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Complex and Symplectic Geometry

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Preface

This volume, which brings together state-of-the-art contributions from a range of experts,is based on the INdAM Meeting "Complex and Symplectic Geometry", held in Cortona from June 12 to 18, 2016 and organized by Daniele Angella, Paolo de Bartolomeis, Costantino Medori, and Adriano Tomassini. A wide variety of research topics of current interest in differential and algebraic geometry are covered in the volume; however, the focus is particularly on complex and symplectic geometry and their cohomological and topological aspects; on complex analysis and related topics, such as Cauchy-Riemann manifolds, Oka theory, and pluripotential theory; on algebraic and complex surfaces; on Kähler geometry; and on special metrics on complex manifolds. The final outcome is a challenging panoramic view of problems connecting a wide area. There is no doubt that the conference encouraged a very fruitful exchange of ideas and initiated important collaborations among the participants.

During the preparation of this volume, our friend Paolo de Bartolomeis sadly passed away. We would like to dedicate this volume to him. All participants at the conference and numerous other mathematicians have had the privilege of spending time and interacting with Paolo. We will all miss him greatly.

Firenze, Italy — Daniele Angella
Parma, Italy — Costantino Medori
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June 2017

Contents

Generalized Connected Sum Constructions for Resolutions of Extremal and Kcsc Orbifolds

Claudio Arezzo

Abstract In this note we review recent progresses on the existence problem of extremal and Kähler constant scalar curvature metrics on complex manifolds. The content of this note is an expanded version of author's talk "Kahler constant scalar curvature metrics on blow ups and resolutions of singularities" given at the INdAM Meeting *Complex and Symplectic Geometry*, Cortona, June 12–18, 2016.

1 Introduction

In [3, 4, 7, 25] and [27] a general existence theory for extremal and Kähler constant scalar curvature (Kcsc from now on) metrics on blow ups at smooth points of extremal and Kcsc manifolds has been developed and various important consequences have been deduced.

For a general introduction to these (and other related) results we will refer the reader to the two surveys [2] and [26].

In this note we will describe how the same technique of "generalised connected sum construction" for extremal and Kcsc metrics can be applied to the problem of resolving isolated singularities keeping the metric canonical (i.e. extremal or Kcsc). We take this opportunity to present the results contained in a number of papers in a unified form and to clarify some relationship and differences.

We will refer to [5, 8–10], and [6] for details and proofs.

The structure of this note is as follows: in Sect. 2 we review the general geometric construction which lies at the base of the problem. In Sect. 3 we discuss how the extremal metrics can be lifted from a singular base to a smooth desingularization (or to a partial desingularization). In Sect. 4 we study the same problem for the Kähler constant scalar curvature equation. This is certainly the most interesting case in that two approaches are possible, one looking at extremal metrics and then imposing the

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vanishing of the Futaki invariant (we will refer to this as to *the algebraic approach*), and a direct PDE approach, *the analytic approach*.

In Sect. 5 we describe a recent extension of these works to the more general case of isolated conical singularities not necessarily of quotient type. Finally, in Sect. 6 we describe a further extension of these result to non-compact manifolds.

2 Resolving Isolated Singularities: The Generalised Connected Sum Construction

We start with an extremal or Kcsc base M with *isolated quotient singularities*, hence locally of the form $\mathbb{C}^m/\Gamma$, where m is the complex dimension of M, and Γ is a finite subgroup of $U(m)$ acting freely away from the origin.

Given such a singular object one would like to replace a small neighborhood of a singular point p and replace it with a large piece of a Kähler resolution $\pi\colon (X_p, \eta) \to \mathbb{C}^m/\Gamma$ keeping the scalar curvature constant (and close to the starting one). For such a construction to even have a chance to preserve the extremal or Kcsc equation it is necessary that (X_p, η) is scalar flat, i.e. it is necessary to assume that $\mathbb{C}^m/\Gamma$ *has a scalar flat ALE resolution.*

Having then fixed a set of singular points $\{p_1, \dots, p_n\} \subset M$ each corresponding to a group Γ_j, and denoted by $B_{j,r} := \{z \in \mathbb{C}^m/\Gamma_j \; : \; |z| < r\}$, we can define, for all $r > 0$ small enough (say $r \in (0, r_0)$)

$$M_r := M \setminus \cup_j B_{j,r}. \tag{1}$$

On the other side, for each $j = 1, \dots, n$, we are given a m-dimensional Kähler manifold (X_{Γ_j}, η_j), with one end biholomorphic to a neighborhood of infinity in $\mathbb{C}^m/\Gamma_j$. Dual to the previous notations on the base manifold, we set $C_{j,R} := \{x \in \mathbb{C}^n/\Gamma_j \; : \; |x| > R\}$, the complement of a closed large ball and the complement of an open large ball in X_{Γ_j} (in the coordinates which parameterize a neighborhood of infinity in X_{Γ_j}). We define, for all $R > 0$ large enough (say $R > R_0$)

$$X_{\Gamma_j,R} := X_{\Gamma_j} \setminus C_{j,R}. \tag{2}$$

which corresponds to the manifold X_{Γ_j} whose end has been truncated. The boundary of $X_{\Gamma_j,R}$ is denoted by $\partial C_{j,R}$.

We are now in a position to describe the generalized connected sum construction. Indeed, for all $\varepsilon \in (0, r_0/R_0)$, we choose $r_\varepsilon \in (\varepsilon R_0, r_0)$ and define

$$R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}. \tag{3}$$

By construction

$$\tilde{M} := M \cup_{p_1,\varepsilon} X_{\Gamma_1} \cup_{p_2,\varepsilon} \cdots \cup_{p_n,\varepsilon} X_{\Gamma_n},$$

is obtained by connecting M_{r_ε} with the truncated ALE spaces $X_{\Gamma_{1,R_\varepsilon}}, \dots, X_{\Gamma_{n,R_\varepsilon}}$. The identification of the boundary $\partial B_{j,r_\varepsilon}$ in M_{r_ε} with the boundary $\partial C_{j,R_\varepsilon}$ of $X_{\Gamma_j,R_\varepsilon}$ is performed using the change of variables

$$(z^1, \dots, z^m) = \varepsilon\,(x^1, \dots, x^m), \tag{4}$$

where $(z^1, \dots, z^m)$ are the coordinates in B_{j,r_0} and $(x^1, \dots, x^m)$ are the coordinates in C_{j,R_0}.

As explained in Section 3 in [2] the right way to put this construction into the familiar general picture of smooth manifolds, is to realise that when varying ε the manifolds obtained are all biholomorphic to each other, as long as $\varepsilon \neq 0$. One should then think of the generalised connected sum construction as a generalised "trivial" deformation of complex structure. This is the reason why we will repeatedly compare our existence results with the more classical analogue results for standard deformations of smooth manifolds when the complex structure is fixed and the Kähler class changes in the Kähler cone.

Before passing to analyse the behaviour of the canonical metrics equations under this generalised concocted sum construction it is important to recall what is know about the existence of the scalar-flat ALE resolution required by the construction from the very beginning. This has turned out to be a very delicate and important problem also related to various problems coming from applications to physics. Building on the first fundamental work by Kronheimer [21] in the case of subgroups of $SU(2)$, and Calderbank-Singer [12] for cyclic subgroups, Lock-Viaclovsky [23] completed the picture in complex dimension 2 by proving the following

Theorem 2.1 *Let $\Gamma \subset U(2)$ be a finite subgroup containing no complex reflections. Then the minimal resolution of $\mathbb{C}^2/\Gamma$ admits scalar-flat Kähler ALE metrics. Furthermore, these metrics admit a holomorphic isometric circle action.*

In fact Han-Viaclovsky [17] went one important step further to study the existence of scalar-flat metrics on deformations of complex structures and proved that for any scalar-flat Kähler ALE surface, all small deformations of complex structure also admit scalar-flat Kähler ALE metrics.

In higher dimensions, even restricting ourselves to subgroups of $SU(n)$, one needs to assume the existence of a Kähler crepant resolution and then apply results by Joyce [20], Goto [16], Van Coevering [28] and Conlon-Hein [14], which at least ensure the existence of our desired structure for any Γ in $SU(3)$.

Up to now, we have constructed a differentiable manifold (or orbifold) endowed with an integrable complex structure. Next step is clearly to build special Kähler metrics on it. It is therefore of crucial importance to compare the local behaviour of the base orbifold metric around a singular point on one side, and the asymptotic behaviour at infinity of η on the model. Of course on the base orbifold everything is standard having restricted ourselves to the case of orbifold metrics.

Proposition 2.1 *Let (M, g, ω) be a Kähler orbifold. Then, given any point $p \in M$, there exists a holomorphic coordinate chart $(U, z^1, \dots, z^m)$ centered at p such that* *the Kähler form can be written as*

$$\omega \;=\; i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \psi_\omega\right), \qquad \textit{with} \qquad \psi_\omega \;=\; \mathcal{O}(|z|^4)\,.$$

If in addition the scalar curvature s_g of the metric g is constant, then ψ_g is a real analytic function on U, and one can write

$$\psi_\omega(z,\overline{z}) \;=\; \sum_{k=0}^{+\infty} \Psi_{4+k}(z,\overline{z})\,, \tag{5}$$

where, for every $k \in \mathbb{N}$, the component Ψ_{4+k} is a real homogeneous polynomial in the variables z and $\overline{z}$ of degree $4+k$. In particular, we have that Ψ_4 and Ψ_5 satisfy the equations

$$\Delta^2\,\Psi_4 = -2s_\omega\,, \tag{6}$$

$$\Delta^2\,\Psi_5 = 0\,, \tag{7}$$

where Δ is the Euclidean Laplace operator of $\mathbb{C}^m$. Finally, the polynomial Ψ_4 can be written as

$$\Psi_4(z,\overline{z}) \;=\; \left(-\frac{s_\omega}{16m(m+1)} \;+\; \Phi_2 \;+\; \Phi_4\right)|z|^4\,, \tag{8}$$

where Φ_2 and Φ_4 are functions in the second and fourth eigenspace of $\Delta_{\mathbb{S}^{2m-1}}$, respectively.

On the contrary the shape of the metric η at infinity turns out to be rather delicate.

Proposition 2.2 *Let (X,η) be a scalar flat ALE Kähler resolution of an isolated quotient singularity of complex dimension $m \geq 3$ and let $\pi : X \to \mathbb{C}^m/\Gamma$ be the quotient map. Then for $R > 0$ large enough, we have that on $X \setminus \pi^{-1}(B_R)$ the Kähler form can be written as*

$$\eta \;=\; i\partial\overline{\partial}\Big(\frac{|x|^2}{2} \;+\; e\,|x|^{4-2m} \;-\; c\,|x|^{2-2m} \;+\; \psi_\eta(x)\Big)\,, \qquad \textit{with} \qquad \psi_\eta \;=\; \mathcal{O}(|x|^{-2m})\,, \tag{9}$$

for some real constants e and c. The constant e is called the ADM mass of the model. Moreover, the radial component $\psi_\eta^{(0)}$ in the Fourier decomposition of ψ_η is such that

$$\psi_\eta^{(0)}(|x|) = \mathcal{O}(|x|^{6-4m})\,. \tag{10}$$

In the case where the ALE Kähler metric is Ricci-flat it is possible to obtain sharper estimates for the deviation of the Kähler potential from the Euclidean one, indeed it happens that $e = 0$. This is far form being obvious and in fact it is an important result of Joyce ([20], Theorem 8.2.3, p. 175).

Since the geometry of scalar-flat ALE spaces is of clear independent interest we point out that in [8] we gave the following improvement of the above classical results in the case of vanishing ADM mass. This turned out to be a crucial step in the proof of our gluing results in these cases.

Proposition 2.3 *Let (X, η) be as in Proposition 2.2. Moreover let $\Gamma \triangleleft U(m)$ be nontrivial and $e = 0$. Then for $R > 0$ large enough, we have that on $X \setminus \pi^{-1}(B_R)$ the Kähler form can be written as*

$$\eta \;=\; i\partial\overline{\partial}\Big(\frac{|x|^2}{2} - c\,|x|^{2-2m} + \psi_\eta(x)\Big)\,, \qquad \textit{with} \qquad \psi_\eta \;=\; \mathcal{O}(|x|^{-2m})\,, \tag{11}$$

for some positive real constant $c > 0$. Moreover, the radial component $\psi_\eta^{(0)}$ in the Fourier decomposition of ψ_η is such that

$$\psi_\eta^{(0)}(|x|) = \mathcal{O}(|x|^{2-4m})\,.$$

In complex dimension similar statements to the above Propositions still hold bearing in mind that the expansion of the potential at infinity takes the shape

$$\eta = i\partial\overline{\partial}\Big(\frac{|x|^2}{2} + e\log(|x|) - c|x|^{-2} + \mathcal{O}(|x|^{-4})\Big)\,. \tag{12}$$

The above two proposition motivate a very natural question:

Problem 1 Are there scalar-flat non Ricci-flat ALE spaces with vanishing ADM mass?

