbolic representation, and it is of interest when this can become negative because then the frame would tip over. Both students who worked through a selection of the tasks felt only obliged to interpret the results if this was called for.

In the sample solutions of the authors, several additional interpretations can be found which are not required explicitly in the tasks. The signs of numerical values can be interpreted in order to check whether the configuration is really static. Sometimes it is interesting whether a quantity does or does not occur in a solution in order to check for independence. This kind of interpretation is easier when working symbolically since otherwise one has to go through the whole computation. If in a symbolical solution functions with restricted domains of definition occur, the meaning needs to be interpreted. If a mathematical procedure provides several solutions, these also have to be interpreted. An interpretation of the assumed behavior is often required in assignments dealing with friction where several objects are involved. There, it has to be determined which parts will slide or roll under which conditions. A special requirement for interpretation can be found in design tasks like the *Bridge Task*. After having produced and investigated an initial design, the results have to be interpreted in order to make adequate changes for improving the design.

16.4.7 Validating the Results in the Situation Model

Since in the assignments the real model is already given, there is no reason to question potential assumptions and simplifications. Therefore, validation is mainly restricted to investigating whether the results make sense in the real model. By investigating the sign of a symbolic expression as in the *Framed Drum Task*, one can check whether the static model is valid after all. If a symbolic result is available, one can also investigate extreme situations, e.g. in the *Framed Drum Task*: What happens if the weight M of the frame is more and more increased? Then, at some stage the contact force should become positive. Sometimes the sign of a result can be used for validation purposes if it is obvious from the mechanical configuration whether a member is under pressure or tension. Moreover, in tasks involving static friction, one can check whether the inequality between normal force and adhesive force is fulfilled such that no movement occurs. Otherwise, "validation" consists of comparing one's own final solution with the one provided.

16.4.8 Exposing the Results

Since the real model is already given in the tasks under investigation, students do not have to reason about and justify any assumptions or simplifications in their exposition. It is essentially expected from the authors that the well-known procedures are performed and documented, and corresponding drawings of geometric relationships and free-body diagrams are provided. At the end a short interpretation should be given if applicable. The two students who worked on a selection of tasks met these expectations.

16.5 Discussion and Conclusions

The investigation has shown that only a subset of the competencies required in the modelling cycle are addressed in the statics assignments in both analysed textbooks which are quite different in style otherwise. Particularly with regard to the earlier phases where the situation and the problem have to be clarified and assumptions and simplifications are made, there are no learning opportunities since the real model is already given. Even the attempt to include an element of simplification by using code words like "smooth" does not fulfil this purpose since students ignore them because they are accustomed to being provided with all relevant information. Regarding the later phases of the modelling cycle, these are also only scarcely addressed. I found two types of tasks with sign interpretation which was part of "routine procedures", but otherwise only one task required a non-routine interpretation. Even the well-performing students working on sample tasks did not feel obliged to interpret results without being asked. Moreover, there is hardly any reason for questioning and validating the real model if this is already provided in the task.

One of the most important competencies needed for successful work on the statics assignments was the ability to set up free-body diagrams by decomposing the configuration and translating the graphical language of mechanics into forces and torques to be included in the FBD. When deciding on which part of a configuration to "cut-free", students already have to anticipate later mathematical work such that there is no strict separation between the "rest of the world" and "mathematics". Biehler et al. (2015) found a similar situation for tasks in the foundations of electrical engineering where application subject and mathematics were rather intertwined. This does not seem to be restricted to engineering education since Niss (2010) illustrated with several examples that an anticipation of usable mathematical concepts is important for successful modelling work: "*implemented anticipation* of relevant future steps, projected 'back' onto the current actions" (p. 55).

The mathematical models that have to be constructed use the quantities and principles that are developed in the textbooks and are illustrated there with several examples. This could be called "modelling with well-known quantities according to well-known principles". Regarding the mathematical work in the models, the role of geometric argumentations was significant. At this stage the mathematical work is disconnected from the application and happens purely in the domain of mathematics. The essential role of geometry distinguishes the use of mathematics in engineering statics tasks from that in the foundation of electrical engineering as described in Biehler et al. (2015).

Otherwise, the results are consistent with the findings by Biehler et al. (2015) with respect to the foundations of electrical engineering. They also confirm the results of the study by Gainsburg (2013) who investigated when (during university studies and professional work) students/engineers acquire modelling competencies. She found that neither the students nor the novices or veterans had an explicit understanding of the modelling process and that the instructors assumed that modelling is rather learned in practice "as a byproduct of extended participation in engineering work rather than from direct, explicit instruction for novices" (p. 272). Correspondingly, she found aspects of modelling competencies which were dealt with in isolation and not addressed as part of the cycle.

The investigation shows that one cannot rely on modelling competencies being acquired in statics. Further research is necessary in order to investigate other subjects of the engineering curriculum like design in order to find out about learning opportunities for acquiring modelling competencies. Based on the current insights, it seems still necessary to address modelling in mathematics, possibly by turning statics tasks into rich modelling tasks (cf. Alpers 2016 for an example).

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Chapter 17
Pre-service Teachers' Levels of Reflectivity After Mathematical Modelling Activities with High School Students

Rita Borromeo Ferri

Abstract Recently there has been a push to train teachers in how to enhance their professional reflection to increase competency, because they do not tend to receive feedback on their teaching from colleagues or students following their teaching practice. Within the seminar, "Modelling Days," at Kassel University, the focus on reflection was promoted explicitly. The explorative study presented in this chapter had the goal to investigate pre-service teachers' levels of reflectivity after modelling activities with high school students. The analysis of 12 written reflections showed different levels of reflectivity, with high levels rarely being reached. The development of a new model on levels of reflectivity, which originated from Hatton and Smith's model, is a further result of the empirical study.

Keywords Mathematical modelling activities • Theory • Reflectivity competency • Teacher education

17.1 Introduction and Background of the Study

Mathematical modelling has become a compulsory part of curricula in several countries in the past 20 years. The necessity to develop programmes for pre- and in-service teacher education for the learning and teaching of mathematical modelling is strongly recommended and still required (Cai et al. 2014). Best practice examples of modelling courses, however, have different foci (for an overview, see Cai et al. 2014) and always show that teachers need time to understand the complexity of mathematical modelling for themselves. The concept of a modelling course which offers a balance between theory and practice can be very successful with regard to

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_17

accessing knowledge needed for teaching modelling (Borromeo Ferri and Blum 2009). There are no well-defined competency criteria which teachers should have to teach modelling in school and to fill the gap Borromeo Ferri and Blum (2009) have created a model of required teacher competencies. The model consists of four dimensions which can also be used for planning modelling courses for pre- and in-service teacher education. These dimensions are (1) theoretical, (2) task, (3) instructional and (4) diagnostic.

The dimensions will not be described here in detail (for that, see Borromeo Ferri 2014). A central goal in the modelling course is to support the pre-service teachers in the development of their reflective competency. This means that teachers should be able to reflect on their teaching (Part II of the course) with regard to using the knowledge learned within the theoretical Part I of the modelling course (e.g. different phases of the modelling process or diagnosis and intervention in critical situations). So Part I and Part II constitute the “Modelling Days” at Kassel University mentioned above by providing a theory-practice balance in the course. Evaluations of this course in the last few years have shown that the opportunities to work practically in Part II was regarded as very helpful for the students’ future work at school and particularly for teaching modelling. So in Part II, pre-service teachers worked as a team of two for three full days together with a group of five high school students. At the beginning of the Modelling Days, groups of learners could choose between one and three complex modelling problems (e.g. traffic lights versus roundabouts – what is the best/optimal arrangement for the traffic?). The pre-service teachers chose a group of learners, and then they commenced their investigation of the modelling problem. At the end of the Modelling Days, the learners presented their results.

After Part II, the pre-service teachers had to submit a written reflection (15–20 pages) based on criteria determined by the lecturer. Some criteria refer to aspects which have to be included at the beginning of the reflection such as theoretical background on modelling, modelling cycle(s), background of teacher interventions (see Leiß 2007; Zech 1998) and descriptions of the student group as they worked with the subject matter analysis of the complex modelling problem they had chosen. As teacher intervention while modelling was the focus, the pre-service teachers learnt about different types of interventions: (1) motivational intervention, (2) feedback intervention, (3) strategic intervention, (4) content-strategic intervention and (5) content-related intervention (Zech 1998). During the 3 days working with high school students, the pre-service teachers had to observe the learners as well as themselves with the focus on student progress and problems encountered by the high school students while modelling. In connection with this, the pre-service teachers were expected to reflect on which kind of intervention they gave and especially what effect their intervention caused on the modelling process of the students.

The goal of this explorative qualitative study was to analyse pre-service teachers’ levels of reflectivity and their reflectivity competency by especially focusing on both the theoretical and practical aspects of modelling. Learning how to reflect on one’s own teaching is a central part of teacher professional development, so the goal was to gain more insight into teachers’ competency on the one hand and to formulate implications for teacher education on mathematical modelling on the other hand. In the following section, a theoretical background of reflectivity competency within pre-service teacher education will be discussed, then the design and results of the study are described, and finally a short discussion closes this chapter.

17.2 Theoretical Background: The Role of Reflections in Teacher Education

Reflections are used in schools and in universities as a documentation tool for critical argumentations mostly with practical experiences. The role of fostering reflectivity competency in teacher education is mentioned, although there is still little empirical research in this area (Abels 2010). Abels emphasized the needs for more research concerning the following three aspects: (1) definition of reflectivity competency, (2) practical implementation in teacher education and (3) research methods for investigating the effectiveness of the concepts. In 1995, Hatton and Smith's research attempted to gain more insight into reflectivity competency of pre-service teachers. Their study was embedded in teacher education with a focus on pre-service teachers undertaking their 4-year Bachelor of Education degree at the Faculties of Arts, Science, Economics and Education. The written reflections of 34 pre-service teachers were analysed concerning the central goal of investigating the nature of reflection in teaching and to define specific forms of reflection. The result of this analysis was the identification of four types of writing, three of which were different kinds of reflection: (1) descriptive writing, (2) descriptive reflection, (3) dialogic reflection and (4) critical reflection. Type 1 is not reflective at all, but type 2 does attempt to provide reasons based often on personal judgement or on pre-service teachers' reading of literature. Type 3 is a form of discourse with oneself, an exploration of possible reasons. Type 4 is defined as involving giving reasons for decisions and events, taking account of broader contexts (Hatton and Smith 1995, pp. 40–45).

Hatton and Smith (1995), in their own work, were able to reconstruct that most of the pre-service teachers wrote descriptive reflections at the beginning of their composition, but then in their writing process changed more to dialogic reflections. Critical reflections were reached very seldom. In this context, Hatton and Smith (1995) tried to investigate the difference between *reflection in action* and *reflection on action*. Reflection in action means the immediate analysis of one's behaviour in the relevant situation and reflection on action happens retrospectively after the action and is aimed at thinking about doing changes in the behaviour in the near future. For Hatton and Smith, reflection in action is the most demanding level of reflection, but on the basis of their data, they could not reconstruct which type of reflection was being used by the pre-service teachers. Hatton and Smith's model formed the basis for analysis in several other studies (e.g. Abels 2010), but there are also similar models developed for investigating levels of reflectivity (see van Manen 1977; Zeichner and Liston 1985) of pre-service and in-service teachers.

For the study presented in this chapter, the model of Hatton and Smith (1995) provided the basis for developing a new model of reflectivity for data within the mathematical modelling context. To meet the needs of this context, the new model should facilitate the analysis of the theory-practice balance of the modelling course to obtain more knowledge about pre-service teachers' reflectivity competency. On the basis of the theoretical background of reflectivity competency within teacher education, the central research questions for the current study were:

- How can levels of reflectivity be described and conceptualized for developing a new model of reflectivity for pre-service and in-service teacher education in mathematical modelling?
- Which levels of reflectivity do pre-service teachers' show within their written reflections when connecting their theoretical knowledge on mathematical modelling with practical experiences of teaching and coaching high school students during modelling activities?

17.3 Design and Methodology of the Study

The course, "Modelling Days," is a well-evaluated modelling course for pre-service teachers in their final semester at the University of Kassel and was developed in its structure, content, methods and assessment by the author of this paper and was based on former experiences at the University of Hamburg (Kaiser and Schwarz 2010). As described in the introduction, the course is divided into two parts in order to enable an appropriate theory-practice balance. In detail, Part I consists of 12 sessions of 90 min and Part II of three full school days (8 a.m.–1 p.m.) working with high school students (aged 16–17 years). After each day, the lecturer and the pre-service teachers had a collective reflection about the modelling activities of the learners and about the interventions they had given. Two months after the course, the pre-service teachers had to submit their written reflections (15–20 pages), which are also the basis for the grading of the course.

All participants agreed that their reflections could be used for this explorative study. The data were collected in the course in 2012 from 34 participants, who chose this course in their final semester of university (fourth year). For the data analysis and model development, an initial sample of 12 written reflections from pre-service teachers who taught the same modelling problem in Part II (see introduction) was used in order to compare them by the task used. As an appropriate method within the field of qualitative research, grounded theory was used (Strauss and Corbin 1990) because it offers good possibilities for exploring new phenomena and allows for generating new pieces of a theory. Data are analysed using a coding procedure (see Table 17.1) and afterwards conceptualized and composed in a new way. The construction of the coding schema depends on the theoretical sensitivity a researcher

Table 17.1 Categories for classifying in the model of reflectivity

Category	Topic
1	Modelling (theoretical and practical references to mathematical modelling)
2	Teacher interventions (teacher's behaviour during modelling activities of learners)
3	Teamwork (illustrations about teacher's work with the other team partner)
4	External circumstances (all information about organizational matters in classroom or school)
5	Goals and motivation (illustration about teacher's own motivation and about learners' motivation while modelling)
6	Own development and perception (all illustrations about learners' and teacher's own perceptions)
7	Appraisal and final results of learners' work (all illustrations about the final result of the modelling problem)
8	Whole working process (all illustrations about teacher's own working process concerning the modelling problem in Part I and of the learners' working process)

has. This means that previous knowledge, including knowledge about the literature in this field, has a substantial influence on building codes. Also in the present study, theoretical approaches have been analysed beforehand, in particular concerning existing models of reflectivity or types of writing. However, in combination with the required aspects, the lecturer gave to the pre-service teachers' instructions for writing their reflection (see introduction), including the depth and the level of reflectivity to be reconstructed.

17.4 Results of the Study

17.4.1 Characterizations of Reflection, Level of Reflectivity and Reflectivity Competency

The following three aspects are presented on the basis of the theoretical background, which gave orientation for the model development:

Reflection: Critical conflict with own behaviour/action in self-experienced pedagogical situations with the goal to learn from these situations and to develop alternatives.

Levels of reflection: These describe the depth of consciousness of critical conflict of own behaviour/actions according to different stages of reflection.

Reflectivity competency: The ability of a (pre-service) teacher to think about and to critically analyse his/her own behaviour/actions in self-experienced pedagogical situations and to learn from these situations consciously.

Table 17.2 Model of reflectivity for teacher education and teacher training in mathematical modelling

Level	Name	Description
0	Descriptive writing	Description of the situation without justification
1	Justified reflection	Justification of own actions
2	Deliberative reflection	Legitimation of own behaviour, description of alternatives of own behaviour and self-criticism
3	Theory-based reflection	Inclusion of theoretical concepts and special literature
4	Perspective reflection	Taking different positions, consideration of a wider context

17.4.1.1 Model of Reflectivity for Teacher Education and Training in Mathematical Modelling

On the basis of the coding procedure, it became evident that categories 1, 2 and 8 (see Table 17.1) were mentioned most within the reflections. Because of the deviation of the present data of this study in contrast to the goals and procedures which were used for developing models of reflectivity in other studies (e.g. Abels 2010; Hatton and Smith 1995), a modified model was developed for the final data analysis and for operationalization, which is also a result of the empirical study (see Table 17.2).

The teachers' reflections were analysed on the basis of the categories (Table 17.1) and classified according to the levels of reflections (Table 17.2). Every single passage was analysed for the different levels by two coders. The Cohen's kappa was 0.73, which is satisfactory. Level 0 was given if a category was only descriptive or if a category was not mentioned within the reflection. Also, if there were only references to theory without connections to their own practical experiences during the Modelling Days, the passage was classified with level 0 and not with level 3. Similarly, to the model of Hatton and Smith (1995), the developed model is hierarchical. If a person wrote reflections on a high level, descriptive passages also became apparent, which were dependent on the context, but overall the amount of the remaining passages was high. To make the levels more concrete, prototype examples are given for all levels in Table 17.3.

17.4.2 High-Level, Middle-Level and Low-Level Reflections on Mathematical Modelling

All written reports demonstrated clear evidence of reflections by the pre-service teachers in their final year. The proportions of coded units in total of all reflections had a strong tendency to level 0 and level 1, with 33.6% coded as descriptive writing and 39.5% coded as justified reflection. Of the coded units, 15% were deliberative reflections, 6.2% were theory-based reflections, and only 5.4% were perspective

Table 17.3 Prototype examples of reflectivity levels

Level	Name	Prototype examples of the teacher's work
0	Descriptive writing	"After I had a discussion with the learners, they worked further on the problem"
1	Justified reflection	"The learners were again confused with all the numbers and assumptions, so I decided to make them reflect upon their previous approach"
2	Deliberative reflection	"I recognized that my interventions were short over the 3 days. Mostly my interventions consisted of one sentence, an exception or a question"
3	Theory-based reflection	"I told my group that another group got results between 9000 and 25000, whereby this was clearly a content-related intervention (see Zech 1998) because I gave the learners concrete ideas for the solution"
4	Perspective reflection	"I had the feeling that Laura and Jenny stagnated in their modelling process. So I conducted a content-strategic intervention, which was appropriate in my opinion. I learned so much about me and my reactions and their effects on learners' modelling process. The possibility to reflect myself and to write it down was not easy, but helped me for my future work"

reflections. On the basis of the analysed data, it became clear that there were high-level, middle-level and low-level reflections on teaching and learning of mathematical modelling, with the majority of reflections being low-level reflections. Low-level reflections are those which were classified overall as level 0 or 1. These reflections are mostly descriptions of several situations. For the most part, these pre-service teachers did not give reasons why certain situations arise and are not able to express alternative actions or to include theoretical aspects. A high-level reflection includes all the points which are missing in the low-level reflections, and in addition it considers different perspectives in a wider context. The overall level of a reflection refers to the frequency of passages belonging to specific levels. For a high-level reflection, the majority of passages would belong to levels 3 and 4, independent of the length of the written work of the pre-service teachers, which ranged between 15 and 37 pages.

Pre-service teachers' reflections which demonstrated a capacity to recognize their own behaviour and actions while teaching modelling but failed to continuously link theory with practice were classified as middle-level reflections (level 2). The levels of reflection concerning the categories (Table 17.1) show how teachers are able to connect theory on mathematical modelling with practical experiences in their reports. In particular, within high-level reflections, this connection is illustrated very well. Still one central question arises: How exactly does the transfer of theoretical aspects on mathematical modelling into practice happen? A follow-on question to this is, of course, how sustainable is it. The latter cannot be answered here, but the written reflections make clear that this transfer is a challenge for pre-service teachers on the one hand, but shows their needs and their success when teaching mathematical modelling on the other hand.

Table 17.4 High-level and low-level reflections

High-level reflection	Low-level reflection
Level 3	Level 0
"The learners talked about their next steps and sometimes needed technical help. So they demanded reactions from me which is called a responsive intervention (see Dann et al. 1999 p. 122)".	"In Part I of the course, we learned about the theory of teacher intervention while modelling. My group worked hard, so I did not help them very much".
Level 4	Level 1
"The learners agreed that their four possible results were not satisfying, although we only had less time on this day. So, I conducted an internal analysis of the situation which pushed me to a content intervention (Leiß 2007). I offered the learners to think about mixing the results they got so far".	"I gave organizational interventions to inform the learners in which working phase they are…Content interventions were rare, because the group was very good".

In Table 17.4, examples are presented of a low-level and a high-level reflection concerning category 2 (teacher intervention) and the theory-practice balance. The first example of the low-level reflection (level 0) shows the way this teacher reflected throughout his written work. Although the connection to theory on teacher interventions was mentioned in his text, there was no transfer to his own teaching of modelling. The second example of the low-level reflection (level 1) shows that the teacher was indeed able to classify his interventions sometimes, but formulating concrete alternatives to his own behaviour or ideas for changing actions in the future was not found in his reflections. Both examples of high-level reflections illustrate the link from practice to theory and clearly show reflection in action ("I conducted an internal analysis"). Authors of high-level reflections independently described their procedure during the 3 days: They made protocols immediately after their interventions to analyse the effect of their interventions for those learners' modelling process. This is a clear sign of reflection in action, which they used directly for thinking about their actions. In combination with elaborate reflection on action, they often reached levels 3 and 4.

17.5 Discussion and Conclusion

The goal of this exploratory study was to analyse pre-service teachers' levels of reflectivity after modelling activities with learners in the context of the course, Modelling Days, at Kassel University. In particular, the Hatton and Smith (1995) model gave orientation for developing a new model of reflectivity with a focus on mathematical modelling, which was used for analysing the data of 12 written reflections. The results show, on average, that the reflectivity competency of pre-service teachers was not high and that increasing their competency should be fostered and integrated as a part of courses on teaching and learning of mathematical modelling.

In order to promote the pre-service teachers' professional development in general, but in particular on mathematical modelling, the opportunity to let them write these reflections seems to be productive. To critically reflect on one's own behaviour or to make a transfer and a connection between the theory and practice of mathematical modelling, and vice versa, was a challenge as well as a learning process for the pre-service teachers. Most of the pre-service teachers in the study were writing reflections of this nature for the first time, because they had not learnt it in their general education courses. So, the teachers engaged in a learning experience on several levels. It would be interesting to observe whether these reflectivity levels would increase within a second reflection.

For implementation of mathematical modelling in everyday teaching, we need teachers who have acquired theoretical knowledge and practical experiences and have reflected themselves already at university. So, a further research question could be: Are teachers, who participated in the course, Modelling Days, and were promoted to reflect, more able to successfully implement modelling in their teaching than teachers who did not get this experience? There are a lot of factors which influence the inclusion or exclusion of mathematical modelling by teachers in their daily work all over the world, but we as researchers and teacher educators in this field should provide teachers with the knowledge, motivation and opportunities for deep thinking about this topic and with themselves – reflection is one possible way.

> Theoretical concepts about teaching and learning of mathematical modelling became so clear during my work with the learners and with my reflections. Concerning modelling, I was very doubtful before, but you have to make the experience – theoretically and practically, this is the key point. (Olga, 7th semester, pre-service teacher for mathematics and sports for secondary school)

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Chapter 18
Context and Understanding: The Case of Linear Models

Jill P. Brown

Abstract The benefits of student engagement with real-world contexts seem to be well accepted by the mathematical modelling and application 'community'. Yet concerns related to difficulties necessarily arising through engagement with the messy real world continue to be raised. This chapter presents a qualitative analysis from a study of Year 9 students and illustrates how engagement with context offers opportunities to demonstrate and deepen genuine mathematical understanding of rate of change. Genuine collaboration and interthinking were found to facilitate the development of mathematical understanding clearly enabled by the real-world context.

Keywords Context • Interthinking • Mathematical understanding • Metacognitive activity • Rate of change • Real world

18.1 Introduction

Perceptions of the relationship between real-world contexts and teaching and learning of mathematics are varied and often contradictory. Within the mathematical modelling and application 'community', dealing with context is seen as an essential element of being able to do mathematics (e.g. Stillman 2002; Villa-Ochoa and Berrio 2015). At times, context is portrayed as a hindrance to learning (Pfannkuch 2011) or framed as a problematic recontextualisation (Jablonka 2007). Within the community, it is well understood that there are differing definitions of modelling and certainly differing emphases in our research which is generally seen as productive (Blum et al. 2007); however, such diversity is at times presented as problematic

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_18

(e.g. Williams and Goos 2013). Others are yet to be convinced of the value of student engagement with real-world contexts in mathematics.

This chapter focuses on illustration and development of understanding related to linear functions used in modelling real situations as it addresses this dilemma. The research question addressed is: *How does context feature as an enabler and displayer of mathematical understanding?* More specifically, how does student engagement with context, in conjunction with student noticing of mathematical knowledge that is unknown, fragile or partially formed, allow the 'development' of this mathematical knowledge through engagement with the task context? *Metacognitive activity* and Mercer's (2000) *interthinking* proved to be useful tools in analysing students' collaborative activity.

18.2 Context

Real-world problems are by their very nature interesting. Two Colombian studies illustrate the situation where a very familiar context was of interest to students. Quiroz et al. (2015) used school flooding due to the overflowing of a nearby river, whereas Villa-Ochoa and Berrio (2015) used local coffee farming. Engagement of Year 10 students with a flooding context allowed "students to see themselves as citizens that can read, think, reflect and propose solutions in their own context" (Quiroz et al. p. 239). Similarly, the context of coffee farming highlighted that not only do applications and modelling allow students to recognise uses of mathematics but also to appreciate the importance of non-mathematical knowledge in solving real-world tasks.

In other cases, the interest in the context comes from the task setter and may or may not evolve into student interest. For example, Brown (2013a) implemented *The Letters Task* in two Grade 6 classrooms, intending the context would be real and of interest to these students. However, when implemented the context was perceived as a *wrapper* for the mathematics (Stillman 2002) and subsequently was thrown away by the student cohort with the intended real-world task degenerating into solving a difficult division problem in the eyes of the students. In a revised task, context could no longer be ignored, and students proceeded to engage with a much more interesting problem, albeit still having difficulties.

Stillman's analysis (2002) of senior secondary students' responses to application tasks included consideration of how contextual demand of tasks impacted on the mathematical demands of the tasks and how this facilitated or impeded student progress on such tasks. She found contextual demand was a gatekeeper to the mathematics of the task or overwhelmed mathematical demand; cognitive demand was sometimes mediated by mathematical expediency and also adapting of familiar methods to meet the mathematical demand of the task. The Year 11 students in Stillman's study worked independently in a clinical setting. As the students in the study reported here worked collaboratively in a classroom setting, it will be interesting to confirm if similar findings are evident.

18.3 Methods

An extended task involving linear models was undertaken by a class of 24 Year 9 students. The students (approx. 14 years old) had access to TI-graphing calculators and laptop computers during task implementations. The study reported here is part of a larger study (Brown 2013b) where student data were collected from this class during four extended tasks and a lesson sequence on linear functions during one school year. Analysis presented here relates to data from the third task. This task was implemented to conclude the lesson sequence on linear functions. A mainly qualitative approach was used in the study in order to best provide a comprehensive picture of what was occurring in the classroom. Following Stake (1995), a case study using an instrumental approach was undertaken. Data sources included audio- and video-recordings, student scripts and observational notes.

Students worked on the task in pairs for two consecutive lessons on day 1 and, in a third lesson, 2 days later (150 min total), when pairs collaborated as groups of four. There were five different parallel versions of the task, involving either the cost of running small village health clinics or the cost of installing safe drinking water wells. The tasks were designed by the classroom teacher, Peter, with the intention that the task context was important and of interest. The different versions were to allow student pairs who worked in relative isolation from others with the same version of the task for the first two lessons to collaborate with those who worked on the same version (e.g. *Health Clinic in Mali*). Task implementation occurred when the school had a week focused on 'classroom outreach', as part of which the school donated money to clinics overseas. In addition, the context was personal for Peter, as he had previously worked in Africa as a geologist prior to his teaching career. One student pair, Kit and Rani, was video- and audio-recorded over the three lessons. Ben and Ken were audio-recorded on day 1. On day 2, Ken was absent, and Ben was audio-recorded in his new group with Amy and Anna. A third pair, Kate and Meg, was also audio-recorded over the three lessons. All recordings captured contributions of other students as the class worked though the task. Student scripts were collected.

18.4 Analysis Framework

These data have previously been analysed using a framework of affordances with a focus on understanding (linear) functions in a technology-rich teaching and learning environment (Brown 2013b). Now these are reconsidered with a focus on identifying the impact of context on understanding. The intention is that in crossing boundaries (Garraway 2010) and revisiting the data from a different perspective, more can be learned. The aim of data analysis is to determine if the use of context allows one to more clearly ascertain the level or depth of student's mathematical understanding. The collaborative nature of task solving as an enabler of deepening current understanding during task solution is also investigated.

In the middle section of the task, students were provided with linear models for costs at two different health clinics (or water wells). They then generated a table of costs for each, examined the table to determine where costs were approximately the same and generated an additional table ‘zooming in’ with their calculators to determine more accurately where costs were equivalent. Next came the question of most interest here. Students were asked for each health clinic: *What is the cost for treating each patient?* In the *Water Well Task*, the parallel question was: *What is the cost for each metre of well depth?*

In responding to these two questions, students have to mathematise the context. It is not possible to understand what the question is asking without doing so. If the students had been asked ‘what is the gradient’ or ‘what is the rate of change’ for a particular linear model, it is probable (as evidenced by lesson sequence data prior to this task) they could do so. However, the question posed by the task required students to firstly recognise that the cost per additional patient is the rate of change of the linear cost model. In situations such as this where students engage with context, their mathematical understanding becomes apparent (to themselves or to a teacher or researcher). The mathematical understanding evident through interaction with the context falls into three broad categories. These are (i) lacking, (ii) apparent fragility and (iii) apparent sound understanding. In situations where understanding is either lacking or fragile, the question arises: When does student interaction with the context allow students to progress in their mathematical understanding?

In analysing student scripts, it became clear that correct written responses often showed no evidence of actual mathematical understanding. Given the collaborative environment, a student recording a correct response may or may not have a sound understanding of the relevant mathematics – rate of change of a linear model. In contrast, an incorrect response, even where no details were shown, could be used to determine what understanding students had in the moment. The video- and audio-recordings proved invaluable in ascertaining how students interacted with the context to determine their written response.

Interthinking has been used by Stillman et al. (2015) in analysing mathematical modelling by senior secondary students. Interthinking is described by Hunter as “pulling students into a shared communicative space” (2012, p. 3), whereby they articulate their thinking, reasoning and beliefs about mathematics and the task context and use their own articulation as well as that of others within their groups in task solving. Group contributions including “erroneous thinking, doubt, confusion and uncertainty” (p. 3) have potential to contribute to collective understanding. Mercer (2000) notes that both dissension and consensus can be a catalyst for progress, as negotiation leads to articulation of thinking and reasoning.

In focussing on collaborative metacognitive activity during problem-solving, Goos et al. (2002) searched for patterns of social interaction that led to metacognitive activity. The relevant metacognitive acts addressing particular aspects of the solution coded in the transcripts included *assessment of execution or appropriateness of a strategy, a result, knowledge or understanding*. Subsequently, Galbraith (2013) categorised as *metacognitive impasse* incidents when no amount of reflective thinking or strategic effort was successful in releasing a blockage in the solution process. These italicised categories will be used to examine collaborative activity during interthinking from the video- and audio-transcripts.

18.5 Analysis of Data

Fifteen students successfully recorded a correct response; however, from their scripts, it was not clear in most cases how the solution was determined. As noted, the written scripts provided little explanation of student thinking. The exceptions were Kate and Meg. Meg wrote, "for each additional patient it costs $19.75, because it's the gradient and that means it's the rate it goes up by". Kate also made her method clear – she calculated $y(2) - y(1)$ and wrote "each additional patient cost $17.50". Thirteen students simply recorded a correct numerical response. In some cases the audio record confirmed this was an indication that the mathematisation of the task context was fully understood. At least five students calculated $C(1)$ (i.e. cost for one patient or metre of well); however for one (Kit) this value was not recorded as her (final) solution. Video evidence confirmed that this was an initial interpretation by some students, but engagement in group interthinking as they tried to make sense of the particular contextual interpretation this involved led to revisions of their thinking until they came to a meaningful contextual understanding that they were able to connect to a full mathematical understanding. Another incorrect approach taken was to find $C(10) \div 10$ as shown by Kit and Rani.

18.5.1 Contextual Demand as Gatekeeper on Mathematisation

When Kit and Rani reached question 7a (*What is the cost for each metre of well depth in Gondar?*), both read the question, Kit then looked at the model they were given for well costs in Gondar: COST = 250 + 17.50*metres drilled. Using her calculator she found $250 + 17.5 \times 1 = 267.50$. Kit's initial interpretation was to find the cost of drilling 1 m. She recorded $267.50 and then noticed Rani had recorded $42.50 [i.e. $C(10) \div 10$]. In the ensuing discussion, it is clear neither student understood the connection between the context and the mathematical model or what mathematical concepts such as gradient represented.

Kit: *How do you know that?*

Rani: It is the same thing as that one but plus 1 not plus 10 … [points to her table showing $C(10) = 425$.] Plus 1, instead of 10. Isn't it just for 1 metre? … What did you get?

Kit: But I got $267.50 *which seems a bit sort of wrong*. [*Assessment of interim result*]

Rani: That seems a bit big doesn't it? [for] only 1m? [*Assessment of interim result* Rani]

Kit: It would be less that that. [*Assessment of interim result* Kit]

Rani: Because half would have been less than that and that would be 5 [Meaning $C(10) \div 2 = C(5)$]. Can't you just divide that by ten? [meaning $C(10) \div 10$] … Can't you just divide that by ten? To get one? [*Assessment of appropriateness of strategy*, Treating cost function as $C(n) = \mathrm{M}n$ assuming a proportional model]

Kit: I don't think it is actually all even. [*Assessment of appropriateness of strategy*. She doesn't think it is a proportional model showing mathematical understanding.]

Rani: What do you mean?

Kit: Well see it goes $42 and 50 cents. … It is low. Oh, okay. [She crosses out $267.50 and replaces it with $42.50.] It is low.

Rani: We can check with the other guys later. We just got to finish first.

Kit: Right. [pause] I think it is sort of wrong. [*Assessment of interim result/ Appropriateness of strategy*] [Reads related question for the second water well] …

Rani: *I don't think we are going to get it out.* [*Assessment of chances of success*] *Do you know what you are doing?*…Do you know what you are doing?… Did you skip (b)?

Kit: Yes. [Nothing is recorded for b.] [*Metacognitive impasse*]

In assessing their understanding and strategic resources to progress, it becomes obvious to them that they have met a *metacognitive impasse*. Although they indicate they will consult the boys, the boys are uncooperative and remain at the front of the room whereas they are sitting at the back. They eventually decide to stop working and not consult the other group as their previous overtures to do so were rebuffed.

18.5.2 Collaborative Engagement with Context: Development of Understanding

When undertaking genuine collaborative work on the task and engaging with the context, Kate and Meg were able to show progression in their mathematical understanding of the model, even though their initial encounter with the question *What is the cost for each additional patient?* may have resulted in a blockage.

18.5.2.1 First Encounter: Context as Blockage

Meg's initial interpretation appeared to be that they were required to show the costs for consecutive numbers of patients beginning with one patient. Kate contextualised this within the domain they have just been exploring to identify where two clinics had the same costs.

Meg: What do they mean, *What is the cost for treating each additional patient at Timbuktu?* [pause] Do you just go one? … Would you just go 1, 2, 3, 4, 5, like so we can find out the additional ones? Instead of like doing 50, 90 and stuff? [50 and 90 refer to the number of patients given in the original problem statement]

Kate: Just go 120, 121, 122. Umm, I have no idea. [*Assessment of resources (knowledge)]*

Meg: What will we do? [*Assessment of resources (strategy)]*

Kate: I don't know. … [pause] I don't get this question. [*Assessment of understanding*]

Meg: No. Let's just skip it. [Long pause] [*Metacognitive impasse*]

The girls decided to leave the question for the time being. In assessing their understanding and strategic resources, it became obvious to them they had met a *metacognitive impasse* as occurred with Kit and Rani.

18.5.2.2 Second Encounter: Negotiating Contextual Meaning

They return to the question in the third lesson when they are joined by Di and Ann. Meg asks: "What about that one?" During the initial exchange, Meg seems to be quite willing to accept Di's interpretation to find $C(1)$ the cost of treating one patient.

Di: [Reads] What is the cost for treating each additional patient? You just, ooh.

Meg: So you times it by one? [meaning let the number of patients be one]

Di: Yes, you times it by one, because *there is only one patient*. Then just get the difference between each cost. [Di calculated $C(1)$ for each clinic.]

Meg: Ooh. Thanks. What is this? One? All right. Cool. Okay.

Although appearing to accept Di's interpretation, Meg was unable to draw Kate into the same meaning as she articulated her thinking whilst working through the calculations.

Meg: I think you just, I am going to put in just one. [Hesitantly following Di's advice]. … Okay, $19.75 \times 1 + 115$, [pause] whatever.

Kate: How come you are doing it times 1?

Meg: Umm, because when you find each additional patient after, like from [pointing to 115 which is the fixed cost for the model], you go up by one.

Kate: [*Unconvinced*] Oh yeah.

Meg: It is hard to explain. Each time it goes up by. Each time it adds on to the 115.

As Meg was having difficulty communicating her understanding to Kate, she directed Kate to ask Ana from another group. Kate asked, "Well okay, so basically n [her symbol for number of patients] is just 1". Unfortunately Ana confirmed Di's misinterpretation as Ana and her partner had calculated the cost for one patient "because there is only one patient". It appeared to this point that all the coming together of the groups had achieved was the spreading of misinterpretation of the context. However, suddenly, Meg articulates a new interpretation when she realised that previously, she was considering she had to find the cost one at a time, whereas now she interpreted it as for ‘each additional one’. Kate asked for clarification (what) and explanation (why) and challenged Meg to articulate her reasoning.

Meg: Wouldn’t it be 20? [*New information*]
Kate: What? [*Clarification question*]
Meg: I thought it asked to *do one*, but [it actually asks] to *do each additional one*. So what if you times it by two and then take away what you got for one. [i.e., [C(2) – C(1)]. Like then it is about 20. It will go up by 20 each time you treat somebody.
Kate: Why 20? [*Explanation question*]
Meg: Because you times. If you treated 2 patients you add. Yeah you get how much it would cost and then you take that away from one [i.e., [C(2) – C(1)].
Kate: Wait, what are you saying? You go 19.75 ×1 + 115. [pause] You do that, right? … Then what? [*Clarification question*]
Meg: Then there is like one, and then there is like two. And there is like the difference. I think the difference there is like 20. … *That is how many if you treat two patients*. That is how much it costs. … when you get up to three it goes up by another 20. Yeah I think [*Assessment of interim result*]

The pair continued this approach and found $C(2) = 154.5$, followed by $C(2) - C(1) = 154.5–134.75 = 19.75$. However, Kate clearly indicated she still did not really understand even after they have worked through the calculations together.

Kate: Yeah. *But what is the answer? Is it that thing?*
Meg: Yeah. 154.5. Yes, it is kind of like, no wait, [long pause]. I got 19.75 for that.
Kate: Maybe you did it wrong.
Meg: Oh no. It is [pause] Oh no, it is 19.75. … I was [incoherent] *yes, I know. Yes!*

18.5.2.3 Coming to Know: Realisation of Meaning

By asking herself the reason the numerical answer was 19.75, Kate suddenly had an “Aha moment” (Mason et al. 1982) when she connected the contextual meaning of $19.75 as the additional cost per patient at the clinic to the mathematical objects she had previously been dealing with, namely, $C = 19.75n + 115$ which was how she had symbolised the given model for the weekly operational cost in Timbuktu. She thus engaged in articulating her sense-making for gradient, making her full mathematical understanding of the concept apparent. Kate chides herself for not having made this meaningful connection, but it soon became apparent that Kate’s understanding was rather fragile.

Kate: How come it is 19.75? *Oh my god! That is the gradient it goes up by each time.* [*New Information*] So it *is* 19.75. So it is like it is just *b* that lifts it up?
Meg: *That is the gradient?*

Kate: Yes, the gradient goes up by, each time it goes up by 19.75! Oh my god!! … Yeah, der, because it is *ax*. [laughs] Der. [laughs]

Meg: Yeah I know. [laughs]

Kate: [working on calculations] [long pause] Meg? … Is what I did here right? Or wrong?

Meg: Nah, it is wrong.

Kate: Oh what? Because if you take that from that, it is 19.75 anyway.

Meg: *Each additional patient costs 19.75, because that is what the gradient is.* The gradient is, whenever you go up by one, no across, then you treat one more patient.

Kate: Yeah I know, I get it.

Meg: So it is 19.75

Kate: For each additional patient. So it is not that [$C(1) = 134.75$]?

Meg: No, that is just how much it costs to treat one patient.

Kate: Yeah, Okay. So this one (7b, C = 17.50n + 390) would just be 17.50.

The interthinking that has been manifested whilst the pair articulated their different understandings of the contextual and mathematical meanings in their 'shared communicative space' has allowed this group to progress their mathematical understanding from engaging with the task context until the mathematics also had meaning for them.

18.5.3 Full Immediate Meaning of Mathematisation of the Context

Ben and Ken were working on *The Village Health Clinic in Angola Task* when they reached Q7a. In this first encounter with the question, the boys' full understanding of the gradient as a means of mathematising the context was evident as Ben's immediate reaction shows. 'We have already answered this question [Q7a]. Oh no we haven't its [records $14.00 and then immediately records $12.50 for the second clinic.]' Ben returned to the question in the third lesson when he was joined by Amy. In this second encounter with the question, the contextual meaning is also apparent. Both clearly considered it a trivial question, with Amy facetiously suggesting it was the hardest question.

18.6 Discussion

Three broad categories of mathematical understanding were evident through interaction with the context. These were (i) a lack of mathematical understanding, (ii) apparent fragility of mathematical understanding and (iii) apparent sound understanding. These were manifested in the three scenarios as: *contextual demand as* *gatekeeper on mathematisation* [(i)], *collaborative engagement with context→development of mathematical understanding* [(i)→(ii)→(iii)] and *full immediate meaning of mathematisation of the context* [(iii)]. Using metacognitive acts as a tool for analysis allowed this researcher a window into the interthinking of the second group in particular, where a true collaborative communicative space was evident.

The articulating of their thinking by Kit and Rani as they conducted multiple metacognitive *assessments of interim results and of appropriateness of strategies* clearly indicates neither fully understood the contextual meaning of cost per metre of well depth. Furthermore, it is clear neither student understood the connection between the context and the mathematical model **nor** what particular mathematical concepts such as *gradient represent*. Rani did not understand her mathematisation as proportional thinking although Kit indicated that she understood that the relationship was not proportional, when she said it is not 'all even'. This pair *did not really use interthinking to bring more meaning*. Their lack of engagement with the context and/or lack of mathematical understanding could be the barriers. In assessing their understanding and strategic resources to progress, it became obvious to them they had arrived at a *metacognitive impasse* where no amount of reflective thinking or strategic effort would be successful in releasing their blockage. For this pair the contextual demand was a gatekeeper on mathematisation.

In stark contrast, collaborative engagement with the context enabled Meg and Kate to shift from an apparent similar starting point to that of Kit and Rani. Meg and Kate provide a paradigmatic example of how interthinking can be used to advance understanding. They progressed from their first encounter, *context as blockage* to a second encounter, *negotiating contextual* meaning, and finally their third encounter, *coming to know – realisation of meaning*. When undertaking genuine collaborative work on the task and engaging with the context, students were able to show progression in their mathematical understanding of the model, even though their initial encounter may have resulted in a blockage. The *interthinking* that was manifested whilst the pair articulated their different understandings of the contextual and mathematical meanings in their 'shared communicative space' allowed progression of their mathematical understanding from engaging with the task context until the mathematics also had meaning for them. Ben and Ken's immediate understanding of both the mathematics and Ben and Amy's link with the context made it unnecessary for them to engage in collaborative discussion.

18.7 Conclusion

The analysis presented here clearly showed that engagement with the real-world context offered opportunities to develop mathematical understanding. However, articulation of thinking, including metacognitive assessments, was not sufficient to realise this potential. Genuine collaborative work involving interthinking and engaging with the task context was necessary to develop mathematical reasoning.

Audio- and video-recording seems essential to ascertain the interthinking and hence development of mathematical understanding. It is hoped the illustration from this study will act as a boundary device (Garraway 2010) that enables the passage of knowledge between insiders and outsiders of the ICTMA community with respect to the value of using real-world context in the classroom.

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Chapter 19
Difficulties in Teaching Modelling: A French-Spanish Exploration

Richard Cabassut and Irene Ferrando

Abstract An exploratory research study based on an online questionnaire in order to better understand difficulties encountered in teaching of mathematical modelling in France and Spain is presented. This questionnaire takes into account biographical variables, mathematics and modelling conceptions of pre-service teachers, teachers, researchers and education inspectors, from primary to tertiary education. Regarding difficulties, heterogeneous conceptions are revealed, and the analysis constructs four clusters, from positive and confident conceptions to negative and lacking confidence. In some cases, the roles of some biographical variables are indicated such as country, age, gender or school level that need to be clarified by future semi-structured interviews in order to offer training and resources in response to the expressed difficulties.

Keywords Modelling • Teaching difficulties • France Spain comparison • Beliefs

19.1 Context of the Study

A new curriculum (Ministère 2015) in France for primary and lower secondary education is discussed. It started in 2016 when modelling became one of the six main components of mathematical activity and practical interdisciplinary teaching through collective projects became compulsory. In this context, it is interesting to look for results of research about the difficulties of implementing modelling activities in France in order to produce resources and training to prepare for the new curriculum.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_19

We did not find such research except Cabassut and Villette (2011) who studied difficulties met by teachers attending a training course on mathematical modelling. This study sought to determine an overview of difficulties felt by French educators to teach mathematical modelling, from primary school to tertiary education.

19.2 Theoretical Background and Design of the Questionnaire

Two previous questionnaires, Maass and Gurlitt (2009) and Borromeo Ferri and Blum (2013), focusing on the teaching of modelling were used. Teacher interviews from a previous project – LEMA project described in Cabassut and Mousoulides (2009), and a review of literature on the difficulties in the teaching of modelling are specified in Cabassut and Ferrando (2015). These enable identification of variables influencing the teaching of modelling.

Regarding biographical variables, Borromeo Ferri and Blum (2013) show differences between teachers studying mathematics as a subject or not. Cabassut and Villette (2011) identify country, age and type of school as split variables. Kuntze (2011) and Borromeo Ferri and Blum (2013) show the influence of experience in teaching. Dorier and García (2013) point to the importance of initial training.

Regarding conceptions and practice about mathematics, we follow Cabassut and Villette (2011), Lee (2012), and Maaβ and Gurlitt (2009). Conceptions and practice about modelling were based on Borromeo Ferri and Blum (2013). We used a framework suggested by Philipp (2007) about conceptions and another by Kaiser (2006) and Mischo and Maass (2013) about beliefs on modelling. Consideration was also given to self-confidence and self-efficacy, as developed by Bandura (1997), helping to express teachers' practice.

Regarding difficulties with modelling, we follow Borromeo Ferri and Blum (2013), Schmidt (2011), Engeln et al. (2013) and Cabassut and Villette (2011). To better understand the specificity of the French situation, we chose a comparative approach (Cabassut 2007) involving a comparison with Spain. This country has a regional organization contrasting with the central organization in France. In addition, teachers' education and training, resource production, syllabus construction and school organization are different.

The questionnaire on difficulties on modelling is composed of 85 multiple-choice questions with 67 four-point Likert scales: 8 on biography, 23 on mathematics' conception and practice, 25 on modelling conception and practice and 37 on modelling difficulties split into six parts: time, assessment, lesson organization, context, student's involvement and resources. It is an online questionnaire, advertised through different national networks, and was completed between February and March 2015 by 231 people, including 124 French and 107 Spanish people. This sample was not constructed on a representative basis but on an exploratory one: people answered the questionnaire voluntarily. An exploratory approach, according to Tukey (1977), means a representative sample is not needed. In any case, our sample included pre-service teachers, teachers, mathematics education researchers, education inspectors, resource writers and teacher trainers. You will find the questionnaire at the following link http://ir.uv.es/fepai/modellingquestionnaire.

19.3 Method of Analysis of Results

The statistical analysis was carried out with SPAD software which provides frequency tables, cross tabulation and cluster analysis. The variables were the different answers to the questions. In the cluster analysis, the *active variables* to build the clusters were the 37 variables on modelling difficulties. To interpret the clusters produced by the software, an active variable was considered as a splitting variable if it was an answer for which the percentage of answers in the cluster was very different from the whole population. Every cluster will be described with these *splitting variables*. The other variables (biography, mathematics conception and practice, modelling conception and practice) are *illustrative variables*. For every cluster, some were *split variables* when the percentage of the answer in the cluster was very different from the whole population. The split variables of every cluster helped to describe the cluster. The splitting and split variables enabled explanation of the heterogeneity and differences that were not shown by aggregated statistics indicators. From the LEMA research (Cabassut and Villette 2011), we kept the split and splitting variables for our questionnaire.

19.4 Results

19.4.1 Frequency Analysis

Biographical variable percentages were 46% Spanish people versus 54% French people and 52% men versus 48% women. For age, people were separated into two categories (younger, older) with a sample median value of 42 which gave 49% of people as older and 51% younger. The same process was applied for the number of years of service in the profession: the median value was 15 from which two categories (less experienced, more experienced) were produced resulting in 44% of people classified as more experienced and 56% as less experienced. For type of job, there were 2% noospherian people (supervisor, inspector, etc.), 23% primary school teacher, 23% secondary school teacher and 23% tertiary education teacher. Regarding the nature of the studies they have followed, 25% of respondents had education studies, 58% mathematics studies and 15% sciences studies.

With respect to modelling or mathematics teaching, 57% considered their teaching conditions as difficult, 80% understood the meaning of "modelling problem" and 58% considered they used in their teaching problems that were simultaneously open, complex, related to the real world, solved with mathematics and an inquiry-based approach (this corresponds to the definition of modelling given in the design of the questionnaire and based on our review of literature). Furthermore, we found that 67% of people were motivated to teach modelling.

A frequency table analysis of the modelling difficulties enabled sorting by the difficulties expressed by more than 50% of the people. The main difficulties were related to time, student involvement and resources. For 70% of the respondents, it is difficult to estimate how long it takes to solve a modelling task. For 58%, it took too much time to prepare modelling tasks for teaching. In addition, 54% of respondents thought that most students do not know how to work out modelling tasks. For 55%, modelling tasks required a lot of extra things that teachers could obtain only at great expense. Lastly, 51% of respondents thought that teachers do not have enough materials for modelling tasks.

It was also possible to identify positive aspects through the frequency table analysis; these were related to evaluation, lesson organization and student involvement. Firstly, 77% thought that modelling tasks promote greatly student autonomy. Secondly, 57% felt able to support students in developing competencies in arguing related to modelling tasks. Thirdly, 56% felt able to use students' mistakes to facilitate their learning in modelling. Fourthly, 50% of respondents considered that modelling tasks promote, at the same time, both low achievers and high achievers.

19.4.2 Cluster Analysis

A cluster analysis produced four clusters. *The first cluster* (74 individuals, 32% of the sample) represented people who had difficulties in mathematics and modelling and were negative towards modelling. In this cluster, much more than in the whole population, they did not feel able to design modelling lessons that could help students overcome difficulties in all modelling steps, to support students in developing competencies in arguing related to modelling tasks, to develop detailed criteria for assessing and grading students' solutions to modelling tasks, to effectively assess students' progress as they worked on modelling tasks, to use students' mistakes to facilitate their learning in modelling, to design their own modelling tasks and to adapt tasks and situations in textbooks to provide realistic open problems. Much more than in the whole population, they agreed with the following statements about modelling tasks: it is difficult to assess the presentation of a solution; it is difficult to differentiate what is correct from what is not correct; assessment takes too much time; it is difficult to assess group work, the solutions found by the pupils or the students are not comparable; the presentation of the solutions is complex; these require complex operations that primary school children cannot cope with; most students do not know what to work out by modelling tasks; it is difficult to manage group work; and modelling lessons are unpredictable.

To describe this cluster with illustrative variables, we meet tertiary education people and trainers much more than in the whole population. Much more than in the whole population, they have difficulties in mathematics teaching (not only in modelling teaching) to use small group work and to assess it, to use an inquiry-based approach, to use open problem-solving and to have heterogeneity; they consider it important in mathematics teaching to apply official curriculum or training programs; they rarely use modelling problems, problems to be solved with an inquiry-based approach, or authentic problems from reality; and they disagree that a modelling problem at school is an open problem.

It seems surprising to find tertiary education or trainee people over-represented. Is it because these people have a demanding theoretical conception of modelling? Is it because they receive university students with a low level in mathematics and in problem-solving? These people seem to consider small group work, an inquiry-based approach and open problem-solving as a difficulty for mathematics teaching not specific to modelling. As a follow-up, we will interview representatives of this cluster to answer these questions.

The second cluster (46 individuals, 20%) represented people who were positive towards modelling and generally did not feel difficulties about modelling. In this cluster, much more than in the whole population, they disagreed that, in a modelling task, it is difficult to assess group work; that most students do not know how to work out modelling tasks; that when teaching modelling, not enough time is left for other learning content; that it is difficult to manage group work by modelling task; that the pupils or students are hard to discipline during modelling activities; that in a modelling task, it is difficult to assess the presentation of a solution of a modelling task; that when pupils or students work on a modelling problem, the environment in the class becomes harder; that modelling tasks require complex operations which primary school children cannot cope with; that it takes too much time to assess modelling tasks; that the lessons are unpredictable by modelling; that in a modelling task, it is difficult to differentiate what is correct from what is not correct; that the presentation of the solutions is complex; and that working on modelling tasks in the classroom is very time-consuming. They felt able to design modelling lessons that help students overcome difficulties in all modelling steps, to develop detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks, to effectively assess students' progress as they work on modelling tasks, to support students in developing competencies in arguing in relation to modelling tasks, to design their own modelling tasks as teachers and to adapt tasks and situations in text books to provide realistic open problems.

To describe this cluster with illustrative variables, much more than in the whole population, there were Spanish people, people motivated to teach modelling and people who considered it easy in mathematics teaching (not only in modelling teaching) to use small group work and to assess it, to use an inquiry-based approach and to have heterogeneity; they often used modelling problems, complex problems, an inquiry-based approach and open problems; they considered it as important to use open problems and to work in small groups. This was the only cluster where a country (Spain) was over-represented. There was a similar result in the study by

Cabassut and Villette (2011). Is there a cultural or institutional explanation why Spanish people are more positive towards modelling? Future interviews will try to explain this. It would be interesting to clarify by interviewing cluster representatives about the relation between the lack of difficulties for modelling and the practice and importance of small group work, the inquiry-based approach and open and complex problems in mathematics teaching.

The third cluster (85 individuals, 37% of answers) represented people who were positive about modelling and neutral on difficulties. In this cluster, much more than in the whole population, they felt able to use students' mistakes to facilitate their learning in modelling and to support students in developing competencies in arguing in relation to modelling tasks, and they agreed that modelling tasks promote greatly students' autonomy and that students acquire a lot of knowledge about the use of mathematics in modelling tasks. Much more than in the whole population, they were neutral to feel able to design modelling lessons that help students overcome difficulties in all modelling steps (e.g. problems in validating), to adapt tasks and situations from textbooks to provide realistic open problems and to develop detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks. Alternatively, they were neutral about the following statements: most students do not know how to work out modelling tasks; it takes too much time to assess modelling tasks; and the lessons are unpredictable with modelling.

To describe this cluster with illustrative variables, much more than in the whole population, there were men, teachers, secondary school teachers and people who have studied mathematics, who knew what modelling meant and who considered that a modelling problem at school is related to the real world. In comparison with the previous cluster, we observed that secondary school teachers and people who had studied mathematics were over-represented in this cluster. Are the secondary school teachers more neutral on difficulties because the implementation in the secondary class is balanced between difficulty and easiness? Why are men over-represented? Interviews will try to explicate these over-representations.

The fourth cluster (24 individuals, 11.5% of sample) represented people who were neutral towards modelling and its difficulties. Much more than in the whole population, these were neutral on the following statements: modelling tasks promote greatly students' autonomy; most students do not know what to work out by modelling tasks; students acquire a lot of knowledge about the use of mathematics in modelling tasks; I feel able to support students in developing competencies in arguing related to modelling tasks; students recognize that often, there is not only one right solution; the pupils or students are hard to discipline during modelling activities; students have difficulty with the fact that there are many different solutions for modelling tasks; modelling tasks promote at the same time both less powerful and more powerful students; I feel able to use students' mistakes to facilitate their learning in modelling; in a modelling task, it is difficult to assess the presentation of a solution of a modelling task; when pupils or students work on a modelling problem, the environment in the class becomes harder; in a modelling task, it is difficult to differentiate what is correct from what it is not correct; when teaching modelling, I am not left enough time for other learning content; I feel able to design modelling lessons that help students overcome difficulties in all modelling steps; the solutions found by the pupils or the students are not comparable; in a modelling task, it is difficult to assess group work; students can use the openness of the tasks to handle them well; and it is difficult to manage group work for modelling tasks.

To describe this cluster with illustrative variables, much more than in the whole population, there were trainee, primary school teachers, people who did not know what "modelling problem" meant, people who did not use the Internet to find modelling problems and people who were neutral to be motivated to teach modelling. Perhaps trainees are over-represented because they have not enough experience to analyse modelling teaching. For primary school teachers, their neutrality is more surprising. Is it because they do not know the concept of modelling? Is it because they are less used to have feedback on such practices? Future interviews will help to explain this.

19.4.3 Conjectures from Chi-Squared Test

Use of Chi-squared tests helped to identify whether there was a significant relation between two variables. Clearly, this allowed us to find some differences related to country or gender. We observed that women felt less able than men to design modelling lessons that help students overcome difficulties in all modelling. Regarding the country, Spanish people felt more able than French people to design modelling lessons that helped students overcome difficulties in all modelling steps. In the LEMA cluster analysis of Cabassut and Villette (2011), we found also that Spanish teachers were more positive about modelling than other teachers. This cultural fact is difficult to explain. There was no significant difference between countries and biographical variables. We also observed some differences in relation to modelling difficulties: people who had difficulties in their mathematics teaching with an inquiry-based approach also had difficulties with modelling in relation to time, evaluation, lesson organization and resources; people who had difficulties in their mathematics teaching with small groups also have difficulties with modelling in relation to evaluation, resources and students involvement; and people who have difficulties in their mathematics teaching with open problems also have difficulties with modelling in relation to students' involvement. Some of these significant results (e.g. about country variable) have to be explained further by interviews. In the following table, we provide the data from the chi square analysis (Table 19.1).

19.5 Conclusion and Perspective

In the studied population, we have pointed out heterogeneity about position on modelling and on difficulties to teach modelling. The majority of people were positive about modelling. Some difficulties about modelling can be explained more

Table 19.1 Results of χ^2 analysis

Variables	df	χ^2	Significance level
Gender I feel able to design modelling lessons that help students overcome difficulties in all modelling steps	2	15.75	0.000
Country I feel able to design modelling lessons that help students overcome difficulties in all modelling steps	2	19.13	0.000
Inquiry-based approach is difficult in mathematics teaching The work on modelling task in the classroom is very time-consuming	4	17.35	0.002
Inquiry-based approach is difficult in mathematics teaching I don't feel able to develop detailed criteria related to the modelling process	4	12.11	0.017
Inquiry-based approach is difficult in mathematics teaching Modelling tasks are unpredictable	4	11.00	0.027
Inquiry-based approach is difficult in mathematics teaching I don't feel able to design modelling tasks	4	19.45	0.001
Small group work is difficult in mathematics teaching It is difficult to assess the presentation of a solution of a modelling task	4	20.63	0.000
Small group work is difficult in mathematics teaching I don't feel able to develop detailed criteria related to the modelling process	4	14.83	0.005
Small group work is difficult in mathematics teaching Most of the students do not know what to work out by modelling tasks	4	11.65	0.020
Open problem-solving is difficult Students have difficulty with the fact that there are many different solutions by modelling tasks	4	14.67	0.005

Note. df means the number of degrees of freedom

generally by difficulties in mathematics teaching. For some people though, difficulties were specific to modelling, especially those related to time, students' involvement and resources. Cluster analysis or chi-squared test results suggest that some variables (gender, country, difficulties in mathematics teaching, type of school, type of education and type of job) could play a role in the difficulties to teach modelling.

In a follow-up study, SPAD software will be used to select ideal examples from different clusters. For every ideal example, we will conduct an additional clarifying semi-structured interview, as described in Cabassut and Ferrando (2015), to confirm the previous conjectures suggested by cluster analysis and chi-squared tests. We propose also to have confirmatory analysis on national representative samples. The samples could be based on the type of school because in France, resources and training productions are strongly related to the type of school (primary, secondary or tertiary education). In the comparison between France and Spain, we found that Spanish people felt more able to design modelling lessons that help students overcome difficulties in all modelling steps. For the other questions, we did not find strong evidence of the role of country variable. Perhaps the comparison with other countries or the additional clarifying semi-structured interviews could change this point of view.

It is important to clearly identify specificities of each country and commonalities in order to avoid generalizations that can hinder the use of modelling. Our study could lead to the design of specific material to prepare teachers to overcome difficulties related with the use of modelling in classrooms. For example, in the frequency analysis, we observed that time was the main domain of difficulties: time necessary to prepare teaching of modelling, yearly time planning of this teaching, time for modelling assessment, and time management of modelling activities for teachers and for students. This means that for resources available to teachers (by considering teachers' training as one of these resources), time is one of the topics to be investigated.

Acknowledgement This research is funded by the University of Strasbourg and the IDEX mission, the Spanish Ministerio de Economía y Competitividad under research EDU2012-35638 project and also Generalitat Valenciana through project GV2016-129.

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Chapter 20
How Students Connect Mathematical Models to Descriptions of Real-World Situations

Dirk De Bock, Nele Veracx, and Wim Van Dooren

Abstract Research has shown that problem posing, in a sense the "inverse activity" of problem-solving, can positively affect students' problem-solving skills. We report the design and results of an empirical study in which the potential positive effect of a specific problem-posing variant, "inverse modelling", (i.e. the selection of a real-world situation given a mathematical model), on modelling was investigated. Eighty 11th grade students were randomly divided into two equal-sized subgroups, one first receiving a modelling task and then an inverse-modelling task. The other subgroup received both tasks in reverse order. Results indicated that inverse modelling did not have an overall positive effect on modelling: Only for affine functions with negative slope, accuracy scores for modelling significantly improved after inverse modelling.

Keywords Affine model • Inverse modelling • Inverse proportional model • Modelling skills • Problem posing • Proportional model

20.1 Theoretical and Empirical Background

The initial idea for this chapter was found in the literature on problem posing. Since the middle of the 1980s, problem posing has received ample attention in the international mathematics education literature (see, e.g. Brown and Walter 1983), leading to vast amounts of research-based findings nowadays (a recent state of the art of research on mathematical problem posing can be found in Singer et al. 2015).

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_20

Basically, problem posing involves the generation of new problems or the reformulation of given problems and can be seen, at least in some sense, as the "inverse activity" of problem-solving. The literature on problem posing is extensive and mentions several potentialities of this activity on students' learning. In the context of this chapter, it is however impossible and also needless to review this literature in detail, so we limit ourselves to a discussion of some important characteristics and perspectives, especially those that are relevant for our research action.

First, although problem posing seems to be the "inverse activity" of problem-solving, it can also be an integral part of it and as such exert a direct positive impact on the problem-solving process and outcome. For instance, asking a student to reformulate a problem in his or her own words can help the student to better understand the problem. Also, in another sense, problem posing can be a valuable heuristic tool for problem-solving. As already recommended by Pólya (1945), in order to solve a challenging problem, it can be helpful to look for an analogous problem that is easier to solve, in the expectation that solving this easier problem may be useful for solving the original problem.

A second potential advantage of problem posing is that it can foster mathematical creativity (Silver 1997). Although it is by no means easy to define mathematical creativity (Sriraman 2009), it seems an important aspect of mathematics learning. There is a parallel with language learning in which the creative interaction of students with language is stimulated by asking them to write stories or essays. Problem posing, the invention of new problems, can be seen as a quite similar activity in mathematics education (Ellerton 1986).

Third, problem posing is seen as a tool to appeal to students with diverse talents and interests. Problems can indeed be created from students' own experiences following their personal interests. According to some authors, problem-posing tasks are also less "intimidating" than problem-solving tasks (Brown and Walter 1983). A problem that has been created by a student is seldom right or wrong; there is always room for discussion and interpretation. Some researchers even state that problem posing can help some students to overcome mathematical fear or anxiety (Moses et al. 1990).

Fourth, and probably the most important in the context of this research, problem posing can positively affect students' problem-solving skills on later tasks (Silver 1994). Problem-posing tasks, as well as other types of alternative tasks such as classification tasks (Van Dooren et al. 2010), can stimulate students to think more deeply about underlying mathematical concepts and relations. In such tasks, students are not prompted to immediately start calculating or applying formulas. As such, they are less inclined (and in some cases even unable) to fall back on routine behaviour (De Bock et al. 2007) and instead think more thoroughly about the kind of mathematics that is applicable. So, problem posing can change the focus of students' mathematical thinking.

In recent years, the mainstream of mathematics education problem-posing research was enriched from the perspective of modelling. Authentic real-world mathematical modelling includes, among other things, that even in schools, modellers themselves are allowed to find the problem situation and to pose the problem(s).

From this viewpoint, problem posing (i.e. the specification or formulation of the problem) in a real-world situation occurs when a problem is formulated in such a manner that it is amenable to mathematical analysis in the sense that mathematisation leads to mathematical models that are solvable. Over the past few years, several members of the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), names including, among others, C. Bonotto, A. Downton, L. English and G. A. Stillman, have argued that a modelling view on problem posing, which is not the generally held view on problem posing in mathematics education research, is an important component of teaching and learning mathematics and also an essential part of mathematical modelling (Downton 2013).

Research at different levels of schooling has however shown that the step from a modeller's understanding of a problem situation into a mathematical model(s) is far from an obvious one (Verschaffel et al. 2000). The general aim of our research is to find ways to make modellers in training more proficient in taking that step. For some problem types, such as division with remainder problems, problem-posing activities have been shown to have a positive impact on taking that step (e.g. Chen et al. 2007). Inspired by the positive outcomes of the research on problem posing and on other alternative tasks in mathematics education, we started thinking about an inverse activity of modelling, namely, the reformulation of a given mathematical model into a real-world situation. We called this activity "inverse modelling", and the aim of the current study was to test the potential beneficial effects of such "inverse modelling" on modelling (in its restricted sense, namely, the formulation of a real-world situation into a mathematical model). In operational terms, our research question can be stated as follows: Does the prior confrontation of student modellers with an "inverse modelling" activity have a beneficial effect on their modelling capacities?

20.2 Method

Eighty 11th graders, most of them aged 16–17, from different secondary schools in Flanders (Belgium) participated in this study. Solving realistic problems and the applicability of basic functions, including those that were involved in this study (see next paragraph), received quite some attention in the participants' previous mathematics courses. All participants followed general education with 3 or 4 h of mathematics per week which is the minimum for general education in Flanders. It was a deliberate choice to work with students who are not in the top streams of education for mathematics because we wanted to avoid ceiling effects. The eighty participants were randomly divided into two equal-sized subgroups. Both subgroups were confronted with a multiple-choice test consisting of a modelling part and an "inverse-modelling" part, but Subgroup 1 received the modelling part first followed by the "inverse-modelling" part. Subgroup 2 received the two parts in the reverse order.

The modelling part consisted of eight items in which a real-world situation was described in words and participants had to connect them with an appropriate model

For a fundraising event, an action committee wants to peel a full container of potatoes. This job will take them several hours. Which formula properly denotes the relation between the number of committee members who collaborate and the time needed to finish this job?

□ $y = ax$

□ $y = ax + b, a > 0$

□ $y = ax + b, a < 0$

□ $y = a/x$

Fig. 20.1 Example item from the modelling part (inverse proportional situation)

that could be either proportional (i.e. of the form $y = ax$), affine with positive slope ($y = ax + b$ with $a > 0$), affine with negative slope ($y = ax + b$ with $a < 0$) or inverse proportional ($y = a/x$). Each model was appropriate for two of the eight given items or situations. The four types of models were used previously by Van Dooren et al. (2013), but in contrast with that study, models were always given in a formula representation. The choice of a formula representation was deliberate. Research has shown that formulas are more difficult for students than, for example, graphs or tables (see, e.g. De Bock et al. 2015). This is not surprising because a formula is a more formal or abstract representation. By choosing formulas, we also hoped to avoid ceiling effects and obtain more variation in our results which facilitates interpretation. Figure 20.1 shows an example item from the modelling part of the test. Test items were specifically constructed for this study, and situations were chosen so that there was always a clear and strong fit with the provided models, although we are aware that models never perfectly fit to a realistic situation. We are also aware that these tasks are not very "rich", nor open-ended, as generally recommended for authentic modelling activities, and, more particularly, we acknowledge that mathematical modelling involves much more than connecting a situation with a formula from a list of given formulas, but for the internal validity of our research, we preferred to use this type of clean, closed-form items.

The "inverse-modelling" part also consisted of eight items, but the participants now had to connect one of the four types of models with a description in words of a real-world situation for which that model was appropriate. In Fig. 20.2, an example item of the "inverse-modelling" part is shown. In both subgroups, the same realistic situations were used, so differences between subgroups could not result from the situations that were used. To prevent that students' choice of a specific model (given a situation) of their choice for a situation (given a model) would depend on their familiarity with the situation (and thus not only on an underlying mathematical model), we used two parallel versions of both the modelling and "inverse-modelling" part. So, we tried to neutralize the effect of specific contexts as much as possible. So, the same context (e.g. about mobile phones) could appear with a certain model in one version of the test but with another model in the other version of the test. Data were analysed by a repeated measures logistic regression analysis, using the generalized estimating of equations (GEE) procedure within SPSS (Liang and Zeger

Choose which one of the following descriptions best fits the formula "$y = ax + b$, $a < 0$".

- □ A taxi company charges for a night ride a fixed fee upon departure and an amount for each kilometre driven. The formula properly denotes the relation between the total price of the night ride and the number of kilometres driven.
- □ A group of friends participates in a gambling game. When they win some money it will be shared equally among the friends. The formula properly denotes the relation between the number of friends and the amount of money each person will receive.
- □ Jennifer buys minced meat at the butcher's shop. The formula properly denotes the relation between the amount of minced meat that Jennifer buys and the price she has to pay.
- □ Thom has a mobile phone subscription, but uses prepaid reloadable cards. Per minute talked the uploaded sum decreases by a fixed amount. The formula properly denotes the relation between the number of minutes talked and the remaining sum on the card.

Fig. 20.2 Example item from the "inverse-modelling" part

1986). This procedure allows analysis of repeated (and thus possibly correlated) categorical observations within series of individuals and to appropriately correct for inferences that can be drawn from such correlated measures. Given the dichotomous nature of the dependent variable (i.e. a particular response alternative is chosen or not), a logistic regression, modelling the probability that a correct response is given, depending on the type of model (proportional, inverse proportional or affine with positive or negative slope) and the condition (first modelling and then inverse modelling or vice versa), is appropriate.

20.3 Results

First, we briefly discuss the results on both the modelling and the "inverse-modelling" task in the whole sample. Table 20.1 shows the accuracy rates for the modelling task. These results are disappointing. Although for all types of situations the appropriate model was most frequently chosen, only half or less than half of the participants made that correct choice. Unexpectedly, the modelling task appeared too difficult for most of the participants. Given the rather low difficulty level of the task, we rather had feared to be confronted with ceiling effects, and we made choices in our design accordingly. Possible explanations relate to the fact that participants were rather weak in mathematics and that they were not sufficiently prepared to this task: The test was taken unannounced, and no specific teaching had preceded the test. Also, the fact that participants were 11th graders, while in Flanders the subject matter about basic functions and their properties and applications is part of the mathematics curriculum in grades 9–10, may have negatively affected the results.

Table 20.1 Accuracy rates (in bold) and other choices (both in %) on the modelling task in the whole sample

		(Chosen) model			
		P	IP	A+	A−
(Given) situation	P	**39**	33	21	8
	IP	22	**50**	19	9
	A+	40	13	**41**	6
	A−	16	31	16	**36**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) situations and models

Surprisingly, the inverse proportional situations were modelled correctly in 50 % of the items wherein such situation was given, which is more than all other types of situations. This result was also unexpected because other studies have reported students' difficulties with decreasing functions in general and with inverse proportional functions in particular (see, e.g. De Bock et al. 2015). The regression analysis indeed revealed a significant effect of the "model" variable ($p = 0.031$), and pairwise comparisons showed that the differences in accuracy scores between the inverse proportional model and the proportional and the affine model with negative slope were significant ($p = 0.026$ and $p = 0.007$, respectively). Also interestingly, 40% of the affine situations with positive slope were modelled proportionally. This relatively high percentage confirms students' well-documented "overreliance on proportionality" (see, e.g. Van Dooren et al. 2008), in particular their difficulties to discriminate between proportional and affine increasing functions (De Bock et al. 2015). An additional error analysis revealed that participants' overuse of proportionality in affine situations with positive slope was significantly higher than their overuse of proportionality in the two types of decreasing situations ($p = 0.000$ and $p = 0.001$ for the affine decreasing and inverse proportional situations).

The accuracy rates, for the "inverse-modelling" task, are shown in Table 20.2. Also, these results are disappointing. Possibly the same explanations can be given as for the weak results on the modelling task. Furthermore, the high reading load for this part – participants had to read and understand $8 \times 4 = 32$ descriptions in words of real-world situations – may also have negatively affected the results. Similar to the results on the modelling task, for all models the appropriate situation was most frequently chosen, but again and unfortunately, less than half of the participants made that correct choice. Moreover, we observed no clear trends in these results. On the basis of the results for the modelling task, one could have expected that the students would confuse proportional and positive affine situations, but this was not confirmed by these results.

Second and most importantly, we discuss the potential beneficial effect of "inverse modelling" on modelling. Therefore, we compare in Tables 20.3 and 20.4 the accuracy rates on the modelling task for the two subgroups, subgroup 1 who started with the modelling task and subgroup 2 who was first confronted with an "inverse-modelling" task. At first glance, it appears that the "inverse- modelling" task had little or no positive effect on the accuracy scores of the modelling task. For

Table 20.2 Accuracy rates (in bold) and other choices (both in %) on the inverse modelling task in the whole sample

		(Chosen) situation			
		P	IP	A+	A−
(Given) model	P	**37**	18	29	16
	IP	21	**36**	20	23
	A+	19	17	**41**	23
	A−	14	11	31	**43**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) models and situations

Table 20.3 Accuracy rates (in bold) and other choices (both in %) on the modelling task in Subgroup 1

		(Chosen) model			
		P	IP	A+	A−
(Given) situation	P	**40**	33	21	6
	IP	24	**58**	16	3
	A+	38	15	**41**	6
	A−	16	43	15	**26**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) situations and models

Table 20.4 Accuracy rates (in bold) and other choices (both in %) on the modelling task in Subgroup 2

		(Chosen) model			
		P	IP	A+	A−
(Given) situation	P	**38**	34	20	9
	IP	20	**41**	23	16
	A+	43	10	**41**	6
	A−	16	19	18	**48**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) situations and models

proportional and inverse proportional models, the trend was even negative. This was confirmed by the results of the regression analysis that did not reveal a main effect of the "condition" variable. So, on the basis of these results, our research question should be answered negatively: "Inverse modelling" does not have an overall positive effect on students' modelling capacities.

Although less important, we note that there was a significant interaction effect between "condition" and "model" ($p = 0.002$). Pairwise comparisons revealed that this interaction effect was due to the accuracy rates for the two types of decreasing models. There was a significant increase of the accuracy rates for the negative affine model ($p = 0.006$), but a significant decrease of the accuracy rates for the inverse proportional model ($p = 0.031$) as a consequence of doing the inverse-modelling task first.

Table 20.5 Accuracy rates (in bold) and other choices (both in %) on the inverse modelling task in Subgroup 1

		(Chosen) situation			
		P	IP	A+	A−
(Given) model	P	**44**	14	31	11
	IP	15	**40**	20	25
	A+	20	21	**35**	24
	A−	16	9	24	**51**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) models and situations

Table 20.6 Accuracy rates (in bold) and other choices (both in %) on the inverse modelling task in Subgroup 2

		(Chosen) situation			
		P	IP	A+	A−
(Given) situation	P	**30**	23	26	21
	IP	28	**33**	20	20
	A+	18	13	**48**	23
	A−	13	14	39	**35**

Proportional (P), inverse proportional (IP), affine with positive slope (A+) and affine with negative slope (A−) models and situations

For the sake of completeness, we also compare in Tables 20.5 and 20.6 the accuracy rates for the "inverse-modelling" task in the two subgroups. Once again, no clear trend was revealed: Accuracy rates increase or decrease, dependent on the type of model involved. Also, the regression analysis did not reveal a main "condition" effect, but there was an interaction between "condition" and "model" ($p = 0.041$). Pairwise comparisons revealed that only the improvement of the accuracy scores for the negative affine model type was significant ($p = 0.028$).

20.4 Conclusions and Discussion

This study did not reveal an overall positive effect of "inverse modelling" on modelling. Only for affine functions with negative slope were accuracy scores for modelling significantly improved after inverse modelling. Possible explanations are that both tasks were too difficult for the participants, who were weak performers in mathematics, who had met the relevant subject matter one or 2 years earlier in their mathematics curriculum and who were not at all prepared for this type of task. In particular, an "inverse-modelling" task, in this study operationalized by asking participants to link descriptions in words of realistic situations to mathematical models, is very likely a kind of task they were never confronted with in their preceding school careers. Moreover, the reading load for the "inverse-modelling" task was high.

We think it would be premature to already abandon the idea of "inverse modelling". Although the results of this study are not encouraging, "inverse modelling" may have some potential for the development of students' modelling capabilities, and this might be shown in follow-up research. So, it could be considered to rerun the current study on a larger scale – the current sample of 80 students spread over two conditions was rather small – and, more importantly, under improved conditions, for example, by using "easier" function representations (graphs or tables instead of formulas), by briefly revising the relevant subject matter and by first solving and discussing an example item together with the participants. For such follow-up research, the reported study may serve as a pilot study.

Since just confronting students with an "inverse-modelling" task in a multiple-choice testing context will probably not suffice to obtain really convincing positive results, it could also be considered to conduct more "ecologically valid" research. Instead of confronting students with "real" situations that were constructed by the researchers, we could think about a research design in which students are invited to think freely about situations in their own environment that can be modelled with specific types of functions. Such open-ended tasks are more likely to elicit deeper thinking processes about the link between functions and real-life situations, whereas the multiple-choice format may have elicited superficial thinking and even guessing, in some students.

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Chapter 21
Mathematical Modelling Strategies and Attitudes of Third Year Pre-service Teachers

Rina Durandt and Gerrie J. Jacobs

Abstract This chapter reports on the thinking and planning strategies of a group of 38 third year mathematics student teachers (preparing to teach Grades 10–12), who were exposed to mathematical modelling for the first time. Participants, in eight comparable groups, attempted a textbook-based traffic flow model-eliciting activity. The open-ended nature of the task, handling intra-group dynamics, construction of appropriate equations and interpretation of findings were the most pressing challenges. Participants' attitudes towards modelling, attained via a post-questionnaire, were very positive, and all appreciated the mathematics-in-the-real-world exposure. Findings of students' planning strategies, experiences and attitudes contributed to a set of guidelines aimed at the integration of mathematical modelling into the pre-service education of mathematics teachers.

Keywords Mathematical modelling • Model-eliciting task • Thinking and planning strategies • Problem-solving • Attitudes towards mathematics • Mathematics student teachers

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_21

21.1 Context and Purpose

Mathematics teachers' knowledge involves components such as knowledge of mathematics, of mathematical representations, of students and of teaching and decision-making (Fennema and Franke 1992). According to Shulman (1986), these components emphasise mathematical content knowledge (MCK) and pedagogical content knowledge (PCK). Mathematical concept formation and learning initially depend on the classroom environment and learner activities, with teachers' attitudes, knowledge, judgements and beliefs impacting on these. Chapman (2002, p. 177) confirms 'It has become an accepted view that it is the [mathematics] teacher's subjective school-related knowledge that determines for the most part what happens in the classroom'. Teacher education programmes therefore have a huge role to play in steering and shaping prospective teachers' subjective school-related knowledge.

Authentic problem-solving is increasingly used in enhancing learners' mathematical competencies and mathematics teachers' PCK and MCK (Buchholtz and Mesrogli 2013). The relationship between mathematical modelling and authentic learning has been proven (Kang and Noh 2012). South Africa's Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education 2011, p. 8) for Grade 10–12 mathematics specifies 'Mathematical modeling is an important focal point of the curriculum. Real life problems should be incorporated into all sections whenever appropriate'. Various researchers (e.g. Ikeda 2013; Ng 2013) caution against the unpreparedness of mathematics teachers in teaching modelling, recommending formal exposure to it during their pre-service education.

The *first goal* of this study is to investigate the thinking and planning strategies of a group of mathematics pre-service teachers, who are exposed to a model-eliciting activity. A *second goal* is to explore their experiences of, and attitudes towards, modelling. Findings will contribute to a set of guidelines aimed at the integration of mathematical modelling into the pre-service education of Grade 10–12 mathematics teachers.

21.2 Literature Perspectives

A review of the literature indicates a disagreement about the potential influence that teacher education has on teacher learning (Boaler 2000; Lampert 2009). Some critics question whether teachers learn anything of value during their pre-service education, while others claim that the effects of these programmes have been reversed once teachers enter more predictable school settings. The authors are of the opinion that the pre-service education of mathematics teachers, especially in the current South African school context, has a fundamental influence on their MCK and PCK. Aligned with this assumption, the theoretical framework that underlies this inquiry relates to two complementary sets of perspectives. The *first* is the Learning to Teach Secondary Mathematics (LTSM) framework (Peressini et al. 2004). LTSM declares that *how* a learner acquires a particular set of knowledge and skills and the specific teaching context in which it happens fundamentally influence what is eventually learned (Greeno et al. 1996). LTSM assumes that teachers' initial knowledge, beliefs and attitudes interact with their work in practice. This implies, in the words of Adler (2000), that mathematics teacher education is 'usefully understood as a process of increasing participation in the practice of teaching, and through this participation, a process of becoming knowledgeable in and about teaching' (p. 37). The *second* set of perspectives is underlined by the zone of proximal development (ZPD), originally defined by Vygotsky (1978) as 'the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers' (p. 33).

A *model* is a visualisation of something that cannot be directly observed via a description or a resemblance (Kang and Noh 2012). Whereas the end-product is known as a model, the cognitive activities preceding it which involve and require reasoning can be labelled as *modelling*. This cyclical process involves a provisional model and a series of interactive activities that should be continually tested and refined in order to improve or verify it (Kang and Noh 2012). *Mathematical modelling* as a process of generating mathematical representations in attempting to solve real-life problems consists of four sequential phases (Balakrishnan et al. 2010), namely, 'mathematisation' (representing a real-world problem mathematically), 'working with mathematics' (using appropriate mathematics to solve the problem), 'interpretation' (making sense of the solution in terms of its relevance and appropriateness to the real-world situation) and 'reflection' (examining the assumptions and subsequent limitations of the suggested solution) (p. 251). Researchers (Kang and Noh 2012) acknowledge three different levels of modelling tasks. Traditional problem-solving fits the description of a so-called level 1 problem. Such problems are already carefully defined, no additional data is required to formulate a model and the problems require specific mathematical procedures. Problems at level 2 have a slight vagueness as insufficient information needed to successfully complete the task is given. Level 3 problems are the most authentic and open-ended type, characterised by unstructuredness and a challenging level of complexity (Ng 2013).

Since 2011, modelling is a prescribed Grade 10–12 mathematics theme, according to the CAPS document (2011). Suitable model-eliciting tasks, with a focus on the process and not necessarily the product (Kang and Noh 2012), are exactly the kind of exposure that students require in striving to attain the envisaged learning outcomes. Research in Singapore (Ng 2013) and South Africa (Julie 2002) reveal that teachers' lack of prior experience in problem-solving and their (sometimes too conventional) beliefs about mathematics are major obstacles, when they are exposed to modelling activities. According to Ng (2013), 'The teachers generally perceive mathematics to be formula-based involving linear track solutions' (p. 346) and imply that they are mostly challenged by model-eliciting tasks. In this regard, preservice teacher education programmes have a pertinent responsibility to fulfil.

21.3 Research Design

21.3.1 Research Paradigm

The *constructivist-interpretivist* paradigm enabled the researchers to collect data on the lived experiences of the participants, via their individual and/or shared exposure to, and involvement in, a modelling task (Creswell 2013). The inquiry also incorporated a quantitative dimension from a *post-positivist* stance (Heppner and Heppner 2004). This dimension enabled the researchers to measure participants' attitudes towards mathematical modelling.

21.3.2 The Model-Eliciting Experiment

An in-class experiment was conducted in May 2014 involving 38 third year mathematics pre-service teachers. They were exposed to a modelling-eliciting activity in groups, and afterwards, their views of their group's problem-solving strategies, as well as their lived experiences and attitudes towards modelling, were gained. The experiment was conducted during one time slot (of almost 2 h) in the timetable. The participants had little formal mathematics teaching experience (approximately 5 weeks of school practice) and had not been exposed to model-eliciting tasks nor to the teaching of such tasks. Proportional stratified sampling was employed to randomly assign them to one of eight groups (of four to six members), in such a way that each group at least had a high(er), a moderate and a low(er) achiever (based on their mathematics marks in the module). The session began with a brief presentation on the goal and nature of the experiment, focussing on modelling, phases of a typical modelling cycle and the ethical measures taken to safeguard the confidentiality of collected data and the anonymity of each participant. Individual written participant consent was obtained, also in respect of their individual feedback, the next day.

A modelling task typically requires participants to ask relevant questions, to identify variables and their relations to a real-world situation, to represent the latter in mathematical 'language' and to propose and to validate a solution (Niss et al. 2007). The eventual purpose of the data collected via the experiment was to deduce an initial set of guidelines aimed at the integration of modelling into the formal education of mathematics teachers at the university in future. The authors shared the views of Niss et al. (2007) that a carefully selected model-eliciting task also requires cognitive, meta-cognitive and affective competencies. A level 3 open-ended modelling task would put these abilities of participants more deliberately to the test. A task on *traffic flow*, an adaption of a textbook problem (Stewart et al. 2012, p. 661), was thus chosen. It involved traffic flow data on a busy section of a city's street network. Participants were requested to recommend the best location for a day-care centre for toddlers.

Traffic Flow Task
The Department of Town and Regional Planning would like to receive a recommendation on the best location for a day-care centre for toddlers. Provide the department with a plan on how they can select the best location. You need to explain the method you used as the department would like to apply this method to other areas. Data collected from the local traffic department gives information on a section of the city's street network (one-way streets and how many cars enter or leave this section of the city via the indicated street in a certain 1-h period):

Both the first and second streets are one-way streets from north to south. Third Avenue is a one-way street from west to east, but Fourth Avenue is a one-way street from east to west. Street corners are identified at Third Avenue and First Street, Third Avenue and Second Street, First Street and Fourth Avenue and Second Street and Fourth Avenue. One hundred eighty cars enter First Street. Seventy cars enter Second Street. Two hundred cars leave First Street. Thirty cars leave Second Street. Two hundred cars enter Third Avenue. Four hundred cars enter Fourth Avenue. Twenty cars leave Third Avenue. Two hundred cars leave Fourth Avenue.

The information in the task is offered in an ambiguous manner and did not explicitly suggest a specific approach nor a well-rehearsed mathematical pathway, presenting it as a level 3 modelling problem according to Kang and Noh (2012). In order to provide possible solution strategies, participants were expected to carefully examine task constraints and systematically analyse information. Taking into account the complexity of the task, the inexperience of the pre-service teachers and the relatively limited time, groups were not expected to come up with well-defined solutions nor to provide their views on the representativity, validity and applicability of their 'answers'. Groups were merely required to report on the strategies and methods that they employed. The experiment and group interactions were carefully monitored, and each group recorded their strategies, processes and suggested solutions on a predesigned worksheet. The researchers initially also planned that each group should critique their suggested solutions. As the experiment unfolded, it was realised this was definitely a bridge too far.

21.3.3 Collection and Analysis of Data

The day after the experiment, individual participant feedback was collected. A self-designed questionnaire was used for this purpose. *Section A* contained demographical items (gender, ethnical group, home language, age and their performance in mathematics at school) used to construct a participant profile. Two additional items gained information on participants' experience of mathematics at school and the reason(s) they were studying towards becoming mathematics teachers.

Section B used selected items from the Attitudes Towards Mathematics Inventory (ATMI, Schackow 2005). The original focus of the ATMI, namely, to detect participants' attitudes towards mathematics as a subject, was geared towards mathematical modelling, but the items were kept intact. Two of the ATMI's dimensions are, namely, *enjoyment* (whether mathematical problem-solving is and the modelling challenge was enjoyable, ten items) and *self-confidence* (expectations about doing well in modelling and how easily the task was performed, 15 items). Participants provided Likert scale-type responses to each of the 25 items, ranging from 1 (*strongly disagree*) through 3 (*neutral*) to 5 (*strongly agree*). Responses were summed, yielding total scores of maximum 50 and 75 for the enjoyment and self-confidence dimensions, respectively. ATMI data were captured and analysed via the Statistical Package for the Social Sciences (SPSS, version 22). A pilot study (involving three mathematics students, who were not participants) contributed to the questionnaire's face and content validity. Cronbach's alpha coefficients were hence calculated, and the coefficients for the enjoyment dimension (0.745), the self-confidence dimension (0.922) and the total ATMI (0.917) revealed high internal consistency.

Section C included four open-ended questions, detecting participants' experiences of the model-eliciting task and of modelling in general. The last question requested concrete suggestions on how participants might be supported during their education in becoming effective modellers and teachers of modelling. Feedback was analysed via the constant comparative method, a directed form of content analysis (Durandt and Jacobs 2013). Appropriate participant views per category, by quoting their direct words, have been integrated into the findings. Particular strategies to enhance the trustworthiness of the qualitative component of the research, in accordance with Creswell (2013), involved *transferability* measures (a thorough description of the experiment's planning and implementation, properties of the participants and the data collection instrument), *dependability* measures (a dense description of the content analysis method) and *credibility* measures (a proper interrogation and triangulation of the findings by both researchers), while the original records were maintained for follow-up purposes.

21.4 Findings and Discussion

The majority of the participants were male (63%), black (76%), indigenous language speaking (74%) and 23 years or younger (61%) and scored 60% or more for mathematics in the examination designed for their final year of high school (79%). Their answers to the question: 'What is the main reason(s) underlying your decision to become a mathematics teacher?' echoed their sentiments to sustain a relationship with the subject mathematics. Main categories of responses were an interest in mathematics and the resulting curiosity and challenges it generates and the

Table 21.1 Distribution of ATMI scores

ATMI dimension			ATMI dimension		
Enjoyment (mean = 44.7)	N	%	Self-confidence (mean = 64.3)	N	%
46–50	13	34.2	68–75	12	32.4
40–45	19	50.0	60–67	16	43.2
36–39	6	15.8	59 or lower	9	24.3
Total (mean = 109.0)			113–125	13	35.1
			100–112	18	48.6
			75–99	6	16.2

opportunity to make a difference to learners in disadvantaged communities (who lack good mathematics education) and to positively contribute to South Africa's educational challenges.

Using Sweeting's (2011) categorisation (pp. 53–54), positive scores on the *enjoyment* dimension (out of 50) have a minimum of 40 and on the *self-confidence* (out of 75) dimension a minimum of 60. A positive total for the two dimensions (out of 125) would thus be minimum 100. Table 21.1 provides a breakdown of participants' ATMI scores on the two dimensions as well as the total scores.

The researchers expected the majority of the participants (as they do want to become mathematics teachers) to portray a relatively positive disposition towards mathematical modelling. More than four fifths of them (84.2%) seem to have *enjoyed* the model-eliciting task, while the activity also boosted the *self-confidence* of three quarters (75.6%) of the group. The score distribution on the *total* ATMI is sufficient reason to describe six out of seven participants' attitude towards modelling as positive to strongly positive. Although the ATMI is a self-rating survey (which is a limiting factor), the strong relationship between a positive attitude towards, and achievement in, mathematics has been documented (Sweeting 2011).