This seemed to the author a very natural guess, a sort of Liouville-type Theorem for scalar-flat metrics. In fact this turns out to be largely false (see Le Brun [22], Section 6, p. 244, and [24], Example 2, Section 6.7) and motivated a beautiful work by Hein-Lebrun [18] which will be precisely quoted and commented in the last section of this note.

Now we have everything in place to ask whether there is a suitable deformation of ω on the base minus a small neighbourhood of a singularity, and of $\varepsilon^2\eta$ on the model minus a big neighbourhood of infinity, which makes the two truncated metrics match on the respective boundaries and to solve on both sides the same PDE (extremal or Kcsc).

3 The Compact Extremal Case

From now on (M, g, ω) will be a Kähler orbifold with isolated singular points, with extremal metric g and extremal vector field X_s. We denote with $G := Iso_0(M, g) \cap Ham(M, \omega)$ the identity component of the group of Hamiltonian isometries and with

$\mathfrak{g}$ its Lie algebra. Moreover, we denote with $T \subset G$ the maximal torus whose Lie algebra $\mathfrak{t}$ contains the extremal vector field X_s. It is a standard fact that the action of T can be linearized at fixed points, more precisely it is possible to find adapted Kähler normal coordinates in some neighborhood U of fixed point p such that

$$\omega = i\partial\overline{\partial}(\frac{|z|^2}{2} + \psi_\omega(z)) \qquad \text{with} \qquad \psi_\omega(z) = \mathcal{O}(|z|^4) \tag{13}$$

and T acts on U as a subgroup of $U(m)$. Clearly, singular points are fixed points for the action of G and hence every $\gamma \in G$ lifts to a $\tilde{\gamma} \in Aut_0\big(\tilde{M}\big)$ so we denote with $\tilde{G}$ and $\tilde{T}$ the lifts of G and of $\tilde{T}$ to $\tilde{M}$ respectively.

We denote by s_ω the scalar curvature of the metric g and by ρ_ω its Ricci form. We denote moreover with $\mathbf{S}_\omega$ the scalar curvature operator

$$\mathbf{S}_\omega(\cdot) \,:\, C^\infty(M) \longrightarrow C^\infty(M)\,, \qquad\qquad f \longmapsto \mathbf{S}_\omega(f) := s_{\omega+i\partial\overline{\partial}f}\,,$$

We denote with $\mu_\omega : M \to \mathfrak{g}^*$ the Hamiltonian moment map for the action of G on M and we say that it is normalized if

$$\int_M \langle \mu_\omega, X\rangle \, d\mu_g = 0 \qquad X \in \mathfrak{g} \tag{14}$$

Using an invariant scalar product on $\mathfrak{g}$ and the natural identifications we can regard μ_ω as

$$\mu_\omega : M \to TM^*\,. \tag{15}$$

The *extremal* equation for ω corresponds to the prescription

$$\overline{\partial}\partial^\sharp s_\omega = 0 \tag{16}$$

and in terms of the Hamiltonian moment map this is equivalent to

$$s_\omega = \langle \mu_\omega, X_s\rangle + \frac{1}{vol(M)} \int_M s_\omega \, d\mu_\omega\,. \tag{17}$$

with X_s a holomorphic vector field on M.

It is now necessary to understand how the above equation changes if we consider another Kähler metric cohomologous to ω. Once we fix a Kähler class $[\omega]$ and we fix a Kähler form $\omega \in [\omega]$ then the *extremal* equation in the class $[\omega]$ is the nonlinear PDE in the unknowns $f \in C^\infty(M)$, $c \in \mathbb{R}$ and $X \in H^0(M, TM)$

$$\mathbf{S}_\omega(f) = c + \left\langle \mu_{\omega+i\partial\overline{\partial}f}, X\right\rangle + \frac{1}{vol(M)} \int_M s_\omega \, d\mu_\omega\,. \tag{18}$$

Remark 3.1 In Eq. (18) it appears an unknown constant c because the perturbed moment $\mu_{\omega+i\partial\bar{\partial}f}$ map is, in general, not normalized. It is hence needed this further degree of freedom to obtain the correct *extremal* equation.

When studying of a PDE, it is a standard procedure to consider the map, between suitable functional spaces, induced by the PDE itself. In the case of Eq. (18) the induced map

$$\mathscr{E} : \mathcal{D} \subseteq C^{4,\alpha}(M) \times \mathbb{R} \times H^0(M, TM) \longrightarrow \mathbb{R} \tag{19}$$

is defined as

$$\mathscr{E}(f, c, X) := s_\omega(f) - c - \left\langle \mu_{\omega+i\partial\bar{\partial}f}, X \right\rangle - \frac{1}{vol(M)} \int_M s_\omega \, d\mu_\omega \, . \tag{20}$$

and it is a matter of fact that is highly nonlinear in its variables and the extremal metrics correspond to the triples (f, c, X) such that $\mathscr{E}(f, c, X) = 0$.

From now on we will work in the T-invariant framework, so we indicate with $C^{k,\alpha}(M)^T$ the subset of T-invariant functions in $C^{k,\alpha}(M)$ and the definition of the map $\mathscr{E}$ in the T-invariant setting is the obvious one i.e.

$$\mathscr{E} : \mathcal{D} \subseteq C^{4,\alpha}(M)^T \times \mathbb{R} \times \mathfrak{t} \longrightarrow \mathbb{R} \tag{21}$$

with $\mathscr{E}$ acting in the same way as above.

The first step is then to understand how the moment map changes as the symplectic form moves in a fixed Kähler class and this is done in the following well known proposition.

Proposition 3.1 *Let (M, g, ω) be a Kähler manifold with a Hamiltonian action of a Torus $T \subset G$ and $f \in C^\infty(M)^T$ such that*

$$\tilde{\omega} := \omega + i\partial\bar{\partial}f \tag{22}$$

is the Kähler form of a T-invariant Kähler metric. A Hamiltonian moment map $\mu_{\tilde{\omega}}$ relative to $\tilde{\omega}$ is

$$\langle \mu_{\tilde{\omega}}, X \rangle := \langle \mu_\omega, X \rangle - \frac{1}{2} JXf \tag{23}$$

With the next proposition we exploit the local structure of map $\mathscr{E}$ in a neighborhood of a zero.

Proposition 3.2 *Let (M, g, ω) be a compact extremal Kähler manifold with extremal vector field X_s, with T-invariant metric g and $\mu_{\underline{\omega}}$ a normalized moment map for the action of G. Let $f \in C^\infty(M)^T$ such that $\omega + i\partial\bar{\partial}f$ is a Kähler metric. If the triple $(f, X, c) \in C^{4,\alpha}(M)^T \times \mathbb{R} \times \mathfrak{t}$ is sufficiently small i.e.*

$$\|f\|_{C^{4,\alpha}(M)^T} + |c| + \|X\|_{C^{4,\alpha}(M)^T} < \mathsf{C} \tag{24}$$

for $C > 0$ sufficiently small, then

$$\mathscr{E}(f, c + \frac{1}{vol(M)} \int_M s_\omega \, d\mu_\omega, X_s + X) = -\frac{1}{2}\mathbb{L}_\omega(f) - \frac{1}{2}\langle \nabla s_\omega, \nabla f \rangle - \langle \mu_\omega, X \rangle + c + \frac{1}{2} JXf + \frac{1}{2}\mathbb{N}_\omega(f) \,. \tag{25}$$

where $\mathbb{L}_\omega$ is given by

$$\mathbb{L}_\omega f \;=\; \Delta^2_\omega f \;+\; 4 \,\langle\, \rho_\omega \,|\, i\partial\overline{\partial} f \,\rangle \,, \tag{26}$$

and $\mathbb{N}_\omega$ is the nonlinear remainder.

Remark 3.2 The immediate consequence of Proposition 3.2 is that if we want to solve Eq. (18) in a small neighborhood of an *extremal* metric then it is sufficient to solve the following equation

$$\mathbb{L}_\omega(f) + \langle \nabla s_\omega, \nabla f \rangle + 2 \,\langle \mu_\omega, X \rangle \;=\; 2c + JXf + \mathbb{N}_\omega(f) \,. \tag{27}$$

We clearly need to produce a right inverse for the linear operator induced by the linear part of Eq. (27). It is indeed necessary to identify the kernel of the induced linear operator and this is done in the following classical proposition.

Proposition 3.3 *Let (M, g, ω) be a compact extremal Kähler manifold and let $P_\omega : C^\infty(M) \to T^*M \otimes TM$ be the differential operator defined by*

$$P_\omega(f) := -L_{J\nabla f} J \,. \tag{28}$$

Then

$$P^*_\omega P_\omega(f) = \mathbb{L}_\omega(f) + \langle \nabla s_\omega, \nabla f \rangle \tag{29}$$

Moreover,

$$\ker(P^*_\omega P_\omega)/\mathbb{R} = \{\langle \mu_\omega, X \rangle \;|\, X \in \mathfrak{g}\} \,. \tag{30}$$

and, if we work with T-equivariant functions, then

$$\ker(P^*_\omega P_\omega)/\mathbb{R} = \{\langle \mu_\omega, X \rangle \;|\, X \in \mathfrak{h}\} \,. \tag{31}$$

where $\mathfrak{h}$ is the Lie algebra of $C_G(T)$, the centralizer of T in G.

In light of the previous lemma, the extremal equation for $\omega + i\partial\overline{\partial} f$ can be rewritten as

$$P^*_\omega P_\omega(f) = 2c - 2 \,\langle \mu_\omega, X \rangle + JXf + \mathbb{N}_\omega(f) \,. \tag{32}$$

The idea of working with T-invariant functions is the key to solve the problem. In fact as already observed in the case of blowing up smooth points in [7, 25] and [27], if we knew how to lift hamiltonian vector fields from the base to the resolution, then the analysis would go through following a rather standard pattern. Unfortunately this property is far from being trivial, and in fact it was known only in very few sporadic examples. The following Proposition proved in [8], gives the complete solution to the problem.

Proposition 3.4 *Let* (X, η) *be a scalar flat ALE resolution of* $\mathbb{C}^m/\Gamma$*, then the centraliser of* Γ *in* $U(m)$ *satisfies*

$$C_{U(m)}(\Gamma) \subset Iso_0(X, \eta)\,. \tag{33}$$

We point out that a consequence of Proposition 3.4 is that η is invariant for the action of any torus in $C_{U(m)}(\Gamma)$ in particular for the action of the special torus $\tilde{T}$ we chose at the beginning. Moreover, as explained in the proof, given a vector field in $X \in \mathfrak{t}$ and denoted with $\tilde{X}$ its lift to X_Γ, we can always find a Hamiltonian potential $\langle \mu_\eta, \tilde{X} \rangle$ such that

$$\overline{\partial} \langle \mu_\eta, \tilde{X} \rangle = \tilde{X} \lrcorner \, \eta\,. \tag{34}$$

Note that a priori there could be obstructions, topological or analytical, to the existence of Hamiltonian potentials as it happens in the asymptotically conical setting. It turns out instead that, because of the special nature of the singularity $\mathbb{C}^m/\Gamma$, it is possible to find, on the resolutions, Hamiltonian potentials of all holomorphic vector fields coming from linear fields on $\mathbb{C}^m/\Gamma$.

It is interesting to point out that the proof in [8] of the above proposition heavily uses the scalar-flat property of the resolution. I believe it would be very interesting to investigate further whether this is really necessary, in order to apply this circle of ideas to other equations.

Now the equation is in the right position, the inverse of the linearised operator satisfies all the required properties to apply the gluing machinery developed in [3] and [4]. The output is then the optimal result, which confirmed the original guess, that resolution of isolated singularities behaves pretty much like the LeBrun-Simanca deformation theory of smooth complex structures.

Theorem 3.1 *Let* (M, g, ω) *be a compact extremal orbifold with* T*-invariant metric* g *and singular points* $\{x_1, \ldots, x_S\}$*. Then there is* $\bar{\varepsilon}$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon})$ *the resolution*

$$\tilde{M} := M \sqcup_{x_1,\varepsilon} X_{\Gamma_1} \sqcup_{x_2,\varepsilon} \cdots \sqcup_{x_S,\varepsilon} X_{\Gamma_S}$$

has a $\tilde{T}$*-invariant extremal Kähler metric.*

Of course the above Theorem can be used to construct large families of new extremal manifolds starting from Kähler-Einstein or Kcsc orbifolds. It has been used also by Apostolov and Rollin [1] with genuinely extremal non Kcsc base orbifolds.

4 The Compact Kcsc Case

When passing from extremal to Kcsc of course we loose the set of (hamiltonian) holomorphic vector fields as possible parameters to adjust an approximate solution into a genuine solution. This drop in degrees of freedom clearly suggests that the problem could become obstructed from the PDE point of view.

At the same time thanks to the foundational work of Futaki [15] and Calabi [11] we can see this obstruction arising also form a more geometrical point of view. Recall that the basic obstruction for the existence of a Kcsc metric in a given Kähler class on a compact manifold is given by the following:

Given an holomorphic vector field V on the resolution $\tilde{M}$, and supposing that V is Hamiltonian with respect to ω with potential ϕ, one can form the Futaki invariant

$$Fut(V,\omega) = \int_{\tilde{M}} (\phi - \underline{\phi}) \frac{\rho \wedge \omega^{n-1}}{(n-1)!}, \tag{35}$$

where ρ is the Ricci form of ω, and $\underline{\phi} = \int \phi\omega^n / \int \omega^n$ is the mean value of ϕ with respect to ω.

The two key properties proved by Futaki are of fundamental importance in this theory: first, $Fut(V,\omega)$ does not depend on ω but only on its cohomology class $[\omega]$. Secondly, if ω is Kcsc, then $Fut(V,\omega) = 0 \ \forall V$. This second part can be seen in the following way: first observe that on a compact manifold there exists a smooth function (which is unique up to a constant and which depends on g) such that

$$s_\omega - \frac{1}{Vol(M,\omega)} \int_M s_\omega dVol_\omega = \Delta_\omega h_\omega$$

and then, given the holomorphic vector field V on M, just prove by integration by parts that

$$F(V,\omega) = \int_M V(h_\omega) dVol_\omega \ .$$

It is in general very hard to compute the Futaki invariant of a given class, but the first property allows to choose the favourite representative to perform the computation and this in turn could be a huge help.

In any case, without entering this huge and beautiful topic, for our central theme what is most important is another key property of the Futaki invariant discovered by Calabi [11]:

Theorem 4.1 *Let (M,ω) be an extremal compact manifold. Then ω has constant scalar curvature if and only if $F(\cdot,\omega) = 0$.*

Since we have just found that the extremal problem is unobstructed, one can then turn the Kcsc problem into the problem of computing the Futaki invariant for the new Kähler class in the generalised connected sum construction very much in the spirit of what Szekelyhidi did for the blow ups of smooth points in [25] and [27].

This is the content of what we did in [10] extending the analysis carried out in [9]. Of course while a pure PDE approach would give at best a sufficient condition for the existence of a Kcsc metric, this more indirect one gives an equivalent one.

The proof of the computation of the Futaki invariant in this setting is significantly more difficult that for blow ups and we avoid now entering into its technical details. Let us just recollect its consequences for our main problem:

Theorem 4.2 *Let (M, g, ω) be a Kcsc orbifold with isolated singularities. Let $\mathbf{p} = \{p_1, \ldots, p_N\} \subseteq M$ be a set of points with neighborhoods biholomorphic to a ball of $\mathbb{C}^m/\Gamma_j$ with Γ_j nontrivial finite subgroup of $U(m)$ such that $\mathbb{C}^m/\Gamma_j$ admits a scalar flat ALE resolution (X_j, η_j) with $e_j = 0$.*

- *Assume $\mathbf{q} := \{q_1, \ldots, q_K\} \subseteq M$ is a set of points with neighborhoods biholomorphic to a ball of $\mathbb{C}^m/\Gamma_{N+l}$ such that $\mathbb{C}^m/\Gamma_{N+l}$ admits a scalar flat ALE resolution (Y_{N+l}, θ_l) with $e_{N+l} \neq 0$. If there exist $\mathbf{a} := (a_1, \ldots, a_K) \in (\mathbb{R}^+)^K$ such that*

$$\begin{cases} \sum_{l=1}^{K} a_l \varphi_i(q_l) = 0 & i = 1, \ldots, d \\ \\ (a_l \varphi_i(q_l))_{\substack{1 \leq i \leq d \\ 1 \leq l \leq K}} & \textit{has full rank} \end{cases} \tag{36}$$

then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and any $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathbb{R}^+)^N$, the manifold

$$\tilde{M} := M \sqcup_{p_1,\varepsilon} X_1 \sqcup_{p_2,\varepsilon} \cdots \sqcup_{p_N,\varepsilon} X_N \sqcup_{q_1,\varepsilon} X_{N+1} \sqcup_{q_2,\varepsilon} \cdots \sqcup_{q_{N+K},\varepsilon} X_{N+K},$$

admits a Kcsc metric. in the class

$$\pi^*[\omega] + \sum_{l=1}^{K} \varepsilon^{2m-2} \tilde{a}_l^{2m-2} [\tilde{\theta}_l] + \sum_{j=1}^{N} \varepsilon^{2m} b_j [\tilde{\eta}_j]$$

where $\mathfrak{i}_l^*[\tilde{\theta}_l] = [\theta_l]$ with $\mathfrak{i}_l : Y_{\Gamma_{N+l}} \hookrightarrow \tilde{M}$ the standard inclusion (and analogously for $\tilde{\eta}_j$),

$$\left| \tilde{a}_l^2 - \frac{|\Gamma_{N+l}| a_l}{4\,|\mathbb{S}^3|} \right| \leq C\varepsilon^{\gamma} \qquad \textit{for } m = 2 \tag{37}$$

$$\left| \tilde{a}_l^{2m-2} - \frac{|\Gamma_{N+l}| a_l}{8(m-2)(m-1)\,|\mathbb{S}^{2m-1}|} \right| \leq C\varepsilon^{\gamma} \qquad \textit{for } m \geq 3 \tag{38}$$

for some $\gamma > 0$.

- *If* $\mathbf{q} = \emptyset$ *and there exists* $\mathbf{b} \in \mathbb{R}^N_+$ *such that*

$$\begin{cases} \sum_{j=1}^N b_j(\Delta_\omega \varphi_i(p_j) + \varphi_i(p_j)) = 0 & i = 1, \dots, d \\ \\ (\Delta_\omega \varphi_i(p_j) + \varphi_i(p_j)))_{\substack{1 \le i \le d \\ 1 \le j \le N}} & \text{has full rank} \end{cases} \tag{39}$$

then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$

$$\tilde{M} := M \sqcup_{p_1,\varepsilon} X_1 \sqcup_{p_2,\varepsilon} \cdots \sqcup_{p_N,\varepsilon} X_N$$

admits a Kcsc metric in the class

$$\pi^*[\omega] + \sum_j \varepsilon^2 \tilde{b}_j^{2m}[\tilde{\eta}_j] \qquad \text{with} \qquad \mathfrak{i}_j^*[\tilde{\eta}_j] = [\eta_j] \tag{40}$$

where π *is the canonical surjection of* $\tilde{M}$ *onto* M *and* $\mathfrak{i}_j$ *the natural embedding of* X_j *into* $\tilde{M}$*. Moreover*

$$\left| \tilde{b}_j^{2m} - \frac{|\Gamma_j| b_j}{2(m-1)} \right| \le \mathsf{C}\varepsilon^\gamma \qquad \text{for some} \qquad \gamma > 0\,, \tag{41}$$

where $|\Gamma_p|$ *denotes the order of the group.*

As observed above the power of relying on the computation for the Futaki invariant in the relevant Kähler classes stays also in the fact that we can get also an almost complete converse:

Theorem 4.3 *Under the assumptions of Theorem 4.2, given* $\mathbf{b} \in (\mathbb{R}^+)^N$ *and* $\mathbf{c} \in \mathbb{R}^N$ *such that*

$$(b_j \Delta_\omega \varphi_i(p_j) + c_j \varphi_i(p_j))_{\substack{1 \le i \le d \\ 1 \le j \le N}} \text{has full rank.}$$

and

$$\sum_{j=1}^N b_j \Delta_\omega \varphi_i(p_j) + c_j \varphi_i(p_j) \neq 0 \, \text{for some } i = 1, \dots, d$$

then there exists $\bar{\varepsilon}$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon})$ *the orbifold*

$$\tilde{M} := M \sqcup_{p_1,\varepsilon} X_{\Gamma_1} \sqcup_{p_2,\varepsilon} \cdots \sqcup_{p_N,\varepsilon} X_{\Gamma_N}$$

has no Kcsc metric in the class

$$\pi^*[\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} [\tilde{\eta}_j] \tag{42}$$

Very much in the spirit of Szekelyhidi's interpretation of the original results of [4] and [7] we can draw some conclusion also about the celebrated Tian-Yau-Donaldson Conjecture for these type of manifolds

Theorem 4.4 *Let (M, ω) be a compact m-dimensional Kcsc orbifold with isolated singularities. Let $\mathbf{p} = \{p_1, \ldots, p_N\} \subseteq M$ the set of points with neighborhoods biholomorphic to a ball of $\mathbb{C}^m / \Gamma_j$ where, for $j = 1, \ldots, N$, the Γ_j's are nontrivial subgroups of $U(m)$ such that $\mathbb{C}^m / \Gamma_j$ admits an ALE Kahler scalar-flat resolution (X_j, η_j). Then the following are equivalent*

(1) *$\tilde{M}$ had a Kcsc metric in the class $\pi^*[\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} [\tilde{\eta}_j]$*
(2) *$(\tilde{M}, \pi^*[\omega] + \sum_{j=1}^{N} \varepsilon^{2m} b_j^{2m} [\tilde{\eta}_j])$ is K-stable.*

5 The Conical Case

It is very natural to think to quotient singularities as a special case of a larger class of singularities which have received great attention both from the analytic/algebraic side and the Riemannian one, namely "conical singularities".

Let us then recall their definition:

Definition 5.1 We say that a CY cone (Y, ω) admits an *asymptotically conical (AC) CY resolution $(\hat{Y}, \hat{\omega})$ of rate $\nu \in \mathbb{R}$ and $i\partial\bar{\partial}$-exact at infinity*, if:

- $\hat{Y}$ is a smooth manifold with trivial m-pluricanonical bundle, and there exists an holomorphic map (crepant resolution) $\pi : \hat{Y} \to Y$ which is a biholomorphism away from $\pi^{-1}(o)$;
- the $i\partial\bar{\partial}$-equation

$$\pi_*(\hat{\omega}) - \omega = i\partial\bar{\partial} g$$

holds on $Y \setminus B(R, o)$, where $B(R, o)$ is a ball on the CY cone of sufficiently big radius R and g a function in $C^\infty(Y \setminus B(R, o); \mathbb{R})$ which obeys the decay condition: for any $k \geq 0$

$$|\nabla_\omega^k g|_\omega = \mathcal{O}(r^{\nu+2-k})$$

with respect to the distance function from the cone apex $r(p) = d_\omega(p, o) \gg 1$. Here the covariant derivatives and the norms are computed using the CY metric cone metric.

Definition 5.2 We say that a normal (pointed) complex n-dimensional non-compact analytic variety (Y, o) is a *Calabi-Yau cone (CY cone)* if it exists a smooth Kähler form ω on $Y \setminus \{o\}$ such that:

- $Y^* := Y \setminus \{o\}$ is isometric to a Riemannian cone $C(L) = (\mathbb{R}^+ \times L, dr^2 + r^2 g_L)$, with L smooth manifold of real dimension $2n-1$ ("the link"), and r the distance function from the cone apex $\{o\}$, so that the Kähler form can be written as $\omega = \frac{i}{2}\partial\bar{\partial}r^2$.
- there exists a nowhere vanishing section Ω of the (pluri)-canonical bundle $K_Y^{\otimes m}$, such that the Calabi-Yau equation

$$\omega^n = c(n)\,\Omega^{\frac{1}{m}} \wedge \bar{\Omega}^{\frac{1}{m}},$$

  holds on Y^*. If $m \geq 2$ the singularity is only $\mathbb{Q}$-Cartier (i.e., K_Y is just a Weil divisor and not a line bundle, but some multiple of it is actually a bundle).

Let X be a normal complex n-dimensional compact analytic variety with $\mathbb{Q}$-Cartier canonical divisor and with isolated singularities. Denote its singular set with $S := \mathrm{Sing}(X) = \{p_1, \ldots, p_l\}$.

Definition 5.3 We say that a variety X as above has *singularities modelled on CY cones*, if the following holds:

- for all $p_i \in S$ there exists a local biholomorphism $\varphi_i : V_i \subseteq Y_i \to X$, where V_i is an open neighborhood of the apex o_i on a CY cone Y_i and $\varphi_i(o_i) = P_i$.

Let ω be a smooth Kähler form on $X^{reg} := X \setminus S$. We say that ω is *conically singular of rate $\mu_i \in \mathbb{R}^+$ at $P_i \in S$* if it is possible to choose a local biholomorphism φ_i as above which in addition satisfies the following propriety:

- the $i\partial\bar{\partial}$-equation

$$\varphi_i^*(\omega) - \omega_{Y_i} = i\partial\bar{\partial}f_i$$

  holds on $V_i^* := V_i \setminus \{o_i\}$, where ω_{Y_i} denotes a fixed Calabi-Yau cone metric on Y_i and f_i a function in $C^\infty(V_i^*; \mathbb{R})$ which obeys the decay condition: for any $k \geq 0$

$$|\nabla^k_{\omega_{Y_i}} f_i|_{\omega_{Y_i}} = \mathcal{O}(r_i^{\mu_i + 2 - k}),$$

  with respect to the distance function from the cone apex $r_i(p) = d_{\omega_i}(p, o_i) \ll 1$. Here the covariant derivatives and the norms are computed using the CY metrics ω_{Y_i}.