Although the first phase of the modelling cycle (*mathematisation*) as described by Balakrishnan et al. (2010) was fairly well mastered by participants, the second phase (*working with mathematics*) resulted in a number of difficulties. During the first phase, all eight groups succeeded in representing the traffic flow problem mathematically. Five groups used more than one format to present the data in a mathematical context. All eight groups also used illustrations (e.g. illustrated in Fig. 21.1a) together with either a histogram (one group) or a double bar graph (one group) or two-way tables (three groups). In phase two, most groups experienced difficulty in introducing variables and in matching them to unknown quantities. Initially, the majority of groups introduced two variables, one for the number of cars entering and another for the number of cars leaving the city's street network. They only realised later that the number of cars entering an intersection (from various directions) must equal the number of cars leaving that intersection. In setting up their mathematical models, four variables (e.g. x, y, w and z) were required. The variables represent the number of cars (from all four directions) travelling along a specific street. Most groups felt really challenged working with four variables. The researchers had to intervene and guided most groups in setting up a first and even a

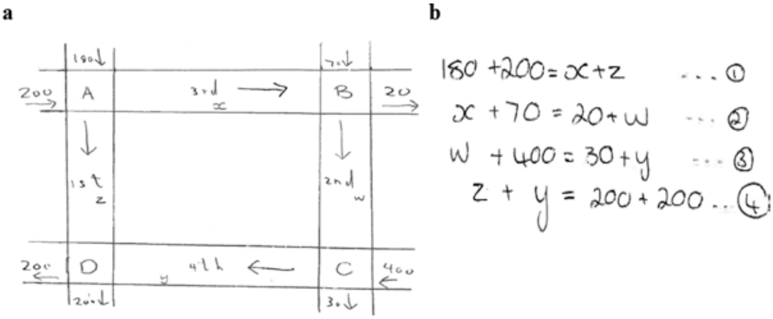

Fig. 21.1 A selected group strategy on (**a**) mathematisation and (**b**) working with mathematics

second equation. All groups could thereafter formulate the third and fourth equations (e.g. illustrated in Fig. 21.1b). Another complication in the second modelling phase was finding appropriate mathematics to solve the equations. Four of the groups attempted to solve the system of linear equations; but only one group eventually provided a probable solution, while another group introduced a more sophisticated mathematical strategy, involving matrices. As a result most groups could not straightforwardly make sense of the solution and got stuck moving from the second phase to the third and fourth phases of the modelling cycle.

A further interrogation of their submitted worksheets revealed that half of the groups made a recommendation as to the most appropriate location of the day-care centre. One group argued in favour of the intersection with the highest traffic flow (being more convenient for working parents), while two groups supported exactly the opposite (an intersection with the lowest traffic volume). Another group juxtapositioned convenience (for parents) versus safety (for toddlers) and thus recommended a medium busy intersection. Only three groups found time to critique their solutions (models) and also made suggestions to improve their own models.

Participants described their lived experiences and also made suggestions on enhancing their ability to approach (and perhaps even solve) model-eliciting tasks effectively in future. Their experiences were dominated by the overwhelming open-ended nature of the modelling task and its consequential challenges. Participants reported that group members struggled to agree on an idea and to get everyone's point of view across. Most groups found it extremely difficult to mathematise the task. Even after formulating and attempting to solve the equations, the interpretation of their findings was still a bit confusing as some participants were not convinced about their validity. Despite the challenging nature of the task, participants acknowledged the opportunity to experience mathematics in the real world. A number of suggestions to assist pre-service mathematics teachers in becoming good modellers and effective modelling teachers were made. The crux revolves around a need for guidelines on how to approach model-eliciting tasks, more frequent exposure to modelling activities (and to examples and their solutions), more group work opportunities and more time on tasks and even to present a lesson on mathematical modelling themselves.

21.5 Conclusion

The literature is filled with references to the positive relationship between mathematical modelling and authentic learning (e.g. Buchholtz and Mesrogli 2013). Mathematical modelling has been a theme in South Africa's Curriculum and Assessment Policy Statement for Mathematics in the Further Education and Training (Grade 10–12) phase since 2011. Not only the underpreparedness of mathematics teachers to teach but also to grasp modelling is a global phenomenon (see, e.g. Ng and Lee 2015). Several calls for the exposure of mathematics pre-service teachers to modelling tasks during their pre-service education have been made (e.g. Tan and Ang 2013). Not only are prospective mathematics teachers expected to model mathematical modelling; but also, they should be able to cultivate a climate conducive towards modelling in their classrooms.

In this study, a group of third year mathematics pre-service teachers was exposed to a model-eliciting task. The thinking and planning strategies of these pre-service teachers in attempting the task and thereafter their lived experiences and attitudes towards modelling were explored. The inquiry revealed that it was not only a very challenging task for the participants, but also it was indeed very difficult for them to link the 'world out there' (reality) to the mathematics of the classroom. The real dilemma was captured in their search to find appropriate mathematics (mathematisation) to solve the problem. Although their first exposure to modelling might have been extremely perplexing, the participants also regarded it as thought provoking, inspiring and motivational. The words of one of the most eager participants perhaps capture the group's attitude towards modelling appropriately: 'We want more, although we realise that it won't come easy'.

In preparing prospective mathematics teachers more optimally to grasp and also to teach modelling, several suggestions were made by the participants. The suggestions will be converted into guidelines focused on enhancing prospective mathematics teachers' abilities to attempt and eventually solve model-eliciting tasks effectively in future. The researchers are of the opinion that, based upon the Learning to Teach Secondary Mathematics (LTSM) framework, in conjunction with the zone of proximal development (ZPD), mathematics pre-service teachers should formally acquire modelling knowledge and skills during their formal education. This should ideally happen in teaching contexts (situations), which not only let them experience for themselves that mathematical modelling but also mathematics teaching is not a formula-dependent, linear-track endeavour but indeed much more authentic, open-ended and, from time to time, even thrilling.

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Chapter 22
Exploring the Notion of Mathematical Literacy in Curricula Documents

Peter Frejd and Vincent Geiger

Abstract The notion of mathematical literacy has gained momentum internationally recently through the influence of international assessment regimes such as the Programme of International Student Assessment (PISA) and national concerns about the ability of citizens to use mathematics effectively in personal, civic and work life. Accordingly, it is to be expected that these concerns should be reflected in relevant curriculum documents. This chapter presents a content analysis of a sample of 12 national curriculum documents in relation to mathematical literacy. The analysis shows that there does not appear to be general agreement about the definition of mathematical literacy within the analysed documents and that the idea of mathematical literacy is represented in a limited fashion.

Keywords Mathematics education • Mathematical literacy • Numeracy • Curricula documents • Content analysis

22.1 Introduction

Members of the International Community of Mathematics and its Applications have been responsible for generating a significant corpus of literature related to mathematical modelling (Geiger and Frejd 2015). However, less attention has been paid by this group to research related to mathematical literacy – another theme within the field of mathematics education that focuses on the use of mathematics in the real world, that is, a perspective within applications of mathematics.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_22

The notion of mathematical literacy (known as numeracy in some international contexts) has been promoted through increasing international attention to the results of the Programme for International Student Assessment (PISA), sponsored by the OECD. Within the PISA framework, mathematical literacy is defined as:

> an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD 2009, p. 5)

This definition positions the application of mathematics as an essential capability for participatory and productive citizenship. Further, the statement emphasises the critical aspect of applying mathematics to the real world when making decisions and judgments.

While PISA's definition is the most widely recognised, other descriptions of mathematical literacy are embedded in curriculum documents across the world and implemented for different purposes in a variety of ways (Geiger et al. 2015; Jablonka 2003; Niss and Jablonka 2014). Such differences appear to be associated with cultural influences and economic and sociopolitical priorities within nations (Jablonka 2003; de Lange 2003). Different curriculum documents also make note of a varying range of basic skills, mathematical concepts and critical capabilities needed for engagement in everyday life and the workplace. Because of the complexity of this situation, Niss and Jablonka (2014) argue that more empirical research is needed in order to document the range of variations in implementing mathematical literacy.

The aim of this chapter is to provide insight into the commonalities and differences between interpretations of mathematical literacy internationally. In addressing this aim, we will attend to the following research questions that guided our study.

- How frequently are terms related to mathematical literacy present in curricula documents across the world?
- How do curriculum documents describe the nature of mathematical literacy?

22.2 Mathematical Literacy

What the term mathematical literacy means varies between the policy documents of educational jurisdictions and also across international assessments such as PISA. Numeracy, for example, may either be conceptualised as the use of basic arithmetic procedures or in terms of problem-solving within authentic contexts (Geiger et al. 2015). Additionally, there exist several closely connected notions, such as *numeracy*, *quantitative literacy*, *critical mathematical literacy*, *mathemacy* and *matheracy* (Geiger et al. 2015; Niss and Jablonka 2014; Stacey and Turner 2015), that add complexity to any discussion in this area. A complicating factor in attempting to outline the characteristics of mathematical literacy in school curricula is that the term is not found in all languages. In Swedish, for example, the words *litterat* (literate) and *illitterat* (illiterate) do exist, but they are not frequently used in everyday conversations and, in particular, are not used in relation to mathematics.

To deal with this complexity, a number of authors have attempted to identify the fundamental characteristics or constituent aspects of mathematical literacy (e.g. Jablonka 2003; Geiger et al. 2015). Jablonka (2003), for example, draws on the literature of mathematics education to identify five perspectives that "attempt to categorise different and, in some cases, conflicting ingredients of mathematical literacy" (p. 80): *developing human capital*, *maintaining cultural identity*, *pursuing social change*, *creating environmental awareness* and *evaluating mathematical applications. Developing human capital* addresses the need for 'all' to have the capability to use academic mathematical knowledge in out-of-school situations, whereas *maintaining cultural identity* links to the capability to use informal mathematical knowledge in social and cultural activities. *Pursuing social change* is a category of mathematical literacy related to an ability to analyse, with mathematics, different aspects of social realities to contribute to and make impact on political debate as an informed and critical citizen. The widespread discussions of global environmental problems give rise to *creating environmental awareness*, which is related to the ability to work in an interdisciplinary manner and to include the use of technology. *Evaluating mathematical applications* is an important aspect of mathematical literacy that includes assessing the reliability and limits of mathematical models.

An alternative view is offered by Geiger et al. (2015), who identified five aspects of mathematical literacy (numeracy) research: *critical views of numeracy*, *numeracy in the workplace*, *the role of technology in numerate activity* and *statistical and financial literacy*. The term, *critical views of numeracy*, relates to the capacity of numerate citizens to participate in society in ways that promote equity, ethical conduct and the greater good – an aspect clearly related to Jablonka's (2003) category of *pursuing social change*. *Numeracy in the workplace* is aligned with Jablonka's (2003) broader category of *developing human capital*. The use of technology is a vital part of workplace practice and is increasingly integral to classroom teaching and learning practice in mathematics, as described in *the role of technology in numerate activity*. The last aspect of Geiger et al.'s (2015) portrayal of numeracy relates to *statistical and financial literacy*. Statistical literacy concerns capabilities related to interpreting, evaluating and communicating statistical information. The ability to carry out financial transactions and make financial decisions is integral to financial literacy. Thus, while there are commonalities in these two perspectives on mathematical literacy, there are also characteristics that do not overlap.

Despite different perspectives, the underlining goal of the mathematical literacy agenda is to promote "awareness of the usefulness of and the ability to use mathematics in a range of different areas" (Niss and Jablonka 2014, p. 391). It is important to acknowledge, however, that the focus of this goal is "the general public rather than with specialized academic training while at the same time stressing the connection between mathematical literacy and democratic participation" (Niss and Jablonka 2014, p. 392).

In summary, there is no broadly accepted definition for mathematical literacy (and related notions). However, most understandings of mathematical literacy coalesce around a focus on building a capacity to use mathematics to participate effectively in society and to contribute in a productive and critical manner.

22.3 Methodology

This study is based on a content analysis of curriculum documents from a range of nations, conducted in order to draw conclusions about the relative visibility and meaning of mathematical literacy internationally. Our approach to the content analysis has two aspects. First, we document the frequency of the term mathematical literacy (and associated expressions). Second, we discuss a range of descriptions for mathematical literacy in different curricula including one example where the characteristics of mathematical literacy receive explicit attention in all subjects across the curriculum and the other where it is embedded in curriculum documents but not explicitly named.

Curriculum documents were analysed because these should include terms and expressions that convey expectations of key student learning outcomes, that is, words that act as signals for what is intended to be taught and learned (Nöth 1990). How teachers, teacher educators, textbook authors and others, with interests in the outcomes of education, interpret the meaning and importance of such signals will impact on what is taught in classrooms. The frequency and placement of these signal words in curriculum documents may also be considered a way of measuring how important they are for a subject.

We followed Robson's (2002) guidelines for conducting a content analysis. In this approach, research questions are first established and relevant documents are selected. It is then important to define the recording unit and construct categories for the analysis, before carrying out the analysis. Consistent with this approach, we selected the 12 curriculum documents from the Swedish National Center for Mathematics Education that were available in English (as both authors are English speakers). This sample is presented in Table 22.1.

We then defined recording units and constructed categories for our content analysis. In attending to the first research question about the frequency of terms associ-

Table 22.1 Sample curricula

Country	Year	No. of pages	Country	Year	No. of pages
Australia	2014	272	Norway	2013	14
China	2004	115	Singapore	2013	40 + 42
Finland	2004	11	South Africa	2004	518 + 306 + 164
India	2006	12 + 10	Sweden	2011	14
Japan	2008	33	England	2013	47 + 9
Korea	2007	59	USA	2010	93

Table 22.2 The analytic scheme

Mathematical literacy and its characteristics	Does the curriculum include descriptions	Key words
Mathematical literacy	Of the words mathematical literacy, numeracy, quantitative literacy, etc.	Literacy and numeracy
Out-of-school context	Of using mathematics in out-of-school contexts (across cultures) such as in everyday practices, workplace, etc.	Every day, workplace, daily living and life
Critical citizenships	Of aims for critical citizenships, like analysing critical aspects of societal realities	Citizen, politics and society
Cultural identity	Of the awareness of informal mathematical knowledge in social and cultural activities	Culture
Interdisciplinary practices	That relate to work interdisciplinary	Interdisciplinary and cross-curricular
The use of technology	Of the use of technology	Technology, digital tools and ICT

ated with mathematical literacy, we defined the recording units, literacy and numeracy as key words. In the second research question, related to the nature of mathematical literacy, we drew on Jablonka's (2003) and Geiger et al.'s (2015) attempts to characterise the field of mathematical literacy to generate the following list of categories with key words, o*ut-of-school context*, *critical citizenships*, *cultural identity*, *interdisciplinary practices* and *the use of technology*, to develop the analytic scheme. Descriptors of these aspects are displayed in Table 22.2. To carry out the analysis, the sample curricula were exported into Nvivo, a qualitative data analysis computer application, to establish word frequencies and to assist with other aspects of the analysis.

22.4 Results

Results are presented in two subsections: Sect. 4.1 addresses the frequency of the *mathematical literacy* and the alternate term *numeracy* and Sect. 4.2 addresses descriptions of mathematical literacy/numeracy.

22.4.1 The Frequency of Mathematical Literacy and Numeracy

The results of the frequency analysis showed that the expression *mathematical literacy* was not found in any curricula. However, the words *financial literacy* were found in curriculum documents from England (Department for Education

Table 22.3 Frequency of the word numeracy

Country	Frequency of numeracy	Country	Frequency of numeracy
Australia	81	Singapore	2
South Africa	7	USA	1
Norway	2		

2013) in a section describing the purpose of studying mathematics. The term *numeracy*, on the other hand, was identified in five different curricula as displayed in Table 22.3.

The Australian curriculum uses the term numeracy more frequently than curricula from other nations. The second most frequent use of the term numeracy is in the South African curriculum, where it is usually expressed in qualified terms such as *emergent numerac*y. The only other occurrences of the term numeracy occurred in the curricula of Norway (2), Singapore (2) and the USA (1).

22.4.2 Descriptions of Mathematical Literacy/Numeracy

An examination of the various definitions of numeracy embedded within curricula of relevant countries revealed considerable variation.

The Australian Curriculum (ACARA 2014) explicitly emphasises *numeracy* as a general capability to be developed in all subjects and each subject syllabus. Our Nvivo analysis also revealed that *numeracy* was found 81 times in the mathematical syllabus. The development of students' *numeracy* capabilities is described as:

> they develop the knowledge and skills to use mathematics confidently across all learning areas at school and in their lives more broadly. Numeracy involves students in recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (ACARA 2014, p.13)

In terms of the aspects of numeracy identified by Jablonka (2003) and Geiger et al. (2015), this definition highlights the role of mathematics in *out-of-school contexts* (real-world contexts) as central to being numerate. Additionally, in other parts of the document, there is reference to capabilities aligned with notions of *critical citizenship*, *cultural identity* and the *use of technology*.

The South African curriculum documents include the terms *emergent numeracy* and *numeracy* in the section for kindergarten but nowhere else. In this document, it is stated that teaching and learning should promote the holistic development of the child, which includes "emergent numeracy [that] includes cognitive development (problem-solving, logical thought and reasoning)" (Department of Basic Education 2004, p. 13). Despite its use, however, South African curriculum documents do not explicitly define the meaning of numeracy.

Norwegian and Singaporean curriculum documents also make limited reference to numeracy. The Norwegian curriculum describes numeracy as:

> Numeracy in Mathematics involves the use of symbolic language, mathematical concepts, methods of approach and varied strategies to solve problems and explore mathematics by taking a point of departure in practical day-to-day situations and mathematical problems. (Utdanningsdirektoratet 2013, p. 5)

In this respect, numeracy is described as a skill related to how mathematics is used in *out-of-school contexts*. The utilisation of mathematics, however, also is related to its strategic employment and not just the use of basic skills.

The Singaporean primary syllabus uses the phrase "basic pre-numeracy skills such as matching, sorting and comparing" (Ministry of Education 2013, p. 10). This reference is related to the use of only basic mathematical skills and is not connected to mathematics in context. The reference to numeracy in curriculum documents from the USA (NGA and CCSSO 2010), on the other hand, is only within the reference list.

For completeness, we also examined an example of a curriculum in which numeracy is not mentioned explicitly but where statements related to learning outcomes align with underpinning aspects of the notion of numeracy. The Swedish curriculum in mathematics for grades preschool to grade 9 is structured around five general abilities (problem-solving, conceptual understanding, procedural fluency, reasoning mathematically and communicate mathematically) for students to develop across six core content areas (understanding and use of numbers, algebra, geometry, probability and statistics, relationships and change and problem-solving). An analysis based on the identified key words revealed a connection to mathematical literacy across a number of aspects of mathematical literacy. For example, reference to *out-of-school contexts* appears several times in the official English translation of the curriculum in statements such as 'teaching in mathematics should aim at helping the pupils to develop knowledge of mathematics and its use in everyday life' (Skolverket 2011, p. 59). Elsewhere, the statement "knowledge of mathematics gives people the preconditions to make informed decisions in the many choices faced in everyday life and increases opportunities to participate in decision-making processes in society" (p. 59) suggests that students may, through appropriate teaching activities, develop *critical citizenship*. The aspect of mathematical literacy described as *cultural identity* is found in relation to the history of mathematics, with statements like "mathematics has a history stretching back many thousands of years with contributions from many cultures" (p. 59).

22.5 Discussion and Conclusion

This study demonstrates that there is no general agreement upon the definition or role of mathematical literacy/numeracy internationally and its prominence in curriculum documents varies significantly across our sample. Within the Australian curriculum, numeracy is explicitly signalled as the responsibility of all teachers, not just mathematics teachers. Other countries, however, pay attention to mathematical literacy in less prominent ways. In the case of South Africa, *numeracy* is seen as a general goal in mathematics education, but no definition is provided. Within

Singaporean curriculum documents, *pre-numeracy skill* is considered a basic capability that young children should acquire before they enter primary education. In Norway, numeracy relates to both a basic arithmetic skill and the ability to use mathematics in complex problems set in different contexts. Numeracy, as an idea within curriculum documents, is clearly less important in documents from the USA as the referral is only from the reference list. While these countries name mathematical literacy/numeracy specifically, others, such as Sweden, have aspects of mathematical literacy implicitly embedded (e.g. *out-of-school context*, *critical citizenships*, *cultural identity* and *the use of technology*) in curriculum documents.

While mathematical literacy/numeracy is defined within the curriculum documents of most countries within our sample, descriptions tend to be limited to the notion of *out-of-school context* with little explicit reference to additional aspects such as those identified by Geiger et al. (2015) or Jablonka (2003). However, a closer examination of the relevant documents, for example, in the Swedish curriculum, shows that a wider range of aspects of numeracy (*out-of-school context*, *critical citizenship*, *cultural identity* and the *use of technology*) are implicit and so less obviously embedded. The presence of these aspects, even if implicit, must challenge teachers to develop learning activities that address goals such as critical citizenship and what mathematics is actually used in everyday situations.

The notion of mathematical literacy, and related terms, is a relatively new area of research (Geiger et al. 2015). This chapter has provided evidence that this issue is complex, as both the definition and role of mathematical literacy in teaching and learning, as outlined by curriculum documents internationally, are not consistent or coherent across different national curriculum documents. How the differing approaches are used across the world in addressing the use of mathematics to support participatory, functional and critical citizenship – mathematical literacy – is worthy of further research.

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Chapter 23
Design and Implementation of a Tool for Analysing Student Products When They Solve Fermi Problems

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Abstract In this chapter, we present a tool for analysing the work of secondary-level students from two different schools when they solve a type of Fermi problem. The tool is based on the characterisation of the concepts, procedures and language used to construct the models. Our results show that the proposed tool is useful to describe the models and to distinguish different aspects between the models produced by students without any previous modelling experience and those obtained by students who were already acquainted with working on modelling activities.

Keywords Modelling-eliciting activities (MEA) • Fermi problems • Secondary school • Real life

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_23

23.1 Introduction

In recent years, there has been rising interest in the introduction of various types of activities involving mathematical modelling tasks in the classroom (Blum 2003; Vorhölter et al. 2014). In our previous work, we have used *Fermi Problems involving Big Numbers* (FPiBN); these entail a contextualisation of the problem and require students to introduce elements of modelling in its resolution (Albarracín and Gorgorió 2013). In this study, we used problems that focus on estimating the number of people or objects that could fit within a defined space. We observed that the students' output from this type of open-ended task can be varied and can result in a wide range of solution strategies (Albarracín and Gorgorió 2014). This poses a classroom-management challenge for teachers (Gallart et al. 2015b) because, when students deal with contextualised problems that involve real-life situations or phenomena, "all mathematical competences are activated during the modelling process" (Gallart et al. 2015a). For this reason, we hereby present an analytical tool to characterise the mathematical models that students have come up with when solving such problems.

We tested the applicability of our analytical tool by studying the results obtained by students aged 15–16 years old after working on a sequence of FPiBN. Two groups of students took part in this study. The first group had already worked on modelling problems in teams the previous year and had even presented and discussed them in class. On the other hand, the second group had no previous modelling experience whatsoever. The results of our study led us to the identification of distinguishing elements between both groups of students according to their modelling experience.

23.2 Mathematical Modelling and Fermi Problems

In this study, we adopt the definition of mathematical model, as proposed by Lesh and Harel (2003). These authors consider that models are conceptual systems used to construct, describe or explain other systems and include a conceptual system and the accompanying procedures. From this definition, we understand that the creation of mathematical models with the purpose of describing or representing a certain phenomenon or reality in an abstract way is a complex process. Indeed, mathematical models include different elements that shape them, such as mathematical concepts, symbolic representations of reality or diagrams, as well as the procedures related to their use, mathematical or not.

The way students elaborate mathematical models in order to solve problems has been an object of discussion, and different points of view are held on this topic (Borromeo Ferri 2006). However, it is generally accepted that modelling processes are cyclic in nature. Therefore, the process is repeated in different iterations that improve previously found models and solutions, adapting to the needs of the formulation of each problem. Students' progress during the modelling process has already been studied in depth (e.g. Blum and Leiss 2007; Haines and Crouch 2010; Matsuzaki 2011, among others). However, their final models have not yet received as much attention. In our work we focus in the final product obtained by the students when they solve Fermi problems. We particularly focus our attention on the concepts and procedures used in their solutions and also in language use.

Many everyday life situations require estimation to gain answers to questions. Estimation is useful in situations in which we do not have access to the resources to give a more precise answer, because we do not have all the information we need or because the effort of offering an exact answer is excessive or unnecessary (Neunzert 2013). So-called Fermi problems are a specific type of activity that requires the simplification and mathematisation of a reality that involves estimation. The definition of Fermi problem by Ärlebäck (2009) is as follows: "Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations" (p. 331). Efthimiou and Llewellyn (2007) characterise Fermi problems from their particular formulation, as always seemingly diffuse, providing little concrete information and few relevant features to direct the solution process. A detailed analysis of the situation presented is needed to break up a Fermi problem into simpler ones that lead to the answer to the original question.

Ärleback and Bergsten (2010) used an analytical tool, a *modelling activity diagram*, to analyse student's productions when they faced Fermi problems. The authors observed that "the processes involved in a modelling cycle were richly represented in groups' solutions" (p. 597) prompting the authors to suggest realistic Fermi problems as a means to introduce modelling in schools. Ärlebäck (2011) states that working on Fermi problems may be useful for introducing modelling into classrooms for several reasons: (1) They are accessible to students of different educational levels and are not dependent on a specific type of previous mathematical knowledge; (2) they force students to structure the information relevant to the problem; (3) they require students to elaborate a resolution strategy specific to the context; (4) since they do not provide numerical data, the students have to estimate several amounts for themselves; and (5) they promote discussion between students.

These reasons led us to consider that Fermi problems, and specifically FPiBN, offer the chance to study students' modelling work, regardless of whether they have previous modelling experience or not. We have previously observed (Ferrando et al. 2017) that the students without experience can develop mathematical models at different abstraction levels that explain the contexts provided in FPiBN, and therefore students should be able to solve FPiBN by developing their own methods.

23.3 Objectives of the Study

In this chapter, we study the work done by fourth-year secondary school students (16 years old) in the solution of two large quantity estimation problems in the classroom. Both were designed as part of a series of problems with the aim of providing situations for the students to develop their own mathematical models and adapt them to new situations with different levels of complexity. The aims of our study were the following:

1. To analyse the mathematical models produced by students when solving FPiBN by using a tool for model characterisation based on Lesh and Harel's (2003) definition of mathematical model.
2. To identify the elements that differentiate the models obtained by students with previous modelling experience and those of students that have none.

23.4 Methods

23.4.1 Design of the Research Study

In order to design the problem sequence used in this study, we considered the integrating elements of *modelling-eliciting activities*. Following the recommendations of Wessels (2014), we have tried to include complex activities, far from conventional problems in Spain and not related to previously defined solution procedures that also involve different real-life situations.

The FPiBN sequence used consists of a series of Fermi problems that tackle the same issue – from a strictly mathematical sense – since all problems require the students to estimate the number of people or objects that can be placed over a certain surface area. The design of the activity is based on a first problem that allows for fieldwork without leaving the school, allowing the students to develop solution methods to be carried out in situ. Following this, the students are asked to attempt a few problems that cannot be solved by experimentation using tools that are within their reach, which brings about the need to transfer the solution strategies of the first problem to other contexts. In this study, we focus on the analysis of the students' work for two problems of the sequence. Problem A is to be solved in the school playground, and problem B is to be worked on later in the classroom. The formulations of these problems are the following:

Problem A: To organise the school year-end festival, it could be nice to bring up a music band and organise a concert in the schoolyard. In this situation, a good question is: How many tickets can we sell to fill the yard during the concert?

Problem B: How many trees are there in Central Park?[1]

[1] This corresponds to the famous park that is in New York City, far away from Valencia or Barcelona.

The work done in the classroom was divided between heterogeneous groups. The students worked in groups of three or four people for several sessions. The experience, we present, has been carried out with a group of 24 students in the fourth year of secondary school from an educational centre in Valencia (referred to as group E1) and another group of 22 students from a school in the province of Barcelona at the same educational level (group E2). Group E1 was divided into seven teams and E2 into six. Our results consist of the written solution processes of problems A and B from six teams of students without previous modelling experience and from seven teams who had dealt with modelling activities before (group E1).

23.4.2 Analysis

In order to analyse the mathematical models generated by the students, we present a tool for the qualitative characterisation of the essential features that define these models. In addition, this tool might provide us with an objective idea of the complexity of the representation of the situation studied.

Following Lesh and Harel (2003), we consider a mathematical model to be formed by concepts and procedures that are interconnected. At the same time, symbols, diagrams and language use can help to convey the mathematical model by adding to the conceptual load of the model. Specifically, we distinguish two kinds of processes. Firstly, we distinguish the processes that allow us to obtain quantitative information of the studied reality or phenomenon which are an essential part of the solution process and that modelling tasks are focussed on studying. Secondly, we distinguish the mathematical procedures that aid in the development of the solution and reach a mathematical solution that can be contrasted with the reality studied.

Our analysis tool is centred upon the qualitative determination of the following elements in the models the students created:

- Concepts (mathematical concepts, relationships between them, patterns, etc.)
- Procedures (for data collection about the studied phenomenon and on the mathematical work required to obtain results)
- Languages (symbolic, written, sketches, diagrams, etc.)

In this study, we have used this analytical tool in order to characterise the models proposed by the students from the two experimental groups when working on problems A and B.

23.5 Results on Concepts

Two fundamental concepts have been identified regarding the conceptual systems of both problems: the notion of *population density* understood as the number of elements comprised within a certain spatial extent and the *iteration of a unit*, based on

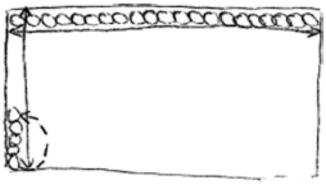

Fig. 23.1 Resolution of problem A carried out by a team from group E2

Table 23.1 Conceptual systems used by groups E1 and E2

	Density		Iteration		Grid		Total	
Problem	E1	E2	E1	E2	E1	E2	E1	E2
A	6	3	1	2	0	1	7	6
B	3	3	4	2	0	0	7	5

Table 23.2 Complex elements detected

	Unusable space		Do not consider unusable space	
Problem	E1	E2	E1	E2
A	7	2	0	5
B	5	1	2	4

determining the surface area taken up by an element (person or tree) and then dividing the total surface by the selected unit. One of the teams in group E2, however, used an alternative concept involving the arrangement of people over a grid. Figure 23.1 shows the scheme used by the students as explanation (see Fig. 23.1).

Table 23.1 shows the results obtained in the analysis of the conceptual systems used by the students. We must point out that one of the teams in group E2 (novice modellers) did not solve problem B. It is worth noting that the students of group E1 (experienced modellers) predominantly used the concept of population density to solve the first problem. On the other hand, they used this concept less in problem B, in which they showed difficulties when applying population density to trees. This was due to the difficulty of expressing the concept of trees per square metre in decimal values.

In our concept analysis, we also observed that some student teams especially included an element that adds complexity to their models. They found that part of the surface in their problems is unusable. They considered that not all the available space in the playground can be taken up by concert attendees and not all the surface of Central Park is covered by trees. In Table 23.2 we show how many working teams have taken into account the aforementioned complex elements, for each of the problems.

The solutions of group E1 for both problems significantly contrast with those of group E2 with respect to concepts used, since the latter use the notion of population density less and only two of the teams without modelling experience considered usable space as a refinement to the conceptual basis of their models.

23.6 Results on Procedures

Regarding the procedures related to data collection, the most used methods are the calculation of surface areas and estimations. The former may be carried out on the basis of measurements (experimental in situ measurements or with digital tools such as Google maps) and in some cases by decomposing the surface into simpler geometrical figures and afterwards applying known formulas. Secondly, as an example of estimation, we could take the estimate calculation of the surface area of a tile or the proportion of unusable surface in a larger area.

In situ measurements are obviously only possible in the resolution of problem A, but it is interesting to observe the differences between the resolutions of both groups. All teams in group E1 performed their measurements with a tape measure except one of them who calculated the surface area by counting tiles. This group claimed explicitly that the strategy used was "counting tiles", but they did not explain how they did it. They just claimed, at the end of the resolution of the problem, that "one tile corresponds to a square metre". In group E2, however, we found two examples of alternative procedures. One of the teams arranged the concert attendees along the vertical and horizontal of the playground (Fig. 23.1), and the other proposed to calculate and approximate the playground's surface area by counting in footsteps.

In problem B, the calculation of the surface area is carried out, in all cases, by performing measurements with Google Earth tools. In the resolution of this problem, we do not find important differences in the procedures carried out by both groups in obtaining the total area of Central Park. However, they did use different procedures to find the surface area covered in trees. Whilst some of the teams calculated the treeless area (more or less precisely), other teams made estimations.

We found differences in data processing-related procedures used by different teams. These procedures were, however, based on equivalent conceptual systems. Specifically, some of the students argued that they used population density to find the number of people in problem A, or trees in problem B, by multiplying it by the (usable) surface area. An illustrative example was the procedure developed by a team in group E1 who calculated the usable area by subtracting surface area of several elements from the total area and multiplying the answer by the population density obtained from experimenting, which happens to be three people per square metre.

> With the measure of the total space that can be used by people and supposing that in 1m^2 three people can fit, we will multiply this space by three people:
> $5100 - 314 - 240 - 200 - 8 - = 4240\text{m}^2$
> 3 persons per m^2 => 12720 persons approx.

This procedure was used by four out of the seven teams in group E1 in problem A but was used by only two out of six teams in group E2. Other procedures used to find the total number from population density were the use of proportions or reasoning based on the equivalence of fractions. Both these methods are minor in problem A (only one team of group E1 used it) and appear in three solutions for problem B.

The students that used conceptual systems related to unit iterations based their resolution on the surface area taken up by each person or tree and obtained the total amount by dividing the area of the premises by the surface area corresponding to the unit.

23.7 Results on Language Use

When focussing our attention on the language used by the students in both problems, we observed some notable differences between both groups. Some students in group E1 (who already had experience in the resolution of modelling problems) used pre-algebraic notation (see Fig. 23.2). This allowed them to obtain formulas that could be generalised (two out of seven teams in group E1 for problem A). However, we did not find this type of pre-algebraic language in the work of group E2.

Literal language was used by all teams in the solution of both problems. It was however used exclusively in 5 of the 12 solutions of problem B, whilst only one of the teams in group E1 solved problem A using only literal language. The rest of the solutions combined literal language with arithmetic language and, in some cases, with graphic language (diagrams) as in Fig. 23.3. Table 23.3 shows the frequency of the different types of language use detected for each problem. As indicated in this table, not all uses of pre-algebraic language allowed for generalisations.

23.8 Discussion

As noted in Gallart et al. (2015a), studies centred upon analysing different stages of the modelling cycle (e.g. Blum and Leiss 2007) provide tools to evaluate the performance of students when facing modelling tasks based on the identification of the competencies that are activated during each of the stages of the solution process. In this chapter, we present as an alternative a tool to characterise the final mathematical models produced by the students, using the definition of mathematical model of Lesh and Harel (2003). The basis of the tool is the determination of conceptual systems, procedures and language types used by the students. From this perspective, we regard the tool proposed in this chapter as *complementary to the analysis of the stages of the modelling cycle* since it may allow researchers to develop an analysis of the output of modelling tasks. On the other hand, this analysis tool can be easily adapted to aid the evaluation of classroom modelling activities, as well as to come up with solution guidelines to help teaching staff in the preparation stage of such activities.

Formula. Total area – non used area . number of people/m²

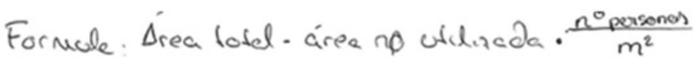

Fig. 23.2 Solution to problem A provided by a team of group E1

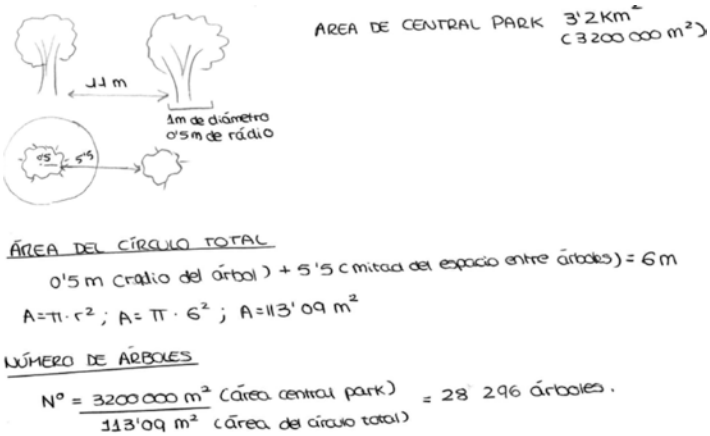

Fig. 23.3 Solution to problem B provided by a team of group E1

Table 23.3 Types of language used in the solutions

	Literal (exclusively)		Use of graphics		Arithmetic language		Pre-algebraic language	
Problem	E1	E2	E1	E2	E1	E2	E1	E2
A	1	0	3	2	4	6	2	0
B	4	1	2	0	3	4	2[a]	2[a]

Note. [a]The numbers marked correspond to use of algebraic language not allowing generalisation

23.9 Conclusions

On the basis of our analytical results, we can state that Fermi problems require students to elaborate models with a high level of detail. When analysing the characterisation of the mathematical models collected from our classroom experience, we observe differences that can be linked to the learning of modelling processes (Gallart et al. 2015a). In previous studies, we have already analysed solution proposals of FPiBN (Albarracín and Gorgorió 2013, 2014). The tool presented in this chapter represents a step forward in the direction of characterising not only initial strategies but also the final products of the solution. Moreover, the analysis tool leads us to confirm the complexity of FPiBN, since their solution led students to use indirect arguments to find the solutions. Maaβ (2006, p. 115) notes that: “Modelling problems are authentic, complex and open problems which relate to reality”. Problem-solving and divergent thinking are required in solving them. Our analysis shows that FPiBN fit perfectly with this definition.

Specifically, the descriptive level attained allowed us to detect differences that showed that students with previous modelling experience tended to formulate models based on more complex concepts, using more rigorous measurement procedures and more elaborate mathematical languages, such as algebraic representations that allow for the generalisation of models. We also observed differences regarding the conceptual systems used in the solution of both problems. Possibly, this may be due to the fact that working with population densities is more natural when dealing with a mass of people – that can conceptually fill or leave vacant a given volume – that are not a static mass, like trees are. However, we consider the proposed tool has room for improvement, especially with regard to the graphics and language used in problem solution. We consider the graphic products of the students deserve a more specific treatment than what we have developed in this study.

Acknowledgements This chapter is the result of research conducted under the framework of projects EDU2012-35638, EDU2013-4683-R, and EDU2015-69731-R (MINECO/FEDED) that have received financial support from the Spanish Ministry of Economy and Competitiveness. The research is also funded by Generalitat Valenciana under projects GV2016-129 and GVPROMETEO2016-143.

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Chapter 24
Implementing Modelling into Classrooms: Results of an Empirical Research Study

Jana Kreckler

Abstract A four-lesson teaching unit to foster global modelling competence (i.e. the ability to undertake a full modelling process and to possess the meta-knowledge of the procedure) in regular school lessons was developed and tested in an empirical study with 332 tenth grade German secondary school students. The goal of the study was to increase global modelling competence independently of influencing factors such as chosen topic of the teaching unit, gender and student report grades. Concerning increase in global modelling competence and motivation of students, hypotheses were formulated and analysed in a pre-post-designed research study. The results show a significant increase in the global modelling competence independent of grade and topic. Concerning motivation, no significant changes could be identified. The chapter is based on Kreckler (Standortplanung und Geometrie: Mathematische Modellierung im Regelunterricht. Springer Fachmedien, Wiesbaden, 2015).

Keywords Global modelling competence • Empirical study • Teaching unit • Secondary school • Competence increase • Motivation • Sustainability • Influencing factors

24.1 Introduction

Mathematical modelling is one of the important mathematical competences that students should gain and develop in school. This is, amongst others, explicitly mentioned in the German education standards (KMK 2003). To many teachers, it is unclear how to put this into practice and they feel insecure about teaching modelling. It is, therefore, essential to find appropriate didactical concepts and teaching units to effectively facilitate the modelling competences of our students in school.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_24

A didactical concept to foster modelling competences in regular school lessons was developed and tested in an empirical study with 332 students of tenth grade in German secondary schools. The goal of the study was to develop a teaching unit which helps to increase the global modelling competence independently of influencing factors such as the chosen topic of the teaching unit and individual factors such as gender and report grades of the participating students. Based on the definition of modelling competence by Blomhøj and Jensen (2003), the focus of the study was laid on the *global modelling competence*, which was defined as the ability to undertake a full modelling process (i.e. to solve given modelling tasks) and to possess the meta-knowledge of the procedure. A four-lesson teaching unit was developed with a holistic and self-dependent approach based on results of the projects DISUM (Blum and Leiß 2007) and ERMO (Brand 2014), amongst others.

Concerning increase in the global modelling competence and motivation of the students during the four-lesson teaching unit, two hypotheses were formulated and analysed in a pre-post-designed research study. A follow-up test was also undertaken 3 months after the teaching unit to test the sustainability of the acquired modelling competence. The global modelling competence of the students was rated by an evaluation scheme of Siller et al. (2015). The theoretical background, the design and the results of the study will be presented in detail in the following sections.

24.2 Theoretical Background

When talking about mathematical modelling, we mean to understand, structure and solve real-world problem situations using mathematical tools and to recognize mathematics in reality (Blum and Leiß 2006). Mathematical modelling competence is demanded to be taught in schools by the curricula in many countries around the world. In the research study described in this chapter, we follow the definition of Blomhøj and Jensen (2003) who define “By mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context” (p. 126). Mathematical modelling competence can be divided into different sub-competences, often represented in so-called modelling cycles (see Blum and Leiß 2005). These sub-competences include to *understand*, *simplify*, *mathematize* and *solve* a given modelling problem, as well as to *interpret* and *validate* the obtained solution with reality.

In the research study described in this chapter, the main focus lay on the *global modelling competence* which was defined to be the ability to undertake a full modelling process and to possess the meta-knowledge of the procedure (Kreckler 2015). The modelling tasks developed for this study were designed following the quality features formulated by Blomhøj and Kjeldsen (2006) who state that a good modelling task should be understandable and reasonable, give an appropriate challenge for independent work, be authentic and include authentic data and be open for interesting modelling results. It is, therefore, important that the tasks are real problems which were posed outside of school and that the data given is authentic, that is not made up by the teacher. To be open for different modelling results, it is also important for tasks to be formulated in an open way, such that different approaches can be chosen and lead to the use of different mathematical tools and solutions.

Looking at the implementation of modelling problems in regular school lessons, two main approaches can be identified (Blomhøj and Jensen 2003). In the *holistic approach*, the students work on complete modelling tasks, running through a full modelling process to find a solution. In the *atomistic approach* on the other hand, students work on separate tasks concerning the subprocesses of a modelling process. Here, the sub-competences are practised individually.

Concerning the implementation of modelling problems, different research studies analysing the effects of modelling can be found. In a study by Gialamas et al. (1999), 97 students of 11th grade were divided into two groups, where one of the groups experienced lessons including modelling problems. The results showed significantly higher achievements of the modelling group with respect to both reality-related and pure mathematical tasks. Similar results were obtained by Dunne and Galbraith (2003) who undertook a small case study with 23 students of eighth grade over a period of 1 year. As in the study of Gialamas et al., the modelling group achieved higher results than the group which had not been working on modelling tasks.

In another study, the DISUM project (Blum and Leiß 2007), the benefit of self-dependent activities when working on modelling problems was empirically shown, as well as a supporting effect of a strategic instrument called solution plan. A solution plan was given to the students which helped them to solve modelling problems more efficiently. It is, therefore, essential to construct teaching units which give the students the opportunity to work self-dependently, that is, independently of the teacher. If problems arise when working on the tasks, different aids can support the learning process.

An empirical comparison of the holistic and the atomistic approach was undertaken in the project ERMO (Brand 2014). The holistic group showed significantly higher achievements concerning both the competence fields "simplify/mathematize" and "complete modelling". Since in the research study (Kreckler 2015) described in the following sections the focus lays on the *global* or *complete* modelling competence, a holistic approach was chosen for the design of the teaching unit which will be described in the following.

24.3 Aims and Design of the Study

The aims of the designed teaching unit included:

1. To integrate real-world problems into an ordinary mathematics class;
2. To increase global modelling competence independently of influencing factors such as the chosen topic of the teaching unit, gender and report grades;
3. To increase motivation with the help of real-world problems.

With the development of a teaching unit that can act as a template to teach modelling for different topics and age groups, it was also aimed to overcome teacher-based difficulties such as large preparation times and insecurities concerning the implementation of modelling into classrooms. Due to another often mentioned obstacle, the lack of time to do modelling, a short four-lesson teaching unit was designed.

24.3.1 Teaching Unit

To fulfil the three aims named above, authentic modelling tasks from the field of location theory (a branch of the mathematical research field of optimization) were chosen for implementation of the teaching unit in the framework of this study. To foster the global modelling competence of the participating students, holistic modelling tasks and self-dependent activities, that is group work, were used. This was based on, amongst others, the results of the projects mentioned above. Instructions and support were given step by step, from a guided to self-dependent learning.

Internal differentiation in class was realized by means of individual support given by the teachers who acted according to the principle of minimal help. This meant the teacher would interact and help the students only when they could not proceed on their own accord. Assistance was then given in a minimal way, starting with motivational help only. If this was not sufficient, the type of aid was increased slowly following the assistance levels formulated by Zech (1996), where the type of help increases from motivational help, feedback and general strategic help to more specific aid with regard to content.

The teaching unit designed consisted of four school lessons (length 45 min each) and was divided into two blocks of two lessons (see Fig. 24.1). In each block, students worked on a task sheet consisting of one complete modelling task. The teaching unit started with a very short introduction about the terms modelling and modelling problem. The students were then divided into groups of three to four students to work on the given modelling problem of the first task sheet. The modelling problem given in the first task sheet was accompanied by instructions of how to proceed in a modelling process to solve the problem. At the end of the first two-lesson block, a class discussion took place. Different approaches and results of the modelling tasks as well as the procedure to obtain a solution were discussed.

The second two-lesson block was structured in a similar way. Again, the students worked in groups on the modelling problem of the second task sheet, and in the end,

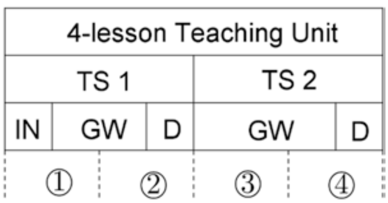

Fig. 24.1 Structure of teaching unit (*TS* task sheet, *IN* introduction, *GW* group work, *D* discussion)

a class discussion led to a summary of approaches, results and important steps in a modelling process. In contrast to the first task sheet, the second task sheet did not include instructions, only a modelling problem to be solved. The modelling procedure learned from the first task sheet needed to be applied here. Figure 24.1 gives an overview of the structure of the teaching unit.

The format of the two task sheets was developed in such a way that it can be used as a template for other real-world problems. Task sheet 1 always consists of one authentic modelling task as well as instructions on how to solve the modelling task step by step. The guidance given follows the steps of a modelling process and gives hints and tips concerning the specific problem:

1. Define simplified assumptions to construct a real model. Tips: […]
2. Mathematize the real model. Choose an appropriate mathematical description of the problem situation.
3. Solve the problem in your mathematical model. Tips: […]
4. Describe the strategy you used to solve the problem.
5. Interpret your mathematical result relating to the original problem.
6. Illustrate strengths and weaknesses of your model. Make suggestions for possible improvements.

Task sheet 2 also consists of one authentic modelling task, but no instructions or hints to solve the problem are given. This means that the students need to adapt and apply the procedure of how to solve a modelling problem (learned in task sheet 1) to a new problem on their own accord.

24.3.2 Study Design and Procedure

After developing the four-lesson teaching unit, two hypotheses were formulated:

1. The *global modelling competence* of the students increases due to the four-lesson teaching unit, independently of gender, report grades and topic.
2. The *motivation* of the students increases due to the four-lesson teaching unit.

In order to analyse these hypotheses, a pre-post-designed research study was chosen to measure the changes in motivation and the global modelling competence. An overview of the setup of the study design is given in the following.

To guarantee uniform implementation of the teaching unit by teachers, teacher professional development was provided at the beginning of the study. This prepared the teachers in a didactical, mathematical and organizational way for the realization of the teaching unit and the corresponding tests. The teachers then carried out a pretest, the four-lesson teaching unit and a post-test with their students. Afterwards, they met again for a review, to discuss the implementation and possible problems that appeared. Three months after the teaching unit, the students undertook a follow-up test to test the sustainability of the acquired global modelling competence.

The participants of the study consisted of 10 schools, 14 teachers and a total of 332 students of tenth grade (aged 15 and 16), 155 male and 177 female students. The data collected included gender and the last report grade in mathematics of each student, a motivation questionnaire in the pre- and post-test and the global modelling competence in the pre-, post- and follow-up test.

The teaching unit was undertaken using two different topics. One half of the teachers implemented the teaching unit with the topic of "sales territories" (task sheet 1, sales territory of supermarkets in Kaiserslautern; task sheet 2, primary schools in Mannheim); the other half of the teachers used a topic of "bus and train stops" (task sheet 1, planning bus stops in Kaiserslautern; task sheet 2, train journeys more attractive with additional stops).

The two task sheets of the first topic "sales territories" will now be described in more detail.

Task Sheet 1: Sales Territories of Supermarkets in Kaiserslautern

In the city of Kaiserslautern, the sales territories of the supermarkets in the town centre are analysed in the context of a market research study. In the town centre, there are six supermarkets, which are marked A, B, C, D, E and F in the given map. The *sales territory* of a supermarket x is defined as the area of all customers, which will probably go shopping at supermarket x. You are now asked to divide the given map into the sales territories of the six supermarkets. Proceed step by step!

Following this modelling task, the instructions shown in Sect. 24.3.1 were given, as well as a map of the town centre with the marked supermarkets.

Task Sheet 2: Primary Schools in Mannheim

The urban administration of Mannheim assigns a primary school to each schoolchild depending on their place of residence. Each child should visit the primary school closest to its home. In the given map of Mannheim, three schools A, B and C are marked. A characteristic feature of the town centre of Mannheim is that the streets are all parallel and perpendicular to each other. Task: Divide the town centre of Mannheim into school districts, indicating which residents are allocated to which school.

All tasks are formulated in an open way and are authentic real-world problems.

24.3.3 Test Instruments

As the main focus of the study lies on the global modelling competence, a holistic approach was also chosen to test its change due to the teaching unit. The students were asked to solve one modelling task each in the pre-, post- and follow-up test:

1. Pretest: Ice cream seller at the beach
2. Post-test: Rescue helicopters
3. Follow-up test: Cell towers

The pretest took place in the preceding lesson of the four-lesson teaching unit and marked the student's individual starting level of the global modelling competence. The post-test took place in the lesson after the teaching unit, while the follow-up test happened 3 months after the teaching unit took place. Each time, the students were given a modelling problem to solve by themselves within 30 min. One task is given in the following.

Rescue Helicopters

In the German Alps, one rescue helicopter is responsible for the five skiing regions Hochschwarzeck, Berchtesgaden, Obersalzberg, Rossfeld and Jenner. (A map was given to the students.) Since rescue missions need to be fast and efficient, a strategic reasonable location needs to be found for the base of the rescue helicopter. Where should the rescue helicopter be positioned? Explain your approach comprehensively.

The solutions of the students were classified by means of the four-step competence model of Siller et al. (2015), which is based on the general formulations of Meyer (2007); see Table 24.1. The achieved level of competence in each test was

Table 24.1 Competence level model

	Modelling competence	
Level	Meyer (2007)	Siller et al. (2015)
1	Execution of an action, largely without reflective understanding	Implementing a representational change between context and mathematical representation. Using familiar and directly recognizable standard models for describing a given situation with appropriate decision.
2	Execution of an action by default	Describing the given situation by mathematical standard models and mathematical relationships. Recognizing and setting general conditions for the use of mathematical standard models.
3	Execution of an action after insight	Applying standard models to novel situations, finding a suitable fit between mathematical model and real situation.
4	Independent process control	Complex modelling of a given situation; reflection of the solution variants or model choice and assessment of the accuracy or adequacy of underlying solution methods.

then compared for all students individually. An additional *Level 0* "No constructive solution approach, no (reasonable) solution" was added in this research study to be able to calculate the change in the global modelling competence for all student solutions handed in.

To analyse the change in motivation before and after the four-lesson teaching unit, a motivation questionnaire by Kuhn (2010) was used. This questionnaire consists of several items which can be divided into three clusters, *intrinsic motivation* (nine items), *self-concept* (nine items) and *correspondence to reality* (eight items), and is rated on a scale from 1 (totally true) to 6 (not true). In the study described in this chapter, five additional items were added to the post-test:

1. How interesting were the modelling problems?
2. How difficult were the modelling problems?
3. How realistic were the modelling problems?
4. In your opinion, what did you learn while working on the modelling problems?
5. Should tasks and topics as discussed during the last four lessons become part of mathematics lessons on a regular basis or would you reject this? Give reasons for your choice.

Questions 1–3 could be answered on a scale from 1 (very interesting/difficult/realistic) to 5 (not interesting/difficult/realistic at all). Questions 4 and 5 required an answer in a written sentence.

24.4 Results

All student solutions of the pre-, post- and follow-up tests were classified by two independent persons. Using these ratings, the inter-rater reliability was calculated. The values of the percentage agreement as well as Cohen's kappa confirmed a uniform evaluation (see Table 24.2).

To verify hypothesis 1, which states that the global modelling competence increases due to the four-lesson teaching unit, a t-test was undertaken. The significance test with paired samples, a significance level of $\alpha = 1\%$ and a critical value of $t_{1-\alpha,q} = 2.326$ ($q > 120$), delivered very significant results. The empirical values of t (pre-post, $t = 21.115$; pre-follow-up, $t = 17.425$) showed a significant increase in the global modelling competence from pre-, to post- and follow-up test. The increase of the global modelling competence from pre- to post-test can also be seen in more detail in Table 24.3. An increase in the competence level by −1 represents students who became worse by one competence level, 0 represents students who kept a con-

Table 24.2 Values of inter-rater reliability

Test	% agreement	Cohen's kappa
Pretest	89	0.715
Post-test	93	0.872
Follow-up test	88	0.712

Table 24.3 Increase of global modelling competence from pre- to post-test

Increase	Quantity	Percentage
−1	6	2.1
0	71	24.6
1	144	49.8
2	65	22.5
3	3	1.0

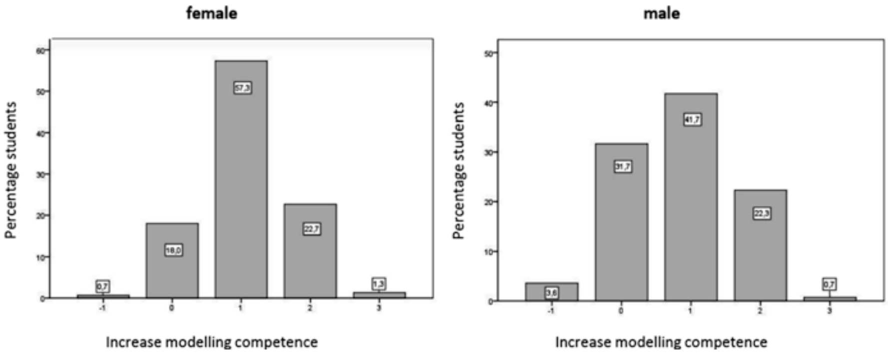

Fig. 24.2 Comparison of the increase in the global modelling competence for the genders

stant level and increases of 1, 2 and 3 represent those students who became better by the respective number of competence levels.

Comparing the increase of each student from pre- to follow-up test, a sustainable increase could be identified. A total of 171 students (70.1%) was able to maintain their competence level even 3 months after the four-lesson teaching unit had taken place.

Concerning the increase in the global modelling competence of each student, three influencing factors (gender, topic of the teaching unit and last report grades in mathematics) were also analysed. No differences were seen concerning topic 1 and topic 2, as well as the report grades. Concerning the gender of the students, minor differences could be identified. Still, the distributions of their increase in the global modelling competence look alike (see Fig. 24.2).

Analysing the changes in motivation of the students from pre- to post-test, the t-test and a regression analysis with the coefficient of determination R^2 showed no significant changes. Hence, no increase or decrease in motivation was detectable. Finally, the open question asked in the post-test, if tasks and topics discussed during the last four lessons should become part of mathematics lessons on a regular basis, was answered with "yes" by nearly two thirds of the students (63.5%).

24.5 Summary

A four-lesson teaching unit to foster global modelling competence in tenth grade was developed based on a holistic and self-dependent approach. The teaching unit was analysed in an empirical research study with 332 students of tenth grade. The focus of the study was on the global modelling competence as well as the motivation of the students. Hypothesis 1 was confirmed, that is, the global modelling competence increased (sustainably) due to the four-lesson teaching unit (concerning problems from the authentic field of location theory). The increase took place independently of the influencing factors grade and topic, while minor differences could be identified concerning the gender of the students. Hypothesis 2 was not confirmed; no significant changes in motivation were detectable. Further questions in the post-test showed that the authentic topics of the teaching units undertaken in the study were interesting and realistic in the opinion of the students. Additionally, nearly two thirds of the students wanted to participate in modelling tasks more often.

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Chapter 25
A Commognitive Perspective on Pre-service Secondary Teachers' Content Knowledge in Mathematical Modelling

Joo Young Park

Abstract This exploratory study examined three pre-service secondary mathematics teachers' content knowledge and views associated with mathematical modelling revealed through engaging in mathematical modelling activities. Using a commognitive approach on mathematical modelling, pre-service teachers' written tasks and discourses were analysed. Data sources were audiotaped discourses among the participants, observation field notes, written tasks, open-ended questionnaires, and reflective journals. Findings suggested that pre-service teachers' content knowledge in the modelling allowed them to fully engage in modelling discourses in verifying a model mathematically as well as critically reflecting on solutions. These pre-service teachers' view on modelling was consistent with a *pragmatic perspective*.

Keywords Commognition • Pre-service teachers • Subject content knowledge • Modelling competencies • Views on mathematical modelling • Modelling process

25.1 Introduction

The Common Core State Standards for Mathematics (CCSSM) in the USA (NGACBP and CCSSO 2010) calls for emphasis on mathematical modelling. The modelling standard appears in each of the other five high school standards of mathematical content and is one of the eight standards for mathematical practice. Although curricula can provide students with opportunities to learn mathematical modelling, it is indisputable that how students acquire modelling skills relies on the quality of classroom instruction. Studies have suggested that teachers require,

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_25

besides other aspects, knowledge about several steps of the modelling process; otherwise the criteria of quality teaching of modelling cannot be fulfilled (Blum 2011; Borromeo Ferri 2014). Hence, mathematics teacher educators are challenged with preparing teachers to understand the intricacies of mathematical modelling.

Within CCSSM, modelling is defined as "the process of choosing and using appropriate mathematics and statistics to analyse empirical situations, to understand them better, and to improve decisions" (NGACBP and CCSSO 2010, p. 72). Among studies on teachers' conception of mathematical modelling, Kaiser and Maaβ (2007) found that some teachers viewed mathematical modelling as the process of creating opportunities for developing solutions, while others focused on the establishment of formulas. These different conceptions can lead to a particular emphasis on how modelling is taught in classrooms (Kaiser and Maaβ 2007).

As curriculum reform has also called for fostering collaborative learning environments to support students' learning mathematics, modelling can be used as a way to facilitate this type of learning environment (Escalante 2010; NCTM 2000). A *commognition framework* (Sfard 2008) has been shown to provide a lens for analysing inter-intrapersonal communication in both social and cognitive dimensions of modelling (Ärlebäck and Frejd 2013).

This exploratory case study is an attempt to understand the nature of future teachers' knowledge in the domain of mathematical modelling as well as their views on modelling by providing them with opportunities for experiencing mathematical modelling activities within a collaborative group. The underlying questions for this study are:

1. How is the pre-service mathematics teachers' content knowledge on mathematical modelling manifested while engaging in modelling activities within a group?
2. How do pre-service mathematics teachers describe mathematical modelling and the role of modelling from a pedagogical perspective?

25.2 Theoretical Framework

25.2.1 Mathematical Modelling and Modelling Process

A framework developed by Galbraith and Stillman (2006) was designed to assess the modelling process, implementation and assessment of mathematical modelling tasks in the secondary classroom. This framework comprises elements of modelling activities that correspond to the respective stages of the modelling process for guiding teachers, researchers, and curriculum designers to anticipate possible student blockages as they transit between the stages of the modelling process (Galbraith and Stillman 2006; Stillman et al. 2007).

The following is the framework for identifying potential sites for student blockages in transitions in the modelling process (Galbraith and Stillman 2006, p. 147):

1. **Messy Real-World Situation → Real-World Problem Statement**
   1.1 Clarifying context of problem
   1.2 Making simplifying assumptions
   1.3 Identifying strategic entities
   1.4 Specifying the correct elements of strategic entities
2. **Real-World Problem Statement → Mathematical Model**
   2.1 Identifying dependent and independent variables for inclusion in an algebraic model.
   2.2 Realizing independent variable must be uniquely defined.
   2.3 Representing elements mathematically so formulae can be applied.
   2.4 Making relevant assumptions.
   2.5 Choosing technology/mathematical tables to enable calculation.
   2.6 Choosing technology to automate application of formulae to multiple cases.
   2.7 Choosing technology to produce graphical representation of model.
   2.8 Choosing to use technology to verify algebraic equation
   2.9 Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation.
3. **Mathematical Model → Mathematical Solution**
   3.1. Applying appropriate symbolic formulae
   3.2. Applying algebraic simplification processes to formulae to produce more sophisticated functions
   3.3. Using technology/mathematical tables to perform calculation
   3.4. Using technology to automate extension of formulae application to multiple cases
   3.5. Using technology to produce graphical representations
   3.6. Using correctly the rules of notational syntax (whether mathematical or technological)
   3.7. Verifying of algebraic model using technology
   3.8. Obtaining additional results to enable interpretation of solutions
4. **Mathematical Solution → Real-World Meaning of Solution**
   4.1. Identifying mathematical results with their real-world counterparts
   4.2. Contextualizing interim and final mathematical results in terms of RW situation (routine complex versions)
   4.3. Integrating arguments to justify interpretations
   4.4. Relaxing of prior constraints to produce results needed to support a new interpretation
   4.5. Realizing the need to involve mathematics before addressing an interpretive question
5. **Real-World Meaning of Solution → Revise Model or Accept Solution**
   5.1. Reconciling unexpected interim results with real situation

5.2. Considering real-world implications of mathematical results
5.3. Reconciling mathematical and real-world aspects of the problem
5.4. Realizing there is a limit to the relaxation of constraints that is acceptable for a valid solution
5.5. Considering real-world adequacy of model output globally

This framework was used for unpacking pre-service teachers' content knowledge in mathematical modelling *while engaging in the modelling process*.

25.2.2 Forms of Teachers' Knowledge in Modelling Competencies

According to Shulman (1986), teachers' knowledge areas are content knowledge, pedagogical content knowledge, general pedagogical knowledge, and knowledge of educational contexts, values and philosophies. This chapter focuses on content knowledge pertaining to mathematical modelling. The mathematical content knowledge of pre-service mathematics teachers, elaborated by Bromme (1992) and Weinert (2001) based on Shulman's (1986) framework, includes the required cognitive activities such as modelling and mathematical content areas such as algebra or statistics (Kaiser et al. 2010). For each phase of the modelling process, the following kinds of modelling competencies are distinguished (Kaiser et al. 2010): sub-competencies for carrying out a single phase of a modelling process like structuring a real-world situation including developing a mathematical model, or validation of a solution, and competence to reflect critically about already executed modelling. Groshong and Park (2016) suggest that teachers' content knowledge in mathematical modelling consists of components of *mathematical knowledge* such as the breadth of mathematical content and skills as well as the application of necessary mathematics needed to solve mathematical modelling tasks and *modelling knowledge* such as the scope of extra-mathematical knowledge required to solve mathematical modelling tasks, knowledge of monitoring progression through the various modelling subprocesses and awareness of various mathematical modelling approaches.

25.2.3 The Commognitive Approach to Studying Learning

The commognition framework (Sfard 2008) provides a socio-cognitive lens to examine the learning processes. Within the framework, learning is viewed as a change in one's mathematical discourse, and participants' discourses are characterized by *word use*, *visual mediators*, *routines, and endorsed narratives* (Sfard 2008). *Word use* refers to participants' use of mathematical vocabulary in their discourses. *Visual mediators* are "visual objects that are operated upon as a part of the process of communication" (p. 133). The visual objects in mathematical discourse include graphs, symbols and diagrams. *Routines* are "a set of meta-rules that describe a repetitive discursive action" (p. 208). *Narratives* refer to any sequences of spoken or written utterances about mathematical objects and relations between objects that participants consider as true or false. *Endorsed narratives* are narratives that participants consider as true, which include definitions, axioms and theorems. With the commognitive approach, teachers' content knowledge in mathematical modelling was examined through analysis of the participants' discourses in each phase of the modelling process identified in the Galbraith and Stillman (2006) framework.

25.3 Methods

25.3.1 Procedure

Using the commognition framework, participants' written tasks and discourses were examined during the modelling process. The participants for this study were pre-service secondary mathematics teachers (Jesse, Steve, and Mary) who enrolled in a required junior-level mathematics course in the undergraduate STEM (science, technology, engineering, and mathematics) education programme. As a new programme, only three students enrolled in the course. Mary and Steve are mathematics majors, whereas Jesse is an interdisciplinary science major. Participants were juniors in the programme. The content course focused on functions and mathematical modelling. The class met twice a week for 75 min. Students worked on eight mathematical modelling tasks on functions during the course. The tasks were selected from Gould et al. (2012). This handbook was designed for teachers' mathematical modelling instruction aligned with CCSSM (NGACBP and CCSSO 2010). The pre-service teachers engaged in mathematical modelling activities as a group and created modelling tasks for lesson plans as a final project. The lesson plans are not part of the analysis for this study.

Data sources for this study were audiotaped discourses among the participants, observation field notes, written tasks, open-ended questionnaires, and reflective journals. Participants' written reports on modelling tasks also served as artefacts to be examined in detail for aspects of the modelling process discussed in class. All audiotaped data were transcribed for analysing discourses to examine the participants' content knowledge in modelling as they passed through each modelling phase. After completing the modelling tasks, the pre-service teachers completed open-ended questionnaires and reflective journals about their modelling experiences and views on teaching and learning mathematical modelling at the end of the semester. The following are some of the questions on the open-ended questionnaires and the reflective journal prompts: How do you define mathematical modelling? What is the purpose of teaching mathematical modelling? What is the goal of your mathematical modelling lesson? Describe the characteristics of the modelling tasks you designed.

25.3.2 Modelling Task

The data comes from analysis of students' work on a modelling task *Bending Steel* (Gould et al. 2012). The task consists of a leading question and a sequence of questions. Students worked on the *Bending Steel* task after having the mathematical modelling introduced and working on two exemplary modelling tasks at the beginning of the semester.

Bending Steel

Railroads are a common source of transportation around the world. Because the tracks are made of metals (often steel), they expand and contract due to change in temperature and various problems arise. Suppose a section of track is fastened down at both ends. The natural process of heating and cooling causes the track to expand and contract. If the track length increases, but is nailed down at both ends, then the tracks should rise off the ground. The tracks may also expand outward along the ground, but this lesson focuses on the case where they expand upward. How can railroad designers design tracks that stay safely on the ground in all types of weather?

Bending Steel Sample Questions

The world's longest railroad sections are about 120 m in length, or about 400 ft, with the typical length in the USA less than 100 ft. Suppose in your city that temperature changes on average about 45 °F (25 °C) from a cold, winter day to a warm, summer day. If the track is 120 m in the winter, the climbing temperature and heat during the summer cause the tracks to swell and increase in length. The linear expansion coefficient, α, for steel is approximately 0.000002 m per degree change in temperature (°C). Use this information to determine how much the track expands in length between winter and summer. Convert your answer to feet and then to inches.

(B) Draw a model of how you think the 400 ft track would look if its length expanded by the amount you found in question 1. Label all the known lengths.

(C) What mathematical shape does your model most closely replicate? Use the properties of that shape to determine how high off of the ground the tracks rise in the summer. Is the result surprising or what you expected?

(D) Based on real-life, physical models, it seems reasonable to model track expansion as the arc of a circle. Draw an arced model below, labelling the original straight length (a chord) and the new curved length. Extend the arc to draw the circle that contains it. Label the unknown radius, r, and central angle, θ, of the circle.

(E) Using the identified values for the radius, r, and central angle, θ, that are required for an arced model of this situation, how high off the ground would the tracks rise? Is the result surprising or what you expected?

25.4 Analysis and Results

25.4.1 Pre-service Teachers' Content Knowledge in Mathematical Modelling

The following are excerpts from the transcripts of pre-service teachers' conversation during modelling. With the commognition framework (Sfard 2008), pre-service teachers' modelling process was analysed based on their use of words, visual mediators, routines and endorsed narratives in mathematical and modelling discourse. The narratives were those discussed or endorsed during the modelling process. Each transition from one modelling phase to another was identified based on Galbraith and Stillman's (2006) framework as seen in Tables 25.1, 25.2, 25.3, and 25.4.

In each transition during the modelling process, the mathematical routines were *closed* (Sfard 2008) by Mary's endorsed narrative. Jesse and Steve's narratives mainly reaffirmed or repeated routines. Jesse and Steve had difficulties in identifying variables and finding an appropriate mathematical model related to trigonometric functions and the arc length during the transition from *real-world problem situation* to *mathematical model*. Mary's endorsed narratives were based on her mathematical reasoning, and she was more flexible than the others in her use of mathematics in verifying solutions mathematically and critically reflecting on found solutions and models. Mary revised her model or equation through the instructor's prompting question (e.g. arcsine function in Table 25.2), but she was able to progress through the various modelling subprocesses. Galbraith and Stillman's (2006) framework was applied to identify teachers' content knowledge displayed in each phase of the modelling process; however, the participants did not go through all subprocesses of each modelling phase for the *Bending Steel* task nor should they as which are relevant depends on the task and its implementation (Stillman et al. 2007).

Table 25.1 Real-world problem statement → mathematical model

Modelling process	Words and visual mediators	Routines	Endorsed narratives
2.3 Representing elements mathematically so formulae can be applied	Steve: "curve", "the height" will be "measured" on the "a half of the arc"	Steve: Measured somewhere on the half of the ground	Instructor: Great
	Jesse: "triangle", <visual mediator>	Mary: Like a rubber ruler, bend it in the middle	Jesse: A half of the arc, divide by 2
	Mary: "in the middle", "isosceles"	Instructor: Does your model look like a familiar mathematical shape?	Mary: In the middle, isosceles
	<Visual mediator> 		

Table 25.2 Mathematical model → mathematical solution

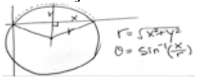
Modelling process	Words and visual mediator	Routines	Endorsed narratives
3.1 apply appropriate symbolic formulae	Mary: I do not know about "y", "formula"	Instructor: What do you know about each variable?	
	<Visual mediator> $r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$	Mary: that's why we need one more equation	
	Mary [revised the formula] 	Instructor: Do you want to try to use a different function?	
		Steve: The height goes up by 1.768 ft	Mary: The height is about 1.8 ft

Table 25.3 Mathematical solution → real-world meaning of solution

Modelling process	Words and visual mediator	Routines	Endorsed narratives
4.1 Identifying mathematical results with their real-world counterpart	Mary: "a huge number". Since "x" is being squared and added 800x	Mary: It is not that far off	Since "x" is being squared and added 800x, for even a little expansion, it causes a great increase in height [from Mary's written task].
4.2 Contextualizing interim and final mathematical results in terms of RW situation 4.3 Integrating arguments to justify interpretations		Mary: The length increases by 1,000 but then the height increases a half of foot	Mary: The arc will look a lot closer to the axis than what we have drawn. The smaller the arc is the more like a triangle it looks [sic]. Versus, if the arc is bigger, and there is a triangle, that's where the missing space is.

25.4.2 *Pre-service Teachers' Conception of Mathematical Modelling*

From the participants' completed open-ended questionnaires and reflective journals, the themes of narratives were revealed in terms of their conceptions of mathematical modelling, views on modelling tasks and role of mathematical modelling from a pedagogical perspective. Excerpts from the participants' completed questionnaire and journals are displayed in Table 25.5.

All three participants' conception of mathematical modelling involved a real-world situation/problem. The pre-service teachers' views on modelling were consistent with a *pragmatic perspective* (Pollak 1969), which focuses on the developing

Table 25.4 Real-world meaning of solution → revise model or accept solution

Modelling process	Words and visual mediators	Routines	Endorsed narratives
5.2 Considering real-world implications of mathematical results	"The arc", "triangle", "triangular model", "the arc model"	Instructor: Which of the models do you think works better and why?	Steve: More accurate.
5.3 Reconciling mathematical and real-world aspects of the problem	"Curvature" of the circle, "straight"	Steve: Easier.	Mary: It is not rising up like a triangle, that is not how nature works, nature works like round figures. The triangle will go above the circle. It would not be inside the circle. Because the curvature of the circle makes up a lot more feet than the line. Line is more direct.

Table 25.5 Pre-service teachers' views on modelling

Questions	Excerpts from participants' responses to journals and a questionnaire
Conception of modelling	Real-world situation that can be described using mathematical languages. (Jesse)
	Representing graphical or numerical equation that can assist you to solve real-life problem. (Steve)
	Mathematical representation of data or situation that simplifies real-world problem. (Mary)
Views on mathematical modelling tasks	Modelling tasks required high-ordered thinking and outside box. (Jesse)
	Good modelling problems give freedom for students to make their own assumptions like real-life problems instead of giving all information or mathematical equations. (Steve)
	It has to be realistic. (Steve)
	Open-ended and challenging. (Mary)
	Requires deep understanding of mathematics and go through modelling process. (Mary)
The goal of teaching mathematical modelling	Learn how to do mathematical modelling, they see how mathematics fits into their daily lives, how you can solve problems quickly and efficiently through mathematical angle. (Jesse)
	Use their knowledge in a different situation to solve a real-world problem. The goal is for students to find mathematics in order to solve a real-world problem and discover relations as well as come up with a model to represent the problem. (Steve)
	Help people to solve real-world problems since you have to make your assumptions and decisions in life, not like mathematics textbooks. (Mary)

ability of learners to apply mathematics to solve practical real-life problems (Kaiser and Sriraman 2006). The participants' common goal of teaching modelling was helping students to learn how to do mathematical modelling and solve real-world problems, which is a similar approach to the modelling as content approach where modelling is content in its own right (Galbraith and Stillman 2006). The purposes of the approach are to develop students' abilities to apply mathematics to problems in their world taking mathematics beyond the classroom and to use the real-world context as a key component (Galbraith and Stillman 2006).

As the participants started with a structured series of questions before encountering the messy real-world situation with the leading question of the *Bending Steel* task, the analysis for this study did not capture how pre-service teachers use contextual knowledge to simplify from the *messy real-world situation to real-world problem statement*. The commognitive perspective highlights how cognitive and social aspects are manifested in modelling activities. Participants' *word use* and *endorsed narratives* indicate their lack of modelling experience when simplifying a model from the real-world situation and verifying a model mathematically. Generalization of the results from this study is neither intended nor possible; however, the study gives insights into the nature of pre-service teachers' mathematical modelling knowledge and conceptions of mathematical modelling and demonstrates a new linking of analytical tools. Further study will examine how pre-service teachers' pedagogical content knowledge of content and students and its relationship with other areas of teachers' knowledge of mathematical modelling are manifested during their modelling instruction.

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Chapter 26
Mathematics Teachers' Learning at the Boundaries of Teaching and Workplace

Giorgos Psycharis and Despina Potari

Abstract This chapter describes how novice and experienced mathematics teachers integrate authentic workplace contexts into mathematics teaching. This goal was inspired by the European MaSciL project and introduced to the teachers in the context of a masters programme in mathematics education. Under an Activity Theory perspective, we use the notions of activity system and boundary crossing to study the process of teachers' professional learning. In particular, we analyse teachers' boundary crossings between two activity systems: mathematics teaching and workplace. Results indicate that collaborative task design and reflection made teachers combine elements from the workplace into mathematics teaching. Different ways of linking reality and mathematics teaching were identified in the modelling process in which the students were asked to be engaged.

Keywords Activity system • Boundary crossing • Workplace mathematics • Teacher education • Professional learning

26.1 Introduction

The introduction of teaching innovations is a central feature of mathematics teacher education programmes. Watson and Mason (2007) report innovations in such programmes that engage prospective and practicing teachers in using research to design, analyse, try out and reflect on the use of tasks with learners. An emerging research area is the exploitation of realistic contexts in mathematics teacher education and the modelling process (e.g. Doerr 2007; Daher and Shahbari 2015). The workplace has not yet been such a context for many research studies, but those that exist indicate potential benefits for teachers of linking the workplace to education. For instance, Nicol (2002) found that a teacher education programme including

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_26

visits to workplace sites helped prospective teachers keep mathematics contextualised when designing student activities. To this end, Triantafillou et al. (2016) studied how practicing teachers integrated workplace activity into mathematics teaching identifying facilitating or hindering factors. Approaches of this type introduce new sites of activity that challenge teachers to move within and between different practices, involving them in crossing boundaries created by different contexts (Akkerman and Bakker 2011). A key challenge then is to support teachers make links between these contexts and address their professional learning. Research focusing on the use of mathematics in the workplace indicates that school mathematics practices and workplace practices differ substantially (Hoyles et al. 2010). Nevertheless, the workplace context has potential for offering authentic situations that provoke problem-solving and modelling (Wake 2014).

Our study is inspired by the philosophy of the MaSciL project,[1] which provides the context for introducing innovative teaching approaches to teachers. Project MaScil aims to promote the integration of inquiry-based learning (IBL) and the world of work (WoW) into teacher education and professional development. The actions taken to achieve these goals included task design based on non-routine workplace situations, the development of communities of teachers and teachers' engagement in designing, implementing and reflecting on mathematics teaching. In this chapter, we use the Activity Theory (AT) and the construct of boundary crossing to study the process of teachers' professional learning when they are challenged to link mathematics teaching to the workplace through modelling.

26.2 Theoretical Considerations

In this section, we first discuss our perspective on the relation between workplace and modelling and then provide a short description of the main constructs we use from AT and boundary crossing.

26.2.1 Workplace and Modelling

Workplace contexts offer rich situations for mathematical modelling. Kaiser et al. (2013) describe a variety of modelling activities that have been designed and used in upper secondary mathematics classrooms on the basis of realistic everyday and workplace contexts. Professionals in workplace settings are involved in modelling, problem-solving and mathematical processes especially in situations where

[1] MaSciL: Mathematics and Science for Life project (see www.Mascil-project.eu) was funded by the European Union seventh Framework Programme (FP7/2007–2013) under grant agreement no. 320693. This chapter reflects only the authors' views, and the European Union is not liable for any use that may be made of the information contained herein.

instruments and technological devices break down (Pozzi et al. 1998). Trial and error, logical exclusion, reasoning and justification, visual inspection and self-monitoring are some examples of processes that are central in workplace mathematical activity (Triantafillou and Potari 2010). In these processes, professionals' actions are mediated by a number of conventions and inscriptions (Noss 2002). Although the variety of processes and tools used and developed in a workplace are essential elements of the modelling process, transforming authentic workplace situations in the classroom for engaging students in mathematical modelling is a complex process. Wake (2015) considers, from an epistemological point of view, the differences between modelling in the workplace and modelling in the mathematics classroom. He points out that in the workplace, a focus on context often deprives the visibility of the underlying mathematics, while the opposite occurs in school. Thus, in the modelling cycle, the initial stage of developing a model of the reality is crucial, and it requires a particular attention to mathematics teaching. Wake (2015) elaborates further how to facilitate the coupling of reality and mathematics in classroom contexts by suggesting that learners need support to develop critically and mathematically informed models of complex realities, to construct and deconstruct real and mathematical models of complex situations and to understand how the structure of models of workplace realities relate to the models of their mathematical counterparts. This perspective is also in the philosophy of MaSciL where teachers are encouraged to attribute to their students the role of the professional who faces a complex reality. In our study, we are interested in the interplay between workplace and mathematics teaching in the teachers' attempt to integrate workplace into their didactical designs, classroom implementations and reflections. The chapter is primarily focused on the teachers' moves horizontally back and forth in workplace situations or contexts so as to see how they can be represented mathematically (horizontal mathematisation). Existing research in mathematics teacher education and modelling shows that, although teachers' understanding of mathematical modelling can evolve through designing, implementing and reflecting on modelling activities, the link between mathematical modelling and 'real-life' contexts in teachers' work requires more research (Anhalt and Cortez 2016).

26.2.2 Activity Theory and Boundary Crossing

The *activity system* is a basic concept of AT in Engeström's (2001) approach. It is collective and tool-mediated, and it needs a motive and an object. Individual and group actions are studied and interpreted against the background of entire activity systems. Activity systems are transformed through contradictions when a new element comes from the outside. Transformations of activity systems are related to interventions that take place and describe phenomena of developmental character. In our study, the focus is on developments in teachers' mathematics teaching in the context of teacher education. Under Engestrom's (1999) perspective, teachers' professional learning can be seen as expansive learning emerging "as practitioners

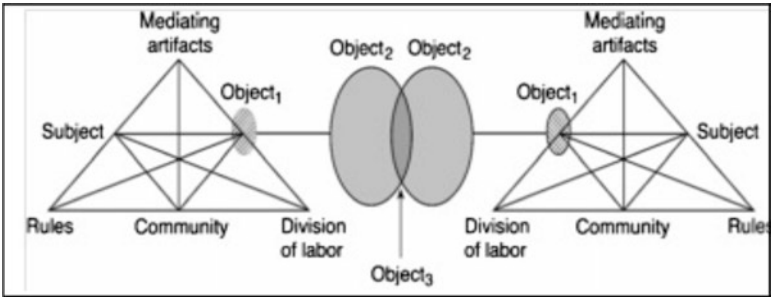

Fig. 26.1 Interacting activity systems (Engeström 2001, p. 136)

struggle through developmental transformations in their activity systems" (p. 7). This learning approach entails a re-conceptualisation of development as a horizontal movement across borders.

Figure 26.1 shows a representation of a third-generation activity in the form of two interacting activity systems. The two triangles indicate the basic dimensions of the second-generation AT with elements: the subject and the object of the activity that is constructed through the mediation of tools, the community in which the subject participates and its rules and the division of labour. Object 1 moves from an unreflected and situational given goal to a collectively meaningful object constructed by the activity system (object 2) and to a potentially shared or jointly constructed object (object 3). In the present study, we distinguish two activity systems: the system of mathematics teaching in which the teachers have been involved and the system of the workplace supported by the MaSciL philosophy.

The third-generation of AT considers learning emerging in dynamic movements between interacting activity systems. This draws our attention to the metaphor of *boundary crossing* (Suchman 1994). We adopt the view of Akkerman and Bakker (2011) for *boundaries* as "sociocultural differences that give rise to discontinuities in action and interaction" (p. 139). People who cross boundaries are often called *boundary crossers*, while a *boundary object* is a single object that has different meanings in several intersecting worlds but retains a common essence. Boundary crossing has been conceived as the efforts of individuals or groups at boundaries to establish or restore continuity in action or interaction across practices (cf. Bakker and Akkerman 2014). Boundary crossing between activity systems has been seen as a way to address learning through four learning mechanisms: identification, coordination, reflection and transformation (Akkerman and Bakker 2011). These mechanisms concern the different ways in which learning can occur when people interact with, move across and participate in different practices:

1. *Identification*: Boundary crossing can lead to a renewed insight into what the different practices concern.
2. *Coordination*: Boundary crossing can also lead to establishing minimal routine exchanges between two practices so as to facilitate transitions.

3. *Reflection:* Reflection involves going deeper into the specificities of two practices and learning to consider one practice by taking on the perspective of the other practice.
4. *Transformation*. Transformation leads to changes in practices or even the creation of a new practice that stands between the established ones.

Little attention has been given to how boundary crossing can be embedded into mathematics teacher education as a way to study teachers' professional learning at the boundaries of multiple practices except for the Wake et al. (2016) study. In the present study, we analyse the interaction of two activity systems focussing more closely on boundary crossing between these systems in order to study the teachers' professional learning in relation to integrating workplace contexts into mathematics teaching.

26.3 Methodology

26.3.1 The Participants

The study participants were mathematics teachers following a masters course that is part of a 2-year masters programme in mathematics education at the University of Athens. Experienced and novice mathematics teachers participated in the course. The experienced teachers came from lower and upper secondary schools, and their teaching experience ranged from 2 to 20 years. The novice teachers were mainly offering private tuition to students to help them with school mathematics. During two academic years (2013–2015), we (as teacher educators) introduced the workplace as a context for task design in the spirit of MaSciL. We encouraged teachers to use MaSciL classroom tasks or develop their own as part of their teacher education activities. Twenty teachers (13 novice and 7 experienced) worked in groups of two to four with MaSciL tasks or others in the same spirit.

26.3.2 The Course and Tasks

The course lasted 13 weeks with weekly sessions of 4 h. The main goals were to support teachers to link research findings and actual teaching, exploit different resources (e.g. digital tools, videos) in their didactical designs and explore the role of context and tools in students' conceptual understanding. In the course, we initially introduced IBL and WoW (e.g. by presenting research findings from the corresponding literature) and engaged teachers in the cycle of design-implementation-analysis-reflection. This cycle began with selecting a MaSciL task or designing an IBL task connected to the WoW. Then participating teachers were asked to read research papers in mathematics education related to

Fig. 26.2 The cylindrical tank at the fuel station

IBL, WoW and students' understanding of specific mathematics concepts related to the task. After this, teachers used the task in the classroom and analysed students' modelling process with an emphasis on the role of the workplace context and tools. Finally, teachers reflected on task design and use in teaching and linked emergent issues with existing research. Teachers' designs were discussed during the course and the analysis of the interventions, and teachers' reflections were presented in the final course session.

Here our focus is on one teacher group. It consisted of two experienced teachers (Elie and Natasa) and one novice (Manos) who designed the task *Fuel Station* in the context of the course. In this task, the students take the role of a worker (Giorgos) who supervises a fuel supply company's filling of cylindrical tanks with petrol at a fuel station (Fig. 26.2). In this scenario, an employee from the fuel delivery company fills an empty petrol tank and claims that the amount needed to fill that tank was 7000 l. This is the value that also appears on the supply company's counter. Giorgos must then verify whether this amount is correct by immersing a scaled (in cm) stick from the top of the tank and taking a measure corresponding to the liquid level. The students explore how Giorgos can do the calculation.

26.3.3 Data Collection and Analysis

The data consisted of (1) teachers' written accounts/journals in which they described their design rationales and implementation experiences, (2) produced artefacts (worksheets, microworlds, etc.), (3) teachers' PowerPoint presentations, (4) selected videos of discussions that took place in the courses and (5) selected teacher interviews. For the analysis, we adopted first a broad, data-grounded approach (Strauss and Corbin 1998) focussed on the group's actions and goals related to each activity system. At a second level, we looked for emerging interactions of these systems and of their objects taking into account boundary crossing and the corresponding learning mechanisms.

26.4 Results

26.4.1 The Two Activity Systems

In Table 26.1, we present an analysis of the two activity systems in which the teacher educators intentionally encouraged the teachers to engage.

26.4.2 Boundary Crossing in the Group

The teachers referred mainly to the elements of the activity system of the workplace. The group's attention was primarily on the ways that an authentic situation could be transformed into a school task. Manos proposed the initial idea for *Fuel Station*, as he had been working part-time at a fuel station at the time. As he put it in his interview:

> I knew that the exact amount of fuel in the station is measured through the use of a scaled stick and ready-made table of values providing the output given the input … We needed to

Table 26.1 Analysis of the activity systems of mathematics teaching and the workplace

Elements	Mathematics teaching	The workplace
Subject	Teachers (the members of the group)	Workers (employees at a fuel station)
Object	Design and implement a classroom intervention integrating IBL/WoW	Carrying out efficiently their professional tasks (e.g. serve clients with fuel)
Tools	Curriculum, textbooks, teachers' and classmates' experiences, MaSciL tasks and MaSciL philosophy	Artefacts developed and used in the workplace (e.g. scaled stick for measuring fuel)
Community	Community of teachers	Workplace community
Rules	Teachers design tasks on the basis of their teaching expertise.	Tasks fulfil needs of the workplace at an operational level.
	Tasks are mainly of closed type and used for practising specific skills.	Tasks are often based on black-boxing processes. It is difficult to unpack the hidden mathematics.
	Students mainly listen to the teacher, respond to their questions and work individually.	Workers work individually/collaboratively to achieve a context-bounded outcome.
Division of labour	The teacher designs lessons individually by implementing the curriculum. This work is not shared with others.	The workers' goal-oriented activity is focussed on performing assigned tasks. The workplace production is evaluated by their supervisor(s).

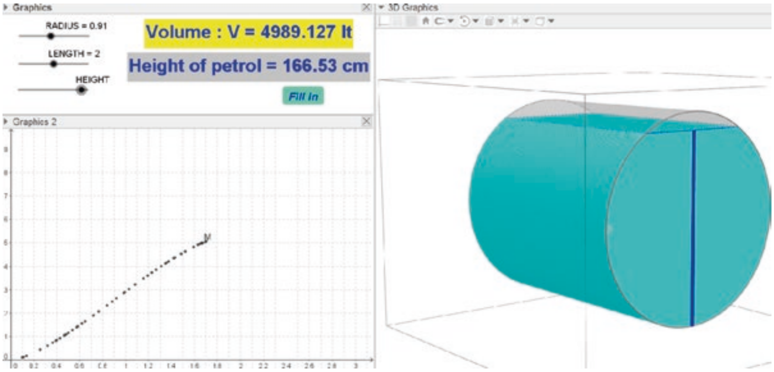

Fig. 26.3 The mathematical model in GeoGebra

> find ways to connect this reality to the school mathematics…Our intention was to see if a design based on an authentic practice would work in a 9th grade class. (Manos's interview)

The teachers decided to implement their task in lower secondary schools where the group's practicing teachers were working at that time. Thus, an additional challenge to be addressed was how to connect the authenticity of the situation to students' prior knowledge. For this, they first analysed task potential to support the connection to the workplace and explored the mathematics underlying the filling of a cylindrical tank (i.e. in terms of functional dependency). The teachers simplified the real context by engaging students in a task where they had to explore how the volume of a cylindrical tank in a vertical position changes in relation to its height. They expected this real model was easier for the students to mathematize as they had the relevant mathematical knowledge (e.g. calculation of volume of a cylinder, proportional relations). Then, they challenged the students to consider the same problem when the cylindrical tank was horizontal as in the authentic situation. For this they simulated the situation using digital tools (i.e. GeoGebra files) to allow students' experimentation with the real phenomenon. In this context, students were able to change tank height with a slider, while the tool gave the corresponding volume values. Students had to choose pairs of height-volume values from the screen and insert into a table on their worksheet. Then, still working with paper and pencil, they placed the corresponding points in a co-ordinate system and sketched a graph describing the functional relation. In a subsequent task, the graph was automatically provided by the software, and students compared it with their own. Figure 26.3 shows the graph in relation to the variation of petrol height in GeoGebra for a given radius and length of cylindrical tank.

Reference to the workplace activity dominated the group's reflections and discussions during the design process. Manos, acting as *boundary crosser* between workplace and teaching, challenged the others' views regarding workplace practices and possible connections to school mathematics. Emerging tensions concerned the different goals underlying workplace actions and those in the school classroom and the difficulty of students' and teachers' familiarity with the authentic context. In their attempts to connect the mathematics within fuel stations to school-based mathematics, the teachers recognised a need to find detailed information about the original workers' practices and the difficulty of conceptualising the specificities of the situation. Further, they appreciated the role of personal workplace experience in the design process, as Manos noted: "If I did not have the experience of working in the fuel station, the activity would not have been successful…It is important to be close to the workplace, to know the subject of the work very well" (Manos's interview). The teachers also recognised the complexity of adapting the problem for the school level by including and excluding real context elements (e.g. excluding reference to physics or chemistry inherent in the situation). Additionally, they appreciated the importance of engaging students in solving workplace problems by use of school mathematics. In their initial attempts to couple school mathematics and reality in the modelling process, they provided a simplified situation model (vertical cylinder) that allowed students to work with known mathematics. Then, a model closer to the authentic situation (horizontal cylinder) brought a new iteration of modelling. This time, the dynamic manipulation of parameters involved in the problem (length, height and radius) supported students in approaching the mathematical model experimentally in close relation to reality.

For this group, boundary crossing was characterised by contradictions between activity systems leading to adaptations of the didactic tools (e.g. tasks) from the activity system of the workplace to that of mathematics teaching. In the reflection phase of their activity, the teachers referred to the role of the workplace in their teaching as can be seen in this excerpt:

> The calculations (i.e., in the table used by workers) have been carried out taking into account elements of physics and chemistry. We chose not to include these elements in the activity. We were interested in the mathematical part. Thus, we did not say anything about the table during the implementation. We just informed the students that checking the volume of fuel in the stations is carried out through the sticks. Therefore, we introduced authenticity and workplace practice in our design without revealing all the tools used in the workplace. We wanted the challenge – namely the tools through which the employee can find out the volume – to emerge through school mathematics. (Manos's interview)

Here the teachers used the ready-made table of values as a *boundary object* intersecting the world of the workplace and teaching. In the workplace, the table is used as a black box to carry out calculations in a routine way. In contrast, the teachers considered the table in the classroom as a mathematical representation. They engaged students in calculating the requested values through mathematical practices involving different function representations (i.e. graph, table). There was a dominance of mathematics teaching over the workplace, since contradictions were resolved by the rules of the dominant activity system and not according to reality. This seems related to teachers' familiarity with school mathematics, limited experiences in integrating the workplace into mathematics teaching and construction of the mathematical model requiring knowledge beyond lower secondary. However, the analysis indicated that the teachers constructed a hybrid situation where school mathematics appears to play a complementary role to workplace mathematics.

26.4.3 Teachers' Professional Learning

The analysis above focussed on the interactions and the corresponding contradictions among the activity systems of workplace and teaching. The teachers' starting point in the group was Manos's work experience at a fuel station. Although the teachers' engagement in exploring the connection between workplace practices and school mathematics was primarily characterised by contradictions, they were able to progressively make links between activity in the workplace and mathematics teaching and enrich students' modelling experiences. It seems that the object of the teachers' activity was re-conceptualised and expanded, as it was more focussed on the ways that innovative workplace practices could be linked to school mathematics and integrated into classroom teaching. With regard to teachers' learning, the analysis showed that different kinds of learning processes were at stake at the boundary crossings between classroom teaching and workplace. *Identification* was evident when the teachers recognised the intersecting practices of the workplace and classroom teaching. *Coordination* took place through their recognition of the need to keep a balance between the authenticity of the workplace and the mathematical inquiry in the classroom. *Reflection* was indicated by the fact that the teachers looked differently at workplace practices by taking into account the ways by which these practices can be exploited in mathematics teaching. Regarding *transformation*, the analysis indicated the emergence of hybrid practices (i.e. teaching characterised by a merging of workplace practices and school practices) in the hope of being better shaped in the future.

26.5 Conclusions

In this chapter, we asked if and how novice and experienced teachers were able to integrate innovative practices such as the workplace into mathematics teaching. In the course, we stimulated boundary crossing in the hope of propelling transformation, in the form of practice integration. The results show that the teachers experienced tensions or contradictions in integrating the workplace into teaching. Mathematics teaching practices dominated workplace practices and school mathematics complemented workplace-related mathematics. The coupling of reality and mathematics that Wake (2015) considers important in integrating workplace into mathematics teaching was approached by teachers through simplification of the authentic situation and the simulation of it with the use of digital tools. However, the boundary crossing from one context to the other was facilitated by a boundary crosser (Manos) who supported further elaboration of the real context and offered insights on how to integrate authentic workplace artefacts into the group's didactical designs. Concerning their professional learning, we identified all learning types in teachers' activity. Identification, coordination and reflection were evident in teachers' attempts to keep a balance between elements of workplace in teaching. Transformation was indicated by teachers' attempts to bring together the objects in the different activity systems, leading to the integration of these practices into a whole. From a methodological perspective, we have exemplified the potential of AT and boundary crossing taking into account the complexity of teacher education and other professional settings.

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Chapter 27
Case Study of Pre-service Teacher Education for Mathematical Modelling and Applications Connecting Paintings with Mathematics

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Abstract The purpose of this chapter is firstly to investigate how graduate teacher education students critiqued and validated mathematical models through connecting paintings and the mathematical domain and secondly to analyse the students' decision-making. A *meta-question* posed by the researchers for the students was to find geometric figures which may be hiding behind paintings in order to explain them to an audience of the Otsuka Museum of Art. Students' critique and validation of the achieved mathematical models for their decision-making resulted from interdisciplinary or extra-mathematical considerations. To realise the *meta-question*, students performed their demonstration to an audience with a diversity of knowledge and skills.

Keywords Pre-service teacher education • Interdisciplinary knowledge • Validation • Painting • Decision-making

27.1 Introduction

Mathematical modelling and applications focus on problem-solving that connects mathematics with the extra-mathematical domain. The relations and boundaries on contexts of each domain are important. Internationally, mathematical modelling and applications have been discussed for more than three decades, and the importance of teaching mathematical modelling has been emphasised. One active area of

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_27

research is in-service and pre-service teacher education (Blum 2015; Kaiser and Schwarz 2006). Cai et al. (2014) noted that "there is no doubt that teachers play an important role in fostering students' learning of mathematical modeling and students' learning of mathematics through engagement in mathematical modeling" (p. 148). From the perspective of teacher education on mathematical modelling, Freudenthal (1973) and Pollak (1979) mentioned that teachers should be well-acquainted with real-world knowledge about modelling contexts and with the associated pedagogical understanding and open-endedness of modelling tasks. Cai et al. (2014), in summarising previous research literature into teacher education in mathematical modelling, focussed on characteristics of teacher education programmes and "interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modelling" (pp. 162–163) finding little such research into the latter.

As a follow-up of pre-service teacher education research, we reviewed chapters in Stillman et al. (2013) and Stillman et al. (2015) (e.g. Biembengut 2013; Hagena 2015; Tan and Ang 2013; Villarreal et al. 2015; Widjaja 2013; Winter and Venkat, 2013), but there was no teacher education research into interdisciplinary or extra-mathematical knowledge requirements for mathematics and the arts. Widjaja (2013) and Tan and Ang (2013), for example, conducted pre-service teacher education studies into modelling activities for pre-service teachers who had no previous knowledge of mathematical modelling. Widjaja (2013) investigated pre-service teachers' awareness of mathematical modelling through the task of *Re-designing a Parking Lot*. Tan and Ang (2013) found that the experience and knowledge gained can help them explicate aspects and nuances of the modelling process with respect to novel modelling tasks.

In our study to be reported here, we conducted a pre-service teacher education programme in mathematical modelling and applications for graduate students who were interested in mathematics education or would like to become mathematics teachers. This programme consisted of 11 lessons (90 min each) over 6 weeks in late 2013 to mid-January 2014 and included eight graduate students. All students had no previous knowledge of mathematical modelling and applications. In addition, they had no experience with lessons that integrated mathematics with other subjects. The purpose of this programme was for participants to gain informal knowledge of mathematical modelling through the activity of connecting between paintings and the mathematical domain and to translate their informal knowledge into formal knowledge by the tutors' instruction. A *meta-question*[1] we provided for students was "To find geometric figures which may be hiding behind paintings in order to explain them from an artistic and a mathematical viewpoint to an audience of the Otsuka Museum of Art[2]".

[1] Niss (2015) did not define it, but seems to call a first problem "meta-question".

[2] The Otsuka Museum of Art is a "Ceramic board masterpiece art museum" with the largest exhibition space in Japan. This museum displays faithfully reproduced paintings to their original colours and size on large ceramic boards. There are more than 1,000 replicas of priceless masterpieces of Western art, from ancient murals to modern paintings. See http:// www.o-museum.or.jp/english/

For this programme, we formed an instructional team of two university teachers, one teacher consultant from Tokushima prefectural general education centre and one specialist and some staff from the museum. The four paintings which the graduate students tackled were the *School of Athens* painted by Raphael Santi and the *Annunciation*, *Last Supper* and *Mona Lisa*, all painted by Leonardo da Vinci. Towards the end of the programme, an approximately 1-h guided tour "Art Meets Mathematics!" was conducted twice at the museum by the student groups.

The purpose of this chapter is (a) to investigate how graduate students critiqued and validated the achieved mathematical models through connecting between paintings and the mathematical domain and (b) to clarify their decision-making through our analysis.

27.2 Practical Frameworks and Method

In order to achieve the purpose of this chapter, we implemented three frameworks in the pre-service teacher education programme of mathematical modelling and applications.

27.2.1 Connecting Between Paintings and the Mathematical Domain

The great artist Leonardo da Vinci chided that "Let no man who is not a Mathematician read the elements of my work" (Prolegomena and General Introduction to the Book on Painting, I:3). We therefore chose four paintings of the Renaissance because geometric figures such as the golden triangle, golden rectangle, root rectangles (see Fig. 27.1a) and so on are more likely to lie hidden in them as these devices are claimed to have been used by the painters of the era (see Posamentier and Lehmann 2012), and Phi (used to designate the golden ratio) "aroused the interest of many mathematicians…during the Renaissance" (Huntley 1970, p. 25).

At the beginning of the programme, we provided students with an application task to find geometric figures which may be hiding behind paintings by use of a hand-held geometrical lens (see Fig. 27.1). Even if students derive some mathematical answers, they have no chance to validate them directly. This is because they cannot ask the painters of the Renaissance about their solutions and little is documented (Posamentier and Lehmann 2012). We hypothesised that students would critique and validate their achieved mathematical models by themselves using some interdisciplinary or extra-mathematical knowledge such as historical facts, arts, the history of Christianity or cultures. We thought this task would lead to independent modelling experience.

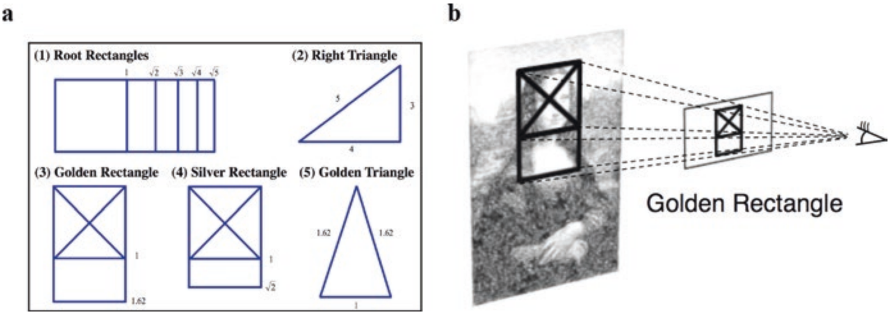

Fig. 27.1 Schematic diagram of the geometrical lens showing (**a**) the transparency and (**b**) the lens in use

27.2.2 Presentation to a Diverse Audience

To realise the *meta-question*, students have to perform their demonstration to an audience with a diversity of knowledge and skills. Consequently, students needed a broad knowledge about the mathematical domain and extra-mathematical domain in advance, and this might need to be learnt. We also hypothesised that the presentation with a correspondingly diverse audience would foster students' abilities for critique and validate the mathematical models. As regards the demonstration of paintings by the mathematical viewpoint, we used a hand-held geometrical lens as a tool for verification of the geometric figures shown in Fig. 27.1, because we assumed that some in the audience would not be good at mathematics.

27.2.3 Instructional Team of Pre-service Teacher Education Programme

We formed our instructional team to support the preparation of the situation to be modelled at the beginning of this programme for students and to facilitate the modelling and application activity of students as mentioned above. The mentors in this team consisted of the university teachers who are researchers in mathematics education, the general education centre teacher consultant who conducted the class that integrated mathematics with paintings at a junior high school and a specialist from the Otsuka Museum of Art who is the expert about paintings. The university teachers supported students to gain informal knowledge of mathematical modelling through the activity connecting between the paintings and the mathematical domain and to translate their informal knowledge into formal knowledge mainly. The teacher consultant supported students to interpret the paintings from a mathematical viewpoint. The specialist from the museum supported students to interpret the paintings from an artistic viewpoint. In addition, all mentors facilitated mathematical modelling and applications activities of mathematisation, de-mathematisation and validation depending on the needs of the student.

27.3 Elements of the Pre-service Teacher Education Programme

This programme of 11 lessons was separated into four stages.

27.3.1 Content of Each Stage

27.3.1.1 First Stage: Acquisition of Basic Knowledge (Three Lessons)

The teacher consultant explained the content of his previous teaching on paintings and mathematics in junior high school. He also taught the usage of the geometrical lens (see Fig.27.1a, b). The university teachers confirmed geometric figures and mathematical content: the golden ratio, silver ratio, golden rectangle and so on (2 December 2013). The specialist from the Otsuka Museum of Art explained writings about the personality of painters and the background of Renaissance art (18 December 2013).

27.3.1.2 Second Stage: Developing Content for Demonstration to an Audience (Two Lessons)

The students in each group attempted to find geometric figures in every one of these paintings. After finding any, each group discussed the content for the demonstration to an audience. As needed, each group accessed some books, articles, the Internet and so on (18 December 2013).

27.3.1.3 Third Stage: Mock Demonstration and Revising (Two Lessons)

Each group conducted a mock demonstration in front of their instructional team twice. After the mock demonstrations, they revised their content of the demonstration taking on-board some feedback from the instructional team, especially from the staff of the museum (20 December 2013).

27.3.1.4 Fourth Stage: Demonstration and Reflection (Four Lessons)

The guided tour on the four paintings was held for an audience at the museum (13 January 2014). After the guided tour, students reflected on the whole 11 lessons. The university teachers translated students' informal knowledge of mathematical modelling and applications into formal knowledge (16 January 2014).

27.3.2 Content of the Demonstration on the Last Supper

From the analysis of responses to a free description-type questionnaire completed by the audience, it became clear that the aim of the guided tour "Art Meets Mathematics!" was accomplished (Saeki et al. 2016).

In this section, we focus on the students' demonstration related to the *Last Supper*. Students explained the situation of this painting with regard to the consternation of the apostles when they heard Jesus saying, "One of you is going to betray me". They explained that this painting is drawn in *a one-point perspective*. Next, they asked the audience to find geometric figures in the *Last Supper* using the geometrical lens. Then they illustrated a silver rectangle, a vanishing point and root triangles with the copies of this painting (See Fig. 27.2).

They asked the audience to find a geometric figure around Jesus in the painting. Some audience members answered, "It is an equilateral triangle". The students accepted their answer and confirmed with an equilateral triangle-shaped teaching tool near Jesus in the painting. Next, they explained their exploration of the reason Jesus was drawn in an equilateral triangle. Firstly, the students noted the historical fact that Leonard da Vinci painted only visible objects and thus did not want to paint a halo. They introduced this notion using the *Holy Mother of the Cave* in the same exhibit hall in the museum. Secondly, they explained their supposition that Leonardo da Vinci tried to express holiness with an equilateral triangle for Jesus and the

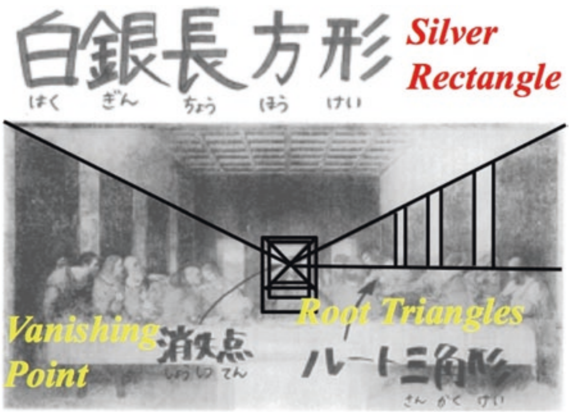

Fig. 27.2 Silver rectangle (Figs. 27.2 and 27.3 are copied pictures from the sketchbook which students used for their demonstration. We have *emphasised lines* and *figures*)

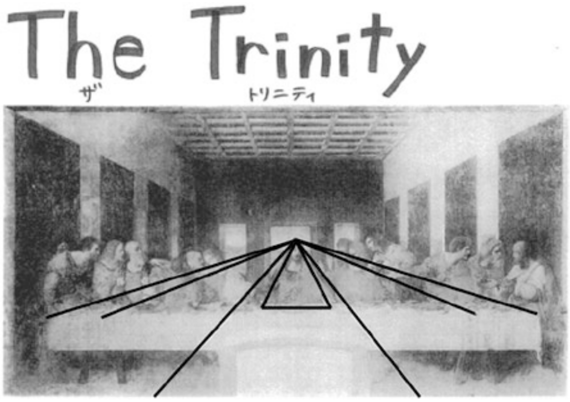

Fig. 27.3 The equilateral triangle and the Trinity

meaning of the Trinity (see Fig. 27.3). Finally, the students explained that the eye of Judas and the right temple of Jesus and a plate between them made the 3, 4 and 5 sides of a right triangle.

27.4 Analysis of Students' Decision-Making on the Equilateral Triangle

In this section, we clarify the reason that the students decided a suitable symbol of Jesus' holiness is an equilateral triangle and analyse the reasons for their decision-making through their preparation for the demonstration at the second and third stages of the pre-service teacher education programme. In order to explain what was happening in terms of modelling in their preparation for the demonstration, we use the modelling cycle and subprocesses that were highlighted by Niss (2015) (see Fig. 27.4). For this analysis and identification, we used videos of the students' activities in these lessons, artefacts, an interview protocol (18 December 2013) and responses of the students to a free description-type questionnaire from the third stage of this programme (20 December 2013).

The *meta-question* we provided for students was, "To find geometric figures which may be hiding behind paintings in order to explain them from an artistic and a mathematical viewpoint to the visitors of the Otsuka Museum of Art". This *meta-question* corresponds to one in the extra-mathematical domain shown in Fig. 27.4.

Preparing for their *meta-question*, they drew lines on a copy of the *Last Supper* accordingly along designs of the table and found that the lines crossed at one point. Through this work, they wondered if any other figures surrounded Jesus (see Fig. 27.3). Then, they came up with a question: "What kind of geometric figure is there around Jesus?" We named this question as Q1, corresponding to one of the *idealised situation cum questions* in Fig. 27.4.

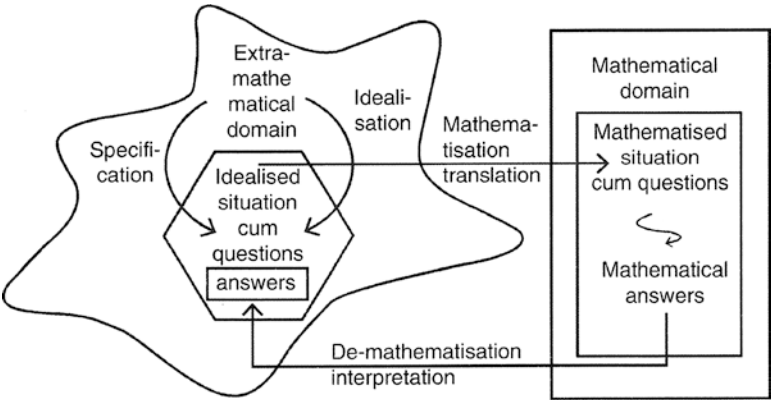

Fig. 27.4 The modelling cycle (Niss 2015, p. 68)

Mathematising Q1, students found these lines are symmetrical by measuring with a ruler. Then they assumed a triangle existed around Jesus. After this, they came up with a question: "What kind of triangle?" We decided this question, Q2, corresponds to one of the *mathematised situation cum questions* in Fig. 27.4.

For the *mathematical treatment* of Q2, they drew a triangle around Jesus on the copy of the painting and measured the lengths of all sides of this triangle. Then, they decided the triangle was equilateral, not a scalene triangle nor an isosceles triangle, as forming the mathematical answer.

De-mathematising the mathematical answer of the mathematical treatment, they raised another question: "Why did Leonard da Vinci try to use the equilateral triangle as a symbol of Jesus?" We labelled this question, Q3, as the "new" idealised question that corresponded to the *idealised situation cum questions* in Fig. 27.4. The students tackled Q3 using Internet content concerning Christianity and equilateral triangles and found the Trinity is regarded as important in Christianity. Then, students decided it was a valid historical fact that Leonard da Vinci had a policy that he did not want to paint a halo as was explained by the specialist in the museum. Additionally, they might have thought that the religious fact that the number *three is* important in Christianity is valid as explained by the teacher consultant.

One student, from the group whose work we analysed, described that "We think that we should be able to explain 'Mathematics hides in art' as well as 'Mathematics hiding in art supports art'" in the student free response questionnaire. We decided this comment indicates *meta-decision-making*.

27.5 Discussion

We conducted a pre-service teacher education programme in mathematical modelling and applications connecting between paintings and mathematics for graduate students. As described previously, the meta-question produced three subsequent questions, Q1, Q2 and Q3. Through analysis of the students' decision-making, we found two points. Firstly, in Q3, students invented their own supposition – "Leonard da Vinci used the equilateral triangle as a symbol of Jesus" – with the Trinity, a historical fact and a religious fact. We thought that Q3 was a trigger for students' critical thinking. Secondly the Trinity, the historical facts and the religious facts are not direct validation of the model of Q3, but a broad-spectrum validation for their decision-making. Through their activity of validation of Q3 for their decision-making, students constructed rich and fruitful contexts of the interdisciplinary or extra-mathematical domain: the Trinity, the historical fact about the no halo policy of Leonardo da Vinci and the religious fact regarding the importance of the number *three*. Before this programme, students had no knowledge of this extra-mathematical domain. In addition, they were *not* Christians. However, to realise the *meta-question*, students performed their demonstration before an audience with a diversity of knowledge and skills. We thought that this *meta-question* motivated the students to acquire a broad spectrum of knowledge of the extra-mathematical domain. Students tackled the meta-question through the activity of connecting the paintings and the mathematical domain. Consequently, we considered that students grasped informal knowledge of mathematical modelling subprocesses as shown by Niss (2015), such as mathematisation, de-mathematisation, validation and so on.

At the end of the fourth stage of this programme, the university teachers explicitly taught knowledge of the mathematical modelling cycle and subprocesses. Furthermore, we used the example of the *Last Supper* and explained their decision-making on the equilateral triangle according to the modelling cycle. The actual task is more of an application of mathematics than full modelling, but it shared some aspects which allowed us as educators to talk about the full modelling cycle. After this explanation, all students reflected on their activity through the modelling cycle noting which aspects were shared with modelling. As a result of analysing the student free response questionnaire by a Grounded theory approach (Corbin and Strauss 2008), it became clear that all students were motivated to develop mathematical modelling and applications materials connecting mathematics and an interdisciplinary domain (Saeki et al. 2016). Consequently, we considered that students have grasped the formal knowledge of mathematical modelling and applications through this programme.

27.6 Conclusion

Decision-making and validating is one of the ten skills categorised by the Assessment and Teaching of 21st Century Skills Project (ATC21S), namely, "critical thinking, problem-solving and decision-making" (Binkley et al. 2012, p. 18). We successfully conducted a pre-service teacher education programme in mathematical modelling and applications connecting between paintings and mathematics for graduate students. All students had no previous knowledge of mathematical modelling and applications. In addition, they had no experience with lessons that integrated mathematics with other subjects. However, students' critique and validation of the achieved mathematical models for their decision-making resulted from interdisciplinary or extra-mathematical considerations. To realise the *meta-question*, students performed their demonstration to an audience with a diversity of knowledge and skills. So, we advocate the necessity and importance of further research for teacher education on mathematical modelling and applications with mathematical ideas in interdisciplinary contexts. We see our result from this study as a small step of mathematical modelling and applications incorporating STEAM, that is, STEM with art (Sousa and Pilecki 2013). As future work, we will analyse the transformation of the students' demonstrations from their first mock ones to those of the guided tours, and we will analyse other groups' demonstrations.

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Chapter 28
Inquiry and Modelling in a Real Archaeological Context

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Abstract This chapter focuses on studying the potentialities of interdisciplinary approaches for mathematical modelling. The research presents the design of a teaching sequence based on an archaeological context—the ruins of a Roman theatre discovered in Badalona (Catalonia)—implemented with 12–14-year-old students in their 2015 course. The aim was to promote inquiry and student modelling competences and to investigate how making multiple disciplines interact could enhance modelling and inquiry processes. An initial historical situation involving the students was presented to deal with a problem integrating an interdisciplinary approach. Mathematical modelling appeared as a central tool in the teaching and learning processes. Furthermore, a constant dialectic between mathematics and history was required to facilitate evolution of the modelling process.

Keywords Inquiry competence • Modelling competence • Interdisciplinary approach • Task design • Teaching devices • Mathematics and history

28.1 Introduction

This chapter focuses on the design, implementation and analysis of a teaching sequence based upon an archaeological context—the ruins of a Roman theatre discovered in Badalona (Catalonia, Spain). The sequence was implemented in the 2014–2015 course with 12–14-year-old students. A starting situation led to the formulation of a specific research question that was at the core of the students' research. They had to mobilise their inquiry and modelling competences in order to answer the questions proposed using an interdisciplinary approach, in the same way as

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_28

previous work such as Sala et al. (2015) and García and Ruíz-Higueras (2010) has shown.

Our design was concerned with integrating the students' work with the subjects of mathematics and history in an interdisciplinary way. The context of the starting situation (and the generating questions)—an archaeological problem—is a novelty because in general, these kinds of proposals, to promote inquiry and/or modelling, are based in a context in the scientific world. Moreover, most of the proposals focus on procedural models that involve numbers (like statistics, exponential or economic models), whereas the focal point of our teaching sequence is a geometrical model. According to the current curricula guidelines (e.g. DECRET 187/2015) about the necessity of integrating mathematical modelling into daily classroom practice at preschool, primary and secondary school education, the features of the proposal presented in this chapter allow students inexperienced in modelling to start modelling without qualms.

There is a common agreement, shared by researchers and new curricula trends (Stohlmann et al. 2016), about the need to introduce students to a mathematical activity oriented to applied problems and modelling. Nevertheless, there still exists a substantial gap between these ideals and innovative approaches and everyday teaching practices. In particular, different local implementations of modelling activities can be highlighted (such as Burkhardt 2011), but its long-term dissemination remains a big problem to be faced. In such a framework, we assume that it is possible to use local contextual experiences coming from other disciplines such as archaeological experiences that promote a permanence and evolution of mathematical practices overcoming the usual 'applicationism' (Barquero et al. 2013), instead of fostering real mathematical modelling practices.

28.2 Theoretical Framework

In the design of the sequence of tasks, different approaches interact. On the one hand, the mathematical and didactic design quality is justified based on the three criteria of didactic 'suitability' proposed by the onto-semiotic approach, EOS (Godino et al. 2007): (a) emotional suitability, (b) epistemic suitability and (c) ecological suitability. The aim is to design a sequence of tasks in which these different suitability criteria are included. For instance, *emotional suitability* can be justified for the task sequence by the fact that students work with data and evidence—real Roman ruins—from their very close context, in their city and next to their school. The mathematical quality (the *epistemic suitability*) can be justified based on the view that the implementation allows students to trigger relevant processes of mathematical activity, in particular processes of mathematical modelling. In turn, *ecological suitability* is justified by the curricula of these secondary school students having a competency-based approach, where the teaching and learning processes provided by the curricula should promote competences to deal with complex and varied real-life situations.

These curricula guidelines are in the same directions as recommendations from other countries and international organisations such as the National Research Council in the USA (NRC 1996). For instance, according to the NRC, teachers should support the development of abilities of inquiry. These inquiry abilities are hardly ever related to modelling perspectives, but sometimes it is difficult to find the differences between both processes (Artigue and Blomhoj 2013). Recent discourse on inquiry in mathematics education focuses on the use of methods and mathematisation processes, promoting the construction of mathematical hypotheses and models, and the need for arguing, valuing and controlling in an appropriate way to solve a contextual problem (Elbers 2003). From our viewpoint, placing mathematical modelling processes at the core of activities involves promoting other kinds of processes important in a rich and functional mathematical activity (understood as mathematical richness of quality processes).

On the other hand, a way to achieve a high epistemic suitability is to design the sequence of tasks using the notion of *research and study path* (SRP) (Chevallard 2015) as a didactic device to facilitate the inclusion of mathematical modelling in educational systems and, more importantly, to explicitly situate mathematical modelling problems in the centre of teaching and learning processes (Barquero et al. 2008). We assume and use the structure of SRP as the main theoretical construct to design the didactic sequence we will present in this chapter:

1. The starting point of a SRP will be a 'lively' generating question with real interest for the community of study (students, teachers and researchers).
2. During a SRP, the study of the generating question will evolve and open many other 'derived questions'. The study of all these questions will lead to successive temporary responses, which would be tracing out the possible 'routes' to be followed in the effective experimentation of the SRP.
3. The teacher will thus have to assume a (possibly) new role of acting as the leader of the study process, instead of lecturing the students.
4. An important dialectic between mathematics and history that will be integrated in the SRP is the task of posing questions and that of the continuous search for answers.
5. Against the temptation of imposing some answers that are acceptable within the educational institution only, the group of students needs to be invited to defend the successive answers they provide.
6. The dialectics between the media and 'milieu' will also be essential to control not only what exiting resources and answers are available 'outside' the classroom (in the media) but also what tools will help us to validate and integrate them in our study.

Last, but not least, in some previous task designs, we had used historical contexts to develop inquiry and student modelling competences. These have involved local historical contexts such as the study of Iberic ruins to give opportunities to introduce algebraic formulae (Vilatzara Group 2003), finding coins leading to the introduction of numerical systems with 10–11-year-old students (Sala et al. 2013) and the study of the consequences of the War of the Spanish Succession and an inquiry into the geometry of some Roman ruins (Sala 2016). We have found that all of these have promoted inquiry attitudes.

28.3 Conditions for the Design and Implementation of Teaching Sequence

28.3.1 Research Questions

We have designed the teaching sequence presented here intending to develop specific student competences: the competence of inquiry and the competence of mathematical modelling. In addition, it is a special sequence because there are several disciplines and subjects linked: Roman history and mathematics. This kind of sequence is an appropriate way to promote the target competences to be developed by students. Moreover, through its implementation, we aimed to study the conditions that could help students to progress and the institutional limitations and constraints that appear; therefore, our main research questions were:

> Could a teaching sequence based on the study of an archaeological problem in a context very close to students (in their city) promote student competence development in inquiry and mathematical modelling?
> Which features and conditions of this kind of teaching sequence facilitate student progress?
> Which were the observed limitations and constraints in the implementation that could prevent student progress?

28.3.2 Teaching Sequence Design and Implementation

The sequence of tasks started with a problematic and real situation—called the *generating situation* S_0—very close to a real extra-mathematical context. This initial problematic situation, which was introduced to the students by the teacher, was a relevant archaeological discovery in their city, Badalona (a city next to Barcelona, in Catalonia): some Roman ruins that could have been a public building. Nowadays in Badalona, many Roman ruins can be visited, and it is a known fact that archaeologists have found some evidence of the ancient population of Badalona. Thus, the teaching sequence was named 'What are these ruins hiding? Investigating the Roman ruins of *Baetulo*' (*Baetulo* is the Roman old name of Badalona).

The experimentation took place at secondary school level, in a high school called Betulia's School. The designed sequence was tested with a group of 30 students (12- to14-year-old). During all sessions, students worked in the same inquiry teams. At the beginning of each session, the teams had to deliver a report of all work done during previous sessions, and there was one team in charge of explaining and defending its report. It was a way to compare and discuss the work done during the entire process and, particularly, a way for the study community to formalise all the

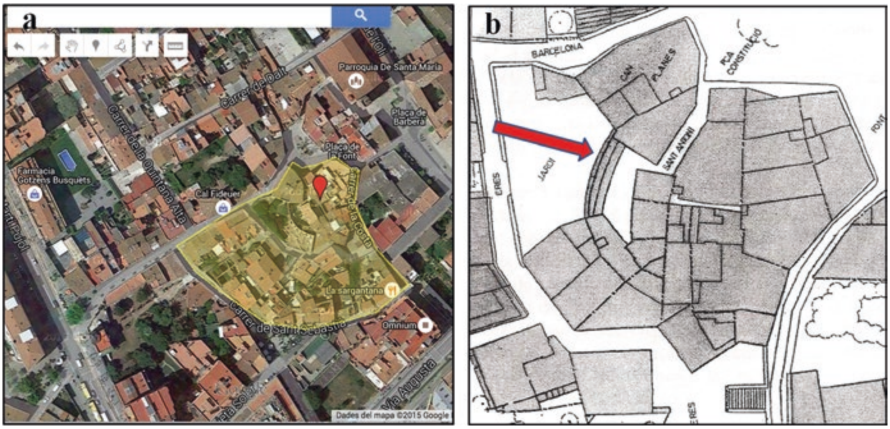

Fig. 28.1 (**a**) Area study (Google maps) (**b**) Detail of the study area; the *arrow* indicates the partial Roman wall discovered (Padrós and Moranta 2001)

questions treated and their successive partial answers. This allowed them to agree on how to continue with the study process.

The teaching sequence was oriented by the idea of reproducing the way researchers in archaeology act. Inspired by the research of Padrós and Moranta (2001), archaeologists of the Museum of Badalona explained their investigation about certain ruins, found some years ago in the centre of the town, in order to discover with which kind of building it could be identified. They summarised their research process with the most important details and their conclusions: the construction investigated could have been an ancient public building, for instance, a theatre.

The students worked in teams and had to investigate—from real data, archaeological reports, canons of Roman architects—what kind of building the discovered Roman ruins could be and its characteristics. They visited the place of the discovery and took photographs and measurements. The current constructions in the zone, houses and streets followed a curious curved shape—easily perceptible in the map of the zone (see Fig. 28.1a, b). This fact indicated that all these constructions were built on top of the ancient constructions or structures. As some research revealed, one of the most important discoveries was a part of a curved Roman wall, a metre and half high (see Fig. 28.1b). The teaching sequence was based on this.

The starting situation, S_0, invited the students to think of these ruins and in their context. This situation also led to the formulation of the initial and generating question:

Q_0: Can the ruins found be a public building? And which kind of public building could it have been?

The students knew it was a partial curved Roman wall discovered by the archaeologists as this was given to them at the beginning of the activity; the shape of this wall could determine the kind of building that it had been part of. They had to find information about all the types of Roman buildings, their shape and their functions.

Fortunately, there were few buildings that had an almost curved shape in one of their parts of the perimeter, namely, theatres, amphitheatres and Roman circuses. This meant the main question Q_0 must be used to derive some other questions, before the students could formulate their hypothesis about which kind of building the ruins were:

Q_1: Which Roman building (theatre, circus, amphitheatre, etc.) shape would concur with the shape of the part of the Roman wall found?

Q_2: What would be the geometrical shape of the whole Roman wall of which the experts had discovered a portion?

The students were organised into cooperative work groups, inquiry teams, of three students to study the questions derived from Q_0, that is, Q_1 and Q_2. From the beginning an inquiry guide was prepared to help students and certain devices that allowed an appropriate work progression. The inquiry guide was designed based on the reports of real investigations carried out by the Badalona Museum professional archaeologist team. This followed a sequenced process introduced by the teachers, which can be differentiated into three stages.

In the first stage, the teacher of history and the first author of the chapter introduced the problem and the starting situation S_0 to the students. All the sources, devices designed, links recommended and the worksheets that they had to follow, were available in the blog designed by the authors: http://ruinesdebaetulo.blogspot.com/. In these first sessions the students started to inquire and search information to understand the context of the problem. Next, they could formulate their preliminary hypothesis and conjectures about which kind of public building the ruins could have been. During this stage, they worked basically with the information about the shape and use of Roman public buildings and the specific archaeological information about the discovery of a Roman wall, helped by indications in the map of the area.

In the second stage, there were different teachers involved in guiding all the sessions (teachers of mathematics, technology, history and Catalan literature). At this stage students carried out developing their research to answer the generating question and their derived questions. They dealt with real data given by a blog; each group had a specific worksheet with some indications, small tasks to introduce the mathematical work and some questions to help progress their inquiry properly. They produced different materials as a result to study, work and deal with the information.

At this time, students were wondering how they could know the shape of the whole Roman wall from the shape of the partial wall discovered. This is a difficult mathematical problem to resolve—to find the geometrical curve from a part of it—because there could be many solutions. However, the dialogue with history limits the possible answers because the Roman buildings only had three relevant shapes: ellipse (amphitheatre), circle (circus) or semi-circle (theatre). Due to the contribution of the historical information, the problem became achievable at the students' level of mathematical knowledge.

A session was held in a square next to the school to discover what kind of curve fitted the discovered wall. The students could work with an exact representation of the part of the Roman wall. They did their fieldwork guided by the teacher of math-

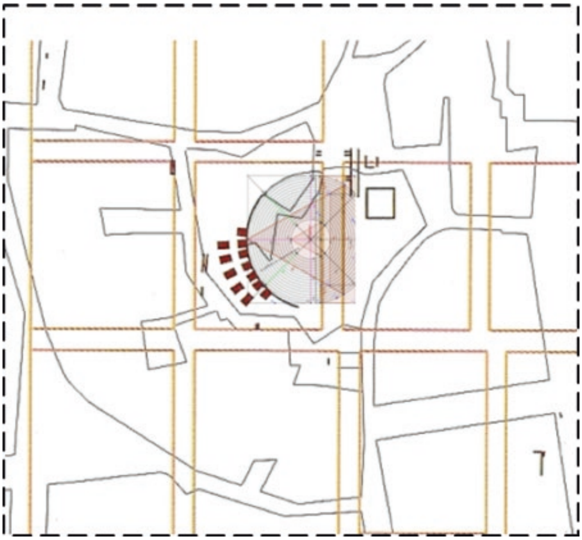

Fig. 28.2 Example of image of the theatre model constructed using GeoGebra, embedded on the map with the ruins marked

ematics and a co-author of the chapter. They could experiment with different materials to construct a circle and an ellipse with the same dimensions as the partial wall. At the end of the session, they could check what the shape of the whole Roman wall was. This location allowed the students to appreciate the likely real dimensions of the Roman building.

After this session, they constructed their theatre model following the canon of Vitruvius (an important Roman architect) on how to build Roman theatres. They conducted this task using the software *GeoGebra*. In order to verify the correctness of their construction and hypothesis, they exported their model as a file image and pasted it onto the map of the area studied (Fig. 28.2).

In the third stage, they could share their doubts with other groups and ask Mrs. Padrós, an important archaeologist from the Museum of Badalona, from their visit to the authentic ruins. During this interview, students could contrast and validate their results about the model selected. This is another example of how the contribution of the history provided students, on one hand, with a way to validate mathematical results with the real world and, on the other hand, to know the experts' work and to notice that it was very similar to the process of inquiry they followed. After this session, they also were able to work on maps with other examples of Roman theatres in Spain and check how the same geometrical model also fits these. At the end of this stage, the students wrote a final report of their inquiry describing their process, the mathematical tools used, the result of verifying their hypothesis, new opened questions, etc. Teachers indicated the required structure of this report via a document that students had to follow. Due to the evidence provided in the written reports of each team, teachers could follow progress and assess inquiry and modelling competences.

28.4 Results and Discussion

In the first stage of the sequence, when the teacher introduced the starting situation S_0, which generated the whole study, the initial questions emerged from the real world in a historical context, closely linked to the students' everyday context. This is an important aspect that had strong influence in the students' motivation to carry on the process of the inquiry to know more about their ancestors. Similar findings have been found by others (e.g. Rivera et al. 2015; Stillman et al. 2013).

The historical context also made easier the generation of derived questions that allowed progress in the process of modelling. The *mathematical world* was constantly in dialogue with the *real-historical world*, which enhanced the apprehension of many derived questions in the sense of the SRP proposal (Barquero et al. 2013). Thus, the archaeological context guided the inquiry and allowed settling on the problem, posing more specific questions, and the pursuit of answers which guided evolution of the SRP. For instance, when students reported about question Q_1, they did not consider all geometrical curve shapes that exist—only the circle and the ellipse—because the historical meanings (that the inquiry teams constantly checked for available answers outside the classroom) indicated that Roman buildings had only three possible shapes. This simplified, for instance, the process of selection of variables, construction of models and selection of the most appropriate models, which correspond to the first steps in the modelling process.

In the second stage, when the students were dealing with question Q_2, they would have to solve a very difficult problem if they had to consider the mathematical context only—to find the whole curve from a part of it; however, due to elements from interaction with the historical context, they followed the modelling process to find the curve considering the only ones that could be possible. The activities to choose the mathematical model that fitted the part of the Roman wall (a circle, in this case) were restricted to two options: they assessed with an ellipse and with a circle. In the end, they found the radius of the part of the wall, which was an arc of a circle. Thus, the RSP designed and explained in this chapter offered the capacity of broadening and articulating different mathematical models and hence gave momentum to the internal dynamics of mathematical processes. In our case, we started with firstly a conic-shaped model related to Roman buildings that evolved into a GeoGebra constructing model based on an applied mathematical model. This resulted in Vitruvian constructions for Roman buildings to develop a conjecture for solving the main question in a research-inquiry approach.

The historical elements (i.e. authentic ruins, the plan of the Roman findings, the Vitruvius canon) provided several tools of validation that normally the mathematical context does not contain. For instance, when the students had built the geometrical model of the theatre (with GeoGebra) following the Roman canon for this type of building, they could check if their scale model fitted with the ruins area on the city map. Also, working with real data and real dimensions (in the fieldwork session in the square near the school) facilitated student understanding of the problem context (cf Rivera et al. 2015). Due to this session, the students could build the model of the Roman theatre following the instructions in the Vitruvius canon with meaning. In addition, in the third stage, when the students talked with the museum archaeologist, they were able to verify that their inquiry and modelling processes were very close to the process followed by the archaeology experts.

Students showed difficulties in using and matching tools from different subjects (history and mathematics) in their project work. A new limitation appeared that, although some teams did very precise calculations in their inquiry, they did not integrate their results into the conclusions of their final report. However, the design of the sequence and all the didactic devices incorporated are crucial to properly facilitate the tasks in order to promote the student inquiry and modelling competences development.

Despite expecting some difficulties to appear, like in other modelling experiences (Stillman et al. 2013), because of the constraints due to the novelty of the general proposal (such as very open tasks at the beginning causing student confusion and demotivation), it was useful for students wondering and posing questions that normally they did not do when faced with other kinds of mathematical exercises more traditionally. For instance, they had to decide what was the goal task, what was the important information and how to organise themselves, amongst other things. However, when they noticed that the historical context could give them much useful information in order to find answers, they had their confidence reinforced in themselves and their motivation revived.

A facilitating condition in the implementation was that teachers accepted a different role as guides of the study process, promoting progressive student autonomy. Thus, we are able to highlight the increasing autonomy assumed by students during the SRP despite that, at the beginning, they showed some objections mainly due to unfamiliarity with these kinds of tasks. This assumption of autonomy is an essential condition to develop an authentic mathematical modelling activity. In this sense, we also observed the importance of always keeping in mind the starting and generating question, which was the thread through the entire study process. The students identified this in their final reports. Finally, our research allowed and described conditions that facilitate functional teaching of mathematics based on modelling and inquiry in extra-mathematics contexts (in this case an archaeological context) so that secondary school teachers (of multiple subjects), with pertinent (and necessary) training, will be able to face the challenge of interdisciplinary teaching.

28.5 Conclusions

The research presented in this chapter describes some of the characteristics of the teaching sequence of ‘What are these ruins hiding? Investigating the Roman ruins of *Baetulo*’, which was designed giving an especial role to inquiry, modelling and the interdisciplinary approach. The historical context contributed to giving students motivation in order to carry on the sequence because it provided them with information to make the problem easier and to generate new questions. We found evidence that the process of mathematical modelling is accelerated when the problem is located in a historical context because this context supplies them with specific information. The historical context promotes the inquiry and contributes to making specific a general mathematical problem in order to facilitate solution. Moreover, the historical elements and the contact with experts from another discipline bring students valuable tools of validation that normally the mathematical context does not. The dialogue between the two disciplines involved made the mathematical question easier to treat because it limited the possible solutions. Considering the historical context to find solutions and to generate questions, the modelling of a difficult mathematical problem became achievable for the mathematical level of the students.

Acknowledgement The research leading to these results has received funding from the Spanish R&D project EDU2015-64646-P, EDU2015-69865-C3-1-R and REDICE16-1520 (ICE-UB).

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Chapter 29
Students' Overreliance on Linearity in Economic Applications: A State of the Art

Daam Van Reeth and Dirk De Bock

Abstract Students' overreliance on linear models is well-known and has been investigated empirically in a variety of mathematical subdomains, at distinct educational levels and in different countries. We present a state of the art of students' overreliance on linearity in economic applications. We illustrate the widespread but sometimes debatable use of linearity in economics, discussing the treatment of demand and supply functions and of the Phillips curve in major economic textbooks. Next, we provide an overview of instances of, and comments on, this phenomenon in the economic education research literature. Typically, the phenomenon is described in the margin of economic studies whose primary focus is elsewhere. Finally, a study having students' overreliance on linearity as its main research focus is discussed in some detail.

Keywords Demand and supply behaviour • Learning macroeconomics • Learning microeconomics • Non-linearity • Overreliance on linearity • Phillips curve

29.1 Students' Overreliance on Linearity

As a major mathematical model underlying various phenomena in real-life and in science, linearity rightfully receives a lot of attention in mathematics education worldwide. However, students' growing experience with linear reasoning and

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_29

their increasing familiarity with linear models during their school careers may have a serious drawback: It may lead to a tendency to use linearity "anywhere" and thus also in situations that are not linear at all (Freudenthal 1983). For example, many students believe that if the radius of a circle is doubled, its area is doubled too (De Bock et al. 1998) or that the probability to get at least one six in two dice rolls is two sixths (Van Dooren et al. 2003). In the mathematics education literature, this phenomenon has sometimes been referred to as the "illusion of linearity". During the last decades, it has been investigated empirically in a variety of mathematical subdomains, at distinct educational levels and in countries having different educational traditions (see, e.g. Van Dooren and Greer 2010). More recently, students' overreliance on linearity was also investigated in physics (De Bock et al. 2011).

Students' overreliance on linearity in the social sciences has been rarely the focus of systematic empirical research. Perhaps the most quantitative of all social sciences is economics. Economists continuously use linear and different types of non-linear models to describe phenomena in their field of knowledge. The tendency to over-rely on linear models is therefore likely to occur in this domain too. An anecdotal but frequently-cited example in that respect refers to a statement of Prince Filip, currently the Belgian King. During a trade mission in China in 2005, the Prince expressed his optimism about the Chinese economy that, at that time, grew at a yearly rate of 14%. He stated: "It means that people's income will double in the next 7 years…" (Huylebrouck 2005). Likely, the Prince thought that he just had to multiply 14 by 7, which gives 98%. A growth of 98% is about 100, thus a doubling. It is an example of linear reasoning in an economic situation of "compound growth", a context in which exponential reasoning is more appropriate.

Below, we present a state of the art of the overreliance on linearity in economics and in economics education. First, we present the results of a small-scale analysis of how two important economic models – demand and supply functions as models for the corresponding consumers' and producers' behaviour and the Phillips curve, modelling the relationship between unemployment and inflation – are represented in major textbooks in this field. Although it is debatable to represent these models linearly, the analysis shows that textbooks often opt for this type of representation. Second, we present the results of a search of the economic (education) research literature conducted in order to find cases in which linear relations were inadequately assumed to grasp economic situations. These cases were typically mentioned in the margin of studies whose primary focus was elsewhere. In one study (De Bock et al. 2014), students' overreliance on linearity was the main research focus. The design and results of that study will be discussed in some detail.

29.2 The Widespread Use of Linearity in Economics Education

In a wide variety of circumstances economists faced with a non-linear function or a function without a closed-form representation, resort to linear approximations of the function under study (Hirschhorn 1986, p. 75). This can be acceptable if the function is not "too non-linear" and if its region is not too large, constraints that are likely to apply in many economic situations. But although the choice of a linear model or "first-order approximation" might be adequate in several situations, it can also lead to an oversimplification of a situation, resulting in wrong conclusions or a poor understanding of a phenomenon under study. This could especially be true in economics education. Ask any undergraduate economics student to draw a demand and supply figure representing a market, and chances are high he will make a diagram with a negative-sloped linear demand curve and a positive-sloped linear supply curve, which assures a unique equilibrium. Ask him if a hyperbolic or even an S-shaped demand curve could do the job as well, and he will very likely be surprised by your question. This observation illustrates how an overwhelming or oversimplified use of linear relationships in economics education can trouble the deeper understanding of important economic concepts. A more thorough understanding of the consequences of overreliance on linearity in economics is therefore absolutely appropriate at this stage. To gain more insight into the presumed dominance of the linear representation in economics education, we analyse in detail two major economic topics that are covered in any textbook. The market for goods and services represented by demand and supply functions is an example from microeconomic theory, while the relationship between unemployment and inflation, represented by the Phillips curve, is an example founded in macroeconomic theory.

There are very relevant didactical reasons to prefer a purely linear representation for depicting a market situation. The simplicity of representing the market by two intersecting straight lines avoids distracting students by complexities that are not relevant in order for an economic principle to be learned. Furthermore, it allows teachers to easily compute exercises and draw graphs, all of which becomes much more complex when non-linear functions are used. In reality, however, producers and consumers will usually not behave linearly. A €1 reduction in price for a box of chocolates will have a different impact if the price goes down from €20 to €19 or, alternatively, from €4 to €3. Besides, assuming a linear consumer demand would imply that, above a certain price level, the quantity demanded would become zero and even negative above a yet higher price. To producers, a linear supply would imply that they are not faced with any capacity constraints and thus could go on producing an indefinite quantity. For these reasons, other curves, for example standard hyperbola branches, having no intersection points with the axes, might be more suitable candidates for modelling consumers' and producers' behaviour mathematically (Ping 2008).

Similar arguments can be given for the Phillips curve, a single-equation empirical model describing the inverse relationship between rates of unemployment and corresponding rates of inflation that result within an economy. This relationship is often represented by a decreasing straight line. However, such a linear relationship between the two variables would imply that for different levels of unemployment, governments could always create lower unemployment at exactly the same cost of higher inflation (and vice versa). Every 1% reduction in unemployment would thus be traded off against the same increase in inflation. In reality, however, measures to fight high or low unemployment rates are likely to have a very different impact on inflation. With high unemployment, there is plenty of surplus labour that could be employed easily without the need to raise wages (and create inflation) very much. But as labour becomes scarcer, firms will find they have to offer increasingly higher wages to obtain the labour they require (Sloman and Garratt 2010, p. 366). Consequently, the Phillips curve could be more properly modelled by a decreasing concave-up curve (Debelle and Vickery 1998).

In Table 29.1 we summarize how both of these concepts are discussed in eight of the world's best sold economics textbooks and in two Dutch textbooks for the tertiary level. It should be clear from the above discussion that there is no theoretical reason or ground to prefer a linear representation over a non-linear one for either one of these concepts. Quite on the contrary: In most cases a non-linear representation probably reflects reality much better. Therefore, the decision to visualize these concepts in economics textbooks as linear or non-linear is purely the result of the author's personal preferences and his (didactical) views on the best possible way to teach students the principles of economics. The data in the table show for each book how many of the officially numbered graphs in the chapters that treat the particular topic are entirely linear. For instance, 25 out of 27 graphs (or 92.6%) that explain demand and supply in the book by Mankiw and Taylor (2014) include linear functions only, and all of the seven graphs that illustrate the Phillips curve are linear.

From Table 29.1, it can be concluded that overall, about two thirds of all the graphs related to demand and supply use linear functions only. There are important differences between the textbooks though. Of the nine textbooks that discuss demand and supply, a first group of five books use linear functions in at least 80% of the graphs. Two of them (Hubbard and O'Brien 2006; De Borger and Van Poeck 2009) even include exclusively graphs with linear functions. Very appropriately, the translated Dutch title of the latter book is, in fact, "Economics in a straight line". Of the second group of four textbooks, two have a very strong focus on non-linear functions with only about 20% of the graphs showing a linear demand or supply (Parkin 2010; Lipsey and Chrystal 2007). The other two textbooks have a more or less balanced use of linear and non-linear curves. Through a contents analysis, we can also roughly distinguish the two different approaches. While the first five textbooks use non-linear demand curves only to explain a couple of specific situations related to the interpretation of the elasticity of demand, the second group of four textbooks explains demand and supply as much as possible from a non-linear point of view and only occasionally turns to a linear approach, for example whenever it enables to make things clearer or it allows easier calculations. We think the latter approach is preferable. Since, as we can see from the chapter numbers mentioned in the table, demand and supply are usually one of the first topics discussed in an eco-

Table 29.1 Graphic representation of "demand and supply functions" and the "Phillips curve" in major economic textbooks

Textbook	Treatment of demand and supply functions	% linear	Treatment of the Phillips curve	% linear
Economics (Mankiw and Taylor 2014)	Chapters 3 and 4: 25 (27) Non-linear functions are used to explain the different values for the price elasticity of demand and to explain how the price elasticity of supply can vary	92.6	Chapter 34: 7 (7)	100.0
Microeconomics (Besanko and Braeutigam 2011)	Chapters 2 and 5: 24 (30) Only to illustrate the link between the choice of the consumer and his demand function, a non-linear approach is used a couple of times in Chap. 5	80.0	/	/
Microeconomics (Parkin 2010)	Chapters 3 and 4: 4 (18) Only for calculating the elasticity of demand and for clarifying the relationship between the elasticity of demand and total revenue a linear demand curve is used	22	/	/
Essentials to economics (Sloman and Garratt 2010)	Chapters 2 and 3: 8 (16) The concept of demand and supply is introduced in a non-linear way, but the further analysis of these functions – e.g. shifts in demand, or price elasticities – is always explained in a linear way	50.0	Chapter 11: 0 (3)	0.0
Managerial economics (Png and Lehman 2007)	Chapters 2 and 3: 13 (14) Only to explain the calculation of the price elasticity of demand, a non-linear function is used once	92.9	/	/
Economics (Lipsey and Chrystal 2007)	Chapters 3 and 4: 4 (24) For two occasions linear functions are used: to explain elasticity of supply and demand and to explain the difference between short-run and long-run demand curves	16.7	Chapter 24: 0 (4)	0.0

(continued)

Table 29.1 (continued)

Textbook	Treatment of demand and supply functions	% linear	Treatment of the Phillips curve	% linear
Macroeconomics (Blanchard 2006)	/	/	Chapters 8 and 9: 1 (1)	100.0
Microeconomics (Hubbard and O'Brien 2006)	Chapters 3 and 6: 19 (19)	100.0	/	/
Economie (Decoster 2010)	Chapters 3 and 4: 9 (19) The concept of demand and supply is introduced in a non-linear way, while in the further analysis, a balanced mixture of linear and non-linear functions is used	47.4	Chapter 23: 2 (2)	100.0
Economie in rechte lijn (De Borger and Van Poeck 2009)	Chapter 2: 15 (15)	100.0	/	/
Overall	121/182	66.5	10/17	58.8

nomics course, a consistent use of linear functions from the first graphs on makes students much more receptive to the illusion of linearity. If, however, students are first taught that demand curves are non-linear in principle and that only for didactical purposes a linear approach is used every now and then, they will be much warier of the risks of linear reasoning. Finally, we noted that practically all of the textbooks consistently use the word demand "curve" or supply "curve", even when using a linear function. This is sometimes explicitly mentioned and motivated. For instance, Sloman and Garratt (2010, p. 32) write: "The term demand 'curve' is used even when the graph is a straight line! In fact, when using demand curves to illustrate arguments we frequently draw them as straight lines – it's easier".

Just like for the analysis of demand and supply, the textbooks show a slight preference for a linear approach in their discussion of the Phillips curve, a much narrower topic than demand and supply. Almost 60% of all the graphs related to the Phillips curve are of a linear nature. However, in each of the five textbooks that treat this concept, all graphs were either linear or non-linear, with three books opting for a linear approach and two books preferring a non-linear approach. As a result of these unambiguous choices, it is hard to evaluate or compare the motives for either choice in more detail. We do notice, however, that the two textbooks choosing a non-linear Phillips curve (Sloman and Garratt 2010; Lipsey and Chrystal 2007) also made a similar choice for the way demand and supply are presented.

29.3 Research Experiments on Students' Overreliance on Linearity in Economic Applications

29.3.1 Examples from Various Economic Studies

Christandl and Fetchenhauer (2009) conducted a series of experiments to examine the accuracy of estimations of long-term economic growth both by experts and laypeople, the factors that influence the accuracy of their estimations and the procedures they use to make the estimations. For an annual growth rate of more than 1%, this long-term growth level was clearly underestimated by both groups, but the underestimation was lower for experts than for laypeople. The authors discuss several causes for the underestimation of the actual economic growth, which essentially can be modelled by an exponential function. A detailed review would lead us too far, but relevant in the context of this chapter is the fact that one third of the participants underestimated long-term economic growth on the basis of a linear model by simply multiplying the time in years by the annual growth rate and thus totally ignoring exponential effects. The authors state (p. 391): "Linear functions are used as a default for situations that require a non-linear approach as long as it is not clear which approach needs to be applied to a particular situation".

Linear reasoning in a context where exponential reasoning is more suitable was also found by Christandl and Gärling (2011), who conducted a series of laboratory experiments with undergraduates on consumers' ability to accurately estimate future price increases in an inflationary economic context. Consumers strongly tended to apply a linear model to extrapolate future prices, and that model was only abandoned when clear counterevidence was provided, definitely showing its inadequacy in this context.

Whereas Christandl and Fetchenhauer (2009) and Christandl and Gärling (2011) focused on people's underestimation of exponential relationships in macroeconomic contexts, Stango and Zinman (2009) investigated this phenomenon in relation to household finance. They apply the term "exponential growth bias" to characterize the pervasive tendency to linearize exponential functions when assessing them intuitively, a bias that can explain two facts in household finance: the tendency to underestimate an interest rate given other loan terms and the tendency to underestimate a future value given other investment terms. The authors argue that this bias affects households' financial decisions: More biased households will borrow more, save less and favour shorter maturities. New empirical evidence was found by constructing a household-level measure of payment/interest bias and correlating it with a wide range of household financial outcomes using nationally representative data in the USA.

Hsee et al. (2003) investigate the impact of a medium – for example, points or money – on people's decisions when they are faced with options entailing different outcomes. In a laboratory study with university students, the authors demonstrate that the presence of a medium can alter people's decisions because the medium creates an illusion of advantage to an otherwise not so advantageous option. One of the presented illusions is called the "illusion of linearity", the effect that occurs because the medium masks an otherwise concave down effort-outcome return relationship by a linear relationship. The authors argue that their work has real economic implications for how points influence consumer choice and how money influences human behaviour.

29.3.2 A Systematic Empirical Study in Economics Education

Recently, students' overreliance on linearity in the domains of micro- and macroeconomics was investigated in a systematic way. For a full report of this study, we refer to De Bock et al. (2014); in this overview we limit ourselves to some key elements. The authors' research question was: Does improper linear reasoning play an important role in business economics students' reasoning about micro- and macroeconomic situations? Because mathematics education research had revealed that problem formulation can influence students' tendency to over-rely on linearity, they decided to add an additional research question: Does improper linear reasoning by business economics students about micro- and macroeconomic statements depend on the way these statements are formulated?

A written test was taken by 92 third year Bachelor students in business economics, aged 20+, at a Belgian university campus. The test consisted of ten statements that the participants had to evaluate as being correct or incorrect (examples are given in Table 29.2). The theory underlying each statement was addressed in courses that participants had followed during their university training so far. Half of the statements were drawn from the subdomain of microeconomics; the other half were drawn from the subdomain of macroeconomics. Within each of these subdomains, there was one statement for which linear reasoning – defined in its narrow "$y = ax$" sense – was appropriate, and there were four items for which such reasoning was not appropriate. They were named the linear and the non-linear items. Half of the non-linear items (in each subdomain) were formulated in a "k times A, thus k times B" format, which is the most natural and straightforward form of "$y = ax$" reasoning. The other half of the non-linear items were formulated in an "$A + k\%$, thus $B + k\%$" format (which is mathematically equivalent with a multiplication by $1 + k/100$). Participants' answers were statistically analysed by means of two repeated measures logistic regression analyses. The first logistic regression was carried out for all experimental items with linearity and economic subdomain as explanatory variables. The second logistic regression was carried out for all non-linear items with item format and economic subdomain as explanatory variables.

The first regression analysis revealed a significant linearity main effect. Linear items elicited more correct answers than non-linear items (respectively, 85.3% and 71.2%). This result is in line with results found in the domain of mathematics. The economic subdomain variable too had a main effect on the percentage of correct answers. The number of correct answers was significantly larger for the micro- than for the macroeconomic items (respectively, 77.8% and 70.2%). This result is prob-

Table 29.2 Examples of items as used in the test

Microeconomics
Linear item
Peter wants to celebrate his 18th birthday. He plans to have an unforgettable evening party, and he wants to take care of everything to the last detail. In order to reduce his costs, he asks guests for a contribution of 10 euros. He was expecting 20 guests, but twice as many come. The total contribution that he will receive from his guests will therefore be twice as large as he had expected.
Example of a non-linear item/"k times A, thus k times B" format
Last year three classmates had a mobile phone. We define the utility of that group as the number of possibilities these classmates have to communicate with each other by mobile phone. This year the number of classmates owning a mobile phone doubled. Accordingly the utility of the group that already had a mobile phone last year (approximately) doubled too.
Example of a non-linear item/"A + k%, thus B + k%" format
Before 2008 farmers in the European Union were obliged to leave 10% of their land fallow. The aim was to avoid overproduction. As a consequence of that 10% fallow rule, agricultural production decreased by (approximately) 10%.
Macroeconomics
Linear item
Today, the exchange rate of the euro and the dollar is 1 US dollar = 1 euro. A Dutch company buys a boat in the USA for 500,000 dollars. One month later a Belgian company buys the same boat at the same price in dollars, but the rate of the dollar has risen to 1 US dollar = 1.1 euro. Both companies pay in dollars to the manufacturer in the USA. The dollar became 10% more expensive, so the cost in euros for the Belgian company is also 10% higher than for the Dutch company.
Example of a non-linear item/"k times A, thus k times B" format
In a certain country a flat tax rate of 20% is applied to all incomes. To bring in more money the government thinks about an increase in the flat tax rate. Next year income tax will be increased from 20 to 40%. This means that in the future, the government will (approximately) double its revenue from income tax.
Example of a non-linear item/"A + k%, thus B + k%" format
After an economic crisis a country enjoys a period of economic growth. The first year after the crisis, the economy is growing by 5% and consequently the total income of the households increases by 5%. Because of that larger income, the total expenditure of the households also increases by (approximately) 5%.

ably due to the more general and more abstract nature of the subdomain of macroeconomics. The second logistic regression analysis revealed a significant item format main effect. Participants performed better (and thus were less inclined to accept an inappropriate linear reasoning) for non-linear items in an "$A + k\%$, thus $B + k\%$" than for non-linear items in a "k times A, thus k times B" format (respectively, 80.4% and 62.0% correct answers).

29.4 Conclusions and Discussion

Although research on students' overreliance on linearity in the domains of economics and economics education is still rare, the phenomenon is acknowledged by several scholars in these fields. Moreover, a small-scale textbook analysis showed that linear quantifications are, at least to some extent, induced by current educational practices. More quantitative but also qualitative research is needed to unravel the nature of students' overreliance on linearity in economics. Why do tertiary level students, who have already thoroughly studied the relationships in question, still succumb to the charms of linear quantifications? Does the intuitive or heuristic character (Kahneman 2002) of the linear model play a major role or can this phenomenon only be explained by school-related practices? Or, do even tertiary level students in economics still have substantial gaps in their economic (pre) knowledge?

Besides the call for more empirical research, also some theoretical work would be useful. When people use linearity to grasp real-world situations that are only approximately linear, at best they obtain reasonable approximations of what really happens. However, when the real-world situation is not linear at all, vast under- or overestimations are the result. Some people are aware they simplify reality, others are not. Research has already shown that people's (over)use of linearity affects their personal economic and financial decision-making (Stango and Zinman 2009). Nevertheless, a more comprehensive view of the individual and societal impact of the (over)use of linear models in different kinds of non-linear (or only approximately) micro- and macroeconomic situations, related to people's economic and financial literacy, would be welcomed.

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Part III
Pedagogical Issues for Teachers and Teacher Educators Using Mathematical Modelling and Applications

Chapter 30
Teaching Modelling and Systemic Change

Hugh Burkhardt and Malcolm Swan

Abstract In this chapter, we set out our views, based on over 30 years of experience, on two related areas: methods and challenges in teaching students the strategies and skills needed to model real problems using mathematics, and approaches to helping education systems make this happen in the classrooms for which they are responsible. We believe that substantial progress has been made with the first of these but much less with the second. We use examples from a sequence of modelling projects to illustrate design principles that we have found powerful for materials to support teaching and professional development. We then discuss barriers in school systems to the implementation of important improvements like modelling, and how they might be tackled.

Keywords Design principles • Design strategies • Design tactics • Professional development • Strategic design • Teaching materials

30.1 Introduction

We begin with a little history that sets the context for the style of work we have developed over the last 35 years. The Shell Centre 'Brief' that one of us (HB) agreed with the university when he became Director in 1976 was broad, simple – and challenging: *To work to improve the teaching and learning of mathematics regionally, nationally and internationally.* The goal was, and is, to work directly to transform practice in classrooms through design and development research, with an 'engineering research' approach based on:

- Finding and analysing promising situations for student investigation,
- Designing new processes, products, and experiences for teachers and learners,
- Developing these products to work well in large-scale use,

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_30

- Articulating values and principles that underpin these designs,
- Analysing 'designs in action',
- Revising and refining theories and designs in the light of these experiences.

The importance of design for this strategy was clear. After a search for people with the unusual design research skills to forward this brief, the other author (MS) was appointed in 1979 to lead this aspect of the work.

A series of modelling-focused projects has followed over the years, building on work from 1964 onwards, described in *The Real World and Mathematics* (Burkhardt 1981, illustrated by MS). We start by describing the structure of the *Testing Strategic Skills* (TSS) project, in order to show the diverse aspects of design that need to be addressed if an innovation is to meet the 'brief' above, that is to fit the complex needs of an education system (in this case, England's).

30.1.1 'The Box Model' of Gradual Improvement

This initiative was developed in the 1980s by the Shell Centre team with the largest UK examination board. We persuaded the board to improve the match between its broad list of 'knowledge and abilities to be tested' and what was actually assessed in this high-stakes mathematics examination for age 16. An unusual strategic design feature was the *gradual change* approach that was adopted. One new task type was introduced each year, with one task on the examination, representing 5% of the 2-year mathematics syllabus and about 3 weeks teaching. Care was taken to remove from the syllabus some topics that took a comparable amount of classroom time.

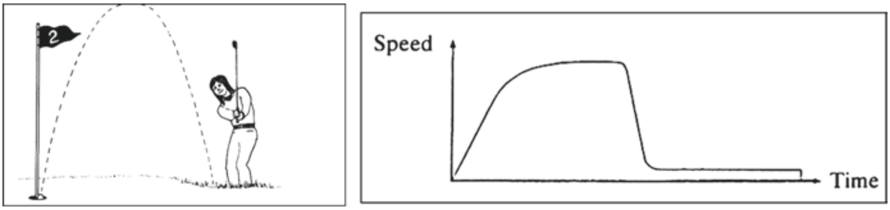
The Shell Centre developed the support, which comprised:

- Five examples of the new type of task, along with scoring guidance and examples of student work at various levels – five exemplars were needed to show the variety to be expected in the 'live' examination;
- Teaching materials for 3 weeks' teaching in the form of a teacher guide and copyable worksheets;
- Materials to support five sessions of in-school do-it-yourself professional development (these included lesson videos and software 'microworlds');

all brought together in a box of materials for purchase by schools, hence the project name. The materials were designed and developed in classroom trials until feedback showed that they enable typical teachers to prepare their students for this new type of task.

The first year's change was the introduction of 15-min tasks that assess nonroutine problem solving in pure mathematics. The following year's module, *The Language of Functions and Graphs* (Swan et al. 1985), focuses on representational skills for modelling real-world situations with Cartesian graphs or with algebra – graph interpretation, model critique, and formulation are all included. Figure 30.1 shows two task exemplars. Note the absence of numbers, designed to ensure students' focus on qualitative understanding.

Translating between the graphs and physical situations



How does the speed of the ball change as it flies through the air?

Which sport will produce a speed v time graph like this?

Fig. 30.1 From *The Language of Functions and Graphs* examination module

This approach proved popular with teachers. They enjoyed the 3 weeks of new teaching, pedagogically challenging but well-supported. They were equally glad to get back to more familiar ground for a while thereafter but said they looked forward to the next year's module. The model died through a government reorganization of assessment – such unintended consequences are a common feature of reorganizations.

30.1.2 A 30-Year Development Programme

Following *Testing Strategic Skills*, the *Numeracy Through Problem Solving* project (NTPS, Shell Centre 1987–1989) developed five modules, each supporting complete 3-week small-group modelling projects. Among the projects that followed were *Extended Tasks for GCSE Mathematics*, *World Class Arena*, *Bowland Maths*, and our US-based *Mathematics Assessment Project* (*MAP*, *2010–2014*). We describe aspects of these in what follows.

While these teaching materials supported teachers in making the shifts of pedagogy that modelling demands, there was also parallel development of materials-based professional development, focused on helping teachers more broadly with the new pedagogical and mathematical challenges. *Bowland Maths* included such a component which was built on later in PRIMAS (*Promoting Inquiry in Mathematics and Science*) and the *Mathematics Assessment Project*, while *Lessons for Mathematical Problem Solving LeMAPS* (*Lesson Study in Mathematical Problem Solving*) has developed the "lesson study" approach pioneered in Japan. Impact on classrooms remains the core goal of Shell Centre projects, now part of a broader range of research across Nottingham's Centre for Research in Mathematics Education, directed by another pioneer in the teaching of modelling, Geoff Wake.

In this chapter, we shall illustrate design principles and tactics with examples from this work. However, the inevitable space constraints of print limit how much we can show, particularly of the visual aspects that are so important in understanding design.[1] To support the interested reader, fuller exemplification is given, section by section, at ictma17.mathshell.org, where fuller descriptions of various Shell Centre

[1] Hence the decision to make *Educational Designer* an e-journal.

modelling projects can also be found. The complete materials can be found at www.mathshell.com.

In the next section, we present a picture of what modelling in middle and high school classrooms can look like by describing the design of modelling activities on three timescales: a single lesson, a four-lesson 'case study', and a 3-week modelling project. In Sect. 30.3, we discuss the pedagogical challenges and how they may be tackled. Section 30.4 looks at the challenges of designing support for the teaching of modelling. Section 30.5 addresses the system-level challenges that, despite the obvious importance of and rhetorical support for modelling, have prevented it happening in most classrooms, discussing how the research community could rebalance its work to contribute more effectively to the large-scale implementation of modelling in classrooms around the world.

30.2 Modelling: What Does It Look Like in Secondary Classrooms?

We begin with a brief look at some aspects of the modelling process. There are many versions of the modelling diagram in this book. We include one from the first publication we developed together, as author and artist (Burkhardt 1981). See Fig. 30.2.

In teaching modelling, the focus is on active student engagement in mathematical aspects of the situation that may help understanding and better decision-making. This should encompass reasoning mathematically and using mathematical concepts, procedures, facts, and tools in describing, explaining, and predicting phenom-

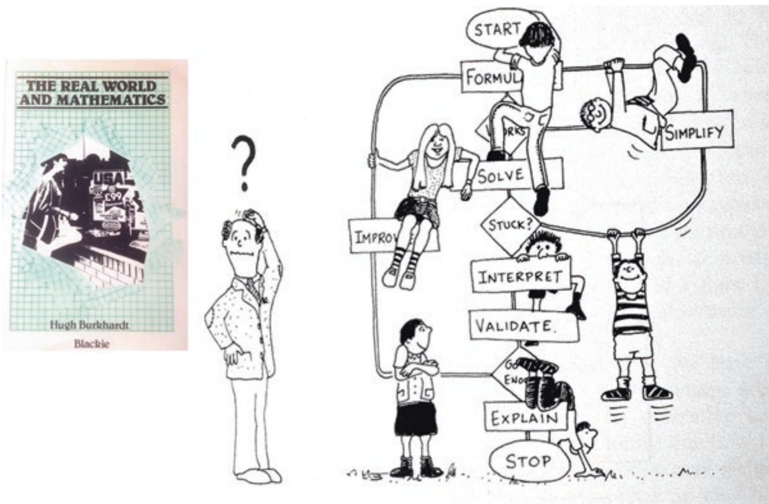


Fig. 30.2 The modelling cycle from Burkhardt (1981)

ena. In particular, the verbs 'formulate', 'solve', and 'interpret' point to the three processes in which students engage as active problem solvers. Formulation is usually the core challenge in modelling.

This much is common ground among the chapters of this book – but unfamiliar turf for both teachers and students in most classrooms. How can we make it accessible and enjoyable? We begin to explain our approach by outlining a few examples.

30.2.1 How Many Teachers? A One-Lesson Modelling Task

There are about 60 million people in the UK.
About how many school teachers do we need?

Tasks like this and the ability to make rough estimates are the essence of modelling. The centrality of proportional relationships, direct and inverse, is typical.

A group of students, working through discussion,[2] converged on the following assumptions and reasoning. (They did not use the symbols, but corresponding words.)

- *Identifying significant variables and making assumptions*

Variable	Symbol	Assumed value
Population	p	60,000,000
Years in school	t	12 years
Lifetime	n	80 years
Class size	c	25

- *Identifying relationships and calculating intermediate variables >> a solution*

Fraction of population at school	$f = t/n$	1/8
School population	$s = p*f$	7,500,000
Number of teachers needed	$T = s/c$	**300,000**

From this outline you can see the modelling reasoning, along with some errors. A *Bowland Maths* (Swan and Pead 2008) unit, *You Reckon*, focuses on 'Fermi problems'[3] like this.

A bank manager says that an armed youth stole a bag containing £5000 in £1 coins and ran away.

The insurance company is suspicious, and wants you to investigate. Could the bank manager be lying? Explain your reasoning carefully! Tell the insurers exactly what assumptions you made.

[2] The video is on the website ictma17.mathshell.org – for the interested reader to analyse.

[3] So-called after the great physicist who loved such 'back of the envelope calculations'.

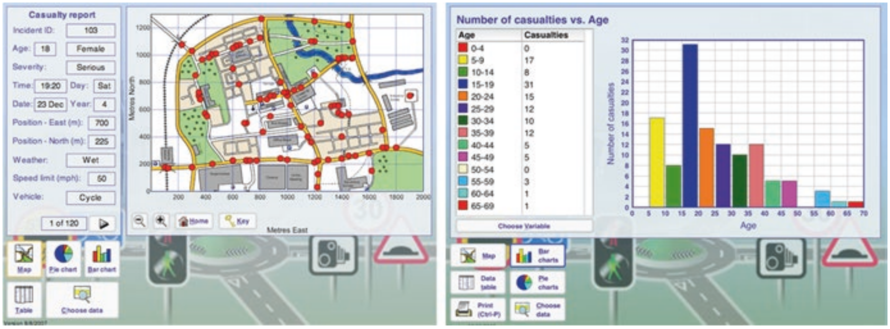


Fig. 30.3 The graphical database

The problems we have found are that outside the school there are a number of accidents to cyclist leaving school.

To sort out this problem we are going to put in 500 metres of cycle lanes and tracks costing £30,000 out of our £100,000.

We could also add 8 road humps costing £8,000 To help the road humps we would need 2 traffic island and pedestrians refuge costing £6,000.

We should get 2 sets of pelican crossing costing £36,000 coming to a total cost of £80,000 this should reduce the accident rate by half. Hopefully helping save pupils lives on bikes every year.

(Student work)

Fig. 30.4 A sample group report

30.2.2 Reducing Road Accidents: A Four-Lesson Modelling Task

In this *Bowland Maths* ‘case study’ (Swan and Pead 2008), students examine police reports, photographs, and a map on 120 road accidents, seeking to identify possible causes and propose cost-effective remedies to the town government. The key tool provided is a computer database of casualty reports along with a variety of ways to present the data (see Fig. 30.3). Students explore the data, choosing subsets ‘filtered’ to focus on specific variables. They try to find patterns in the accident locations, times, weather conditions, vehicle usage, and so on.

The students are also given costs for various measures that reduce accidents: pedestrian crossings, road humps, and cycle lanes. The student reports, as in Fig. 30.4, showed a high level of engagement and of analytic modelling – some in the form of PowerPoint presentations for ‘the council’. The unit was popular with both teachers and students.

30.2.3 Numeracy Through Problem Solving: 3-Week Modelling Projects

The *Numeracy Through Problem Solving* project developed five modules: *Design a Board Game*, *Produce a Quiz Show*, *Plan a Trip*, *Be a Paper Engineer*, and *Be a Shrewd Chooser* (Shell Centre 1987–1989). The rationale was the teaching, learning, and assessment of mathematical literacy á *la PISA* – but here through tackling substantial real problems more seriously than is usual in school. The goal was that students see themselves as consultant designers, planners, and decision-makers who put their designs and plans into action. Each module was designed to take 10–20 h over 3–6 weeks, depending on the depth the teacher chose to address the issues involved.

The examination board provided certification at three levels: basic, standard, and extension. Basic level was teacher-assessed using tasks built into the modules at each stage, with written examinations papers for the higher levels. This model provided an opportunity to assess problem solving with 'controlled transfer distance', since all students had the same preparatory experience: working through the module. Standard level assessed transfer to closely related situations (e.g. other board games for *Design a Board Game*), while extension level assessed modelling in other real-world contexts requiring similar mathematics – that is, more distant transfer. The common design, set out in detail in the student booklet, supports four stages of student work in small groups over about 3 weeks of daily mathematics lessons:

Stage 1: Looking at existing examples. Students analyse existing products or processes (e.g. for *Design a Board Game*, we provide five badly designed board games for them to play, analyse, and critique). This process familiarizes the students with the essentials of the challenge. In this way, they identify criteria and possible structures for successful products.

Stage 2: Planning an approach. Students brainstorm ideas, select one to work on, study the techniques they need, and plan their work.

Stage 3: Carrying out the plan. Students work through their plan and then make a prototype product.

Stage 4: Presenting and evaluating the product. Students present the product or enact the process. The groups test and evaluate each other's work.

In each case, key critical guidance is delayed until the students have (or have not) themselves recognized a need (e.g. for parental permission letters in *Plan a Trip*). The core goal, that the activities actually happen, is normally achieved: the games are made and played; the TV-like game shows for each group take place, with the rest of the class as participants and audience; the class goes on the trip to a place they together chose and planned; and so on. When interviewed afterwards, students in the trial made clear that the contrast with their other mathematics lessons was stark and welcome.

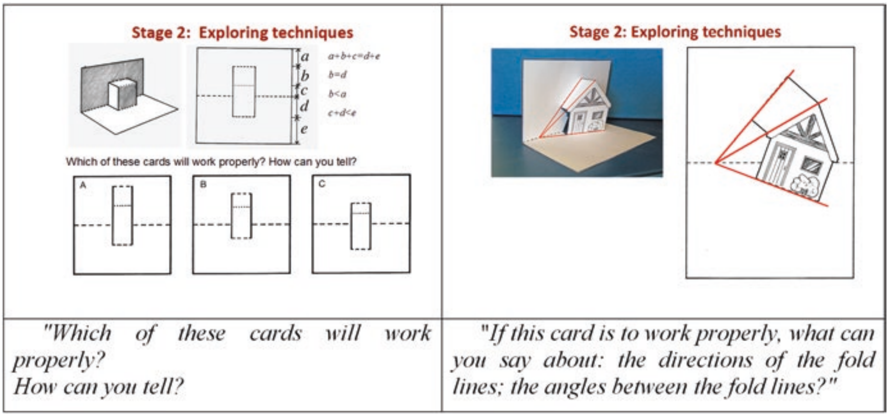


Fig. 30.5 *Be a Paper Engineer* investigations

Mathematically, though all modules involve a broad spectrum of mathematical reasoning, *Design a Board Game* emphasizes geometry and probability, *Produce a Quiz Show* demands real-time scheduling and statistical fairness of questions, *Plan a Trip* is all about money, time, and distance, while *Be a Shrewd Chooser* focuses on money and data handling. We will look in a bit more detail at *Be a Paper Engineer*, which combines three-dimensional geometry and algebra. It realizes this design strategy in the following way:

In Stage 1, students share the work of making 30 boxes, envelopes, and pop-up cards from two-dimensional 'nets' provided on card or paper. They bring examples from home as well. They then classify them all according to elements in their design.

In Stage 2, students select a type of box or card they are interested in and explore the mathematical principles involved in more depth. They tackle a series of investigations and challenges and write up their findings. Figure 30.5 above shows two of the questions that students investigate (the answers, shown here, are not given to the students!). The left-hand figure is essentially two-dimensional (viewed from the side), and the answers, arrived at by most students, are standard parallelogram theorems that are normally taught directly, rather than devised by investigation. In the three-dimensional right-hand example, it is challenging to show that, for the pop-up to work, the three lines must intersect on the fold line.

This investigative stage raises all the questions the student groups will need to think through in *formulating* their model in Stage 3:

- Identify specific questions: '*How can I make a card that pops up like this?*'
- Make simplified drawings: '*Let's simplify this card so we can see its structure.*'
- Represent mathematically: '*How can we draw this 3D shape in 2D*?'
- Identify significant variables: '*Which lengths/angles are important here?*'
- Generate relationships: '*For the card to work, how must the lengths be related?*'

In Stage 3, making their own originals, students brainstorm ideas, and then each makes a rough version, followed by a final version of the object they choose – that is, the *solving* phase of modelling: making and carrying out a plan and then monitoring progress.

In Stage 4, going into production, students devise step-by-step instructions for making their product – then test the instructions by asking someone else to follow them. This involves *interpreting* results (e.g. '*Can you interpret John's instructions for making the box?*') and *evaluating* the solution (e.g. '*Can you reconstruct the card from John's instructions?*'). All the products are then put on display for the class to analyse and evaluate.

Numeracy Through Problem Solving modules were adopted with great enthusiasm in a substantial number of classrooms but not in the majority of the examination board's schools. The issues that led to this include:

- *Teacher expectations*. Teachers found that the modules took them outside what they understood to be mathematics – some saw it as a fine cross-curricular activity.
- *Demand on class time*. Five modules of 10–20 h were too much for many teachers, unwilling to lose time from consolidation and practice of procedures.
- *Pedagogy* was very different from the standard 'demonstrate and practise' approach.
- *No high-stakes examination*. Unlike the *Testing Strategic Skills* modules, which were part of a change in the examination that all students take at age 16, the *NTPS* assessment was separate and different in kind.

As a result, teachers tended to use the modules mainly with 'low attainers'. The roots of the scheme in 'numeracy', its emphasis on practical activity, and 'the classroom time taken from preparing for the exam' made some teachers reluctant to use it with more able students – though trials had shown the challenges the modules present to all students and the quality of work high performers can produce. The development of an optional GCSE syllabus based on the modules largely met the last bullet point and increased uptake.

30.3 Pedagogies for Modelling

The main approaches to teaching mathematics may be grouped as in Fig. 30.6. To teach modelling effectively requires a collaborative pedagogy, with the students taking responsibility for their learning and teachers adopting new roles, facilitative rather than directive. The 'adaptive expertise' that this requires takes most mathematics teacher into territory outside the comfort zone of their well-practised mode of working. Recognizing the need, we have developed ways of supporting this kind of teaching, more recently under the title of 'formative assessment', through materials both for lessons and for professional development activities.

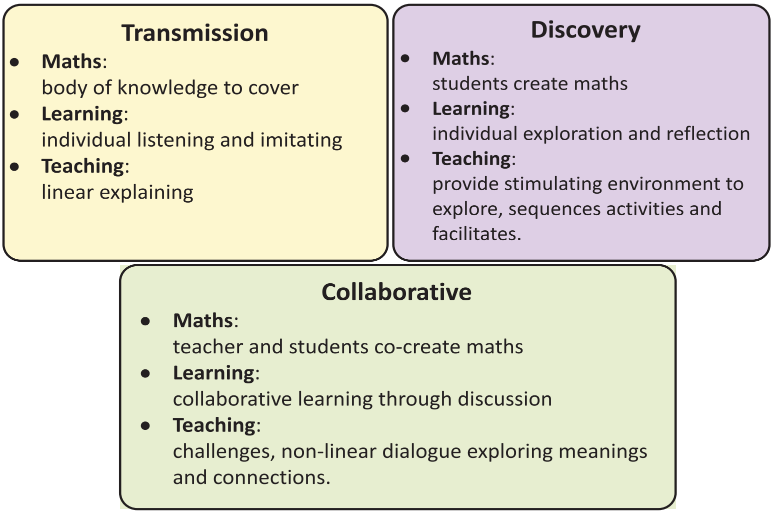
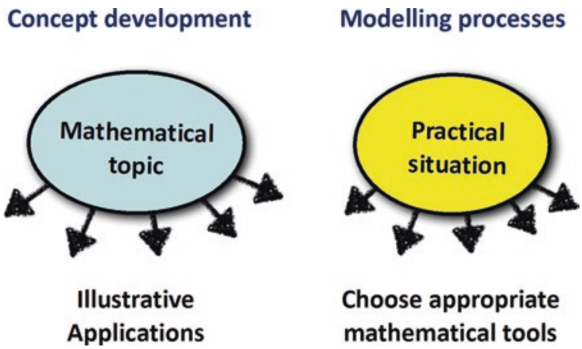


Fig. 30.6 Alternative pedagogies

Fig. 30.7 Alternative lesson goals



In the *Mathematics Assessment Project* (2010–2015), we faced the challenge of seeing how far teaching materials could enable typical teachers of mathematics in supportive school environments to move to high-quality formative assessment of the kind that the research review of Black and Wiliam (1998) had shown to be so effective in advancing student learning. Formative assessment can be described as "students and teachers, using evidence of learning to adapt teaching and learning to meet immediate needs minute-to-minute and day-by-day" (Wiliam and Thompson 2007).

The project developed 20 formative assessment lessons ('FALs') for each US grades 6 through 10. There are two quite different types (see Fig. 30.7): about a third of the lessons are on *problem solving*, mostly modelling, while the rest are focused on *concept development*. The distinction is important because the foci and the quality criteria are different. Modelling looks for power in terms of understanding the

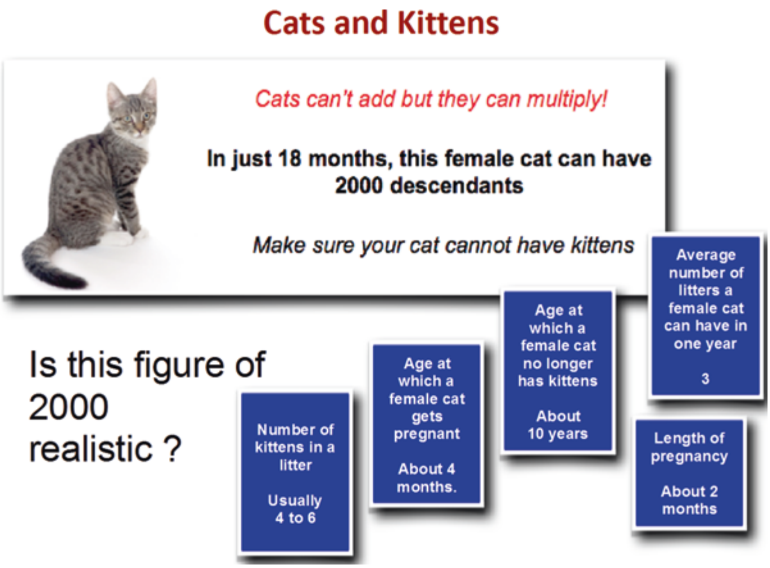


Fig. 30.8 *Cats and Kittens* – the task

practical situation, choosing and using whatever mathematical tools you find useful for this. Concept development is focused on understanding a specific piece of mathematics; practical contexts can illustrate and thus reinforce this. It is important that both teacher and students understand this distinction – a student model using elementary mathematics can be, and often is, more powerful than one that tries to use more advanced concepts.

Both kinds of 'FAL' proved popular, with over seven million lesson downloads from map.mathshell.org, and effective in forwarding both student learning and teacher pedagogy. We shall illustrate the design of the modelling FALs using *Cats and Kittens* (see Fig. 30.8), as the example. The mathematical challenge is mainly in the complexity of the situation and the data and in finding a representation to handle it.

In a prior lesson, the problem is presented to the students who tackle it individually and unaided. This enables the teacher to assess the student work,[4] looking for common issues, then to prepare qualitative feedback. To help the teacher, each FAL has a 'common issues' table, as in Table 30.1. Note that the suggested teacher interventions are in the form of questions or general suggestions, so that the student still 'owns' the solution.

The main lesson structure is typically as follows: After a bridging introduction, the teacher reintroduces the main task. Students respond to the prepared questions by reviewing and revising their *individual* solutions. In the next phase, the students,

[4]Not to score it! Research and our student interviews clearly show that getting numerical scores terminates student interest in understanding and improving their, and others', reasoning.

Table 30.1 Common issues for *Cats and Kittens*

Issue	Suggested questions
Has difficulty starting	Can you describe what happened in the first 5 months?
Does not develop a suitable representation	Can you make a diagram or table to show what is happening?
Work is unsystematic	Could you start by looking at the litters from the first cat? What would you do after that?
Develops a partial model	Do you think the first litter of kittens will have time to grow and have litters of their own?
Does not make clear or reasonable assumptions	What assumptions have you made? Are all your kittens born at the beginning of the year?
Makes a successful attempt	How could you check this using a different method?

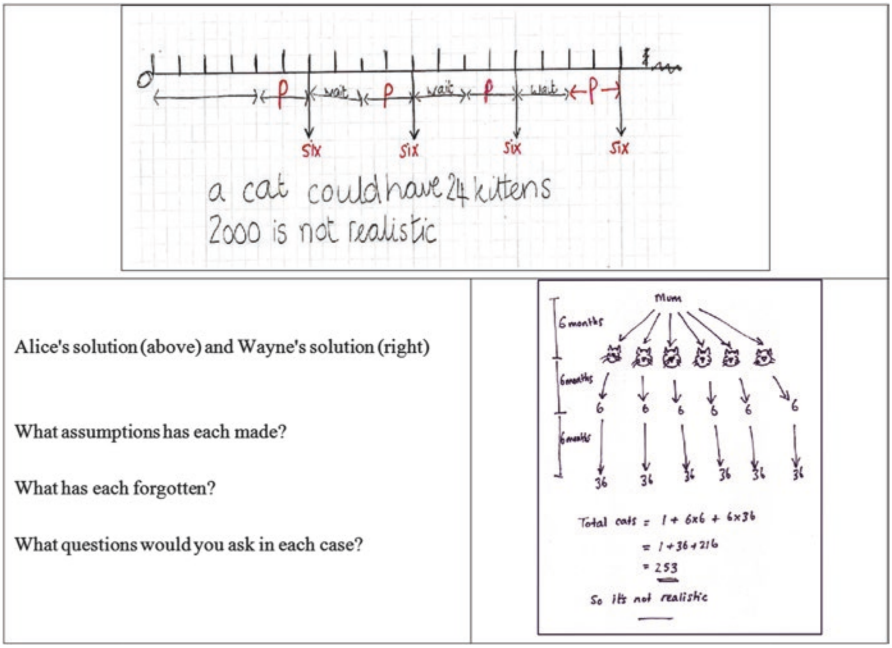


Fig. 30.9 Sample student work for critiquing

working in small groups, compare their approaches – particularly, in this problem, the representations they have devised. From this discussion, they produce a poster showing a joint solution, completing the inherent peer assessment. The posters are displayed promoting an *intergroup discussion*. Groups compare approaches, justifying their own and recognizing others. Each group now analyses and critiques sample student work we provided (see Fig. 30.9). This leads them to discuss approaches they may not have considered. The groups then work to improve their solutions to the problem. *Whole class discussion* follows, seeking to combine a review of what has been learned with discussion of the processes, assumptions and their implications, and alternative representations, their strengths and weaknesses.