Finally, we say that (X, ω) is a *conically singular cscK variety* if, in addition to the previous proprieties, the Kähler form ω satisfies the constant scalar curvature (cscK) equation

$$Sc(\omega) := i\, tr_\omega \bar{\partial}\partial \log \omega^n = c \in \mathbb{R}$$

on the regular part X^{reg}.

It is important to underline that it is possible (and very interesting) to allow a more flexible condition on the maps φ_i in the above definition, requiring that they are not biholomorphisms but only asymptotic to biholomorphisms. This seemingly innocent relaxation in fact would generate in the gluing procedure we are describing very hard technical problems that the authors are not able to solve.

Also, the existence of a canonical metric with the above asymptotic behaviour near the singularities (or in fact with any prescribed behaviour…) were not known until very recently. Only in 2016 Hein and Sun in a remarkable piece of work could produce many such examples [19].

In [5], Spotti and the author have applied the generalised connected sum construction described in the previous sections and showed that does indeed produce new smooth Kcsc metrics at least in the case of no holomorphic vector fields.

Theorem 5.4 *Let (X, ω) be a complex n-dimensional with $n \geq 3$ conically singular Kcsc variety with discrete automorphism group. Assume that the Kcsc metric ω on X^{reg} is asymptotic to a metric Calabi-Yau cone at rate $\mu_i > 0$ near $p_i \in Sing(X)$ with $i = 1, \ldots, l$, and that the Calabi-Yau cone singularities admit asymptotically conical Calabi-Yau resolutions $(\hat{Y}_i, \omega_{\hat{Y}_i})$ of rate $-2n$ and $i\partial\bar{\partial}$-exact at infinity.*

Then there exists a crepant resolution $\hat{X}$ of X admitting a family of smooth Kcsc metrics $\hat{\omega}_\lambda$, for a parameter $\lambda \in \mathbb{R}^+$ small enough. Moreover, the sign of the scalar curvature of the Kcsc metrics $\hat{\omega}_\lambda$ on the resolution is the same as the one of the metric ω and, finally, the cscK manifolds $(\hat{X}, \hat{\omega}_\lambda)$ converge to the singular Kcsc space (X, ω) in the Gromov-Hausdorff topology as λ tends to zero.

In particular we get the following (see [13] in dimension three).

Corollary 5.5 *Let (X, ω) be a conically singular Ricci flat CY variety satisfying the hypothesis of the main Theorem 5.4. Then on the crepant resolution $\hat{X}$ we have a family ω_λ of smooth Ricci flat CY metrics converging to the singular space (X, ω) in the Gromov-Hausdorff topology.*

6 The Non Compact ALE Case

Another very natural direction in which it is interesting to extend the previous result is when one tries to apply the same machinery starting from a non-compact manifold. One of the main reasons is of course that constant scalar curvature manifolds can be effectively used to manufacture in various ways good initial data seta for the Einstein Constraint Equations and their properties, both geometrical and analytical, serves as an important inspiration also to guess the general behaviour of such solutions.

From this point of view understanding the deep nature of the ADM mass and its possible constraints is of course of great importance. In this respect focusing on ALE non compact manifolds is very natural since these are the models used in General Relativity to represent isolated physical systems. Of course for such manifolds the only admissible constant for scalar curvature is zero, hence we are basically using what in the previous constructions were local models, this time as base spaces for our construction.

It is a simple observation that such manifolds do not carry bounded holomorphic vector fields, so it is reasonable to expect that the Kcsc problem is unobstructed both for blowing up smooth points and desingularize isolated quotient singularities.

In a joint work with Spotti [6], we observed that the this is indeed the case:

Theorem 6.1 *Let $\Gamma \lhd U(m)$ finite acting freely on $\mathbb{S}^{2m-1}$, let (X, ω) be a scalar-flat ALE Kähler orbifold such that there is a compact $K \subset X$ such that*

$$X \setminus K \simeq (\mathbb{C}^m \setminus B_R)/\,\Gamma\,. \tag{43}$$

where $\simeq$ stands for biholomorphic.

- *Let $q \in X$ be a smooth point, then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ Bl_qX, the blow up at q of X, carries a scalar-flat ALE Kähler metric $\tilde{\omega}_a$ in the class*

$$[\tilde{\omega}_a] := [\omega] - \varepsilon^{2m-2} a [c_1(\mathcal{O}(E_q)] \qquad \forall a > 0 \tag{44}$$

  with E_q the exceptional divisor. Moreover $m(Bl_qX, \tilde{\omega}_a)$, the mass of Bl_qX, is a small perturbation of $m(X, \omega)$ i.e.

$$\lim_{\varepsilon \to 0} m(Bl_qX, \tilde{\omega}_a) = m(X, \omega)\,. \tag{45}$$

- *Let $q \in K$ be a singular point with nontrivial (finite) local orbifold group $G \lhd U(m)$. Suppose there is a scalar-flat ALE Kähler manifold (Y, η) that is a resolution of $\mathbb{C}^m/G$. Then there is $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ $\tilde{X}$, the orbifold obtained by gluing topologically $X \setminus q$ and Y, carries a scalar-flat ALE Kähler metric $\tilde{\omega}_a$ in the class*

$$[\tilde{\omega}_a] := [\omega] + \varepsilon^{2m-2\alpha} a [\tilde{\eta}] \qquad \forall a > 0 \tag{46}$$

  with $\alpha = 1$ if $m(Y, \eta) \neq 0$ and $\alpha = 0$ if $m(Y, \eta) = 0$ and $[\tilde{\eta}] \in H^2(\tilde{X}, \mathbb{R})$ induced by $[\eta]$ via the natural embedding in $\tilde{X}$ of a neighborhood of the exceptional locus of Y. Moreover $m(\tilde{X}, \tilde{\omega}_a)$, the mass of $\tilde{X}$, is a small perturbation of $m(X, \omega)$ i.e.

$$\lim_{\varepsilon \to 0} m(\tilde{X}, \tilde{\omega}_a) = m(X, \omega)\,. \tag{47}$$

The proof of previous Theorem is nothing more than a check that the compact analysis passes to the relevant weighted analysis on non compact ALE spaces. What is really interesting, both geometrically and for its physical importance, is to study the behaviour of the ADM mass under this construction. In particular in a recent beautiful paper Hein and LeBrun have proved the following results:

Theorem 6.2

- *The mass of an ALE scalar-flat Kähler manifold is a topological invariant, determined entirely by the smooth manifold, together with the first Chern class of the complex structure and the Kähler class of the metric.*
- *There are infinitely many topological types of ALE scalar-flat Kähler surfaces that have zero mass, but are not Ricci-flat. Indeed, for any $l \geq 3$, the blow-up of the $\mathcal{O}(-l)$ line bundle over $\mathbb{C}P^1$ at any non-empty collection of distinct points on the zero section admits such metrics.*

It seemed then natural to conjecture that a refinement of Theorem 6.1 could produce scalar flat non Ricci-flat ALE metrics with zero mass on the blow up of any scalar flat ALE manifold with negative mass, provided the position of the points is special. In particular the trivial example of the blow up of the flat space (even though it is a real pity not be able to write down a PDE proof of the existence of the Burns-Simanca metric !) indicates that if we indicate by

$$\bar{\varepsilon}_q(X) := sup\{\bar{\varepsilon} \text{ s.t. Theorem 6.1 holds for } \varepsilon < \bar{\varepsilon} \text{ at } q \in X\}$$

our main result can be expressed in the following form

Theorem 6.3 *For any scalar-flat ALE X, if p is sufficiently far away from q in the ω-distance on X, then*

$$\bar{\varepsilon}_q(X) \leq C\bar{\varepsilon}_p(Bl_qX) ,$$

for some constant $C = C(X)$.

Of course this estimate implies the desired generalisation of Hein-LeBrun construction, with the difference of loosing control on the number of points to be blown up to reach zero mass, gaining the freedom of choosing any base ALE space to start with.

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Ohsawa-Takegoshi Extension Theorem for Compact Kähler Manifolds and Applications

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Abstract Our main goal in this article is to prove an extension theorem for sections of the canonical bundle of a weakly pseudoconvex Kähler manifold with values in a line bundle endowed with a possibly singular metric. We also give some applications of our result.

1 Introduction

The L^2 extension theorem by Ohsawa-Takegoshi is a tool of fundamental importance in algebraic and analytic geometry. After the crucial contribution of [18, 19], this result has been generalized by many authors in various contexts, including [1, 2, 6–8, 10, 13, 17, 20, 22, 23].

In this article we treat yet another version of the extension theorem in the context of Kähler manifolds. We first state a consequence of our main result; we remark that a version of it was conjectured by Y.-T. Siu in the framework of his work on the invariance of plurigenera.

Theorem 1.1 *Let* (X, ω) *be a Kähler manifold and* $\mathrm{pr} : X \to \Delta$ *be a proper holomorphic map to the ball* $\Delta \subset \mathbb{C}^1$ *centered at* 0 *of radius* R*. Let* (L, h) *be a holomorphic line bundle over* X *equipped with a hermitian metric (maybe singular)* $h = h_0 e^{-\varphi_L}$ *such that* $i\Theta_h(L) \geq 0$ *in the sense of currents, where* h_0 *is a smooth hermitian metric and* φ_L *is a quasi-psh function over* X*. We suppose that* $X_0 := \mathrm{pr}^{-1}(0)$ *is smooth of codimension* 1*, and that the restriction of* h *to* X_0 *is not identically* ∞*.*

Let $f \in H^0(X_0, K_{X_0} \otimes L)$ *be a holomorphic section in the multiplier ideal defined by the restriction of* h *to* X_0*. Then there exists a section* $F \in H^0(X, K_X \otimes L)$ *whose*

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restriction to X_0 *is equal to* f*, and such that the following optimal estimate holds*

$$\frac{1}{\pi R^2}\int_X |F|^2_{\omega,h}dV_{X,\omega} \leq \int_{X_0} |f|^2_{\omega,h}dV_{X_0,\omega}. \tag{1}$$

We note that the volume form $|F|^2_{\omega,h}dV_{X,\omega}$ is independent of choice of the metric ω, and $dV_{X_0,\omega}$ is the volume form on X_0 induced by the metric $\omega|_{X_0}$.

If the manifold X is isomorphic to the product $X_0 \times \Delta$ and if the line bundle L is trivial, then it is clear how to construct F. If not, the existence of an extension verifying the estimate above is quite subtle, and it has many important applications. The result above is proved by combining the arguments in [6, 13] and [22]. Comparing to [6, 13], the new input here is that we allow the metric h of L to be singular, while the ambient manifold is only assumed to be Kähler. This general context leads to rather severe difficulties, mainly due to the loss of positivity in the process of regularizing the metric h which adds to the intricate relationship between the several parameters involved in the proof. We use here the arguments in [22] to overcome the difficulties.

Before stating the main result of this paper in its most general form and explaining the main ideas in the proof, we note the following consequence of Theorem 1.1 by an idea of H. Tsuji. It is a generalization of [4, Thm 0.1] to arbitrary compact Kähler families, which follows from our main theorem and the arguments in [13, Cor 3.7].

Theorem 1.2 *Let* $p : X \to Y$ *be a fibration between two compact Kähler manifolds. Let* $L \to X$ *be a line bundle endowed with a metric (maybe singular)* $h = h_0 e^{-\varphi_L}$ *such that* $i\Theta_h(L) \geq 0$ *in the sense of currents, where* h_0 *is a smooth hermitian metric and* φ_L *is a quasi-psh function over* X*. Suppose that there exists a generic point* $z \in Y$ *and a section* $u \in H^0(X_z, mK_{X/Y} + L)$ *such that*

$$\int_{X_z} |u|^{\frac{2}{m}}_{\omega,h}dV_{X_z,\omega} < +\infty.$$

Then the line bundle $mK_{X/Y} + L$ *admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric on the generic fibers of* p*.*

We note that the original proof of the theorem above in the projective case does not go through in the Kähler case. This is due to the fact that in [4, Thm 0.1] the authors are using in an essential manner the existence of Zariski dense open subsets of X.

We will state next our general version of Theorem 1.1; prior to this, we introduce some auxiliary weights, following [6, 13].