From a design perspective, using sample student work can be powerful in many ways:

- To encourage a student that is stuck in one line of thinking to consider others.
- To enable students to make connections.
- To compare alternative representations, including more powerful ones.
- To compare hidden assumptions and their effect.
- To encourage metacognitive behaviour.
- To draw attention to common errors.
- To encourage criticality without fear of criticism.
- To become more aware of valued criteria for assessment.

Strategically, it moves students into 'teacher roles' (Burkhardt et al. 1988) – a design tactic that reliably increases the depth of classroom discussions and of learning.

30.4 Support for Teaching Modelling

So far we have described how teaching materials can be designed and developed to help teachers face the challenges of teaching their students not just to learn standard models but to become active modellers. This support needs to be complemented by effective professional development, for which there are alternative designs, including:

- '*Training*' *models* based on transmission of information by an 'expert'. These are useful mainly for raising awareness of needs and opportunities; teachers find there is a gulf between the advice and their day-to-day classroom practice that is difficult to bridge. We see this as an 'inappropriate design load'.
- '*Experiential course*' *models* provide a sequence of sessions, mediated by a provider, that offer teachers opportunities to explore ideas in their own classrooms and report back.
- '*Embedded*' *professional learning communities* move from a finite treatment approach to one of long-term development through shared activities. Teachers take over responsibility for setting their own 'action research' goals, collaboratively and systematically studying them in their own and each other's classrooms, often with outside support from materials and/or invited experts.

For teaching modelling, there is a mismatch between the numbers of teachers needing professional development support and of those who have the expertise to lead that support. We have worked to see how far materials can fill the need – an interesting design challenge. We have designed experiential professional development sessions to complement our teaching materials, first in the *Testing Strategic Skills* modules, more recently as part of our work on *Bowland Maths* and subsequent US and European projects. The Bowland Professional Development Modules

(Swan and Pead 2008) each focused on a specific pedagogical challenge that teaching modelling involves:

- Tackling unstructured problems: *Do I stand back or intervene and tell them what to do?*
- The projects and mathematics: *Where is the maths?*
- Fostering and managing collaborative work: *How can I get them to stop talking and start discussing?*
- Using technology effectively: *How can I get them to stop playing and start thinking?*
- Questioning and reasoning: *How can I ask questions that improve thinking and reasoning?*
- Assessing modelling processes: *How do I assess progress?*
- Involving students in self and peer assessment: *How can students help each other to progress?*

Each module has the same three-part 'sandwich' structure:

- **Introductory session:** Teachers work on problems provided, discussing specific pedagogical challenges that are the focus of the module. They then watch a video of other teachers using these problems and together plan a lesson using given materials.
- **Into the classroom:** Teachers all teach the planned lesson in their own classroom.
- **Follow-up session:** Teachers describe and reflect on what happened, discuss video extracts, and plan strategies for future lessons.

As with the materials, the lessons are based on a variety of modelling task types:

- *Plan and organize*. Find optimum solution subject to constraints.
- *Design and make*. Design an artefact or procedure, and test it.
- *Model and explain*. Invent, interpret, and explain models.
- *Explore and discover*. Find relationships and make predictions.
- *Interpret and translate*. Deduce information and move between representations of data.
- *Evaluate and improve*. Review and improve an argument, a plan, or an artefact.

We have space here to illustrate two of these (see Fig. 30.10).
Examples of the core task-type *model and explain* include:

Heat Kills Toddlers. Babies must never be left in locked cars on hot days. They quickly dehydrate. Explain why toddlers dehydrate more rapidly than adults.

Traffic Jam. In a 12-mile traffic jam on a two-lane freeway, how many cars are there? If drivers have a 2-s reaction time, how long will it take to clear?

Explore and discover tasks include computer-based microworlds like that in Fig. 30.3.

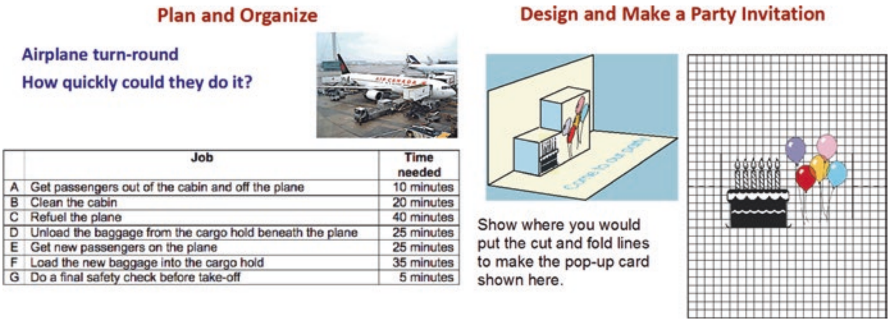


Fig. 30.10 Tasks for professional development

30.5 The System-Level Challenge

We now know how to enable typical teachers to teach much better mathematics, notably modelling, much more effectively. So why doesn't it happen in classrooms?

It is easy to identify some direct causes. Policy initiatives are often misguided – for example, reducing class size ignores the research evidence and is a uniquely expensive move. Initiatives are usually badly designed by policy makers, with outcomes far from their intentions – for example, short tests with artificial tasks aim to reduce testing time but lead to many wasted days of 'test prep' that do not help students learn mathematics. Mistakes in design like this lead to unintended consequences that are not only predictable but often predicted – and usually avoidable.

But from a system perspective, this analysis is too easy – because *politicians and policy makers are part of the system we seek to improve*. Yet the situation is different in other research-based fields, like engineering or medicine. All healthcare systems have economic and other challenges for politicians to manage, but they do not interfere at a technical level – imagine a health minister deciding "We are going to move over to all-acupuncture-based health care". It just doesn't happen – but things like "You can't solve problems until you've mastered a lot of maths" are commonplace. Making education more like medicine in this sense would be a big step forward. How might we tackle this challenge? What are the barriers and affordances we face? What are the most promising things to try, short term and longer term? That is the focus of this final section; without proven well-engineered answers to these questions, the potential of most children will remain unrealized.

30.5.1 The Big Picture

In mathematics, modelling is generally accepted as important. Everyone recognizes that all real improvement in student learning depends on what happens in classrooms. But education systems are complex, with players at different levels having different pressures and priorities. Politicians' priority is to stay in power and thus, short term, to avoid negative reactions to any initiatives. There is a fundamental 'timescale mismatch' here – ministers of education know they will have moved to other roles long before the impact of any initiative can be evaluated, so plausibility to influential non-expert opinion is their key need. Superintendents and school principals face daily urgencies that demand their attention. Pressures are passed down the line by these 'powers' to teachers who are expected to deliver, with support that fails to match what the new demands really require – the teachers are then blamed for perceived failures. (The problem is exacerbated because teachers' representatives never say, as doctors' do all the time: 'We don't know how to do that yet' or 'It will take these resources and this time to achieve that'.) Parents, not surprisingly, are mostly puzzled bystanders who regard their own school experience as the proper norm.

Finding ways of making a change that is positive for all these groups is a huge design challenge – and one that policy makers do not even recognize as such. Consequently, the challenges of large-scale improvement are underestimated. The potential contribution of good engineering (Burkhardt 2006) is ignored. The very concept of alternative models of change is still seen as novel. As a result, there is no established way to design, develop, and implement improvement. However, from our experience some things are clear.

30.5.2 Strategic Design

We now know quite a lot about *strategic design* (Burkhardt 2009), those aspects of design that concern the 'fit' of an initiative to the system it aims to serve. All key constituencies need to accept, and preferably to support, the change and what it requires of them. Experience suggests that this needs a coherent combination of *incentives* that make the change valuable for those involved, *pressure* to carry through the new challenges it presents, and well-engineered *support* that enables them and their colleagues to do so and to do it well. Alignment among these three elements is vital – but rare. Too often, for example, a desirable change in curriculum (incentive) is implemented without changing the high-stakes test (pressure) and with teaching materials and/or professional development (support) that are not good enough to enable typical teachers to achieve the changes that are involved. However good some of these elements may be, poor quality in others will undermine the change. Poor strategic design is a major cause of the low impact, with outcomes far from intentions, that is so common.

30.5.3 Issues and Opportunities

Any initiative needs to make choices and, as with any design challenge, there are trade-offs (Burkhardt 2009). Here we list a few of the issues that should be considered in designing an improvement initiative.

- *Pace of change* is a variable that is rarely considered in the design of initiatives in education. Yet it should be obvious that a policy initiative that asks large numbers of professionals across a complex system to change their well-established practice in challenging ways will take time. We believe that carefully paced gradual change with well-aligned support, exemplified by 'the Box Model' outlined in Sect. 30.1.1, is much the *fastest* approach in the medium term. (The gradual improvement approach is, of course, standard in medicine.) So far policy makers, at least in the UK and the USA,[5] prefer 'Big Bang' approaches, seeking to 'fix' a perceived problem; the outcome is invariably disappointing, combining disruption with little change at classroom level.
- *Ambition* v *robustness* in planning an improvement initiative is a related issue. Shall we take a big step, trying to move to where we want to go? In which case, few will follow. Or shall we take a small 'digestible' step? In which case, it seems a lot of effort for a small gain. (Gradual change models combine the virtues of both, with small steps moving over time towards the 'ideal' – which will surely change over time.)
- *The development process* needs to be *formative*, so that the final product is better than the first draft design. For this, the trialling process must be *iterative*, ending with teachers and students that are reasonably representative of those the strategic design aims to serve. The feedback needs to be rich, detailed, and structured enough to tell the design team what happened and thus reliably to guide revision.
- *Engines for further improvement* need to be built into any initiative. Even the most successful implementation will decline in quality over time. Nonroutine problems will become routine. Practical considerations, often about costs, will 'simplify' a rich examination with several components into a more routine test. To counter this 'downward drift', a structure for ongoing improvement is critical to survival.
- *Methodology*. The methods we have used in Shell Centre work (Burkhardt 2006) are fairly standard in engineering and medicine. They may be summarized as:

  research > design > development > robust product.

  The process in more detail is to create designs from research insights and prior examples; circulate draft designs for comment and discussion; prepare a draft version for trialling; observe the trials, getting structured rich, detailed, descriptive feedback from observers; revise the draft, guided by the feedback; iterate the trials and revision until outcomes match (perhaps revised) goals; release to the user community; and monitor.

[5] Some of the countries in East Asia that anglophone policy makers most admire, like Singapore, take a more measured approach, with serious engineering of the initiative and the support.

- *Cost*. This process is more expensive than the 'authoring' model, mostly used in education, which relies on the authors' communicating their experience to other users in writing with a minimal development process. Shell Centre lessons cost around $30,000 each, which may seem expensive. Strategically, this is not so. Even to revise the whole curriculum (~15,000 h) every decade in this way would cost less than 1% of the cost of running the education system in a reasonably large country.

We illustrate these general points with a specific case: the design challenges of combining high-quality and high-stakes in testing; though a rare combination, it can be done (ISDDE 2012) and is a crucial part of getting important elements of curriculum like modelling implemented in classrooms on a large scale. *What You Test Is What You Get*. The key design questions include: Does this test assess the intended curriculum in a balanced way? Does it encourage good teaching? Does it produce reasonably reliable scores? Usually, in test design, only the last is taken seriously[6] – as a result, the effects of large-scale testing on the teaching and learning of modelling have usually been disastrous.

How has this situation arisen and been justified? The psychometrician's view that 'tests are just measurement' ignores the dominant influence of high-stakes test tasks on teaching and learning activities in most classrooms. But this fallacy is used to justify cheap tests that 'correlate well with other measures'. Criterion-based testing has a similar effect. Both drive you to test bits of mathematics separately, which tells you nothing on how well the student can construct the chains of reasoning that doing mathematics actually involves.

30.5.4 Systemic Barriers and Levers

Changing education on a large scale in the way we have discussed here is not easy – indeed it remains an unsolved problem in most countries. What are the strategic barriers and what promising ways are there to get around them?

30.5.4.1 Policy

There are some things that increase the probability of influencing policy decisions. Work with policy makers, where you can get access to them. The core objective is to get them to see educational initiatives as serious design and engineering problems that cannot be resolved just through discussion, but need expert design input from conception to delivery. To summarize, the key issues to get discussed are alternative

[6] But with different criteria in different subjects. For essays in the humanities, substantial 'mark-remark' variation is accepted; mathematics is proud of its scoring 'accuracy', which is only achieved by excluding tasks that demand substantial chains of autonomous student reasoning, as is essays – or in assessing modelling.

models for implementing the change, its strategic design, which teams have the skills and capacity for the detailed design and engineering, and a selection process. *To satisfy the different needs and timescales of politicians and educational improvement, the design should be long-term but with visible short-term achievements.*

30.5.4.2 Research

Reviewing the nature and effectiveness of research in education is sobering (Schoenfeld 2002; Burkhardt 2016). How well do we perform? In serving ourselves, the research community does pretty well. Papers get written and published, graduate students get PhDs, and academic staff are appointed and promoted. But when policy makers think they have a problem in education, do they (as in medicine) turn first to the research community? Rarely. What is needed to have more impact? The key elements in successful research-based fields of practice, like medicine or engineering, are (Burkhardt and Schoenfeld 2003) a body of generally accepted knowledge, a system for turning research insights into effective tools and processes, and rich evaluative feedback to guide the next stage of research and development.

Education has none of these in full working order. Why not? A key factor is the academic value system, which favours new ideas over reliable research, new results over replication, disputation over consensus building, small studies over team research, first author over team member, and journal papers over tools for improving practice. *These priorities are the reverse of what is needed to drive system improvement*.

30.5.4.3 Progress

What changes might meet the challenge? We believe that moving forward at system level will require a practice-focused research enterprise that:

- Builds collaborations for tackling big issues, which need big projects;
- Focuses on developing and evaluating specific well-engineered exemplars;
- Does evaluation-in-depth – on what happens as well as student test outcomes;
- Builds bodies of reliable results with evidence of their range of validity;
- Identifies and publicizes successes.

That is the next step towards a ‘big education’ effort that matches the challenges we face. We have recently moved to develop ‘system change’ level tools that guide improvement programmes. But in this new and crucial domain, all is still to play for.

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Chapter 31
Mathematical Modelling as a Professional Activity: Lessons for the Classroom

Peter Frejd

Abstract This chapter presents a discussion about similarities and differences between working with mathematical modelling in 'school' and mathematical modelling as a 'professional task' in the workplace based on empirical and theoretical research studies. Issues discussed concern goals; technology; division of labour, communication and collaboration; model construction, including the application and adaption of predefined models; projects; and risks involved in using the models. Based on this discussion and examples from innovative teaching practices, approaches to simulate modelling as a 'professional activity' in educational settings are explored and exemplified with a role-play activity.

Keywords Modelling • Modeller • School activity • Professional activity • Innovative teaching methods • Role-play

31.1 Introduction

Mathematical modelling and models are used for various purposes in school and in society. Descriptions from mathematics syllabuses across the world indicate that the use of modelling activities in the mathematics classroom may contribute to developing students' understanding of how and why mathematics is used in the everyday and in the workplace, at least if the modelling problems are chosen adequately (e.g. Brasil 1997; Department of Basic Education 2004; Department for Education 2013; NGAC 2010; Ministry of Education 2013; Skolverket 2012). While mathematical modelling has been described as "the most important educational interface between mathematics and industry" (Li 2013, p. 51), there are indications, however, that it is not emphasised in current teaching practices at upper secondary school (e.g. the preface in Stillman et al. 2015) nor is the coordination between school and working

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_31

life strong enough (Damlamian et al. 2013). There is an international consensus among researchers in mathematics education that "an agenda for action is needed containing short- and long-term activities that strengthen the relation between industry and mathematics education at school" (Kaiser et al. 2013b, p. 269).

To identify activities that strengthen the connection between industry and school as well as the teaching of mathematical modelling, it has been suggested (e.g. Drakes 2012) to 'mirror' or simulate parts of expert modellers' working practice in teaching practices. This would involve, for example, spending a large proportion of learning/teaching time on formulating the problem and validating the solution, activities currently not given much space in teaching practice. Drakes (2012) also argues that "students would ... benefit from seeing real modelling done by experts. Seeing experts deal with being stuck is informative, and helps change the belief that experts simply rely on intuition" (p. 207). Other researchers suggest activities that include a more complete simulation with realistic characteristics of some workplace practice for students to become proficient modellers (e.g. Burghes 1984; Heilio 2013). As an example, Heilio (2013) argues:

> For successful transfer of mathematical knowledge to client disciplines the theme of mathematical modelling is a crucial educational challenge. The lectures, books and laboratory exercises are necessary, but the actual maturing into an expert can only be achieved by 'treating real patients'. (p. 224)

As described in the quotation above, the transfer between mathematics used in different workplaces and mathematics taught and learned at school is not always straightforward. Mathematics at the workplace can be more complex and includes specific technologies, social, political and cultural dimensions not found in educational settings (Damlamian et al. 2013; Noss and Hoyles 1996; Wedege 2010).

Drawing on empirical and theoretical research studies, this chapter will discuss *similarities and differences* between working with mathematical modelling in 'school' and mathematical modelling as a 'professional task' in the workplace. Based on this discussion and examples from innovative teaching practices, approaches to simulate modelling as a 'professional activity' in educational settings will be explored.

31.2 Working with Modelling in School and as a Professional Task in the Workplace: Similarities and Differences

The literature discussed in this section is deliberately selected to demonstrate similarities and differences between the two practices rather than for the purpose of presenting a comprehensive and exhaustive review. There is a diversity of theoretical perspectives on mathematical modelling as a school activity in mathematics education literature (e.g. Blum et al. 2007; Garcia et al. 2006; Geiger and Frejd 2015; Jablonka and Gellert 2007; Kaiser and Sriraman 2006). This plurality is natural considering the different social and cultural realities in which research is being

Table 31.1 Aims of mathematical modelling teaching approaches according to Kaiser and Sriraman (2006)

Name of approach	Aims
Realistic or applied modelling	Solving real-world problems
Contextual modelling	Subject-related and psychological goals
Educational modelling	Modelling as a didactical tool
Socio-critical and sociocultural modelling	Critical understanding of the surrounding world
Epistemological modelling	Theory-oriented goals

carried out with different objectives and in different traditions. In addition, this shows that there are many different teaching approaches related to modelling.

One way to characterise teaching connected with applications and modelling is based on the classification of the variety of approaches developed by Kaiser and Sriraman (2006), illustrating various perspectives that are strongly influenced by particular theoretical backgrounds (Table 31.1).

The amount of available literature focussing on school modelling is plentiful, but research literature focussing on how professional modellers work is more limited (e.g. Gainsburg 2007a, b; Spandaw 2011; Willemain 1995). In this chapter, the descriptions of modelling as a professional activity draw mainly on results from Frejd and Bergsten (2016) and to some extent on Drakes (2012) and Gainsburg (2003). This sample of literature focusses on individual modellers, who explicitly would argue that mathematical modelling is central for their profession.

To structure the discussion, I will set up three phases to compare: *pre-modelling*, *modelling* and *post-modelling* (cf. Frejd and Bergsten 2016). The first phase, pre-modelling, concerns the goal of the activity and who delivers the tasks. The second phase, modelling, relates to the actual activity 'how students and modellers work', and finally, the third phase, post-modelling, concerns how the models will be implemented and what risk there is involved in using the models.

31.2.1 Pre-modelling

A modelling activity in a mathematics classroom can serve two aims: Either to develop modelling competencies and give students experience in developing models or to develop a broader mathematical ability (i.e. as a didactical tool to learn mathematics and embed mathematics that has already been learned) (e.g. Blum and Niss 1991). Teachers or researchers usually set the modelling problems for the students, and there is a diversity of knowledge taught such as elementary arithmetic with base ten blocks (Speiser and Walter 2010) or exponential and power functions with algal bloom problems (Geiger 2013).

According to Frejd and Bergsten (2016), mathematical modellers in the workplace receive orders or problems from clients, who may come from government

Table 31.2 Differences in pre-modelling as a school versus a professional activity

School activity	Professional activity
The models are developed to serve as a basis to learn modelling or to broaden mathematical ability	The models are developed to serve as a basis for decision-making
Teachers provide the problem	Clients provide the problem

bureaus, other companies or employers and supervisors within the firm. The goals of modelling as a professional activity include description and simulation of a phenomenon for prediction (to make prognoses about the future), design (improving objects) or construction (objects). The models are developed to serve as a basis for decision-making (Frejd and Bergsten 2016).

Table 31.2 indicates some differences between school and professional modelling activities in terms of aims. The goals to simulate and predict relate to prescriptive modelling (Niss 2015), which is not a frequent activity in today's classrooms according to Niss. "In prescriptive modelling the ultimate aim is to pave the way for taking actions based on decisions resulting from a certain kind of mathematical considerations, in other words 'to change the world' rather than only 'to understand the world'" (Niss 2015, p. 69). However, there exist examples where students make models in the classroom for decision-making. Galbraith (2013), for example, reports about a 12-year-old girl who made a mathematical model in the classroom that convinced her parents that she could buy and take care of a pony.

31.2.2 Modelling

31.2.2.1 School Activity

Modelling as a school activity takes different forms depending on how modelling is conceptualised and realised in curricula documents, textbooks and assessment modes, as well as depending on teachers' and students' views of the notion of mathematical modelling. Other factors impacting are, of course, the social, cultural and historical dimensions in which teaching and learning takes place. To capture some characteristics of modelling as a school activity, the discussion of the selected sample of literature draws on Kaiser and Sriraman's (2006) categories presented in Table 31.1.

A *realistic or applied approach* to modelling may be characterised by students often working in teams trying to solve 'realistic' problems, such as designing a wind park, pricing for internet booking of flights or location of bus stops (Kaiser et al. 2013a). It may take the students from 1 day to several weeks to complete the tasks using different resources finishing by presenting a poster, a PowerPoint or a written report. The aims are to develop abilities such as collaboration, communication and solving realistic problems.

The current teaching practice in Sweden commonly relates to *contextual modelling*, to solve word problems. The situation in Sweden may be characterised by students solving textbook tasks individually. In an analysis of textbooks for upper secondary school, it was concluded that models and modelling do not play a central role and modelling problems for students to solve relate predominantly to adapting and using predefined models (Frejd 2013a).

Educational modelling focusses on modelling as a tool to promote learning of mathematics and concept development. This approach has different goals from the other approaches, namely, to structure the learning process and develop understanding of concepts, even if the activity as such can be in the format of any of the other approaches. This perspective puts the educational goals into the foreground and issues such as how modelling tasks are assessed in relation to learning objectives in curricular documents is a central part of the modelling activity (examples of different modes of modelling assessment are found in Frejd 2013b).

A news clip from Brazil about distributing 5 kg of seeds to farmers was a starting point for a modelling activity in Barbosa (2006), categorised as *socio-critical and sociocultural modelling*. Students in grade 7 in a rural area of Brazil were organised in groups to discuss whether families had different needs and therefore should receive different quantities. These discussions facilitated a critical awareness among the students about the role of mathematics and modelling in society.

Epistemological modelling is often related to the Anthropological Theory of Didactics (Chevallard 1991). One example is found in Garcia et al. (2006). Based on theoretical tools from Anthropological Theory of Didactics, they reformulated the mathematical modelling process in this theory and applied the developed theoretical approach to analyse a teaching sequence about 'savings plans' for a trip. They concluded that the analysis gave valuable information about the characteristics of the modelling process in terms of a set of praxeologies (a theoretical term for 'know-how' and 'know why').

31.2.2.2 Professional Activity

Modelling as professional activity may be characterised in different ways. In Frejd and Bergsten's (2016) case study of nine professional modellers, three differently structured modelling activities were identified: *data-generated modelling*, *theory-generated modelling* and *model-generated modelling*. For seven of the nine participants, the activity of data-generated modelling played a prominent role in their work. The other two model constructors were mainly engaged in theory-generated modelling. Model-generated modelling was a common activity among all participants in their working practice. In Frejd and Bergsten (2016), the different structures of the three modelling activities are visualised by diagrams, *modelling activity schemes*. These activities will now be briefly described.

In *data-generated modelling*, the modeller first receives the problem from a client. There is communication between the modeller and the client about clarifying, adapting and reformulating the problem. Larger projects often include interdisciplinary competencies, which require communication between experts. The data are treated as a fundamental aspect of the modelling activity and therefore have a central position in the process, influencing how the model is going to be developed and impacting on the specific problem formulation. The data are also used to identify processes, variables, conditions and constraints of the phenomena together with computer support. However, constraints, processes and facts are also found in discussions with clients and experts.

The identified variables and processes are then formulated in mathematical terms and computer codes. The model is calibrated with data within a computer environment, and computer support is also used for the evaluation/simulation process when inputs and outputs are tested, validated and compared with given data, outcomes of experiments or expert opinions. The expert opinions and clients' interests must be taken into consideration in the process of determining an acceptable or reasonable solution.

The problems in *theory-generated modelling* are provided by clients and consist of sets of (theorised) equations such as differential equations. To solve the problem, modellers reformulate it and set up mathematical models of equations with approximations. The modellers often communicate with other experts and work collaboratively to develop an optimal way to solve a problem. After the reformulation of the given task, the modeller translates the mathematical model to a computer model, solves the computer model and interprets and evaluates the result and finally evaluates the validity of the computer model. Similarly to the case of the data-generated modelling, the clients' main interest in an acceptable solution is its *usefulness*, whereas the communication with other experts is more focussed on the *effectiveness* of the models (cf. Jablonka 1997).

The activity to apply already-developed models on a problem, *model-generated modelling*, is a part of all modellers' working practice. Reasons for applying already-developed models may be time constraints and that there exist models that take care of types of standardised problems. The identification of how existing models can be adapted draws on working experience or is an outcome of communication with other experts. The modellers reported in Frejd and Bergsten (2016) also emphasised the use of computers and dedicated software that include some types of established models. Based on the evaluation and validation of the outcome of the application, these standardised models might be adapted and applied again. Communication with other experts and clients about the usefulness and the effectiveness of the applied model may take place as a consequence of the validation process. Finally, the issue that concerns what constitutes an acceptable solution is similar to the data-generated or theory-generated modelling.

The aims of the three modelling activities are summarised in Table 31.3. Other key aspects found in the three activities relate to communication, collaboration and the use of computer support. Communication and collaboration between clients, operators and other experts are vital parts of the modelling work that concern issues such as clarifying the problem and identifying solution strategies in discussions about the usefulness and the effectiveness of the model and in the process of determining a reasonable solution. Computer support also influences the modelling activity to a large extent, for example, in terms of identifying variables and processes,

Table 31.3 Mathematical modelling as a professional activity (Frejd and Bergsten 2016, p. 29)

Name of activity	Aims
Data-generated modelling	The work of gathering, interpreting, synthesising and transforming data as the underlying base for identifying variables, relationships and constraints about a phenomenon used in the development process.
Theory-generated modelling	The work of setting up new equations based on already 'theorised' and established physical equations, followed by the activation of computer resources for computational purposes to solve the new equations with the aim being to obtain information about the 'theorised' equations.
Model-generated modelling	Models are constructed by identifying situations on which some mathematics or established mathematical models can be directly applied.

calibrating the model, solving computer models, simulating different processes and evaluating and validating the outcome.

31.2.2.3 Professional Versus School Activity

There seem to be several similarities between modelling as a professional activity and a school activity, such as collaboration, communication and adaption and development of models. However, there are also differences identified.

Much of the professional modellers' work is based on knowledge and experiences reaching far beyond what can be found in a secondary mathematics classroom. In the professional activity, computer support and programming play a major role in the development of models which is also the case for the evaluation or simulation process when inputs and outputs are tested, with software that is not frequently found in educational settings. The division of labour in school is used to support students' learning in relation to learning objectives, while in the professional activity the division of labour is based on individual skills or the distribution of work. From the three types of modelling activities discussed above, *model-generated modelling* seems to be the activity (in a very simplified form) which most resembles activities in schools. For example, in textbook descriptions of modelling in upper secondary school in Sweden, the use of already-defined models is emphasised as a central aspect of modelling (Frejd 2013a). Examples and projects, found in research literature in mathematics education, that include parts of data-generated modelling, in a simplified version, are common (e.g. Blum et al. 2007). Theory-generated modelling, however, presents more of a demanding task for teachers to implement in school mathematics.

31.2.3 Post-modelling

There are also major differences between the professional and the educational contexts in terms of objectives and consequences of the modelling activity (cf. Wake 2014, p. 272). For example, in the classroom mathematical models constructed by students are seldom put to use in a context of practice or in other ways that involve risks. In professional practice, however, the situation is different as there are risks when models are put into practice; people may, for example, lose money or be injured. According to a finance modeller in the case study by Frejd and Bergsten (2016, p. 23), there is a range of potential risks involved when using models, not only due to a 'blind' trust in the model:

> Of course there are risks of using models and sometimes one talks about model risks and that is exactly that you have missed something, that you use the model in a context where it should not be used. Or you use it even if the conditions are not fulfilled, or that the assumptions maybe worked when you made the model [...] the customer has paid money today to get it back as a pension after twenty years and when you get there, no money is left.

There may also be ethical problems due to the new information that the use of the models may present such as some farmers might be banned from raising some types of animals if the maximum time on animal transportations were to be regulated, an outcome described by the biology modeller, after optimising travelling time to different slaughter houses.

31.2.4 Summary of Similarities and Differences

Based on the literature review, *differences* between mathematical modelling in school and professional practice are identified in how modellers, teachers and students work within terms of the goal of the modelling activity, the risks involved in using the models, the use of technology, division of labour and the construction of mathematical models. In addition, *similarities* are identified, described as important aspects of modelling work in the different practices, such as communication, collaboration, projects and the use of applying and adapting predefined models.

These major differences between modelling work in educational and workplace contexts seem to indicate that mathematical modelling in school will remain an unreachable goal in terms of coherence to professional practice. However, this comment could be made also about mathematics in general. In order not to lose key elements of modelling as a professional activity, the results presented here point to the necessity of having access to knowledge about how professional modellers work. While modelling as it shows in the workplace can never be fully 'mapped' into the mathematical classroom, it may nevertheless be possible to simulate such activity.

31.3 Innovative Teaching Methods: Simulation, Gaming and Role-Playing

Using innovative teaching methods, such as simulation, gaming and role-playing, might be a way to bridge the gap between modelling as a professional activity and as a school activity. However, the terminology – simulation, gaming and role-play – is mixed and confused in literature (Armstrong 2003). To clarify this, van Ments (1999, p. 3) describes simulation as "a simplified reproduction of part of a real or imaginary world", whereas gaming is "a structured system of competitive play that incorporates the material to be learned". Finally, role-play is "a make-believe representation of some real-life event, carried out in order to help participants [who play a role] get better at managing the event itself" (McGuire and Priestley 1981, p. 87).

The descriptions above of the terminology may be used as working definitions. Nevertheless, from the point of view of a critique, many questions must be addressed, such as: How simplified may the reproduction be and still be a simulation? and What is not a real or imaginary world or can anything be simulated? Closely related to simulations is the notion of *authenticity*. Vos (2011) discusses authenticity in relation to simulation of professional modelling practices. She defines authenticity as a social construct for which the community agrees on its qualifications and defines four criteria for a modelling task to be authentic:

- The task should not be created for educational purposes.
- It should have some connection to out-of-school practice.
- It should be binary in the sense that all, or aspects of the problem, or the way of working is *either* authentic or not.
- The task should be actor independent, meaning the task should be certified by expert actors (stakeholders, modelling researchers) and be authentic to 'all' actors (students and teachers) involved in the activity.

Drawing on the description of authenticity by Vos (2011), together with innovative teaching methods, in particular simulation (reproducing realistic aspects of working practice) and role-playing (playing the role of professional modellers) seem to have potential for reducing the gap between modelling in school and modelling at the workplace. Gaming as a structured system of competitive play (van Ments 1999) is another method that can be used. However, this chapter will only discuss and present examples of teaching practice that focus on simulations, in Sect. 31.3.1, and role-playing, in Sect. 31.3.2.

31.3.1 Examples of Simulations of Professional Modelling Practice

An extracurricular activity is presented by Vos (2015), with the aim of enculturation of students into modelling as a professional practice. Students from upper secondary school visited the Science Park of the University of Amsterdam where they had the opportunity to meet a manager working for the National Dutch Railway Company through live video conferencing. This was the introduction of a simulation activity where the managers gave the students a modelling problem to design a railway timetable. The problem included both simplifications and constraints such as some trains did not stop at all stations; passengers should have the right to access a train within a maximum waiting time and have time for transit; and there should be a safety distance between departing and arriving trains. To solve the problem, the students learned about graph theory and used ICT, a software that was specially designed for the extracurricular activity and simulated the software that the modellers used originally. Most aspects of the activity may be described as a simulation of a realistic scenario, but some aspects may be described as authentic. For example, the students tried to solve one single complex problem during a full day without finding a reasonable solution and thereby associated doing research with frustration. This experience was also certified by a researcher at the university who described his own struggling and his need for endurance while doing research. Another authentic aspect found in Vos (2015) was the enlightenment of students of the use of mathematics in extra mathematical contexts, such as improving real timetables for railway companies.

A second example of a simulation activity is found in Edwards and Morton's (1987) study of a boardroom meeting between a management panel and a modelling team. The students in the modelling team had worked on a project for some time (a whole term or 1 week) and were expected to present their findings, with the suggestion that they needed more money to continue their research. Whether the students would receive any money depended on the outcome of the meeting. To reduce the gap between education and workplace practice and make the simulation activity more authentic, non-mathematical experts were invited. For example, a sheep farmer was invited when a group of students were going to present their work on a sheep farm model. This action made the students adapt their language and explain and defend their model to a client that was not necessarily familiar with the mathematics used, which is common in modelling as a professional practice (Frejd and Bergsten 2016). Nevertheless, it is stated by Edwards and Morton (1987) that the farmer was not impressed by the mathematics but fascinated by the work the students had done.

31.3.2 Role-Playing

31.3.2.1 General Aspects of Role-Playing

The students in Edwards and Morton's (1987) paper seem to play the role of modelling experts. Role-playing has been known as an activity for education since 1940 (Williams 2014). It has gained momentum in different subjects and occurs most frequently in the humanities (van Ments 1999) but also in business (Armstrong 2003) and healthcare education (Nestel and Tierney 2007). Role-playing emphasises decision-making (e.g. Belova et al. 2013), which is a central aim within professional modelling (Frejd and Bergsten 2015). Research literature about role-playing in mathematics education tends to focus on primary school education, while research into role-play and mathematics with older students seems to be non-existent (Williams 2014). There are research results indicating that role-playing in mathematics education is an activity that facilitates students' learning. Williams (2014, p. 3) describes the relevance of role-play to the learning of mathematics in the primary classroom as follows:

> that role play is useful for mathematical learning and that it is possible to engage in complex mathematics through role play. I argue that the potency of role play is its ability to suspend disbelief and engage children as participants in a community of learners. My study also concludes there is potential for developing children's mathematical awareness and metacognition through reflecting on role play.

Arguments are put forward that role-playing is powerful, is motivating, provides meaningful contexts and increases students' skills in collaboration and communication (Griffiths 2010; Ginsburg 2009; van Ments 1999; Williams 2014). In addition, in tackling mathematical content within role-play, children have been observed to think mathematically and engage in activities of gathering, ordering and analysing information as well as making conjectures (Williams, 2014). All these aspects are emphasised as central aspects also in professional modelling practice (Drakes 2012; Frejd and Bergsten 2016; Gainsburg 2003).

The basic idea with role-playing is:

> to give students the opportunity to practice interacting with others in certain roles. The situation is defined by producing a scenario and a set of role-descriptions. The scenario gives a background to the particular problem or environment and indicates the constraints which operate. The role-descriptions give profiles of the people involved. (van Ments 1999, p. 9)

Implementing role-play activities into mathematics education requires the teacher to consider several issues. Table 31.4 gives some guidelines of how to teach with the use of role-playing.

Table 31.4 Guidelines for teaching role-playing

Guidelines	Explanations
1. Objectives	Define why you want to include role-playing in mathematics education. What content should be covered? How much time should be spent? What do you expect of your students?
2. Choose context and roles	Decide on a problem in relation to the chosen content and the setting of the activity. Define the goals of the characters and prepare background information about them.
3. Introduction	Explain to the students why they take on this role-playing exercise and stress what you expect them to learn. Introduce the problem, the characters and the setting.
4. Student preparation/ research	The students will need time to get into their roles and learn about their characters as well as get more information in relation to the problem.
5. The role-play	The actual activity.
6. Debriefing	The teacher may offer an opportunity to ask the students what they have learned during the lessons and give comments and corrections of errors that have occurred during the role-play.
7. Assessment	Assessment modes for the activity depend on the goal of the activity and many types of assessment modes may be used in relation to modelling activities (see, e.g. Frejd 2013b).

Retrieved from: http://serc.carleton.edu/introgeo/roleplaying/howto.html

31.3.2.2 A Role-Play Activity Emphasising Aspects of Modelling as a Professional Activity

In a case study by Frejd (2013a), modellers suggested that mathematics teachers could invite people from the workplace (or other organisations) to present how they work to increase the motivation for the study of mathematics. To invite a person from the workplace, like a manager from a railway company (Vos 2015), to the mathematical classroom to set the scene for the role-play activity could serve three purposes. First, the person may act as the client; second, he/she may certify the authenticity of the modelling problem; and third, the person can explain how mathematical models are used in his/her profession.

In modelling as a professional activity, the models are developed to serve as a basis for decision-making. One example could be to invite a local politician with the following problem (within a Swedish context):

Wolves

"The long-term goal of the predator policy in Sweden is to achieve and conserve a healthy population of wolf, bear, wolverine, lynx and golden eagles" (p. 14). "The aim is to create a good balance between the predator population and the impact it causes on business, public and individual interests" (p. 16) (Swedish Ministry of the Environment 2012/2013; SOU 2012/2013:191, my translation).

How many wolves should we have in Sweden?

31.3.2.3 The Wolves Example: Methodology

The *Wolves* example, inspired by the work of the biology modeller in Frejd and Bergsten (2016), is developed from a government proposal (SOU 2012/2013:191) about sustainable predator policy. The problem can be regarded as a problem "clearly not created for educational purposes" (Vos 2011, p. 721). This means that the problem was originally developed for out-of-school purposes; it is authentic to anyone, which can be certified by different stakeholders and modellers (Vos 2011).

One idea could be to split the teaching group into smaller teams and let them choose to play the characters of modelling experts associated with the Swedish Association for Hunting or associated with the Swedish University of Agricultural Sciences, with an aim to develop a model to be used for making a decision on how many wolves there should be in Sweden. The two consultation bodies have clearly different views on how many wolves there should be in Sweden as described in the government proposal (SOU 2012/2013: 191). The Swedish Association for Hunting argues for 200 wolves, whereas the Swedish University of Agricultural Sciences argues for 1250–2000 (SOU 2012/2013:191).

Collaborative or cooperative group work is a common school practice in many subjects, but role-playing as a part of mathematics education where team members have specific characters seems not to be frequently used. That is the case at least in the Swedish upper secondary school (Frejd 2015). Placing the students in teams could, to some extent, divide the workload between them and define more specific characters to play, thus making a division of labour. One student may search for historical data on the wolf population and aspects that effect the population like inbreeding, poaching, the amount of prey, etcetera. Another student may search for data on the impact that population growth has on business, public and individual interests and what it costs. A third student may focus on the mathematical relations, and a fourth student prepares the report and the presentation. However, what is significantly important is the collaboration and communication with the aim that all students learn about all parts of the issue discussed. The modelling work of the students may include aspects of data-generated modelling, theory-generated modelling and model-generated modelling. The students are involved in the data-generated modelling when they identify the data and parameters, variables, constants and processes to develop models and maybe they also consider the quality of data. The involvement of statistical data may be displayed and analysed with the use of technology, which is an essential part of modelling in workplace practice. Aspects of theory-generated modelling could also be a part of the students' work if the students read something about predator-prey relations and about differential equations. These differential equations could, for example, be visualised (and solved) with technology. To apply already-defined models, model-generated modelling, is a part of modelling work and may include some economic models, statistical models or differential equations, to name but a few.

The end of the project may be organised as a political debate inspired by Edwards and Morton's (1987) idea to "simulate a boardroom meeting between a management panel (a mixture of technical and non technical managers) and a modelling team" (p. 53). The goal of the political debate for the client, the politician, is to make a decision on the number of wolves, based on the modelling teams' (students') oral and visual (e.g. PowerPoint) presentations as well as based on their written project report. The political panel may include the politician, experts and teachers. This will imply that the students need to communicate and adapt their language, which was an important part of the modellers' workplace, as well as explain the mathematical models they have used.

Regarding the assessment of the activity, it is necessary for the teacher to be explicit about the meaning and the goal of modelling and its relation to assessment criteria. In the example above the teacher may gather information about the students' performances from multiple resources, such as written reports, oral presentations and students' abilities to play the role of a certain character.

31.4 Conclusions

I end this chapter by revisiting the title, *Mathematical modelling as a professional activity – Lessons for the classroom*, and conclude that this chapter explored two issues: first, what *lessons for the classroom* can we as educators and teachers learn from professional modellers and their way of working with mathematical modelling and, second, in what way is it possible to design *lessons for the classroom* that might allow the practice of professional modellers.

There are several principles that teachers may rely on in their mathematics teaching practice, depending on what they see as the goals of mathematics teaching and what use of mathematics they consider important for students to learn. One such principle is that modelling ability is developed through the practice of *doing* modelling as a professional practice (Burghes 1984; Heilio 2013), meaning the students should be exposed to 'realistic' modelling problems in their education. The question is: How realistic, in terms of similarities to the professional practice, can we be in a classroom situation at upper secondary school, due to the differences presented in Sect. 31.2.4? The knowledge required for modelling as a professional practice is not accessible in upper secondary school, because the modellers' work is based on collaboration with other experts, knowledge of advanced mathematics, specialised knowledge of other fields, the use of technology and programming and years of modelling experience. Other differences identified relate to the goal with mathematical modelling, the risks involved in using the models and the division of labour. These differences show that modelling in professional practice can never be fully 'transposed' into the mathematical classroom. Nevertheless, it is possible to simulate parts of the modelling activity as a professional practice in the classroom. These activities can emphasise aspects not only identified as similarities such as communication, collaboration and projects but also other aspects of modelling such as *explorative modelling* or to *analyse* (given) models (cf. Barbosa 2009; Doerr and Pratt 2008; Jablonka 1996) that concern the importance of having an awareness of assumptions, quality of data and risks linked to the models such as 'blind' uses of mathematical models.

One teaching and learning activity that could have a potential to simulate modelling as a professional practice is role-play. It can provide meaningful context-related professional practice to be used in the classroom and increase students' skills in collaboration and communication (Griffiths 2010; Ginsburg 2009; van Ments 1999; Williams 2014) if it is designed and implemented properly. However, according to Williams (2014) the focus of research literature is on role-playing in primary mathematics education. There is thus a need for doing research into role-play and mathematics with older students. Therefore, I end this chapter with a sample of issues to be addressed in research regarding role-play and modelling, as starting points for future research studies:

1. What challenges and opportunities are there of using role-play for teaching and learning modelling?
2. To what extent is it possible to simulate professional modelling practice with the use of role-play?
3. How does role-play in a modelling context impact on attitudes, motivations and/or beliefs about mathematics?
4. How can assessment of modelling be organised in the activity of role-play?

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Chapter 32
Modelling Task Design: Science Teachers' View

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Abstract A seminar was organised to promote interdisciplinary work between teachers from different departments at a university in order to design pedagogical situations to teach mathematical modelling in the first years of higher education. This chapter presents the views of teachers from different disciplines related to skills needed to work with models in physics, biology and mathematics. The participants discussed how models are present in their courses. They also highlight the students' skills that are relevant to success in working with models in sciences. The main findings were that interpreting graphs and structuring a model with information from other models are considered relevant competences by the sample of science teachers. The discussion on designing teaching activities also revealed the university teachers' conceptions regarding mathematical modelling and models.

Keywords Interdisciplinary • Modelling • Models • Mathematics and science teachers

32.1 Introduction

Models and modelling are central elements that join and structure mathematical and science knowledge but unfortunately there is a separation in teaching the topics associated with them (Hestenes 2010). In the context of science education, Kapur (1982) discusses the mathematics courses students must take and highlights the differences in how they work with models. He states that, while models in science are seen from the heart of the discipline itself, projecting strategies for interpretation and explanation, in mathematics courses, the teaching is oriented towards the application of techniques and memorisation of procedures, creating obstacles for

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_32

students when they study subjects relevant to their profession. This aspect represents a challenge in mathematics education, but where should the study of mathematical models and content of the discipline converge? This brings us to the dilemma that represents learning applications and mathematical modelling: Must we attend to the demands of the discipline or should we put more focus on in-depth learning of modelling within the requirements of mathematical rigour? As mentioned by Kapur, knowing mathematical models is neither sufficient to learn to model nor is it sufficient to fulfil the requirements for working with models in the context of the sciences. One question that emerges is how models are used in the context of learning the sciences. The present study contributes to the research in this area by giving a perspective on how some university teachers in different sciences use mathematical models in their classes and what skills they expect from their students.

Several aspects contribute to increasing the gap between teaching sciences and mathematics. One is the curriculum where the development of modelling skills is not favoured. This aspect leads mathematics teachers to experience difficulties in implementing modelling activities (Kawasaki and Moriya 2011). Consequently, by studying the subjects of mathematics in the early years of higher education, students do not acquire the skills to work with mathematical models in the context of sciences or the skills to relate mathematical and scientific concepts. Moreover, mathematics courses are traditionally structured to obey the mathematical paradigm, emphasising learning and understanding of mathematics properties and their foundation, instead of focusing on developing mathematics skills and tools that students will need to succeed in learning in science courses. As Frejd (2012) concludes, more research is necessary to know what types of modelling activities are done in the sciences. This is why we seek to identify, in the context of designing teaching activities, what is the view of science teachers regarding the use of models in their courses. In addition, what are their interests related to working with models in advanced courses? The answers to these questions are relevant, since the construction and use of mathematical models forms part of the foundation of developing knowledge in the sciences. In addition, understanding the sciences implies that students are able to create and interpret several models used in their discipline. By knowing what the sample teachers expect of students about working with models in their courses, we identify some additional aspects that should be considered in the design of modelling tasks.

Many researchers have proposed ways to teach models in the sciences or mathematics, but few studies have considered the transition from studying mathematics to studying science courses (Bock and Bracke 2013). Ashmann et al. (2006) state that one of the goals to be achieved is:

> enhancing science and mathematics content understanding and problem-solving skills, understanding the connections between the two disciplines, developing pedagogical content knowledge that enables a diverse student population to achieve a depth of conceptual understanding, developing forms of authentic assessment, and implementing teaching practices consistent with the standards movement. (p. 191)

Assuming that task design should be developed in an interdisciplinary way, taking into account the learning needs related to building and interpreting mathematical models to work not only in mathematics, we conducted a seminar aiming to reflect on the design of pedagogical situations that support students in transitioning from studying mathematics to studying the subjects corresponding to their degree tracks.

32.2 Conceptual Framework

The term "model" can be understood in different ways among different disciplines. In physics, we may refer to a model when we talk about the atomic structure of materials or we may refer to it when representing the motion of a body. Models can be pictorial, graphical, numerical or algebraic. They may be present in verbal, visual or mathematical form. Their representation determines the type of information that the model offers. Models also have different functions as descriptors, means of explanation and prediction, strongly depending on the discipline in question. A model can be static, dynamic, deterministic, stochastic, qualitative or quantitative (Justi and Gilbert 2003). Depending on the discipline and the goal, one may give more or less importance to the accuracy between the model and the corresponding target. A model can be too simple or too complex in relation to the context of use. Some characteristics of the context can be emphasised and others ignored in order to build a model. Models in the sciences are different depending on the system that they embody. A model can be a tool to make inferences regarding a phenomenon; while empirical data and the interpretation supported by the model can be complementary (Hestenes 2010).

Working with models in the sciences means making connections between extra-mathematical and mathematical knowledge, bringing to light different uses and meanings given to mathematical objects. Along this line, Michelsen (2006) pointed out that in mathematics students think of a graph as a representation of a function most often considering one variable. In the sciences however, a graph represents a relationship between quantities, and the dependence between two or more variables is more evident. Regarding variables, Michelsen (2006) highlights that in mathematics the variable has several meanings, but students frequently think of a variable as a symbol to be manipulated. In the sciences, a variable is most often considered a name for a changing quantity or a value that in many cases can be measured. In this case, a variable is more related to the functional relationship between varying quantities. To build or to analyse models means to abstract the most relevant information on the situation and separate it or add it to a mathematical expression. This in turn requires understanding and insight (Kapur 1982).

32.3 Methodology

One way to have an approach to understand how models are used in teaching sciences is by observing what modelling means for science teachers and what they expect from students in their courses, where the teacher's conceptions about models and the nature of learning mathematics play an important role (Ernest 1989; Frejd 2012). This leads us to frame the study in the context of qualitative research, specifically a case study (Stake 1999). The data analysed comes from the discussion held during a task design seminar, which involved the participation of four professors and researchers in mathematics, physics and biology-physiology. The focus is on these topics because of the different uses of models seen in some of the most commonly used textbooks in higher education (e.g., Guyton and Hall 2006; Leithold 1998; Young and Freedman 2009). In physics, building mathematical models is more favoured, whereas in physiology more attention is placed on interpreting models, and in mathematics courses, modelling and models appear with classical problems to show how to apply some mathematical concepts. The analysis is centred on how these professors work with their models and therefore what they understand the models to be. An initial analysis was done by a researcher that did not participate in the seminar, and it was later validated by researchers specialised in mathematical modelling.

The seminar took place during a semester at a Chilean university, where there is a growing interest in the design of interdisciplinary activities for teaching mathematics. Two aspects were addressed: (1) the participants shared their views on models and how they use mathematical models in science courses and (2) they worked together on designing learning tasks. The results of the first phase of research are presented here; this consisted of the observation of the current use of models by the participants. The second part of the research is currently in development; it consists of implementing and testing the activities.

The following section focuses on discussions held by a physicist (Kim), a biologist (Saúl) and a mathematician (Peter). They are all teachers with at least 5 years of teaching experience. These participants were chosen for the analysis because they were more involved in explaining the way models are used in their classes. They also expressed their views regarding the students' skills that should be encouraged in order to be able to solve problems and answer questions in non-mathematical contexts. Discussions at the seminar were videotaped and transcripts were written up for analysis. Sets of paragraphs related to models and their uses were set as reference units. The researchers took part as moderators and occasionally asked questions to encourage the professors to explain their ideas regarding what they would like students to do in their classes in relation to models, what types of models are used in their classes and what is required from students when working with models.

32.4 Data Analysis

This section describes the teachers' views of working with models. Saúl focussed on the use of models in the context of a physiology course, Kim made references to modelling in the context of teaching physics and Peter described the general context of a mathematics course and the role of modelling in it.

Saúl noted that in his discipline, it is important that students develop the ability to interpret and match relationships between the information provided by different graphical models. He considered the graph as a relationship between quantities (Michelsen 2006). With questions of the sort: What does the graph represent about the pressure and volume related to the process of respiration (Fig. 32.1a)? And, what information can be obtained about blood flow (Fig. 32.1b)? He explained that his students should be able to identify a relationship between trans-pulmonary pressure (which is the difference between the pressure in the alveoli and the pressure on the outer surfaces of the lungs) and the change in lung volume. He also pointed to how the information about the blood flux should be interpreted from a graph. When the transversal area of the artery is small (left side of Fig. 32.1b), blood velocity is higher, and blood pressure is also higher. When the transversal area increases (middle part of Fig. 32.1b), pressure and volume decrease. This fact may be explained by making reference to the continuity equation in fluid mechanics, which states that at two points in the same pipe, the volume of fluid passing through must be equal even if the diameter of the pipe changes; therefore, the fluid velocity is different at each point. Also, supported in Fig. 32.1b, Saúl highlighted that students have to be able to make predictions when regular conditions in the body change, for example, what happens if the blood flux is turbulent.

In the cases described by Saúl, identifying the dependency between the information provided by the graphs is a relevant factor for success in the task. It is evident how knowledge regarding change, volume and pressure is represented by means of a graphical model (Michelsen 2006). The model works as a tool to relate information and make assumptions. From Saúl's arguments, we identified three approaches to the use of models: for interpreting, for establishing relationships between information provided by graphs and as a means of prediction. He is focused on developing the ability to understand and interpret the information provided by models. The models that he refers to are represented by graphs, which are usually found in textbooks. He also states that students frequently encounter difficulties in working with graphs and in making sense of the information provided by them.

Kim was interested in explaining some phenomena by using models from physics. Her point was to show the students how the models appear. She used pictorial illustrations to represent a movement phenomenon. She considered models as a means of explanation and prediction (Justi and Gilbert 2003). She started describing a situation: "if we have a particle on a curve ... Where the speed is tangent to the path, so it does not deviate from the path, some force must act on it that is directed toward the centre". In order for the particle to remain on the path, there must be an

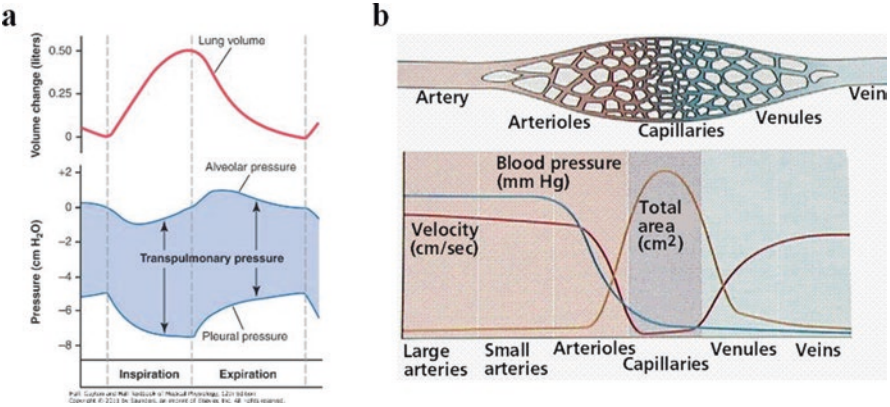

Fig. 32.1 (**a**) Relationship between trans-pulmonary pressure and volume. (**b**) Blood pressure, velocity and transversal area (Guyton and Hall 2006, p. 472) (Used with permission.)

equilibrium between forces: $\mu \mathrm{mg} = \frac{mv^2}{R}$. She proceeded to do some algebraic operations to explain that the maximum velocity of the moving particle on a curve (Fig. 32.2a) should be $v^2 = \mu g R$. As can be seen, there are several mathematics and physics concepts involved, such as the sum of forces $\sum F_x = \mathrm{ma}_R = m\frac{v^2}{R}, \sum F_y = n - \mathrm{mg} = 0$ and work with vectors. These concepts have to be put together in order to obtain an answer in a physics context. The role of time as an implicit variable in plotting orbits to describe the behaviour of some phenomenon also arises (Fig. 32.2b).

Kim also stated that building mathematical models experimentally can be useful to show some regularities in the behaviour of phenomena. To support her statement, she described how a mathematical model is built by identifying that the period in a ballistic pendulum is linearly dependant on the distances L and D (Fig. 32.2c). Her interest was to highlight the linear behaviour by using the logarithm as a mathematical tool.

In this case, it can be seen that work with models is associated with a process of building mathematical models as a means of explaining the behaviour of some phenomena and demonstrating physics laws. The models are presented in pictorial, graphical and algebraic ways. Building mathematical models in physics involves making connections between the structures of knowledge represented by each model.

Peter described how models are presented in his courses. He was focussed on how mathematics can be applied in solving problems. Models are considered as an application of mathematical concepts (Ernest 1989). In the case of functions and

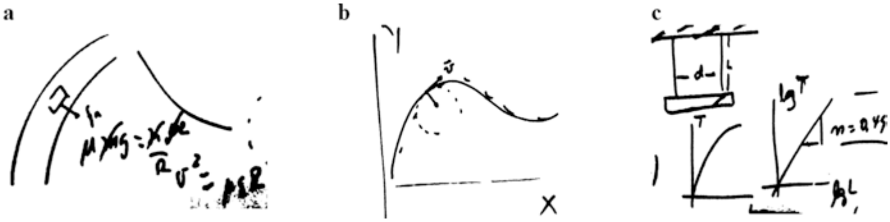

Fig. 32.2 (**a**) Analysing the movement of a car. (**b**) Considering the time as an implicit variable. (**c**) Linear relationship between the period on a pendulum and the length $T = \text{k}l^{0.49}$

integral calculus for engineering students, Peter mentioned some examples that he uses in his courses:

> "I take a sphere and make a hole, then the question is how much is the volume?"
>
> "There is a farm and in it there are 60 trees. These trees produce 70 fruits each tree. If I plant more trees, the number of fruits per tree decreases by five. How many trees should I plant to obtain the largest production?"

Regarding the second task, Peter stated that three representations should be presented so that students are able to understand the problem. He wrote the function $P(x) = (60 + x)(70-5x)$ and plotted the graph associated with it. In doing so, he highlighted that students should be able to identify relevant issues in the parabola, such as vertex and the relationship between the vertex and solutions of the equation. He noted that his interest was in studying the second-degree equation and its interpretation as a known mathematical object, focussing on its properties. Peter also stressed that students should be able to understand the symbolism of mathematical language in order to work with mathematics and to apply their mathematical knowledge to solve problems: "I try to make this mix of theory, demonstration and application, but not much demonstration because they don't have the mind set for that".

Peter presented a hypothetical context as a means of building a mathematical model and then addressed the study of the objects derived from it. Models are seen as a tool to study mathematical characteristics of the objects, where extra-mathematical knowledge is not considered. Reflection on the validity of the model and its limitations was not considered.

With these examples, we have shown different ways in which the sample of university teachers of sciences and mathematics uses models and the mathematical activity that they promote. There is an evident gap between the skills that are required in each case. In the sciences, interpreting a model and understanding concepts related to the discipline are complementary. In physics, concepts related to the movement of a body (Newton laws) and mathematical concepts and skills (representation and manipulation of relevant information in algebraic form) are mobilised simultaneously. In physiology, interpreting the information provided by graphs acquires relevance through knowledge of the body's homeostasis mechanism. In contrast, in mathematics, almost all of the work is developed in the world of mathematics, and more relevance is given to the study and analysis of the mathematical characteristics of the objects.

There are also evident differences in the goals of working with models. In teaching mathematics, in this case, work with models means introducing some mathematical concepts. The context is usually simplified in advance by this teacher, and the students' work is far from science concepts and decision-making. In physiology, working with models means obtaining information from the graph and giving meaning to it in order to explain some aspects of the body's mechanisms. However, students rarely build mathematical models, and the teacher reported not knowing how graphical models in the books are built as they are obtained through experiments. In physics, the approach given to the study of models depends on whether it is theoretical or experimental physics. In the first case, the teacher stated that working with models means finding explanations for the behaviour of certain phenomena. In experimental physics, she argued that using experimentation to demonstrate why a phenomenon can be described by a specific mathematical model is an important point.

The expectations of the teachers regarding their students' capabilities with models are strongly motivated by their beliefs about teaching and learning (Ernest 1989; Frejd 2012). The following statement represents a part of the comments of the science teachers showing the importance of establishing relationships between mathematical and extra-mathematical knowledge: "To interpret a model it is necessary to know something about mathematics and a lot of the relationships between mathematical models and the context".

There were differences between the participants' ideas of what a model is. For Saúl, it can be seen that a graph is a model, from which qualitative information can be extracted, as shown in Fig. 32.1b. In the case of Kim, the situation is similar, but graphs are analysed in more detail and explicit formulas arise. Peter considers mathematical expressions and uses them to make graphs, as another form of representation. In showing how models are taught in mathematics courses, the gap between learning mathematics and learning the sciences is more evident. In mathematics, modelling is related to the development of algorithmic methods, while in the sciences, modelling is close to qualitative techniques for interpretation and explanation.

Thus, the particular teachers' views on working with models and the relevant abilities needed for success in each course were evident in their discussions of how to design teaching tasks. Of course, the results presented here are one interpretation based in sample teachers' comments of how models are present in science and mathematics courses and what kind of skills are necessary to work with them. No general conclusions can be derived from the results.

32.5 Conclusions

The results of the present study show some aspects that might need to be considered in supporting the design of teaching activities that help students link knowledge from mathematics courses to other sciences. From here, it should be taken into account how students deal with the relationships encapsulated by the models presented in the sciences and the (mental) models they have built on their way through mathematics courses. Accordingly, these teachers agree that models in physics and physiology have a more pragmatic character in the sense that necessarily undertakes some processes to obtain answers and solve problems more efficiently. Different approaches to working with mathematical models, together with the pragmatic character they acquire in practice, should be considered as a way to converge the learning of mathematics and the sciences. As Frejd highlighted in Chap. 31 (this volume), we must look at how, and for what purposes, experts use mathematical models in their fields of work in order to find ways that help us develop the necessary skills in higher education students.

Given the big differences between the skills that are promoted by working with models in mathematics and the skills required to work with models in the sciences, more seminars like the one reported here should be conducted in order to build a structure of knowledge that allows us to design teaching activities to bridge the gap between the teaching of these disciplines. One of the main objectives to be addressed is to implement changes in early mathematics courses, taking a similar approach, to help students develop the skills needed to work with models in advanced courses. An example of what kind of discussion could be approached is by observing the differential equations $\frac{dN}{dt} = aN\left(1 - \frac{N}{k}\right) - H$ and $\frac{dN}{dt} = aN - cN^2 - H$. Although mathematically both have the same meaning, in practice they can represent different information in a context of population growth.

In designing teaching activities, it also should be taken into account that in biology the results come from the qualitative interpretation of graphical models. To explain the behaviour of phenomena, graphical and numerical models are manipulated. These models have been obtained experimentally and generally the students work with models that are presented in a textbook. In physics, mathematical results are compared with observations. To explain the behaviour of phenomena, known algebraic models are manipulated, and it is sometimes necessary to structure the knowledge provided by different systems.

In addition to the material produced, we trust that this seminar can generate changes in the teachers' beliefs about teaching and learning mathematics and their own role in teaching practices. In regard to this, Haines (2011) pointed out that teachers should be "in touch with the real world" (p. 350). We add that teachers should also be in touch with the content that is taught in other courses where students have to put into practice their abilities to work with models and modelling. In response to this, the participants of the seminar are now testing some teaching activities involving work with mathematics and analysing some physics experiments where interpreting a plot is a relevant skill to be strengthened.

Acknowledgements This work was supported by CONICYT/FONDECYT/POSDOCTORADO/ No.3150317 and FONDECYT REGULAR, No. 1151093.

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Chapter 33
Modelling as Interactive Translations Among Plural Worlds: Experimental Teaching Using the Night-Time Problem

Toshikazu Ikeda and Max Stephens

Abstract This case study examines the advantages in interpreting students' modelling from the perspective of interactive translations among plural worlds. This perspective has greater pedagogical potential than a simplified perspective which treats mathematical modelling as involving transitions only between two fixed worlds – a real world and a mathematical world. Experimental lessons with the *Night-Time Problem* for Japanese 10th grade students were held over a period of 100 min, using a structured investigation. As a result, the following two advantages are exemplified. The first enables a teacher to direct attention to intermediate models which can help students build further abstract models, and the second focusses attention on meaningful contradictions to help students to verify/critique/modify their original models.

Keywords Model • Modelling • Plural worlds • Interactive translation • Night time • Experimental lesson

33.1 Background

Different instructional approaches to modelling have been analysed and classified on an international level (e.g., Blum and Niss 1989; Kaiser 1991; Kaiser and Sriraman 2006; Lesh et al. 1986). Among these, we focus on two fundamental traditional trends, namely, a pragmatic and a scientific/humanistic trend (Kaiser 1991). In the case of the pragmatic trend, people mathematize in order to enable formal

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_33

processing using mathematical systems, and it is mathematization aimed at the creation of mathematical models that is emphasized. A dualism between the *real world* and the *mathematical world* underlies this approach. On the other hand, in the case of the scientific/humanistic tendency, the emphasis is placed not on a mathematical model, but on the model as a medium for promoting mathematization. As Freudenthal (1991) noted:

> According to my terminology, a model is just the – often dispensable – intermediary by which a complex reality or theory is idealized or simplified in order to become accessible to more formal mathematical treatment. … I lay so much stress on the role of the model as an intermediary because people are all too often unaware of its indispensability. Much too often mathematical formulas are applied like recipes in a complex reality that lacks any intermediate model to justify their use. (p. 34)

Both cases have in common the idea that a model functions as a space for thinking that enables problems to be contemplated in a separate space. However, the intermediate model for Freudenthal seems to be a model built between a real world and purely mathematical world in order to overcome the difficulty of mathematization. This distinguishes Freudenthal's usage from the idea of a pragmatic trend in which mathematical modelling is interpreted in terms of two fixed worlds (dualism). The idea of an intermediate model leads us into a new perspective where mathematical modelling is interpreted as an interactive translation, not between fixed two worlds (between a real world and mathematical world), but actually among plural worlds.

Therefore, how can we identify plural worlds about students' activities in modelling, how do students really perform regarding interactive translations (changed representations), and what are the advantages to interpret students' activity in modelling as interactive translations among these plural worlds? This chapter uses the *Night-Time Problem* to concretize these interactive mathematical translations among plural worlds and illustrates these through actual students' responses during a modelling lesson.

33.2 Design of the Study

At first, we adopt the stance that mathematical modelling can be interpreted as interactive translations, not between fixed two worlds, but among plural worlds. How can we distinguish these different worlds? If we can represent an original action (or operation) with a new action (or operation) explained with elicited properties, we can interpret that two corresponding actions (original and new) each exist in different worlds. For that we need a concrete illustration.

33.2.1 A Concrete Approach to Interactive Translations Among Plural Worlds

In our concrete approach, we propose to introduce Japanese 10th grade students to the *Night-Time Problem* (Ikeda 2013), in which night time *y* (hours) at latitude *x* (° N) can be estimated and investigated by observation in the real world, and then by using a globe, by drawing/measuring geometrical figures, by constructing a formula, and by creating a graph involving *x* and *y*. Potentially five different worlds can be applied. In the first case, students encounter the simple question, why does the daytime at higher northern latitude become so long close to midsummer? Or conversely, why does the length of summer night time decrease as latitude north increases? Students can consider this problem by manipulating a model globe. Manipulation in a concrete operational world can assist students to translate from a real world into the *geometrical operational world*. Our goal is to analyse the *Night-Time Problem* using plural worlds or perspectives and to assist students to translate from one to the other. In our concrete example, students will analyse three specific cases, namely, the length of night time at latitude 66.6° N (latitude of the midnight sun), at latitude 0° (equator) and at latitude 33.3° N (half in between) all at the same time, close to midsummer.

At the beginning, students may rely on a simple proportional model in a *graphical representational world* and give a prediction of 6 h. This simple graphical/proportional representation allows them to consider visual images of the mathematical relationships. In order to check this answer, two types of ideas or approaches can be used by the teacher. One is to draw a geometrical model with a side view in order to set out the positional relation between globe and the supposed sun by taking account of the fact that the axis of the Earth is tilted. Then, by drawing a side view of the Earth, students can be asked to predict the answer by measuring the ratio between the length of night time and daytime in this *geometrical operational world*. Then by using the internet, students can use data from the *real world* to check the actual length of night time. Even at this stage, two types of contradictions can occur: the first between a graphical/proportional representation of the world and a *geometrical operational world* and the other between a *graphical representational world* and real world data. By limiting the problem to two fixed worlds (a *real world* and *mathematical world*), these two types of contradictions cannot be identified and utilized by the teacher in responding to students' different approaches.

However, there is still a contradiction between the result derived from the geometrical model and the actual length of night time. This may cause students to modify their model in two ways. The first arises from a comparison between a *geometrical operational world* and a *concrete operational world*. The second arises from a comparison between a *concrete operational world* and the *real world*. Translation from three dimensions into two dimensions can help to focus on the first comparison; and idealization from the *real world* into the *concrete operational world* can help to focus on the second comparison. Comparing these two results derived from two different representations can help to facilitate important interactive

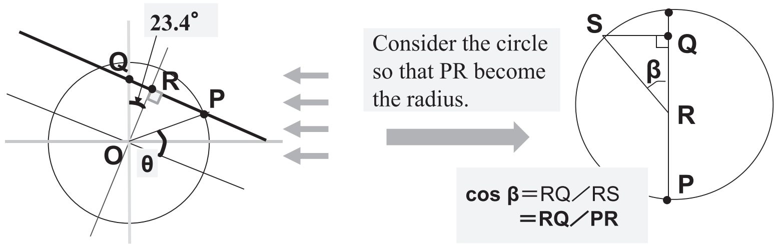

Fig. 33.1 Formulation of the night-time function

translations between plural worlds. Clarifying these two types of comparison based on the distinction among plural worlds is crucial for how the teacher assists students' approaches, with the aim of shifting the model from a side view figure to one based on a circle viewed from the North Pole, taking into account the tilt of the Earth's axis as 23.4°.

In order to represent the general night time according to the north latitude, it is necessary to develop and become familiar with a functional model using trigonometry. This is the *symbolic operational world* which allows us to consider the phenomenon algebraically. At first, by reflecting on and utilizing the solution methods in the *concrete operational world*, we can make a geometric model as base. In Fig. 33.1 (left), P is north latitude θ, ∠QOR = 23.4° because of midsummer and ∠ORP = ∠R. By considering the circle so that PR becomes the radius as shown in Fig. 33.1 (right), night time is formulated as $2/15 \cdot \beta$ ($=24 \times 2\beta/360$) where $\cos \beta$ is RQ/RS = RQ/PR. By the way, as $\tan \theta$ is RO/PR and tan 23.4° is RQ/RO from Fig. 33.1 (left), the following relation is derived.

$$\cos \beta = \frac{\mathrm{RQ}}{\mathrm{PR}} = \frac{\mathrm{RO}}{\mathrm{PR}} \cdot \frac{\mathrm{RQ}}{\mathrm{RO}} = \tan \theta \cdot \tan (23.4)$$

Therefore, night time y (hours) is formulated by using north latitude x (°) as follows:

$$y = \frac{2}{15} \cos^{-1} \{\tan (23.4) \cdot \tan x\}$$

A benefit of using technology to explore the behaviour of this trigonometric function is that its graph provides a simultaneous correlate to the *Night-Time Problem* in southern latitudes as well as understanding that a difference of one degree in latitude in high-latitude countries such as Norway and Sweden has greater impact on the hours of night time than in lower latitude countries such as Japan.

33.2.2 Experimental Teaching for 10th Grade Using the Night-Time Problem

The following case study is intended to illustrate how a teacher and students can approach mathematical modelling, as a series of interactive translations among plural worlds, by exploring how each representation brings its own affordances and constraints. A teaching experiment was conducted with 10th grade students at a public high school in Yokohama City, Japan. Twenty eight high school students indicated their intention to attend the lesson in Yokohama National University. The experimental lesson, based on the *Night-Time Problem*, took place for 100 min and consisted of the following three phases (P1, P2, P3).

P1: To understand the reasons underlying the phenomenon of the midnight sun and compare it with the night time at latitude 33.3°N, midway between 0° N and 66.6°N. (Latitude 33.3° N actually corresponds to that of Kochi City in Japan and 66.6° N is close to that of Bergen City in Norway which will be featured later in the study.) Simple feelings about the real world and insights derived from manipulating the globe are expected to be derived from the students.

P2: To consider how to verify any predictions proposed by the students. An investigation by drawing/measuring a geometrical model or checking the real night time at latitude 33.3°N in Japan is expected to be carried out by students.

P3: To examine and to build up the further geometrical model by reflecting the result of P2.

33.2.3 Analysis of Interactive Translations Among Plural Worlds

Until now, trigonometric functions have not been taught to Grade 10 students in Japan. Therefore, this experimental lesson illustrates how students can translate interactively among four worlds, namely, using data from the *real world*, a *concrete operational world* based on manipulating a globe, a *graphical representational world* which allows them to consider a visual image of the variables involved and a *geometrical operational world* which allows them to consider by drawing and measuring. The aim of the lesson is to illustrate how deliberately planned interactive translations among four worlds can lead students to consider more deeply and to develop more accurate or more general models. Specifically, we need to analyse how students translate interactively among four worlds and their willingness to develop further models.

The following four questions were presented to students by the teacher in order to analyse their interactive translations among four worlds:

Q1 posed during P1: How many hours of night time do you estimate at latitude 33.3°N?

Q2 posed before P2: How could you check your estimate of the night time at latitude 33.3°N?
Q3 posed before P3: If your expected result was mismatched with the result of another world, what might be the reason?
Q4 posed after P3: What further problems do you want to consider next?

33.3 Implemented Teaching with the Night-Time Problem

Actual teaching took place according to the following three parts: (1) presenting the situation generating this real-world problem and explanation about the reason for the midnight sun, (2) considering the night time of latitude 33.3°N and (3) reconsidering any mismatches between results by drawing/measuring the side view figure and the real night time of latitude 33.3°N.

33.3.1 Situation Generating a Real World Problem

The following situation utilizing several photos (Fig. 33.2) was presented to students at the introduction:

> We went to Bergen city in north Europe at midsummer. Bergen city is located at latitude 60° N in Norway as shown in Fig. 33.2 (left). We arrived late in the evening of 19 June and had a dinner at 9:30 pm. After dinner, we went out from the restaurant and were surprised to see that sun had not set yet as shown in Fig. 33.2 (right). The time by my watch had passed 10:30 pm. Why is daytime so long here in Bergen? In Japan, it is pitch-dark at 10:30 pm. I heard from someone about the Midnight Sun at higher Northern latitudes. But I don't understand why.

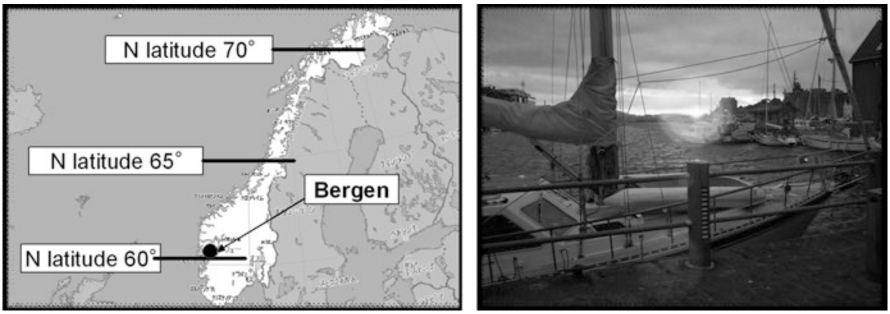

Fig. 33.2 Bergen City in Norway (at 10:41) on 19 June

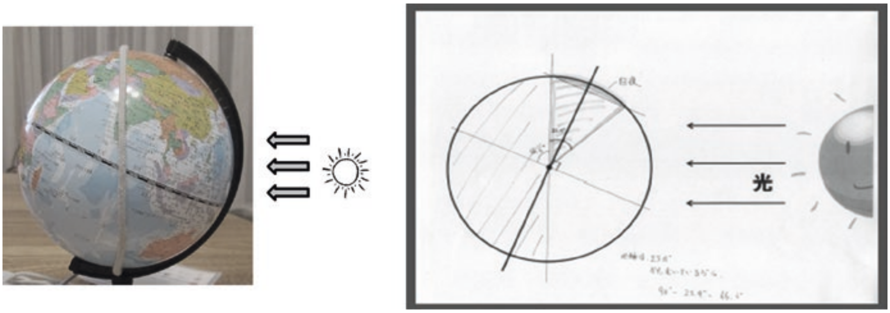

Fig. 33.3 Explanation of midnight sun by a student

After presenting the situation above, the teacher asked students to explain why a midnight sun happens at higher northern latitudes. Students worked together with a friend next to them by manipulating the globe. Students observed the positional relation between the globe and the supposed sun at midsummer (Fig. 33.3 left) and prepared a geometrical figure to explain why the midnight sun phenomenon happens (Fig. 33.3 right). Manipulation in this *concrete operational world* assisted students to translate from a *real world* into the *geometrical operational world*. It also confirmed that analysing the *Night-Time Problem* using multiple representations is possible.

After sharing the reason for the midnight sun in a class, the teacher asked the students: “Above what degree of latitude North does the Midnight Sun happen?” By starting from the fact that the axis of the Earth is tilted at 23.4° away from the North Pole, two pairs of students explained the answer. As the result, they concluded that the midnight sun happens from latitude 66.6°N at midsummer when there are zero hours of night time.

33.3.2 Considering Night Time of Latitude 33.3 ° N

Next, the teacher asked the students: “What degree of latitude North allows the night time to be 12 hours?” Students answered easily that the night time in areas close to the equator (where latitude is 0 °) is 12 h. So, the teacher asked further “How many hours of night time will be in other areas, for example, in places between 66.6°N degree and the Equator?” In this phase, question 1 “How many hours do you estimate the night time at latitude 33.3°N?” is asked for students. Most students (22 students, 78.6%) answered 6 h by making the assumption that there is a simple linear relationship between length of night time and latitude north. The teacher invited one of those who answered 6 h to explain their reason. The student drew a straight line in a graph and gave this as the reason.

Table 33.1 Summary of students' ideas on how to check their initial estimates

Responses of students	Number of students (percentage)
Considering by drawing the figure	14 (50.0%)
Measuring the night time at latitude 33.3°N in the real world	9 (32.1%)
No answer	5 (17.9%)

At this stage, the teacher posed question 2: "How could you check your answer about the night time at latitude 33.3°N?" A summary of students' ideas on how to check their initial estimates is given in Table 33.1.

Half the students wrote "Consider by drawing the figure" and 32% of students wrote "Measuring the night time at latitude 33.3°N in the real world". Among students who wrote "Considering by drawing the figure", only nine students could propose any concrete solution method such as "Check the ratio of the length of night time at latitude 33.3°N to the length of equator radius in the side view figure. If the ratio is ½, it is ok". On the other hand, three students (10.7%) wrote a rough direction toward a solution and two students (7.1%) wrote quite simple ideas such as "measuring a length" or "drawing a figure".

In the lesson, the teacher picked up two types of answers in order, namely, "Considering by drawing the figure" and "Measuring the night time at latitude 33.3°N in Japan" and let students explain them. Two types of comparisons occurred: the first between a *graphical/proportional representation of the world* and a *geometrical operational world* and the other between a *graphical representational world* and *real-world* data. These two types of comparisons cannot be utilized by the teacher in responding to students' different approaches unless the teacher helps students to identify multiple perspectives.

33.3.3 Reconsidering Mismatch Between Result by Drawing/ Measuring Side View Figure and Real Night Time of Latitude 33.3° N

After sharing both ideas of comparisons, the teacher let students draw the side view figure of the Earth and had them measure or calculate the ratio of night-time length to 1 day time (or half day time). Although there was a little variability among students' results, most students had an answer around 8.5 h. At this stage, the teacher guided students to an alternative method, namely, by checking the real night time at latitude 33.3°N in Japan. By using internet data for Kochi City, which has latitude 33.3°N, students found that night time in midsummer for Kochi City is actually 9 h 40 min. Students encountered a mismatch between 8.5 h (derived from measuring or calculating the ratio by drawing the side view figure) and 9 h 40 min (derived from the real time in the internet). This prompted students to reflect and re-examine the implicit factors that may have been overlooked. At this stage, Question 3 was

Table 33.2 Students' reasons to explain the mismatch

Responses of students	Number of students (percentages)
Don't rely on a side view figure of the Earth, but view the circle from the North Pole	13 (46.4%)
The Earth is not a complete sphere	8 (28.6%)
No answer	5 (17.9%)

posed by the teacher: "If your result is mismatched with the result of another world, what is a possible reason?" The results of students' ideas are as shown in Table 33.2.

Nearly half the students (13 students, 46.4%) identified the need to consider the problem not from the side view figure of the Earth, but viewed from a circle from the North Pole. However, the remaining half could not see the point being made. Eight students (28.6%) referred to the reason that the Earth is not a complete sphere and five students (17.9%) had no idea about this question. From this result, we can see that students modified their model in two ways. The first arose from a comparison between a *geometrical operational world* and a *concrete operational world*. The second arose from a comparison between a *concrete operational world* and *real-world* data. Translation from three dimensions into two dimensions could help students to focus on the first comparison; and idealization from the *real world* into the *concrete operational world* could help them to focus on the second comparison. Utilizing these plural representations assisted students' modelling activities. Finally, the teacher asked a student who reconsidered the problem by viewing a circle from the North Pole to explain the reason for the mismatch. This student explained that the ratio of the length of night time to the length of night and daytime is different from the ratio of the night angle to the angle of 1 day (360°) as shown in Fig. 33.4 (left). This student drew a circle of centre O with diameter AC in the side view of the Earth and measured ∠DOE (141°) of night time (Fig. 33.4 right). This supported an answer of 9 h 24 min. Most students seemed to understand why there was the mismatch and the necessity to shift the model from the side view figure into the circle viewed from the North Pole.

At the end of the lesson, the teacher asked students what kinds of problems could be considered next. Table 33.3 shows the categories of problems suggested and the number of problems in each category of problem. Seventy-five percent of students were able to write at least one problem to be considered next. It is clear that developing a more general model about the *Night-Time Problem* could be promoted through a lesson based on the same context in an upper grade where students could see how a generalized solution becomes more precise by applying trigonometric methods (Ikeda and Stephens 2011).

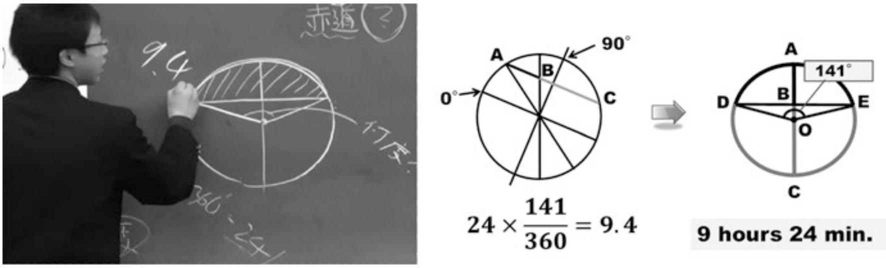

Fig. 33.4 A student considers the circle viewed from the North Pole

Table 33.3 Categories of problems that students posed at the end of lesson

Three categories of problems	Number of problems (percentage)
Problems concerned with generalization	23 (67.6%)
How many hours of night time are there at other northern latitudes?	3
What is the northern latitude where the night time is 6 h?	3
How can we graph the relationship between night time and northern latitude? What kinds of function generally?	7
How does night time change in the course of a year?	6
How about the night time in southern latitudes?	4
Problems concerned with solution methods	5 (14.7%)
It is not clear why the mismatch happened.	3
Is there any method to consider the night time only in the side view figure of the Earth?	2
Problems concerned with error	6 (17.6%)
Is there any other factor which caused the error?	2
How can we treat the error?	4

33.4 Conclusion

This chapter proposes a perspective in which modelling is interpreted as interactive translations among plural worlds and how the utilization of models in plural worlds is intended to assist students to make progress in mathematical modelling. We concretized the interactive translations among plural worlds using the *Night-Time Problem*. From this case study, we have argued that plural worlds are not simply the result of arbitrarily changing mathematical representations, but arise fundamentally as a result of comparisons and contradictions between competing perspectives. How these comparisons or contradictions can deepen students' modelling activities is only possible if we see the modelling process as operating in plural worlds and not simply from two fixed worlds (a real world and mathematical world). These provide indispensable clues to promote modelling activities.

However, when a teacher analyses a prospective modelling task, it is necessary to identify *in advance* what these different worlds are in order to be effective in promoting students' modelling. In other words, it is important to identify the affordances and constraints that make each world meaningful or not for students. Those that are not meaningful for students should be postponed or omitted. Those that give rise to comparisons and contradictions clearly have greater potential for deepening modelling.

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Chapter 34
The Dual Modelling Cycle Framework: Report on an Australian Study

Janeen Lamb, Akio Matsuzaki, Akihiko Saeki, and Takashi Kawakami

Abstract The aim of this study was to investigate how 23 students from one Year 6 class in an Australian primary school engaged with two modelling tasks using the dual modelling cycle framework. This framework is designed to assist students who do not find a solution to a modelling task by introducing a second similar yet simpler modelling task in a second cycle. Students participated in 2 × 60 min lessons over 2 days. Results indicate they benefitted from the modelling approach theorised by the Dual Modelling Cycle Framework. While students demonstrated an inability to find a solution for the first task, they were fully engaged in Task 2. They enjoyed this cognitively demanding yet stimulating approach that provided all students with opportunities to participate in an orchestrated discussion where they were able to find solutions for Task 1 and justify their findings using evidence from their concrete models.

Keywords Dual Modelling Cycle Framework (DMCF) • Oil Tank Task • Toilet Paper Tube Task • Primary school

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_34

34.1 Introduction

The Australian Curriculum Mathematics is designed to develop capabilities necessary for all Australian school-age students to fully engage in daily life (ACARA 2015, para. 1). In order to achieve this, "The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, logical reasoning, analytical thought and problem-solving skills" (para. 2). With this curriculum, there have been calls for teachers to change their pedagogical approach to embrace the intent of the Australian Curriculum Mathematics (Galbraith 2013). The work by Stillman and Galbraith (1998) continues to be relevant to supporting such a change as they argue for an emphasis on context to ensure sense making is promoted, and it is here that the lesson launch is important (Jackson et al. 2012). In addition, tasks need to be cognitively demanding (Lampert et al. 2013) yet suitable for the differentiated classroom (Boaler and Staples 2008). Pulling aspects of the lesson together is a skillful orchestration of the discussion (Stein et al. 2008) where students are "pressed" to make connections while justifying their perspectives with evidence. While recognising these and other previous research, Galbraith (2013) called for an emphasis on mathematical modelling as one way to create balance within conventional classroom mathematics, in an effort to support Australian teachers as they go about implementing the Australian Curriculum Mathematics.

34.2 Theoretical Framework

Mathematical modelling is widely used with realistic problem-solving contexts as a way to empower modeller independent use of mathematical knowledge in thoughtful and creative ways. This approach requires opportunities for multiple solution paths with the orchestration of discussion around the *best* solution in comparison to the conventional approach to mathematics problem-solving that looks at *the* solution. This approach has been captured by the cognitive theoretical framework developed by Blum and Leiß (2007, p. 225) where modellers move through a cycle of steps that requires them to access both the real and mathematical worlds. This single modelling cycle is sufficient if modelling is proceeding successfully. While many researchers draw on this model, research does indicate that students will move between the real and mathematical worlds while in the process of finding a solution (e.g., Stillman and Galbraith 1998; Matsuzaki 2007, 2011). When this process stalls and modellers do not know how to proceed to find a solution, one way forward is for them to be guided to a *similar yet simpler modelling task* that will aid the development of a solution for the original problem. In this chapter, we explore a theoretical extension to Blum and Leiß's (2007) model with a view to facilitating the teaching of mathematical modelling that considers a diversity of modeller abilities. Here Saeki and Matsuzaki's (2013) extended theoretical modelling framework, the Dual

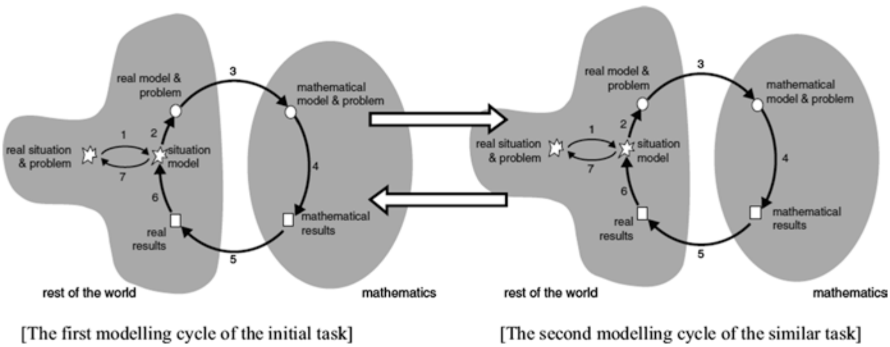


Fig. 34.1 Dual modelling cycle framework (Saeki and Matsuzaki 2013, p. 94)

Modelling Cycle Framework (DMCF) (see Fig. 34.1), is designed to cater for a diversity of learners.

The theoretical propositional basis of the DMCF is that it requires two tasks, the initial task, Task 1, which is located in the first modelling cycle, and Task 2, which is located in the second modelling cycle. When students cannot progress their solution to Task 1, they are guided by their teacher to move to cycle 2 where they are introduced to a similar, yet simpler task (Polya 1945). The selection of the second task is critical as its role is specifically designed to develop student understanding that will assist with the solution of Task 1. The intention with the DCMF is therefore that by moving from the initial modelling task, Task 1, to a similar and simpler modelling task, Task 2, they are more likely to experience success in both modelling cycles.

Research by Matsuzaki and Saeki (2013) identified that teachers play an important role in facilitating switching between cycles and tasks to ensure successful outcomes for all students. Their research implemented experimental modelling lessons with undergraduate students in Japan that led to the identification of three stages in the DMCF: (1) transition from the first modelling cycle to the second modelling cycle, (2) modelling within the second modelling cycle, and (3) transition from the second modelling cycle back to the first modelling cycle. Kawakami et al. (2012, 2015) moved the use of the DMCF from undergraduate students to Year 5, elementary school students in Japan. It was the use of the DMCF in the elementary setting that captured the interest of Australian researchers, as this framework was seen as a way to assist teachers to implement the Australian Curriculum Mathematics answering Galbraith's (2013) call for greater use of mathematical modelling and at the same time cater for a wide diversity of student ability (Lamb et al. 2014). The research questions that guide this research are: *How do students in this Australian school, who are experiencing difficulty with Task 1, respond when their teacher switches to Task 2? And, how does this influence student modelling response to Task 1?*

34.3 Research Design

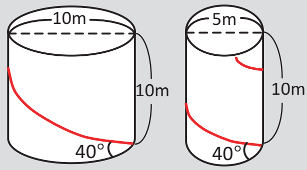
One primary school in Brisbane, Australia, participated in the DMCF project. Although this school was a sample of convenience, it is also typical of most primary schools within Brisbane with year levels from Prep to Year 6. Participants described in this paper involved 23 students (age 11 or 12). The students participated in two lessons (60 min × 2) over 2 days in the second week of their school year. All researchers named on this chapter attended the lessons. These two lessons were designed to cycle through the three stages of the DMCF identified by Matsuzaki and Saeki (2013). The tasks the students completed are outlined in Fig. 34.2, and these tasks were the same tasks that had been completed previously in the Japanese studies.

The tasks were designed to assist in developing student understanding of the geometric structure of an ordinary helix on the outside of a cylinder (see Fig. 34.2 below). The delivery of the lessons used the following structure as in the Japanese studies (Matsuzaki and Saeki 2013; Kawakami et al. 2012, 2015). Initially a picture of oil tanks was shown where the tanks had differing diameters and a spiral staircase from the ground to the top. The photograph included several fire trucks with firemen and engineers in discussion at the foot of the oil tanks. The context was presented as the firemen needing to know which spiral stair would get them to the top first, as they needed to climb to the top of one of the tanks as quickly as possible to cool them because they were in danger of exploding. It was clear to all that there were several types of oil tanks with their heights equal but their diameters different. The students were asked, "Were the lengths of the spiral stairs on these oil tanks the same or not?" It was explained that the angle of the spiral stairs around each tank climbed at 40°. Task 1 in Fig. 34.2.2 was then presented and those participating were asked to produce 2D drawings of the 3D model. Following this modelling, Task 2, the *Toilet Paper Tube Task*, was introduced. The purpose of this task was to model the oil tank, but this model permitted the toilet paper tube to be cut up along the slit to assist in identifying a second 2D model. After this task, the students were asked to again consider Task 1.

Collected data included lesson video-recordings, iPad audio-recordings of each group's discussion, each student's worksheets, lesson artefacts and field notes. Lesson artefacts included digital images of student modelling, while field notes were kept by researchers noting any critical insights or issues as they emerged throughout the lessons. These data were analysed in two ways. First the analysis looked for evidence of student independent engagement with each modelling cycle, their transition to the second modelling cycle and how the second modelling situation informed the first, and if this led to enhanced potential in mathematical proficiency. Second, the predicted models that the student would draw for Task 1 were the rectangular model and the parallelogram models, with the expectation that most will draw the rectangular model as this had been the case when Japanese students had attempted this task (see Kawakami et al. 2015). Analysis of Task 2 was expected to focus on the parallelogram model where student mathematics to explain the rela-

Oil Tank Task (TASK1)

There are several types of oil tanks. Their heights are equal but their lengths of diameters are different. Is the length of the spiral stairs on these oil tanks equal or not? As the angle of the spiral stairs climbed at 40° for each.



Toilet Paper Tube Task (TASK2)

It is impossible to open along the actual spiral stair of the oil tank. We can use a toilet paper tube as a similar shape to an oil tank as it can be opened along its slit to show the 2D shape. Consider what the shape of an opened toilet paper tube would be.

Fig. 34.2 Teaching material based on DMCF (Kawakami et al. 2015, p. 197)

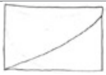
tionship between the parallelogram model and the rectangular model would be drawn out.

34.4 Results and Discussion

34.4.1 Student Experiences with the Modelling Tasks

While the 23 Year 6 students enthusiastically engaged with Task 1, only 11 students drew the anticipated model, the rectangular model. This result meant that the researchers had to modify their analysis protocol for Task 1. For the 11 rectangular models, each was drawn with a curved line to represent the stairs. See Table 34.1 for models A and B being variations in the rectangular model. Note Model A did not indicate reaching the top of the oil tank and Model B did not accurately represent the transition of the wrap around spiral stairs from front to back. The remaining 12 students reproduced the 3D model (Models C and D) suggesting that they did not know how to produce 2D drawings from the 3D models. Our initial interpretation of Models C and D was that the students had reproduced the problem. On greater reflection, this model and that of Model D, do include a 2D net of the oil tank, but also include additional features. Clear evidence of the front and back view of the stairs in Model D suggests students' earlier learning of orientation where they have been required to visualise and draw the view from the top, front, back and sides of various shapes. This finding resulted in a reclassification where Models C and D were classified as examples of an *orientation model* which we consider is in the *grey zone* incorporating some aspects of the 2D model and some aspects of an orientation model. Nonetheless, it was evident that these models were not going to assist the students to provide a solution to the problem as all students had experienced some form of difficulty with Task 1. The teacher then intentionally switched the students over to Task 2, the similar but simpler task.

Table. 34.1 Student Task 1, drawings of 3D model of the oil tanks – Models A, B, C and D

2D models		Orientation models	
Model A	Model B	Model C	Model D
			

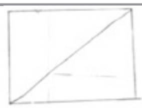
As a result of Task 1, part of the intentional switching to Task 2 was to get the students to predict, through visualisation, what a toilet paper tube would look like when cut along the slit. A toilet paper tube was selected as it is a similar shape to the oil tank, and the slit can readily represent the spiral stairs assisting visualisation. Also, it is easy to cut the toilet paper along the slit to disclose the shape.

Table 34.2 displays the variety of students' 2D models. Again, the students did not provide the models predicted by the researchers, and as a consequence, the analysis had to change again. Worthy of note is that three students drew models that were a combination of 2D and 3D models. The model displayed in Table 34.2 seems to indicate that the students have provided a side view producing a further orientation model. The 3D model is close to the parallelogram model but the student has drawn it in 3D. Both these models were analysed as being within the *grey zone* incorporating 2D and orientation features.

No students used mathematics to establish the relationship between the *Oil Tank Task* and *Toilet Paper Tube Task*. To assist in finding the relationship between these tasks the students were given two toilet paper tubes and asked to cut one toilet tube straight up from bottom to top to confirm the rectangle model of the oil tank and to cut around the slit of the second toilet paper tube to confirm the parallelogram model. These activities concluded the first lesson.

The second lesson commenced by reinforcing the rectangle and parallelogram models from the day before by using large concrete materials to model the oil tanks. One model was cut straight up from bottom to top to produce the rectangle model and the other along the slit to produce the parallelogram model. Following this activity, the students were intentionally switched back to Task 1 and asked, "Are the staircases the same length or not?" In trying to solve this problem, it was noted in the researchers' field notes that the students enthusiastically engaged in collaboratively constructing models to represent the 5 m and 10 m diameter oil tanks. When they cut these models up they were able to provide evidence they needed to convincingly argue through an orchestrated discussion, that the staircases on the tanks were the same length. Following this realisation the students were again stretched by being asked to explain the relationship between the rectangle and parallelogram models. Using the concrete models created at the beginning of the lesson, the students were able to overlay the parallelogram model of the 5 m diameter oil tank over the 10 m diameter model to prove that the staircases were the same. They were also able to prove empirically this result using the rectangle models by cutting and moving sections so that the stairs aligned. Moreover, a discussion was then made pos-

Table 34.2 Toilet paper tube models for Task 2

2D				2D and 3D	3D
Parallelogram model	Close to parallelogram model	Rectangle model	Other	Grey zone	Other
					
3	12	1	1	3	2

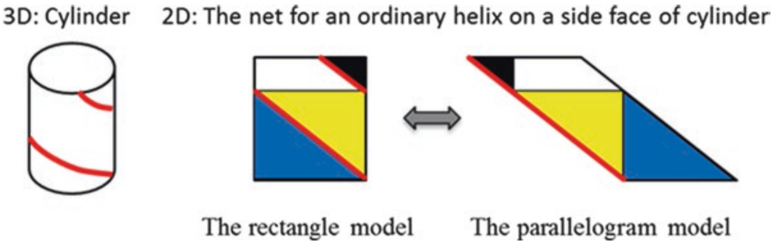


Fig. 34.3 Models to explain the same outcome (Saeki et al. 2016, p. 1748)

sible where students could argue why both the rectangle and parallelogram models produce the same result. The models displayed in Fig. 34.3 were used to assist in this explanation process.

34.5 Conclusions

There are several important findings related to the use of the Dual Modelling Cycle Framework. First, very few students were able to correctly complete either Task 1 or Task 2 by producing mathematically correct models. This result is different from the Japanese students who were able to draw on their findings from Task 1 to support their solution for Task 2 (see Kawakami et al. 2015). As we worked very hard to understand the models produced by the participating students, we developed new categories to allocate to student work. We believe that the students' previous study of 2D and 3D shapes has been influenced by work with orientation where the students have been required to visualise and draw the view of different shapes from the top, front, back and sides. This realisation lead to the reclassification of responses as representative of the new categorisation in our analysis protocol, the *orientation model* in the *grey zone* where students' responses seemed to incorporate some aspects of the 2D model and some aspects of the *orientation model*. We believe that this finding also supports the work of Stillman and Galbraith (1998) where they argue that Australian teaching of mathematics places a heavy emphasis on context. This emphasis on context was naturally continued by the students in this study where many elected to not only draw the net but also include features of the oil tank from different orientations, see for example Models C and D. This finding contrasts significantly with the Japanese approach to teaching mathematics where the focus is very much on the mathematics of the tasks with less focus on the context.

Second, when the students were intentionally switched to Task 2, as is the intent of the DMCF, again their prior experiences of orientation influenced their work. These cognitively demanding tasks (Lampert et al. 2013) resulted in teaching that focussed student attention on concrete models using toilet paper tubes where they successfully produced both the rectangle and parallelogram models. This approach captured every student's interest (Boaler and Staples 2008) and gave them the understanding and the confidence to return to Task 1 and respond to the question, "Were the lengths of the spiral stairs on these oil tanks the same or not?" The teacher's intentional switching back to Task 1, when the students had a fuller understanding of the two models to solve this task, resulted in their being able to provide evidence for their solution and make connections between the models in an orchestrated discussion as described by (Stein et al. 2008). The students were able to persuasively present their arguments that the staircases were the same length using evidence from their concrete models.

Third, we can confirm that the DMCF supports students who do not know how to solve an initial modelling task, but were able to advance their modelling of this task by modelling a similar but simpler task, Task 2. As a result of the students engaging with both tasks they developed a more enlightened mathematical understanding of an ordinary helix on the outside of a cylinder than they would have by doing only one of the two tasks. This approach to promote switching between Task 1 and Task 2 allowed students to solve Task 1, the *Oil Tank Task*.

The success experienced by students in this research by moving between Tasks 1 and 2 has led us to recommend the DMCF as a suitable mathematical modelling framework that should be introduced to Australian teachers as a way to address the diversity of modeller abilities and at the same time, realise the intent of the Australian Curriculum Mathematics.

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Chapter 35
Implementing Mathematical Modelling: The Challenge of Teacher Educating

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Abstract This chapter reports efforts to assist a group of intermediate and high school mathematics teachers in developing knowledge of mathematical modelling and its implementation in the school curriculum. Over nine academic months, the teachers engaged in 25 h of professional development during which they worked on modelling tasks and discussed implementation issues. Results indicate that teachers' level of comfort with mathematical modelling increased though they remained concerned about how to manage short- and long-term demands of curriculum, prepare students for skills-based standardized tests, and guide learners' discussions without violating their autonomy. Challenges faced by the teacher educators included managing teachers' diverse mathematical backgrounds and limiting direct instruction on mathematics.

Keywords Professional development • Teacher knowledge • Design

35.1 Introduction

The expectation that teachers will help school learners develop mathematical modelling skills has gained visibility in the United States with the adoption of Common Core State Standards of Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010). Arguably, one of the least understood expectations among the set of CCSSM Standards (Gould 2013), and one of the most conceptually demanding domains of knowledge to nurture due to its complexity (Meyers 1984), mathematics teacher educators face the challenge of preparing teachers to first develop an understanding of the intricacies of mathematical modelling and then helping them define ways to implement it effectively. While currently prominent in the United States, this challenge has been

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_35

recognized by the international research community for quite some time (e.g. Lingefjärd 2000). Scholars concerned with teaching and learning of mathematical modelling continue to insist on the need to explore models and programmes that might assist teachers to meet implementation challenges (Doerr 2007) as well as documenting conditions that such efforts might impose on teacher educators (Cai et al. 2014). The work reported resides at the intersection of these two domains of need.

In this chapter, efforts towards designing mathematical modelling experiences for a cohort of 25 intermediate and secondary mathematics teacher leaders from eight different low-performing school districts will be described and analysed. Factors considered when selecting tasks for use with the participants, a few of the tasks used, and particular issues teachers faced as they engaged in learning about the modelling process and implementing modelling tasks in their classrooms will be the focus. Lastly, challenges faced in our work as teacher educators will be identified and suggestions offered for future research.

35.2 Context and Participants

The teacher participants in this work were members of a larger group of 85 teacher leaders spanning grades K-10 involved in a professional development (PD) programme that aimed to enhance mathematics learning in low-performing school districts across the states of Ohio and Michigan. The larger project, Mathematics Coaching Program (Brosnan 2016), provides free content-specific PD for teacher leaders (coaches) hired by school districts and offers monthly PD sessions (2 days a month) for a period of nine academic months during which the participants engage in mathematics activities that facilitate learning inquiry-based instruction and ways of assisting classroom teachers in developing such pedagogies. We worked with 6–10 grade level teachers some of whom also had instructional responsibilities that consisted of teaching two or three sections of classes and spent the remaining work hours coaching other mathematics teachers in their schools. This group asked if specific sessions could be designed for them during which they could focus solely on mathematical modelling and its implementation.

In preparation for our work, a questionnaire was administered asking participants to first express their perceptions of mathematical modelling and then to offer examples of modelling tasks that were suitable for classroom use at different grade levels. The results indicated that the teachers had a fragile understanding of mathematical modelling as either a content strand or a conceptual domain. Consistent with Gould's report (2013), a majority of the teachers interpreted mathematical modelling as solving application problems, building physical models, or using representational media for illustrating concepts. Only three individuals offered examples that resembled modelling tasks (population growth model, calculating amount of mortgage in the presence of various interest rates, and determining optimal travel route). The remaining examples were standard textbook applications in which the major demand was using specific algorithms. The questionnaire results were critical in establishing conceptual goals for the programme we planned to implement. Due to teacher leaders being able to meet for only approximately 3 h a month, sessions and their content were planned to accommodate for this constraint.

35.3 Task Selection: Conceptual, Analytical, and Pedagogical Considerations

Researchers have identified a number of skills that teachers must possess in order to effectively implement mathematical modelling and navigate student learning in modelling tasks. A key ingredient includes knowledge about modelling tasks along with the type of mathematical and extra-mathematical knowledge needed for implementing them (Blum 2011, 2015). Additionally, pedagogical skills such as scaffolding techniques that ensure successful extended student engagement in modelling cycles (Blum 2015), ways to manage students' progression towards development of increasingly more sophisticated models (Doerr and Lesh 2011), understanding and responding to students' work, and the diverse approaches they may use to solve problems (Doerr 2007) have been identified as pivotal to effective incorporation of mathematical modelling in educational settings. Niss and colleagues (2007) further proposed that educational programmes designed for teachers must embody modelling experiences that match those expected of them to teach. In light of these recommendations, our fundamental goals for the participants included:

1. Developing a deeper understanding of the mathematical modelling process and its intricacies,
2. Experiencing how inquiry instruction interacts with learning of mathematical modelling,
3. Developing an understanding of connections among various subject areas and how they may come to aid in development of models that respond to conditions of the task,
4. Discriminating between mathematical modelling as a process and solving routine application problems,
5. Distinguishing between models as objects and mathematical models,
6. Understanding how productive mathematical thinking could be nurtured through use of modelling tasks.

Acknowledging that the development of mathematical modelling knowledge for teaching demands substantial experience with both content and the model building process (Lingefjärd 2007), we had to set boundaries for what the tasks, collectively, would meet. Following the recommendations of Meyers (1984) as a starting point, we wanted:

1. The level of tasks to be accessible for all teachers,
2. Problems to be independent so that the teachers would have an opportunity to work on a variety of modelling tasks,
3. To include a variety of subject matter (physics, biology, social sciences) and mathematical tools,
4. To use the type of task situations that often do not reach school curriculum,
5. To focus on mathematical modelling as opposed to the frequently used data modelling tasks (e.g. Doerr and Lesh 2011).

Most importantly, we wanted the tasks, collectively, to provide the teachers with authentic modelling experiences. Considering this, task selection and design became a critical part of our work. Although we found model-eliciting activities (Lesh et al. 2003) and their associated principles to be foundational to the design process, since our primary concern was expanding teachers' knowledge of mathematical modelling for teaching, we conceptualized additional criteria to accommodate for the teachers' particular needs as identified by recent research on teachers' perceptions of, and experiences with, modelling contexts (Gould 2013; Lingefjärd 2000, 2007). We considered it important that the problems selected for the PD sessions be *useable* and *relevant*, *extendible*, and *revealing*.

Usability and *relevance* of tasks used during the sessions would enable teachers to connect what they experienced and learned during PD to their classroom work (Sowder 1998). Since the teachers were non-homogeneous in terms of not only their mathematical background but also the courses and grade levels they taught, we wanted problems to elicit *extendibility* that would allow teachers to experience how the same task could be refined and restated to address a larger scope of mathematical tools and accommodate different student populations. Lastly, we wanted tasks to be *revealing* in three distinct ways. First, the tasks were to allow teachers to experience the full mathematical modelling cycle as an iterative process of interpreting, refining, and validating in an effort to *reveal* how more precise models could be constructed by accounting for more variables. Second, the tasks had to confront teachers' thinking in order to *reveal* aspects that were challenging for them during the modelling process so as to allow us to create continuity and coherence towards planning PD activities for future work. Third, tasks had to *reveal* connections among various content standards currently in place in the United States (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010). With mathematical, pedagogical, and research objectives in place, we consulted several sources. See Table A.1 (Appendix) for sample tasks and associated objectives that were used during the sessions.

Lastly, in operationalizing the activities and guiding teachers' development, we based our instruction on Blum and Leiβ's (2007) modelling cycle; the model was used as a means to frame teachers' thinking about the modelling process. We hypothesized that such grounding would serve three purposes: (1) providing teachers a cognitive tool to monitor their own modelling practices so as to self-regulate their actions, (2) creating a communication tool that allowed the group a common language through which they could initiate and expand discourse surrounding teaching and learning of mathematical modelling, and (3) equipping teachers with a pedagogical tool for gauging and assessing learners' work.

Teachers were emailed one task per month approximately 2 weeks prior to each PD session. This allowed time for them to think about and examine the tasks. Teachers were encouraged to implement them in classrooms and to bring examples of student work to the sessions. The teachers were also introduced to the modelling cycle (Blum and Leiβ 2007) during the first session. This cycle was revisited each month and in analysing teachers' and their students' work. Each PD session was in two parts. During the first part, teachers shared their work on the assigned tasks, compared and contrasted answers, and engaged in refining their solutions. This provided us with the opportunity to challenge their artefacts (Lesh and Doerr 2003). These discussions also granted us the space to introduce how different mathematical tools the teachers might not have considered, either independently or collectively, could be used to construct more robust models. The second part of a session was devoted to discussion of issues teachers raised from either their task implementation or the implementation attempts of teachers they coached.

All sessions were videotaped and transcribed. Each PD session was divided according to discussion foci. Particular themes pertaining to issues teachers raised or elements with which they struggled were noted and traced throughout the data. In tracing teacher growth of mathematical modelling competencies, four indicators were relied on: (1) sustained time on task, (2) number of self-initiated attempts at validating models, (3) number of self-initiated attempts at generalizing solutions, and (4) frequency of reflective actions exhibited in tackling tasks. These criteria were selected from the literature on modelling (Blum 2015) and inquiry-based instruction (Pollak 2003). Our discussion of the teachers' work, challenges they experienced and articulated, and dilemmas we encountered in the PD is based on our analysis of group deliberations.

35.4 Results

35.4.1 Teachers' Challenges

Analysis of the content of the discussions during the PD sessions, along with the data collected from surveys at the beginning and end of the year, revealed growth in teachers' knowledge of mathematical modelling. Further, three prominent sets of challenges teachers faced surfaced: mathematical, pedagogical, and epistemological.

35.4.1.1 Mathematical Challenges

Modelling tasks are by definition vague. An integral component of modelling activities is the generation of *what if* and *what if not* questions about the problem and its mathematical model(s) (Pollak 2003). There are no precise rules in mathematical modelling and no correct answers (Swetz and Hartzler 1991). Model adequacy is based on how accurately it describes and predicts the behaviour of the real system. This feature of the modelling process distinguishes it from other forms of mathematical work and is most challenging for students and teachers (Lingefjärd 2000).

Teachers were uncomfortable making assumptions or isolating specific variables to reduce parameters when confronted with the modelling process. Similar to the report by Lingefjärd (2000), the teachers struggled to select from among variables they deemed as appropriate to solve the problem. As a result, they were compelled to suggest that a problem could not be solved due to the large number of variables needing consideration. Teachers also struggled with whether they should use a formula in solving a problem or in model building processes if they were unable to explain or to derive it. Once teachers believed a model to be intuitively sensible, they were reluctant to refine the model to make it more precise. This issue was particularly paramount when the context demanded a certain degree of intuitive or experiential knowledge or when the need for precision was not immediately apparent. Past experiences, such as usefulness of approximations and importance of estimating, hindered teachers' desire to consider alternative interpretations of the givens of a task. Thus, helping them understand when to estimate or be precise became a challenge. Such efforts were not always successful. Although teachers complied with our requests, they were not particularly convinced of their relevance.

35.4.1.2 Pedagogical Challenges

It is well accepted that knowledge of learners' thinking, their approaches, conceptions, and misconceptions is an important part of mathematical knowledge to teach (e.g. Doerr 2007). Schoenfeld (2010) referenced this domain of knowledge as teachers' vision: the ability to anticipate potential student approaches and plan for dealing with those approaches to ultimately guide students towards desired learning objectives.

During the first four PD sessions, teachers repeatedly displayed the desire to know what to anticipate of students' work. They were curious to know what would be difficult or easy for students and concerned about how much information they would need to provide so as to structure learners' work. We were pressed for explanations on how to implement modelling tasks in the context of mathematical topics they were teaching each month. They were also sceptical of how modelling tasks could assure that students mastered the skills they were expected to cover. These concerns echo two major tensions associated with teaching of mathematical modelling in education. On the one hand, they highlight the chasm associated with distinct goals of mathematical modelling as a vehicle and modelling as content (Kaiser and Maaβ 2007). On the other hand, they reveal the need for offering venues for assisting teachers to develop an understanding of how to balance short- and long-term outcomes of a modelling-based curriculum in the presence of high-stakes testing and also assessing learners' performance (Schmidt et al. 2011). Blum (2015) previously identified the absence of coherent and substantial modelling resources as a major barrier to teachers' implementation efforts. Our participants' concerns further punctuate the need for systemizing efforts towards development of materials that enable teachers to manage, effectively, short- and long-term goals of their required curriculum.

35.4.1.3 Epistemological Challenges

The literature characterizes mathematical modelling to consist of cycles of analytical work (see, e.g. Borromeo Ferri 2006), including:

1. Examining the situation and setting up a problem to be solved,
2. Identifying variables in the situation and selecting those that are essential,
3. Creating a model that best describes the relationships among the variables using geometric, graphical, tabular, algebraic, or statistical representations,
4. Formulating conclusions,
5. Interpreting the results for accuracy and relevance,
6. Refining the model through validating its potential to account for all relevant variables,
7. Testing model generalizability to other similar situations.

The process of developing useful models often involves a series of iterative testing and revision cycles (Blum 1991). These descriptions clearly distinguish mathematical modelling from solving application problems in which task parameters are frequently well-defined, relevant theories are established and reinforced, and pertinent concepts to be used are easily identifiable. In mathematical modelling, the individuals draw from previous experiences and knowledge in an attempt to seek algorithms relevant to the task. Unlike applications, modelling tasks do not have correct answers; the adequacy of a constructed model is evaluated based on its accuracy, descriptive realism, precision, robustness, and generalizability (Meyers 1984).

These criteria stress the importance of iterations in modelling as means to increase sophistication by accounting for the greatest number of variables. Assessment of many criteria can be subjective since task interpretation and what is perceived as a realistic solution rely heavily on the modeller's judgement (Pollak 2003). "What is usually missing is the understanding of the original situation, the process of deciding what to keep and what to throw away, and the verification that the results make sense in the real world" (Pollak, p. 650). In making these decisions, personal experiences of modellers shape how individuals interpret and solve a task. The use of classroom-taught mathematical concepts may not be seen as necessary. Additionally, misconceptions about the real-world phenomenon studied can influence what modellers consider and what is perceived as plausible. Participants struggled throughout the year to reconcile these epistemological tensions, as both mathematical modellers and teachers of mathematical modelling. The tension associated with mathematical modelling teaching in the presence of these obstacles is particularly real to teachers when assessing appropriateness of their interventions. This issue, raised in the past (e.g. Niss et al. 2007), persists as one needing further inquiry.

35.4.2 Reflections on Teacher Educator Efforts and Challenges

Aside from having to reconcile the real tensions that the teachers experienced and expressed throughout the year, the teacher educators faced two specific challenges: managing the teachers' diverse mathematical backgrounds and the amount of instruction to be provided so as not to compromise the spirit of inquiry we had hoped to establish.

35.4.2.1 Managing Diverse Mathematical Backgrounds

We deliberately chose a task sequence that would allow all teachers to engage in modelling and provide an opportunity for all teachers to learn new mathematics. Many tasks spanned the content of algebra, geometry, probability and statistics, and calculus. However, creating a harmonious climate for mathematical modelling and for collaborative learning was difficult at the beginning. Teachers with lower mathematical backgrounds assumed their ideas too trivial to be shared. Frequently when the approaches used demanded mathematics beyond the level/concepts they taught, teachers became disengaged and reluctant to re-engage with the task to increase its efficiency. On the other hand, teachers with more advanced mathematical backgrounds were compelled to rely on algorithms and procedures they knew and not to consider alternative representational modes that were appropriate to be used in the model building process.

35.4.2.2 Avoiding Show and Tell

One of our major objectives was to demonstrate how inquiry instruction interacted with mathematical modelling that learners are expected to do. As such, our intent was to create the same climate during the PD sessions that we hoped teachers would enact in their classrooms. Although we had anticipated and planned for occasions to provide instruction on how specific mathematical concepts and tools could be used during the modelling process, the need for presenting explicit guidance became paramount throughout. The need for intervention ranged from guiding teachers through what variables to include or exclude to reduce task complexity, disregarding superficial and intuitive responses, to judging robustness of models they presented. It was particularly challenging to resolve the chasm between teachers' collective agreements on their intuitive responses to tasks and the need to engage them in refining and validating modelling phases. Teacher tolerance for large error margins due to estimation hindered attempts to establish more precise task solutions. These challenges might not be unique to mathematical modelling but they interfered with a productive modelling process.

35.4.3 Teachers' Reactions

On an end-of-the-year survey, teachers were asked to identify their gains from the experiences during the year. Similar to Maaß and Gurlitt's (2011) work with European teachers, the teachers expressed greater self-efficacy towards teaching mathematical modelling and greater understanding of its process. The teachers became more aware of how modelling tasks interacted with the curriculum and ways these could be used to enhance learners' mathematical thinking. Due to experience in using different subject areas to solve the same mathematical task, the teachers claimed feeling greater comfort with how to introduce different representations throughout the year so that these would serve as problem-solving heuristics during learners' modelling process. A deeper understanding of the role and importance of precision when working on mathematical modelling and model building tasks was gained. Significant positive changes in sustained time on task, number of self-initiated attempts at validating models, and number of self-initiated attempts towards generalizing solutions indicate greater maturity in their modelling practices. While results are promising, quantifying teachers' growth was not possible due to lack of instruments allowing tracing the development of modelling cognition over time.

35.5 Final Comments

In our work, tasks were designed and sequenced in a manner that would raise teachers' knowledge of modelling as both a content strand and a vehicle for teaching mathematics (Kaiser and Maaß 2007). This selection and sequencing was informed by our knowledge of mathematics, perception of mathematical needs of teachers, and the sorts of tasks that could facilitate coherent professional growth. We acknowledge our choices are vulnerable since curriculum development was neither a focus of our research nor one which was informed by literature. Published reports of efforts towards meeting these same goals are limited, particularly if these experiences are not embedded in university coursework. Further, scholarly elaborations are needed in unpacking how the mathematical coherence and continuity that teachers need when managing short- and long-term curricular obligations might be met. This demands more theoretical and empirical accounts of teacher educators' decision-making, as identified by Cai et al. (2014).

Data indicated that teachers became more comfortable with modelling processes, as learners, over the course of the year. The problem sequence used as well as the detailed mathematical discussions during sessions appears to have contributed to what teachers claimed to have gained mathematically. Inclusion of the modelling cycle in guiding participants' thinking, both as learners and teachers of mathematics, also appeared effective since teachers frequently referenced it to reason and argue, to articulate their own actions or characterize students' modelling efforts. These results propose two venues for further inquiry. First, additional research is needed to better articulate ways the demands of mathematical modelling-based instruction might be gauged with a greater focus on the quality of mathematics shared or gained longitudinally. This might assist in developing a research-based teacher learning trajectory specific to mathematical modelling. Second, detailed theoretical descriptions of how teachers might be guided in developing pedagogical tools for navigating classroom implementation and assessment of effectiveness of their own practices merit attention.

Appendix

Table A.1 Tasks and objectives

Task	Objectives
Rainfall problem (Bocci, F. (2012). *European Journal of Physics,* 33, 1321. doi:10.1088/0143-0807/33/5/1321) It is about to rain; you have to walk about 1 km between your car and class. You don't have an umbrella but decide to take a chance and walk the distance. Suppose that it now starts to rain heavily and you don't turn back; how wet you will get? What do you have to do to avoid getting too wet?	Making assumptions Full modelling cycle Use of trigonometry and calculus
Basketball problem (Barrett, G., Bartkovich, K., & Compton, H. (1999). *Contemporary pre-calculus through applications* (2nd ed., p. 275). New York: Glencoe/McGraw-Hill) As the star player of a basketball game stands at the free throw line, the announcer states that he had hit 78 % of his free throws that year. The star player misses the first shot and makes the second. Later in the game he is fouled for the second time. As he moves to the free throw line, the announcer states that he had made 76 % of his free throws so far that year. Can you determine how many free throws this player had attempted and how many he had made that year?	Precision and accuracy System of linear inequalities
Spaghetti problem (D'Andrea, C., & Gomez, E. (2006). The broken spaghetti noodle. *The American Mathematical Monthly*, *113*(6), 555–557) If a piece of spaghetti is broken at two randomly chosen points, what is the probability that the three pieces, placed end-to-end, can form a triangle?	Probability theory Geometric modelling

(continued)

Table A.1 (continued)

Task	Objectives
Financing college education (Dossey et al., (2003). *Mathematics methods and modelling for today's mathematics classroom* (p. 97). Pacific Grove, CA: Thomson Learning) You plan to invest part of your pay check to finance your children's education. You want enough money in the account to be able to draw $1,000 a month, every month for 8 years beginning 20 years from now. The account pays 0.5% interest each month. (a) How much money will you need 20 years from now to finance one child's education? Assume you stop investing when your first child begins college. (b) How much must you deposit each month over the next 20 years	Dynamic systems Finding equilibrium points
Population growth in China (Reading source: http://content.time.com/time/world/article/0,8599,1912861,00.html) What considerations needed to be made to implement the one-child policy in China in 1980 and then to relax the one-child policy in 2013 (families can have two children if one parent is an only child)?	Population growth Analytic solutions
Ping-pong ball problem (Starfield, A.M., Smith, K.A., & Bleloch, A.L. (1990). *How to model it: Problem solving for the computer age*. New York: McGraw-Hill) How many ping-pong balls would fit in your living room? Extensions: How many miniature footballs would fit in your living room?	Full modelling cycle Refining models Precision
Establishing a new international airline hub (Reading source: http://en.wikipedia.org/wiki/Airline_hub) Airline hubs are airports that an airline uses as a transfer point to get passengers to their intended destination. Suppose you are approached by a newly established airline and asked to offer them a plan for which hubs to use. What considerations need to be made to determine the location of an international airport hub in the United States?	Defining variables Strategy development Data-based decision-making

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Chapter 36
The Velocity Concept: The History of Its Modelling Development

Regina Dorothea Moeller

Abstract In mathematics education, there are applications of physics that refer to the real world. These applications have a long history of being mathematically modelled and can be considered as one of the central cross-curricular topics. They often involve several mathematical fields and sometimes show a long process of modelling activities. One of these applications is the velocity concept. Its genesis refers to a centuries-old search within the context of motion and for more than 150 years has shown new technical applications.

Keywords Phenomena • Velocity concept • Teacher education • Physics

36.1 Introduction

The subject of modelling processes in mathematics classes, especially at the secondary level, appears with a strong interest for its realization only during the last 20 years or so (Blum 2008). This recent development has enormous benefits towards a broader understanding of mathematics, its various applications together with their teaching strategies. Along with this process, teachers have undergone a change in their attitudes towards mathematics and its modelling procedures and also in their pedagogical content knowledge (Kuntze et al. 2013, p. 322).

This pedagogical development in modern mathematics classes, with a strong emphasis in some parts of western education, happened on top of a modelling process that was initiated about 300 years ago. It has shaped our understanding of today's reality enormously and also the role that mathematics plays in it. In the past, scientists used mathematical models long before the establishment of didactics of mathematics as an independent science that publicized the need to put a focus on modelling. Galileo Galilei was one of the first who perceived that the complexity of

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_36

reality needed to be simplified in order to be able to describe it with mathematical tools. Von Weizsäcker (1964, p. 107) pinpoints this idea:

> Galilei took a big step, daring to describe the world the way we do not experience it. He installed rules which in the form he expressed them we never experience it in real life. Therefore these simple mathematical rules could never be confirmed by a single observation. Thus he opened a way for a mathematical analysis which divides the complexity of reality into single elements.

Consequently, Galilei chose already those quantities which were substantial to describe real processes and constructed mathematical relationships which imitated his observations most closely – that is, he modelled.

Realizing this historical development influenced the scientific world in such a strong manner, it seems necessary to make students aware of this thinking process and reflect about its impact in mathematics classes. Classroom work using elementary model representations constitutes a phenomenological acquaintance which can be undergone with one of the early scientific conceptions, that is velocity. Students come with their everyday perceptions into class – they run in sports classes, they see cars moving and they cycle themselves. These experiences can be used as Freudenthal (1973) has pointed out. One can start with a small aspect of reality, that is, with a "text problem" or otherwise with an inverse modelling, to show the benefit of mathematics and to discover a piece of mathematics. He considered thought experiments for students as a means to invent a "piece of mathematics". To observe and analyse their behaviour and to deduct pieces of information from the thought experiment would be a useful and, at the same time, a responsible project of mathematical didactics. By undertaking this classroom endeavour, mathematics is linked with the "experienced reality of the learners … which becomes the skeleton onto which mathematics develops" (Freudenthal 1973, p. 77). Also Klein focussed on the content and its understanding in a similar way. For an inspiring and descriptive lesson, it is always necessary to build on the perceptions and experiences of the students. Klein (1939, p. 227) says: "one needs to build onto the concrete visual perception and only slowly bring forth logical elements into the focus".

One could even say that today's focus on modelling procedures hinders a perspective that entails a rather rich view on how mathematics has been used to model reality for a long period of time. As Clements (1989, pp. 23ff) emphasizes, there is a strong separation between the mathematics used and the issues to be modelled. Especially within this context, he distinguished between experienced modellers and those who are learning to model. Also, Noss and Hoyles (1996) pointed out there is a difference in the level of abstraction within each context to be modelled. These different levels of abstraction can already be regarded as parts of modelling procedures. The final outcome may suggest a separation of abstract mathematics from the concrete real world. It also makes apparent that the modelling cycle with its distinction between mathematics and reality, or even the rest of the world, is not always a valid interpretation (Möller 2014).

This chapter emphasizes a specific conceptual example that offers a window into the historical modelling scene. The definition of the concept of velocity was made a long time ago in the centre of a very significant modelling process between reality and the two involved subject matters of mathematics and what we call physics today (earlier it was natural philosophy). This modelling outcome has been taken as a natural view of reality with respect to velocity which, in everyday life (in contrast to the concept of speed which has a rather colloquial meaning), is usually not recognized as the end product of a modelling process. In school, there is not much in-depth teaching of this concept, neither in mathematics nor in physics classes. In physics classes, one waits for the quantities length and time span and the repertoire of the function concept to introduce the formula $v = s/t$ at once without a lot of experiments. In mathematics classes at the secondary level, the subject matter of analysis is in focus and velocity problems are looked upon as pure applications. To date, this concept has not been given the importance that it deserves in the history of science nor under the focus of modelling and educational possibilities.

This situation is not much different in other European mathematics classes from the German situation. For example, the Dutch curriculum shows a similar approach to the German one: on the secondary level with the tools of notions of limit. In France, there is a similar situation since at the end of primary classes, the velocity concept is a matter of application. All in all, the concept of velocity is not looked upon as a historically developed modelling of a phenomenon.

Given the rich historical background which is taken unconsciously for granted, it is most fruitful to look carefully at how the concept of velocity is introduced in today's mathematics classes even at the elementary level. It is argued that the way velocity is shown in textbooks does justice neither to the learning of the students nor to the understanding of the concept of velocity. Indeed, some aspects of teaching methods could even hinder a clear understanding of the modelling perspectives involved.

36.2 Actual Phenomenon in Elementary Mathematics Class

Text problems, like the one shown in Fig. 36.1 accompanied by a small text like "Situations with velocity – create tables like the one which is shown!" (taken from a mathematics textbook), underpinned with pictures suggesting movement, can often be found in German fourth grade classes. It is the first time students read about velocity, and it therefore stands as a classroom introduction to this phenomenon.

Several pictures are given that show people or machines such as cars or trains each with information about the respective velocity such as the train covers 100 km in an hour. The assigned task is to fill out tables in which the students write the distances for different time spans: 1 h, 2 h and so forth. (Obviously the numbers in the table do not refer exactly to the examples but only show some kind of proportionality.) Above the assigned tasks, one can read the title velocities. One can observe at once that like in other cases in which an introduction into the field is in the focus of interest – here a start with a series of different observable motions and a lead to the question of how they can be quantified – there is an emphasis on the computational

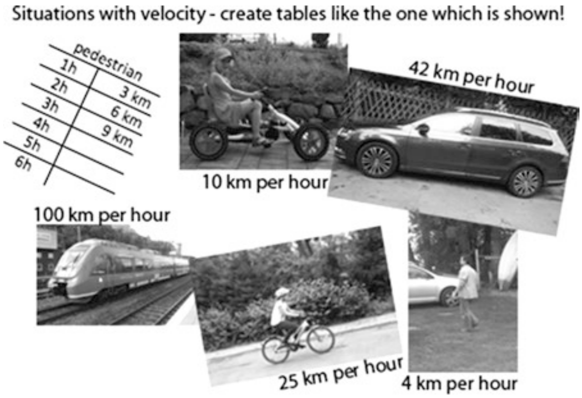

Fig. 36.1 Velocity problem

aspect under which the ideas that have led to the possible computations have disappeared (Doorman and van Maanen 2009). Instead of an approach pinpointing the quantities in question, there is a given table to be filled out by the students.

This kind of problem can be considered as an anticipation. There are at least three other kinds of anticipation tasks at the elementary level: one refers to the ratios such as a half, one quarter and three quarters, and the second refers to the decimals within the context of quantities. The third is the appearance of tables in application problems. In these problems, prices of products are given – like 1 kg of apples cost 75 c – and the question is how much is paid for 2 kg/3 kg/5 kg. These three kinds of anticipations occur in mathematics classes because of the application principle. Students see these kinds of numbers and kinds of questions in their daily life, and mathematics classes respond to this phenomenon by introducing these numbers and tables without giving a rigid mathematical reasoning.

What kind of anticipation is made when students solve this kind of velocity problem? Since the students are to fill out tables, which are a representation of functions, one could argue that functions have arrived in elementary mathematics classes due to Felix Klein (1905) who made functions a subject matter in mathematics classes at the secondary level. It also could be understood as an example of an (anti-)didactical inversion (Freudenthal 1983, p. 305 ff.). In any case, it is obvious that this is not following the historical development, and it is not showing an elementary approach which is possible at this stage of mathematics classes. Since this kind of velocity problem is not really an application task, as only tables need to be filled out, one could argue that such tasks give a mathematical way to compute velocities which the students can observe in their daily lives because our modern world presents this phenomenon.

Having in mind the way quantities are introduced in mathematics classes at the elementary level (Griesel 1984, i.e. finding representatives, studies of comparisons with chosen measurement objects and afterwards with agreed-upon measurement objects, learning the standardized measurement unity and finally solving application problems), the velocity problems focussed on do not show any such procedure although velocity is the first composed quantity that students encounter. Since velocity is a mathematical concept, one should expect a sequence of steps that lead to a definition. A possible approach would be the didactical triangle of Bruner (1960) in which he argues for an approach that encompasses the enactive, iconic and symbolic level. Alternatively, another didactical theory of learning concepts is given by Vollrath (1984) who outlines in general what kind of steps lead to an understanding of mathematical concepts.

Another concern is the fact that the concept of velocity can be looked upon as a real mathematical modelling procedure (Blum and Leiß 2007) where observable movements can be measured in two dimensions: length and time span. It could be arranged as a project for students in which the definition of velocity is the end product of their investigative endeavour. This point also leads to the question: why does a fundamental phenomenon like the concept of velocity lack any modelling approach in textbooks? It comes to mind that Newton might not have thought primarily of velocity as a function since he was still following Galilei's proportional theory. Indeed, did he define velocity within a text written in Latin?

36.3 The Historical Process of Modelling

The concept of velocity is one with a long tradition, similar to the history of calculus (e.g. Doorman and van Maanen 2009). The concept arose out of the concept of movement which already the Greek philosophers were aware of.

Although a lot of the work of Archimedes (287–212 BC) concerning mechanics has been transmitted to us over the centuries and gives an idea of his wide-ranging mathematical understanding, we have no clear idea what he understood by velocity. On the other hand, verified work has been passed on to us of Aristotle (384–322 BC) investigating the phenomenon of motion qualitatively and verbally (Aristotle 2015). Aristotle distinguished three types of motion: motion in undisturbed order, such as the celestial spheres, the "earthly" motion of objects rising and falling and the violent motion of bodies that needs an impulse (cf. Hund 1996, p. 29). Although his remarks touched the phenomenon of velocity, his conceptions were proved wrong later on (Hund 1996, p. 30): "Aristotle came close to the concept of velocity when in the sixth book, the words 'faster' (longer distance in the same time, same route in shorter time) and the 'same speed' are explained."

Before the next step on the journey to velocity was carried out by Galilei (1564–1642), Nicole of Oresme (1330–1382) used a graphic representation of changing qualities. Moving towards a functional terminology of velocity, Oresme (ca. 1320–1382) sought the help of experiments to assign a rate of change to certain intensities and concluded: "All things are measurable with the exception of numbers" (Pfeiffer and Dahaene-Dalmedico 1994, p. 228ff.). The difficulties mathematicians had at that time are well summarized in this source.

As Weisheipl (1985) points out, Galilei (1564–1642) struggled with the Aristotelian concept of nature which he thought of as an active principle: "Nature is a source not only of activity but also of rest" (Weisheipl 1985, p. 22). This view has an impact on his understanding of motion. He still pondered over the idea of Parmenides, "all change is illusion", and that of Heraclitus, "everything is flux". Galilei's work can be understood as being at the brink of the Aristotelian view of nature coming to the understanding that one cannot always refer to intuitive conclusions based on immediate observation because they sometimes are misleading (cf. Einstein and Infeld 1938). Later, Galilei used experiments to argue for the statement that there is a quadratic dependency between the distance travelled and the falling time of an object.

Finally, Galilei succeeded in a better understanding of the concept of velocity, as he did not rely on his direct perception, as noted by Einstein and Infeld (1938, p. 17):

> The means of scientific evidence was invented by Galilei and used for the first time. It is one of the most significant achievements, which boosts our intellectual history .... Galilei showed that one cannot always refer to intuitive conclusions based on immediate observation because they sometimes lead to the wrong track.

It was Galilei who decided to consider mechanics as part of mathematics (Koyré 1998, p. 73). The consequence of this decision was to substitute reality of daily experiences for an only imagined reality of geometry. This can be considered as a modelling of reality. Later Newton (1642–1727) defined velocity in his *Principia* using the concept of force that initiates motion. He formulated the principle of inertia as: "Every body preserves in its state of rest or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it" (Weisheipl 1985, p. 69).

Newton introduced the concepts of absolute time and absolute space and opened up the exact relationship between force and motion: The power does not get the motion upright (Aristotle), but it causes its change (acceleration) (Newton 2016). While Aristotle argued by observation, Newton made an abstraction, as he looked upon length and time as not necessarily bounded materially (Newton 2016). Leibniz (1646–1716), with his focus on variables, developed the differential and integral calculus also considering the idea of (planetary) movements. His version of the dependency of force and mass is what we normally learn in mathematics classes.

In today's linguistic usage, we understand motion as a change in position in the (Euclidean) space over a certain period of time. Lengths and time periods are conditions for the quantification of such motions. On this basis, the *(average) velocity* is defined as the quotient of the distance travelled and the time required. Even Piaget (1996, p. 69) referred to this circular argument: "Speed is defined as a relationship between space and time – but time can be measured solely on the basis of a constant velocity". For him, the concepts of space, time and speed are mutually dependent.

36.4 Mathematical Aspects of the Velocity Concept

The reasoning of Aristotle and Galilei was based on their observations of linear motions. However, both had also planetary motions in mind. For motions on a curved path, two different aspects are needed: the direction and the quantity of a velocity vector. It is this distinction which led, in modern terms, to a vector description and thus to a further clarification of the concept of velocity, which is thus a generalization of the concept of velocity on a straight line. Solid bodies moving on a straight line have the same speed, in the same direction and the same quantity. Since then, the following statement is true: the change in force and velocity are vectors with the same direction (Einstein and Infeld 1938, p. 38). We note an idea of congruence, because all statements that apply to velocities along curved paths must also apply to linear trajectories.

The cause of this observation is given by an idealized thought experiment, which confirmed the theory (cf. Einstein and Infeld 1938, pp. 25 ff.); this is yet another idea that came into effect only at the time of Galilei. Since then, the mathematical language has been used in physics to reason for not only qualitative but also quantitative conclusions. As soon as one engages in quantitative calculations, one deals with quantities. With respect to the concept of velocity, there are the dimension (the quotient of distance and time) and the measured value (an element of real numbers), which are a combined physical quantity.

Griesel (1973, p. 55 ff.) analysed the subject matter of quantities at the primary level (length, weight, time periods) as technical background for the didactics of quantities. This presentation, however, does not fit the quantity of velocity (and is not mentioned there) because it requires a description as an element of a vector space, which can be higher than one-dimensional. However, Freudenthal (1973, p. 188) argued that one can interpret measure indications as function symbols, an idea that was not previously addressed in classes. Another functional aspect occurs in two ways with the concept of velocity: The distance-time function taking into consideration the difference quotients to the average speed and the transition to differential quotient to instantaneous velocities that are themselves functions again, namely, the velocity-time functions.

It has taken over 2000 years for the concept of velocity to be defined consistently out of the concept of motion – in today's usage, this is an act of mathematical modelling. It is therefore a prime example of a mathematical and interdisciplinary concept development, with both mathematical and physical – mechanical – representations throughout history. The intuitive conclusions drawn by Aristotle led to difficulties and proved much later untenable. Only an idealized thought experiment led eventually to a verifiable physical theory. The concept of velocity is an example of mathematical modelling of qualitative knowledge and observation with more potential as there are quantitative statements and other insights. The knowledge of such phenomena, the resulting misconceptions and the trodden paths of knowledge are essential components of mathematics education and exemplify scientific processes.

The genesis of the concept also shows a potential for didactical perspectives of mathematics education. Despite the scarce representation of this topic in mathematics classes (in many curricula of the German provinces, e.g. the concept of velocity is only mentioned once at the secondary level), there is a diversity of ideas which can be reflected.

36.5 Didactic Consequences for Mathematics Classes

The genesis of the concept of velocity was influenced by its interdisciplinary character for centuries. The later development of Leibniz using variables in formulas has influenced our learning in mathematics and physics classes. Taking this into account, mathematics education should not neglect the role that mathematics has played in shaping the explanation of the world of physics of today. Both subjects have influenced each other's individual conceptions. It is therefore a question of epistemological knowledge and, consequently, not only the task of physics education to define velocity or to avoid it in the classroom at all, for instance. In mathematics teaching, the teaching of the velocity concept possesses a great potential with several important functions:

1. Since it is a conception of highly relevant historical influence, it needs to be understood in the classroom as such. That means there is no need to use tables or other teaching materials or devices in order to be able to quickly come to answers. Instead one can use pupils' everyday experiences. That puts their experiences into the middle of the learning process and "picks them up where they stand".
2. Doing this there is a high motivation for the students to think about it. Time is needed for the students to verbalize their understanding of motion, of quickness and later of velocity. In light of the use of language in mathematics classes, Vygotsky (1986) elaborated on the relationship between everyday experiences and logical reasoning. Everyday experiences may interfere with scientific reasoning which, in the case of the concept of velocity, occurred historically for quite some time because of the lack of understanding of the invisible forces. It is therefore necessary for pre-service and in-service teachers to learn to make a distinction between the appearance of motions and their scientific reasoning.
3. Taking into account that under the current relations of applications of mathematics education, the concept of velocity is a prime example of mathematical modelling of the everyday phenomena of motions which can be quantified normatively, allowing measurements and calculations. According to the current understanding, these are necessarily factual and methodological skills.
4. Basic everyday experiences relate to the phenomenon of velocity and can be used at several grade levels by taking up Vollrath's idea and taking seriously measurement processes as a basis of experience (Vollrath 1980). This can be addressed in propaedeutic (i.e. pre-theoretical) form at the primary level and in a quantitative manner at secondary levels.

5. As the first combined quantity, the concept of velocity can be addressed in lower secondary education on many occasions. Before coming to the usual "algorithmization" (Doorman and van Maanen 2009), teaching this concept allows for teaching the fundamental idea of measuring. This is a valuable and vital contribution to the implementation of the rules in the educational standards and curricula and guiding principles would be met. The implementation of the items listed would be an important part in promoting mathematics education.

This discussion of the velocity concept in the context of the stage scheme (cf. Vollrath 1984) shows the potential that this concept already has in mathematics education. From this rather elaborated standpoint, the potential for the interdisciplinary character can be developed further in more detail.

The four steps in the concept development mentioned above could enlighten teacher education students in their knowledge of this topic and also give them an appropriate background to approach the topic in an ethical attitude (Ernest 2012). It is essential for their teaching to have a solid grip on knowledge and the cultural heritage of the subject matter they are to teach.

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Chapter 37
Developing a Mathematical Modelling Task for All Students

Edel Reilly

Abstract This chapter describes an authentic mathematical modelling task designed by a learning support teacher for a Year 7 pre-algebra class co-taught with a regular Year 7 grade mathematics teacher. The task integrated two parts of the Year 7 mathematics curriculum: area and percent. Students were asked to calculate the cost of decorating (flooring and painting) the inside of their dream house. Using prices from a local building supply store, students had to decide what type of flooring and wall covering they could afford. Using a constructivist framework, this chapter provides an overview of authentic task development and the role such tasks play for students who struggle with mathematics, particularly students with special needs.

Keywords Constructivism • Authentic tasks • Students with special needs • Instruction

37.1 Introduction

Today's mathematics teachers face a bewildering range of learners in every class: students identified as gifted learners or needing learning support, students who have not been identified as needing support but who struggle with many mathematical concepts, and learners who have a variety of learning styles and abilities. As each mathematics lesson builds on the previous one, mathematics teachers aim to reach all students; but with so many different levels of mathematical abilities in one class, teachers often feel lost. Mathematics teachers need strategies to help them reach all students in today's inclusive classrooms. The diversity of the student population and the expectation that all students can achieve high standards require a shift in

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_37

instructional practices and design. In order for students with disabilities, as well as all students who struggle with mathematics, to be successful, they need access to meaningful and rigorous curricula that meet high standards.

Classrooms that have students who struggle in mathematics tend to focus on computational skills and recollection of mathematical facts. In a small Australian study, Vincent and Stacey (2008) found that in some of the best-selling mathematics textbooks, "the balance is too far towards repetitive problems of low procedural complexity that require little more than using procedures" (p. 102). Many textbooks are designed to walk students through the problem-solving process without allowing much room for students' exploring their own approach to solving problems. Students can then become reliant on these step-by-step processes which remove the students' creativity for problem-solving. In many instances, the problems provided for students to solve have little or no connection with the students' everyday lives, and so students' struggles are compounded as they work to decipher what the problems are asking, to identify a solution strategy, and to evaluate the appropriateness of their answers.

Being able to answer the age-old question "When are we ever going to need this?" is something that comes to mind when one starts to think about meaningful and challenging problems to which all students can relate. While there is no denying that class time needs to be devoted to making sure students master the fundamentals of the mathematics (basic facts, computations, algorithms, or formulas), time also needs to be spent on activities that allow students to practise the mathematics they are learning in meaningful ways. The fourth mathematical practice standard from the Common Core State Standards (CCSSM) calls for students to be able to "apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010). This certainly follows from the work of Lesh et al. (2002) who point out that problems need to go beyond short answers to specific questions. Activities should be posed as open-ended problems that are designed to challenge students to solve complex, real-world problems (Vos 2013).

37.2 Theoretical Framework

37.2.1 Constructivism

Constructivists argue that students construct knowledge out of experiences. The theory of constructivism is grounded in the pioneering work of the educational theorist Jean Piaget (1972, 1990). Piaget argued that when people have new experiences, they either assimilate those experiences into their current framework for understanding the world or they change that framework to accommodate the new experience they have had (Auger and Rich 2007). Building from a constructivist framework, then, teachers need to provide students with learning experiences to help them gain knowledge. This means teachers need to shift their role from that of lecturer to facilitator (Brewer and Daane 2002). Teachers need to help students engage in meaningful learning activities in order for the students to construct an understanding of the world. It is thus not enough to tell students what a mathematical concept is. Instead, teachers must provide students with an opportunity to work with a mathematical concept in order to construct a full understanding of that concept.

"The challenge in teaching is to create experiences that engage the student and support his or her own explanation, evaluation, communication, and application of the mathematical models needed to make sense of these experiences" (Math Forum 2015). As students develop mathematical models, they are able to construct personally meaningful understandings of the mathematical concepts they are studying. This makes the use of modelling in mathematics an effective way for teachers to apply constructivist theories in the classroom.

37.2.2 Modelling with Mathematics

According to Lesh et al. (2002), modelling with mathematics includes "simulations of real life problem solving situations (that) require more than a few minutes to complete" (p. 41). All students can benefit from working on real-world mathematical problems as these applications make the mathematical concepts more meaningful. For instance, being asked to find the perimeter of a rectangle whose length is 5 and width is 4 is a common question in a middle school mathematics class working on a measurement unit, but the real-life application of these mathematical ideas is different. There are instances when one would need to find the perimeter of a rectangle, and so students need to be taught why such a task is valuable. So a mathematics teacher could contextualize that problem by saying a person might decide they want a garden, and so the size of the piece of land to be devoted to the garden has to be considered as well as the amount of fence needed to keep unwanted animals from eating the vegetables. In the real world, consideration also has to be given to the cost of purchasing the fencing which in many cases might determine the final size of the garden. Cost will also be impacted by the type of fencing to be used. It is this application of the mathematical concept being studied that the mathematical practice "model with mathematics" is addressing. All students given access to information need to be able to make assumptions, analyse relationships, and draw conclusions.

37.2.3 Authentic Tasks for All Students

When looking at authentic tasks and why they should be used, Kramarski et al. (2002) defined authentic tasks as conveying common contexts "for which there is no ready-made algorithm" (p. 226). Students need to be able to see where the mathematics they study fits into the world in which they live. When completing an authentic task, all students should not be providing the same answer. Students should be able to make decisions about what they are presenting as the product of their task. Kramarski et al. go on to discuss the need for school mathematics to prepare students for the economic demands of society.

McDuffie et al. (2011) argue that mathematical learning tasks need to be designed so that all students, even students with special learning needs, can master appropriate skills. In many cases, published instructional materials need to be adapted so that they can be used for teaching meaningful and relevant mathematics to all students. McDuffie et al. have identified several criteria for adaptation that make tasks meaningful. Of relevance here are use of a familiar context, supplementing foundational gaps, and incorporating overarching goals.

Use of a Familiar Context Capturing and maintaining the attention of all students is critical for any teacher in any discipline. Beswick (2011) lists several reasons for paying attention to context. These include a utilitarian purpose of meeting the economic needs of society and helping students understand those societal needs. In addition to societal needs, appropriate context also helps students understand mathematical concepts, develops an appreciation for the nature of mathematics, and improves disposition towards learning mathematics. Using appropriate context is particularly important for teachers who have students who struggle as these students often lack the intrinsic motivation to study a concept long enough to understand it (Sartawi et al. 2012).

All students need to be actively engaged in learning. Sometimes this is difficult for students with disabilities or unmotivated learners. Teachers need to carefully guide class discussions about topics being presented. Strategically seating students in the classroom can play an important role in ensuring lively discussions. Students should be encouraged to talk to their shoulder partner or their elbow partner so that everyone has the opportunity to make their own connections or share their thoughts. Group discussions can help students make suitable connections to the mathematics problems. Learning needs to be made more student-centred and less teacher-centred. Finding meaningful contexts in which to place mathematics lessons requires planning on the teacher's part. All students need opportunities to see that concepts studied in mathematics class have relevance and value beyond school.

Supplementing Foundational Gaps The amount of prior knowledge students have is instrumental in their gaining understanding of new topics. It is important to build on prior knowledge for students who struggle with mathematics in order to help them gain better understanding of new concepts. In addition, when students are asked to apply mathematics in authentic settings, teachers can provide more effective and meaningful feedback (Fyfe et al. 2012). It is through this feedback that teachers can fill in any foundational gaps in the students' understanding of concepts.

Incorporating Overarching Goals Teachers should work to focus lessons on essential questions and big ideas. When a class is co-taught by a regular education teacher and a learning support teacher, it is important that all students in the class be helped to work towards those essential questions. Each student needs to be able to answer those questions, and while all students might not be at the same place at the end of the unit, all students need to have made adequate progress towards their goals (Voltz et al. 2010). According to Small (2010), two widely held beliefs continue to dominate mathematics instruction: that all students should work on the same problem at the same time and that each mathematics question should have a single answer. Teachers have to find a way to meet the needs of a broader range of students with varied and rich activities while at the same time meeting a standards-based curriculum. Modelling and application tasks are such activities.

37.3 Methodology

The purpose of the study was to determine the effect that adaptation of an authentic task, in the sense described above, had on students with learning support and those who struggle with mathematics. A qualitative approach was used in this action research. An authentic mathematics task (Fig. 37.1) was designed for students in a Year 7 pre-algebra class. The task was broken into three parts and guidance provided along the way. The purpose of the task was to help students connect mathematical formulas to a real-world application. There were 15 students assigned to the class. Six students in the class had individualized educational plans due to a learning disability, and an additional five students were identified as performing below basic level according to their most recent standardized state assessment.

37.3.1 Data Collection

For the task, students were asked to research and compare costs of decorating a new home. Following completion of three subtasks, students were asked to fill out an open-ended survey where they were asked several questions regarding their experience with the activity. In addition to the survey, students were also asked to submit their plans. A rubric was created to evaluate the students' work. Since the class consisted of students who struggled with mathematics and students who were closer to grade level, the rubric focused on the mathematical tasks rather than the number, size, or shape of the rooms. The rubric required students to round their prices to the nearest cent. Students were also graded on two worksheets they were given to keep track of the prices for the wall coverings and the floor coverings. Another section of the rubric focused on students' understanding of the money saved with help that was supplied by parents. Students had to demonstrate that they knew how to find a discount using a given percentage. They were also required to show for which room donated cans of paint were to be used. Students needed to subtract these savings from the total before adding 6% sales tax. Finding the tax was another area that was

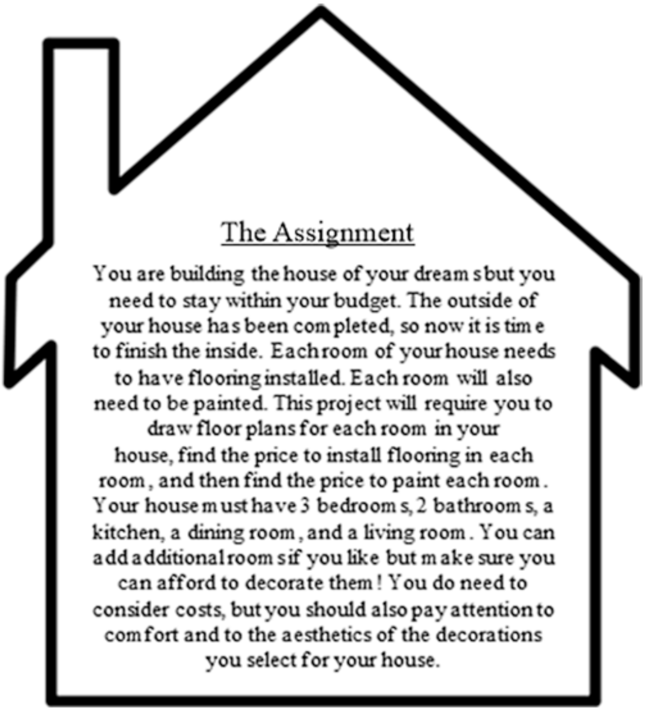

Fig. 37.1 *Furnishing Your Dream Home*

graded and addressed on the rubric. Even though each project varied, the overall goal was understanding the mathematics it took to complete the task at hand, and that is what the students were evaluated on in order to meet the purpose of the research, that is, to determine the effect this authentic task had on students with learning support and those who struggle with mathematics.

37.3.2 Task Implementation

The first part of the project began with a whole class discussion about the costs of decorating a home. Following this discussion, the students were assigned to work with a partner on the essential question: What mathematics concepts are needed to work through the task? At this point students began to ask, “How do we find out how big each room is so we know how much carpet to get?” Students discussed area and how to calculate it. They sketched various floor plans and discussed ways to find the area of rooms that were not always rectangular. Rather than the teachers provide them with a worksheet, the students themselves thought about different shaped rooms and then found the area of them (see Fig. 37.2a). Following the sharing of examples of floor plans, students created wall designs with windows and doors (see Fig. 37.2b).

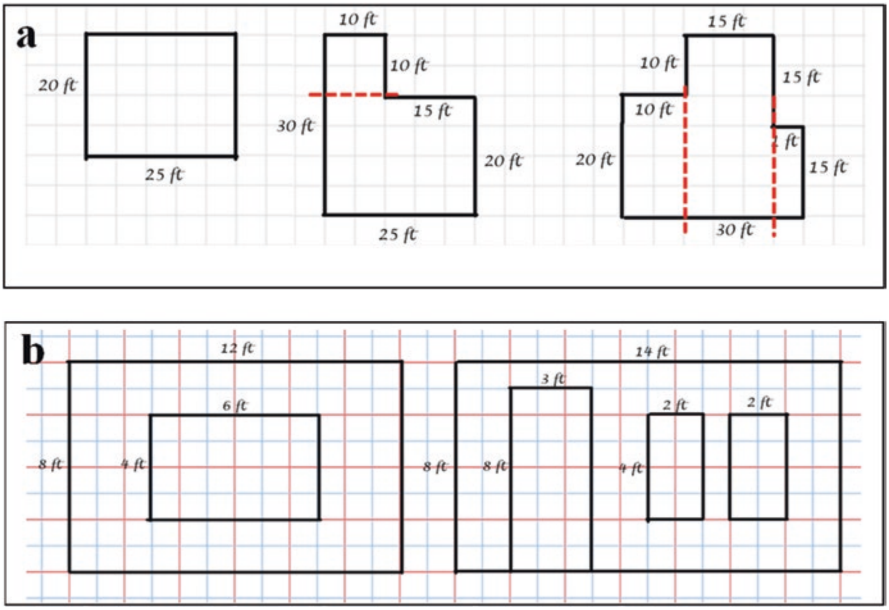

Fig. 37.2 Sample (**a**) student created floor plans to find area (**b**) wall plans including windows and doors

Students were given the opportunity to share their own thoughts on how many windows and doors they could include. They also viewed other students' creations. More importantly, this sharing of plans encouraged students to think carefully about the kind of home design they wanted for their own assignment. Students at different ability levels could create different house plans allowing all students to find the area for their own distinct layouts.

Once students were comfortable finding the area of polygons on floor and wall plans, they were introduced to the second part: designing their own house plan. Students were asked to draw floor plans for each room in their house. They had to find the floor area of each room in order to calculate how much flooring material would be needed. Based on their ability level and the work done previously on finding area, students came up with a variety of floor plans.

In order to have realistic prices, students used data from the website of a local home improvement store to select preferred type of flooring for each room. This information was then used to calculate the cost of flooring for each room (see Fig. 37.3). At this point, a discussion occurred regarding what it meant to find the cost of flooring and paint. Students had already discussed the idea of different types of flooring and paint. In order to help manage the choices, students were given four types of carpet to choose from (Frieze, Fashion Forward, Textured, and Berber) and four types of non-carpet floor covering (vinyl tile, commercial vinyl tile, luxury vinyl plank, and luxury vinyl tile). The students discussed the prices of flooring and paint using the original designs they had created during the initial discussion of

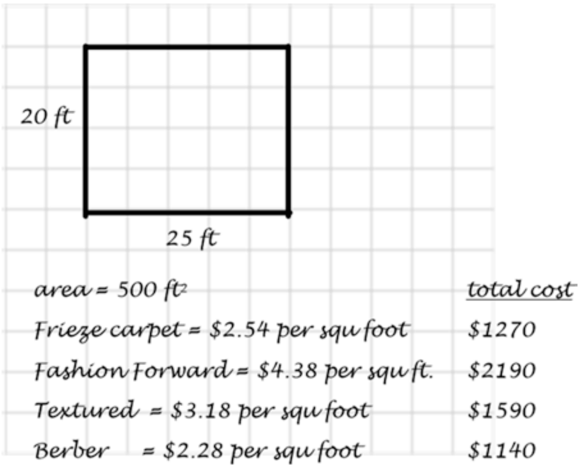

Fig. 37.3 Sample price listing for one floor plan

finding area. Discussion also included carpet preferences: For example, while Berber carpet was the cheapest per square foot, is it a carpet you would like to have in your bedroom? For each floor covering selected and number of gallons of paint identified, students had to calculate the total cost and then add 6% tax. They were given two tables to complete in order to manage their numbers—one for finding flooring costs and the second for calculating painting costs.

As with any real-world project, things happen that change costs, so for the third part of the task, students were informed that their parents had offered to pay for 35% of the flooring of one room. Students had to select the room their parents would help pay the cost to cover. This gave students an opportunity to continue to work with percent and to perhaps consider if it would be worth choosing a more expensive floor covering since now they would have a little more money in their budget. Another assignment addition was that the students were to receive, from a friend, two gallons of interior wall paint donated to the project. They had to decide for which room the paint could be used and to calculate new costs with the parent help and paint donation.

37.4 Findings

Results from the students' survey show that all students liked the initial class discussion where they were given the opportunity to talk about the purpose of the task. Students mentioned that, while they had been in home improvement stores so they were familiar with the context, they had not given any consideration to the variety of materials that are available to customers. All students agreed that they would never consider what they are studying in their mathematics classes as something that is needed in the future without the opportunity to work on a project like this. Students also reported that they liked the individual nature of the task. It was

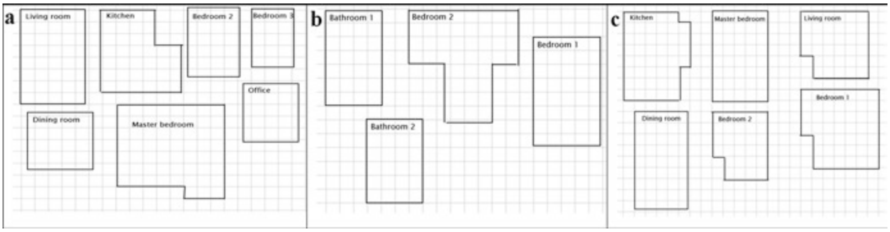

Fig. 37.4 Sample plans from (**a**) student 1 (**b**) student 2 (**c**) student 3

important to them that they had the opportunity to design a floor plan and calculate the costs that were different from everyone else's. The students liked the opportunity to be able to look at what others were doing and to see the differences in the design and product choices being made. Finally, students felt that working on their own design, as they worked towards the overarching goal of the lesson, allowed for them to receive more individualized instruction from their teachers, supplementing any foundational gaps. Since everyone was working on a different plan, teachers would go from student to student providing assistance when needed rather than a large group lecture which, these students pointed out, does not often work for them.

Three different levels of student work are provided from the plans and analysis of the other work submitted so evidence can be gleaned for what was able to be achieved on the task by students with learning support and those who struggle with mathematics. Figure 37.4a shows work from a regular student in the class. While there were some minor mathematical errors in her work mostly to do with rounding, she demonstrated a good grasp of the concepts being studied. She was willing to include more than just rectangular-shaped rooms in the master bedroom and the kitchen, and she also included an additional room, the office, which was not required for the project. Figure 37.4b is a plan from a student who was identified as needing learning support. While the student did not include all the required rooms in the plan, she was able to demonstrate her understanding of finding area. She also successfully calculated the cost of flooring and painting the rooms she had designed. One room had a compound shape, and the student calculated its area correctly. The final plan (Fig. 37.4c) came from a Year 8 learning support student in the Year 7 class. While there were some mathematical errors throughout her project, she was still able to correctly calculate the cost of decorating her house. Her diagrams demonstrate that she was willing to try some rooms that were not rectangular in shape although the irregular shape she chose was similar for each room that had that shape.

37.5 Conclusion

According to Felton et al. (2015), "modeling helps learners develop habits of mind" (p. 343). When given the opportunity to talk about the mathematics needed for a particular task, all students are taking ownership of their learning. This is particularly beneficial for struggling learners as talking can give them the confidence to pursue the task by making the context more familiar. Rather than being given a worksheet of various rectangular shapes and being asked to find the area, the students in this class created their own shapes for which they would find the area. These shapes had a real-world context; students had a reason to calculate their areas. Having students research flooring and paint prices meant they saw real-world applications of the mathematics curriculum their school district follows. As students were progressing through the tasks, the teacher was able to fill in any concepts that were missing from the students' mathematical background. As the students were selecting what to work on themselves, those students who did not need teacher assistance had the opportunity to be creative when furnishing their house. Students who struggled with the mathematical concepts did not have to complete the same design as their more capable peers. Instead, struggling students could create a design that was manageable for them and which they could calculate with teacher assistance. The key to an assignment like this is that while all students worked towards the overarching goal of the lesson, each student could make individual progress. The use of the task adaptation criteria of McDuffie et al. (2011) has afforded this outcome for this study as evidenced above.

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Chapter 38
The Hidden Benefits of Mathematical Modelling for Students with Disabilities

Rina Scott-Wilson, Dirk C.J. Wessels, Helena M. Wessels, and Estelle Swart

Abstract The impact of learning via mathematical modelling tasks on students with disabilities in a remote part of Australia was studied to answer the question: Can students with disabilities learn through mathematical modelling tasks? Daily mathematics lessons were substituted with a set of modelling tasks for 1 month. A design-based research methodology with a neo-Vygotskian design philosophy and Feuerstein's theory of structural cognitive modification was coupled to three intensive case studies to monitor how these students with disabilities responded to modelling tasks. Findings showed evidence of engagement and meaningful mathematical learning. "Hidden benefits" of modelling for students with disabilities were the development of literacy, social skills practice around collaboration and social negotiations, and support for the development of potentially more robust thinking operations.

Keywords Modelling tasks • Students with disabilities • Neo-Vygotskian design philosophy • Feuerstein's cognition processes • Hidden benefits

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The original version of this chapter was revised.
An erratum to this chapter can be found at https://doi.org/10.1007/978-3-319-62968-1_53

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_38

38.1 Introduction

The intention of this study is to open the discourse on what an appropriate mathematical modelling pedagogy (Jacobs and Durandt 2017) looks like for students with disabilities. As teachers go about developing their unit plans term by term, creating and implementing mathematical modelling tasks are not typically at the forefront of their minds when it comes to students with disabilities. This is hardly surprising in the wake of the strong emphasis on explicit instruction for students with disabilities (Ellis 2005; Mitchell 2014). To some extent, it is to be expected when considering the conspicuous gap in research on the application and usefulness of modelling for students with disabilities. The current philosophy of inclusion and social justice in schools makes it an opportune time to reflect on what it is about modelling, or students with disabilities, that warrants these visible silences in practice and research.

Before continuing, it is worthwhile addressing the documented challenge of the vast scope of strengths and vulnerabilities, diagnoses, and educational profiles that emerge when researching and writing about students with disabilities (Norwich 2013). The high levels of diversity amongst this cohort make it difficult to synchronise local and global data into a more coherent and comparative understanding of mathematical development for students with disabilities. This study focused on students with disabilities who met the outcomes of a diagnosed cognitive impairment and low adaptive functioning in several areas and who need an individual educational adjustment plan and intensive support to access the curriculum. These criteria are taken from the policy documentation of the Northern Territory Education Department (Department of Education and Child Services 2012), where the study took place, and are arguably still very wide for clinical research applications. On the other hand, they authentically reflect the complexity of fitting students with disabilities into preset categories for research purposes.

38.2 Theoretical Orientation

The Elements for Inclusion stipulated by the United Nations Educational, Scientific and Cultural Organization (UNESCO 2005) influenced the design of the study. When comparing UNESCO's Elements of Inclusion (i.e. all students having access to formal curricular activities, engaging in collaborative learning, achieving in subjects, and having their barriers to access identified and addressed) to the current modelling opportunities for students with disabilities, it becomes clear that these inclusion criteria are being obscured by the lack thereof. In the interest of social justice, there is a call for a new dialogue – an epistemological space that responsibly connects students with disabilities with modelling tasks at a developmentally sensitive level.

According to authors of modelling (Blomhøj and Jensen 2003; Doerr and Pratt 2008), modelling is about interpreting and finding solutions to everyday life situations mathematically through building and testing models. A complex problem is placed in a culturally meaningful real-life setting. Learners work collaboratively to identify the problem; imagine, create, and implement a solution; and then evaluate and modify it through feedback. The primary objective is to use contextualised mathematics tasks that are experientially real to learners as a stimulus from which to generate formalised and decontextualised mathematical principles. The intention of the study was to introduce learning through modelling tasks, which is considered to be different from using word problems (Schroeder and Lester 1989). Using word problems is understood to involve the teaching of decontextualised mathematical procedures beforehand, followed by a contextualised scenario which must be solved by producing a right answer using the learnt skills, given rules, and practised operations. In contrast, the approach to modelling used in this study allowed no teaching beforehand. Learners were given the problem and had to learn the mathematics through solving the problem. They had to draw on their own strengths and intuition, develop their own strategies, and adjust them following conversations with peers while working together in small groups, and feedback from presenting their ideas to other groups, to find a suitable solution.

The design of the modelling tasks and the expectations for the learners and the nature of their models were influenced by authors who have investigated early modellers and the qualities of the models produced by this cohort (Brown and Stillman 2017; English 2006; Lehrer and Schauble 2000). These authors agree that early modellers or young modellers will produce models that will express idiosyncratic and unstable conceptual systems, influenced by personal experience. For example, Lehrer and Schauble (2000) describe how typically “beginner” models will be more fragmented and one sided, ignoring rival models, missing important objects and their relations, and reverting to familiar personal aspects. Moreover, they argue that early models may include physical models using concrete objects or representative models such as drawings and maps rather than hypothetical-deductive models (Lehrer and Schauble 2000). Where complex tasks are structurally quite long to decompose, modelling for developmentally young learners can still include a form of scaffolding to guide the learners through the task deconstruction process, without overriding their intuition or imposing strategies onto them Paolucci and Wessels (2017). On balance, considering the developmental level of the students, far more elementary, emergent and early forms of models, which differ in nature, content, and process to their more sophisticated adult counterparts, were anticipated. The focus of early modellers is on the progressive development of models and their associated mathematical, cognitive, and social abilities instead of on mastery. Moreover, whereas a recognised goal in modelling is to produce a model that is generalisable and reusable, the goal of this study was to go beyond prototypes, for example, by designing a grid reference as a type of generalisable product, to embody Brown and Stillman’s (2017) interpretation of modelling as an approach to life – not only handling a mathematical problem but as a way of, and tool in, understanding the world. Zawojewski (2010) reminds us that mathematical content and the development of higher-order thinking should be considered as developing interactively. Feuerstein’s theory of structural cognitive modifiability (Feuerstein et al. 2010) was used to identify which cognitive operations may need strengthening to assist learners to participate in modelling tasks.

38.3 The Study

The research was conducted in a remote setting in the Northern Territory of Australia in a middle years' state school for students from Year 7 to Year 9, where the unit for students with disabilities occupied a wing of the school. The development of the study was strongly influenced by the *Index for Inclusion* (Booth and Ainscow 2002). The first three stages of the *Index for Inclusion*, namely, researching existing knowledge and deepening the inquiry, getting to know various aspects of the school and the community in-depth before deciding on priorities for development, and then matching these priorities to the school's developmental plans, were developed over 2 years before the implementation in the classroom. The primary research question was investigating if there was evidence of mathematical learning taking place when students with disabilities engage in modelling tasks. Simply put, *could students with disabilities learn through mathematical modelling tasks or not?* Data for students with disabilities in respect to modelling is so scarce and for learning taking place through direct teaching so strong (Ellis 2005) that it could not be assumed at the start of the research that learners would in any way benefit educationally from mathematical modelling. This information had to be established as a baseline through the research. Moreover, considering that students with disabilities typically learn at a slower rate than their mainstream peers and that they are commonly already developmentally behind in their mathematical milestones, it was thought unethical to introduce a long study which may compromise their learning should the modelling tasks not suit their needs. For this reason, the school agreed to one unit or a 5-week investigation period (half a term). For a month, learners worked on three modelling tasks each day during their usual mathematical lesson (varying between 40 and 80 min per day). Students spent approximately 6 days on each task, working through the cycles of feedback and refinement. The study adhered to ethical requirements that were developed collaboratively by an Indigenous community advocacy group, a government school-based ethics group and an academic ethics review committee. An Indigenous disability advocate was asked to work directly with the students and to mediate between the students and the teacher to control for teacher-student power imbalances.

While the students with disabilities were working on the modelling tasks, the teacher-researcher was working through the usual cycles of a design-based research (DBR) of design, implementation, reflection, and adjustment and deriving broader principles for practices such as described in Reeves' (2006) work. The DBR processes were coupled to three in-depth case studies allowing for a detailed description of the students' responses to modelling tasks. Specific attention was given to designing and differentiating modelling tasks to support the learners' strengths and to create tasks which were developmentally appropriate, age appropriate, and culturally sensitive. Care was taken to identify and address barriers to learning before and during the investigation. For this purpose, the modelling tasks were evaluated through external collaboration and consultation with a fellow disability practitioner, a cultural advisor, a disability advisor, and a mathematics education expert during the design phase before implementation.

The class consisted of eight students with disabilities who participated in the modelling tasks. From these, a subset of three cases was selected on the basis of maximum variance in terms of range of disabilities (autism spectrum disorder, foetal alcohol spectrum disorder, and global developmental delay), in addition to cognitive impairment, in terms of gender (male and female), and in terms of mathematical attainment (high and low performers in mathematics at school). Indigenous students were included in the study, with an ethical agreement to focus only on matters related to disability. Given that a qualitative research methodology was used, subjective research bias was covered through aspects such as collaborative monitoring, seeking negative evidence, and triangulation. Data were collected and triangulated through interviews with the students, work samples, and voice and video-recordings of the learners participating in their groups. Data from the case studies were analysed with respect to pre-determined questions: What evidence of learning can be found in the analysis of learners' reasoning and representations over time? What strengths and assets emerge from the learners during the activities? What barriers emerge and how were these addressed? Which of the primary cognitive functions as identified by Feuerstein emerge and which remain absent? This was coupled with data from the student interviews on how much learning the students themselves thought they were gaining from the tasks, and lastly, the modelling activities were assessed against a current programme evaluation model.

38.4 The Modelling Tasks

In accordance with the Vygotskian/neo-Vygotskian ideals, the three modelling tasks were developed with two aspects in mind, namely, helping learners acquire the appropriate mathematical content and strengthening vulnerable cognitive operations that will assist in mathematical learning. The mathematical content was taken from the Year 1 to 3 descriptors of the Australian Curriculum: Mathematics (ACARA 2010) based on the school data that all students in the study tested at a Year 1 level for mathematics, except for one student who tested at a Year 3 level. The mathematical units taught by the school for the period of the study were location, direction, and shape. Learners worked in small groups of two to three for each task.

For the first task, students planned a *Scavenger Hunt*. This task involved deciding on a treasure within a specific budget and finding a suitable location to place the treasure at school, representing the information in a basic map, and then working out directions from their current location to the selected treasure spot. Cognitive operations listed under Feuerstein et al.'s (2010) elaboration phase were included such as being able to think forward by planning all aspects of the hunt and using logical evidence as support for their plans. For the second task, students were given a combination lock with a dial designed to look like a bomb. Their task was to *Diffuse the Bomb* by working out the code that unlocked the mechanism. The code was the numbers on the dial, the direction of the turns, and the number of turns to get to the numbers while demonstrating the meaning and importance of words such as "clockwise" and "anticlockwise" and being able to identify and describe one-half and one-fourth turns. All operations under the input phase of Feuerstein et al. (2010) were addressed, including being accurate in their recording of the data and being able to use more than one source of information simultaneously by combining numbers, the distance of each turn, and the number of turns. The third task, *Mystery Location*, was to create a map of the school from an aerial view of the site in Google Earth, first by building a top view representation of the school with blocks and then by drawing it. Students presented their designs from their groups and then selected the most accurate representation of the school by debating choices with others. The next step was to overlay the selected map with a grid reference system of their own design. Students were asked to use coordinates to show key positions around the school and then give the coordinates of particular locations of their choosing to the other team. Based on the grid reference system and the coordinates provided, the second team had to work out the "mystery" locations and fly a remote controlled helicopter there. All Feuerstein et al.'s (2010) cognitive operations from the output phase were included. Examples include being able to consider another person's point of view in the debate on which map to choose, giving thoughtfully worded responses to justify their own choice of design, and persevering with all tasks.

Several authors such as Lesh and Doerr (2003) discuss modifying versions of modelling cycles for students. In the study, students participated in an adapted version of the typical phases of modelling of creating and testing a model of a given situation, receiving feedback on it, and revising it based on the feedback. To illustrate their learning, students had to develop a model, namely, directions to a treasure, instructions on how to disable the bomb, and a grid reference system, and then present their model to the second group. They worked in small groups of two to four, solving the problems together. The tasks were open ended, but some scaffolding was provided in the third task by directing learners to first build models of the school layout with blocks. The adaptation was that the second group's *enactment* of their directions provided them with feedback on the accuracy of their model. If the second group could not find the treasure or unlock the combination, the first group had to consider whether their model contained inaccuracies that needed correction and make the necessary adjustments or whether the second group was not following the given model and provide them with appropriate feedback.

38.5 Discussion

The discussion focuses on firstly evidence for the primary research question that students are learning mathematics aligned with the national curriculum standards and secondly student development of more robust cognitive structures and essential literacy and life skills.

38.5.1 Evidence of Mathematical Learning

Modelling is about the learning of mathematics. The first recognition was that learners engaged with the tasks. Evidence of mathematical learning was taking place insofar as students were achieving the specific learning objectives matched to their measured levels and slightly above. For example, Fig. 38.1 shows that a student whose initial data placed her at a Year 1 phase met the outcomes for this stage by using directional words and phrases. The next step for the student is to include distance in her model. Additional examples in the main study (Scott-Wilson 2014) showed students meeting Year 2 and Year 3 criteria partially or fully as seen in their directions for the combination lock, students' attempts to create a top view of the school, and in their own grid reference designs.

38.5.2 Learning Through Mathematical Modelling

The first indicator that students are beginning to model according to authors such as Lehrer and Romberg (1996) is that they develop referents that are separate from the object(s) in reality. In each of the tasks, learners in this study produced a product that described and contained elements of the real world, but that was separate from it. As predicted by Lehrer and Schauble (2010), their products were not delivered as a sophisticated hypothetical-deductive explanation but as physical objects such as representing a bird's eye view of the school by building the model with blocks, then by drawing it, and finally by creating a grid reference. Through a Vygotskian lens (Vygotsky 1931), the social interactions and the products or early models being produced by these activities were exposing them to a range of psychological tools through various sign and symbol systems, including drawings, maps, grids, spoken and written language, figurative language, and mathematical symbols. These developments support a well-recognised aspect in literature, namely, that in creating these referents or emergent models, students acquired familiarity of multiple representations of the same information to show their thinking processes.

Modelling is a social initiative. In this context, learners particularly struggled with the notion of "interthink" (Mercer 2002) – using dialogue to produce models together. At the beginning, they typically tried to solve the problem on their own and then called for the teacher-researcher to explain their individual solutions to her, looking for confirmation. Throughout the process, learners were actively encouraged to take their ideas back to their team and to present it there and were given pointers on how they could possibly work together. For example, learners were reminded that to work out the dialling code for the combination number lock, one learner could turn the dial and report the numbers, another watching the back of the device to see at which point it unlocked the rotor, and the third one recording results. For Vygotsky (1931), collaborative processes are important precedents of cognitive development as these lay the groundwork for intellectual adaptation. As the study progressed, early markers showed a shift in that students were beginning to engage

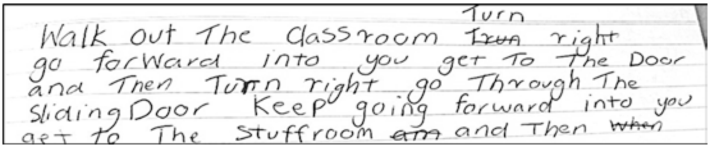

Fig. 38.1 Example of student work meeting mathematical outcomes differentiated to her data

in meaning making processes and were trying to co-construct cultural knowledge and practices with others from the group. To illustrate, the student in the extract below is beginning to work with his peers, asking them questions and assimilating their responses.

Student B: Where is the garden (of the school on Google Earth)?
Peer: Look at the date. That was 2011. Even my home looks very different now to then.

Once their model had been created in their groups, students had to share it with the other groups, which meant considering the views of other classmates at an even broader level and justifying their own thinking to a wider audience, thereby developing communication skills such as negotiating and dealing with disagreements.

Student A: I have a question. Where is the assembly hall?
Member of another group: It did not fit in the picture so we left it out.
Student A: I can see the picture perfectly in the other picture. So that picture looks a bit better than that one. The assembly hall is a big thing.
Member: It is our group's turn not your group's turn.
Teacher: No, that group has the right to question your group.
Student A: So where is the assembly hall?
Member: [Getting upset] It's none of your business.

Vygotsky (1931) argued that higher functions of thinking begin in the group life of children in the form of arguments and that reflective behaviour is generated in students as a result of arguments. He believed that as students increasingly internalise these types of experiences they develop authentic thinking. Part of the argumentation process is recognising that ideas need to be revisited and re-edited as errors become obvious. As explained earlier, given that the learners were not strong in language, at times, learners acted out the instructions of others, instead of talking about their ideas in front of the class. To illustrate, learners had to give their model of the combination lock code to the other group and then watch as the other group enacted the model by turning the dials. As the other group got stuck and could not open the lock, learners had to consider whether the other group was indeed following the instructions correctly or whether their model contained possible errors. One group edited their model of the instructions three times before the other group was successful in "disarming the bomb".

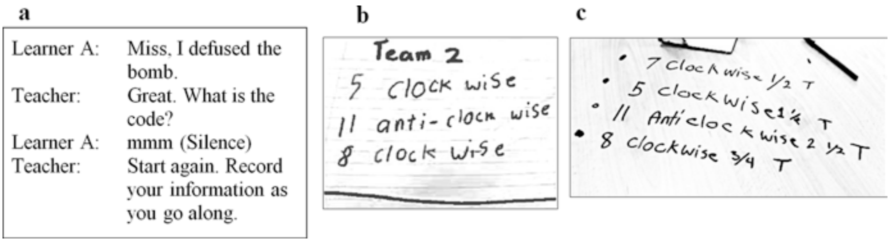

Fig. 38.2 Student's work showing progression in strengthening cognitive function of combining multiple pieces of data simultaneously: (**a**) no record of data, (**b**) two pieces of data combined, (**c**) three pieces of data combined (number on dial, direction of turn, amount of turn)

A significant spin-off for learners trying to represent, then share their models with others, was language development. Spelling and writing progressed as students were recording their mathematical ideas through writing (see Fig. 38.1). Students also engaged with the symbolic representations of mathematical ideas, trying to make sense of given symbolic representations, or to represent their own thinking symbolically, in simulated physical and lived environments, for example, following directions around the school. This attempt to connect everyday activities to mathematical challenges embodies some of Brown and Stillman's (2017) desire to see modelling extending into, and becoming an approach in, and to life.

Student following a clue: Miss, it says turn 90° [ninety degrees] right. That's funny. We should turn 900 times right! What the heck?

Figure 38.2 shows that, aside from reaching mathematical objectives, the students' emergent ability to work with multiple sources of information simultaneously was being developed and strengthened. From a Feuerstein et al. (2010) perspective, these types of developments are evidence of the strengthening of vulnerable cognitive functions which in time will support further mathematical learning. This aligns with Zawojewski's (2010) ideal of working with content and cognition interactively.

38.6 Conclusion

Evidence from this study suggests that modelling activities afford students with disabilities learning of mathematical concepts when tasks are differentiated to match their measured level of performance. At the same time, the qualitative nature of this research implies that the results cannot be generalised. The main premise of this chapter was to argue the case that instead of being dismissive of modelling as a learning approach for students with disabilities, it would be worthwhile to conceptualise and research a mathematical space alongside direct teaching that allows students with disabilities curricular access to modelling activities with a real-world application. Social justice requires a continual interplay between working with individuals by strengthening their modelling capacity in its social, cognitive form, and symbolic forms, and it requires challenging the larger education system to research and adopt a broader framework of intervention which include an appropriate modelling pedagogy for students with disabilities.

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Chapter 39
Scaffolding Complex Modelling Processes: An In-Depth Study

Peter Stender, Nadine Krosanke, and Gabriele Kaiser

Abstract The support of students during their work on complex modelling problems is an ambitious process, especially if the students work as autonomously as possible. Scaffolding as a theoretical construct to describe how teachers should act in these situations, so that students can solve the problem as independently as possible, has proven to be adequate for empirical studies. In the research project presented, the activities analysed were those of future teachers working as tutors supporting students working on complex problems over 3 days. The tutors were educated beforehand in pre-service teacher seminars and had learned special scaffolding measure activities for this small group work. Based on an analysis of videotaped modelling processes, examples of successful and unsuccessful teacher activities are analysed. Finally, examples of appropriate strategic scaffolding measures are presented.

Keywords Complex modelling problems • Modelling days • Scaffolding • Scaffolding measure • Teacher intervention • Minimal support • Adaptivity

39.1 Introduction

Already for decades, the competency to solve real-world problems with mathematics is emphasised as one of the core competencies in mathematics education in many curricula around the world, and various approaches for the implementation of mathematical modelling in schools are proposed. Although there exists consensus on the relevance of mathematical modelling in schools, modelling examples still do not play a high role in everyday teaching in many parts of the world, amongst other reasons, due to the fact that teaching and learning processes become more difficult and less predictable and the design of the learning environment is more ambitious

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_39

(for an overview, see Kaiser 2016). Especially, when complex, authentic modelling problems are treated, the role of the teacher becomes highly demanding. It is still an open question, how the teacher can, and will, support students within their modelling processes, especially when independent working of the students is to be fostered as called for in many curricula. As previous studies have shown, adaptive support by teachers guiding the students on their own way can seldom be identified in classrooms (Leiss 2007).

In the following, we will describe a study, in which students solved complex modelling problems within a learning environment fostering their independency. The students tackled the problem together with other students working in small groups, only tutored by (future) teachers, educated for adaptive support. As the theoretical framework for the present study, the scaffolding approach has been chosen, which seems to be especially appropriate for the kind of adaptive teacher activities necessary for independent modelling activities.

39.2 Theoretical Background of the Study

39.2.1 Scaffolding as Theoretical Basis of the Study

The concept of *scaffolding* was originally introduced by Wood et al. (1976), who described it as a form of fostering a problem-solving process of a single child by a single tutor. As the aim of this scaffolding process, they described that the child solved a problem as independently as possible and received support from an experienced person only in situations where independent work due to non-existing knowledge or skills of the child was not possible. Within a problem-solving phase, the tutor intensified or reduced her interventions, depending on the child's ability to work further on independently or not (Wood et al. 1976, p. 92).