Notations 1 *Given $\delta > 0$ and $A \in \mathbb{R}$, let $c_A(t)$ be a positive smooth function on $(-A, +\infty)$ such that $\int_{-A}^{+\infty} c_A(t)e^{-t}dt < +\infty$. Set*

$$u(t) = -\ln\left(\frac{c_A(-A)e^A}{\delta} + \int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1\right),$$

and

$$s(t) = \frac{\int_{-A}^{t} e^{-u(t_1)}dt_1 + \frac{c_A(-A)e^A}{\delta^2}}{e^{-u(t)}}.$$

Then $u(t)$ and $s(t)$ satisfy the ODE equations:

$$\left(s(t) + \frac{(s'(t))^2}{u''(t)s(t) - s''(t)}\right)e^{u(t)-t} = \frac{1}{c_A(t)} \tag{2}$$

and

$$s'(t) - s(t)u'(t) = 1. \tag{3}$$

We suppose moreover that

$$e^{-u(t)} \geq c_A(t)s(t) \cdot e^{-t} \qquad \text{for every } t \in (-A, +\infty). \tag{4}$$

Remark 2 If $c_A(t) \cdot e^{-t}$ is decreasing, then (4) is automatically satisfied. Moreover, by the construction of $u(t), s(t)$, we know that

$$\lim_{t \to +\infty} u(t) = -\ln\left(\frac{c_A(-A)e^A}{\delta} + \int_{-A}^{+\infty} c_A(t_1)e^{-t_1}dt_1\right) < +\infty \tag{5}$$

and

$$|s(t)| \leq C_1|t| + C_2 \tag{6}$$

for two constants C_1, C_2 independent of t.

In this set-up, by combining the arguments in [13] and [22], we can prove the main result of the present paper:

Theorem 1.3 *Let (X, ω) be a weakly pseudoconvex n-dimensional Kähler manifold and E be a vector bundle of rank r endowed with a smooth metric h_E. Let $Z \subset X$ be the zero locus of $v \in H^0(X, E)$. We assume that Z is smooth of codimension r and $|v|_{h_E}^{2r} \leq e^A$ for some $A \in \mathbb{R}$. Set $\Psi(z) := \ln |v|_{h_E}^{2r}$.*

Let L be a line bundle on X equipped with a singular metric $h := h_0 \cdot e^{-\varphi}$ such that $i\Theta_h(L) \geq \gamma$ for some continuous $(1,1)$-form γ, where h_0 is a smooth metric on

L. We assume that there exists a sequence of analytic approximations $\{\varphi_k\}_{k=1}^{\infty}$ *of* φ *such that*[1]

$$i\Theta_{h_0\cdot e^{-\varphi_k}}(L) \geq \gamma - \frac{\omega}{k}. \tag{7}$$

We suppose that there exists a continuous function $a(t)$*on* $(-A, +\infty]$, *such that* $0 < a(t) \leq s(t)$ *and*

$$a(-\Psi)(\gamma + id'd''\Psi) + id'd''\Psi \geq 0. \tag{8}$$

Then for every $f \in H^0(Z, K_X \otimes L \otimes \mathcal{I}(\varphi|_Z))$,[2] *there exists a* $F \in H^0(X, K_X \otimes L)$ *such that* $F|_Z = f$ *and*

$$\int_X c_A(-\Psi)|F|^2_{\omega,h} dV_{X,\omega} \leq e^{-\lim\limits_{t\to+\infty} u(t)} \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}, \tag{9}$$

where the weight $|\Lambda^r(ds)|^2$ *is defined as the unique function such that*

$$\int_Z \frac{G}{|\Lambda^r(dv)|^2} dV_{Z,\omega} = \lim_{m\to+\infty} \int_{-m-1\leq \ln|v|^{2r}_{h_E} \leq -m} \frac{G}{|v|^{2r}_{h_E}} dV_{X,\omega} \qquad \text{for every } G \in C^{\infty}(X).$$

Remark 3 As already pointed out in [13], by taking $E = \mathrm{pr}^* \mathcal{O}_\Delta$, $v = \mathrm{pr}^* z$, $A = \ln R^2$, $c_A(t) \equiv 1$ and letting $\delta \to +\infty$, Theorem 1.3 implies Theorem 1.1.

We comment next a few results at the foundation of Theorem 1.3. The original Ohsawa-Takegoshi extension theorem [19] deals with the local case, i.e. X is a pseudoconvex domain in $\mathbb{C}^n$. The potential applications of this type of results in global complex geometry become apparent shortly after the article [19] appeared, and to this end it was necessary to rephrase it in the context of manifolds. As far as we are aware, the first global version is due to Manivel [17]. We quote here a simplified version of his result.

Theorem 1.4 ([17, Thm 2]) *Let* X *be a* n*-dimensional Stein manifold, and* E *be a holomorphic vector bundle over* X *of rank* r *with a smooth metric* h_E. *Let* $Y \subset X$ *be the zero locus of* $s \in H^0(X, E)$. *We assume that* Y *is smooth and of codimension* r. *Let* Ω *be a* $(1, 1)$*-closed semi-positive form on* X *such that*

$$\Omega \otimes \mathrm{Id}_E \geq i\Theta_{h_E}(E)$$

in the sense of Griffiths, i.e., $\Omega \otimes \mathrm{Id}_E - i\Theta_{h_E}(E)$ *is semipositive on the vectors* $\xi \otimes s \in T_X \otimes \mathrm{End}(E)$ *for every* $\xi \in T_X$ *and* $s \in \mathrm{End}(E)$.

[1] If X is compact, such approximation always exists, cf. [9, Chapter 13].

[2] $\mathcal{I}(\varphi|_Z)$ is the multiplier ideal sheaf on Z associated to the weight $\varphi|_Z$.

Let (L, h_0) be a line bundle on X equipped with a smooth metric h_0, such that there exists a constant $\alpha > 0$ satisfying

$$i\Theta_{h_0}(L) \geq \alpha\Omega - rid'd'' \ln |s|^2_{h_E}.$$

Then for every $f \in H^0(Y, K_Y \otimes L \otimes (\det E)^{-1})$, there exists a section $F \in H^0(X, K_X \otimes L)$ such that $F\,|_Y = f \wedge (\wedge^r ds)$ and

$$\int_X \frac{|\,F|^2_{\omega,h_0}}{|s|^{2r-2}_{h_E}(1+|s|^2_{h_E})^\beta} dV_{X,\omega} \leq C \int_Y |f|^2_{\omega,h_0} dV_{Y,\omega}, \tag{10}$$

where C is a numerical constant depending only on r, α and β.

Remark 4 Theorem 1.4 can be easily generalized to the case when X is a weakly pseudoconvex Kähler manifold and the weight function $|s|^{2r-2}_{h_E}(1 + |s|^2_{h_E})^\beta$ can be ameliorated by $|s|^{2r}_{h_E}(\ln |s|_{h_E})^2$, cf. [9, Thm 12.6].

One of the important limitations of Theorem 1.4 is that the metric h_0 is assumed to be smooth. Indeed this is unfortunate, given that in the usual set-up of algebraic geometry one has to deal with extension problems for canonical forms with values in pseudo-effective line bundles. A famous example is the invariance of plurigenera for projective manifolds [20]: one needs an extension theorem under the hypothesis that the metric h_0 has arbitrary singularities. We remark that the proof of the extension theorem used in the article mentioned above is confined to the case of projective manifolds. Thus, in order to generalize [20] to compact Kähler manifolds, the first step would be to allow the metric h_0 in Theorem 1.4 to have arbitrary singularities.

Among the very few results in this direction we mention the important work of L. Yi. In order to keep the discussion simple, we restrict ourselves to the setup in Theorem 1.1. Let $\mathcal{I}_+(h) := \lim\limits_{\delta\to 0+} \mathcal{I}(h^{1+\delta})$. Yi [22] established Theorem 1.1[3] for sections f which belong to the augmented multiplier ideal sheaf $\mathcal{I}_+(h)$. Guan and Zhou [14] (cf. also Hiep [15]) showed that $\mathcal{I}_+(h) = \mathcal{I}(h)$. Thus, the conjunction of these two results as well as the optimal extension [13] establish Theorem 1.1. The proof of our main theorem is mainly based on the arguments in [13] and [22].

Remark 5 In the situation of Theorem 1.3, if we take the weight function $c_A(t) \equiv 1$, then we have Theorem 1.1. There is another weight function which might be useful. If we take $c_A(t) = \frac{e^t}{(t+A+c)^2}$ for some constant $c > 0$, thanks to Remark 2, (4) is satisfied. Using this weight function, [13, Thm 3.16] proved an optimal estimate version of Theorem 1.4 and its Remark 4. Thanks to Theorem 1.3, we know that [13, Thm 3.16] is also true for weakly pseudoconvex Kähler manifolds under the approximation assumption (7).

[3] Yi [22] proved it in a more general setting.

2 Proof of Theorem 1.3

Proof of Theorem 1.3 The constants $C_1, C_2, \cdots$ below are all independent of k. The proof follows closely [13] and [22]. To begin with, we introduce several notations. In the setting of Theorem 1.3, for every $m \in \mathbb{R}$ fixed, we can define a C^1-function b_m on $\mathbb{R}$ such that

$$b_m(t) = t \text{ for } t \geq -m \qquad \text{and} \qquad b''_m(t) = \mathbf{1}_{\{-m-1 \leq t \leq -m\}}.$$

Then

$$b_m(t) \geq t \qquad \text{and} \qquad b_m(t) \geq -m-1 \qquad \text{for every } t \in \mathbb{R}^1. \tag{11}$$

Let s, u be the two functions defined in the introduction. Set $\chi_m(z) := -b_m \circ \Psi$, $\eta_m(z) := s \circ \chi_m$ and $\phi_m(z) := u \circ \chi_m$. Thanks to (5) and (6), we have

$$|\phi_m(z)| \leq C_1 \tag{12}$$

and

$$|\eta_m(z)| \leq C_2 |\chi_m(z)| + C_3 \leq C_2 \cdot \min\{2r |\ln |v|_{h_E}|, m+1\} + C_3. \tag{13}$$

Set $\lambda_m(z) := \frac{(s')^2}{u''s-s''} \circ \chi_m$, $h_k := h_0 \cdot e^{-\varphi_k}$ and $\widetilde{h}_{m,k} := h_k \cdot e^{-\Psi-\phi_m}$. By (2), we have

$$c_A(\chi_m) \cdot e^{-\chi_m+\phi_m} = (\eta_m + \lambda_m)^{-1}. \tag{14}$$

The proof of the theorem is divided by three steps.

Step 1: Construction of smooth extension

We construct in this step a smooth section $\widetilde{f} \in C^\infty(X, K_X \otimes L)$ extending f such that $(D''\widetilde{f})(z) = 0$ for every $z \in Z$ and

$$\int_X \frac{|D''\widetilde{f}|^2_{\omega,h_0}}{|v|^{2r}_{h_E}(\ln |v|_{h_E})^2} \cdot e^{-(1+\sigma)\varphi} dV_{X,\omega} \leq C_1 \cdot \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega} \tag{15}$$

for some constant $\sigma > 0$.

In fact, let (U_i) be a small Stein cover of X and let (χ_i) be a partition of unity subordinate to (U_i). Thanks to [14], there exists a $\sigma > 0$, such that

$$\int_{U_i \cap Z} |f|^2_{\omega,h_0} e^{-(1+\sigma)\varphi} dV_{Z,\omega} \leq 2 \int_{U_i \cap Z} |f|^2_{\omega,h} dV_{Z,\omega}.$$

Applying the local Ohsawa-Takegoshi extension theorem (cf. for example [9, Thm 12.6]) to the weight $e^{-(1+\sigma)\varphi}$ on U_i, we obtain a holomorphic section f_i on U_i

such that

$$\int_{U_i} \frac{|f_i|^2_{\omega,h_0}}{|v|^{2r}_{h_E}(\ln|v|_{h_E})^2} \cdot e^{-(1+\sigma)\varphi} dV_{X,\omega} \le C_2 \cdot \int_{U_i \cap Z} \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}. \tag{16}$$

Set $\widetilde{f} := \sum_i \chi_i \cdot f_i$. Then

$$(D''\widetilde{f})|_{U_j} = D''(\sum_i \chi_i \cdot (f_i - f_j)) = \sum_i (\overline{\partial}\chi_i) \cdot (f_i - f_j) \text{ on } U_j.$$

Combining this with (16), (15) is proved. We have also $(D''\widetilde{f})(z) = 0$ for every $z \in Z$.

Step 2: $\mathbf{L^2}$ estimate

Set $g_m := D''((1 - b'_m \circ \Psi) \cdot \widetilde{f})$. We claim that

Claim There exists a sequence $\{a_m\}_{m=1}^{+\infty} \subset \mathbb{N}$ tending to $+\infty$, γ_m and β_m such that

$$D''\gamma_m + (\frac{m}{a_m})^{\frac{1}{2}}\beta_m = g_m, \qquad \lim_{m\to+\infty} \frac{m}{a_m} = 0, \tag{17}$$

and

$$\overline{\lim_{m\to+\infty}} (\int_X \frac{|\gamma_m|^2_{\omega,\widetilde{h}_{m,a_m}}}{\eta_m + \lambda_m} dV_{X,\omega} + C\int_X |\beta_m|^2_{\omega,\widetilde{h}_{m,a_m}} dV_{X,\omega}) \tag{18}$$

$$\le e^{-\lim_{t\to+\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}$$

for some uniform constant $C > 0$. The proof of the claim combines the estimates in [13] and [22]. We postpone the proof of the claim in Lemma 2.1 and first finish the proof of the theorem.