The term scaffolding has been extended and adapted over time in a variety of ways, which is described in the extensive survey paper by Van de Pol et al. (2010) with an overview on the current discussion on scaffolding. Their model relies on three important aspects of scaffolding, namely, *contingency* (responsiveness, tailored, adjusted, differentiated or calibrated support), *fading* (gradual withdrawal of the scaffolding) and *transfer of responsibility* to the learner, while diagnostic strategies play an important role in the whole process. The development of the ability to take over responsibility only develops over a longer period of time, and according to this, *fading* should be seen as a long-term process.

39.2.2 Teacher Activities to Promote Independent Student Activities

An approach developed by Zech (1996) within the problem-solving discussion proposed a step-by-step approach to support students with minimal help at five different levels:

1. Students are motivated only in a general way.
2. Positive feedback is given based on successful intermediate results.
3. Strategic support is given which takes the form of hints that refer on how to proceed without addressing content-related issues.
4. Content-related strategic support is offered; these are interventions, which also relate to the procedure, but content-related issues are involved.
5. A content-related intervention is completely related to the content of the task and contains the core of the solution.

In this differentiation, the first two supportive measures mainly encourage the students, while the last three interventions give support related to solution methods or the content of the task. Of these three interventions, the strategic support plays a prominent role as the students are only supported to find a way to go on, but the solution itself must still be developed by the students themselves. The use of strategic support is for the intention of partly independent work by the students and is one possible approach to realise the aspects of scaffolding (fading and transfer of responsibility) and even contingency if the strategic supports rely on subtle diagnosis.

39.2.3 Mathematical Modelling in School

As already mentioned, mathematical modelling plays an important role in mathematics education all over the world. There exist various conceptions of mathematical modelling; in this study, we understand mathematical modelling as comprising the following important steps from the real-world situation into the mathematics: a real-world problem is coming up, which has to be understood and simplified. This leads to a real-world problem, which is then translated into a mathematical model. Mathematical work within the mathematical model leads to mathematical results, which are translated back into the real world. The real-world result is validated, whether the result answers the original problem adequately or not. If this is not the case, the modelling cycle has to be carried out again until a satisfactory solution is produced. This approach can be visualised with a modelling cycle (Fig. 39.1), which can serve a metacognitive means for classroom activities.

While working through the modelling cycle different sub-competencies are needed (for a detailed description, see Kaiser and Brand 2015), amongst others:

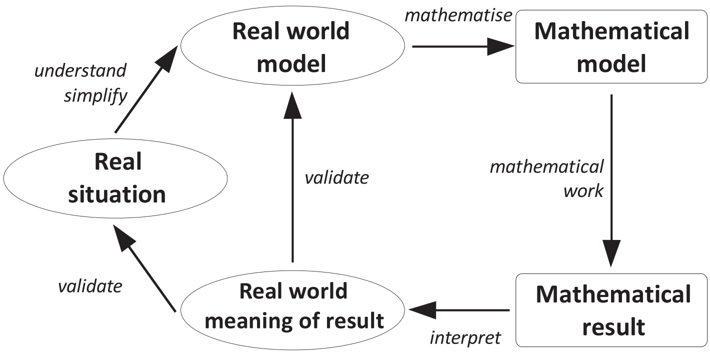

Fig. 39.1 Modelling cycle (Kaiser and Stender 2013, p. 227)

- Competence to understand a real-world situation,
- Competence to develop a real-world model,
- Competence for establishing a mathematical model out of a real model,
- Competence to solve mathematical problems within a mathematical model,
- Competence to interpret mathematical results in a real-world model or a real-world situation,
- Competence to validate the solution in the real-world model or the real situation and, if necessary, do another loop in the modelling process.

In addition, the metacognitive competence to understand your own work and to control your own work is central. For acquiring these competences, it is essential that students work independently on their own. However, when students begin modelling independently, the task to do everything on their own is too complex, so monitoring by a teacher is indispensable. The previously described approaches to scaffolding are an important concept for the implementation of this kind of support.

39.3 Research Aim, Design of the Study and Modelling Problem Used

The aim of this study is to identify empirically appropriate teacher interventions for scaffolding in situations where teachers are tutoring students who are solving complex, realistic modelling problems. As the research environment, the so-called 'modelling' days were established, in which the students work independently supported by tutors. The 'modelling days' are a learning environment where students (15 years old) work for 3 days on one single modelling problem chosen by themselves, and the work takes place in small groups of students. Two future teachers having been educated in special master seminars acted as tutors for two groups.

In the in-depth study presented here, students and teachers worked on the following problem: *Roundabout Versus Traffic Light*. At what kind of an intersection can more cars pass a crossing? This problem allows different approaches. If there are any intersections nearby the school, traffic counts at these crossings could be done, but analytical considerations can also be carried out, which was suggested in the observed learning groups. Two fundamentally different assumptions for the work on the problem can be made:

- The maximum number of cars passing the intersection or roundabout depends on how fast the vehicles drive *through* the intersection area.
- The maximum number of cars passing the intersection or roundabout depends on how fast the vehicles drive *into* the intersection area.

In a first approach, it makes sense to assume the maximal possible symmetry in the situation, that is, from all directions come the same number of cars and the drivers want to go in all directions with equal probability, with velocities and accelerations being the same for all cars. The crossing is a simple four-road intersection, where the traffic light gives way only for one direction at a time. The restrictive assumptions can be reduced during the modelling process, to obtain a more sophisticated solution.

For the case of the traffic lights using the first approach, students have to identify the possible ways through the intersection and then calculate the time a car needs to pass the crossing in three different directions that means with different radii of curvature. These calculations are mostly done with constant speed in the first approach considering acceleration in further work. Students may then calculate the average time to drive through the intersection based on three times calculated before. Using this average time and estimated times for red, yellow and green phases, the number of cars passing the intersection in a certain time is calculated and seen as the capacity of the intersection.

In the second approach focusing on the time until a car is entering the intersection, one has to calculate the starting time of a whole queue of cars, when the traffic light switches to green, which leads to the number of cars that can enter the intersection during one green phase. This calculation deals with constant and accelerated movements and the time a car has to wait until the necessary distance to the car before occurs. One also needs to take into account that the cars at the end of the line drive with constant speed according to the speed limit after a phase of acceleration. The processes in the roundabout are quite complex and can be simulated, which leads to a deeper understanding of the roundabout process (for further details, see Stender and Kaiser 2016).

39.4 Methods

In an in-depth study (Beutel and Krosanke 2012), the entire working process of one group of students was analysed and reconstructed. The reconstruction provided important knowledge about the students' solving process and about teachers' behaviour in modelling processes. The effects of interventions in the context of the complete modelling process were analysed, distinguishing short-term and long-term effects. Based on transcribed video-recordings, the material was analysed using the methodological approach of qualitative content analysis (Mayring 2015).

The segmentation of the entire solution process at various levels, which is the result of the analyses, has been visualised over time (see exemplarily in Fig. 39.2). To ensure reliability of the methodical approach two researchers coded the material separately and discussed non-conformities.

The main categories of analysis were developed inductively and deductively (these are shown in the seven rows of Fig. 39.2):

- Thematic topics related to the modelling problem (code: paraphrase of the work);
- Type of intersection the students are working on (codes: students are working on the traffic light or on the roundabout or on both comparing them);
- Subtopic (code: according to subtopic coding scheme developed inductively);
- Phases of the modelling cycle (codes: six phases of the modelling cycle);
- Timeline in minutes (Fig. 39.2 shows an example lasting 12 min);
- Working behaviour (codes: independent student activities, reluctant participation, nonworking phases);

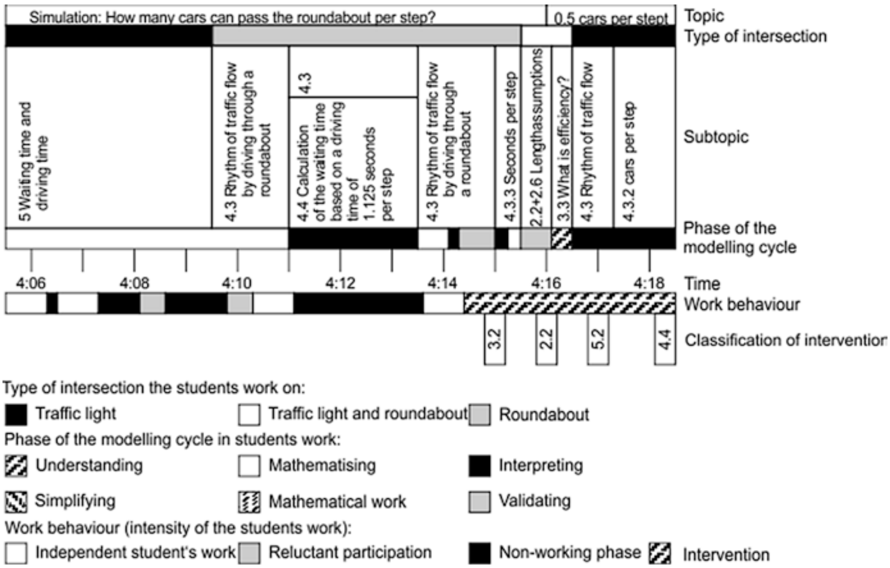

Fig. 39.2 Visualisation of the reconstructed modelling process

- Classification of the intervention (code: according to coding scheme of intervention, e.g. invasive or responsive, non-verbal intervention, motivational help, feedback help, general-strategic help, content-oriented strategic help, content help, organisational matters, discipline problems);
- Intervention as a trigger of metacognitive processes such as procedural and/or declarative (not included in the visualisation shown in Fig. 39.2).

The visualisation helps to identify specific characteristics of the solving process by considering only one level of the reconstruction at a time. On the other hand, the different levels can be considered at the same time to identify interactions between the students' modelling process and tutors' behaviour. Due to the restriction of the observation to only one small group of students with five students, the generalisability of the results is limited.

39.5 Results

In the following, we will concentrate on the interactions between students' solution processes and tutors' behaviour.

39.5.1 Effects of Tutors' Interventions on the Working Behaviour of Students

The analyses show that non-verbal interventions had no negative impact on the way students work in the group. This means that just the fact that tutors looked for the kind of work the students were doing without speaking to them, but noticed by the student, did not disturb the students' work.

Mostly success can be observed, that is, students take up their work again after a standstill; however, several special incidents happened. For example, in the middle of the second day of modelling, a long period of nonworking occurred, which arose after a special intervention. The students had completed their first modelling cycle, and the tutor wanted them to validate their solution and to start a new modelling cycle. The tutor therefore pointed to the validation phase in a modelling diagram hanging on the wall telling them that they were in that phase of the modelling process and asked them to think about the results once more and then start again. This intervention was followed by long nonworking phases of the students and a dominance of intervention phases compared to independent activities. Apparently, the students were not able to improve their model by going through the modelling cycle another time or were not motivated enough to restart the modelling process, maybe because they felt their previous work was not valued enough. This incident highlights that restarting a modelling cycle after validating the results achieved is a complex phase within modelling activities and needs sensitive teacher interventions. Furthermore, this incident shows that considering only the working behaviour directly after an intervention as an indicator of successful assistance is not differentiated enough. It must be taken into account that interventions may have as a long-term process other effects, showing the success of minimal and adaptive support over time. In the following, we report the effects of teacher interventions differentiating between short-term and long-term effects of the tutors' interventions.

39.5.2 Short-Term and Long-Term Effects of Interventions

Two interventions that took place within the first hour of the modelling process observed seem adequate for the analysis of long-term and short-term effects. The students were working on understanding the modelling problem and on making assumptions about the flow of traffic at crossroads. The analysis of the incident shows that the students were unable to assess whether they were allowed to make assumptions and, accordingly, for which variables of the specific modelling problem it was adequate to make assumptions.

Eric: Are we allowed to come up with numbers? (all students looking to the tutor)
Tutor: Yes. So what kind of numbers do you want to come up with?
Eric: Oh, well – there is an intersection and from the right the cars come from the countryside driving into the town. That are eight percent. Okay?
Tutor: If you all agreed on it this is okay in the first run.
Eric: Yes (nodding).
Tutor: You together have to decide that.
Eric: So eight percent (addressing his group) – even it's not so good for calculation?
Anna: No.

This dialogue led to various assumptions by the students. Analysis of students' activities immediately after this intervention showed the response could be assessed as successful, because the students continued to work and developed assumptions which they could not have made before. Finally, the students' assumptions led to a real-world model, which was too complex to solve, so the concentration on work decreased and the intervention was rated as unsuccessful. This last nonworking phase led to another intervention: The students presented their work, and the tutor expounded the problems of the students' assumptions, not by rating them but in a way that the students were in need of an explanation of the assumptions made. This is what we call a *strategic intervention* and in terms of scaffolding transfers responsibility of the work to the students. Within this explanation, the students realised that their assumptions had a high complexity and were thus encouraged by the tutor to simplify them further. Due to this intervention, reasonable assumptions of the general conditions at one crossroad were developed, but the students also estimated the number of cars that pass through the roundabout per minute. As this influential factor needs to be calculated and not estimated, this was not yet an adequate simplification of the real situation and led to a phase of uncertainty. The effect of this intervention can again be rated differently, namely, as success in the short term but with problematic long-term consequences. From the perspective of scaffolding, the students took over responsibility and worked independently for a while, so this strategic help was successful with respect to scaffolding.

There are also interventions which were completely ignored in the short term by the students but were taken up in the long run. For example, prepared material for the simulation of the roundabout was handed out by the tutor, but the students did not use it immediately. After a short while, the material was used independently and the results were very important for the modelling process. This intervention had hardly any short-term effects but had a long-term success.

To summarise, these results indicate that the success of teachers' interventions can be evaluated differently considering short- or long-term developments. The tutors have to consider these different aspects while intervening, which points to the difficulty of real adaptive interventions. In an attempt to develop a definition of success of interventions at different levels, metacognitive processes, which have been triggered by an intervention, have proven to be important. Effects at the declarative level (concerning learning strategies, person and task characteristics) as well as at the procedural level of metacognition can be identified. Strategic interventions are mostly the trigger of such processes, but they can also be an effect of content-related help and feedback. If feedback was identified as the trigger of metacognitive processes, feedback had been given in combination with a content-related help.

39.5.3 Improvement of the Competence 'Simplifying'

The students dealt with complex modelling problems for the first time, that is, they were modelling novices. This meant, amongst other things, that they were not used to dealing with complex modelling problems. As already mentioned, they had especially no experience with adequate simplifying. Many difficulties in understanding which assumptions and simplifications were adequate and which were not can be identified. After the interventions described in Sect. 39.5.2, 11 other interventions concerning simplifying could be identified. In more than half of these interventions, only already existing assumptions were presented, and no new ones were developed. Even in other interventions observed, mainly no simplifying activities could be identified. However, on the second day, the students gained competency in simplifying and to know which assumptions may be adequate during a modelling process. For example, the assumption of how many cars can pass the roundabout per minute, which had been used on the first day, was explicitly rejected on the second day. To conclude, low external influence – that is, only strategic help by the tutor – and a long time of working autonomously on one problem seem to be adequate to increase the competency to simplify in the group and thus the transfer of responsibility to the students was successful.

39.5.4 Focus on the Intervention: 'Present Status of Work'

The intervention *to present the status of the work and the results* is the intervention, which was identified as given the most in the present study. As in previous studies (Kaiser and Stender 2013), its high potential for students' metacognitive activities and for teachers was confirmed. One consequence of this particular kind of intervention can be the promotion of reflecting and structuring the current results and current activities. Success was identified in terms of progress in the modelling process by the following effects: the solution of a partial problem, the realisation of the importance of obtained results and thus their inclusion into the modelling process and the verbalisation of previously intuitive insights. For example, the students formulated the result of their mathematical simulation of traffic flow on a roundabout, which is essential for the solution process, within the intervention *present status of work* for the first time. In another situation, the students tried to use their previous results to solve the interim problem: 'How many cars can pass through the roundabout per minute?' After a long time of nonworking, the intervention *present status of work* helped the students to remember an important result, which had already been obtained earlier. A short time later, the students were able to solve their interim problem adequately.

39.6 Conclusion

To summarise, the study presented confirms the usefulness of special kinds of intervention as a diagnostic tool for teachers. The study confirms the high relevance of diagnosis activity in order to give adaptive assistance to the students' problems. The importance of diagnosis-based support could be identified in exemplary interventions. For example, a wrong diagnosis led to the breakdown of the work by the students, although the work may have made sense in terms of working independently. In addition, a misdiagnosis may lead to an inadequate assistance by the tutor, which might cause much confusion with negative impact on the working behaviour or may lead to boredom by students, because the tutor refers to aspects already considered sufficiently by the students. In conclusion, the intervention 'present status of work' can be considered by teachers as a powerful scaffolding measure at the beginning of every intervention in complex modelling processes, because it has the potential for a positive impact on the solution process in various respects.

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Chapter 40
Long-Term Development of How Students Interpret a Model: Complementarity of Contexts and Mathematics

Pauline Vos and Gerrit Roorda

Abstract When students engage in rich mathematical modelling tasks, they have to handle real-world contexts and mathematics in chorus. This is not easy. In this chapter, contexts and mathematics are perceived as complementary, which means they can be integrated. Based on four types of approaches to modelling tasks (ambivalent, reality bound, mathematics bound or integrating), we used task-based interviews to study the development of students' approaches while the students moved from grade 11 to 12. Our participants were ten Dutch students. We found that their approaches initially were either ambivalent, reality bound or mathematics bound. In subsequent interviews, the preference was maintained, and in the end, the approaches of four students were integrating. Both a reality bound and a mathematics bound preference could lead to a more advanced integrating approach.

Keywords Case study (multiple) • Complementarity (of mathematics and real-world contexts) • Context (task) • Derivative • Ideal types (of dealing with contexts) • Integration (of mathematics and contexts) • Interpreting (a formula) • Perspective on modelling (cognitive) • Longitudinal research • World (rest of the world)

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_40

40.1 Introduction

In mathematical modelling, students have to deal with real-world contexts on the one hand and mathematics on the other hand. The variety of prompts within a task activates students' knowledge of the context or their knowledge of mathematics or both. As a result, students' thinking and acting will be very dynamic and diverse.

Borromeo Ferri (2010) studied patterns in students' approaches to modelling problems, finding that students followed their own *modelling routes*. Borromeo Ferri related students' modelling routes to their learning styles, which revealed an underlying preference to task approaches. Busse (2011) also studied patterns in students' approaches to modelling tasks. He found four different types of approaches of how students dealt with the real-world context within a modelling task. Students' approaches could be ambivalent, reality bound, mathematics bound or integrating. For example, an approach was considered *reality bound*, if only extra-mathematical concepts and methods were applied. An approach was considered *mathematics bound* if the real-world context was treated as a mere decoration, and the task was solved exclusively by mathematical methods.

These four types of approaches that Busse identified are *ideal types*. Ideal types are intellectual constructions emerging from interpretative research, whereby categories are developed to describe and analyse phenomena in reality (Bikner-Ahsbahs 2015). In his study, Busse determined a hierarchy between the ideal types, with *ambivalent* at the lowest cognitive level, *reality bound* and *mathematics bound* at an intermediate level and *integrating* at the highest level. There is not a hierarchy between reality bound and mathematics bound (see Fig. 40.1). Both Busse (2011) and Borromeo Ferri (2010) found that patterns in problem solving could differ between students and between tasks. Therefore, the four ideal types are neither attributes of a student nor of a task, but they are a characterisation of how a particular student deals with a particular task.

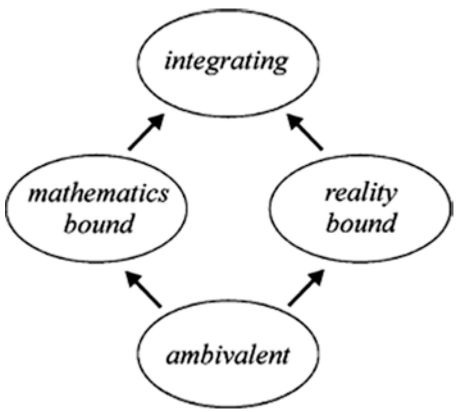


Fig. 40.1 Ideal types of dealing with a real-world context within a modelling task (Busse 2011)

Our study takes a longitudinal perspective on modelling. Instead of researching students at just one moment in their educational career, we were interested in their growth or lack thereof. The deeper aim of this research is to obtain a better insight into how students deal with real-life contexts and mathematics, what blockages and opportunities occur when students move from contexts to mathematics and back and how students develop modelling competencies. To study this, we assumed that Busse's ideal types are a characterisation of how a particular student deals with a particular task at a particular moment in time. By keeping task and students as constants, and having time as independent variable, we had as research question: how do students' problem-solving approaches when characterised by Busse's ideal types develop over time?

In mathematical modelling tasks, the dynamics of dealing with real-life contexts and mathematics occurs in particular during the phase of mathematising and the phase of interpreting. The study presented here only deals with the activity of interpreting.

40.2 Theoretical Background

Pollak (1979) conceptualised how mathematical modelling is an activity that takes place in two disjoint spheres: in mathematics and 'the rest of the world'. With 'the rest of the world', he meant all outside mathematics including nature, society, everyday life and other scientific disciplines. Other authors followed this description (e.g. Blum 2002). However, this distinction can be challenged, because mathematics can be found scattered within nature, society, everyday life and other scientific disciplines. So, it may not always be possible to clearly distinguish between the different spheres. Also, if in modelling we move between the two spheres, where are we when we are in a transition between the two? Below, we discuss the nature of this distinction.

In this chapter, we will speak of contexts instead of 'the rest of the world'. By contexts, we mean the real-life situations described in mathematical modelling tasks. A context can be more or less close to reality, and this context may be recognised and understood by students in different ways.

Pollak's (1979) original terminology suggests a dichotomy of contexts and mathematics, that is, contexts and mathematics are mutually exclusive and cannot overlap. This dichotomy is confirmed by Busse's (2011) findings, in which some students were more mathematics bound, while others were more reality bound. However, the higher-achieving students were able to integrate mathematics and contexts. This observation is confirmed by Vos and Roorda (2007), who used the term *reconciliation* of mathematics and context for a similar case, in which one of the smarter students manages to see the context through the mathematics and vice versa. Thus, a distinction between mathematics and contexts requires the option that they can be integrated.

In this chapter, we take contexts and mathematics as being complementary. Complementarity is a notion with origins in the work by Niels Bohr, who worked on a dilemma in physics, needing to integrate two conceptions of light: one as a particle and the other one as a wave. The two notions offer different ways of understanding light; they are not mutually exclusive, and they can support each other. As such, complementarity differs from notions such as dichotomy or duality. In the educational setting of mathematical modelling, complementarity of mathematics and contexts means that the two are different, but that they can be integrated and then strengthen each other. This fits Busse's (2011) ideal types, in which the highest cognitive level is termed *integrating*.

40.3 Methods

We carried out a longitudinal multiple case study with a detailed analysis of work by individual students (Yin 2003). While the students moved from grade 11 to grade 12, we administered three task-based interviews (Goldin 2000) at successive moments. In each interview, we used several tasks which were not shown to the students beforehand. The tasks were rotated between interviews, and not all tasks were used in all interviews. The study described in this chapter was part of a larger study (Roorda 2012; Roorda et al. 2015). It is based on one task, which deals with derivatives and interpreting these within a context. More details about the task are described below.

The first interview was held in the third month in grade 11, a few weeks after the mathematics teacher had introduced derivatives. Interview 2 was held 6 months later, and Interview 3 was held a year later. Between the first and the final interview, derivatives were a recurring topic in mathematics lessons and for some students in their elective subjects (physics or economics) as well. We observed that the curriculum between interviews focused primarily on calculations and did not contain interpretation tasks such as the one used in the interview. To enable comparison across interviews, exactly the same task was used, as small changes in a task can yield large differences in students' approaches. The time interval of 6 months was considered sufficient to limit inter-interview effects.

40.3.1 The Task

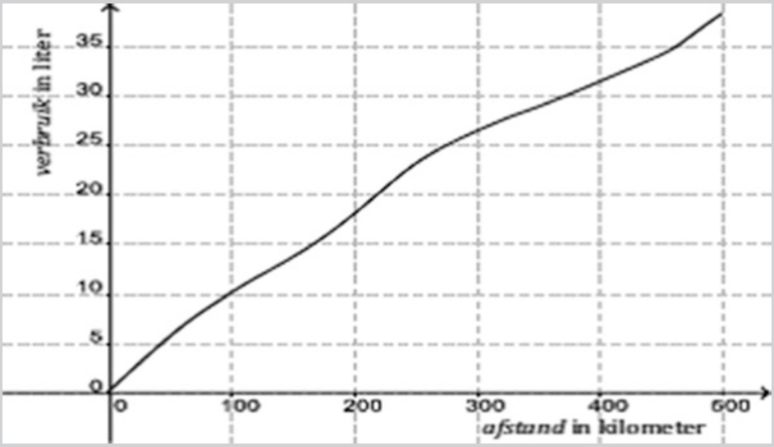
We adapted a task from Kaiser-Messmer (1986), which is set in the context of cars, petrol consumption and the distance driven. Central is a function $V(a)$ for the volume of petrol (in litres) that depends on the travelled distance a (in km). The word for distance in Dutch is *afstand*; hence, a is used for this variable. The task is rich in resources: there are different mathematical representations (graph, table), and students can address different aspects of the derivative: the average rate of change on an interval (with data from the table), the rate of change in a point, a tangent, slope, limits and so forth. Also, students can reason about the real-life context: the average petrol consumption over a distance of h kilometres.

Petrol

In a car, a measuring system was installed, which measures the petrol consumption of the car every 10 km. During a trip of 500 km, the measurements were written down. In the table, you see some of the measurements during this trip. The travelled distance is *a* (in km) and the petrol consumption is *V* (in litres).

a (km)	10	20	30	50	100	200	300	400	500
V (litres)	1.3	2.7	4.0	6.4	10.3	18.3	26.6	31.2	39.7

The measurement points were plotted into a graph by drawing a smooth line through the points.



What is the meaning of $\frac{V(a+h)-V(a)}{h}$ in this situation? (*h* is a value, which you can choose.)

Differences from the original task from Kaiser-Messmer (1986) are as follows: (1) To make the task more realistic, we added details to the context by describing a system for measuring the petrol consumption. (2) We added a table to increase variety in mathematical representations. (3) We removed a second question about the interpretation of the limit for $h \to 0$ of the same difference quotient, because this would give a cue about *h* possibly being small. This would hinder us from observing students' spontaneous reflections about limits.

The *Petrol Task* has a number of specific features. (1) Function *V*(*a*) is not given as a formula with variable *a*, from which volume *V* can be calculated. (2) The task is about interpretation and not about standard mathematical activities such as calculating or solving. (3) One can give an interpretation of the difference quotient without knowledge of the derivative. (4) The task context can be regarded as realistic (recognisable, possibly existing in real life) but inauthentic (there is no evidence of an actually existing car with such a measuring system). (5) The formula (a difference quotient) has h as additional variable (or parameter) to V and a; therefore, three symbols need to be considered, while the table and the graph suggest only two dimensions.

40.3.2 Participants, Interview Protocol and Data Analysis

We selected ten pre-university students (six boys, four girls), who took mathematics at an advanced level. The mathematics teacher had indicated one student as weak, four as average and five as good. In our study, weak students are underrepresented because we looked for students who most likely would move up from grade 11 to grade 12 without delay. The study was carried out at two schools to reduce inter-student communication about the tasks between interviews. The students' pseudonyms are Andy, Bob, Casper, Dorien and Elly from School I and Karin, Maaike, Nico, Otto and Piet from School II.

The interview started by asking the student to solve the task. During the solving, the interviewer did not interfere. If a student was silently thinking for over a minute, he or she was asked for an explication. To enhance the reasoning and interpretation process, the interviewer would ask students about the effect of the size of h in the formula. This hint could offer students the opportunity to reason about a limit. The interviewer would *not* use words that directed towards mathematical concepts, such as 'derivative', 'differentiation', 'rate of change', 'tangent' or 'slope'. By avoiding these words, we did not lead students to more mathematics than the task already did. In case a student would reason completely in terms of the situation (cars, petrol consumption, distance travelled), an additional question was whether the student had seen the formula (i.e. a difference quotient) before.

Both authors independently analysed the transcripts of the interviews and the written answers to the task, thereafter reaching an agreement on labelling students' problem-solving approaches using Busse's ideal types. We identified utterances as being more reality bound, when a student spoke about average consumption. We identified utterances as being more mathematics bound, when a student spoke about aspects of the derivative, such as rates of change, slope and decreasing difference intervals. Additionally, we coded students' expressions on a simple scale: accurate and clear – somewhat accurate or clear – unclear.

40.4 Results

Below we report on four students and their approaches to the *Petrol Task* in the three subsequent interviews. We selected these because of their illuminating differences. The approaches of the six others are reported in detail in Roorda (2012). At the end of this paragraph, we synthesise the findings across all ten students.

40.4.1 The Case of Nico

In the first interview, Nico started by saying: "*So, the steeper the line goes, the more is his petrol consumption per kilometre*". This was a correct interpretation of the graph, but not of the difference quotient. Thereafter, he interpreted $V(a+h)$ as multiplication $Va+Vh$. He remarked that he had no idea about the meaning of h. When prompted by the interviewer for a meaning of the formula, he said: "*It is the average consumption of the car, of course, what else would you want to calculate?*", but he did not link this correct statement to the formula.

In Interview 2, Nico started by thinking that $V(a)$ is a multiplication, but then corrected himself spontaneously and recognised that $V(a)$ is the petrol consumption after a km, and rewrote the formula into $V(a) + V(h) - V(a)/h$, then $V(h)/h$ and then wrote: *V with 1 unit h on average*. He explained this as the consumption after 1 km. After being prompted to further explain, he took numbers: at 100 km the consumption is 10 l. The value 10/100 is 0.1 litre per kilometre, and according to Nico, this was the average consumption. When the interviewer asked about the effect of the size of h in the formula, Nico reasoned that it does not matter, because h/h is equal to 1.

In Interview 3, Nico used the table to calculate 39.7/500 and 1.3/10 (these numbers are $V(500)/500$ and $V(10)/10$) and said that the consumption is not constant, "*otherwise the graph would be straight*". He went on to interpret the difference quotient as: the consumption at h divided by h. Thereafter, he said that it was about a route: "*It is the extra distance h that one travels, and that divided by h (…), so h is the consumption per kilometre h. So the formula means what the consumption is in kilometres h on a certain kilometre* [points at different points in the graph] *on that route. Approximately I think*". He then wrote: *the consumption per kilometre during distance h*.

We interpreted Nico's utterances in all interviews as being reality bound, because he mainly talked in terms of consumption and distances. We interpreted his explanation in Interview 3 as being reality bound and quite clear and correct.

40.4.2 The Case of Elly

In Interview 1, Elly wondered what h could be: "*I don't understand at all what my h is*". She inserted numbers by taking $a = 10$ and $h = 4$ and said: "*It will become 10 + 4 – 10 divided by 4, but what this means, no idea*". She clearly could not interpret the function notation. In Interview 2, Elly said: "*I don't understand what this h is, and why you can choose it*". She used numbers from the table and wrote: 1.3(10 + 10)−1.3(10)/10. She obtained 1.3 and said: "*I get a number I already had*". Again, she could not interpret the function notation. In the final interview, Interview 3, she changed the h in the formula into an x and said: "*Then I will not think all the time that h is the height or something*". She wrote 1.3 (10 + 3)−1.3 (10)/10 and said:

"*I don't get what they want with this formula.... what it means, and for what you can use it. No idea*".

In all interviews, Elly interpreted the notation $V(a+h)$ as multiplication $Va+Vh$. Not once did she relate the formula to a rate of change, nor to an average petrol consumption. In all interviews, we considered her as mathematics bound, unclear and inaccurate.

40.4.3 The Case of Bob

In Interview 1, Bob took $a = 40$ and said: "*Here you could have the consumption 40 and here the consumption 40 plus a certain value*". He then said that the formula was about the average consumption in litres per kilometre.

In Interview 2, Bob took the petrol consumption at distances 200 and 300 and said: "*It is the petrol consumption between two points of the distance travelled.... how much he used while driving those 100 km*". He said that the formula is like $V_{\text{end}} - V_{\text{start}}$ divided by the travelled distance: "*Yes, in fact this is the average consumption per km*".

In Interview 3, Bob first interpreted the formula as $V(h)/h$, but changed this because already a km has been travelled. He drew a line with points 0, a and $a + h$ and indicated that it is the consumption between a and $a + h$: "*It is the consumption per kilometre within this piece*". When prompted to explain the role of h, he said: "*I think it often is 1, then you will have the consumption on one moment, that is more precise (..) for example you take a = 400 then you will know how much he uses from 400 to 401, that is approximately the consumption on 400. That has something of a limit from mathematics in it, then you can make h smaller like 0.001 or something*".

In all interviews, Bob's approach to the task was reality bound, as he used terms such as average consumption per kilometre and litres per kilometre. From the first interview onwards, he interpreted the formula as a difference of consumption between two points, and from the second interview onwards, this difference was divided by the distance. In Interview 3, he related the formula to limits, which we interpreted as – somewhat – integrating.

40.4.4 The Case of Dorien

In Interview 1, Dorien recognised the formula: "*We did this in the chapter on derivatives (..) with adding small values, first 0,3 and then 0,03 and then you came closer every time*". She thought the formula was about litres of petrol used, but she could not explain this.

In Interview 2, Dorien said: "*With this formula I had to calculate the slope, and later also thederivative. This formula was used for the proof for another, faster*

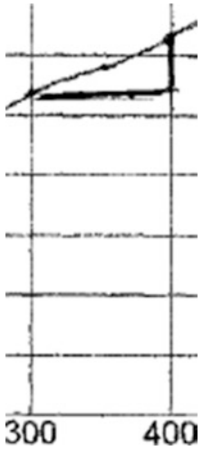


Fig. 40.2 Dorien's illustration of a slope into the graph in Interview 2

formula, and then we had to use the other one, and not this one anymore". She explained that the formula has to do with limits, by saying: "*I recognize it from how the formula is built, that h was first larger, and then you could make it smaller and then you reached a limit, and that was a number that you never reached, that was the slope in one point*". She also said that the formula is "*how much litre is used per km*", explaining: "*If you take for example 300 and 400, then you will know the slope, and that is how many litres is used per kilometre*", and she drew Fig. 40.2.

In Interview 3, Dorien first said that the formula is about limits and that she is a little allergic to them. She learnt them before they did the derivative. She explained that the formula is a $\Delta y/\Delta x$. She also explained it as a derivative, which can calculate how many litres are used per kilometre. It is "*some kind of speed of petrol consumption in fact, in litres per kilometre*". She also connected the formula to gradients and explained the limiting process: "*If you take h smaller and smaller, then h becomes nearly zero. That is called a limit, and it became more precise. I know exactly that it was on that page, it was the first paragraph*".

In the first interview, Dorien's approach was mathematics bound, and she could not explain the formula well within the context. From the second interview onwards, her approach was integrating, explaining the formula both mathematically and within its context.

40.4.5 Synthesis of Results

Table 40.1 gives an overview of students' approaches to the *Petrol Task* in the three sequential interviews. The first two students, Andy and Nico (see Sect. 40.4.1), maintained a reality bound approach throughout all interviews, and their statements became more accurate and clear. The next four students, Elly (see Sect. 40.4.2), Maaike, Casper and Piet, maintained a mathematics bound approach throughout all

Table 40.1 Results of students' approaches being reality bound or mathematics bound

	Andy			Nico			Elly			Maaike			Casper			Piet			Karin			Bob			Otto			Dorien		
Interview #	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
Reality bound	o	+	*	o	o	*													o		*	o	*	*	o		o		*	*
Mathematics bound							o	o	o	+	+		+	+	*	+	*	*			o			+	o	+	*	o	+	*

* Accurate and clear
+ somewhat accurate or clear
o unclear

interviews. From these, Casper and Piet became more accurate and clear. The next two students, Karin and Bob (see Sect. 40.4.3), started with reality bound approaches, and these became integrating. The final two students, Otto and Dorien (see Sect. 40.4.4), started with mathematics-bound approaches, and these became more integrating in Interview 3.

Table 40.1 shows that in the first interview, all students' approaches are either mathematics bound or reality bound, with the exception of Otto (not reported here): his approach is ambivalent. In the subsequent interviews, the students maintain their preference, and their statements become more accurate and clear. In the final interview, four students have somewhat – integrating approaches.

40.5 Conclusion and Discussion

Our study was guided by the research question: How do students' problem-solving approaches when characterised by Busse's ideal types (ambivalent, reality bound, mathematics bound or integrating) develop over time? Our results show that the approaches to the *Petrol Task* can be associated with all four ideal types and that students' approaches can develop from one ideal type to another. In the course of a year, while the students followed the same curriculum about derivatives, the development of their approaches followed different paths. Not one student had a mathematics bound approach in one interview and reality bound in a subsequent interview, or vice versa. All students' approaches were first either reality bound, mathematics bound or ambivalent. An integrating approach could be observed with students, who earlier had a mathematics bound or a reality bound approach. This confirms Busse's hierarchy, in which integrating has a higher cognitive level than both mathematics bound and reality-bound approaches (see Fig. 40.1). An integrating approach was independent of the initial preference.

We cannot confirm Busse's hierarchy with ambivalent approaches at the lowest level. The weakest student in our study, Elly, had a mathematics bound preference, albeit unclear and inaccurate. She took $V(a+h)$ as multiplication in all interviews, and this inability to recognise a function notation probably hindered her progress in learning about derivatives. This may explain the absence of growth in her approaches to the *Petrol Task*.

Also, we see that in the first and second interview, not one approach is integrating. We see students grow: their vocabulary becomes more accurate, they become more flexible in using different mathematical representations and their confidence grows. After the introduction of derivatives, it takes the best students, Bob and Dorien, a year to reach the integrating level. This confirms that it is not easy for students to integrate contexts and mathematics in modelling tasks and that learning to integrate these takes time: at least a year.

Busse's (2011) ideal types proved extremely useful to analyse students' different approaches to tasks and how their preferences develop. Also, the ideal types can assist teachers to analyse students' approaches and develop instructional methods to encourage the uptake of complementary approaches. The framework shows that contexts and mathematics are not disjoint spheres, but that students can integrate these.

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Chapter 41
Exploring Aspects of Creativity in Mathematical Modelling

Helena M. Wessels

Abstract The demands of the twenty-first century require a new focus on identification and nurturing of mathematical creativity, an important key to personal and global success. This chapter reports on an investigation of student teachers' notions of creativity and how creativity can be fostered in school students, as well as an analysis of the creativity evident in their group solutions to a mathematical modelling problem. A questionnaire, a mathematical modelling problem and interviews were used to generate data analysed qualitatively. The findings show participants' intuitive conceptions of creativity are in line with the main aspects of creativity discussed in the literature – *fluency*, *flexibility*, *novelty* and *usefulness* – and that creativity in the solving of the modelling task was influenced by suitability of the task for the specific cohort.

Keywords Creativity • Mathematical modelling tasks • Student teachers • Task suitability

41.1 Introduction

The demands of the twenty-first century require a new focus on the identification and nurturing of mathematical creativity, which can be regarded as an important key to personal and global success (Marshak 2003). References to the necessity of developing creativity can be found in mathematics curricula all over the world, and many teachers would like to develop mathematical creativity in learners, but do not always have a clear conception of what this construct entails and how to develop it.

The original version of this chapter was revised.
An erratum to this chapter can be found at https://doi.org/10.1007/978-3-319-62968-1_53

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_41

Some questions then come to mind: What are student teachers' notions of mathematical creativity and how can it be developed? How creative are student teachers themselves when they are solving mathematical problems?

41.2 Theoretical Orientation

Sternberg (2006) attributes the initial development of the field of creativity to efforts of Guilford (1950) and Torrance (1974). Creativity most commonly is associated with creations in art, music and science, not with mathematics. Mathematical creativity was however already mentioned for the first time in 1902 by Poincaré (Sriraman 2004) but it is only in recent decades that research on the topic started emerging (Manuel 2009).

Definitions of creativity abound. Four themes in definitions of creativity emerge in the literature (Fetterly 2010): process and product, individual and societal, originality and utility and radical novelty and orthodox novelty (Gardner 1993; Plucker and Beghetto 2004; Sternberg and Lubart 2000; Wu and Chiou 2008). Runco (1993, 1999) refers to creativity as a multifaceted construct which includes convergent and divergent thinking as well as a questioning attitude, problem posing, problem-solving, motivation, self-expression and self-confidence. Sheffield (2000) offers seven criteria for the identification of creativity: depth of understanding, fluency, flexibility, originality or novelty, elaboration or elegance, generalisation and reasoning and extensions. Other definitions address the process or product of creativity and in most of these three major components surface: fluency, flexibility and originality (Torrance 1974). In the field of mathematics, views of creativity refer to a thinking process revealed in the three products mentioned above: fluency, flexibility and novelty (Sriraman 2005). *Fluency* is related to the number of different correct answers, solutions or new questions posed; *flexibility* is linked to different categories of answers and the ability to change focus to other solution paths during problem-solving; and *originality* or *novelty* is the uniqueness of solutions or questions posed. A fourth component of creativity, *usefulness* or *utility*, is included in definitions of Anabile (1996), Feist (1998) and Sternberg and Lubart (2000). *Usefulness* refers to "the relevance, adaptability and reusability of solutions in other real world situations" (Wessels 2014, p. 6).

Several authors link creativity and problem-solving or modelling (Biembengut and Vieira 2013; Burghes 2015). Chamberlin and Moon (2005 p. 38) describe creativity as the "domain-specific thinking processes" used in nonroutine problem-solving in their article on the use of model-eliciting activities as a tool to develop creativity. Silver (1997) argues that it can be productive to regard creativity as a means to foster more creative approaches to mathematics, specifically through problem-posing and problem-solving tasks. Sriraman (2005) also alludes to the link between mathematical creativity and problem-solving, referring to novel, unusual and insightful outcomes resulting from problem-solving. In a mathematical modelling approach, complex open-ended tasks are used, which presents an ideal opportunity to foster creative thinking in learners. Dan and Xie (2011) found in their research that there was a strong positive correlation between mathematical modelling skills and creative thinking levels, while Mann (2006) argues that the solving of open-ended contextual problems points to mathematical creativity. Wessels (2011, 2014) points out that creative thinking enables multiple entry points into mathematical modelling problems as well as to a variety of solution strategies on different levels. Solving such complex, open-ended problems support the development of *fluency*, *flexibility* and *novelty* in learners, as well as their consideration of the *usefulness* of solutions. Since model-eliciting activities promote fluency, flexibility, novelty and consideration of usefulness, these activities can be used as an ideal tool in developing mathematical creativity.

41.3 Method

The purpose of the study was twofold: to explore student teachers' theoretical and pedagogical conceptions of creativity and identifying aspects of creativity in their mathematical modelling processes and products. The research was conducted during the researcher's visit to a European university as part of a staff exchange opportunity. All ethical requirements were adhered to.

41.3.1 Participants

A convenience sample at a large European university was used in this mixed method study. Twenty-six masters students (student teachers), enrolled for a seminar on mathematical modelling, voluntarily participated in the first session of the semester. Topics in the seminar included theory about aims of modelling, modelling competencies, characteristics of modelling problems, appropriate teacher interventions and assessment of modelling. The student teachers all prepared to teach mathematics as a subject in school grades ranging from grades 1 to 13, in primary and secondary school as well as in special and vocational schools.

41.3.2 Instruments

Three instruments were used: a questionnaire individually completed, a mathematical modelling problem solved in groups and focus group interviews. The questionnaire collected biographical data of participants and their notions of creativity, as well as how to identify and develop it. The mathematical modelling task, *Making Money* (see Appendix), was used to explore aspects of creativity in student teachers' solution processes and final models. The task is about an entrepreneur who had nine vendors selling popcorn and cold drinks at a play park during summer. Recommendations had to be made about which six vendors she should rehire for the next summer, based on their sales and the number of hours they worked during slow, steady and busy shifts over 3 months. Semi-structured group interviews probed participants' experiences during the solving of the mathematical modelling problem.

41.3.3 Data Collection

Data were generated in three phases: the questionnaire and modelling task were administered on the same day, and 1 week later, group interviews were conducted. The questionnaire was in English and was administered during the first contact session of the seminar on mathematical modelling. Only one student in the group of 26 had been involved in mathematical modelling before enrolling for the modelling seminar. As the researcher was not fluent in the mother tongue of the participants, the seminar lecturer and a PhD student in mathematics education were available to interpret when needed. In the first session, the questionnaire was completed individually after which participants solved the mathematical modelling problem, *Making Money,* in five groups. Although an attempt was made to group students according to the grades which they were preparing to teach in, it was not possible due to the variety of school types and grades students were preparing to teach in. Students in one group also insisted to solve the problem in their group of friends. Four of the five groups were audio- and videotaped while solving the mathematical modelling problem. One group gave consent only for audiotaping. Videotaped focus group interviews were conducted in English 1 week after the first round of data collection. Only three groups consented to be interviewed. During the interviews students' experiences while solving the modelling problem were explored as well as the influence of these experiences on their views of the suitability of modelling tasks for school learners with regard to mathematical level and context.

41.3.4 Analysis and Results

41.3.4.1 Analysis of the Questionnaire

A framework comprising the four criteria for mathematical creativity from the literature guided the analysis of the questionnaire and student teachers' mathematical models. The section of the questionnaire that explored participants' notions of creativity comprised the following four questions:

Q1: What is mathematical creativity?
Q2: Do you regard yourself as mathematically creative? Motivate your answer.
Q3: How would you know whether a learner is mathematically creative?
Q4: How can mathematical creativity be developed in school learners?

Responses to these open-ended questionnaire items were categorised according to emerging themes and compared with the four criteria for mathematical creativity. In the discussion of the analysis of the questionnaire, examples of participant responses for each category are given in brackets.

Participants' responses to Q1 (What is mathematical creativity?) could, with the exception of three anomalous responses, be categorised according to the four criteria for creativity: *fluency* (responses referred to different solutions, quantity of solutions), *flexibility* (responses referred to varying approaches, translating of a problem to another context, using different representations, etc.), *originality* (responses referred to new approaches, own approach, etc.), and *usefulness* (using mathematics to solve real-world problems, making sense).

In Q2, Do you regard yourself as mathematically creative?, more than half of the students answered ‘yes’. In one group, students might have influenced each other as four out of the five student teachers put their cross on the line between the yes and the no, and one of them also ticked ‘yes’. In another group, two student teachers also indicated an answer between ‘yes’ and ‘no’.

Most of the student teachers’ answers about their own creativity implicitly refer to fluency, flexibility and novelty. Almost all student teachers who do not regard themselves as creative or gave an ‘in-between’ answer (halfway between ‘yes’ and ‘no’), in some way, indicated that they are not fluent, referring to solving tasks in only one way, or not being able to think of more than one way to solve a task. One student teacher who gave an ‘in-between’ answer commented that creativity is dependent on the topic and the task. Of those who regard themselves as creative, six referred to *fluency* (coming up with different answers to solve a problem) and five to *novelty* (referring to new ideas, own ideas). One student referred to *flexibility*, saying that when she cannot find a result with a procedure, she would try another way. Not surprisingly, there were no references to the *usefulness* of processes or products of creativity because school and university students are not often challenged to consider the relevance, adaptability and reusability of their solutions for other real-world situations.

The four criteria for creativity were also alluded to in participants’ responses to Q3, How would you know whether a learner is mathematically creative?, but four additional categories emerged in the analysis: *the use of mathematical knowledge* (can use knowledge from different parts of mathematics/only uses knowledge which is part of the issue/using knowledge from previous lessons), *kind of tasks* (through problem-solving tasks, thinking tasks or challenging problems/using open tasks – no prescribed method given, etc.), *classroom discourse* (able to describe solution/talk to others, explain their work), and *personal abilities, characteristics and dispositions*. Responses in the last category include references to *objectivity* (when you can get a distance to the problem), *cognitive engagement* (when he thinks the task over), *confidence* (when he confidently starts working on the solution of the problem), *motivation* (if the person is motivated), *perseverance* (does not give up), *assisting others* (can explain to learners who do not understand other ways how a task can be solved), and *enjoyment* (enjoy solving the task).

Categories that emerged in responses to Q4, How can mathematical creativity be developed in school learners?, include references to *tasks* (open tasks, modelling tasks, understanding tasks), *teaching* (general teaching, broaden thinking, no prescribed methods, give freedom and more time for learners to create own solutions, non-judgemental, appreciate efforts, emphasize connections, motivate learners), *classroom organisation and management* (learning environment, group work, discourse), and *curriculum* (mathematical creativity as topic in schools, special education, theatre courses).

Table 41.1 Description of solution strategies and recommendations of the five groups

Group	Solution strategies
1	Average hours in different shifts; income/h for different shifts; total average income/h; weighted average, rankings
2	Income/h for different shifts – months separately; average income/h for different shifts; total average income/h; full time, best overall turnover and very good in slow and steady; part time, good overall turnover in busy and steady
3	Income/h for months separately and overall; formulas, graphs and regression lines; consider only busy and slow shifts; full time, highest hourly income; part time, less hours worked, still passable turnover
4	Average income/h in different shifts; total average income/h; colour coding in table; case studies of all ten workers to motivate choices for full and part time
5	Income/h in different shifts; total average income/h; full time, highest average; part time, next 3

41.3.4.2 Analysis of Modelling Process and Solutions for Task

Analysis of solutions to the mathematical modelling problem explored aspects of mathematical creativity in student teachers' modelling processes and products. To ensure reliability of analysed data, two PhD students checked the researcher's translations and interpretations of participants' written work as well as of discussions during the interviews.

The solutions to the modelling problem were analysed to establish how the thinking and interpretations of the five groups differed (Table 41.1). Solutions and solution paths were also scrutinised to determine elements of creativity in the processes and products of the modelling activity.

Solution paths were quite similar with all groups calculating income generated per hour: some for different shifts and others for different months. In some cases, average income generated per hour was calculated for individuals or overall, while some groups calculated both. Two of the three groups who participated in the interviews confirmed that they had just one idea to start with, while the other group had two ideas, but after some time, they discarded one idea. This points to low creativity regarding *fluency*, even though one group were somewhat more fluent than the other two. The five groups all developed their original ideas in different ways, showing *flexibility*. Groups 1, 3 and 4 used *novel* ideas: group 1 used weighting to rank the employees, while group 3 used formulas and graphs to determine full-time and part-times employees. Group 4 used a colour-coded table and detailed case studies of all nine current employees to determine positions for the next year. All groups created *useful* models for determining positions for the next season.

41.4 Discussion of Findings

The purpose of this study was to investigate student teachers' notions of creativity as well as their own creativity that was evident in their solutions to a mathematical modelling task. Analysis of the open-ended responses to the questionnaire provided useful insights into their conceptions of the construct of creativity, and analysis of their solutions to the mathematical modelling problem highlighted the importance of task selection in the fostering and identification of mathematical creativity.

Although creativity as a topic was not formally covered in any of their courses, student teachers' intuitive notions of what creativity is coincided with indicators for creativity described in the literature and my chosen framework (Sriraman 2004; Torrance 1974; Sternberg and Lubart 2000). Student teachers implicitly or explicitly referred to the four main aspects of creativity found in the literature, *fluency*, *flexibility*, *novelty* and less so to *usefulness*, in their descriptions of what they understand creativity to be and their motivations why they regard themselves as creative or not. The participants added cognitive, motivational and teaching aspects to these indicators for identification of creativity in learners and for developing creativity in the classroom. These findings correspond with a study by Aljughaiman and Mowrer-Reynolds (2005) in which teachers referred, amongst others, to original ideas, inventiveness (*novelty*), problem-solving and divergent thinking (*fluency* and *flexibility*) and aesthetic and linguistic products (*usefulness*) in their definitions of creativity. Cognitive, creative and motivational aspects including solving complex problems and creating elegant solutions, flexible thinking and using a wide range of mathematical strategies in nonroutine ways, determination in performing difficult tasks and willingness to face learning challenges are also alluded to in Amit's (2014) discussion of what academic talent comprises.

In solving the modelling problem, *Making Money*, all five groups combined the two data sets of hours worked and money generated. Participants' *fluency* was not high, although one group was somewhat more fluent than the others. All five groups demonstrated *flexibility* by elaborating on their initial strategies to come up with a final strategy to determine candidates for part-time and full-time employment for the next season. Three groups applied different *novel* ways to construct their final models, while all five groups generated adaptable *useful* models, although one group's model was mathematically complicated and would have been difficult for the entrepreneur to use if she was not well educated in mathematics.

In a research study with undergraduate students described elsewhere (Wessels 2014), the *Making Money* task was solved by 48 groups of student teachers preparing to teach 6- to 9-year-old learners. Solutions were analysed for creativity using the four main characteristics of creativity. Levels of student teachers' creativity in this study varied widely, ranging from models that were not useful, displaying little creativity, to very useful sophisticated models that were adaptable in other contexts.

Reasons for the difference in findings lie in sample size as well as the mathematical background, maturity and experience of the participants. Furthermore, the specific modelling task has been more suitable for school students and undergraduate student teachers than for postgraduate student teachers, as the level of mathematics that the postgraduate students had been exposed to was much higher than that of the other cohorts. The last cohort confirmed in the interviews that the task was not mathematically challenging for them but that interpreting their mathematical solution in the real-world situation was more challenging than they expected – they found it difficult to recommend which vendors should be rehired for the next season. The groups also expressed their surprise that all groups did not recommend the same vendors for part-time and full-time employment in the next season.

The scope of this study was limited by the fact that data collection had to take place during the exchange opportunity of the researcher to the host university and the choice of modelling problem that student teachers had to solve was therefore limited by the time available to collect data. The sample size was determined by the number of students in the seminar on mathematical modelling of the host lecturer at the university. The fact that the researcher has a limited command of the mother tongue of participants slowed down communication and analysis of data. This problem was mitigated through the assistance of PhD students with interpretation and translation. The fact that there was no time for the different groups to present their solution processes and models to the other groups deprived participants of the opportunity to develop an understanding of other possible solution paths and models and therefore limited the benefits of the task for this cohort.

41.5 Conclusion

Student teachers' intuitive notions of creativity displayed conceptions of all four main characteristics of mathematical creativity described in the literature. Their solutions to the modelling task elicited flexibility as well as novel and useful models, but their fluency was restricted as the task was not mathematically challenging enough.

Mathematics teachers' awareness of the importance of creativity in mathematics and how it can be identified and developed in learners needs to be raised. An awareness and appreciation for mathematical creativity are crucial in preparing students to become citizens equipped for the challenging world when they finish school. The development of mathematical creativity is often hampered by typical textbook tasks that do not allow for creativity. More challenging tasks that have the potential to elicit creativity are needed. Mathematical modelling problems have potential to foster creativity (Chamberlin and Moon 2005), but careful problem selection for a specific cohort or grade is crucial as creativity can be task dependent (Leikin and Lev 2007). Creativity can also be hampered by instruction favouring taught procedures and not independent thinking and self-constructed strategies. If teachers are more aware of, and knowledgeable about, what mathematical creativity entails and how it can be fostered, they might be able to "view creativity not as the domain of only a few exceptional individuals but rather as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (Silver 1997, p. 79).

Appendix: Model-Eliciting Task: Making Money

During the last summer holidays, Maya started a concession business at Wild Days Amusement Park. Her vendors carried popcorn and drinks around the park, selling wherever they can find customers. Maya needs help deciding which workers to rehire next summer.

Last year Maya had nine vendors. This summer, she can have only six – three full time and three part time. She wants to rehire the vendors who will make the most money for her, but she does not know how to compare them because they worked different numbers of hours. Also, *when* they worked makes a big difference. After all, it is easier to sell more on a crowded Friday night than a on a rainy afternoon.

Maya reviewed her records from last year. For each vendor, she totalled the number of hours worked and the money collected – when business in the park was busy (high attendance), steady and slow (low attendance) (see table). Please evaluate how well the different vendors did last year for the business and which three should she rehire full time and which three she should rehire part time.

Write a letter to Maya giving your results. In your letter, describe how you evaluated the vendors. Give details so Maya can check your work, and give a clear explanation so she can decide whether your method is a good one for her to use.

HOURS WORKED LAST SUMMER

	NOVEMBER			DECEMBER			JANUARY		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
Maria	12.5	15	9	10	14	17.5	12.5	33.5	35
Kim	5.5	22	15.5	53.5	40	15.5	50	14	23.5
Terry	12	17	14.5	20	25	21.5	19.5	20.5	24.5
Jose	19.5	30.5	34	20	31	14	22	19.5	36
Yusuf	19.5	26	0	36	15.5	27	30	24	4.5
Thandi	13	4,5	12	33.5	37.5	6.5	16	24	16.5
Robin	26.5	43.5	27	67	26	3	41.5	58	5.5
Tony	7.5	16	25	16	45.5	51	7.5	42	84
Willy	0	3	4.5	38	17.5	39	37	22	12

MONEY COLLECTED LAST SUMMER (IN SOUTH AFRICAN RAND)

	NOVEMBER			DECEMBER			JANUARY		
	Busy	Steady	Slow	Busy	Steady	Slow	Busy	Steady	Slow
Maria	690	780	452	699	758	835	788	1732	1462
Kim	474	874	406	4612	2032	477	4500	834	712
Terry	1047	667	284	1389	804	450	1062	806	491
Jose	1236	1188	765	1584	1668	449	1822	1276	1358
Yusuf	1264	1172	0	2477	681	548	1923	1130	89
Thandi	1115	278	574	2972	2399	231	1322	1594	577
Robin	2253	1702	610	4470	993	75	2754	2327	87
Tony	550	903	928	1296	2360	2610	615	2184	2518
Willy	0	125	64	3073	767	768	3005	1253	253

(Source: Lesh et al. 1997, p. 67)

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Chapter 42
Mathematical Modelling in Dutch Textbooks: Is It Genuine Mathematical Modelling?

Bert Zwaneveld, Jacob Perrenet, Kees van Overveld, and Tijn Borghuis

Abstract In this chapter, we analyse the two most frequently used Dutch mathematics textbooks for upper secondary schools in order to determine to what extent the tasks in these textbooks meet the criteria we have set for genuine mathematical modelling: does a modelling task have a modelling purpose, and do the students have to perform characteristic modelling activities? The criterion of having a modelling purpose stems from a modelling course in tertiary education by the last two authors. For the characteristic modelling activities, we used the research of the first two authors. Only a very small percentage of the analysed tasks meets the criteria. So, there is hardly any genuine mathematical modelling in the two textbooks, although it is explicitly mentioned in the formal curriculum.

Keywords Genuine mathematical modelling • Dutch secondary mathematics textbooks • Modelling purposes • Characteristic modelling activities • Problem solving and mathematical modelling • Conceptual development and mathematical modelling

42.1 Introduction

In this chapter, we focus on mathematical modelling in upper secondary education. After a global overview of the relevant theoretical frameworks, we restrict ourselves to the Dutch situation. As a starting point, we take how modelling is formulated in the examination programme for mathematics, subsequently called the curriculum.[1] We explore, describe and analyse how the two most frequently used textbooks

[1] In the Netherlands, the Dutch government establishes what is called the examination programme: the content of the central examination.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_42

Sub-domain A2: Research skills
2. The candidate can analyse a given problem situation and translate it into a mathematical model, use mathematical solving techniques within this model, and give meaning to the resulting solutions in terms of the problem situation.

Fig. 42.1 The part of the mathematical curriculum about modelling (Examenprogramma wiskunde 2007 [translated from Dutch])

implement the modelling part of the curriculum. We focus on the modelling purposes of the tasks and on the modelling activities of the students. For the analysis, we use two sources: the modelling education at Eindhoven University of Technology of Van Overveld and Borghuis (2013) for the modelling purposes and a paper of Perrenet and Zwaneveld (2012) for modelling activities. In Sect. 42.2.2, we describe these two sources in more detail.

42.1.1 Modelling in the Dutch Mathematics Curriculum

In Fig. 42.1, we present the relevant part of the Dutch mathematics curriculum. Modelling *purposes* are not mentioned in Fig. 42.1 and also not in the further specifications which we have left out. Besides the research skills of Fig. 42.1, there are many 'pure math' objectives in the curriculum. It mentions explicitly that these objectives are examined in combination with the research skills of Fig. 42.1.

We shall describe how mathematical modelling is implemented in order to draw conclusions about the question: To what extent do Dutch students in upper secondary education have to perform genuine mathematical modelling activities? We define *genuine* mathematical modelling by performing one or more of the characteristic modelling activities to context-based problems with one or more modelling purposes. This definition is the result of several theoretical frameworks, as described in the next section.

As an example, which comes close to what we mean by 'genuine modelling', we present a slightly adapted part of a task in one of the textbooks. One less important task is omitted. The purposes are analysis and explaining. The modelling activities are interpreting, mathematising (conceptualising is almost completely done by the authors), iterating and solving.

Mathematics A, Volume 6, Section 5-4, Setting Up Formulas, Task 26 [Translated from Dutch]

In 1977, researchers found a small group of long-eared owls in a village in Limburg. Following this group, it turned out that these owls early in spring sit and that the young fly from half July. From counts held end of June each year, one saw that the population grew. The results of the counts are set in a coordinate system with single-logarithmic axes and draw a as fitting as possible line through the measuring points.

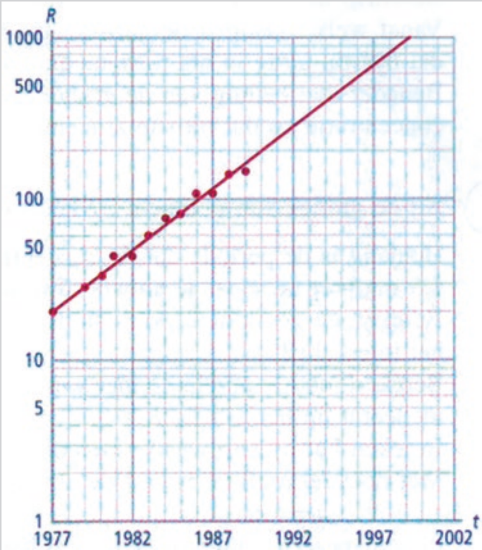

Figure 1 (**a**) The researchers concluded that the number of long-eared owls increased each year with a stable percentage. Why could they draw this conclusion?
(**b**) Give a formula by which the number of long-eared owls R can be calculated t years after 1977 and calculate also the percentage of the yearly increase.
(**c**) End of June 1991, one counted 205 long-eared owls. Show that this is less what could be expected by the formula of part b.
The explanation of the biologists was that the number of breeding places was too restricted for the increasing number of owls. One researcher suggested that the number of long-eared owls from 1989 was well predictable by the formula $R = a - b \cdot 0{,}6^t$, R the number of long-eared owls and t the number of years after the end of June 1989. He chose a and b so that the formula resulted in178 owls in 1989 and 205 owls for 1991.
(**d**) Calculate a and b using the data of 1989 and 1991

42.2 Theory

From a theoretical point of view, we consider the following points relevant for the implementation of modelling in a curriculum: the relationship between problem solving and modelling, the integration of mathematical conceptual development and modelling and the demands of modelling tasks. We give a summary of our eclectic choice.

42.2.1 Problem Solving and Mathematical Modelling

During the second half of the last century, problem solving mainly focused on solving purely mathematical problems; see, for instance, the work of Pólya, Silver and Schoenfeld. In the 1970s, there was a shift towards the teaching of mathematics with context-based problems, which led to applying mathematical concepts and techniques to these problems, but also to modelling. This was started mainly by Freudenthal and is now well known as realistic mathematics education (Wijaya et al. 2015). In the last decades, the focus is more and more on problem solving by mathematical modelling. Lesh and Zawojewski (2007) argued that in the twenty-first century, citizens and workers are confronted with complex systems, like new technologies for communication, co-operation and conceptualisation. They plead the teaching should not be about modelling as such, but conceptual development integrated into problem solving should be central.

English and Sriraman (2010) concluded that there is a lot of research in problem solving, related to modelling, but for several reasons, mainly lack of coherence between these studies, there is not much progress. They concluded that there are, roughly speaking, three approaches to problem solving: content-driven problem solving (mathematics first, application later), problem-driven concept development (mathematical content is developed during problem solving) and heuristics/strategy-driven problem solving (isolated courses in problem solving). Van Streun (1987) investigated in his dissertation two approaches which he called: *math first, then applying* (closely related to content-driven problem solving) and *conceptual development and modelling integrated* (closely related to problem-driven concept development). His conclusion was that in the latter approach, the students learned mathematics better. English and Sriraman concluded that none of these three approaches apart is successful, and recommend, besides a combination of these approaches, an approach in problem solving, where mathematical modelling is dominant.

42.2.2 Conceptual Development and Mathematical Modelling

Lesh et al. (2003) argued that the cognitive processes during mathematical modelling are similar to those when learning mathematics: conceptualising, describing, explaining, visualising, sorting, representing, refining, modifying, integrating, extending, building upon existing knowledge as well as switching between concrete and abstract, simple and complex, intuitive and formal and situated and decontextualised. They also placed problem solving and modelling opposite to each other. In problem solving, there is almost always one answer, the solving process runs through succeeding stages and the explanation comes after the process. During a mathematical modelling process, the reasoning, explaining and predicting happen, mostly, in an interactive and iterative way.

Here is an example of the integration of mathematical conceptual development and modelling. Primary schoolchildren had to decide where the English government should establish a settlement in Australia in the seventeenth century; they had five options, and for each possible location, there were data available about the environment and infrastructure. The children developed the idea of ranking the five locations based on weights they gave to the data of each location. Gravemeijer (2007) calls this *emergent modelling*:

> a model is the result of an organizing activity. It is in the process of structuring a problem situation that the model emerges. Within this perspective, the model and the situation modeled co-evolve and are mutually constituted in the course of modeling activity. Thus, when we characterize modeling as a process of mathematization by which the situation is being structured in terms of mathematical relationships, the distinction between the model and the situation modeled dissolves.

42.2.3 Demands for Modelling Tasks

We refer to two issues of *The International Journal on Mathematics Education* about mathematical modelling (former *ZDM*), issues 38(2) and 38(3), published in 2006, which precede the work of English and Sriraman. The editors of these issues, Kaiser et al. (2006, p. 82), admit that “obviously the theory of teaching and learning of mathematical modelling is far from complete” and focus towards a didactical theory for mathematical modelling. The papers from these two issues about the main components of the modelling process, the criteria for ‘authentic’ modelling problems, the key elements in modelling competency and the principles for designing modelling problems are relevant to our analysis.

42.2.4 Purposes of Modelling

Van Overveld and Borghuis (2013) in their lecture notes about mathematical modelling for all first-year Bachelor students at Eindhoven University of Technology stress the importance of the purposes of modelling. They list as purposes: explanation, prediction 1 (when?) and 2 (what?), compression, inspiration, communication, unification, abstraction, analysis, verification, exploration, decision, optimisation, specification, realisation and steering and control. An example of compression is the work of Kepler who compressed the data of Tycho Brahe into his laws of the movements of the planets. An example of unification is the work of Newton who unified, for instance, the fall of an apple and the movements of the planets into his universal law of gravitation. More about these modelling purposes one can find in Perrenet et al. (2017) in this volume (see Chap. 12).

42.2.5 Modelling Activities

The curriculum excerpt of Fig. 42.1 mentions four steps of a modelling process: analysing, translating into mathematics, using solving techniques and giving meaning. These are comparable to the modelling activities of Perrenet and Zwaneveld (2012): conceptualising, mathematising, solving and interpreting. They also mention five other modelling activities: verifying, validating, iterating, communicating and reflecting. Of these activities, only verifying and communicating are not mentioned in the specifications of the research skills of Fig. 42.1.

42.2.6 Research Question

From the mentioned papers about modelling, we conclude that teaching and learning modelling are very complex, because a lot of aspects have to be taken into account, like the teaching objectives, for example both conceptual development and modelling, and the choice of the problem situations; important criteria are the presence of a modelling purpose, the modelling activities the students have to perform, and the supporting tools. The most important tool for supporting the teaching and learning in mathematics classes is the mathematics textbook. In the context of the Dutch curriculum, authors of these textbooks can, roughly speaking, choose a design where the mathematical concepts are applied to problem situations or where conceptual development and mathematical modelling are integrated. But students have to learn genuine mathematical modelling. Our research question is as follows: *To what extent is mathematical modelling in Dutch textbooks genuine mathematical modelling, that is performing one or more characteristic modelling activities to context-based problems which have one or more modelling purposes?*

42.3 Methods

42.3.1 Selection of the Learning Materials

In the Dutch secondary school system, one of the streams is pre-university education. Students in the last 3 years have to choose one out of four profiles. We restrict this analysis to the textbooks for students with profile Economy and Society, because the central examination here is the only one where all the five assignments are about a given context-based problem situation.

The two most frequently used mathematics textbooks in the Netherlands (about 90% of the schools use one of these books) are Number and Space (in Dutch: *Getal en Ruimte*, further abbreviated as GR) and Modern Mathematics (in Dutch: *Moderne Wiskunde*, further abbreviated as MW). We used in this study the tenth edition (2013) of both GR and MW. Both textbooks present the mathematical concepts and accompanying skills as described in the curriculum. In GR, it is mostly the mathematical concepts that are presented first, afterwards they are applied to a variety of context-based problems. In MW, the mathematical concepts are mostly integrated into context-based problems. We took a sample consisting of every second task of all the context-based tasks of GR and MW.

42.3.2 Analysis of the Selected Learning Materials

To analyse how mathematical modelling occurs in these textbooks, we first scored the tasks on the nine characteristic activities mentioned by Perrenet and Zwaneveld (2012). Then we scored these tasks according to the purposes mentioned by Van Overveld and Borghuis (2013). The first author interpreted and scored all the modelling tasks with respect to characteristic modelling activities and modelling purposes. For the tasks he was not sure, about a quarter of the tasks, the second author also scored these independently. After discussion, we agreed with respect to the modelling activities. The fourth author was consulted when clarification of the purposes was necessary. Here, we also agreed.

42.3.3 An Example of the Scoring

We scored the *Day Proceeds* task as: purposes are explaining (subtask a), analysing (subtasks b and c), and optimising (subtask d); the characteristic modelling activities are conceptualising (subtasks a, b and e), mathematising (subtasks a and b) and solving (subtasks c, d and e).

Day Proceeds

A manufacturer sells 400 items per day for € 28 per item. If he lowers the price to € 20 then the sale increases to 1200 items. He assumes € 1500 fixed costs. The variable costs are € 16 per item. The price p in euros is a linear function of the sold number q.

(a) Show that it follows that $p = -0.01\ q + 32$.
(b) Give the formula of the day proceeds R in euros as a function of q.
(c) Calculate algebraically for which price R is equal to €24,000.
(d) Calculate algebraically the maximal profit W per day in euros. How much is the price then?
(e) As a consequence of a reorganisation the fixed costs per day change, while all other data stays the same. What are these fixed costs per day if the maximal profit is € 6000 per day?

42.4 Results

There are about 1000 tasks in the textbooks, of which 542 tasks are context based. With respect to the occurrence of the characteristic modelling activities, we identify conceptualising, mathematising, solving, interpreting, verifying, validating, reflecting, iterating and communicating. In Table 42.1, we present the number of occurrences in the 542 analysed tasks.

About 500 of the analysed tasks have, at times, no characteristic modelling activity, sometimes one up to six. We counted 992 times a characteristic modelling activity.

In Table 42.2, we present how frequently the characteristic modelling activities occur in each task. For instance, about 55% of the tasks have exactly one modelling purpose, while tasks with 0, 5, 6, 7, 8 or 9 activities hardly occur.

Next, we analyse to what extent the purposes occur. We only established: explanation, prediction 1 (when?), prediction 2 (what?), analysis, optimisation and decision. None of the other possible purposes are identified. See Table 42.3.

Table 42.1 Occurrence of the characteristic modelling activities in the 542 analysed tasks

Characteristic modelling activity	Number	Percentage
Conceptualising	172	32
Mathematising	169	31
Solving	526	97
Interpreting	96	18
Verifying	9	2
Validating	13	2
Iterating	3	1
Communicating	1	0
Reflecting	3	1
Total number	992	

Table 42.2 Frequencies and percentages of the number of characteristic modelling activities per task

Number of modelling activities	0	1	2	3	4	5	6	7, 8, 9	Total
Frequency	2	297	74	129	35	4	1	0	542
Percentage	0	55	14	24	7	1	0	0	100

Table 42.3 Modelling purposes occurring in the 542 analysed tasks

Purpose	Frequency	Percentage
Explanation	70	13
Prediction 1	16	3
Prediction 2	8	2
Analysis	81	15
Optimisation	47	9
Decision	43	8
Compression, inspiration, communication, unification, abstraction, verification, exploration, specification, realisation, steering and control	0	0
Total	265	

Table 42.4 Frequencies and percentages of the number of modelling purposes

Number of modelling purposes	0	1	2	3	≥4	Total
Frequency	314	193	33	2	0	542
Percentage	58	36	6	0	0	100

We also analyse how often a task has 0, 1, 2 or 3 modelling purposes. More than three do not occur. In Table 42.4, we present these results. For instance, about 35% of the analysed tasks have exactly one modelling purpose. Note that 228 tasks have one or more modelling purposes.

42.5 Conclusions and Discussion

Our first conclusion, based on Table 42.1, is that in almost 100% of the context-based tasks solving, for example applying mathematical concepts and techniques, is by far the main activity; conceptualising and mathematising follow with about 30% for each. Interpreting gets some attention: 18%. All the other five modelling activities hardly occur or are fully absent. From Table 42.2, we see that about 70% of the tasks have 1 or 2 modelling activities, about 30% have 3 or 4 of these activities, 5 or 6 activities hardly occur and 0 or more than 6 never occur.

Our second conclusion is about the presence of modelling purposes. Table 42.3 shows that analysis and explanation occur mostly: 15% and 13%, respectively. They

Table 42.5 Three categories of the 228 modelling tasks with a modelling purpose

Modelling category	0 or 1 characteristic activity: 'no modelling'	2, 3, or 4 characteristic activities: 'grey area' or 'small modelling'	More than four characteristic activities: 'genuine modelling'
Percentage (%)	33	66	1

are followed by optimisation and decision, both about almost 10%. Predictions 1 (when?) and 2 (what?) have an occurrence of less than 5%. The other ten modelling purposes do not occur at all. From Table 42.4, we conclude that about 60% of the context-based tasks do not have a modelling purpose.

Our answer to the research question, in short *To what extent is mathematical modelling in Dutch textbooks genuine mathematical modelling?* is that many of the tasks do not meet the criteria: characteristic modelling activities to problem situations with a modelling purpose. This is comparable to the conclusion of Wijaya et al. (2015, p. 60) about the Indonesian situation in secondary education:

> the mathematical procedure to be applied is more or less given and students do not have to identify an appropriate mathematics procedure to solve the tasks and consequently they are not getting enough experience to develop their ability to transform a context-based task into a mathematical problem.

Vos (2013, p. 487) observed a comparable result in the Dutch examination papers: "More than 90% of the tasks merely asked for calculations leading to a number answer". The students' activities were loosely related to the given non-mathematical problem. She used the adjectives mechanistic and reproductive for the mathematising activity. Also, from the point of view that conceptual development and modelling should be integrated, the results are not very hopeful. Dividing the 228 analysed tasks with a modelling purpose into categories, genuine modelling (more than four characteristic modelling activities), no modelling (0 or 1 activity) and a 'grey area' in between (two, three or four activities), we get Table 42.5. We characterise the grey area as 'small modelling': there is a modelling purpose, and there are characteristic modelling activities, but mostly the mathematical model is given or very obvious, for example. a linear or exponential function. Conceptualising is missing.

From the high percentages of the solving activity in both textbooks, we conclude that in these textbooks, practicing mathematical skills is more important than conceptual development integrated with mathematical modelling. But one cannot expect that all the context-based tasks will be tasks evoking genuine mathematical modelling. We guess that the ratio between conceptual development, practising mathematical skills and mathematical modelling should be like 1:2:1.

One should take into account that in secondary mathematics education where problem solving is a main didactical characteristic, and learning to model a main educational objective, the authors have to find the balance between the teaching of the mathematical concepts and techniques of the curriculum and the teaching of mathematical modelling. Modelling tasks in school have always two kinds of goals: the modelling purpose of the task and the educational goals (Vos 2011). Table 42.5 can also be interpreted, more positively, as there is in about one third of the context-based tasks conceptual development without modelling activities, but about two thirds of the tasks combine conceptual development with a restricted number of modelling activities.

42.6 A Recommendation for Textbook Improvement

We recommend to improve the position of mathematical modelling by using more the design of MW rather than GR. Start each chapter with a modelling problem, by which the students can also experience what mathematical concepts and skills will be treated. At the end of a chapter, as part of the mixed tasks, many problems might be presented with a specific focus on genuine modelling as such: students learn that the purpose of a modelling activity directs the modelling process, and they experience the relevance and importance of the characteristic modelling activities. In between, the students learn the concepts and techniques, they practise these and they apply these, every time both in purely mathematical- and context-based tasks.

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Part IV
Influences of Technologies on Modelling and Applications

Chapter 43
Initial Results of an Intervention Using a Mobile Game App to Simulate a Pandemic Outbreak

Peter Frejd and Jonas B. Ärlebäck

Abstract This chapter presents the design and results from the first iteration of a classroom activity using the context and simulation provided by a commercial game app for smartphones and tablets. The aim was to study students' experiences of an intervention entailing the game app in order to inform a more mature design with the long-term goal of being able to develop principles to design and implement modelling activities using game apps. An analysis focusing on the interplay between the designed intervention environment and students' work from two upper secondary classes resulted in information in terms of affordances to inform the redesign. In addition, students experienced the activity as interesting and engaging, but with significant gender differences. The results inform a discussion of the role of new technology and simulation within classroom teaching and learning of mathematical modelling.

Keywords Modelling • Simulation • Technology • Mobile app • Game

43.1 Introduction

Scientific and technological inventions have fundamentally changed the way we live our lives and modernised our society. Mathematics is one discipline where new technological advances in particular have affected ways of working. It is now possible to do more advanced and complex calculations, visualisations and simulations – often needed when mathematics is applied to solve realistic or authentic real-world problems, a generic problem-solving process we refer to as *modelling*.

In industry and academia, professional modellers often use different types of digital technology (programming, computer-aided computation and ICT tools) in their work to run simulations to investigate the behaviour of evolving models, their

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_43

limitations and flaws (Frejd and Bergsten 2016). Furthermore, a more frequent use of simulations in connection with different types of modelling activities has been advocated in school settings (Neunzert 2013). Recent technological advances resulting in faster computers and easy-to-work-with interfaces now offer real possibilities to integrate simulations as an integral component of modelling activities in schools. In particular, this has the potential to be realised through the use of the steadily increasing number of easy accessible applications to mobile devices (apps). The use of simulations and games on mobile devices with "experimentally" realistic features seem to engage students in learning (Wijers et al. 2010). However, research on the use of applications to mobile devices in mathematics education is still sparse, and more empirical, as well as theoretical, studies are called for (Drijvers 2012). This chapter is intended to contribute to this line of research, in that we seek to investigate how educators can make use of new technologies to include simulations as a more integrated and realistic part of modelling activities in schools to support students' learning and doing modelling.

43.2 Theoretical Framework

In this chapter we use Gibson's (1979) notion of affordances as the framework for describing and analysing the potential for students' actions in learning environments. Although affordances are widely used in research literature, there exist multiple definitions and an agreed and shared definition of affordances is hard to find (Brown et al. 2004; Hammond 2010; Scarantino 2003). According to Gibson (1979), affordances are relationships between objects and individuals in an environment that facilitate interactivity. Affordances mediate the opportunities for what might happen (*happenings*) or what an individual will be triggered to do with a particular intention (*doings*) in an activity within an environment (Scarantino 2003). However, affordances of themselves do not automatically imply that the happenings and doings will occur. The individual's abilities such as skills, motivations and metacognition are also conditions for some happenings and doings to be realised. Scarantino (2003) differentiates between *surefire affordances* and *probabilistic affordances.* Surefire affordances are guaranteed to trigger certain happening and doings, whereas probabilistic affordances have a likelihood to trigger their associated happenings and doings with a given probability ($p < 1$).

Gibson (1979) used a specific type of linguistic form to refer to affordances, achieved by adding the extension "*-ability*" to a given *verb phrase*: [verb phrase]-ability. For example, in the context of ICT, a software environment may provide affordances, such as *Data display-ability* (offerings of plots of numerical data) and *Function view-ability* (offerings of particular views of functions) (Brown 2015). However, as Hammond (2010) points out, one might argue that in an ICT context the two affordances above may be related to either *symbolic properties* in the environment or *physical properties.* According to Hammond, "affordances arise because of real physical and symbolic properties of objects. Affordances provide both opportunities and constraints" (2010, p. 216). Here, physical properties are similar to hardware and concrete objects in an environment, whereas symbolical properties capture higher-level software instantiations in terms of interfaces and texts.

We are motivated by Hammond (2010) who argues that "there is a strong case for using the term affordances in discussing ICT ... [as] Gibson gives distinctive insights into the relationship of tool and user and points us to the right question: how do user and tool come together?" (p. 215). This will inform us about students' experiences with a game app and so is useful for identifying principles for design and implementation of modelling activities. We follow Scarantino (2003) who proposes the following conceptualisation of affordance bearers which stresses the temporal aspect of the manifestation of affordances: "given background circumstances C, an organism O *can at t* engage in an event that qualifies as a doing or happening M and involves X, then X is at t an affordance-bearer with manifestation M relative to O in circumstances C" (p. 958, italics in original).

43.3 Aim and Research Questions

The aim of this chapter is to present the results from the first iteration of a teaching experiment centred around students engaged in group work with a game app to develop principles to design and implement modelling activities. Departing from the circumstances, a prototype classroom activity C, we sought to improve the design so that it is more likely that the happenings and doings M triggered by the perceiving of affordances offered in C support students' learning towards designated learning objectives. In other words, we wanted to investigate students' experiences of working with the game app within the intervention environment and to compare those experiences with the corresponding intended outcomes for the designed prototype. Our hypothesis in the redesign process was that alignment of students' impressions and experiences with the designed intervention would increase the chances of students' productively perceiving affordances. To gain insight into how to improve the intervention prototype accordingly, we address the following research questions:

1. What are students' first impressions of the intervention prototype in terms of students' expressed experiences, and what do they suggest to be changed for the redesign?
2. What areas of mathematics and mathematical content do students associate with the activity?

43.4 Methodology

For the design, implementation and redesign of the intervention in the teaching experiment, we adopted an Educational Design Research (EDR) paradigm, that is "the systematic analysis, design and evaluation of educational interventions with

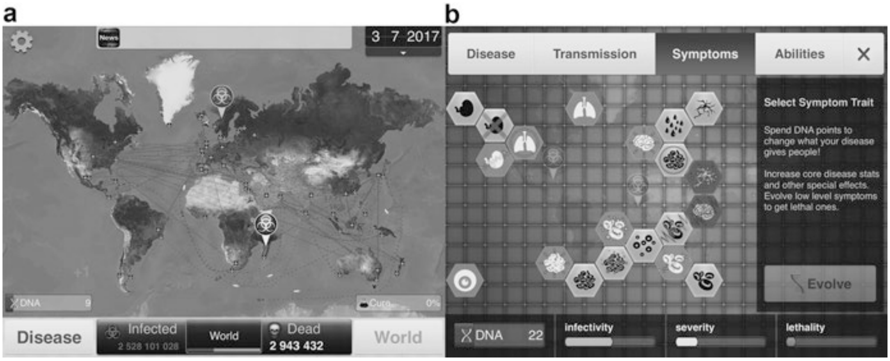

Fig. 43.1 Screen shots: (**a**) the *World tab*, (**b**) the *Disease tab* for evolving the disease

the dual aim of generating research-based solutions for complex problems in educational practice, and advancing our knowledge about the characteristics of these interventions and the processes of designing and developing them" (Plomp 2013, p. 16). This choice provided us with both guidance for intervention design and connected with our overall goal of identifying principles to design and implement modelling activities using game apps. The EDR-based methodological approach systematically *analyses*, *designs* and *evaluates* interventions (Plomp 2013). To begin we briefly describe the game apps, *Plague* and *Infection*, the two free games for mobile devices used in our study.

43.4.1 The Game

The goal of both games, *Plague* and *Infection,* is to create and manipulate a virus and try to evolve it into a *pandemic*. Ultimately, the goal of the games is to wipe out the world population as fast and effectively as possible. The two games are for all practical purposes identical in design, function, controls and game interface, so we illustrate using *Plague*. When starting a new game, one needs to select a plague type and the difficulty level to play the game and to name the virus. The game simulation begins when a player places an infected patient zero in the country where the outbreak will take place.

The taskbar at the bottom of the screen in Fig. 43.1a provides the gamer with statistics such as how many people are infected and the number diseased at any given time. There is also a news banner and calendar at the top of the screen in Fig. 43.1a making it possible to track elapsed time and dates. In addition, there is information which tells the gamer about the progress of a cure that is being developed as well as the number of so-called *DNA points* collected. DNA points essentially form the game's currency which is gathered in different ways during the game, for example, by touching the circular symbols in the world map in Fig. 43.1a.

At the bottom of the screen, the gamer can also access the *Disease* and *World tabs*. In the Disease tab, see Fig. 43.1b, DNA points can be used to alter and manipulate the virus. Circumstances may be altered and evolved, such as virus transmission, symptoms induced and resistance to drug treatments. The alterations affect virus infectivity, severity and lethality. The gamer has to develop a strategy to make the virus widespread and to alter it in order to make it as lethal as possible to win the game (i.e. wipe out all human life worldwide). The World tab gives the gamer information about how the virus is spreading and how the cure being developed progresses and data on the number of infected and diseased persons.

43.4.2 Design of the Intervention Prototype

To design the intervention prototype, an epistemological analysis of the game app was carried out based on the learning goals in the Swedish mathematics curricula. We focused on the goals: *use digital tools*, *use of mathematics for exploring realistic scenarios*, *interpret and use different representations of data such as tables and graphs* and *investigate the role of exponential functions in a given context* (Skolverket 2012), which we consider important components of classroom teaching and learning of mathematical modelling involving exponential phenomena using technology and simulation. Departing from these learning objectives, we then turned to identifying affordances of the environment (C) provided by the game app (X) that potentially would provide the opportunity for an individual (O) to interact in the manifestations (M) of the enactment of these affordances realising the intended learning outcomes. Based on the authors' experimenting and experiences with the game app the following affordances related to the efficient use of the app for modelling were identified:

- *Simulate with-ability* (an opportunity to simulate a realistic scenario)
- *Multi-solution-possibility* (opportunities for students to get different outcomes)
- *Interactivity* (opportunities for hands-on activities)
- *Data-generate-ability* (opportunities to use different data such as time, number of infected, number of deaths)
- *Plot-ability* (opportunities to get plots of data)
- *Multi-analyse-ability* (opportunities for data to be analysed in different ways dependent on curricular goals such as learning exponential *functions (exponential analyse-ability)* and logistic growth (*logistic growth analyse-ability*)
- *Efficient-time solve-ability* (an opportunity to complete the game in approximately half an hour)

In addition, taking the relative complexity of the game app as well as potential variation in student familiarity with this type of game into account, the intervention prototype (C) was designed to include two game-related resources for students: a *manual* describing the game and how to play it and *instructions* for the actual classroom activity.

The manual consisted of a two-page handbook with historical facts about pandemics that have swept over the world, descriptions of how to download the games and basic information and strategies on how to play. The main intended function of the manual was to prepare students before the activity in the classroom, so they would be familiar with the game and have more time for collecting and analysing data in class.

The instructions also included historical facts about pandemics to set the scene, but foremost imperatives to structure the activity, for example, *play the game at the easiest level*; *take data points in terms of number of infected and dead in the world every other week*; *record the data in an Excel spreadsheet*, *play until you win or lose*; and *draw a graph of how the number of infected and dead evolved as the game progressed*. In addition, the instructions contained 11 tasks for students to investigate and address. Examples of student tasks were: *Draw a sketch of the graph of number of dead and infected and describe the spread of the plague. Are the number of infected decreasing and if so why? Are there any connections between the graphs? Describe the spread of the plague during month 0–6 and 7–12*. *When is the risk greatest to get infected?* Instructions were also provided to assist students with little experience in using Excel to explore a data set with how to use it for making tables and graphs.

43.4.3 Implementation of Intervention Prototype

The intervention prototype was implemented and evaluated in two upper secondary classrooms (grade 10) in May and June 2014. The students were given the manual a couple of days before the implementation and asked to familiarise themselves with the game. The students were also instructed to bring tablets or mobile phones to class on the day of the implementation. The teachers for the two classes independently prepared the activity by bringing laptops with Excel and the instructions to their classrooms. After a short introduction to the activity by the teacher, the students followed the instructions and worked in pairs to collect data that they analysed using Excel. The students then worked with the questions in the instructions. At the end of the 90-min activity, the students individually completed an evaluation form.

43.4.4 Evaluation Instrument

To evaluate the intervention prototype, the students were given an evaluation form at the end of the class. The instrument consisted of 29 items divided into three items (short answers) about background information (gender and type of device used), eight items (Likert scale) relating to students' previous experiences of mobile apps used in education, four items (Likert scale) about students' expressed experiences about working on the activity and ten items (Likert scale) focusing on the clarity of the instructions. The Likert scales used were five-point scales ranging from strongly disagree (1) to strongly agree (5). There were an additional four open questions: *What mathematics do you think this activity includes? What have you experienced as positive/negative with the activity? What do you think should be changed next time?* as well as a question asking for any other comments or thoughts about the intervention prototype. In this chapter, we report from the four Likert questions about how the students expressed their experiences of engaging in the intervention prototype and the three open questions above. The fourth question did not provide any further insights in relation to the other questions.

43.5 Results and Analysis

The analysis of the 16 female and 14 male students' answers in the two classes on the evaluation form regarding their expressed experiences about the activity in the intervention prototype being *fun*, *interesting*, *easy* and *engaging* is summarised in Table 43.1. All 30 students answered all questions except one male student who did not provide an answer to the question of whether he thought the activity *Pandemic* was easy.

From Table 43.1 one can generally conclude that female students' expressed experiences of the activity *Pandemic* was less fun, less interesting, less easy and less engaging (mean scores ranging between 2.50 and 2.75) compared to the male students' expressed experiences (mean scores ranging between 4.07 and 4.36). Independent t-tests show that the differences in the scores between female and male students are statistically significant in all four questions (Fun: $t(28) = -3.70$, $p = 0.001$; Interesting: $t(28) = -3.38$, $p = 0.001$; Easy: $t(27) = -4.59$, $p = 0.000$; and Engaging: $t(28) = -3{,}94$, $p = 0.000$). Viewing the categories in Table 43.1 as triggers potentially affecting the probabilities for realising happenings and doings related to probabilistic affordances, this shows a discrepancy between affective tendencies towards the activity in the prototype which needs to be considered in the redesign if all students are to productively engage in modelling.

The students' answers to the first three open questions were generally short and unambiguous. Hence, the answers could easily be grouped into categories based on

Table 43.1 Students' expressed experiences about the activity

		I think the activity pandemic was:			
Gender		Fun	Interesting	Easy	Engaging
Female	Mean (M)	2.69	2.75	2.56	2.50
	Std. deviation	1.621	1.183	1.315	1.366
Male	Mean (M)	4.36	4.07	4.46	4.14
	Std. deviation	0.745	0.917	0.776	0.864
Total	Mean (M)	3.47	3.37	3.41	3.27
	Std. deviation	1.525	1.245	1.452	1.413

Table 43.2 Students answers to the first three open questions summarised in categories

Questions	Categories with frequencies of responses	
What mathematics do you think this activity includes? ($n = 19$)	11 statistics	3 nothing
	10 diagrams	3 draw diagrams with ICT
	6 tables	2 interpret diagrams
	5 graphs of functions (1 related to exponential graphs)	1 I don't know
		1 use of models
What have you experienced as positive/negative with the activity? ($n = 23$)	16 fun	5 the app in itself
	14 too many data points and too little time	5 more instructions
	11 everything was good	4 to use excel
	6 relation to real life	3 no clear aims with the activity
	6 different activity	2 easy
	6 boring	1 difficult to get going
		1 group work
What do you think should be changed next time? ($n = 18$)	Shorten the time with the game	
	Nothing should be changed	
	Should not be used	
	The teacher should be more detailed when introducing the activity	

the topics raised. If a student's answer entailed more than one topic, it was coded to belong to multiple categories. The result of this grouping is summarised in Table 43.2.