We use (18) to estimate $\int_X c_A(-b_m \circ \Psi) \cdot |\gamma_m|^2_{\omega,h_{a_m}} dV_{X,\omega}$. By (11) and (14), we have

$$c_A(-b_m \circ \Psi) \cdot e^{\Psi+\phi_m} = c_A(\chi_m) \cdot e^{\Psi+\phi_m} \le (\eta_m + \lambda_m)^{-1}.$$

Therefore

$$\int_X c_A(-b_m \circ \Psi) \cdot |\gamma_m|^2_{\omega,h_{a_m}} dV_{X,\omega} \le \int_X \frac{|\gamma_m|^2_{\omega,\widetilde{h}_{m,a_m}}}{(\eta_m + \lambda_m)} dV_{X,\omega}. \tag{19}$$

Combining this with (18), we get

$$\overline{\lim_{m\to+\infty}} \int_X c_A(-b_m \circ \Psi)|\gamma_m|^2_{\omega,h_{a_m}} dV_{X,\omega} \le e^{-\lim_{t\to+\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h_0} e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}. \tag{20}$$

Thanks to (20), by passing to a subsequence, we can assume that the sequence

$$\{\gamma_m - (1 - b'_m \circ \Psi)\widetilde{f}\}_{k=1}^{+\infty}$$

converges weakly (in the weak L^2-sense) to a section $F \in L^2(X, K_X \otimes L)$.

Step 3: Final conclusion

We first prove that F is holomorphic and satisfies (9). In fact, thanks to (12) and (18), we have

$$\int_X |\beta_m|^2_{\omega,h_{a_m}} e^{-\Psi} dV_{X,\omega} \leq C_3 \tag{21}$$

for some uniform constant C_3. Since $\frac{m}{a_m}$ tends to 0, (17) and (21) imply that $D''(\gamma_m - (1 - b'_m \circ \Psi)\widetilde{f})$ tends to 0. Therefore $F \in H^0(X, K_X \otimes L)$.

As $\{\varphi_k\}_{k=1}^{+\infty}$ is a decreasing sequence, for every $k_0 \in \mathbb{N}$ fixed, we have

$$\int_X c_A(-b_m \circ \Psi)|\gamma_m|^2_{\omega,h_{k_0}} dV_{\omega,X} \leq \int_X c_A(-b_m \circ \Psi)|\gamma_m|^2_{\omega,h_k} dV_{\omega,X} \tag{22}$$

for every $k \geq k_0$. Combining this with (20), we get

$$\varliminf_{m\to+\infty} \int_X c_A(-b_m \circ \Psi)|\gamma_m|^2_{\omega,h_{k_0}} dV_{X,\omega} \leq e^{-\lim\limits_{t\to+\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h_0} e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}$$

Applying Fatou's lemma to the above inequality, we obtain

$$\int_X c_A(-\Psi)|F|^2_{\omega,h_{k_0}} dV_{\omega,X} \leq e^{-\lim\limits_{t\to+\infty} u(t)} \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega},$$

and (9) is proved by letting $k_0 \to +\infty$.

Let $\{U_i\}$ be a Stein cover of X. To finish the proof of the theorem, it remains to prove that $F|_{U_i\cap Z} = f$ for every i. Since β_m is $\overline{\partial}$-closed, on the Stein open set U_i, we can find a function w_m such that $\overline{\partial} w_m = \beta_m$ and

$$\int_{U_i} |w_m|^2_{\omega,h_{a_m}} e^{-\Psi} dV_{X,\omega} \leq C_4 \int_{U_i} |\beta_m|^2_{\omega,h_{a_m}} e^{-\Psi} dV_{X,\omega} \leq C_4 \cdot C_3.$$

for some uniform constant C_4. Then

$$F_m := (1 - b'_m \circ \Psi) \cdot \widetilde{f} - \gamma_m - (\frac{m}{a_m})^{\frac{1}{2}} \cdot w_m$$

is a holomorphic function on U_i and $F_m \rightharpoonup F$ on U_i. As $F_m|_{U_i\cap Z} = f$ by construction, we know that $F|_{U_i\cap Z} = f$. The theorem is proved. □

We complete here the proof of Theorem 1.3 by establishing the claim in Step 2.

Lemma 2.1 *The claim in Theorem 1.3 is true.*

Proof Step 1: Approximation

Since b_m is not smooth, we construct first a smooth approximation of b_m. Let m, k be two fixed constants. Set

$$v_\epsilon(t) := \int_{-\infty}^{t}\int_{-\infty}^{t_1} \frac{1}{1-2\epsilon}\mathbf{1}_{\{-m-1+\epsilon<s<-m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$

$$-\int_{-\infty}^{0}\int_{-\infty}^{t_1} \frac{1}{1-2\epsilon}\mathbf{1}_{\{-m-1+\epsilon<s<-m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$

where $\rho_{\frac{\epsilon}{4}}$ is the kernel of convolution satisfying $\mathrm{supp}(\rho_{\frac{\epsilon}{4}}) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. It is easy to check that $v_\epsilon(t)$ is a smooth approximation of $b_m(t)$. Set

$$\eta_\epsilon := s(-v_\epsilon \circ \Psi), \qquad \phi_\epsilon := u(-v_\epsilon \circ \Psi), \qquad \widetilde{h}_\epsilon := h_k \cdot e^{-\Psi-\phi_\epsilon}$$

and

$$B_\epsilon := [\eta_\epsilon i\Theta_{\widetilde{h}_{m,k}} - i\partial\overline{\partial}\eta_\epsilon - i(\lambda_\epsilon)^{-1}\partial\eta_\epsilon \wedge \overline{\partial}\eta_\epsilon, \Lambda_\omega],$$

where Λ_ω is the contraction with respect to ω. Then $\eta_\epsilon, \phi_\epsilon, B_\epsilon$ tend to $\eta_m, \phi_m, B_{m,k}$.

Step 2: L^2 estimate

By using the estimates in [13, p. 1180], we know that

$$B_{m,k} := [\eta_m(i\Theta_{\widetilde{h}_{m,k}}(L)) - id'd''\eta_m - \lambda_m^{-1} id'\eta_m \wedge d''\eta_m, \Lambda_\omega] \tag{23}$$

satisfies

$$B_{m,k} \geq (b_m'' \circ \Psi)\cdot[\partial\Psi \wedge \overline{\partial}\Psi, \Lambda_\omega] - \frac{\eta_m}{k}\,\mathrm{Id}\,.$$

Combining this with (13), we have

$$B_{m,k} \geq (b_m'' \circ \Psi)\cdot[\partial\Psi \wedge \overline{\partial}\Psi, \Lambda_\omega] - \frac{C\cdot m}{k}\,\mathrm{Id}\,.$$

for some uniform constant C. Therefore, for every form $\alpha \in C_c^\infty(X, \wedge^{n,1}T_X^* \otimes L)$, we have[4]

$$\|(\eta_\epsilon + \lambda_\epsilon)^{\frac{1}{2}}(D'')^*\alpha\|^2_{\widetilde{h}_k} + \|(\eta_\epsilon)^{\frac{1}{2}}D''\alpha\|^2_{\widetilde{h}_k} \geq \langle B_\epsilon\alpha, \alpha\rangle_{\widetilde{h}_k}. \tag{24}$$

[4]We refer to [13, 5.1] for a detailed calculus.

and

$$\langle (B_\epsilon + \frac{C\cdot m}{k}\,\mathrm{Id})\alpha, \alpha\rangle_{\widetilde{h_\epsilon}} \geq (v''_\epsilon \circ \Psi)\langle [\partial\Psi \wedge \overline{\partial}\Psi, \Lambda_\omega]\alpha, \alpha\rangle_{\widetilde{h_\epsilon}} \tag{25}$$

By applying a standard L^2-estimate (cf. Appendix), we can find γ_ϵ and β_ϵ such that

$$D''\gamma_\epsilon + (\frac{m}{k})^{\frac{1}{2}}\beta_\epsilon = g_m \tag{26}$$

and

$$\int_X \frac{|\gamma_\epsilon|^2_{\omega,\widetilde{h_\epsilon}}}{\eta_\epsilon + \lambda_\epsilon} dV_{X,\omega} + \frac{1}{2C}\int_X |\beta_\epsilon|^2_{\omega,\widetilde{h_\epsilon}} dV_{X,\omega} \tag{27}$$

$$\leq \int_X \langle (B_\epsilon + \frac{2C\cdot m}{k})^{-1} g_m, g_m\rangle_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

By letting $\epsilon \to 0$, we can find $\gamma_{m,k}$ and $\beta_{m,k}$, such that

$$D''\gamma_{m,k} + (\frac{m}{k})^{\frac{1}{2}}\beta_{m,k} = g_m \tag{28}$$

and

$$\int_X \frac{|\gamma_{m,k}|^2_{\omega,\widetilde{h}_{m,k}}}{\eta_m + \lambda_m} dV_{X,\omega} + \frac{1}{2C}\int_X |\beta_{m,k}|^2_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega} \tag{29}$$

$$\leq \int_X \langle (B_{m,k} + \frac{2C\cdot m}{k})^{-1} g_m, g_m\rangle_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

Step 3: Final conclusion

We first estimate the right hand side of (29). By the construction of g_m and (25), we have

$$\int_X \langle (B_{m,k} + \frac{2C\cdot m}{k})^{-1} g_m, g_m\rangle_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega} \tag{30}$$

$$\leq \int_X (b''_m \circ \Psi)\cdot |\widetilde{f}|^2_{\omega,\widetilde{h}_{m,k}} dV_\omega + \frac{C\cdot k}{m}\int_X (1 - b'_m \circ \Psi)|D''\widetilde{f}|^2_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

Since $(1 - b'_m \circ \Psi)(z) = 0$ on $\{\Psi \geq -m\}$, we have

$$\int_X (1 - b'_m \circ \Psi)|D''\widetilde{f}|^2_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega} \leq \int_{\{\Psi\leq -m\}} |D''\widetilde{f}|^2_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega}.$$

We use the following key estimate [22, Lemma 3.1]: by Hölder inequality, we have

$$\int_{\{\Psi\le -m\}} \frac{|D''\widetilde{f}|^2_{\omega,h}}{|v|^{2r}_{h_E}} dV_{X,\omega} \tag{31}$$

$$\le \Big(\int_{\{\Psi\le -m\}} \frac{|D''\widetilde{f}|^2_{\omega,h_0} e^{-(1+\sigma)\varphi}}{|v|^{2r}_{h_E}(\ln|v|_{h_E})^2} dV_{X,\omega}\Big)^{\frac{1}{1+\sigma}} \cdot \Big(\int_{\{\Psi\le -m\}} \frac{|D''\widetilde{f}|^2_{\omega,h_0}(\ln|v|_{h_E})^{\frac{2}{\sigma}}}{|v|^{2r}_{h_E}} dV_{X,\omega}\Big)^{\frac{\sigma}{1+\sigma}}.$$

As $D''\widetilde{f} = 0$ on Z by construction, we have

$$\lim_{m\to+\infty} \int_{\{\Psi\le -m\}} \frac{|D''\widetilde{f}|^2_{\omega,h_0}(\ln|v|_{h_E})^{\frac{2}{\sigma}}}{|v|^{2r}_{h_E}} dV_{X,\omega} = 0.$$

Combining this with (31) and (15), we obtain

$$\lim_{m\to+\infty} \int_{\{\Psi\le -m\}} \frac{|D''\widetilde{f}|^2_{\omega,h}}{|v|^{2r}_{h_E}} dV_{X,\omega} = 0.$$

As a consequence, we can find a sequence $a_m \to +\infty$ such that

$$\lim_{m\to+\infty} \frac{m}{a_m} = 0 \qquad \text{and} \qquad \lim_{m\to+\infty} \frac{a_m}{m} \int_{\{\Psi\le -m\}} |D''\widetilde{f}|^2_{\omega,\widetilde{h}_{m,a_m}} dV_{X,\omega} = 0. \tag{32}$$

Applying (32) to (30), we obtain

$$\overline{\lim_{m\to+\infty}} \int_X \langle (B_{m,a_m} + \frac{2C\cdot m}{a_m})^{-1} g_m, g_m\rangle_{\omega,\widetilde{h}_{m,a_m}} dV_{X,\omega} \tag{33}$$

$$\le \overline{\lim_{m\to+\infty}} \int_X (b''_m \circ \Psi)\cdot |\widetilde{f}|^2_{\omega,\widetilde{h}_{m,k}} dV_{X,\omega} \le e^{-\lim_{t\to+\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}.$$

Finally, we take $\gamma_m = \gamma_{m,a_m}$ and $\beta_m = \beta_{m,a_m}$ in (28). Then (29) and (33) imply

$$\overline{\lim_{m\to+\infty}} \Big(\int_X |\gamma_m|^2_{\omega,\widetilde{h}_{m,a_m}} (\eta_m + \lambda_m)^{-1} dV_{X,\omega} + \frac{1}{2C}\int_X |\beta_m|^2_{\omega,\widetilde{h}_{m,a_m}} dV_{X,\omega}\Big)$$

$$\le e^{-\lim_{t\to+\infty} u(t)} \cdot \int_Z \frac{|f|^2_{\omega,h}}{|\Lambda^r(dv)|^2} dV_{Z,\omega}.$$

The lemma is proved. □

3 Applications

3.1 Some Direct Applications

As pointed out in Remark 3 in the introduction, Theorem 1.3 implies that

Corollary 3.1 *Let* (X, ω) *be a Kähler manifold with a proper map* $\mathrm{pr} : X \to \Delta$ *to a ball* $\Delta \subset \mathbb{C}^1$ *centered at* 0 *of radius R. Let* (L, h) *be a holomorphic line bundle over X equipped with a hermitian metric (maybe singular) h such that* $i\Theta_h(L) \geq 0$. *Suppose that* $X_0 := \mathrm{pr}^{-1}(0)$ *is a smooth subvariety of codimension* 1. *Let* $f \in H^0(X_0, K_{X_0} + L)$. *Then there exists a section* $F \in H^0(X, K_X + L)$ *such that*

$$\frac{1}{\pi R^2} \int_X |F|^2_{\omega,h} dV_{X,\omega} \leq \int_{X_0} |f|^2_{\omega,h} dV_{X_0,\omega} \tag{34}$$

and $F|_{X_0} = \mathrm{pr}^*(dt) \wedge f$, *where t is the standard coordinate of* $\mathbb{C}^1$.