The areas of mathematics and the mathematical content the students associated with the activity in the intervention prototype mainly related to statistics manifested in the categories *statistics*, *diagram* and *tables* (see Table 43.2). These categories can be viewed as pointing to a *descriptive statistic apply-ability,* providing opportunities in the intervention prototype for triggering happenings and doings related to teaching and learning objectives in statistics. Other categories we associated with affordances are *plot-ability* (draw diagrams with ICT), *function view and interpret-ability* (identify type of function and interpret functions) and *model use-ability* (the use of models). However, only one student explicitly related exponential functions as mathematical content in the activity, which was one of the learning objectives of the intervention prototype that would be a desired outcome as a modelling activity.

The category summarising most of the answers to the question: *What have you experienced as positive/negative with the activity?* was *fun*, which included statements about finding the activity engaging and exciting. *Fun*, alongside the categories *everything was good*, *easy* and *boring*, relates to students' general willingness to engage in the activity.

Some of the categories capture the students' experiences in relation to symbolic properties of the game app. For example, six students stressed *the relation to real* *life* in that the activity entailed dealing with a context they considered to be from outside the classroom of mathematics relating to how diseases spread. We identified and refer to this as a *creating reality-ability* affordance. A physical property of the game app articulated was that the app consumed too much battery. Five students' answers expressed frustration and discontent related to properties of *the app in itself* that it was difficult to pause the simulation in order to record data, as well as finding the sound of the game annoying and tiring.

Looking at the intervention prototype more broadly, but in terms of qualities that would affect the utility of the learning activity for modelling, the students expressed *use of Excel* as a good experience as well as being positive to engaging in *group work* (*collaborative learning-ability*). Five students expressed negative or mixed feelings about the challenge to come up with a game strategy needed to win the game. Others ($n = 14$) expressed the opinion that they were instructed to *get too many data points* and that *there was too little time* to finish the activity in the designated time.

The students' answers to the question: *What do you think should be changed next time?* were in line with many of the comments in the answers depicted above and did not present any further information on our goal of elaborating principles for a modelling app implementation.

43.6 Conclusions, Discussion and Implications

Overall, students were positive towards the intervention prototype in terms of the activity being fun, interesting, easy and engaging. This signals a general willingness to engage in a modelling activity where the use of the app is an integrated part. However, a significant divide and difference between female and male students' experiences were noted. These findings are in line with results from previous studies on students' attitudes towards using simulations and games on mobile devices (Wijers et al. 2010) and that technology, in particular, engages boys' interest (Schreiner and Sjøberg 2004). One potential reason for the difference expressed with respect to gender might be how the game context was used in the activity, with the aim to terminate humanity. A way to try and come to terms with this aspect in the redesign could be to clarify the aim with the activity and stress aspects of modelling: for the students to explore how mathematical models are used to predict how infectious viruses spread around the world and what assumptions and parameters are used to calculate the effects of mass vaccination programmes. In addition, emphasising that the playing time is secondary in relation to learning objectives may trigger happenings and doings associated with the intended modelling goal manifested in the *model use-ability*, *exponential function apply-ability* or refocus the activity stressing *descriptive statistic apply-ability.*

The intervention prototype stressed *efficient-time solve-ability* (playing time is approximately half an hour per game), but the result indicated that data collecting was experienced as tedious and boring. Engaging students in solving complex authentic modelling problems (*creating reality-ability*) in collaboration with others (*collaborative learning-ability*) requires students to reflect and discuss different strategies (*multi-analyse-ability)* and solutions (*multi-solution-possibility*), which capture some key ideas about work habits for modelling in the workplace (Frejd and Bergsten 2016). For the activity to be more authentic, we could increase the students' autonomy and in the redesign let the students decide and make decisions on how much data to collect, which is a central aspect of modelling activities. Other affordances stressed in the intervention prototype such as *simulate with-ability* (an opportunity to simulate a realistic scenario)*, interactivity* (opportunities for hands-on activities)*, data-generate-ability* (opportunities to use different data such as time, number of infected, number of deaths) and *plot-ability* (opportunities to get plots of data) are not only central within this particular design but also more generally emphasise the connection between modelling, new technologies and simulation.

Recent advances in technology and students' increased access to digital tools have changed the possibilities to engage in realistic simulations (apps) and have added new dimensions to the teaching and learning of modelling. Based on our results, we suggest that the affordances identified and discussed in this chapter could be used as a first set of tentative principles for designing and implementing modelling activities using game apps. However, more research is needed to further develop these principles so that the potential and role of new technology and simulation can be explored and put to productive use within everyday classroom teaching and learning of mathematical modelling.

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Chapter 44
Modelling and Simulation with the Help of Digital Tools

Gilbert Greefrath and Hans-Stefan Siller

Abstract This chapter undertakes a theoretical and empirical examination of modelling with digital tools in mathematics instruction. First, modelling processes that integrate the use of digital tools are considered from a theoretical point of view. With the help of several significant examples, the varying added benefits of digital tools in modelling and simulations are clarified and put into perspective within the theoretical discussion. The relationship between modelling and simulation is also clarified. To complement the theoretical discussion, a qualitative, empirical study, examining *what activities* students actually perform when using a digital tool, Geogebra, for working on modelling tasks and *where* these activities are located within the modelling cycle is reported.

Keywords Technology • Digital tools • Computer • Qualitative empirical research

44.1 Introduction

Mathematical models and simulations in mathematics instruction have gained importance, due to, amongst other things, the current prioritisation in educational standards, the continuing development of digital tools and the increased use of models and simulations to solve real-world problems. They have been a growing subject of attention and didactic discussion, because many problems facing the world today are becoming more and more complex, increasing the need for mathematical

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_44

models in order to obtain the most objective possible decision criteria. Simulations become especially important thanks to the relatively inexpensive and sufficient computer capacities that are available. In our view, another reason is that (simple) simulations are generally more cost-effective than real experiments. Yet, it is also possible to test scenarios that could, or should, never occur in reality, for example, the spreading of oil in the Atlantic or the effects of a major nuclear disaster in Tokyo. This approach can also be used effectively for educational purposes without such simulations actually going as far as the complex issues of the real world. The usefulness and necessity of simulations in the context of mathematical modelling can already be shown on a small scale.

Modelling activities in mathematics instruction are also subject to the influence of digital tools. Digital tools can be of great assistance for teachers and learners alike, particularly in connection with real-world problems and the discussion of those. Simulations are also of interest in the context of using digital tools for modelling in mathematics instruction. "Simulation, the creation of an analogue of a real world situation, is increasingly used in many areas of education and training. Aircraft simulators for pilot training, business games for managers, ... are all examples of simulations" (Clements 1984, p. 319).

A *simulation* thus serves to investigate an operation, a process or an experiment with the help of mathematical models. To this end, technical, biological or economic content is modelled and simulated with the help of mathematical models – usually in conjunction with a digital tool. This means that simulations are inseparably linked to mathematical modelling. The topic of modelling and simulation is thus of great interest for mathematical didactics, particularly because it testifies to the social relevance of mathematics.

44.2 Theoretical Framework

44.2.1 Modelling with Digital Tools

The German educational standards for mathematics at primary and secondary level and for higher education entrance qualification describe mathematical modelling as a competency. Digital tools can support various processes of, and for, modelling. This is described in more detail in the German educational standards (KMK 2015). The potential of digital tools is emphasised there. Particular value is seen:

- In the discovery of mathematical relationships...
- In promoting the understanding of mathematical relationships, not least by means of diverse visualisation possibilities.
- In the reduction of schematic processes and the processing of large amounts of data.
- In the ... reflected use of control options. (p. 13)

Concerning the *discovery of mathematical relationships*, digital tools are especially important in mathematical experiments. Such experiments are carried out on a real or mathematical model, say, in connection with population developments, traffic situations or the functioning of technical devices.

44.2.1.1 Example Rescue Helicopter

For the best-possible stationing of a rescue helicopter (Greefrath and Weitendorf 2013, p. 198), data based on the accident frequency in a certain area can be acquired with the help of a spreadsheet, and a suitable site for the helicopter can first be found experimentally. With the help of dynamic geometry software, the real-world situation can be translated into a geometric model. For the case of a helicopter and three accident sites, we can fall back on known geometric objects. Continuous models, in which each site is suitable as the helicopter site, and combinatorial models, in which only certain sites are permissible, can be distinguished for several sites (Ortlieb et al. 2009, p. 88).

Digital tools can be used to create different representations, it is possible to switch between representations relatively simply, and several, interactively linked representations can be generated on the display simultaneously (Weigand and Weth 2002, pp. 36–37).

It is possible to *reduce schematic processes*, especially in conjunction with computer algebra systems (CAS) as seen in this next example.

44.2.1.2 Example Milk Packaging

When making calculations in the context of optimal packaging problems, a milk package, for example (Böer 1993), rational functions arise, in which the zeros of the first derivative can no longer be determined exactly with the methods of school mathematics without the aid of digital tools. In this case, a CAS can be used to determine and visualise the derivative function and to calculate its zeros. Furthermore, the calculations can also be performed graphically and numerically. In addition, it is possible to discuss the complex, real-world issue – say, taking the flap of the packaging into account – in the mathematics classroom (cf. Deuber 2005).

The verification and control of solutions obtained is an essential mathematical activity. Digital tools can support these control processes, for example, with the help of graphic visualisations of numerical calculations, when solving equations, in terms of conversions or when working with discrete and functional models.

44.2.1.3 Example Bacterial Growth

In the growth of a bacterial culture, the increase in bacteria over a period of time can be assumed to be proportional to the existing population and to the time elapsed, at least neglecting further interactions with the outside world or the limits of the growth process due to physical conditions. The extent to which this theoretical model can be applied to a real growth process can be checked graphically or analytically by calculating a regression line.

The various functions of digital tools in mathematics instruction are applied to modelling problems at different steps in the modelling cycle. Control processes, for example, are generally assigned to the last step of the modelling cycle. Calculations are made with the help of the generated mathematical model, which is usually a function, for example, in analysis. Some of the potential uses of digital tools in a modelling process are shown in Siller and Greefrath (2011); as in Geiger (2011, p. 312), the view on how technology can be utilised at several steps in a modelling cycle can be found. In addition to this integrated presentation of the use of digital tools in modelling, the literature also has modelling cycles that represent technology as a separate area between the mathematical model and mathematical results (Daher and Shahbari 2015; Maki and Thompson 2015; Savelsbergh et al. 2008; Siller and Greefrath 2010).

Regardless of this depiction of the integration of digital tools into the modelling cycle, there are still many open questions concerning the use of digital tools in modelling, as Niss et al. (2007) show in detail:

- How should technology be used at different educational levels?
- What implications does technology have for the range of applications and modelling problems that can be introduced?
- How is the culture of the classroom influenced by the presence of technological devices?
- When does technology potentially kidnap learning possibilities, e.g. by rendering a task trivial, and when can it enrich them? (Niss et al. 2007, p. 24)

44.2.2 *Simulations*

Simulations are an opportunity to answer questions about real-world problems. In some cases, they are the only way to handle a problem. They can be carried out using real, mathematical models or computer models. Simulations are used to collect data that can be applied for a variety of purposes. One possibility is to obtain information about the simulated system. Another is to use the data to optimise the model used. This can be achieved by comparing the data acquired from the simulation with the real data. In such cases, the simulation is a part of the modelling cycle for the development of a suitable model of a real situation (Sonar 2001). This is depicted in Fig. 44.1.

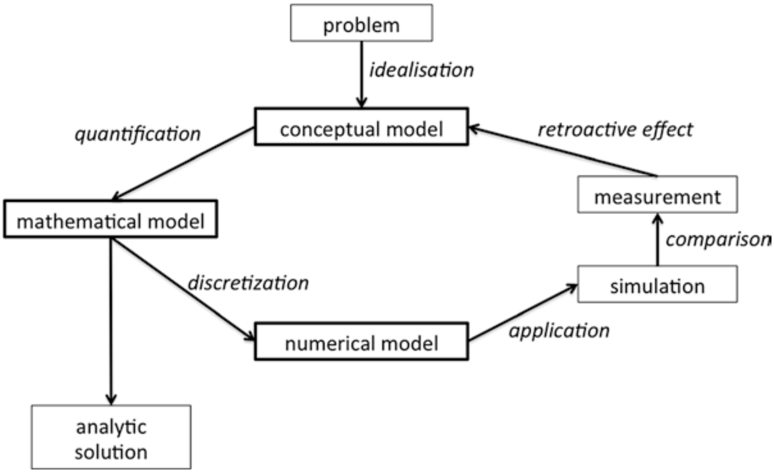

Fig. 44.1 Simulation as part of the modelling cycle (Sonar 2001)

The various activities described within the modelling cycle become especially clear in simulations aided by digital tools. Simulations involve experimentation with mathematical models – usually in conjunction with a digital tool. Yet simulations can also promote the construction of models in the same way. Simulations can support the design of various, elaborate models, particularly when an associated reference to reality is examined (cf. Siller 2015). In such cases, a real situation is selected as a starting point, and a mathematical description of the situation is generated for further investigation with the help of models.

44.2.2.1 Example Mountain Ascent

The calculation of an energy-optimised mountain ascent (Bracke et al. 2015) first requires a model of the terrain, which must be generated from available data points. Since we are starting with an extraordinarily large volume of data, we must consider how the digital tool will process these data and whether a prior reduction of the data points is necessary and sensible for further work. In the case of an energy-optimised climb, the path on the collected terrain model can be implemented by further mathematical considerations, so that alternatives can be simulated on this basis.

Extra-mathematical issues, real-world circumstances in particular, can be integrated into mathematics instruction by modelling activities supported by a simulation. In the process, learners acquire not only mathematical skills but also such interaction with mathematics also develops their ability to interpret and assess. Mathematical discussion of real-world problems is also integrated.

The mathematical models used in simulations can be viewed as a simplified representation of reality taking only a few, primarily objective (sub)aspects into account. This interpretation permits a diverse variety of model variants. Two model types can be identified as being particularly important, the *descriptive model* and the *normative model*. Descriptive models describe, explain or predict (real) processes. With them, we attempt to generate a representation of aspects of reality that is as accurate as possible under certain perspectives. In contrast, normative models stipulate certain facts or processes to the user. Descriptive models are highly suitable for learning mathematics and modelling, because they are always designed with specific intentions in mind. In any case, the mutual relations between *mathematics* and the *rest of the world* are at the heart of every modelling project (Pollak 1979). It is indispensable that we take up a real-world issue and engage ourselves with it.

In order to create a simulation, decisions must be made as to the type of simulation. These decisions concern time dependency and predictability. If a time dependency is considered, we refer to it as a *dynamic simulation*. Since the focus is on a time-dependent change, we can refer to such simulations as "carrying out experiments with models of dynamic systems" (Krüger 1974, p. 24). For example, we can take the flow of traffic on a street with several traffic lights and consider the optimisation of light changes as a function of time. There are also simulations in which time does not pay a role. These are called *static simulations*. Another characteristic of simulations is their predictability. If a simulation is employed to find a value several times in a row with randomly varying results, we call it a *stochastic simulation*. In contrast to these stochastic simulations, random aspects do not arise in *deterministic simulations*. For example, if we investigate the movements of a fairground ride, which always run identically under the same starting conditions, this would be an example of a deterministic simulation (cf. Greefrath and Weigand 2012).

44.3 Design of the Study

44.3.1 Research Issues and Instruments

On the basis of the theoretical background, the following research issues were examined:

- At which steps in the modelling cycle can digital tools be employed?
- What activities are performed with the digital tools in the modelling process?

To investigate the solution processes of students, an open *Pirate Task* created by Laakmann (2005, p. 86) with a reference to the real world was used:

Pirate Task

On a foggy November morning a patrol boat sets sail from the safe harbour, to track pirates. The conditions for this are very bad, because the estimated visibility is only about 500 m. Nevertheless, the commander orders the boat to head Northeast. The boat leaves the port at 7 o'clock in the morning. At the same time a pirate ship with a mast height of about 45 m sets sail toward the Southeast. It has a speed of about 10 knots. As the patrol boat is leaving the port, the pirate ship is located 7 km to the north of the port and 2 km to the east of the port. The patrol boat makes approximately 15 knots and is one and a half times as fast as the pirate ship. Will the pirate ship be spotted?

This task can be solved in different ways. In this case, the added benefits of using digital tools are apparent. For example, the task can be handled with the help of dynamic geometry software, a spreadsheet or a computer algebra system. The resulting mathematical models differ, resulting in different perspectives that must be taken into account (Siller and Greefrath 2010).

44.3.2 Participant Selection and Study Implementation

For the study, students were selected who had already worked with the dynamic geometry software, Geogebra, in the classroom for three lessons before the study started. The qualitative study was carried out at a secondary school in Münster (Germany) at the end of year 10. Four pairs of students were observed as they performed their tasks. Every student pair worked in a separate room with a researcher and a video camera. The students were instructed to solve the problem using Geogebra without any further aid. The students' efforts were recorded using a video camera.

44.3.3 Data Analysis

To evaluate the observations, the videos of the observations were completely transcribed. The transcripts contain all verbal exchanges. The subsequent coding process took place in several steps. The first step was to develop the categories. To do this, conceptual terms were assigned to the individual statements of the students in the context of *open coding*. These terms were discussed and modified over several rounds (Strauss and Corbin 1990). The objective was a description of the solution process independently of the concrete task at hand for the students so that these categories could be reapplied to later studies and solution phases compared on the basis of these categories. The second step was to confirm and verify the categories. The following categories were developed for when digital tools were used on the basis of the material available:

- Draw: drawing simple geometric objects into a coordinate system (e.g. points, lines, sections, circles)
- Visualise: drawing in or moving points or segments in order to represent previously found values graphically (e.g. moving points in order to create a segment of previously determined length)
- Construct: drawing more complex geometric objects and configurations, using intermediate steps/auxiliary lines to do so (e.g. angle bisectors)
- Measure: finding the distances between points, the lengths of segments, the sizes of angles or the gradients of lines and segments
- Experiment: changing the parameters, conditions or assumptions of a drawing and observing the effects
- Calculate: making calculations with a handheld device or a software-based computer
- Researching: researching information on the internet (e.g. the meaning of the word “node”) (Vehring 2012)

The partial competencies of modelling in the cycle of Blum and Leiss (2007) were used as the basis for categories for the solution processes within the modelling cycle to address the second research issue.

44.4 Results of the Study

With the help of the categories for the solution process within the modelling cycle, the individual modelling paths of the student pairs could be plotted. At the same time, the categories could be used to find the corresponding activity for the use of digital tools. The results of the student pairs thus showed the modelling path of the pair within the modelling cycle including the use of digital tools.

Figure 44.2 shows the path of student pair A. The other three student pairs had similar paths. Digital tools were employed primarily between the situation model, real model, mathematical model and the mathematical results in the modelling cycle. In addition, tools were also used directly between the situation model and the mathematical model, where there is actually no direct connection shown in this modelling cycle.

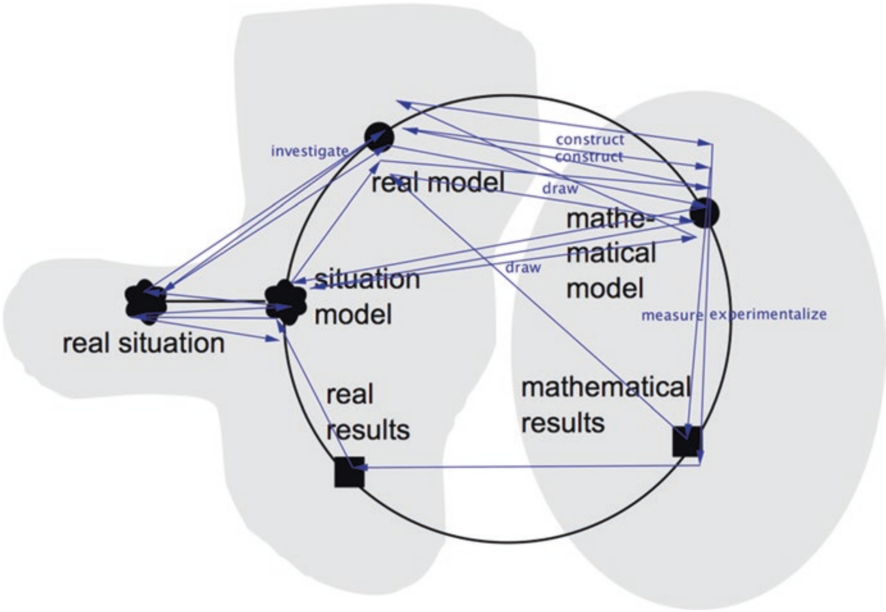

Fig. 44.2 Modelling path and digital tool use, student pair A

44.5 Discussion

The study is a contribution to the discussion of how modelling with digital tools can be investigated and illustrated best. The observations show that digital tools can indeed be utilised at different steps in the modelling cycle. However, if we took into account only the partial competencies described in the modelling cycle with respect to the ascertained use of digital tools, it was shown that the students did not use the tools while interpreting and validating the mathematical results, for example, using another visualisation or checking the results by repeating the steps. Of course, it is not to be expected that the students will go through the steps in the order of the modelling cycle, but we assume that all partial competencies are necessary for the solution. There could be a number of reasons for the missing interpreting and validating with digital tools, which could be associated with the students themselves or the selected tasks.

The study shows that the modelling cycle used describes the utilisation of digital tools in modelling processes meaningfully, as does that of Geiger (2011). A modelling cycle that represent technology as a separate area (cf. Siller and Greefrath 2010), locating it exclusively between the mathematical model and the mathematical solution, does not describe these modelling processes sufficiently. The learners use the tools in very diverse ways for researching, constructing, drawing, calculating, measuring, experimenting and visualising. To reinforce these activities in the classroom, we recommend as well, the use of simulations that naturally link modelling with the use of digital tools. However, further research with a broader range of data and other example problems are required.

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Chapter 45
Mathematical Modelling for Engineering Diploma Students: Perspectives on Visualisation

Hanti Kotze, Gerrie J. Jacobs, and Erica D. Spangenberg

Abstract This inquiry aims to determine the influence of mathematical modelling on engineering diploma students' visualisation when solving differential equations (DE) with a computer algebra system (CAS). In CAS environments, students usually struggle to interpret numerical tables and computer graphs derived from symbolic DEs and often leave interpretative questions unanswered. Participants comprised 80 second year vocational engineering diploma students at a comprehensive university. Students' abilities to make contextual connections between different representations through a model-eliciting task were assessed using content analysis. By reversing the curricular approach, most participants constructed a meaningful DE that deepened understandings of the world in which they modelled. The modelling environment stimulated development of adequate schema through experimentation with paper-and-pen and CAS technologies.

Keywords Computer algebra system • Differential equations • Engineering diploma students • Visualisation • Mathematical modelling • Multiple representations

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_45

45.1 Introduction

Engineering students are increasingly being exposed to a technology-rich environment where conceptual knowledge and technological skills need to be integrated. To interpret and analyse computer-generated graphics are vital skills in engineering applications but demand conceptual understanding (Hjalmarson et al. 2008). When foundational mathematics relies extensively on paper-and-pencil graphs, students develop "traditional symbolic lenses" (Nardi 2014, p. 208) and often lack "visual literacy" (Natsheh and Karsenty 2014, p. 119). The transition from paper-and-pen techniques to a computer algebra system (CAS) is a nontrivial process, and visualisation cannot simply be assumed. In particular, engineering students in Singapore are able to solve difficult differential equations (DE) symbolically but fail to validate CAS solutions (Soon et al. 2011). In their study these authors used a mathematical modelling approach to emphasise the real-world contexts of DE through visualisation with non-stereotyped problems. Stillman et al. (2013, p. 22) point out that for engineering students, redesigning tasks so as to make modelling more visible can be useful and important in making "students more successful modellers and designers" and more appreciative of the applicability of modelling activities.

School curricula in South Africa do not promote visualisation opportunities and/or the use of CAS calculators. This can cause visualisation deficits for engineering students when first exposed to a CAS environment. As observed in Singapore (Soon et al. 2011), many engineering diploma students almost thoughtlessly jump straight into a CAS task and fail to (1) relate DE as models of real-world phenomena, (2) connect and contextualise numerical and graphical representations with symbolic equivalents and (3) interpret and reflect on computer-generated tables and graphs. These failures are compounded when analytical approaches are taught in isolation from other equivalent representations, an unfortunate stereotype that still prevails in South Africa. This chapter explores the influence of a researcher designed modelling task, that is in the form of a model-eliciting activity (MEA) (Lesh and Doerr 2003), on engineering diploma students' visualisation when solving DE using CAS. The following research question was posed: What is the influence of a first-time encountered mathematical modelling task on the development of engineering diploma students' visualisation when solving DE using CAS?

45.2 Theoretical Perspectives

45.2.1 Visualisation with Cognitive Technology

Scholars have attached different meanings to the concept of visualisation. In generic terms, visualisation is the act of forming a mental image (Zimmerman and Cunningham 1991). For centuries, teachers have produced visual images to illustrate and explain mathematical concepts. Since the mid-1980s, the notion of visualisation became strongly associated with representations of computer objects.

Emergent technologies in mathematics education spurred momentum into the research of the cognitive value afforded by visualisation. The centrality of visualisation must therefore go beyond the mere perceptual and also involve a deep engagement with the conceptual (Arcavi 2003). Tools such as paper-and-pen and CAS are cognitive technologies since it can help to transfer internal images created in the mind to external images that can then be analysed and reflected upon. The end product of visualisation is not the visual object itself, but the meaning-making of the underlying concept. It is believed that visualisation of conceptually rich images is cognitively more demanding than analytical procedures (Arcavi 2003). For this reason, many school students are reluctant visualisers and find it difficult to reconcile mental images with digital images (Brown 2015). In-depth understanding and proper mathematical meaning comes with fluency between different presentations of a concept. This study adopts the definition of Arcavi (2003, p. 217); *visualisation* is

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

45.2.2 Theoretical Lens

APOS theory is used in this study as the theoretical lens to analyse the development of visualisation in a modelling task. According to Buchholtz (2013, p. 103), theory as a lens enables researchers to "view" or to "observe" in order to formulate perspectives accordingly. Dubinsky and McDonald (2001, p. 276) developed APOS theory which refers to the mental construction of "actions, processes, objects and schemas" in order to make sense of phenomena. They showed how APOS theory can be used in a cooperative technology-rich environment and found that the ability of students to mentally construct concepts through actions, processes, objects and schemas can indicate their success or failure of making sense of situations. According to Dubinsky and McDonald, when one student can progress to a certain point and another student cannot, APOS theory can direct the researcher to the mental construction of actions, processes, objects and schemas that the first student apparently has made but the other student has not.

APOS theory appraises learning of new mathematical concepts through actions on existing processes, objects and schemas. When actions make sense and become meaningful, students can internalise such actions and construct new internal processes. Consequently, new objects can be formed through the encapsulation or conversion of these internal processes. All these actions, processes and objects are reorganised to form a new mental schema. Following Piaget, Dubinsky and McDonald (2001) acknowledge that the learning of new mathematical concepts also requires existing schemas in order to construct new schemas. A new process can therefore be constructed out of existing ones by either *coordinating* two or more processes or through the *reversal* of a process.

45.3 Empirical Design and Method

The study is exploratory in design, and qualitative content analysis was used to analyse documents with process coding (Saldaña 2009). Data were themed with process coding in search of actions, processes, objects and consequences thereof. The extent to which processes and objects were connected and the role these connections played in the modelling of schema were analysed. Content analysis revealed *how* processes unfolded; this was complemented with the researchers' field notes to understand *why* certain actions and decisions were taken.

45.3.1 Engineering Mathematics and Current Curricular Approach

Engineering Mathematics 1 and Engineering Mathematics 2 are compulsory modules in the National Diploma of Engineering offered at universities of technology and comprehensive universities (offering both diplomas and degrees) in South Africa. These modules focus predominantly on paper-and-pen techniques and require students to use a standard (non-graphic) calculator; moreover, group work is not promoted. Unlike at most other institutions in South Africa, the University of Johannesburg's (UJ) offering of Engineering Mathematics 3 not only comprises theory lectures but also involves a weekly Mathematica session of 150 min. Engineering departments are particularly appreciative for students' competencies in Mathematica acquired in Engineering Mathematics 3. Students first learn the Mathematica syntax and programming principles which can then be applied to solve DE with the Euler and Runge-Kutta methods from first principles. The Mathematica curriculum is shaped by routines requiring students to solve a given DE; numerical tables and graphs are then generated and compared with analytical solutions. Engineering programmes do not include statistics or regression analysis.

45.3.2 Participants

The participants in this inquiry were a cohort of 80 Engineering Mathematics 3 students at UJ. Recent Mathematica test scores were analysed to purposively group five to six students together. The task required prerequisite knowledge and skills and was therefore performed towards the end of semester one of 2015 when students were better acquainted with Mathematica and DE theory (Galbraith and Stillman 2006). The task took place during the weekly Mathematica session, and this was the first time these students had encountered mathematical modelling at university.

Table 45.1 World population data between 1900 and 2000

t (in years)	Population *P* (in millions)	*t* (in years)	Population *P* (in millions)
1900	1,650*	1960	3,040
1910	1,750	1970	3,710
1920	1,860	1980	4,450
1930	2,070	1990	5,280
1940	2,300	2000	6,080
1950	2,560	2020	??

Note: The asterisk (*) indicates an uncertainty about the accuracy of this data point

45.3.3 The Task

A *National Geographic* article[1] was used to develop a population model; data were provided for the twentieth century (see Table 45.1). As this was an open-ended task, any mathematics, software, textbooks and internet sources could be used. The task aimed to introduce students to new model-eliciting processes to potentially assist in their understanding and visualisation of DE as concrete models of real-world phenomena (and not made up by the lecturer). New processes were intended to supplement the limited processes of the current curriculum. Groups had to (1) develop a DE that models the world population over the last century, (2) validate the model, (3) recommend a more appropriate initial condition (the population in 1900) and (4) predict the world population in the year 2020. These subtasks involve "a complete modelling cycle, including formulating the problems mathematically, integrating real-world data and mathematical manipulation" (Soon et al. 2011, p. 1026) as well as "the opportunity to use visual images generated by technological tools" (Brown 2015, p. 432). Group reports were collected; these included paper-and-pen documents and electronic documents.

45.4 Findings and Discussion

45.4.1 Analysis of a Single Case

Group 14's solution (Fig. 45.1) is used as an exemplar to formulate perspectives on visualisation in the sense of Arcavi (2003). In a first attempt to construct a model, this group mathematised the real-world data of Table 45.1 visually. This required the given data (*object* 1) to be translated (*action* 1) to a graph (*object* 2) using technology (*process* 1). To perform action 1, two internalised *processes* must be evoked: (1) programming the given data (Fig. 45.1a) as a Mathematica matrix and (2) using Mathematica syntax and commands to generate a graph with an applicable increment and interval. When actions on these *existing* processes are meaningful, a *new*

[1] http://ngm.nationalgeographic.com/2011/01/seven-billion/kunzig-text

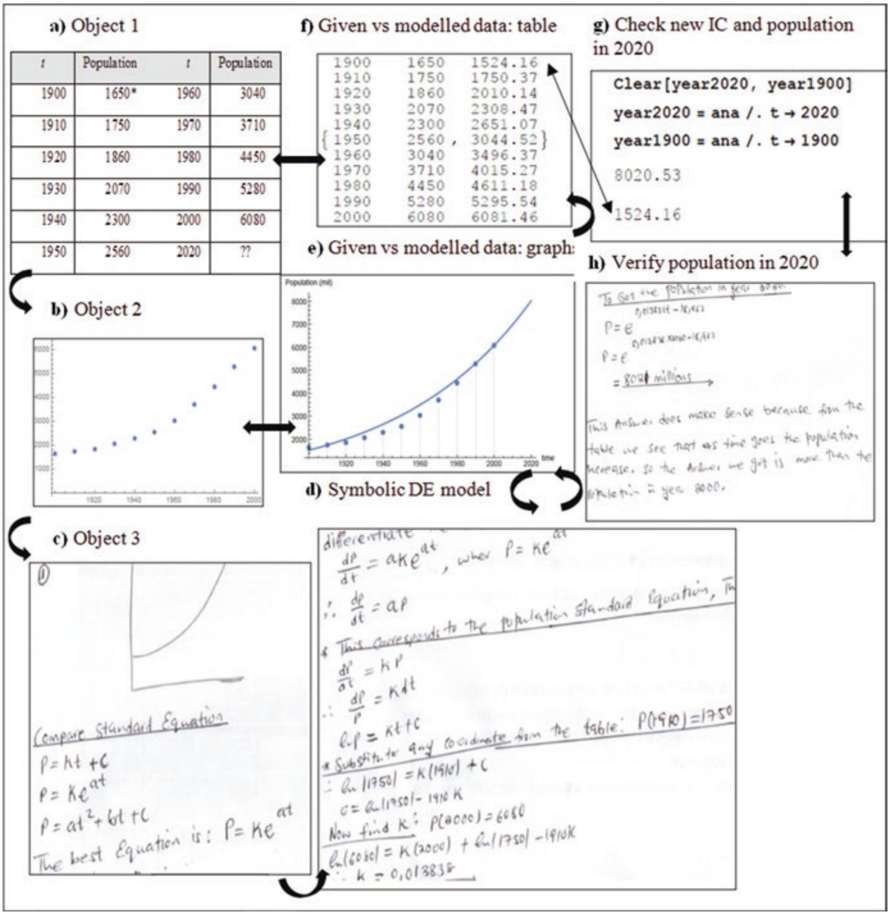

Fig. 45.1 Model development and justification of Group 14. (**a**) Object 1. (**b**) Object 2. (**c**) Object 3. (**d**) Symbolic DE model. (**e**) Given vs modelled data: graphs. (**f**) Given vs modelled data: table. (**g**) Check new IC and population in 2020. (**h**) Verify population in 2020

process is internalised and its end product is the graph in object 2 (Fig. 45.1b). This intrinsically visual model now had to be mathematised symbolically (*process* 2).

To this end, students have to conjure up a repertoire of existing *objects* (e.g. exponential function) and *extract* the *existing* processes from which each *object* was formed. The *process* to construct (*action* 2) the symbolic exponential function $P = ke^{at}$ (Fig. 45.1c) therefore involves reflection on the qualitative properties of the visual model created in *object 2*. This decision-making process is evident when Group 14 listed their options, namely, a linear, quadratic or exponential function. They argued in favour of the exponential model since it “would be more realistic, population cannot go up on the other side” [referring to the shape of a parabola]. Their argumentation was visually induced to sustain the appropriateness of the visual model true to the context of the real-world data (Lesh and Doerr 2003). To develop (*action* 3) the symbolic DE (Fig. 45.1d), this group evoked two internalised processes, namely, differentiation and separation of the variables (*process* 3). Although the symbolic derivation of Fig. 45.1d may seem trivial, these students did not necessarily reconcile the formula $P = A\ e^{kt}$ with the population growth model $dP/dt = k\ P$. A probable explanation could be that students are inexperienced in (1) manipulating from the specific to the general (Lesh and Doerr 2003) and (2) contextualising the process (of differentiating $P = A\ e^{kt}$) with the object (the resulting DE). At this juncture, Group 14 has used the real-world data to construct, debate, select, manipulate and purposefully control processes and objects to generate a general DE model. They felt confident (*schema* 1) that "this corresponds to the population standard equation" (Fig. 45.1d).

APOS theory asserts two criteria for a *new process* to be constructed from existing ones. Firstly, through the *coordination* of two or more processes and secondly through the *reversal* of a process. Both these criteria were realised in Group 14's solution. The coordination of theory and technology induced new processes in the construction of the model. Instead of being given a DE model as source, students had to reverse the process. In this task, the source was the real-world data in Table 45.1 that had to be converted into a DE model. "When the roles of source register and target register are inverted within a semiotic representation conversion task, the problem is radically changed for students" (Duval 2006, p. 122). This task involved the *construction* of the DE model, the reversal of *solving* a DE which is the focus of the curriculum. Group 14 captured the essence of Arcavi's (2003) definition of visualisation: they could use paper-and-pen techniques and technological tools to employ new actions and new processes, create new objects and rearrange these to form a *new* sense-making mental schema: how to *construct* a DE model from real-world data.

The integration and rearrangement of processes are however not the only requirements of visualisation in the sense of Arcavi (2003). It is now argued that the reflection upon effective communication and understanding of graphs and tables in the context of the model must also be demonstrated (Nardi 2014). To this effect, three more subtasks were included: validating the model, suggesting a more appropriate initial condition (IC) and predicting the population in 2020. Bearing in mind that the solution of a DE depends on the IC, the data from Table 45.1 had to be used with caution due to the uncertainty expressed about the population count in 1900. Group 14 intentionally *omitted* this circumspect IC and used another data point $P(1910) = 1750$ (Fig. 45.1d) to calculate the integration constant c. In fact, they stated that one can "substitute any coordinate from the table". This group followed internalised processes requiring only one more data point to calculate the growth parameter k in the DE $dP/dt = k\ P$. In principle, each data point employed would result in a different k-value, thus producing each time a different model. The availability of ten data points (from Table 45.1) was a novel situation for students who are accustomed to solve DE where only two data points are provided. Group 14 decided to experiment with different data points and "tried many equations", iteratively generating different graphs "over and over" (Fig. 45.1d, e). For each k-value computed, the resulting model was plotted in Mathematica together with the real-world data. When an inadequate overlap between the proposed model and the real-world data was observed, the process was repeated. Eventually, they chose the more distant data point $P(2000) = 6080$ (Fig. 45.1d) and discovered that an improved model would largely depend on the choice of data points. Their experimentation with the parameter k was a new and challenging process through which students witnessed graphically how sensitive models are to changes in parameters. Mathematica-generated graphs afforded students to visually compare "multiple models for the same data" (Brown 2015, p. 432) that served as visual proof for getting closer to a more appropriate fit. Figure 45.1d–f shows the detailed symbolical, numerical and graphical validation of Group 14. Their best model is seen in Fig. 45.1e which suggests an improved initial population of 1542 million people in the year 1900 (Fig. 45.1f). This involved another reversal of the usual process. Instead of *using* a given IC, this group intentionally circumvented the flawed IC to *extrapolate* it from their model. Their results were presented, validated and unambiguously communicated with "substantial intermediate work" (Nardi 2014, p. 217). Mathematica codes were cross-checked with paper-and-pen calculations (Fig. 45.1f–h) to estimate the population in 2020 as 8,021 million people. The relevance of their model is confirmed with "the answer makes sense" (*schema* 2). For this group, curricular boundaries were challenged through modelling processes that were empowered by Mathematica technology, thus enabling them to forge new understandings about DE as real-life phenomena. After all, "knowing routines for solving differential equations symbolically … is a very different process from being able to visualize a solution or a family of solutions" (Dubinsky and Tall 1991, p. 237).

It could be argued that a more sophisticated regression procedure would produce a more accurate model. However, even within the curricular limitations, the evolving actions and processes that led to the symbolic formation of the model afforded these students the opportunity to explore the volatility of model building in a contextually rich task. In the study of DE, students learn to appreciate that numerical solutions are approximations, produced by Mathematica as discrete data points. When these data points are compared with the continuous analytical solution, there might not necessarily be a perfect overlap between solutions since numerical solutions ultimately depend on the step size and particular algorithm used. Perhaps in conformity with this notion, Group 14 did not problematise the discrete real-world data points in Fig. 45.1e that did not perfectly fit their (continuous) model. For them, the essential qualitative features of the real-world data could be accommodated in their model.

45.4.2 Group Results

It is encouraging that 7 of the 15 groups could successfully (1) generate new processes and new objects on paper and with technology; (2) make use of symbolic, numeric and graphic representations to contextually validate the real-world data; (3)

present their results via multiple representations; (4) suggest an improved IC to the DE model; and (5) extrapolate the model via multiple representations to predict the population in the year 2020. Whereas most students usually answer the interpretative questions incorrectly or leave them unanswered in the normal curriculum approach, this task provided learning opportunities for acquiring visually advanced understandings (Arcavi 2003) that enabled these groups to interpret, reflect on and communicate the reality of the real-world problem which they modelled.

Group 10 was visually unaware of the lack of fit of their exponential model. They used the *first two* data points in Table 45.1 to calculate the growth parameter k. Due to the close proximity of the two selected data points, the model followed a trajectory that deviated from the real-world data over time. This group could not successfully combine the graphs to be displayed simultaneously (Brown 2015) due to a programming error. Due to a lack of such graphical evidence, they could not validate the appropriateness of their two-point symbolic model. This underscores the power of technological affordances in the visualisation and endorsement of the symbolic model in the modelling task.

Five groups plotted the real-world data in Excel to find a trend line that best fits the data. With the click of the right mouse button, three of these five groups obtained the exponential model $P = 1447\ e^{0.0136\,t}$ which was subsequently differentiated. Essentially, the most demanding part of the modelling cycle was outsourced to Excel. This was an unexpected strategy since Excel is not used in the Mathematica sessions. Two possible scenarios come to mind: as many as 10 % of registered students for this module came from other African countries, it is therefore possible that these international students' knowledge of Excel and/or regression analysis benefitted certain groups. Alternatively, the first year Computer Skills module introduces students to Excel where they learn how to *chart* data. Nevertheless, with a mere mouse click, these groups *allowed technology to symbolise* the model and thereby forfeited the prospect of doing important mathematical work inspired by visual reasoning. It is noteworthy that only one of the three groups who tendered a trend line-generated model could fully reflect on the symbolic meaning of their model. Evidently, the model $P = 1447\ e^{0.0136\,t}$ would imply a new *updated* IC $P(0) = 1447$ but the original (flawed) IC $P(0) = 1650$ was retained. Two of the groups who generated an Excel trend line opted for a quadratic model "because in the exponential model, some values [data points] were excluded by the computer, the polynomial however included all the given data [points] into the plot" (Group 13). Such opposing degrees of interpretation illustrate the cognitive incoherence imposed by technology which undermines symbolic understandings and, in turn, affect the contextual relevance of the model. This is in line with Nardi (2014) who states that once a visual representation has been created, one cannot assume that its meaning is transparent or that its understanding is coordinated (Nardi 2014).

45.5 Conclusion

The aim of this chapter was to explore the first-time impact of a task in the form of a MEA on the visualisation of engineering diploma students modelling in a CAS environment. When students come from visually deprived teaching and learning backgrounds into a technology-rich setting, they are often underprepared for the cognitive challenges imposed by visualisation. Visualisation in the modelling task was theoretically underpinned by students' actions, processes, objects and schemas. A limitation of the study is that only one task was assessed. Still, students could experience the crucial interplay between theoretical and technological processes when constructing a DE. While the real-world data prompted visual mathematising, paper-and-pen processes supported opportunities for visual exploration. Through trial-and-error, manipulations were performed with paper-and-pen and connected with CAS objects, thus awarding *both* cognitive technologies equal status. Students' understanding of DE was advanced when potential models were questioned and visualisation supported the mathematisation process (Brown 2015). In the absence of paper-and-pen work, students were unaware of contradictions and conflicting models which undermined the sense-making of the real-world context. During the MEA, a sense-making process evolved in the work of students who displayed a higher tolerance towards model acceptance in response to rigorous visual validation. The task created opportunities to contextualise the role of DE for future engineers, in particular, opportunities to develop and enhance visualisation, especially where cognitive processes are outsourced to CAS.

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Chapter 46
Interactive Diagrams Used for Collaborative Learning Concerning Mathematical Models of Motion

Elena Naftaliev

Abstract This chapter investigates how students addressing the same modelling activity presented as three different interactive diagrams (IDs) participated in collaborative learning processes and developed modelling analysis competency. Three ID settings were designed as an animation of multiprocess motion but differed in their pedagogical functions. The students explored sets of characteristics of the mathematical models in the diagrams to analyse related phenomena presented as a real model and develop meaning of the mathematical models regarding the phenomena. Shared knowledge was developed when students engaged in a reflective activity concerning other group members' reasoning and instruments involved in the collaborative process. Analysis showed choosing and combining models from different IDs reflected personal choice to anchor the inquiry in more familiar IDs.

Keywords Interactive diagrams • Collaborative learning • Mathematical models of motion • Animation

46.1 Introduction

In recent years, modelling competencies and their promotion have been discussed widely by mathematics educators and researchers (Kaiser and Brand 2015). Niss and Højgaard (2011) defined modelling competency as the ability to analyse and build mathematical models concerning other areas. This competency involves, on the one hand, model analysis ability – the ability to "de-mathematize" existing mathematical models, that is, being able to interpret model elements and results in terms of the real situation which they are supposed to model. On the other hand, the competency also involves being able to perform active modelling in given contexts,

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_46

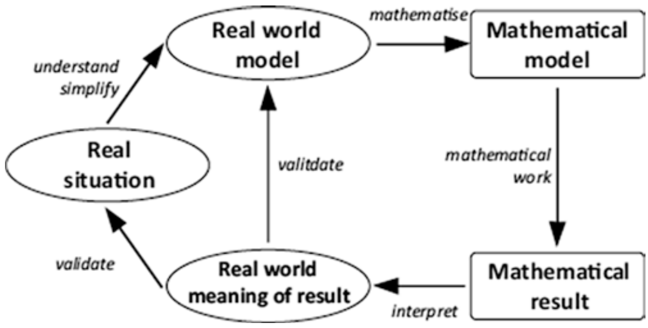


Fig. 46.1 Modelling cycle (Kaiser and Stender 2013)

that is, "mathematizing and applying it to situations beyond mathematics itself" (p. 58). Active modelling contains a range of different sub-competencies and is usually illustrated by a modelling cycle which presents the steps, the sub-competencies and the order in the process. One version of the cycle is presented in Fig. 46.1.

Modelling activities in mathematical education are usually characterized as independent activities in which students solve a modelling problem on their own (Stender and Kaiser 2015). The implementation of independent modelling activities is an ambitious educational purpose with many difficulties. Little empirical knowledge about efficient ways of supporting students in independent modelling processes exists (Kaiser and Brand 2015). To resolve the issues related to the successful engagement with modelling activities, we need innovative methods in pedagogical practices and research on the issue (Stillman et al. 2015).

To help learners construct mathematical representations of reality, the teaching-learning processes need to include the development of tools that will serve them in practice. Two approaches to teaching-learning mathematical modelling are (1) to learn by constructing models and (2) to learn by using models (Schwartz 2007), but the two perspectives should not be seen as being in contrast with each other. Students who do not have experience with mathematical models will probably not benefit greatly from constructing their own models, if indeed they can learn to do so at all (Schwartz 2007). At first, learners tend to explore models by modifying their parameters. Next, they are asked to modify the models themselves, providing them with the original and many similar models with which to work. Finally, students may be asked to devise models of phenomena *independently.* Pedagogic artistry, or the art of executing the teaching-learning process well, lies in helping students move through this sequence in ways that are appropriate to their current understanding of mathematical modelling.

Modelling activities in mathematics have changed in the last decades due to technology development. Sriraman and Lesh (2006) took a critical view about conceptions of mathematical modelling in the modern era and argued for the need to develop new research initiatives with students in experiments involving the simulations of complex systems. Using technology to develop interactive curriculum materials, such as interactive textbooks, provides a captivating, engaging tool which encourages learners to explore mathematical models and to devise their own models as suggested by the learning sequence in Schwartz (2007). It is especially important that, while students learn about dynamic processes, such as motion, and about the mathematical models of the processes, the materials be represented in a similarly dynamic way by animations and interactive models in order to reinforce their knowledge development (Schwartz 2007; Yerushalmy and Naftaliev 2011).

Current technology allows a variety of interactive tools, examples and representations to design interactive diagrams (IDs), which are small units of multimodal interactive text (in e-textbooks or other materials) and are important elements in e-textbooks. For example,[1] an ID focused on motion may include the following components: a wide range of real and mathematical models of motions and a wide repertoire of linking tools and choices of activation of the models. The findings of our previous research show that similar modelling activities with different IDs, which were designed for different pedagogical functions, should be considered as different learning settings (Naftaliev and Yerushalmy 2011, 2013, 2017; Yerushalmy and Naftaliev 2011). This research aims to investigate how students who had been asked to address similar modelling activities presented by different IDs participate in collaborative learning processes and develop model analysis competency.

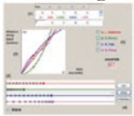
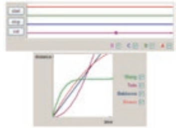
46.2 Semiotic Analysis for Pedagogical Functionality of ID Functions

There are profound differences between the traditional paper page in mathematics textbooks and the new page that derives its principles of design and organization from the screen and the affordances of technology. This issue requires scholars to develop a lens for analysing pedagogical design and teaching-learning processes with IDs. Naftaliev and Yerushalmy (2017) have developed and elaborated a semiotic framework for the pedagogical functionality of IDs that allows an informed discussion of the subject. There are three functions of IDs in the framework: the presentational function, the orientational function and the organizational function. The *presentational function* focuses on what and how this is being illustrated by the ID. The reader may act within the context of the given example and change it or create other similar examples. Three types of examples are widely used: *random, specific* and *generic*. The *orientational function* relates to the type of relationships that the text design attempts to set between the viewer and the text. IDs can function both as schematic and as accurate metric representations in the sense that they can reveal their details.

The *organizational function* looks at the system of relations defining wholes and parts and specifically at how the elements of text combine. IDs can be designed to

[1] http://visualmath.haifa.ac.il/en/linear_functions/raste_of_change_of_linear_functions/200_meter_dash

Table 46.1 Comparative view on the IDs' design

	Video clip 	Illustrating ID 	Elaborating ID 	Guiding ID 
Video clip, real model and its components				
Video clip	✓	x	x	x
Animation	x	✓	✓	✓
"Run", "stop"	✓, ✓	✓, ✓	✓, ✓	✓, x
Choose components	x	✓	✓	✓
Timer	✓	x	✓	x
Examples		Generic	Generic	Generic (motionless component)
Mathematical models				
1D graph		x	✓	x
2D graph		x	✓ (schematic and/or metric)	✓ (schematic)
Table of values		x	✓	x
Links between the models		x	✓	✓ (partial)

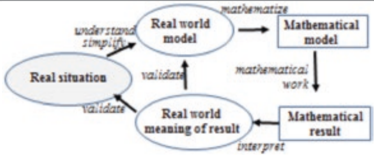
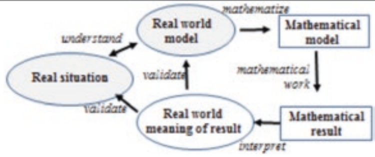
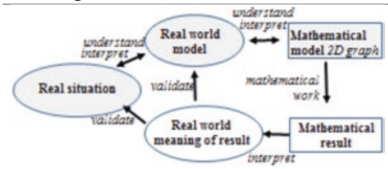
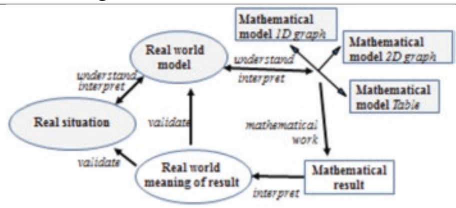
function in three different ways: illustrating, elaborating and guiding. *Illustrating IDs* are simply operated unsophisticated representations. They are intended to orient the student's thinking to the structure and objectives of the activity by usually offering a single representation and relatively simple actions. For example, an illustrating ID may have a limited degree of intervention by activation of controls in the animation (Table 46.1). At any time in the animation (see column 3), users can freeze the positions on the track, continue the run or initialize the race. *Elaborating IDs* provide the means that students may need to engage in activities that lead to the formulation of a solution and to operate at a metacognitive level. The important components in the design of the *elaborating IDs* are rich tools and linked representations that enable various directions in the search for a solution. For example, the same animation that serves as an illustrating ID can be part of an elaborating ID when set within other tools and representations. The ID provides four adjacent, linked representations: a table of values that represents distance and time, a two-dimensional graph of distance over time, a one-dimensional graph which traces the objects' positions at each time unit and an animation (Table 46.1). The variety of linked representations and rich tools in this elaborating ID enables various options in viewing the ID: as a schematic and/or as a metric diagram, as discrete information and/or as a continuous flow of information. The term *guiding IDs* is used in relation to guided inquiry. Guiding IDs are designed to call for action in a specific way that supports the construction of the principal ideas of the activity and may serve to balance constraints and open-ended explorations and support autonomous inquiry. In addition to providing resources that promote inquiry, they also set the boundaries and provide a framework for the process of working with the task. For example, the guiding ID (Table 46.1) was designed around a known conflict about a time-position graph describing a "motionless" situation over continuously running time. The ID consists of two representations of the motion: an animation and a hot-linked position-time graph. The graph and the animation are only partially linked: motion occurs simultaneously on the animation and on the graph, but there is no colour match, so the identification process requires extracting data from the animation and the graph in order to link them. The following constraints contribute to making the task an interesting challenge: a given set of objects running at different types of motion which included motionless object, schematic representations, continuous motion that cannot be stopped but rather viewed continuously and partial links between representations.

46.3 Research Design

The activity, which asked the students to analyse a motion situation, was first illustrated by a video clip and subsequently as an illustrating, elaborating or guiding ID. Each ID includes a real model, an animation of simultaneous multi-motion and two IDs include mathematical models of the motion (Table 46.1). On both the video clip and the animation, users could watch at all times locations on the runway, continue the run or initialize the motion. The three IDs varied by the design choices concerning what was included in the given example and how it was represented and controlled. Regarding what to include in the example, the animation was designed around simultaneous multiprocess motions, to include motion situations known to be challenging, such as nonconstant rate-of-change and "no motion" situations, as well as surprising situations such as an "unexpected win". Considerations of how to design these choices were driven by the semiotic functions framework. Comparative decisions were made about the variety and type of models, the control features, and the linking features to support development of modelling sub-competencies by students (Tables 46.1 and 46.2).

The activities with guiding and elaborating IDs include mathematical and real models and ask for exploration of the models and links between them (Table 46.2). The elaborating ID enables interaction between the students and the various mathematical and real models and so supports the process of learning by experimenting with the models. The design of the guiding ID creates opportunities to interpret the given mathematical and real models by supporting the processes of identifying the visual and kinematic conflict around motionless situation and resolving it. The illustrating ID invites the students to interact with the given real model and to devise mathematical models of the phenomena independently (Table 46.2). The activity with the video clip requires engagement in the full-scale modelling process. If the activities with the elaborating and guiding IDs support development of model analysis ability, the activities with illustrating ID and video activity invite interaction with the phenomena and performance of active modelling in the given contexts.

Table 46.2 Comparative view of the modelling cycles and required task sub-competences

<table>
<tr><th>Video</th><th>Illustrating ID</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><th>Guiding ID</th><th>Elaborating ID</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Based on Kaiser and Stender (2013)
Grey items indicate given components in the activities

The study generated 16 interviews with 12 students: 12 personal interviews and 4 group interviews. The 14-year-old students volunteered to participate in after-school meetings. Each interviewee met the interviewer twice. The first meeting included an individual interview. The second meeting was a group interview. All interviews were video-recorded. Each participant followed a three-step procedure that enabled examining and tracking the role of IDs in the students' knowledge development process concerning mathematical models of motion. At the first stage, the students were given a preliminary task presented as a video clip and designed to evaluate their knowledge and solution techniques. At the second stage, the students were given a task that was presented as an ID. At the third stage, the three students who had been asked to address similar tasks that included different IDs shared their work and participated in a group discussion. The students were asked to describe the technique they used in their solution and to be involved in a conversation regarding other students' techniques. The students could use all the diagrams they worked with in the previous stages.

The students' personal engagement processes as they interacted with the IDs are discussed in previous work (Naftaliev and Yerushalmy 2013; Yerushalmy and Naftaliev 2011). This chapter focuses on the collaborative learning concerning mathematical models of motion.

46.4 The Social Construction of Knowledge in a New Pedagogical Setting

This section includes an analysis of one group's engagement processes to present the social construction of knowledge in a new pedagogical setting. The group (Elad, Helena and Or) had one student for each of the three types of IDs: illustrating, elaborating and guiding. At the first stage, in their individual work with the video activity, the learners put emphasis on getting the story right, which required attending to details such as the runners' body motion: "When they ran, they moved their body a little bit back and their feet a little bit forward and… this maybe gave them, I think, more acceleration. And in the end the one that was on the left won. They all made almost the same movements; just that there were some that started running and some that jumped out later and some that jumped a little sooner". The video clip kept the students too close to the situation and prevented them from thinking in the abstract.

At the second stage, Elad, the student who worked with the illustrating ID, started by activating the animation. Throughout the process, he stopped the animation several times. During each pause, Elad examined the runners' respective positions and described the changes in speed. Elad described each runners' changes in speed with reference to their relative positions at specific moments. He mistakenly interpreted continuous change of speed by comparing relative positions. For example, he argued that passing another runner must have meant speeding up, whereas, in reality, the runner maintained a constant speed. To cope with the challenge, Elad resorted to a failed attempt at drawing graphs by himself to complete the diagram. Helena, working with the elaborating ID, started by activating the representation and tools in the ID. She learned about the wide variety of options and representations available in the ID, but there was no evidence showing developing knowledge concerning mathematical models of motion processes. Or, working with the guiding ID, began his work by identifying a visual and kinematic conflict: while all seven dots moved on the graphs, one of the dots in the animation stopped and remained still. To resolve this conflict, he focused on discrete events much like Elad, using discrete events to match the motions described in the animation and graph extracting discrete motion characteristics such as average speed, time and distance. He successfully matched the dots yet failed to resolve the conflict.

At the third stage, in the group discussion, Or decided to open the conversation with the question which remained unsolved in his individual work (Fig. 46.2). He demonstrated the problem while activating the guiding ID with which he worked. Elad and Helena were intrigued by the question, and it became the goal of their collaborative work. They began by familiarizing themselves with the options of the ID and examples presented in it to resolve the conflict. When they were unsuccessful resolving the conflict using the guiding ID and realized their diagrams were different, Helena suggested using representations and tools from her elaborating ID to accomplish the goal they defined for themselves. Each time, she suggested adding only one option from the elaborating ID. They used it firstly to develop meaning

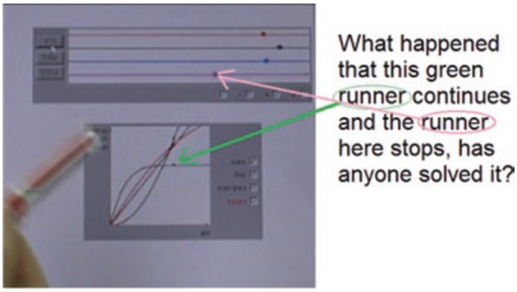


Fig. 46.2 "…has anyone solved it?"

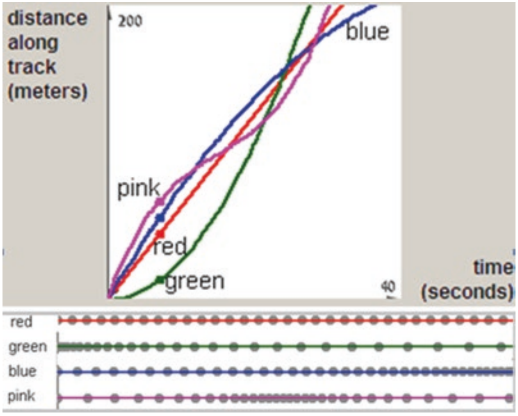


Fig. 46.3 2D and 1D (traces) graphs recorded while running the animation

regarding the motion presented in the elaborating ID. Then, they used the ideas which they developed to resolve the conflict using the guiding ID. The following presents the process which took place in the last step of their work in which they successfully resolved the conflict.

Following a suggestion from Helena, the students activated the animation with traces, resulting in the generation of a 1D graph of the motion (Fig. 46.3). While running the animation and generating of a 1D graph, they read the race from the traced motion using the size of the spaces between the traces as a gauge for speed:

Helena: Press on traces. You see! Where they are stopping?
Or: Ahh… Yes, it describes every time point.
Elad: It describes the steps, the distance of the steps.
Helena: Here, you see the black starts [green] to advance more.
Elad: Pink starts with greater steps. If the traces describe the steps then here he starts to slow down as the time goes on and here it stays at the same speed.
Helena: And the black [green] is really fast.
Elad: But in the end he speeds a bit. The black [green] almost doesn't, he starts with slowness, as the time goes on, his steps only enlarge.
Helena: The red doesn't change… and the red. At the same speed.

Elad: And the red, like I told you in the beginning, remember? That the red is always at the same distance, at the same speed, the same steps. And the blue at the beginning until the middle at the same speed, same steps and towards the end he starts to slow down.

Following the interpretation of the 1D graph as describing speed, the students checked whether this option was available in the guiding ID. Once they verified it was not, they returned to work with the elaborating ID. They began by interpreting the 2D graph based on the 1D graph in static mode with which they became familiar. In the end, they were able to describe the speed by using only the 2D graph.

Helena: Wait, in his [Or's] diagram there is it [the traces]? Check.

Or: Check. …No.

Helena: It's interesting what happened with the pink in his [Or's] diagram [Guiding ID].

Or: I think that this [the Elaborating ID] is the best.

Helena: The red is running at the same speed. The black in the beginning runs really slow, and then he ups his speed more and more [they closed the 1D graph and continued work only with the 2D graph]. The blue runs really quick and then he starts to slow down. The pink runs fast, in the middle he slows down and then in the end again he runs fast.

Once they have succeeded in interpreting the 2D graph in the elaborating ID, they were able to resolve the conflict they had about the motionless process presented by the guiding ID:

Or: Yes. So, as the line is steeper, then his speed is... ehh... it is steep and… that's it, I see that in the end it turns into a straight line, plane, something like this. That means that he slowed the speed and even stopped in place.

Elad: If this shows distance, then it means that the distance here does not change.

The episode describes the students' exploration concerning the description of speed in the mathematical models in four stages: (1) analysis of a dynamic mode of 1D graph which was linked to the running animation, (2) analysis of a static mode of 1D graph, (3) analysis of shapes of 2D graphs and (4) analysis of a motionless process represented by 2D graph.

46.5 Discussion

The chapter focuses on the role of similar modelling activities presented by different IDs in supporting students' collaborative work. It describes their development of model analysis competency. This is especially relevant when they explore models in relation to new mathematical concepts with which they are not familiar. Students do need to have enough experience with models to understand the point of mathematical modelling, that is, its “language” (Schwartz 2007). Once such representations exist in the cognitive “baggage” of learners, they also become a tool for mathematical modelling (Wilensky 1999).

The students explored sets of model characteristics in the IDs to analyse the related phenomena presented as a real model and developed meaning of the mathematical models regarding the phenomena. They interpreted the real and mathematical models by using the elaborating ID, pointing to the speed, time and distance as continuous variables. Then they used the ideas which they had developed to analyse the characteristics of motion presented in the guiding ID. At the end of the discussion, the mathematical models in static mode prompted them to mentally recreate and describe the motion processes.

The development of shared knowledge occurred when the students engaged in a reflective activity concerning other members’ reasoning and instruments involved in the collaborative process. As a result of the collaboration, students generated an interactive text: they posed a new question, decided what component from what ID to bring to discussion, decided on the sequence between the components, defined the role of each component and created a representation of the data. All this was done to accomplish the goal they posed to themselves, thus building meaning concerning mathematical models of motion. The analysis clarified that choosing and combining representations from similar tasks, which were designed as different IDs, reflected students’ personal choices to anchor their inquiry in the more familiar ones. The interactive texts became an instrument supporting development of shared knowledge concerning the mathematical models and characteristics of the kinematic phenomena.

Acknowledgements I thank Michal Yerushalmy and Judah Schwartz for discussing these ideas. This research was supported by I-CORE Program of the Planning and Budgeting Committee and The Israel Science Foundation (1716/12).

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Chapter 47
Using Modelling and Tablets in the Classroom to Learn Quadratic Functions

Miriam Ortega and Luis Puig

Abstract In this chapter, we present teaching material to work with a quadratic function and the meaning of its parameters through the mathematical modelling of a real-life phenomenon: the relation between the time and the height of a ball during a complete vertical rebound and fall. The teaching material uses electronic tablets to collect and process real data in the classroom. After analysing a year 11 implementation, we note that a qualitative analysis of the phenomenon and the families of functions and the students' prior knowledge about these functions are key elements to manage and control the modelling process, especially, to choose the model and to interpret the results in terms of the phenomenon. Considering the limitations of the first design of the teaching material, we present the elements that we have incorporated into a new design in order to improve it.

Keywords Modelling process • Problem-solving • Function • (Upper) secondary education • Real data • Technological tool

47.1 Introduction

Many investigations in mathematics education (e.g. Almeida and da Silva 2015; Villa-Ochoa and Berrío 2015) point out the importance of introducing modelling in education to show the relation between mathematics and the real world to students. Despite all of this research, the use of modelling in classrooms is still a pending issue (Blomhøj 2013). This is due to a lack of resources and support material for teachers and the necessity to change their model to manage teaching practices in classrooms, among other issues.

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_47

Another factor that allows students to relate mathematics to "reality" is the use of real data. As Stacey et al. (2000) suggested in the discussion document of the ICMI Study, *The Future of the Teaching and Learning of Algebra*, it is important to introduce the use of real data in classrooms in order to help students to manipulate algebraic expressions. One means of doing this is through the use of technology (Brown 2004). In particular, the National Council of Teachers of Mathematics tools and technology principle (2014) states that "an excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (p. 4). Geiger (2011) affirmed that the use of technological tools enriches the modelling process, as mathematical routines and processes, students and technology are engaged in partnership while solving a problem. Therefore, the incorporation of technological tools into the classroom to solve modelling tasks is an interesting element to consider.

In this chapter, we show the design of teaching material to work with the quadratic function through the mathematical modelling process using real data obtained with electronic tablets. Subsequently, the results of the implementation are presented and an improved design is suggested as a consequence of reflecting upon the results of this evaluation and the limitations observed in the first design.

47.2 Background

From an educational perspective, Julie and Mudaly (2007) identified two different points of view of how mathematical modelling can be used in education: as content itself or as a vehicle to promote and motivate students to learn other mathematical content. In our case, we use mathematical modelling as a vehicle to support the students' learning of the concepts of families of functions and the meaning of the parameters through the modelling process. A modelling process can be represented using a cycle, not for indicating the way in which the students follow the process but to schematise the different phases through which the students can pass. One of the cycles that give a general vision of the phases that can be followed to model a phenomenon is the one suggested by Blum (2011). For that reason, we will use it to design specific questions to guide the students through the modelling process towards the acquisition of this content.

On the other hand, taking a definition of problem that includes what Butts (1980) called a "problematic situation", a modelling process can be conceived as a particular case of a problem-solving process. For that reason, the students will need to be competent not only in modelling but also in problem-solving. Schoenfeld (1985) established what he called "components of knowledge and behaviour" that explain the performances of students in problem solving, including among these components the management and control of the process. Puig (1996) proposed a competence model for problem-solving, in which management and control elements were included. He argued that for being competent, it is not enough to be aware of the need for managing the process, but it is necessary to know how to manage the specific process of a class of problems. In a later study, Puig and Monzó (2013) affirmed that a qualitative analysis of the phenomenon and the families of functions that might model the phenomenon is key to manage the process of solving a modelling problem, so we included these metacognitive elements in the design of the materials to lead the students through the modelling process. This is in keeping with the work of many other researchers who have incorporated metacognitive aspects into the design of modelling support materials such as task booklets (see, e.g. Stillman 2011).

Therefore, our research aim is to analyse which phases of the modelling cycle are influenced by the qualitative analysis of the characteristics of the phenomenon, the families of functions and the students' prior knowledge of these.

47.3 Materials and Method

47.3.1 Design of the Teaching Experiment

We present here the teaching material for the study of the family of quadratic functions through mathematical modelling of a physical phenomenon and the use of electronic tablets, characterised by the inclusion of management and control elements of the modelling process and the use of real data obtained directly in the classroom using the tablets. The phenomenon studied is the relation between the time and the height of a ball dropped from a certain height, restricting the model to the first rebound and the subsequent drop of the ball, that is, from the moment that the ball touches the ground for the first time until it touches it again. As can be seen in the statement, taking into account the modelling cycle of Blum (2011), the real situation is given already simplified because the variables that should be studied, the time and height of the ball, are specified in the statement.

Bounce of a Ball
A ball is dropped vertically from a certain height, and when it touches the ground, it bounces several times until it stops. We want to study the relation between the height and the time of the ball in each moment considering only the first rebound and subsequent drop, that is, from the moment that the ball touches the ground for the first time until it touches it again.

From the perspective of kinematics, this is a case of uniformly accelerated rectilinear motion. This is because, if friction is neglected, the ball is subjected to only the action of gravity and it starts from rest. As a consequence, the phenomenon is described by a quadratic function when the variables distance and time are related. The function describing such motion is $y(t) = y_0 + v_0 \cdot (t - t_0) + 1/2 \cdot g \cdot (t - t_0)^2$ where y is the distance travelled by the ball, t is the time in each instant and y_0, v_0 (null in our case), g and t_0 are constants. Our students are not expected to find the function expressed in this canonical form but to find that the function that models the data is a quadratic.

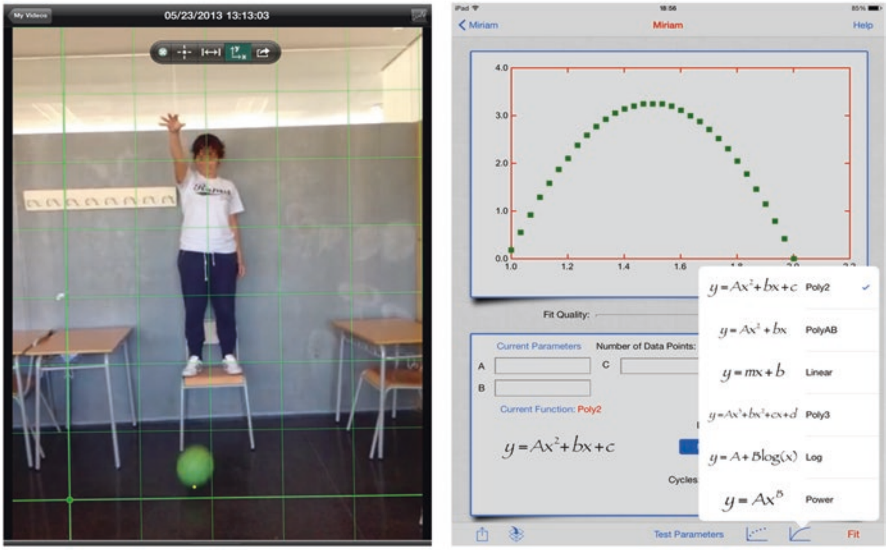
The teaching experiment was carried out in a natural group of 16 year 11 students. They had not dealt with modelling problems before, but they had previous knowledge about families of functions and the meaning of the parameters acquired in previous courses by using new technologies.

With regard to the implementation of the material, first the students had to answer several questions related to the characteristics of the phenomenon studied. After that, they had to reproduce the phenomenon in classroom and record a video of it using iPads® in order to take real data to find a model that describes the phenomenon and to answer questions to study it in more detail.

The experiment was carried out in a total of three sessions of 55 min each. In particular, during the first and the second session, the teaching material was implemented in the classroom, and, in the last one, four interviews were conducted by the researchers. During the teaching sessions, the students were working in pairs because, according to Schoenfeld (1985), this encourages the verbalisation of what they are doing, thinking or wanting to do.

In the first session, the students were given a worksheet to study qualitatively the phenomenon and the family of functions that could model the phenomenon before representing it and collecting the data. In particular, they were asked to draw a sketch of the graph that they expected to model the phenomenon, to choose the family of functions that best fits the graph from a given list and to explain the reasons for their choice. This previous qualitative analysis of the phenomenon and the families of functions is a metacognitive element that we included in the design to lead the students through the modelling process. Subsequently, they simulated the phenomenon studied and recorded it using Video Physics® (see photo Fig. 47.1a), which allowed them to obtain a set of points, where the coordinates represented the relation between the time and height of the ball in each instant.

In the second session, the students had to introduce the coordinates of the points into the app Data Analysis® in order to choose the function that fitted best to them from a given list of possible function models, $y = A\,x^2 + b\,x + c$, $y = A\,x^2 + b\,x$, $y = mx + b$, etc. (Fig. 47.1b), and to obtain a model of the phenomenon. Moreover, they were given another worksheet with more questions concerning the calculation of the domain of the function (items 5 and 6, Fig. 47.2) and the interpretation of some data in relation to the phenomenon studied (item 7, Fig. 47.2) in order for the students to analyse some of the characteristics of the function and to validate the adequacy of the model chosen. Specially, in item 5, they were asked to calculate some images of the function where values for the independent variable were outside the domain, and



a b

Fig. 47.1 Sequence of screenshots of (**a**) Video Physics® and (**b**) Data Analysis®

in item 6, they were asked to explain if the images obtained showed what really happened, to make the students realise that the regression function only represents the function that models the phenomenon in the interval in which it has been defined, that is, during the first rebound and subsequent fall. These metacognitive elements were included to make the students think about the qualitative analysis of the phenomenon and the families of functions. To answer these questions, they could use the app Free GraCalc®, which functions in a similar manner to a graphing calculator.

Finally, in the last session, two pairs of students were selected, considering the results obtained from the analysis of their answers and their performance in the previous two sessions, to participate in a case study. Specially, we designed a diagnostic interview with the aim of finding out the origin of the students' performances but incorporating several teaching elements to make the students think more carefully about some of their answers.

All the teaching experiment was conducted based on the reinterpretation made by Roth and Radford (2011) of Vygotsky's zone of proximal development idea that goes further than what Freudenthal (1983) called the "process of guided reinvention". It was done in order to guide the students through questions and suggestions for the acquisition of the knowledge related to the particular characteristics of the function and the phenomenon studied and the meaning of the parameters involved.

5. At which height is the ball when $x = 0.76$, $x = 1.1$, $x = 0.11$ and $x = 100$?

6.a. In general, do you think the answers above show what truly happens? Why?

b. What are the data that don't show what you expected? Why?

7.a. At which values of x (time) does the ball touch the ground? Explain how you got the solution.

b. At which values of x (time) does the ball reach the maximum height? Explain how you got the solution.

Fig. 47.2 Questions related to function domain (items 5 and 6) and data interpretation in real situation (item 7)

47.3.2 Data Collection and Data Analysis

Data collection was carried out in pairs because this was how the students worked during the whole teaching experiment. Therefore, the nature of the data came from students' collaborative work.

Firstly, data collected from teaching sessions in the classroom were obtained from worksheets, tablets and classroom diaries in which we recorded questions that students asked during the teaching experiment and other aspects that could be interesting for the research purpose. Regarding the analysis of the data, first we proceeded to extract the information from the data collection instruments in order to put it together. In particular, we wrote a text for each pair of students considering the information from all the sources. After that, we extracted the pieces of text in which it was noticeable that the students used their prior knowledge about the families of functions and the qualitative analysis to model the phenomenon. Finally, we categorised these elements according to the phases of the modelling cycle in which the students used them or referred to them.

Secondly, data from interview sessions were obtained using a video camera, which allowed us to obtain not only the conversation between interviewer and students but also the students' actions and gestures. To analyse these data, first we obtained the written protocol by transcribing the interviews and adding comments about gestures and images of the videos. Later, we made comments on the written protocol interpreting the meaning of the discourse to make sense of all the data collected. Afterwards, we organised the comments to make a rational reconstruction, that is, a narration of the students' behaviour with the aim of making sense of the whole text (Puig 1996). This allowed us to elaborate a listing of observations pair by pair according to the influence of the qualitative analysis and other elements on the students' performances. Finally, we categorised these elements according to the phases of the modelling cycle in which the students used them.

Once we obtained the categorisation pair by pair of the phases in which students' prior knowledge and qualitative analysis of the phenomenon influences the modelling process, from both teaching experiment and interviews, we proceeded to observe if there were results that appear in most of pairs studied and then we elaborated a listing of them.

47.4 Results and Discussion

After analysing the data, we found that the qualitative analysis of the phenomenon and the families of functions and the students' prior knowledge about the families of functions and the meaning of the parameters are crucial elements in two key moments: when the students have to choose the function to model the phenomenon and during the interpretation of the characteristics of the function in relation with the phenomenon.

As can be seen in Fig. 47.3, when asked to select the function to model the phenomenon from the given list in Data Analysis®, the students of pair 2 chose the quadratic function $f(x) = a\,x^2 + b\,x + c$. In order to know the reason of their choosing, during the interview, we asked them why they had selected that one, so they explained that they knew that the graph had a parabolic shape because they had studied carefully the phenomenon before taking the data, so "*x* would have a square". Therefore, they based their answer on the previous qualitative analysis to recognise the shape of the graph that the phenomenon would have. In addition, they used their knowledge about families of functions to relate the graph of the parabola to the characteristic of having an x^2 in its algebraic expression as the term with maximum exponent. These were not the only pair who used their knowledge about the characteristics of the families of function to choose the algebraic expression that they thought would model the phenomenon. For instance, as the transcript shows, during interview, students of pair 3 explained that they chose $f(x) = a\,x^2 + b\,x + c$ because "it is not a straight line, it is a parabola, so the formula will be that".

Moreover, the students of pair 1 not only used the qualitative analysis to choose the function but also their prior knowledge about the meaning of the parameters of the quadratic function. In particular, when they were asked why they chose the function $f(x) = ax^2 + bx + c$ in Data Analysis® instead of $f(x) = a\,x^2 + b\,x$, they explained that it was because the graph that they had drawn "didn't cross the (0,0)", referring

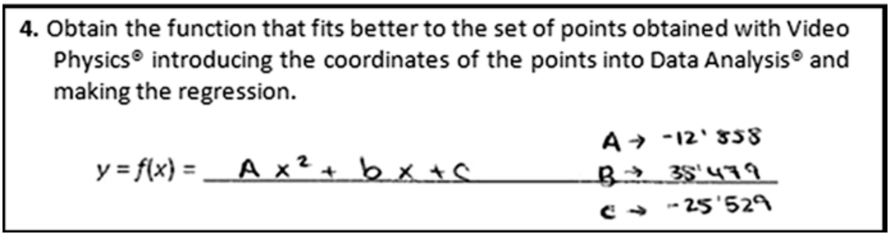


Fig. 47.3 Pair 2's model

to the sketch of the graph that they had drawn on the first worksheet and using their knowledge about the meaning of the parameter *c* in this formula for the function.

We also could observe the influence of the qualitative analysis of the phenomenon and the students' prior knowledge during the interpretation phase of the function chosen as a model in most of the cases. In examining the responses to item 7b (Fig. 47.2), it is possible to see how students explained that the ball reaches the maximum height "in the midpoint between the two times obtained in the previous exercise" which are the moments when the ball touches the ground, "because it is an approximate place where the vertex of the parabola would be". So, they used their knowledge about the qualitative properties of the function to relate the maximum height of the ball to the vertex of the parabola, the highest point of the function. Moreover, they consider the property of symmetry of the quadratic function because they take into account that the vertex is the point between two points that have the same height. Furthermore, many students used what they did during the previous qualitative analysis to justify their answers, while they were interpreting the function in relation to the phenomenon. For example, students of pair 8 explained in item 6b (Fig. 47.2) that "as we had seen on the beginning [referring to the qualitative analysis of the phenomenon] that it has to be a parabola that started a little after zero, it has no sense to calculate the images of big values like *f*(100) or images of values close to zero like *f*(0,11)".

47.5 Discussion of Elements to Improve the First Design

As a consequence of the observation made during the experiment and the results obtained, we found certain elements that it is possible to change in order to obtain an improved design that allows us to analyse whether the qualitative analysis and the students' prior knowledge influence other phases of the modelling process in future implementations.

First of all, we have found a lack of information about the students' prior knowledge of the characteristics of families of functions and the meaning of the parameters in the first worksheet, which has not allowed us to know in detail the origin of their answers. This has led us to include some questions during the interviews to find out more information about these, but this is useful only in the case of the students interviewed. For instance, we could not determine the reasons that led students to draw the graph in a certain form and position with respect to the axes. Similarly, we could find out which algebraic expressions were known by them and which graphical representations. We also observed a lack of information from the second worksheet, which had not allowed us to know if the students' prior knowledge and the qualitative analysis influenced other phases of the modelling process.

Secondly, it should be pointed out that the technological tool does not always help in the mathematical reasoning of the students as we mentioned in the introduction because sometimes it helps so much the students that it allows them to obtain what they want without having thought very much about it. That is, the students transfer some competences to the technological tool used. This occurs in particular when the app Data Analysis® provides directly the regression function and the students do not necessarily have to reflect on the meaning of all the parameters related to the graph. In this case, they only need to reflect on the meaning of the parameter c of the function $y = ax^2 + bx + c$ because the app gives the option to choose between $y = ax^2 + bx + c$ and $y = ax^2 + bx$.

47.6 Incorporation of New Elements into the Design

Taking into account the results and as a consequence of reflecting upon the observation in the classroom, we decided to incorporate some new elements into the design of the teaching materials.

On the one hand, we have incorporated more questions where the students can explain and justify the decisions that they make. At the same time, this will allow us as researchers to obtain more information about the elements that influence their answers. For example, after having drawn the sketch of the graph in the first worksheet, we ask why the graph has that shape and why it is in this position with respect to the axis in order to know the elements that make them give a particular answer. We also incorporated a question to determine which algebraic expressions they know and their graphical representations. In the second worksheet, we added new questions that point to metacognitive elements to guide the students through the modelling process but what also will allow us to know if they use the qualitative analysis of the phenomenon and their prior knowledge in other phases of the process. For instance, a question has been included to know in factors their choice of function as a model is based.

On the other hand, we set other questions in order to require students to interpret the meaning of the parameters in relation to the graph to find the function. For that purpose, we have included a question where the students will have to use another app, Desmos®, to represent the points obtained and to transform the function $y = x^2$ to fit them from observing the effect of the different values of the parameters a, b and c of the function in the canonical form $y = a(x - b)^2 + c$ in the graph. Moreover, we have incorporated other questions to make the most of the learning opportunities that emerge as a consequence of the nature of the activity to work other content and taking into account the possibilities that the tablets offer. Specially, we have posed a question to ensure the students obtain the function using Data Analysis® as well, which gives them a function in the canonical form $y = a\,x^2 + b\,x + c$, and to make them compare both functions and, consequently, the values of the parameters of both of them. With this task, the students can learn the algebraic transformations between canonical forms with sense, not as a mechanical task on how they usually do and verify the model that they have obtained by using Desmos®.

47.7 Conclusion

In this chapter, we have presented the results of implementing a teaching material to work the quadratic function through mathematical modelling of a real phenomenon and using electronic tablets. With respect to the research aim, we have confirmed, as Puig and Monzó (2013) suggested previously, that the qualitative analysis of the phenomenon and the families of functions are key elements throughout the modelling process. Specially, we have observed that the qualitative analysis as well as the students' prior knowledge are crucial in the phases of the modelling cycle (Blum 2011) in which the students have to choose the function used as a model and to interpret the model in relation to the phenomenon. Therefore, these metacognitive elements can help the students through the modelling process, and it is necessary to incorporate them in the design of tasks of similar nature.

Taking into account the observation in the classroom and the results of the implementation of the first design, we have incorporated new items to give the students the possibility of explaining their answers in more detail and to make the most of the opportunity to work with algebraic transformations between canonical forms of the same function and the meaning of the parameters of a specific canonical form with sense, not as a mechanical transformation out of context. This will allow us in future implementations to obtain more information about how students use their prior knowledge of the characteristics of families of functions and qualitative analysis of the phenomenon being studied to answer the questions and, as a consequence, analyse if these elements influence other phases of the modelling process, which is the main objective of our study.

Acknowledgements This research has been funded from the research programmes EDU2015-69731-R (Ministerio de Economía y Competitividad/FEDER), GV/2016/129 and GVPROMETEO2016-143 (Conselleria d'Educació, Investigació, Cultura i Esport) and the PhD grant BES-2013-063826 (Ministerio de Economía y Competitividad).

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Chapter 48
Mathematical Modelling in a Long-Distance Teacher Education in Brazil: Democratizing Mathematics

Daniel Clark Orey and Milton Rosa

Abstract This chapter describes how a group of mixed ability students used long-distance education technologies to develop mathematical models in relation to their experience with nationwide protests related to sudden and steep rises in transportation costs during June 2013 in Brazil. Mathematical modelling became a teaching methodology that focused on the development of a critical and reflective efficacy engaging students in a contextualized teaching-learning process that allowed them to become involved in the construction of solutions of social significance. Pertinent theories related to critical mathematical modelling in the context of long-distance technologies are outlined. This approach allows for the democratization of higher education in Brazil by democratizing mathematics through the development of the modelling process in virtual learning environments.

Keywords Critical and reflective dimension • Critical and reflective mathematical modelling • Long distance education • Teacher education • Technology • Virtual learning ernvironment

48.1 Introduction

Nationwide, Brazil is in the process of upgrading teacher competencies and the training of new teachers on a massive scale that is making a difference in the quality of life in many schools and communities. To increase access to a wider audience, we make use of freeware and Moodle as the platform; this has enabled the *Universidade Aberta do Brazil* (Brazilian Open University) system to democratize and increase

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_48

access to higher education. The study of new educational and methodological proposals promotes social change resulting from contemporary scientific and technological development.

The need to update and upgrade professional development for teachers on a massive national scale created new institutional solutions, methods, and resources in order to meet the demand for specialized teacher education programs such as in mathematics. Long-distance teacher education programs for prospective teachers in Brazil are offered in regions that traditionally have had limited to no access to higher education and professional development opportunities. These programs were developed because traditional face-to-face teacher training programs could not meet the tremendous immediate need or allow people the time required by traditional instruction to earn degrees.

This context allowed the *Universidade Federal de Ouro Preto* (UFOP), which is the Federal University of Ouro Preto, to offer *seminars in mathematical modelling* in long-distance mathematics at the undergraduate level. These courses are entirely developed in environments mediated by technology and the Internet. Freitas (2016) stated that these virtual learning environments help to provide the development of students' critical and reflective dimensions during the conducting of the mathematical modelling process. These environments enable the development of relevant discussions on themes chosen by the students for the elaboration of their modelling projects.

The development of the activities in these courses was conducted by using the interactional tools among teachers, tutors, and students found on the Moodle platform. In this regard, the *Centro de Educação Aberta e a Distância* (*CEAD*), which is *the Open and Distance Educational Centre* at the *Universidade Federal de Ouro Preto*, has come to integrate instruction, technology, digital media, content, and pedagogical methods in order to reach a diversity of students. At just this university alone, there are over 2000 students enrolled in four undergraduate major courses including Mathematics, Geography, Pedagogy, and Public Administration. Enrollment represents 16% of UFOP students who live in three states: Bahia, Minas Gerais, and São Paulo in 33 long-distance learning centers named *polos* that are attached to the university.

48.2 The Role of Long-Distance Education

In Brazil, pushback in regard to long-distance education is evident, especially in relation to its implementation in higher education. The *Brazilian Open University* was developed with the mission of providing access to higher education to a population of prospective learners who have not traditionally had access to higher education. *Article 80* of *Law No. 9394/1996*, which is the guidelines and basis for Brazilian education, states that the government must encourage the development and diffusion of distance education programs at all educational levels.

Over the past few decades, and in many diverse locations in Brazil, distance education has grown quickly. Beginning initially with the use of mail-order courses, it transitioned to include radio and television. Once associated with mail and printed materials, it facilitated the dissemination and democratization of access and has now moved to the Internet and includes a diversity of MOOC offerings. It has become a key element in democratization and now allows access to education and professional development opportunities once only given in face-to-face and in elite contexts. In Brazil, it has allowed a portion of the population that traditionally was denied access to participate in public education and to advance.

In long-distance education and virtual learning environments, students and teachers are in different both temporal and physical locations (Moore and Kearsley 2005). Although this modality of education might hinder traditional teacher-student relationships, strangely enough it allows students who had never had access to professors or teachers to gain intimate and close contact with their instructors. Long-distance technologies and technological resources answer the need of a population who deserve initial or continuing education opportunities. Long-distance education allows for both educators and learners to break barriers related to time and space by promoting interactivity and information dissemination. Many long-distance education environments are open systems composed of flexible mechanisms for participation and decentralization, with control rules discussed by the community and decisions taken by interdisciplinary groups (Rosa and Orey 2007).

This approach allows interactions between teachers who prepare instructional materials and strategies, with tutors, who, in our case, provide hands-on face-to-face assistance at *polos*. In this educational modality, tutors are tasked to assist students in their activities and tasks, guiding them in their doubts, helping them learn to use search tools and libraries, and offering help in writing and basic mathematics skills. This assistance is very important during the development of the modelling process. These interactions are triggered by lessons that are delivered on *platforms* that enable the use of technology and the teaching and learning of specific mathematics content in the elaboration of mathematical models. These features have enabled the development of a variety of educational methodologies that utilize web interaction channels and aim to provide needed support in the achievement of mathematical modelling curricular activities.

48.3 Critical and Reflective Dimension of Mathematical Modelling

In the last three decades, critical and reflective mathematical modelling as a method for teaching and learning mathematics has been a central theme in mathematics education. In teacher education programs, this is a way to rebuild or restore part of the fragmented knowledge students acquired during their previous mathematics learning experiences. The critical and reflective dimension of mathematical modelling has become one of the most important lines of research for processes of mathematics teaching and learning in Brazil. This supports forms of teaching and learning of mathematics aimed at solving real-world problems that makes use of critical mathematical modelling as a methodology that values and enables connections between mathematics and reality.

However, this aspect is not always reflected in the teaching practices of educators. Much of the literature related to mathematical modelling and its socio-critical perspectives contributes to the development of both critical and reflective teachers and presents us with opportunities for the meaningful learning of mathematical concepts by students in virtual learning environments (Rosa and Orey 2015), which allows for the exploration of issues related to the context and interest of students and, thus, provides meaning for mathematical content under study.

By using critical-reflective mathematical modelling processes, educators encourage the examination of a variety of ways in which students develop and use certain mathematical procedures so that they learn to identify and propose solutions to problems faced in everyday life. This process of collecting and organizing data in order to develop an opinion is crucial to the development of an informed, active, and critical citizenship. One of the necessary pedagogical practices for transforming the nature of mathematical teaching is the deployment and implementation of this perspective in long-distance mathematics undergraduate courses. Interpreting and understanding these phenomena are due to the power provided by critical-reflective mathematical modelling, which occurs through the analysis of the applications of mathematical concepts during the development of mathematical models in the virtual learning environments. Because the solution for modelling a problem situation includes the understanding of how ideas and mathematical concepts are designed in the preparation, analysis, and resolution of models, the process of developing mathematical models is not a neutral activity. It is important that mathematical results obtained in this process are linked to the reality of the learners themselves (Barbosa 2006).

During the process of elaborating a model, it is necessary to describe, analyze, and interpret phenomena present in reality in order to generate critical and reflective discussions about different processes for the resolution of models, which are prepared by learners. Thus, it is important to enable true reflections of reality, which become a transformative action that allows students to practice both explaining and sharing understandings and develop abilities to organize, manage, and find solutions to problems that present themselves (Rosa and Orey 2015). Both critical and reflective data-focused discussion triggers a cycle of acquisition of mathematical knowledge from reality through the process of mathematical modelling. In this process, students develop skills that help them to process information and define essential strategies to perform actions that aim to transform reality. This kind of discussion provokes students the ability to comprehend and debate the implications of mathematical results, which flow from the resolution of problem and situations (Rosa and Orey 2015).