By the same arguments as in [4, A.1], Corollary 3.1 implies the following result:

Corollary 3.2 *Let* (X, ω) *be a Kähler manifold and* $\mathrm{pr} : X \to \Delta$ *be a proper map to the ball* $\Delta \subset \mathbb{C}^1$ *centered at* 0 *of radius R. Let L be a holomorphic line bundle over X equipped with a hermitian metric (maybe singular)* $h = h_0 \cdot e^{-\varphi}$ *such that* $i\Theta_h(L) \geq 0$ *in the sense of current, where* h_0 *is a smooth metric and* φ *is a quasi-psh function on X. Suppose that* $X_0 := \mathrm{pr}^{-1}(0)$ *is smooth of codimension* 1. *Let* $f \in H^0(X_0, mK_{X_0} \otimes L)$. *We suppose that*

$$\int_{X_0} |f|^{\frac{2}{m}}_{\omega,h} dV_{X_0,\omega} < +\infty$$

and there exists a $F \in H^0(X, mK_X \otimes L)$ *such that*

$$F|_{X_0} = f \otimes \mathrm{pr}^*(dt^{\otimes m}) \qquad \textit{and} \qquad \int_X |F|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} < +\infty,$$

where t is the standard coordinate of $\mathbb{C}^1$. *Then there exists a* $\widetilde{F} \in H^0(X, mK_X \otimes L)$ *such that*

$$\widetilde{F}|_{X_0} = f \otimes \mathrm{pr}^*(dt^{\otimes m}) \qquad \textit{and} \qquad \frac{1}{\pi R^2} \int_X |\widetilde{F}|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq \int_{X_0} |f|^{\frac{2}{m}}_{\omega,h} dV_{X_0,\omega}. \tag{35}$$

Proof The proof given here follows closely [4, A.1]. Set

$$C_1 := \int_{X_0} |f|^{\frac{2}{m}}_{\omega,h} dV_{X_0,\omega} \qquad \text{and} \qquad C_2 := \frac{1}{\pi R^2} \int_X |F|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega}.$$

If $C_2 \leq C_1$, then F satisfies (35) and the corollary is proved. If $C_1 < C_2$, since F is holomorphic, we can apply Corollary 3.1 with weight

$$\varphi_1 := \frac{m-1}{m} \ln |F|^2_{\omega,h_0} + \frac{1}{m}\varphi$$

on the line bundle $(m-1)K_X \otimes L$, and obtain a new extension F_1 of f satisfying

$$\frac{1}{\pi R^2} \int_X \frac{|F_1|^2_{\omega,h}}{|F|^{\frac{2(m-1)}{m}}_{\omega,h}} dV_{X,\omega} \leq \int_{X_0} \frac{|f|^2_{\omega,h}}{|f|^{\frac{2(m-1)}{m}}_{\omega,h}} dV_{X_0,\omega} = \int_{X_0} |f|^{\frac{2}{m}}_{\omega,h} dV_{X_0,\omega}. \tag{36}$$

By Hölder's inequality, we have

$$\frac{1}{\pi R^2} \int_X |F_1|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq (\frac{1}{\pi R^2} \int_X |F|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega})^{1-\frac{1}{m}} \cdot (\frac{1}{\pi R^2} \int_X \frac{|F_1|^2_{\omega,h}}{|F|^{\frac{2(m-1)}{m}}_{\omega,h}} dV_{X,\omega})^{\frac{1}{m}}.$$

Combining with (36), we have

$$\frac{1}{\pi R^2} \int_X |F_1|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq (C_2)^{1-\frac{1}{m}} (C_1)^{\frac{1}{m}}. \tag{37}$$

We can repeat the same argument with F replaced by F_1, etc. We obtain thus a sequence $\{F_i\}_{i=1}^{+\infty} \subset H^0(X, mK_X \otimes L)$, and

$$\frac{1}{\pi R^2} \int_X |F_i|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq (\frac{1}{\pi R^2} \int_X |F_{i-1}|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega}) \cdot (C_1)^{\frac{1}{m}} \tag{38}$$

If there exists an $i \in \mathbb{N}$ such that F_i satisfies (35), then Corollary (3.2) is proved. If not, thanks to (38), we have

$$\frac{1}{\pi R^2} \int_X |F_i|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \searrow C_1. \tag{39}$$

By passing to a subsequence, F_i tends to a section $\widetilde{F} \in H^0(X, mK_X \otimes L)$, and $\widetilde{F}|_Z = f$. By Fatou lemma, (39) implies that

$$\frac{1}{\pi R^2} \int_X |\widetilde{F}|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq C_1.$$

Corollary 3.2 is proved. □

3.2 Positivity of m-Relative Bergman Kernel Metric

We first recall the definition of m-relative Bergman Kernel metric (cf. [4, A.2], [3, 16, 21]). Let $p : X \to Y$ be a surjective map between two smooth manifolds and let (L, h_L) be a line bundle on X equipped with a hermitian metric h_L. Let $x \in X$ be a point on a smooth fiber of p. We first define a hermitian metric h on $-(mK_{X/Y} + L)_x$ by

$$\|\xi\|_h^2 := \sup \frac{|\xi(\tau(x))|^2}{(\int_{X_{p(x)}} |\tau|_{\omega,h_L}^{\frac{2}{m}} dV_{X_{p(x)},\omega})^m}, \tag{40}$$

where the "sup" is taken over all sections $\tau \in H^0(X_{p(x)}, mK_{X/Y} + L)$. The m-relative Bergman Kernel metric $h_{X/Y}^{(m)}$ on $mK_{X/Y} + L$ is defined to be the dual of h.

Although the construction of the metric $h_{X/Y}^{(m)}$ is fiberwise and only defined on the smooth fibers, by using the positivity of direct image arguments, [4, Thm 0.1] proved that:

Theorem 3.3 ([4, Thm 0.1]) *Let $p : X \to Y$ be a fibration between two projective manifolds, and let ω be a Kähler metric on X. Let $L \to X$ be a line bundle endowed with a metric (maybe singular) h such that $i\Theta_h(L) \geq 0$. Suppose that there exists a generic point $z \in Y$ and a section $u \in H^0(X_z, mK_{X/Y} + L)$ such that*

$$\int_{X_z} |u|_{\omega,h}^{\frac{2}{m}} dV_{X_z,\omega} < +\infty.$$

Then the line bundle $mK_{X/Y} + L$ admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric (with respect to h) on the generic fibers of p.

An alternative proof of Theorem 3.3 is given by using the optimal extension proved in [13, Thm 2.1, Cor 3.7]. We should remark that, if φ_L has arbitrary singularity, the proof of in [4, Thm 0.1] uses the existence of ample line on X. Therefore the assumption that p is a projective map is essential in the proof of Theorem 3.3 in [4, Thm 0.1]. However, as pointed out by Păun, since the optimal extension proved in Corollary 3.2 is without projectivity assumption, we can use Corollary 3.2 to generalize Theorem 3.3 to arbitrary compact Kähler fibrations, by using the same arguments in [13, Cor 3.7]. For the reader's convenience, we give the proof of this generalization in this subsection.

To begin with, we first prove the following lemma, which uses the recent important result [14].

Lemma 3.4 *Let φ be a psh function on a Stein open set U. Set:*

$$\mathcal{I}_m(\varphi)_x := \{f \in \mathcal{O}_x | \int_{U_x} |f|^{\frac{2}{m}} e^{-\frac{\varphi}{m}} < +\infty\}.$$

Then $\mathcal{I}_m(\varphi)$ is a coherent sheaf.

Proof We first prove the lemma under the assumption that φ has analytic singularities. In this case, Let $\pi : \widetilde{U} \to U$ be a resolution of singularities of φ, i.e., $\varphi \circ \pi$ can be written locally as

$$\varphi \circ \pi = \sum_i a_i \ln(|s_i|) + O(1),$$

where s_i are holomorphic functions on $\widetilde{U}$ and $\bigcup_i \mathrm{Div}(s_i)$ is normal crossing. We suppose that $K_{\widetilde{U}} = K_X + \sum_i b_i \cdot E_i$ and $\sum_i a_i \cdot \mathrm{Div}(s_i) = \sum_i c_i \cdot E_i$. Let k_i be the minimal number in $\mathbb{Z}^+$ such that $k_i \cdot \frac{2}{m} > \frac{c_i}{m} - 2b_i - 2$. It is easy to check that $\mathcal{I}_m(\varphi) = \pi_*(\mathcal{O}(-\sum_i k_i \cdot E_i))$. Therefore $\mathcal{I}_m(\varphi)$ is a coherent sheaf.

We now prove the lemma for arbitrary psh functions. Thanks to [9, 15.B], we can find a sequence of quasi-psh φ_k with analytic singularities and a sequence $\delta_k \to 0^+$, such that

(i): φ_k decrease to φ.
(ii): $\int_{\{\frac{\varphi}{m} < \frac{(1+\delta_k)\varphi_k}{m} + a_k\}} e^{-\frac{\varphi}{m}} < +\infty$ (cf. [9, proof of Thm 15.3, Step 2]) for certain constant a_k.

As a consequence, we have $\mathcal{I}_m((1+\delta_k)\varphi_k) \subset \mathcal{I}_m(\varphi)$. Since we proved that

$$\mathcal{I}_m((1+\delta_k)\varphi_k)$$

are coherent, by the Noetherien property of coherent sheaf, $\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k)$ is also coherent and

$$\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k) \subset \mathcal{I}_m(\varphi).$$

To prove the lemma, it is sufficient to prove that for every $f \in \mathcal{I}_m(\varphi)$, we can find a $k \in \mathbb{N}$, such that $f \in \mathcal{I}_m((1+\delta_k)\varphi_k)$.

Let f be a holomorphic germ of $(\mathcal{I}_m(\varphi))_x$. Then

$$\int_{U_x} |f|^2 e^{-\frac{\varphi}{m} - \frac{2(m-1)\ln|f|}{m}} < +\infty,$$

for some neighborhood U_x of x. By Guan and Zhou [14], there exists some $\delta > 0$, such that

$$\int_{U_x} |f|^2 e^{-\frac{(1+\delta)\varphi}{m} - \frac{2(1+\delta)(m-1)\ln|f|}{m}} < +\infty.$$

Replacing U_x by a smaller neighborhood U'_x of x, we have

$$\int_{U'_x} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta)\varphi}{m}} < +\infty. \tag{41}$$

We take a $k \in \mathbb{N}$, such that $\delta_k < \delta$. Thanks to (i) and (41), we have

$$\int_{U'_x} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta_k)\varphi_k}{m}} < +\infty.$$

Therefore $f \in \mathcal{I}_m((1+\delta_k)\varphi_k)$ and the lemma is proved. □

We now generalize [4, Thm 0.1] to arbitrary proper Kähler fibrations. The proof is almost the same as [13, Cor 3.7].

Theorem 3.5 *Let $p : X \to Y$ be a proper fibration between two Kähler manifolds and let ω be a Kähler metric on X. Let $L \to X$ be a line bundle endowed with a metric (maybe singular) $h = h_0 \cdot e^{-\varphi}$ such that $i\Theta_h(L) \geq 0$ in the sense of current, where h_0 is a smooth metric and φ is a quasi-psh function on X.*

Suppose that there exists a generic point $z \in Y$ and a $u \in H^0(X_z, (K_{X/Y})^m \otimes L)$ such that

$$\int_{X_z} |u|^{\frac{2}{m}}_{\omega,h} dV_{X_z,\omega} < +\infty \qquad \text{and} \qquad u \not\equiv 0.$$

Then the line bundle $(K_{X/Y})^m \otimes L$ admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise m-Bergman kernel metric (with respect to h) on the generic fibers of p.