In this regard, critical and reflective mathematical modelling is considered as an artistic, indeed a poetic process, because during the elaboration of a model, modelers develop a certain creative sense of intuition or creativity that enables the interpretation of data. Hence, students experience and work in a motivating virtual learning environment so that they are able to develop and exercise their creativity, reflection, and criticality during the modelling process of generation, analyses, and production of knowledge. Mathematical modelling in a virtual learning environment becomes a place in which students are invited to inquire, investigate, and work with real problems as well as use the mathematics they know, as a language for understanding, simplifying, and solving these situations in an interdisciplinary fashion by using technological tools available in this environment (Freitas 2016).

The *Brazilian National Curriculum for Mathematics* developed in 1998 states that students need to develop their own autonomous competencies to solve problems, make decisions, work collaboratively, and effectively communicate their ideas. This approach helps students face challenges posed by society by turning them into flexible, adaptive, reflective, critical, and creative citizens. This aspect emphasizes the role of mathematics in society by highlighting the necessity to analyze the role of critical thinking in relation to the nature of mathematical models as well as the function of modelling that solves everyday challenges (Rosa and Orey 2015).

Therefore, when Brazil erupted in protests in June 2013, it seemed the perfect opportunity to use the crisis related to transportation in order to develop mathematical modelling curricular activities. By having 31 *polos* with diverse student populations and distinct social contexts, it seemed a rich opportunity to work with the critical and reflective dimension of mathematical modelling. This approach helped students' development of creativity and criticality that allowed them to apply different tools to focus on the problem and data in their own context to solve problems faced in their daily lives in order to elaborate mathematical models related to proposed transportation themes.

48.4 Mathematical Modelling in Virtual Learning Environments

Mathematics is often referred to as a language, but it seems that it has become a language taught almost entirely focusing on its grammar and without giving leaners the opportunity to communicate mathematically. It is not until learners reach advanced mathematics that the few who survive this process are afforded the opportunity to engage in communicating and creating new ideas using the beauty and power found in the language of mathematics. It is no wonder then that most people detest mathematics. To them mathematics is stuck in endless skill drills in the use of mechanical and mathematical grammar without being able to write or communicate in this synthetic but powerful language.

In Brazil, a strong culture of inquiry has developed in the mathematics education community by using critical and reflective mathematical modelling in which students are encouraged to reflect upon, engage in, debate, and dialogue to resolve problems they find in their own contexts. For example, data gleaned from a course offered to mathematics majors in mathematical modelling used this historic event regarding the 2013 Brazilian nationwide demonstrations. This context allowed 150 students in ten *polos* in the states of Minas Gerais and São Paulo to develop their competency in the mathematical modelling process in the virtual learning environment. This approach also helped them to study in depth the rise in bus fares in their communities in Brazil as well as to share their findings with fellow students, faculty, and tutors.

In this context, in June 2013, early in the seminar on mathematical modelling, Brazil erupted in mass demonstrations against the growing problem of corruption and overspending in relation to preparation for the 2014 World Cup tournament. Just in the small college town of Ouro Preto, 10,000 people marched from the university campus to the main square of the city. What sparked this national mass movement was a sudden spike in fares in urban transportation systems. What may seem to those who do not use mass transit as something minor (20 cent rise) created a difficult problem for many who live in the large metropolises of São Paulo, Rio de Janeiro, Salvador, Brasilia, Fortaleza, and Belo Horizonte. Some long daily commutes became R&30 (about US&12) round-trips five or six times a week and for many became untenable.

Normally, a week is devoted to bringing consensus with students and generating a number of themes, and to make use of this particular historic circumstance, the instructor consulted with the tutors and students, and together they agreed that transportation would be the theme. Eight *polos* were participating in the seminar. The instructor asked the tutors at each *polo* to organize students into smaller working groups of four or five students. Over a period of 5 weeks, students were led through the steps. The work groups were required to post evidence of their work online and on YouTube. Synchronous virtual classes were held. Mathematical modelling lessons were transmitted through videoconferences. Lessons were organized and activities and projects were posted in the Moodle platform. Discussion forums were conducted in order to prepare students to develop the modelling process. By the end of the 16-week seminar, four synchronous meetings were developed in order to discuss the elaboration of mathematical models by each group of students. The course calendar, the description of the seminar, the terms of the proposed activities, and the dates and times of synchronous activities were published in the virtual learning environment. Every 2 weeks, there were activities and questions to be worked on by the students and sent to the tutors and the professor through links in the Moodle platform.

For example, one group in one of the *polos* that had students attending from 5 (five) different towns decided to problematize the situation of public transportation in these locations. In order to do so, they posed the following research question: *What is the fair price of a bus ticket by considering the per capita income of the population of each town?* Therefore, during the development of the modelling process, this group of students interviewed people in each town to obtain information about the percentage of their salary they spend in public transportation and about the services provided by the bus companies such as delays, mechanical problems, and longtime travel. They also interviewed public officers to obtain information about the *per capita income* as well as the percentage of the population of each town that uses public transportation.

Thus, based on the results of the interviews, students started the mathematization process by finding out that people spent, approximately, 7% of the *per capita income* of each town in public transportation. They were also able to determine that, approximately, 30% of the population used public transportation. In this context, in order to elaborate the mathematical model that represented this problem situation, students also considered that people, in Brazil, may use public transportation 2 (two) times a day (to go to work and to return home) and 24 days a month by only considering working days from Monday to Friday. Consequently, students determined the following mathematical model:

$$\mathrm{TP} = \frac{\mathrm{PCI} \times 0.07}{48} \qquad \begin{array}{l} \mathrm{TP} = \text{ticket} \\ \textit{mathrmprice} \\ \mathrm{PCI} = \text{per capita} \\ \textit{mathrmincome} \end{array}$$

This group of students applied this mathematical model to understand the public transportation in one of the towns in this *polo* in which the bus ticket cost R$ 2.50. Buses were available from 5 a.m. to 11 p.m. with high frequency in the schedule, except on the weekends and holidays; however, customers complained that there was a long travel time from one bus stop to another. The *per capita income* of this town was 2.70 minimum salaries, which was at that time R$ 678.00; thus, students determined that 2.70 × 678.00 = 1830.60. By applying the formula, they found out that

$$\mathrm{TP} = \frac{1{,}830.60 \times 0.07}{48} = 2.67$$

Students in this group concluded that the bus tickets' fares were compatible with the *per capita income* of the town; however, the transportation service provided to the population needed improvement. Public transportation is necessary, but this need does not generate the imposition of excessive tariffs, which are disproportionate to the service provided to the population in this town. At the end of the modelling process, each group of students filmed the presentation of their project and posted it on YouTube with the link for everyone to see on the Moodle platform.

This example emphasizes the role of mathematics in society by claiming that it is necessary to develop student reasoning about the nature of mathematical models and its function in order to help them to critically and reflectively analyze, understand, and comprehend phenomena in the surrounding world (Kaiser and Sriraman 2006). It also shows that although geographically distant from the students during the development of the course, the professor and tutors effectively used Moodle, YouTube, and other freeware tools to be connected. Pedagogical and didactic strategies were used to promote tutorial interactions with professors, tutors, and learners, in order to contribute to the process of teaching and learning mathematical modelling. The resources used were found in the discussion forums and videoconferences, which made possible the development of dialogue between all participants in the virtual learning environment.

This virtual learning environment allows for the development of discussion forums concerning teaching practices in the critical mathematical modelling process and the elaboration of questions about the pedagogical and technical aspects of this process. It also allows for the integration of students, tutors, and professors to deliver messages, the provision of summaries of content of the course, conduction of technological and pedagogical monitoring such as sending messages to all participants and participation in discussion forums, and technical support for students and tutors and access reports in the virtual learning environment. Consequently, students' engagement with a sociocultural context helped them to be more involved in meaningful dialogue and activities. Thus, context allows the use of a *dialogical constructivism* because the source of knowledge is based on social interactions between students and environments in which cognition is the result of cultural artifacts in these interactions (Rosa and Orey 2007).

This critical-reflective dimension of mathematical modelling has provided us with concrete opportunities for our students to discuss the role of mathematics as well as the nature of their models and can be understood as a language used to study, understand, and comprehend problems faced by their own community. Hence, mathematical modelling is used to analyze, simplify, and solve daily phenomena in order to predict results or modify the characteristics of these phenomena (Rosa and Orey 2015). Similarly, in accordance with Freitas (2016), an important proposition of mathematical modelling is to favor the development of a sense of data-based criticality and reflection by the students through the elaboration of projects that demonstrated their applicability in problem situations in everyday life.

Developing strategies through technological tools provided by this virtual learning environment encourages students to explain, understand, manage, analyze, and become critically reflective on all parts of this system. This approach optimizes pedagogical conditions for teaching and learning so that students understand a particular phenomenon in order to act effectively and transform it according to the needs of their own community. Due to perceived needs of the students during the course, the professor created supplemental materials and short video lessons in order to lead students step by step in the modelling process, so they were able to improve their performance in carrying out the modelling-proposed activities. Therefore, it is important to highlight the design of the use of digital technologies in the development of this long-distance course such as the use of videoconferences and discussion forums, which are not frequently present in face-to-face environments. Hence, the virtual learning environment helped the professor to guide the selection of techniques and procedures students used during the conducting of the modelling process.

For example, videoconferences enabled the integration of students, tutors, and the professor for socialization and clarification of questioning that allowed for the development of a collaborative environment for sharing experiences on the proposed themes and promoted students attendance in the *polos* to conduct their modelling projects. Freitas (2016) stated that the use of a videoconference is effective because it has sufficient teaching resources for conducting synchronous classes. In this perspective, knowledge is translated in a dialogical way so these technological tools can be used as instruments to help students to critically think about problems they face daily.

In addition to promoting interaction, the professor prepared teaching materials as well as posted information about the structure and policies of the activities available in this environment, leading students toward an understanding of critical and reflective dimensions of mathematical modelling, which exposes them to a wide variety of themes and techniques. According to Rosa and Orey (2015), as part of this process, questionings in the virtual learning environment through postings on the forums and the use of videoconferences help students to discuss, explain, reflect, and make predictions about the phenomena under study through the elaboration of models that represent these situations. In this context, "critical thinking of the students is emphasized as central goal of teaching. Therefore reflexive discussions among the students within the modelling process are seen as an indispensable part of the modelling process" (Kaiser and Sriraman 2006, p. 306).

Thus, the purpose of this modelling process becomes the ability to develop critical and reflective skills that enable teachers and students to analyze and interpret data together, to formulate and test hypotheses, and to develop and verify the effectiveness of mathematical models. In so doing, the reflections become a transforming action, seeking to reduce the degree of complexity of reality by choosing a system that can represent it (Rosa and Orey 2015). For example, the results of the study conducted by Freitas (2016) showed that this active, interactive, and collaborative participation in the virtual learning environment made possible the development of discussions in the forums that allowed students to clarify any questions regarding the tasks and the development of models as well as in encouraging them to develop an autonomy to prepare their own projects.

48.5 Final Considerations

The study of new methodological proposals becomes relevant because it originates with the ideas regarding social changes resulting from ongoing continuous contemporary scientific and technological developments. In order to enable teaching methods by using structured learning materials and existing technological resources, long-distance learning was developed, which is a form of planned learning that normally occurs outside of traditional school and learning environments (Moore and Kearsley 2005).

Over the last three decades, critical mathematical modelling, as a teaching and learning methodology, has been one of the central themes in mathematics education in Brazil and has come to offer a way to rebuild or restore what has become for many, a fragmented and meaningless mathematical knowledge. Mathematical modelling then becomes a teaching methodology that engages our students in a contextualized teaching and learning process and which can allow them to become involved in the construction of solutions of social significance (Rosa and Orey 2015).

This critical dimension of mathematical modelling is based on the comprehension and understanding of reality, which allows students to learn how to reflect, analyze, and take action on their reality using technological tools provided in a virtual learning environment. Thus, with discussion forums and videoconferences, professors, students, and tutors are empowered to critically analyze interactions enabled by these tools, which contributed to the critical-reflective development of the elaboration of mathematical models in this virtual learning environment.

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Part V
Assessment of Mathematical Modelling in Schools

Chapter 49
Six Principles to Assess Modelling Abilities of Students Working in Groups

Piera Biccard and Dirk Wessels

Abstract This chapter sets out how the six instructional design principles for model-eliciting activities (MEAs) can be reworded and may serve as principles for assessing modelling abilities of students working in groups. The chapter explores some modelling assessment ideas and explains how the six principles form a framework for a holistic evaluation of group modelling. A design research study investigated the modelling competencies of grade 7 students working in a group. The assessment unit was that of the group as a whole and not of individual students. It was found that the six principles reworded as an assessment framework enabled the authors to evaluate significant aspects of model-eliciting activities such as model construction, reality integration, quality of documentation, self-evaluation, development of prototypes for thinking and generalisation.

Keywords Assessing competencies • Modelling • Design principles • Design research • Primary school students

49.1 Introduction

Mathematics education has much to gain from including mathematical modelling activities in the classroom. According to Niss et al. (2007, p. 19), modelling can make "fundamental contributions" to a student's development of mathematical competencies. Measuring mathematical modelling is a much more complicated task as it involves not only the solution but also logical reasoning, linguistic competency

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_49

and previous knowledge of the students and their attitudes (Lingefjard and Holmquist 2005). Assessing or measuring student abilities or understandings is undertaken for different reasons and to meet various needs and designed in accordance with instructional principles. The various dimensions of modelling as an integrated whole, as well as the process and the products of modelling, need to be assessed. To meet this need, we sought a framework that encompassed the broad aims of teaching modelling and the specific competencies of the modelling cycle.

Modelling can be considered as mathematical learning through problem contexts. According to Gijbels et al. (2005, p. 31), "many educators and researchers have advocated new modes of assessment to be congruent with the education goals and instructional principles of problem based learning". Since the goals and instructional principles of modelling are different from traditional teaching, a new assessment framework is needed. According to Cohen (1987), when instruction and assessment are aligned, results are greatly improved. It is therefore important in assessing modelling to align the instructional principles to assessment. This chapter intends to meet some of these aims and to assist in providing an assessment framework for model-eliciting tasks that take place in group situations. Model-eliciting tasks are formulated in such a way that the students have to produce a model in response to the task. The model goes beyond a short response to very specific questions (Lesh and Doerr 2003). We considered it important to find assessment guidelines for group work situations since this is the authentic modelling environment.

49.2 Dimensions of Modelling Assessment

Biggs (1996) suggests a model of instruction that includes students being placed in situations that are likely to elicit the necessary learning and that assessment tasks address the same performances that are stated in the curriculum. Students should be evaluated in an authentic assessment environment (Baxter and Shavelson, cited in Gijbels et al. 2005). An assessment framework needs the following characteristics or features:

- To be aligned with teaching principles,
- To reveal both strengths and weaknesses in student thinking,
- To be based on authentic tasks in an authentic environment.

We wanted to develop a framework that could advance the goals of modelling and problem-centred learning. Problem-centred learning is mathematical learning that takes place through solving rich contextual problems. From a modelling perspective, the following principles are relevant:

- Assessment for modelling is to be aligned with modelling teaching (instructional) principles. The assessment of modelling would parallel the aims of modelling.
- Assessment is to be based on a holistic approach to modelling. This means the entire process and products of modelling should be included in the framework.

Frejd (2013) noted that assessment in modelling can be distinguished as formative or summative. We decided that a formative assessment framework would enable us to focus on the student group as a unit of analysis. An evaluation of previous work on modelling assessment provided a starting point for the development of such an assessment framework. Modelling competencies have been measured in studies using multiple-choice questions (Kaiser 2007). This enables one to assess individual modelling competencies. Clatworthy (1989) developed an assessment rubric that was used in a modelling course to assess modelling competence and provided each student with feedback which assisted in developing positive attitudes. Clatworthy concluded that the development of reliable methods for assessing modelling remained a challenge. In a study by English and Fox (2005), a tool was developed for a single modelling problem to describe student modelling; whilst in another study (English 2007), cycles of mathematical development displayed by a group of students at primary school level were addressed. Chan et al. (2012) developed a rubric to assess modelling competencies. Their specific task instruction was aligned to the rubric. We wanted to start with principles that could be a generic starting point for modelling assessment and the starting point for the development of task-specific rubrics.

Jensen (2007) suggested a multidimensional approach to assessing mathematical modelling competencies. Jensen proposed that the three dimensions provide vocabulary for discussing quality in performance and as such offer a more valid but less reliable alternative to mark schemes. However, we felt that these three dimensions may be too broad to use for single modelling activities. Niss (1992, p. 355) suggested that we ought to assess students' work on the entire process of modelling in all its phases. Many of the studies above focussed on individual phases of the modelling cycle or individual competencies. In an endeavour to merge the above assessment ideals, a qualitative, multidimensional approach informed by instructional principles that would provide the necessary vocabulary to discuss quality performance or evaluate the entire modelling process was deemed necessary. We wanted a framework that allowed us to unpack a group's entire modelling experience. More significantly, such a framework could allow one to develop assessment protocols (e.g. rubrics) for modelling that teachers could use in the classroom at a later stage. Frejd (2013), in his extensive literature review on assessing mathematical modelling, found that very few studies involved theoretically based case studies. We hope to add to the growing literature on framework-based case studies.

49.2.1 Six Principles for Instructional Design and Modelling Assessment

Whilst selecting tasks for the study, six principles of instructional design (Lesh et al. 2000) were found to be used extensively in task design. These principles enable the transformation of existing problems into model-eliciting activities (MEAs) or the creation of new model-eliciting tasks. These principles ensure that tasks qualify as MEAs. The six principles are the reality principle, the model construction principle, the self-evaluation principle, the model-documentation principle, the simple prototype and the model generalisation principle.

The instructional design principles can be reworded so that they assess groups working on modelling tasks. These questions, rewritten as assessment guidelines or principles, can be transferred into a mathematics classroom since they allow teachers or researchers to focus on the essential products and processes of modelling. Reworded, the six principles for assessing modelling are:

1. To what extent *does the group* make sense of the real-life situation?
2. To what extent *does the group* construct a model?
3. To what extent *does the group* judge that their ideas, responses and models are good enough?
4. What is the quality of the documentation *that the group produces* when modelling?
5. To what extent *does the group* produce a solution that is a metaphor (a prototype) for interpreting other situations?
6. To what extent *does the group* develop a shareable, generalisable model?

A study by Yildirim et al. (2010) used four of the six design principles to assess engineering students' modelling abilities in actual teamwork situations. The authors designed a five-point scale rubric for principles 3, 4, 5 and 6. We wanted to include all the principles in the evaluation process so our research question (and the focus of this chapter) is: How can the six instructional design principles for model-eliciting activities be used to assess students working in a group?

49.3 The Study

Design research was the overarching methodology followed in this study (Biccard 2010). This means that a three-phase teaching experiment (Bakker 2004) was followed: a *planning phase* where the tasks and instruments were prepared, a *cyclical teaching experiment* where students solved the tasks and a *retrospective analysis* after each cycle that allowed the researcher to analyse the data and prepare for the next teaching phase. Twelve students were purposively selected for the study and worked in three groups of four students. They were selected based on their school results. For this chapter, only the results of one group working on one task are presented and discussed. The group chosen fared better than the other groups at solving the modelling problems. They comprised students whose results for mathematics were above average for the previous year. Student ages ranged from 11 to 13 years. They worked once a week (after school hours) for 60–90 min. Over a 12-week period, they worked on three MEAs. These students had not been exposed to modelling problems before. All contact sessions were audio-recorded and transcribed. The transcriptions were analysed and coded according to the six principles. Groups worked with minimal researcher (Author 1) intervention. The researcher interacted with groups periodically to question them on what they were doing but not to direct them. At the end of each task, students presented their solutions to the other groups. These sessions were video-recorded. The tasks used were sourced from existing modelling literature and related to proportional reasoning. In the transcripts, R stands for researcher, whilst other letters are identity codes for the four students in the group: T, A, J and S.

49.4 The Results

The results presented are from the recorded transcripts of one group as they worked through the *Sears Catalogue* MEA (adapted from Lesh et al. 2000). An outline of this task is provided for readers who are unfamiliar with the task.

Sears Catalogue
Hello, my name is Sipho and I need some help with a problem. My parents are really unreasonable. My sister, Karabo, is 10 years older than me. When she was in Grade 7 her pocket money was ZAR30 per month. I also get ZAR30 per month. With ZAR30 I cannot buy as much as she could 10 years ago. To prove this I collected some information about prices now and 10 years ago. What I need from you:

Use my price information to determine how much pocket money today would be the same as ZAR30 10 years ago. Write a report for me to give to my parents, describe your method and your conclusions. Show that you accurately figured out how much money gives me the same spending power as ZAR30 did 10 years ago. Explain your method so other children in similar situations can use it to figure out what their allowances should be. Remember, my parents do not like emotional or illogical arguments.

49.4.1 To What Extent Did the Group Make Sense of the Real-Life Situation?

The modelling task elicited student thinking about *real life* and *mathematics*. Knowing *what* real-life knowledge can be woven into a model is a complex process. Students may benefit from more experience in their day-to-day mathematics of working with real situations. Students are required to translate real ideas and processes into mathematical entities. Mathematisation, therefore, lies at the heart of modelling.

The group integrated their own knowledge into their understanding of the problem and the need to produce a model for the parents:

T: Think of the tuckshop, for a hot dog. It used to be ZAR4-50 and now it's nearly ZAR10.

[...]

T: I don't think a parent wants to come home each month and calculate prices from 1999 and 2009 just to get the amount that the child must get?

In discussing one of their ideas, one student considered that it may not be "fair" to the sister:

T: If I was the sister and I had to find out [that they calculated an amount that was too high], I would kill you if it was wrong.

When selecting which data to use from the catalogues, they looked at what it is that children do/should buy with pocket money:

S: No, this is a need and not a want.

J: So maybe we should look at only the stuff that a child should buy, because I think the parents should buy the stationery; the child shouldn't buy stationery with his pocket money. Maybe he wants to buy games and trainers, the nice stuff?

[...]

J: We should scratch those [needs] out.

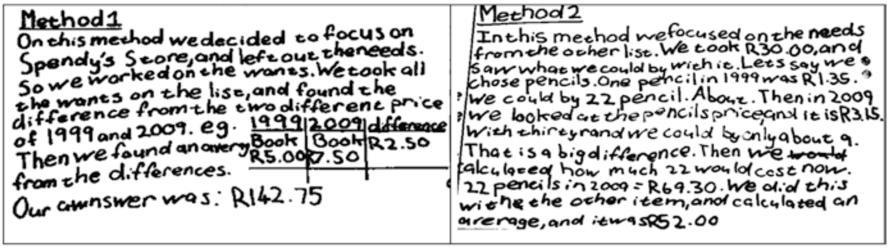
Making sense of a real-life situation affects different modelling competencies. Here they used their real-life experiences to simplify the problem and to justify which data they would be using in generating the model. This also assisted them in validating their model in the presentation sessions. Their real-life knowledge assisted them in advancing and refining their thinking about the situation. Just talking about the real situation without making an impact on the model is not sufficient.

49.4.2 To What Extent Did the Group Construct a Model?

Modelling tasks require that students develop a simplified model to explain their thinking about another situation. Jablonka (1997, cited in Frejd 2013) found the most crucial part of assessing modelling lies in the quality of the mathematical model. As with all MEAs, the group underwent several reformulations and revisions as their thinking about the problem matured. Their initial formulation was to look for a pattern:

S: How do you start this?

T: Maybe we should find a pattern?



Method1
On this method we decided to focus on Spendy's Store, and left out the needs. So we worked on the wants. We took all the wants on the list, and found the difference from the two different price of 1999 and 2009. eg.

1999	2009	difference
Book R5.00	Book R7.50	R2.50

Then we found an avery from the differences.
Our awnswer was: R142.75

Method2
In this method we focused on the needs from the other list. We took R30.00, and saw what we could by with it. Lets say we chose pencils. One pencil in 1999 was R1.35. We could by 22 pencil. About. Then in 2009 we looked at the pencils price and it is R3.15. With thirty rand we could by only about 9. That is a big difference. Then we ~~would~~ calculated how much 22 would cost now. 22 pencils in 2009 = R69.30. We did this with the other item, and calculated an average, and it was R52.00

Fig. 49.1 Model development.

J: Maybe we should look at the difference at the stationery store and see a pattern.

A: Is there a pattern?

They went through several formulations and revisions before deciding on their model (Fig. 49.1). A second model was also produced (Method 2), and they could not decide which model to use – so they opted for averaging the results from both models. Their model explains their assumptions and simplifying procedures and also gives an example of how they worked. The model should describe, explain or predict elements of the real situation through a process of extracting essential features that can be mathematised.

49.4.3 To What Extent Did the Group Judge That Their Own Ideas, Responses and Models Were Good Enough?

The focus of this question is on the level of metacognition in terms of the students judging their own ideas. The students' ability to judge their own ideas and responses means that they are able to move themselves ahead since modelling is characterised by a feeling of uncertainty because there are too many roads to follow (Blomhøj and Jensen 2007). Whilst three group members were working on an early idea, one group member saw the flaw in their argument:

J: I don't understand? Did you start with a price from 1999, and then times it by 2 and then times it by 3 and then minused the times by 2 from the times by 3 and you came up with ZAR1 again … but that's an obvious answer!

J: We used a whole paper just for this, an obvious answer!

However, having taken this route and having seen the error in their reasoning, they were able to refine their thinking and move towards their next model. In their presentations, the researcher questioned the large difference in their answers from their two methods:

R: Didn't it worry you that the one method gave you so much more than the other?

J: No, this list [method 1] is based on wants and wants cost more than needs.
Focussing on their own self-evaluation allowed us insight into their mathematical thinking as well as tracking their progress through model development.

49.4.4 What Was the Quality of the Documentation That the Group Produced When Modelling?

The group produced rough working sheets whilst solving the task as well as presentation sheets for their oral presentations. There were no prescriptions as to what had to be included in these sheets or their format. The working sheets were messy and haphazard. The group seemed to think that only the final solution was important, placing less value on their working sheets compared to their presentation sheets. Working sheets did not always show the progression of group ideas nor did they contain sufficient evidence of some group competencies. The presentation sheets contained only what the group had filtered for use in the presentation. Lesh et al. (2000) remind us that objects, relationships, operations, principles and representational systems will be revealed through MEAs. Students may represent these in a variety of ways. Capturing all these is problematic during classroom modelling sessions, but can be elicited from groups when they present their solutions.

Smith (cited in Derry et al. 1998) describes three types of products produced when students work in groups: *tangible*, *intangible* and *ephemeral*. These descriptions remind us that various forms of representation are necessary for modelling, from temporary sketches or conversations to the rough notes, tables and data to the final model. These products are valuable to students, teachers and researchers in the modelling process as they leave “auditable trails of documentation” (Lesh and Doerr 2003, p. 31) that can be used to assess the modelling process and the model itself. This group was strong verbally and produced written documents only when directly requested to, for example, a letter to the parents. Many of their products were relegated to “rough work”. This may be due to their previous experiences in mathematics classrooms where expectations are for a final answer only.

49.4.5 To What Extent Did the Group Generate an Effective Prototype?

This section relates to the group’s ability to extract from their modelling process the essence of the mathematical structure of the problem. The group was using very simple ideas in their solution processes; nevertheless, these simple ideas were well understood by the students and could be used by them. The mathematisation process that they followed in creating two methods for the problem is similar to the suggested products set out in Lesh et al. (2000). Simplicity of computation must not be confused with simplicity of mathematising. Iversen and Larson (2006) found that students use simple mathematics in complex ways. "Group decisions about what to do and where to go utilizing only incomplete information and extensive use of heuristics and simple but robust concepts and procedures" (Lesh et al. 2008, p. 125) is the norm whilst working on modelling tasks. Lesh et al. (2000) state that the situation (and therefore the student's response) should focus on the important idea and should not involve unnecessary complexity. In the *Sears Catalogue* problem, the central idea is that of reasoning proportionally from price indexes to their effect on the spending power of money over 10 years. During their presentation of their model, the group used simple concepts to build a significant prototype of the concept "average difference". This becomes the prototype that they may use when confronted with structurally similar problems in the future:

S: We took all the "wants" on the list from one store and we found the difference between the two prices [1999 and 2009]. We added them together and worked out an average which was ZAR142-75.

A: But we worked on his needs and worked on a different method.

J: We took ZAR30 and saw what we could buy [in 1999]. Then we looked at how much do you need to buy the same amount in 2009. Then we worked out the average difference between the two.

49.4.6 To What Extent Did the Group Develop a Shareable, Generalisable Model?

Modelling involves more than finding the solution to a given situation. In the task instructions, a reusable or generalisable model is required which means the mathematical model should be usable in similar situations or using other data. Student abilities to create a generalisable model mean that they are working at higher abstraction levels. It means that students have fully understood the real situation, have mathematised the problem with the mathematical knowledge and concepts that they have at their disposal and are now able to place and "run" their model in unfamiliar conditions. This group remained aware of the need for a generalisable model; however, they struggled to produce one:

T: I wouldn't want to get ZAR30 if my sister got it [ten years ago].

A: It's not only him, it's for anyone else.

In their letter to the parents, they also stated: "The method we have come up with can be used on any prices of items".

The generalisable model relies on the quality of the situation model. When this model is weak or not fully understood, then the task of creating a generalisable model becomes more difficult. For this task, the group worked deeply in the situation model but they were not confident about their situation model. They also excluded large parts of the data in their situation model, which may have contributed to their difficulty in producing a generalisable model. When asked during the presentation sessions about their generalisable model, they were able to explain the qualitative modelling process they went through.

49.5 Discussion and Conclusion

The six instructional design questions were used to gauge group modelling abilities. This assessment framework provides a holistic, integrated and practical approach to assess students' modelling abilities. It also provides an avenue for developing further research on student modelling abilities. The framework could also be formative in that it may guide a novice teacher to what to look for when groups of students solve MEAs.

The six principles' framework assisted us in gauging milestones in the group's modelling development. The questions cover areas such as reality, construction, reflection, representations, constructing prototypes and generalisability. Moreover, using these principles allows one to weave metacognitive competencies and cognitive competencies together and to focus on what students working in groups are achieving holistically in their modelling endeavours. Although we focussed on group modelling, it may be possible to evaluate individual modelling abilities using this framework. Further research could focus on producing a rubric from each of these questions.

The framework provides a suitable way to unpack the entire modelling episode to assess a group's entire modelling process and the product produced. The questions are broad enough to be applicable to all modelling problems, not only in terms of student assessment but also for practitioner reflection. The questions form a bridge between tasks and assessment of these. The six areas may provide alternative lenses to assess modelling as well as vocabulary for teachers and researchers to converse about modelling assessment. Furthermore, the framework can be seen as a "bottom-up" assessment approach – it is designed for classroom use but could extend to inform professional development, curricular change and research in mathematical modelling. Further research into these six areas of modelling ability and how to assess these abilities in groups and individuals is necessary.

Acknowledgements The financial assistance of the National Research Foundation (NRF) towards the research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the authors and should not be attributed to the NRF.

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Chapter 50
Assessing Mathematizing Competences Through Multiple-Choice Tasks: Using Students' Response Processes to Investigate Task Validity

Brikena Djepaxhija, Pauline Vos, and Anne Berit Fuglestad

Abstract In this chapter, we report on multiple-choice tasks for assessing mathematizing competences of grade 9 students. The task format is complex, consisting of two layers. In the first layer, students are asked to consider a holistic modelling problem. In the second layer, they are asked for an atomistic competence (making assumptions, assigning variables, etc.) related to the same modelling problem. We conducted a qualitative study to investigate the validity of these tasks based on students' response processes. Eight students worked in pairs solving the tasks collaboratively. The results show that all students were able to handle the layered task format. They reflected meta-cognitively on the holistic modelling problem, but none of them started solving it in itself. All students considered the remainder of the task, which made them focus on a specific mathematizing activity.

Keywords Assessment • Holistic assessment • Mathematizing • Lower secondary students • Multiple-choice tasks • Task format • Task validity • Validity based on response processes

50.1 Introduction

Assessment is an inseparable part of education. Assessment is needed for reporting students' learning outcomes to parents, policy makers, teachers, school leaders and students themselves. Also, assessment is used in educational research. For the

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_50

assessment of mathematical modelling education, different modes of assessment have been developed: project-based assessment, written tests, portfolio, contests and so forth (Frejd 2013). These modes draw on two different approaches: (1) a holistic approach, which asks students to accomplish a complete modelling problem, and (2) an atomistic approach, which asks students for separate competences needed to accomplish only part of the modelling.

The assessment of modelling competences is still considered a challenge (Niss et al. 2007), and since the 1990s, research has been carried out to develop reliable and valid modes of assessment, looking for a balance between holistic and atomistic modes (Kaiser and Brand 2015). On the one hand, holistic modes such as project-based assessments are time-consuming and raise questions on reliability of results (Vos 2007). Written tests with holistic modelling problems raise questions on validity of results, for example, when students 'get stuck' in the beginning of the modelling process, they cannot carry out the subsequent activities, which then cannot be captured for evaluation (Stacey and Turner 2015). On the other hand, atomistic modes of assessment fail to capture the complexity of modelling competences. For example, it is possible to assess a specific modelling activity, but it is not possible to capture how students connect this activity to other modelling activities (Reit and Ludwig 2015). Along these lines, holistic assessment is considered as ideal but complex, while atomistic assessment is considered as limited but practical and informative (Haines and Crouch 2010). Generally, a balance between the two approaches is recommended for the assessment of modelling (Blomhøj and Jensen 2003).

Our research builds on Haines et al. (2000), who developed a test for students in higher education consisting of multiple-choice tasks. They identified distinct modelling competences and developed multiple-choice tasks that assessed these separately. Students' overall score on all tasks aimed to report on their achievement and progress in modelling at large. The multiple-choice test was fruitfully used in research (Frejd and Ärlebäck 2011).

In our research, we studied the feasibility of a similar multiple-choice test, but then fitting students at the level of grade 9. In the design of tasks, we focused on the first activities that one undertakes in modelling. We will indicate these activities by the umbrella term *mathematizing*, which are activities between facing a modelling problem, and making a translation to a mathematical model in order to reach a solution for the problem.

Because the new multiple-choice tasks aim to assess mathematizing competences, we call these tasks *mathematizing multiple-choice tasks* (MMC-tasks). The format of an MMC-task is characterized by two 'layers'. See *MMC-task format*. In the first layer, the MMC-task contains a holistic modelling problem. To make it stand out visually, it is written within a box. The second layer of the MMC-task contains: (1) instructions to focus and reflect metacognitively on the modelling problem, (2) an atomistic question to ask students for a specific mathematizing competence, and (3) a number of alternatives.

MMC-task format
Consider the problem below (read it well!).

$$\text{Modelling problem} = \text{problem situation} + \text{holistic question}$$

Think about yourself and how you can solve this problem.
Atomistic question asking for a specific mathematizing competence

(A) *Alternative 1*
(B) *Alternative 2*
(C) *Alternative 3etc.*

At the second layer, we used two instructions surrounding the modelling problem. These instructions are: *Consider the problem below (read it well!)* and *Think about yourself and how you can solve this problem*. The two instructions aim to guide students not to solve the modelling problem but to focus and reflect metacognitively about it. The instructions also aim to give students time to recognize that to solve the MMC-task, they only need to answer the atomistic question by selecting one of the alternatives.

50.2 Theoretical Framework

50.2.1 Mathematizing Competences

Blum and Leiss (2005) describe mathematical modelling with an idealized modelling cycle consisting of subsequent activities: understanding, simplifying/structuring, mathematizing, working mathematically, interpreting and validating. Borromeo Ferri (2006) has argued that it can be difficult to distinguish among these activities, because these differ between students and tasks. Therefore, in the present chapter, we use the term mathematizing to describe the transformation activities from 'the modelling problem' to 'the mathematical model'. Thus, mathematizing comprises all activities before a student starts on the purely mathematical work. In our research, we focus on four observable mathematizing activities: (1) making assumptions, (2) asking clarifying questions, (3) assigning variables, parameters and constants, and (4) formulating mathematical statements (creating a formula, expressing a range for a variable, etc.). These four activity categories were also discerned in Haines et al. (2000).

Different authors have defined modelling competences referring to a cyclic representation of the modelling process (Blomhøj and Jensen 2003; Maaß 2006; Niss et al. 2007). In their definitions of modelling competences, the terminology converges to cognitive competences. However, other competences, such as metacognitive competences, are also needed in modelling, as pointed out by Maaß (2006).

Thus, we use a broad definition of *modelling competences*: they comprise students' abilities to consciously carry out all modelling activities. With the focus on *mathematizing competences* in the present chapter, we define these as students' ability to consciously carry out all activities needed to construct a mathematical model for a given modelling problem.

50.2.2 Response Processes Validity

Tasks developed for an assessment must be valid and reliable: they must assess what they intend to assess, and the results produced must be consistent, repeatable and independent. In this chapter, we focus on validity only. For studying validity, Krathwohl (1998) distinguishes a range of sources, such as *the content* (does the task assess the proper content?), *relations to other variables* (is the task similar to other tasks with the same intent?), *the consequence of assessment* (does the task have other consequences than the intended ones?), *face validity* (does the task appear to be valid?) and *response processes* (do students display the intended mental processes, abilities and skills?).

For researching the validity of the MMC-tasks, we opted to study it qualitatively, based on students' response processes. By observing students while working on the tasks, we could generate evidence concerning the fit between students' actual and the expected performance (Krathwohl 1998). For researching task validity, Pellegrino et al. (2001) suggest examining different parts of a task. In the case of the MMC-task, we opted to investigate two aspects: (1) the instruction clarity and (2) the task assessing purpose. Within the MMC-tasks, the instructions are statements that are meant to activate metacognitive processes while solving the task. Therefore, we investigated whether these instructions were clear and promoted the intended processes. Also, we investigated whether the MMC-tasks in themselves assessed the intended mathematizing competences. The research question was: To what extent are the MMC-tasks valid regarding instructions clarity and task assessing purpose?

50.3 Methods

The study was operationalized by studying students while working in pairs on MMC-tasks. The participants in the study were eight grade 9 students (approx. 15 years old) from a lower secondary school in Albania. The students were average achievers in mathematics according to their teacher. The students participated voluntarily, and they will be identified by pseudonyms (Eva and Iris, Rea and Tom, Tea and Emma, Max and Ben).

A two-step method was used to investigate students' response processes: observation followed by a retrospective report (Ericsson and Simon 1993). In the observation phase, the students worked in pairs solving MMC-tasks collaboratively. In the retrospective report phase, which took place immediately after students' work on each task, students described what they did while working on the MMC-task. Both phases of data collection were video recorded. After transcribing, the data were analysed for clarity of instructions and task assessing purpose.

We designed six MMC-tasks. For the modelling problem in the first layer, we used PISA problems for mathematics, because these are problems with a real world origin, and they have been designed to match 15-year-old students (OECD 2014). The three PISA problems were: Rock Concert, Pizzas, and Distance, and each was used to develop two MMC-tasks. The multiple-choice alternatives were developed based on students' answers from an empirical study, in which grade 9 students were observed while working on these three PISA problems (Djepaxhija et al. 2015). The mathematizing competences addressed by the six resulting MMC-tasks were: assumption making (two MMC-tasks), asking clarifying questions (one MMC-task), assigning variables, parameters and constants (two MMC-tasks), and formulating mathematical statements (one MMC-task). Two MMC-tasks are displayed in this chapter. The MMC-task *Rock Concert* aims to assess students' competence on assigning variables, parameters and constants. The MMC-task *Pizzas* aims to assess students' competence on formulating mathematically a model that fits the problem (find the relations between the variables, parameters and constants and then express it through a mathematical statement).

Because of the work intensity in the interview (students had to solve the tasks together and to report retrospectively), we could only administer three tasks per student pair. Therefore, we distributed the six tasks over two sets. Each set comprised three different PISA problems and addressed three different mathematizing competences. Each set was taken by four students.

Rock Concert

Consider the problem below (read it well!).

> For a rock concert, a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing. Which is the total number of people attending the concert?

Think about yourself and how you can solve this problem.
Pick out two pieces of information that you need to answer the problem.

(A) There will be 12 rock stars performing.
(B) The field size is 5000 square metres.
(C) The price of the ticket is 1000 ALL.
(D) The density of the fans in the field is four persons per square metre.
(E) The average age of fans is 30 years old.

Pizzas

Consider the problem below (read it well!).

> A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 300 ALL. The larger one has a diameter of 40 cm and costs 400 ALL. Which pizza is better value for money?

Think about yourself and how you can solve this problem.
Which one of the following options would you choose to answer the problem?

(A) I would compare the prices of the pizzas. Then I would choose the pizza which has the cheaper price.
(B) I would compare the diameters of the pizzas. Then I would choose the pizza which has the bigger diameter.
(C) I would divide the pizzas' diameters by their prices. Then I would choose the pizza which gives me more for less money.
(D) I would calculate the area of both pizzas. I would divide the pizzas' areas by their prices. Then I would choose the pizza which gives me more for less money.
(E) I would calculate the volume of both pizzas. I would divide the pizzas' volumes by their prices. Then I would choose the pizza which gives me more for less money.

50.4 Results

Below, we report on all twelve cases (four student pairs, each working on three MMC-tasks). Selected episodes are related to the two aforementioned tasks.

50.4.1 Clarity of the Instructions

Students were introduced to the MMC-tasks through a sheet of paper with the task written on it. In all cases, they started to read individually, and in silence. An example of this is the following episode with the reactions of Tom and Rea to the MMC-task *Pizzas*:

[Tom and Rea are given the task]

Both: …silence… [Tom and Rea are reading in silence and individually]

Tom: I am yet at the first sentence to get more from it…can we draw these ones [two pizzas]

Rea: Yes we can draw…make them with the same thickness.

Tom: Yes they can both have it x.
Rea: Write on it [on the small pizza] the diameter 30 cm and 300 ALL…and 40 cm and 400 ALL [on the big pizza].

Tom and Rea start by reading in silence and individually. Thereafter, Tom stresses that he is reading slowly in order to get more information from the text. Then, he asks Rea if they can visualize the features of the pizzas, which they cooperatively do. Tom's insistence on the reading process can be interpreted as his attempt to become familiar with the modelling problem (in the first layer) before moving to the next step on the task. The students' sketch of the pizzas can be interpreted as their attempt to check and show the quality of their reading process.

The above episode is illustrative of what we met in all data. The instruction *Consider the problem below (read it well!)* prompts reactions such as: reading in silence and individually, insistence on the reading (and re-reading) to get familiar with the problem, and listing of features of the modelling problem.

The second instruction *Think about yourself and how you will solve this problem* also triggers a reaction of silence, as shown in the following episode of Tea and Emma:

[students read the second instruction]
Both: …silence…
Emma: Let's think for a while…

[then Emma and Tea start investigating the alternatives]

Emma and Tea fall silent yet again after reading the instruction *Think about yourself and how you will solve this problem*. This silence is broken by Emma who openly asks her friend to think individually for a while. Thereafter, they start investigating the alternatives. From the silence, we cannot observe whether they are reflecting metacognitively, but when they start to investigate the multiple-choice alternatives, Emma and Tea have realized that they are not going to solve the modelling problem (in the first layer) but they will consider the remainder of the task.

In the data of all twelve cases, the instruction *Think about yourself and how you will solve this problem* generates silence, and thereafter they all start dealing with the multiple-choice alternatives (see next section). Not one of the students starts solving the modelling problem in the first layer of the MMC-task. All of them consider the question and the alternatives asking for a mathematizing competence in second layer. We consider this as a hint towards students' metacognitive reflection on the intentions of the task.

50.4.2 Assessing Purpose of the MMC Task

After reading the instructions and the modelling problem, all students considered the question that refers to a specific mathematizing competence and investigate the given alternatives. The following episode shows Ben and Max's work on the task *Rock Concert*:

Max: …we should consider that the 12 rock stars performed there. It means that we can find how many people are in the field using the area and the density and then we subtract the 12 rock stars to find the number of fans.

Ben: No, it does not mean that, because here [in the problem] it is not given that the stage is inside the field…it is not reasonable to consider the first alternative [alternative A] because it says that the field is reserved for the fans only.

In this episode, Max invites Ben to consider the constant '12 rock stars' (alternative A), and he anticipates the role of this constant to the mathematical model to be constructed (subtracting the number of rock stars from the number of people in the field). Simultaneously, he brings in the relevance of 'the area of the field' and 'the density of people in the field'. However, Ben disagrees by pointing out that the constant '12 rock stars' does not fit to the problem as 'the field is reserved only for the fans'.

The episode shows how students focus on the mathematizing activity of assigning variables, parameters and constants, as intended. They investigate the relevance of a given constant in connection to the problem and the mathematical model to be constructed using both their mathematical and extra-mathematical knowledge. Below, we offer an episode, in which Eva and Iris discuss the multiple-choice alternatives in the MMC-task *Pizzas*:

Eva: About the alternative A, to compare the prices of the pizzas, the one that has lower price has also smaller size, the diameter is smaller. The one that that has higher price has also bigger size, the diameter is bigger…the same for the alternative B, the one that has bigger diameter has also higher price, and the one that has smaller diameter…

Iris: … has also lower price. While we found the alternative C as more logical because comparing the diameter to the price we can choose a sizeable pizza and with a reasonable price.

Eva: If we divide the pizzas' diameters by their prices, we can find how much it [the unit of each] does cost …and we can find which pizza gives us more for less money…the alternative D, since it is a situation we meet in our everyday life to find the area, and thereafter to divide it by the price, will be impractical…the same to the alternative E to find the volume.

Iris: We think it is the alternative C.

In this episode, Eva and Iris consider the alternatives one by one. Eva, who is leading the thoughts, starts by rejecting alternatives A and B, because these statements neglect the price-size relation. The girls select alternative C, because here the relation price-size is considered. According to them, a division between 'the diameter and the price' for both pizzas will give them 'how much the unit of each pizza costs', which then can be used for a decision. Eva excludes alternatives D and E, because according to her, these mathematical statements do not fit to the real context of the problem.

This episode provides evidence that Eva and Iris focus on the mathematizing activity of formulating a mathematical model. They are using both their extra-mathematical and mathematical knowledge to investigate the relevance of different mathematical operations in connection to the context of the problem. Moreover, this episode shows metacognitive reflection when they think about their own thinking in a pizzeria context. They recognize their different ways of thinking under different circumstances.

In all twelve cases (six MMC-tasks, each carried out by two pairs of students), the students' responses show that the MMC-tasks made them focus on the mathematizing activity, which was intended to be assessed. All pairs of students investigated one by one the given alternatives. They used their extra-mathematical and mathematical knowledge to investigate the alternatives' relevance for this specific activity. In addition, they also investigated the alternatives' relevance in connection to the modelling problem and to the mathematical model to be constructed.

50.5 Conclusion and Implications

In this chapter, we reported on a validity study of six MMC-tasks that we developed to assess mathematizing competences. The MMC-task format has the specific feature of centring around a holistic modelling problem, while asking atomistically for a separate mathematizing competency related to that problem. We investigated whether grade 9 students were able to cope with this format by analysing students' response processes on: (1) the clarity of instructions and (2) the task assessing purpose. The results show that students were well able to distinguish between the holistic modelling problem (first layer) and the atomistic part of the task (second layer). They could handle the format of an MMC-task, and the activities prompted by the tasks were the intended mathematizing activities.

It is evident that the instruction, *Consider the problem below (read it well!)*, prompted students to focus on the reading process of the modelling problem. This instruction is an effective entry for students to engage in the task. It is also evident that the instruction, *Think about yourself, and how you can solve this problem,* works as intended. In some cases, it made students reflect metacognitively immediately after they read it. In most cases, there was no such clear evidence, but we observed students reflect metacognitively later while dealing with the remainder of the task. In none of the cases did the students start to solve the modelling problem (at the first layer). They all considered the question and the alternatives asking for a mathematizing competence.

The results show that the MMC tasks can validly assess grade 9 students' mathematizing competences. As intended, they made students work on a specific mathematizing activity atomistically. The students understood that they were asked to select the best alternative, which would lead towards an appropriate mathematical model for the problem. They investigated the relevance of each alternative in connection to the modelling problem and to the mathematical model to be constructed. Students' responses show that they used both their extra-mathematical and mathematical knowledge while dealing with the MMC-task. The students found the scope of the choices (given alternatives) sufficient. None of them suggested a new alternative that would help them to answer the task. These results show a balanced focus of students on the process and on the product of the MMC-task, which indicates that they have a notion for the holistic elements of an MMC-task as well.

The validity of the MMC-tasks that we developed offers opportunities for the future. The format can be used to develop more MMC-tasks, which then can be used in tests similar to the test developed by Haines et al. (2000), to assess students' mathematizing competences. Also, the study shows that the MMC-tasks generated meaningful discussion among the students on how they can mathematize a real-world problem. Thus, the MMC-tasks can be used within classroom practice. Students' group work on such tasks, whether or not guided by a teacher, can foster meaningful learning about specific mathematizing activities, as part of solving modelling problems. Discussing MMC-tasks in groups can assist the development of students' competences to carry out single steps in the mathematizing process. An atomistic approach to the modelling, as prompted in the MMC-tasks, fits our target group, who are beginners in modelling. At the same time, the MMC-tasks include a holistic modelling problem, and as such they convey a clear message that atomistic activities are part of a larger whole.

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Part VI
Applicability at Different Levels of Schooling

Chapter 51
How to Build a Hydrogen Refuelling Station Infrastructure in Germany: An Interdisciplinary Project Approach for Mathematics Classrooms

Irene Grafenhofer and Hans-Stefan Siller

Abstract This chapter is based upon an interdisciplinary modelling project about alternative energies carried out with high school students. In this project, we focus on modelling a road map for hydrogen refuelling stations that can differ in conditions (costs, energy demand, etc.), precision of the model and variety of (mathematical) tools students choose. We use hydrogen, which is a current topic in European politics, in our study as a matter for (mathematical) discussion and modelling with students in an interdisciplinary context. This qualitative study explores this mathematical modelling problem related to the interdisciplinary learning environments with a focus on time reduction for students' research. We also investigated the influences the interdisciplinary context has for using extra-mathematical knowledge to solve this problem.

Keywords Interdisciplinary modelling project • Real-life context • Extra-mathematical knowledge

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_51

51.1 Introduction

The acceptance of integrating modelling into the classroom among (aspiring) mathematics teachers is classified as high (cf. Siller et al. 2012). Here we want to consider one important aspect of these modelling experiences – the extra-mathematical context or real-life background where students try to find their solution. We refer here to the model of the modelling process as suggested by Blum and Leiß (2007) and to the indication of extra-mathematical knowledge within this model by Borromeo Ferri (2011). A study by Siller and Meckel (2015) showed that 43% of the whole modelling project time goes into researching the real-life background (e.g. geographical knowledge) that is significant and worth studying more precisely. So, in our qualitative study, we want to focus on extra-mathematical knowledge that is used when students are working in an interdisciplinary context (e.g. mathematics, physics and chemistry). There are a lot of possibilities of interdisciplinary teaching strategies (cf. Stadler 1999); for example, one given topic is taught in two or more academic subjects. That means, you learn the same thing from a different point of view. Furthermore, interdisciplinary teaching also can mean that you do some preparation in one subject to integrate these ideas into another one. This is important when you, for instance, need some chemical information about hydrogen to solve a mathematical problem. In our study, we used the topic of alternative energy that has an obvious interdisciplinary background in mathematics, physics and chemistry. Considering this background, we want to examine if the given problem – alternative energy with focus on hydrogen – in combination with this interdisciplinary teaching helps students to obtain extra-mathematical information to solve the problem.

51.2 Setting and Methodology of the Study

The qualitative study concerning mathematical modelling introduced here was conducted as part of modelling days (3–4 days) at three secondary level II schools (two secondary schools and one vocational school) in Rhineland-Palatinate (Germany). Here, students occupied themselves independently in an interdisciplinary context (mathematics, physics and chemistry) with the subject of alternative energy by modelling a comprehensive, cost-optimized hydrogen refuelling station network in Germany.

In the process of our study, we consider four different student groups: two groups (eight ninth and tenth grade students divided into two groups) whose teaching staff had explicitly emphasized the interdisciplinary aspect and had thus generated references for physics and chemistry suitable to the subject of hydrogen and its use as a fuel. Therefore, all students of ninth and tenth grade were divided into three big groups (mathematics, physics and chemistry groups), where they had to obtain information about hydrogen from a mathematical, physical and chemical point of view and its use as an alternative energy. For our study, we focused only on the

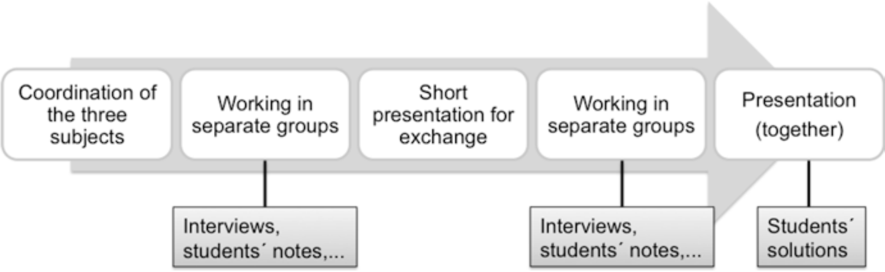

Fig. 51.1 Methodology of the study

mathematical group. Following this, the students gave some presentations for exchange of ideas. Afterwards, they worked again in groups with the possibility to talk to other students who were specialized in one field. At the end, they presented their models and solutions (Fig. 51.1). In the other student groups (nine 11th grade students divided into two groups), we saw the students' interdisciplinary activities without specific evidence of the integration of other academic subjects. So, these students did not have any instruction in physics and chemistry and worked for around 3 days on their own.

During these modelling days, two research questions were investigated:

- In which way can we integrate and organize the given problem – looking for an optimal network of hydrogen stations in Germany in an interdisciplinary learning environment?
- Which influences from this interdisciplinary environment can we find in students' solutions?

For answering the research questions, we especially focused on the organization for the interdisciplinary learning environment. First, we had to choose the topic, planned how to give instructions to the groups and chose which questions they should answer. Afterwards, we observed the students during the extra-mathematical preparation and followed the solution process of each group. We conducted some interviews with students, and to gain more detail, we collected ideas, citations, notes and finished models by the students (Fig. 51.1). Then we compared each group to the others, in order to find out how they differed from each other and if we can identify influences resulting from the interdisciplinary teaching.

51.3 Problem Statement

The reduction of carbon dioxide emissions through alternative energy is a current topic in European energy policy, which can be found at the European Commission homepage. Primarily, hydrogen as a carbon-dioxide-neutral energy supply is, amongst other things, a source of hope in this context, because hydrogen can be produced from various energy sources (sun and wind energy, petroleum, natural gas, etc.). Meanwhile, many companies are bringing their hydrogen cars into mass production, and initial sales figures (early 2015) bear witness to potential great customers' interest (cf. Frankfurter Allgemeine 2015). Hydrogen cars not only have a larger range than traditional electrical cars, but also, they can be refuelled within a few minutes. However, conversion to hydrogen also requires a change to the refuelling station infrastructure, and consideration must be given to how such a hydrogen refuelling station network would look like, which is cost-effective all over Germany and for companies, so that the end customer is not deterred from buying due to the lack of refuelling stations as occurred during the introduction of e-cars. Below, such a refuelling station network is modelled and the implementation introduced as part of the modelling days. The following task was issued:

What Does an Optimal Network of Hydrogen Refuelling Stations for Germany Look Like?
A hydrogen car drives its electrical motor with hydrogen, which is the reason for only water steam arising. Hydrogen is thus an energy source with 0% CO_2 emission and is highly interesting from a political perspective in Germany with reference to the promised energy transition. A nationwide, cost-optimized network of refuelling stations is necessary to supply hydrogen (at the moment, somewhat more than ten pilot refuelling stations exist in Germany).

51.4 Understanding and Reduction of the Problem

We put ourselves in the position of a hydrogen-car owner for a better understanding of the problem. Vehicle owner A lives in the vicinity of German city B, which could also be located in the area surrounding Germany. He can drive, at most, 400 km with his fully fuelled hydrogen car and would like to reach as many cities in Germany as possible. In this connection, vehicle owner A always takes the quickest connecting routes, namely, the motorways. Owner A drives to the first big city, which is already 150 km away on federal highways, and can only refuel on the connecting routes between cities. From the perspective of a new hydrogen refuelling station operator, it is desirable to reach as many customers as possible. That means locating in the vicinity of large cities, if possible drawing on the existing rest-stop infrastructure to reduce costs, and providing a comprehensive refuelling station network with a minimum number of refuelling stations.

51.4.1 Data Collection and Initial Assumptions

The initial modelling approach, which we presented above, delivers a very simplified real model for developing a refuelling network in Germany. The conception for realizing the model is refined through extensive data collection and the assumptions resulting from it. First, important information that must be clarified in preparation by the students is itemized:

- Germany's large cities and the largest airports are possible initial locations.
- Their connection routes are considered taking Germany's motorway network and existing motorway refuelling stations into account.
- At first, only Euclidean distances should be worked with, because it is assumed that the approach to already existing refuelling rest stops or additional large cities is sufficient. This assumption should be challenged during the validation process.
- A maximum range of about 150 km must be guaranteed on these motorway routes. This is especially true, when you consider that approach ways are also necessary before driving onto the motorway and that more fuel is consumed on the latter. Since calculations are being made with Euclidean distances and motorways do not run directly along the quickest path between cities, a refuelling station should cover a 75 km radius.
- Additional cities such as those lying on Germany's outer border (Dresden, Aachen, Passau, Kiel, etc.) must be included primarily in regard to the goal of total coverage.
- Initial estimates relative to the market penetration of hydrogen cars vary in the lower, single-digit percent range (Welcome to HyWays project n.d.).
- On the homepage of the Heute-Mobil automobile club, you can look up the number of refuelling stations in the metropolitan areas of the ten largest cities in Germany (2013 status). For example, Berlin has 254 refuelling stations, 2% of which would be about five hydrogen refuelling stations in Berlin that are already in operation or under construction. This way, the number of refuelling stations in each large city can now be estimated.

Based on the research, the following assumptions can now be formulated, which can be drawn upon for realizing refuelling station plans. These can also be used to validate and evaluate student assumptions and results:

1. Germany's ten largest cities and the six largest airports (e.g. Munchen FH) are selected as the initial locations.
2. Motorways between these locations are interpreted as Euclidean distances.
3. Dynamic geometry software (e.g. GeoGebra) should be used to represent the map of Germany.
4. Each hydrogen refuelling station covers a circle with 75 km radius.
5. The refuelling stations should supply car drivers all over Germany.
6. The market penetration of hydrogen cars will be 2% in the year 2020.
7. Large cities need more than one refuelling station depending on population density.

51.4.2 Mathematical Approach via a Facility-Location Problem

In the present problem, we have two or more cities from where we want to reach a refuelling station as quickly as possible. These cities are always given as points $S_{\mathrm{xm}}(a_{m1}, a_{m2})$ on a digitized map (with a_{m1}: x coordinate of the mth city, a_{m2}: y coordinate of the mth city), on the connecting line of which the location $X(x_1,x_2)$ is sought that minimizes the distance to one of the two cities. This can be solved geometrically or computationally for the connection of two or more cities. Mathematically, a facility-location problem is involved here, an optimization problem of which the calculation for arbitrarily many cities will now be briefly introduced (cf. Hamacher 1995).

The Euclidean distance between a city and an arbitrary location can be calculated with $l_2\left(S_{\mathrm{xm}},X\right)=\sqrt{\left(a_{11}-x_{11}\right)^2+\left(a_{12}-x_{12}\right)^2}$. For this, the minimumof the following facility-location function is to be determined, which seeks a location that minimizes the maximum distance to the sites: $g(X) \coloneqq \max_{1 \leq m \leq M} l_2(S_{\mathrm{xm}}, X)$. The following facility-location problem results from the goal of minimizing the maximum distance min $g(X) := \max_{1 \leq m \leq M} l_2(S_{\mathrm{xm}}, X)$. For the geometrical determination of an optimal location between two cities, the solution is obtained by constructing the route midpoint $M\left(\frac{a_{11}+a_{21}}{2},\frac{a_{12}+a_{22}}{2}\right)$ via the perpendicular bisector of the segment joining the given points, because the location thus lies exactly in the middle with equal separation between the two cities. Geometrical determination of the optimal location for three cities proceeds using the same principle via the intersection of perpendicular bisectors of the segments joining adjacent cities to constructing the circumcentre.

We now consider the following problem as an example: it is desired to find the optimal location between Nuremberg (Nürnberg), Leipzig and Dresden. The coordinates generated from the geometrical solution are now compared to possible locations in reality on the corresponding motorway, for example, in Google Maps. Thus, only the Bayreuth (X_1) and the motorway interchange in Hof (X_2) locations are eligible. We define the set of possible locations as C:= {X_1(684.477, 5535.756), X_2(707.37, 5577.55)}. The cities for which the optimum centre location should now be found are S_x:= {Nuremberg S_{x1}(650.51, 5479.78), Leipzig S_{x2}(718.47, 5691.06), Dresden S_{x3}(811.49, 5656.19)}. Now the Euclidean separations of the cities and the possible locations are each calculated: $l_2(S_{x2}, X_1) \approx 159$ km and $l_2(S_{x3}, X_1) \approx 175$ km.

Thus, the maximum Euclidean distance for the Bayreuth and Hof location is

$$g\left(X_1\right):=\max l_2\left(S_{\mathrm{xm}},X_1\right)=\max\left\{65.5;159;175\right\}=175\text{ km}$$

$$g\left(X_2\right):=\max l_2\left(S_{\mathrm{xm}},X_2\right)=\max\left\{97.2;114;130.5\right\}=130.5\text{ km}$$

From this, the minimum as a result of the facility-location problem yields $\min_{X \in C} g(X) \coloneqq \{175; 130.5\} = 130.5\text{ km} = l_2(S_{x3}, X_2)$

The optimum location would thus be Hof with a maximum Euclidean distance of 130.5 km.

51.4.3 Representation in Dynamic Geometry Software

The way this initial real model is visualized must be considered next. All of the geographical data is illustrated with a dynamic geometry software program (here GeoGebra) below (assumption 3). The coordinates of the ten largest cities (assumption 1) are first input as points. Then, the connecting routes between all of them are input (assumption 2). Circles where the radii can be arbitrarily selected are subsequently drawn around the locations. Since Euclidean distances are significantly smaller than those via the street network, 75 km was therefore selected as the radius (assumption 4). Thus the area coverage of the first refuelling stations on the connecting routes and across Germany can be roughly estimated (Fig. 51.2). Locations between two cities are then generated by forming central vertical lines. Now, you attempt heuristically to "drive off" all of the important connecting routes on the motorways and to create as few locations as possible over the range.

It can be easily seen in the Berlin-Munich connection that a good approximation to further large cities or existing refuelling stations results (here Nuremberg, Hof and Leipzig) with the help of the intersection points of the central vertical lines with the Berlin-Munich route or by solving a facility-location problem (see mathematical approach). The street network (motorway A9) is now compared to these via Google Maps and the corresponding locations selected. Coverage is made visible by drawing the circles with a radius of 75 km (assumption 5). This procedure is now conducted for all of the main connections (Fig. 51.2). Cities near the border (e.g. Aachen, Kiel, Dresden, etc.) are subsequently checked with reference to reachability and integrated into the list of locations, if necessary. Finally, the market penetration and population density are incorporated (assumptions 6 and 7). After that, five refuelling stations are calculated for Berlin, four for Hamburg, and so on, and missing refuelling stations are added. The result in GeoGebra from the preceding considerations is illustrated in Fig. 51.3.

51.4.4 Comparison with Already Existing Plans

Using this heuristic approach, about 50 refuelling stations are a possible result (Fig. 51.3) for nationwide expansion, taking already existing refuelling stations into consideration (2015 status). The Clean Energy Partnership also published a map, which suggests 50 new refuelling stations. The locations deviate only a little from Fig. 51.3. Apparently, the heuristic procedure of approaching with Euclidean distances and optimizing via facility-location problems is completely adequate for arriving at a plausible initial result.

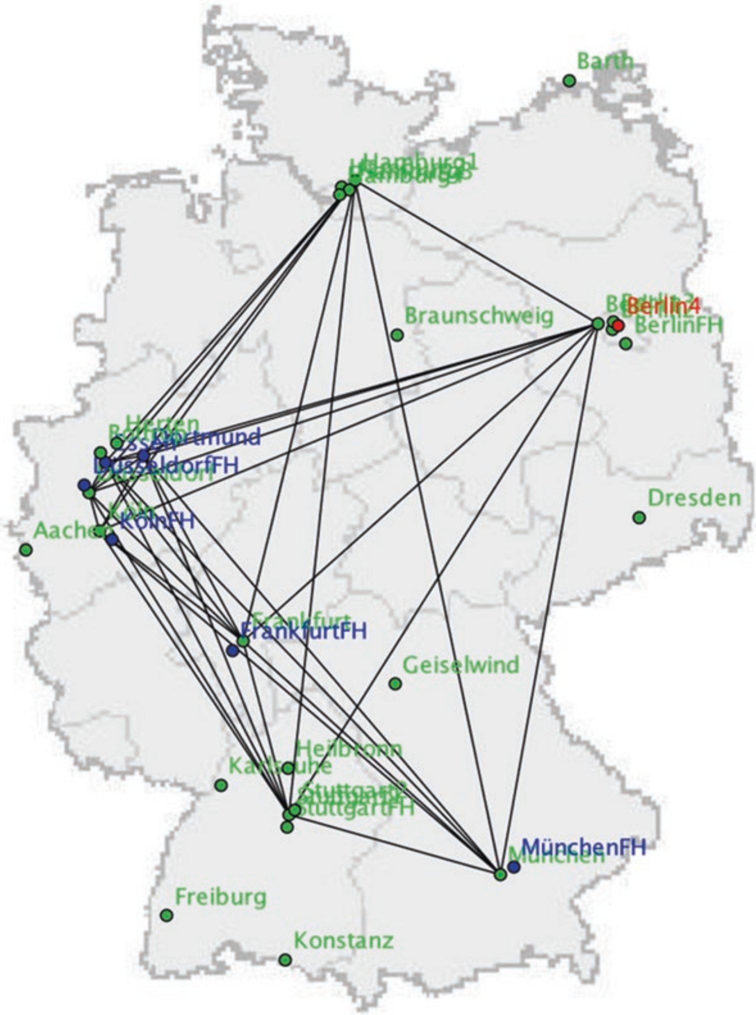

Fig. 51.2 Pre-existing locations, locations under construction and new locations

51.5 Student Solutions and Discussion

At the beginning the students tried to find out more about the topic by posing the following questions in their real-life background. First, they thought about everyday life (e.g. of commuters) and asked themselves: Who will drive hydrogen cars? Where will people drive hydrogen cars? As a second step, they tried to find out more about the market situation of hydrogen cars in different ways by posing other questions:

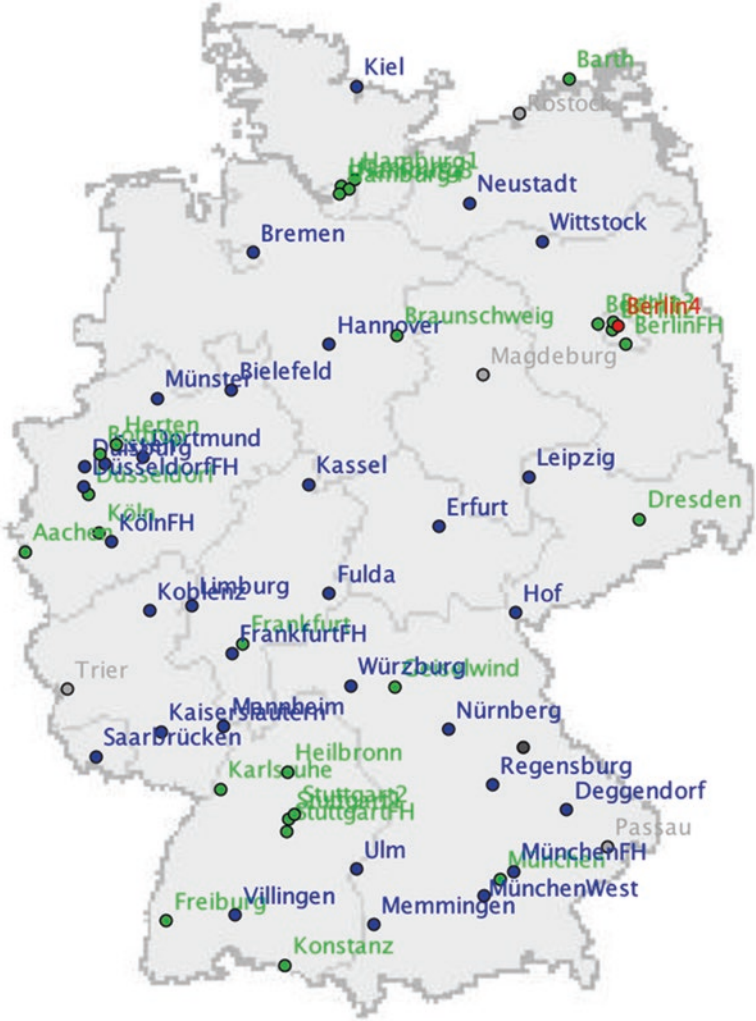

Fig. 51.3 New locations

- They studied car-selling platforms: Are there any used cars offered and what do they cost? Can we conclude any demand from that?
- They had a look at the prices of different new and used hydrogen cars, which leads to the questions: Who could afford it? Thus, how many buyers can we expect in which time period?
- The students searched for companies that sell these cars: How many types are on the market? What can we conclude from these facts?

This process took more than 1 day for all groups, which is comparable to the study by Siller and Meckel (2015). Regarding the extra-mathematical context, all groups (two with the interdisciplinary context and two groups without it) searched for answers by studying economic and geographical contexts and the situation of population density, like we have shown in our model previously. Two groups with the interdisciplinary background preferred this geographical way, because the students had learnt how to create an optimal geographical network for companies in Germany in geography half a year ago. So, instead of ideas coming from the interdisciplinary background, they took this information into account. A reason for this motivation might be that the learning environment in geography had a greater influence in keeping the knowledge in mind and using it in different fields. There are a lot of reasons why teaching environments are more or less efficient (cf. Klieme et al. 2006). In fact, we did not consider this in our study. Another group (without the interdisciplinary background) tried to find out more about the profitability of hydrogen refuelling stations, by calculating the time period of the return on invested capital that was a topic in an economic subject and can also result from various reasons as above.

Considering our first research question, we can say that the topic of alternative energy can be integrated very well into interdisciplinary teaching, but what we had expected in the interdisciplinary groups was thinking about issues such as properties of hydrogen, using (existing) hydrogen pipes for distribution or the different types of refuelling stations that was taught by the teachers. Instead, we have learned from this that it is important to connect science subjects and to do so more often. We can also see that very open questions like in this modelling project lead to interesting solutions (as also shown in Siller 2015). However, if we want to influence solutions with certain ideas, we have to pose very precise questions. Concerning our second research question, we can clearly see in these students' models that the interdisciplinary processes and preparations had no influence on the modelling processes of these two particular groups.

51.6 Conclusion

What we can learn from this study is that it is hard to predict the (mathematical) outcome of a modelling and interdisciplinary activity at school. Even if we try to influence the modelling process with specific interdisciplinary learning environments, students try to find their own – perhaps more comfortable or realistic – way for them, by using their real-life extra-mathematical knowledge. In our case, students mentioned that they were influenced by knowledge from other subjects they were interested in. So, they found more connections to geography. In fact, it is really hard for students to find connections between science subjects like mathematics, physics and chemistry. The possible reason for this lack of students' capability could be the missing interdisciplinary activities in these subjects (cf. Maier 2006) and the complexity of science topics like alternative energy. Hence, for the next time we should integrate more subjects, teachers should be asked about their knowledge, which can be used for the problem to give students more possibilities and to reduce the time for researching the background. Furthermore, if we want to integrate modelling activities into regular classes, we have to consider time problems. As we saw here in this study, it takes a long time thinking about the extra-mathematical context, which is impossible during normal classroom lessons that take about 45 min. Therefore, we have to think about reorganizing these modelling processes, for instance, by connecting subjects in an interdisciplinary way or separating tasks into small "pieces". As a result of this study, next we want to find out how we can ensure interdisciplinary activities happen in combination with modelling at school during regular lessons and which influences these opportunities for interdisciplinary instruction have on students' modelling processes.

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Chapter 52
Authentic Mathematical Modelling Experiences of Upper Secondary School: A Case Study

Kerri Spooner

Abstract The purpose of the research project presented in this chapter was to see if it was possible for authentic mathematical modelling, based on the characteristics and behaviours of a real-world modelling team, to be carried out at secondary school level. From the author's previous opportunity to work as a member of a professional modelling team, an authentic mathematical modelling experience for secondary school students was developed and researched. Classroom activities were created and trialled with a group of 16- and 17-year-old New Zealand students. Data were collected from the learning activities. The results demonstrated that an authentic experience of the process of mathematical modelling is possible at secondary school.

Keywords Authentic mathematical modelling experience • Modelling process • Realistic learning experiences • Real-world modelling team • Secondary school students

52.1 Background

An opportunity for the author to explore authentic mathematical modelling as a member of a professional mathematical modelling team was used to develop and research a realistic mathematical modelling experience for secondary school students. This experience encompassed activities representative of the process a real-world modelling group would engage in. The study addressed the question: Is it possible for authentic mathematical modelling, based on the characteristics and behaviours of a real-world modelling team, to be carried out at secondary school level? In this chapter, mathematical modelling, the current situation of modelling in secondary schools, what the research from this study shows about the potential

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1_52

situation and how this can impact on the likely understanding and capabilities of secondary school students entering university will be discussed.

Mathematical modelling is the process by which we represent a situation in useful mathematical terms (Dym 2004; Wake 1997). It involves moving from reality to mathematics. Modelling is process orientated with the objective being to find mathematics that makes sense of the situation (Stillman et al. 2016). A formed mathematical model is a description of the behaviour of real devices, objects and situations written in the language of mathematics (Dym 2004; Wake 1997). The broad stages of an authentic mathematical modelling experience are forming a modelling group; establishing a shared understanding of the problem; undertaking necessary research of the context and what is known; defining a mathematical direction for the model; identifying the essential aspects of the situation; mathematically interpreting the essential aspects of the situation for the model; constructing equations or other mathematical representations of the model; critiquing, modifying and refining the model mathematically and otherwise; applying the model; repeating the cycle to improve the model; and reporting on the model (Dym 2004; Tam 2011; Treilibs et al. 1980).

Modelling differs from applications in that it is more open and complex. Applications tend to be object orientated, involving examples of contexts where the mathematics to apply is already predetermined and some can use artificial contexts (Stillman et al. 2016). To further complicate matters, the term modelling is used in different ways in education. Stillman et al. (2008) classified some approaches being currently coined as mathematical modelling. These are the use of contextualised examples, curve fitting, modelling as vehicle and modelling as content. All of the approaches, except for modelling as content, overlooked the complete mathematical modelling process and ignored the contextual background when generating a solution and/or the mathematics to be used in the solution was predetermined – all behaviours that are needed for an authentic mathematical modelling experience (Stillman et al. 2016). In contrast, modelling as content better reflects the principles of realistic authentic modelling. Here the objective is to go through the full process of modelling, determine the mathematics needed for the solution and ensure that the solution is located within the context of the problem (Stillman et al. 2008, 2016). Teaching based on this is known by researchers as "the realistic perspective on the teaching and learning of mathematical modelling" (Blomhøj 2009, p. 3).

Only recently, as late as the 1970s, has a move to formalise mathematical modelling in education occurred (Dindyal and Kaur 2010; Kaiser et al. 2010). Now in the twenty-first century, most parts of the world have curriculum statements and resources available (Blum et al. 2007; Frejd 2013). The 2007 New Zealand curriculum, for example, states "'forming and using a model'; 'relating findings to a context' and 'problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts'" (New Zealand Qualifications Authority 2013) as evidence that aspects of modelling are now part of that curriculum. Sweden's current curriculum document captures the essence of modelling and states "Interpret a realistic situation and design a mathematical model, as well as use and assess a model's properties and limitations" (Skolverket 2012, p. 2).

Although research has been done on the teaching of mathematical modelling at secondary school level (Frejd 2013; Blum et al. 2007), little has been conducted in New Zealand. Most of the research in New Zealand has been done on modelling eliciting activities (MEAs). (See Yoon et al. (2010) for an example.) Outside of New Zealand, work has begun on the impact of authenticity of tasks on mathematical modelling development (Palm 2007), educational gains from authentic modelling (Boaler 2001), influence on learning from different styles of teaching modelling (Lege 2007), competencies involved in the modelling process (Maaß 2006), blockages students experience working with modelling (Galbraith and Stillman 2006) and related challenges for teaching of mathematical modelling (Ikeda 2007; Kaiser et al. 2006). Some key findings to come out of this research are a task should be as authentic as possible (Alsina 2007; Palm 2007), students need to be actively modelling (Lege 2007) to be aware of student difficulties with the modelling process (Maaß 2006) and working in groups helps develop modelling abilities (Maaß 2006).

There are curriculum statements, resources and research to support the teaching of mathematical modelling but what is happening in our classrooms? Ikeda (2007) overviewed case studies in eight different countries and observed that there is a lack of "doing" modelling at secondary school. Biembengut and Hein (2010) stated in Brazil "despite the growing interest in modelling we see that many math teachers still do not use mathematical modelling as a classroom teaching practice" (Biembengut and Hein 2010, p. 482). In New Zealand, fragments of modelling occur in classrooms. What is being done would not necessarily be classified as authentic modelling nor satisfy MEA criteria (Caroline Yoon, conversation Auckland University 2016). Teaching experience within the New Zealand educational system supports this as there is limited exposure in New Zealand to mathematical modelling and its processes. Students tend to be exposed to "parts" (e.g. curve fitting) as opposed to the "whole" process.

This is not to say that mathematical modelling is not being taught. Most of the records of examples, including those for New Zealand (Caroline Yoon, conversation Auckland University 2016), are reported as part of academic research projects and teaching experiments. See Ang (2013), Biembengut and Hein (2010), Maaß (2006), Yanagimoto and Yoshimura (2013) and Yoon et al. (2010) for some cases.

One main reason modelling is not being embraced by mainstream mathematics teachers is the teacher belief that modelling is too difficult and hard to manage in the classroom (Armstrong and Bajpai 1988; Maaß 2005; Schmidt 2011). There is a need to address teaching styles and peer group attitudes that are more conducive to modelling (Borba 2011; Maaß 2005; Schmidt 2011). Lack of teacher experience with modelling (Biembengut and Hein 2010; Ikeda 2007) and lack of time to effectively teach modelling (Ang 2013; Biembengut and Hein 2010; Schmidt 2011) also have a significant impact.

52.2 The Research Project

In 2011, the author had the opportunity to work in a real-world mathematical modelling team (RMT) at the Centre for Maths in Industry, Massey University, Albany, New Zealand, where a genuine mathematical modelling process was experienced first-hand. This led to asking if this experience was possible to be reproduced for a secondary school student. Encouraged by Maaß's (2006) research project showing students as young as 13 years old modelling, a research project based on realistic mathematics education theory (Cobb et al. 2008; De Lange 1996; Gravemeijer 1994) was developed. The purpose of the project was to determine if authentic modelling based on the characteristics of a real-world modelling team was possible at secondary school. The project involved creating a student hypothetical learning trajectory (HLT) for the process of mathematical modelling. The HLT was developed by analysing the key components of the RMT modelling process experienced, exploring what these significant parts would look like to a 16–18-year-old secondary school student and then designing and developing classroom learning activities to experience these components. The activities were trialled with a Year 12 average-ability class of eighteen 16- and 17-year-old New Zealand students. The mathematical goal of the activities was for students to experience, remember and be able to recreate, in basic form, the processes involved in the mathematical modelling of a real situation, with particular focus on identifying the essential aspects for model formation.

A *conjectured learning process* was developed as part of the HLT. It describes the thinking and learning the students might engage in for the different stages of the mathematical modelling process. The different stages being: forming a group, establishing a shared understanding, defining a task, recognising and identifying the essential aspects of the situation, forming the model, testing the model and improving the model and overall process. These are considered to produce a realistic learning experience being developed from analysing the observed key components and behaviours of the RMT, features of research developed mathematical models and research and literature supporting these stages (Dym 2004; Tam 2011; Treilibs et al. 1980). The *conjectured learning process* for the stage, *recognise and identify the essential aspects of the situation*, for example, was defined as identifying what we know about a situation, identifying what we want to find out about a situation, recognising and identifying important factors (essential aspects) of a situation, using lists to identify important factors, identifying and classifying factors as important and unimportant and recognising assumptions.

Three mathematical modelling classroom activities were developed based on the main points emerging from literature and the developed HLT. Modelling was to be executed in groups of three using real situations relevant to the students. A minimum of 5 hours was allocated with the teacher acting as a facilitator to guide the groups.

Each classroom activity had its own goals and problem situation. The contexts (dropping a phone, kicking a goal, visibility of light from a lighthouse) were chosen for their relevance to students' personal life. The context of the first activity was chosen specifically for its significance to a student with the view of providing motivation to engage in the modelling process. In addition, the second and third activities were selected based on the teacher's knowledge of a range of different possible models for the situations. The guidance given by the teacher during the activities was based on the processes and behaviours of the RMT observed and documented by the researcher.

The activities differed from MEA-type activities currently being used in New Zealand. MEAs are set up with careful thought to allow for easier formulation of the model in contrast to these activities. Even though MEAs are set in reality, the prompting and questioning are fictional and can be contrived. They are structured and manipulated to have the elements necessary for model construction, generalisation and ease of interpretation present in the activity (Lesh and Doerr 2003). In contrast, the activities the author used took the form of messy real-world situations. No manipulation of the situation was carried out to make the model more accessible. The structure of the activity was in the prompts given by the teacher to direct students through the stages of the process. It is believed these qualities ensured the authenticity of the experience.

52.2.1 Classroom Implementation

The activities were trialled with the author/researcher as the teacher. The context of the first activity was chosen as a relevant topic of concern for the students and therefore engaging. It was anticipated that one lesson would be spent using this topic, with the focus being on the process and not on producing a specific solution to the problem.

Activity 1: "You've dropped your phone. Arrgghhh….. Will it break?"
Goals:

- To introduce the process of mathematical modelling
- To set up student modelling groups (SMG) for the sequence of planned learning activities
- To establish how these groups will operate

The context for the second activity was chosen as the author was familiar with different models and their development for the best position to take the kick when kicking a rugby goal. This placed the author in a good position to be able to provide guidance for this context. Members of the class were rugby players meaning the context was familiar and relevant. The aim of the activity was to go through the discussion cycle for all stages of the process. Groups would also work independently between discussions with the teacher providing guidance on the process where appropriate. Again the focus was on the process not on the actual solution. It was planned to spend three lessons on this situation.

Activity 2: "Your rugby team has just scored a try. You are responsible for taking the conversion. Where on the field will you place the ball for the kick?"
Goals:

- To mimic as closely as possible the experience of being a member of a mathematical modelling team
- To experience the process of attempting to develop a mathematical model (solution)
- To experience and gain knowledge of some of the tools of mathematical modelling, in particular, but not limited to, software, tables, physical laws and known mathematics

Activity 3 is an assessment using the problem situation: "You are sailing to the Great Barrier Island and have just passed Tiri-tiri Matangi Island. Tiri-tiri Matangi is home to what was the last manned lighthouse in Auckland's Hauraki Gulf. The lighthouse is 21 m in height with its base standing 91 m above sea level. As you pass you wonder how far out to sea the light from Tiri-tiri Matangi will stay in sight?" The assessment asks how a student would go about forming a model to provide a solution for the problem. The assessment then presents different parts of models that could be used to form a solution and asks students to critique the models, thinking about what assumptions have been made, how the model might work and what might be done next. The last part of the assessment asks students to talk about what aspects they liked and did not like about the modelling activities. The assessment was to determine how much of the "process" of mathematical modelling the students had internalised and to see what solutions the students would come up with.

Activity 3: Tiri-tiri Matangi Light
Goals:

- To determine how much of the process of mathematical modelling the students assimilated
- To understand what parts students found easy or difficult
- To find out whether they can identify the underlying ideas about the process of mathematical modelling

52.3 Data Collection and Analysis

Data were collected from students writing diaries, sitting the assessment and undertaking interviews that were conducted between 4 and 6 weeks after the completion of classroom learning activities. Diaries were written in the last 5 min of each lesson to provide a written first-person account or record of what was taught or occurred in each lesson. The assessment provided information on students' recall of the process, application of mathematical modelling and aspects they did and did not enjoy. The interviews provided information on what was learnt and could be recalled at a later date.

The HLT provided the framework to analyse the data. It was analysed first by individually analysing the diaries, assessment and interviews for statements and evidence of aspects and stages of the process of mathematical modelling. The data from the diaries were then looked at collectively to establish an account of what was taught during the lessons. The data from the assessment and interviews were also collectively examined to establish what was learnt by the group from the activities. Table 52.1 shows the evidence collected for the experience of *recognise and identify the essential aspects of the situation* stage of the mathematical modelling process collected from the diaries, assessments and interviews. The number(s) in brackets refer to the student who provided the evidence.

52.4 Results

The data collected showed that collectively all aspects of the mathematical modelling process were taught to, and learnt by, the group. All features were mentioned across all three forms of data collection. Techniques for, and benefits of, effectively working in a group were experienced by all students. "Working in groups pooling ideas was beneficial and made the process easier" and "I learnt how to get along in a team" are comments reflective of student evidence. Half of the students recognised understanding the background and context of a problem is important with a typical comment being, "Research and discussion part of your model as it gives you the foundations of your model". Defining the problem was not strongly recalled with only 17% mentioning this in interviews. Processes associated with identifying essential aspects were mentioned in their dairies and assessment by 82% of students. All students recalled strategies for identifying essential aspects in the interviews. There was evidence that all students acquired strategies for formulating a model with students mentioning, recalling and discussing approaches in the dairies, assessment and interviews. They enjoyed this stage, though 50% revealed they found it difficult with one student commenting that it "was like figuring out all the ingredients to make a cake just not knowing how to put them all together to actually make the cake". Testing the model provided motivation for developing the model. It was observed that ideas for testing the model came easily to students. A useful

Table 52.1 Evidence for modelling stage from diary, assessment and interviews

Stage of modelling process	Diary	Assessment	Interview
4. Recognise and identify situation's essential aspects	Discussing and looking at all of the factors that affect the situation (1,2,3,4,5,7,9,10) Classifying things that affect the situation (what we already know about a situation and what we want to find out – what we know, what we want to find out and what we can control) (3,4,7,8,10,14,17) Discussion and identification of important variables (1,4,5) Brainstorming what things affect the situation (4) Measure factors, measuring factors (2,3,11) Brainstorming and making assumptions (1,4,5,7,8,11) Controlling certain aspects of the situation, what things can be controlled (making assumptions) (5,7,11) Ignoring certain aspects (making assumptions) (1)	Brainstorming, asking questions (4) Find out what factors affect the situation (2,3, 4,5,6, 9,16,18) Factors already known about the situation (3,10,11,13) Factors want to find out (10,13) Make assumptions (2,3,12,18), identify factors need to know (2,9); Identify important factors (9,18); What things would be ignored and/or assumed (2,5,10) Stated and made assumptions (2,4,9,10)	Brainstorming, discussion to identify all factors that can affect the situation (3,4,6,9,10,11,13,19) Identify relevant factors (2,6,9) Identify variables need to find (1,4) Ignore and eliminate factors (2,3,11) What things do we already know about the situation (2,3,4) What things do we need to find (2,3,4,9) How to measure variables (9) Classifying/sorting factors (3,10,19) Rank variables in order of importance (6,13,19) Made assumptions (2,3,13,19) Simplify variables (4) Simplify situation (4,10,11)

model needing to reflect reality was noted by 65% of students, suggesting they comprehended the importance of the context in modelling. Interview data indicated students saw modelling as a team activity where members contributed individually and communally. No evidence was found for improving the model.

52.5 Discussion

The allocated time did not allow for students to improve their models indicating more time is needed for the full process. The 16- and 17-year-old students were able to recognise and use strategies to identify essential aspects of a situation confirming the teaching sequence and discussion cycle for identifying essential aspects of the situation were successfully implemented. Once essential aspects were identified, students had difficulty in forming models. Defining the problem was not strongly recalled, possibly due to this part of the process being teacher led and the problem definitions accepted as a group.

Limitations of the data included the possibility that students wrote or told the researcher what they thought she wanted to hear instead of their honest thoughts. Even though the author's goal as a teacher is always to provide an environment of free expression, students are aware of the authority a teacher's position carries.

What are the implications for tertiary education? The evidence-based recommendation is for first year modelling courses to be instituted that provide an experience of the whole process of modelling. If this is administratively too difficult, it is suggested that there be a move towards more broad, open-ended modelling problems. Full "modelling days" could be set aside as part of course requirements. A colleague from Auckland University has instigated this approach with the formal exposure of the modelling processes being covered in lectures prior to the modelling day. Teaching tools and guidance are needed for identifying essential aspects of the situation. Special attention, including good guidelines and time, is needed in providing instruction for forming models.

52.6 Conclusion

There appears to be a gap in authentic modelling experiences for secondary students in New Zealand. Modelling that is being done is mainly for research purposes and is not necessarily considered to be authentic or realistic. The purpose of this study was to determine if it was possible for secondary students to experience authentic modelling based on the behaviours and characteristics of a real-world modelling team as a way to fill this gap. The participants in this study were average-ability 16- and 17-year-old New Zealand students. They worked in groups as they took part in two authentic open-ended teaching activities and an assessment designed to experience the process of mathematical modelling. Data for research were collected in the form of diaries, assessments and interviews. Though these findings cannot be extrapolated to make conclusions for all students of diverse abilities, and different backgrounds without further studies, the main findings of this study showed an authentic modelling process is achievable within the restricted secondary school classroom environment. The classroom activity developed and trialled was effective for teaching the process of mathematical modelling. Students were able to recognise and use strategies to identify the essential aspects of the situation although there were gaps in students' abilities to construct models. It is recommended that a minimum of 7 hours of curriculum time is needed to allow for a full experience of the process of modelling. An implication of the study for tertiary education is the recommendation for first year mathematical modelling courses to include experiences of the whole process of mathematical modelling.

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Errata to: Mathematical Modelling and Applications

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Errata to:
G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1

This book was inadvertently published with an incorrect name as Dirk Wessels in the author group of Chapter 38.

This has now been corrected throughout the book as follows:

Rina Scott-Wilson, Dirk C.J. Wessels, Helena M. Wessels, and Estelle Swart

The affiliation information of the author **Helena M. Wessels** has also been updated in Chapters 38 and 41 as follows:

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The online version for the original chapters can be found at
https://doi.org/10.1007/978-3-319-62968-1_38
https://doi.org/10.1007/978-3-319-62968-1_41
https://doi.org/10.1007/978-3-319-62968-1

G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1_53

In addition, the affiliation information of the author **Estelle Swart** has also been updated in Chapter 38.

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Refereeing Process

To ensure the scholastic quality of the chapters in this book, all proposed chapters submitted to the editors for consideration in a timely manner have been subjected to a strict impartial and peer review independent of the editors. Only those chapters that have been accepted after this process of review and revision as being of appropriate quality (given differences of geographical location) and consistent with the intended aims of the book have been included. The editors wish to thank the following international reviewers:

Luis Albarracín Universitat Autònoma de Barcelona, Spain
Burkhard Alpers Aalen University of Applied Science, Germany
Jonas Bergman Ärlebäck Linkoping University, Sweden
Nélia Amado Universidade de Lisboa, Portugal
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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*,
International Perspectives on the Teaching and Learning of Mathematical
Modelling, https://doi.org/10.1007/978-3-319-62968-1

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G.A. Stillman et al. (eds.), *Mathematical Modelling and Applications*, International Perspectives on the Teaching and Learning of Mathematical Modelling, https://doi.org/10.1007/978-3-319-62968-1

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