Proof By Lemma 3.4, $p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ is coherent. Using [12] (cf. also [5, Thm 10.7, p. 47]), there exists a subvariety Z of Y of codimension at least 1 such that p is smooth on $Y \setminus Z$ and for every point $t \in Y \setminus Z$, we have

$$\dim H^0(X_t, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi)|_{X_t}) = \operatorname{rank} p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi)),$$

where $\mathcal{I}_m(\varphi)|_{X_t}$ is the restriction of the coherent sheaf $\mathcal{I}_m(\varphi)$ on X_t. By local extension theorem, we know that $\mathcal{I}_m(\varphi|_{X_t}) \subset \mathcal{I}_m(\varphi)|_{X_t}$. As a consequence, for every Stein neighborhood U of $t \in Y \setminus Z$, the fibration $p : p^{-1}(U) \to U$ and the point t satisfy the conditions in Corollary 3.2.

Let $h^{(m)}$ be the fiberwise m-Bergman kernel metric on $p^{-1}(Y \setminus Z) \to Y \setminus Z$ (cf. construction in the beginning of this subsection). For every $x \in p^{-1}(Y \setminus Z)$, we now estimate the curvature of $h^{(m)}$ near x. Let e be a local coordinate of $(K_{X/Y})^m \otimes L$ near x. Let

$$B(z) := \sup \frac{|u^0(z)|^2}{(\int_{X_{p(z)}} |u|^{\frac{2}{m}}_{\omega,h} dV_{X_{p(z)},\omega})^m}, \tag{42}$$

where $u = u^0 \cdot e$ and the "sup" is taken over all sections $u \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$. Thanks to (40), to prove that the curvature of $h^{(m)}$ is positive near x, it is sufficient to prove that $\ln B(z)$ is psh near x.

For every fixed point z near x, we can find a section $u_1 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ such that

$$B(z) = \frac{|u_1^0(z)|^2}{(\int_{X_{p(z)}} |u_1|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega})^m}.$$

Let Δ_r be a one dimensional radius r disc in Y centered at $p(z)$, and Δ'_r be a one dimensional disc in X passing through z and $p(\Delta'_r) = \Delta_r$. Thanks to Proposition 3.2, there exists an extension of u_1: $U_1 \in H^0(p^{-1}(\Delta_r), (K_X)^m \otimes L \otimes \mathcal{I}_m(\varphi))$, such that

$$\frac{1}{\pi r^2} \int_{p^{-1}(\Delta_r)} |U_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega} \leq \int_{X_{p(z)}} |u_1|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega}. \tag{43}$$

Set $\widetilde{u}_1 := U_1/(dt)^m \in H^0(p^{-1}(\Delta_r), (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ and $\widetilde{u}_1^0 := \widetilde{\frac{u_1}{e}}$, where t is coordinate of Δ_r. By the definition of $B(z)$, we have

$$\frac{1}{\pi r^2} \int_{\Delta'_r} \ln B(x) p^*(d't \wedge d''t) \geq \frac{1}{\pi r^2} \int_{\Delta'_r} \ln \frac{|\widetilde{u}_1^0(x)|^2}{(\int_{X_{p(x)}} |\widetilde{u}_1(x)|_{\omega,h}^{\frac{2}{m}} dV_{X_{p(z)},\omega})^m} p^*(d't \wedge d''t)$$

$$\geq \frac{1}{\pi r^2} \int_{\Delta'_r} \ln |\widetilde{u}_1^0|^2 p^*(d't \wedge d''t) - \frac{m}{\pi r^2} \ln \int_{p^{-1}(\Delta_r)} |U_1|_{\omega,h}^{\frac{2}{m}} dV_{X,\omega}.$$

Combining this with (43) and the holomorphicity of $\widetilde{u}_1^0$, we obtain

$$\frac{1}{\pi r^2} \int_{\Delta'_r} \ln B(x) p^*(d't \wedge d''t) \geq \ln B(z).$$

Therefore, $\ln B(x)$ is psh in the horizontal direction. By the convexity of $\ln |u^0(x)|$ and the construction of $\ln B(x)$, $\ln B(x)$ is also psh in the fiberwise direction. Therefore $\ln B(x)$ is psh on $p^{-1}(Y \setminus Z)$ and the curvature of $h^{(m)}$ is semi-positive on $p^{-1}(Y \setminus Z)$ (in the sense of currents).

Using the arguments in [4, A.2], we now prove that $h^{(m)}$ can be extended to the whole X. We first express $h^{(m)}$ locally as the potential form $e^{-\varphi_{X/Y}}$, where $\varphi_{X/Y}$ is a quasi-psh function outside the subvariety $p^{-1}(Z)$. By the standard results in pluripotential theory, to prove that $h^{(m)}$ can be extended to X, it is sufficient to prove the existence of a uniform constant C such that

$$\varphi_{X/Y} \leq C \qquad \text{on } X \setminus p^{-1}(Z). \tag{44}$$

Let U be a small open set in X. Let B be the function on $U \setminus p^{-1}(Z)$ defined by (42). Thanks to (40), to prove (44), it is equivalent to prove that B is uniformly bounded on $U \setminus p^{-1}(Z)$. For every $z \in U \setminus p^{-1}(Z)$, we can find a $u_2 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ such that

$$B(z) = |u_2^0(z)|^2 \qquad \text{and} \qquad \int_{X_{p(z)}} |u_2|^{\frac{2}{m}}_{\omega,h} dV_{X_{p(z)},\omega} = 1,$$

where $u_2^0 := \frac{u_2}{e}$. Using Proposition 3.2, we can find an extension $\widetilde{u}_2$ of u_2, such that

$$\int_{p^{-1}(p(U))} |\widetilde{u}_2|^{\frac{2}{m}}_{\omega,h} dV_{X,\omega} \leq C_U,$$

where the constant C_U depends only on U. By mean value inequality, we know that $|u_2^0(z)|$ is controlled by a constant depending only on C_U. The theorem is thus proved. □

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Appendix

For the reader's convenience, we give the proof of (26) and (27), which is a rather standard estimate (cf. [9, Prop. 12.4, Remark 12.5], [11] or [22]).

Set $g := g_m$, $\eta := \eta_\epsilon$, $B := B_{\epsilon,k}$ and $\delta := \delta_k$ for simplicity. Let Y_k be a subvariety of X such that φ_k is smooth outside Y_k. Then there exists a complete Kähler metric ω_1 on $X \setminus Y_k$. Set $\omega_s := \omega + s\omega_1$. Then ω_s is also a complete Kähler metric on $X \setminus Y_k$ for every $s > 0$.

We apply the twist L^2-estimate (cf. [9, 12.A, 12.B]) for the line bundle $(L, \widetilde{h}_k)$ on $(X \setminus Y_k, \omega_s)$. Thanks to (24) and [13, Lemma 4.1], for every smooth $(n, 1)$-form v with compact support, we have

$$|\langle g, v\rangle_{\omega_s}|^2 \tag{45}$$

$$\leq \Big(\int_{X\setminus Y_k} \langle (B+\frac{2C\cdot m}{k})^{-1} g, g\rangle dV_{\omega_s}\Big)\cdot\Big(\|(\eta+\lambda)^{\frac{1}{2}} D''^{*} v\|^2_{\omega_s} + \frac{2C\cdot m}{k}\int_{X\setminus Y_k}\langle v, v\rangle dV_{\omega_s}\Big)$$

Set $H_1 := \|\cdot\|_{L^2}$, where the L^2-norm $\|\cdot\|_{L^2}$ is defined with respect to the metrics ω_s and $(L, \widetilde{h_k})$. Let H_2 be a Hilbert space where the norm is defined by

$$\|f\|_{H_2}^2 := \frac{2C \cdot m}{k} \int_{X \setminus Y_k} |f|_{\widetilde{h_k}}^2 dV_{\omega_s}.$$

By (45) and the Hahn-Banach theorem, we can construct a continuous linear map (cf. for example [9, 5.A])

$$H_1 \oplus H_2 \to \mathbb{C},$$

which is an extension of the application

$$((\eta + \lambda)^{\frac{1}{2}} D''^* v, v) \to \langle g, v \rangle_{\omega_s}.$$

Therefore, there exist f and h such that

$$\langle g, v \rangle_{\omega_s} = \langle f, (\eta + \lambda)^{\frac{1}{2}} D''^* v \rangle_{\omega_s} + \frac{2C \cdot m}{k} \langle h, v \rangle_{\omega_s}$$

and

$$\|f\|_{\omega_s}^2 + \frac{2C \cdot m}{k} \|h\|_{\omega_s}^2 \leq \int_X (\langle B + \frac{2C \cdot m}{k})^{-1} g, g \rangle dV_{\omega_s}$$

Let $\beta := 2C(\frac{m}{k})^{\frac{1}{2}} \cdot h$ and $\gamma := (\eta + \lambda)^{\frac{1}{2}} f$. Then

$$g = D'' \gamma + (\frac{m}{k})^{\frac{1}{2}} \beta$$

and

$$\| \frac{\gamma}{(\lambda + \eta)^{\frac{1}{2}}} \|_{(X \setminus Y_k, \omega_s)}^2 + \frac{1}{2C} \|\beta\|_{(X \setminus Y_k, \omega_s)}^2 \leq \int_{X \setminus Y_k} \langle (B + \frac{2C \cdot m}{k})^{-1} g, g \rangle dV_{\omega_s}$$

Then (26) and (27) are proved by letting $s \to 0^+$.

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Teichmüller Spaces of Generalized Hyperelliptic Manifolds

Fabrizio Catanese and Pietro Corvaja

Abstract In this paper we answer two questions posed by Catanese (Bull Math Sci 5(3):287–449, 2015), thus achieving in particular a description of the connected components of Teichmüller space corresponding to Generalized Hyperelliptic Manifolds X. These are the quotients $X = T/G$ of a complex torus T by the free action of a finite group G, and they are also the Kähler classifying spaces for a certain class of Euclidean crystallographic groups Γ, the ones which are torsion free and even.

1 Introduction

The classical hyperelliptic surfaces are the quotients of a complex torus of dimension 2 by a finite group G acting freely, and in such a way that the quotient is not again a complex torus.

These surfaces were classified by Bagnera and de Franchis ([2], see also [14] and [3]) and they are obtained as quotients $(E_1 \times E_2)/G$ where E_1, E_2 are two elliptic curves, and G is an abelian group acting on E_1 by translations, and on E_2 effectively and in such a way that $E_2/G \cong \mathbb{P}^1$.

In higher dimension we define the Generalized Hyperelliptic Manifolds (GHM) as quotients T/G of a (compact) complex torus T by a finite group G acting freely, and with the property that G is not a subgroup of the group of translations. Without loss of generality one can then assume that G contains no translations (since the subgroup G_T of translations in G would be a normal subgroup, and if we denote $G' = G/G_T$, then $T/G = T'/G'$, where T' is the torus $T' := T/G_T$).

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The name Bagnera-de Franchis (BdF) Manifolds was instead reserved in [12] and [5] for those quotients $X = T/G$ were G contains no translations, and G is a cyclic group of order m (observe that, when T has dimension $n = 2$, the two notions coincide, thanks to the classification result of Bagnera-De Franchis in [2]). BdF manifolds of small dimension were studied in [12] and [13].

Before stating our main theorem, recall first of all that the Teichmüller space $\mathcal{T}(X)$ of a compact complex manifold X is the quotient

$$\mathcal{T}(X) := \mathcal{CS}(X)/\mathcal{D}iff^0(X)$$

of the space of complex structures on the oriented differentiable manifold underlying X, which are compatible with the natural orientation of X, by the diffeomorphisms of X which are isotopic to the identity.

Recall also that a $K(\Gamma, 1)$ manifold is a manifold M such that its universal covering is contractible, and such that $\pi_1(M) \cong \Gamma$.

We have then:

Theorem 1 *Given a Generalized Hyperelliptic Manifold X, X is Kähler and its fundamental group* $\pi_1(X)$ *is a torsion free even Euclidean crystallographic group* Γ *(see Definitions 2 and 13).*

Conversely, given such a torsion free even Euclidean crystallographic group Γ*, there are GHM with* $\pi_1(X) \cong \Gamma$*; moreover any compact Kähler manifold X which is a* $K(\Gamma, 1)$ *is a Generalized Hyperelliptic manifold.*

The subspace of the Teichmüller space $\mathcal{T}(X)$ *corresponding to Kähler manifolds consists of a finite number of connected components, indexed by the Hodge type of the Hodge decomposition. Each such component is a product of open sets of complex Grassmannians.*

2 Euclidean Crystallographic Groups

Definition 2

(i) We shall say that a group Γ is an abstract Euclidean crystallographic group if there exists an exact sequence of groups

$$(*)\ \ 0 \to \Lambda \to \Gamma \to G \to 1$$

such that

(1) G is a finite group
(2) Λ is free abelian (we shall denote its rank by r)

(3) Inner conjugation $Ad : \Gamma \to Aut(\Lambda)$ has Kernel exactly Λ, hence Ad induces an embedding, called **Linear part**,

$$L : G \to GL(\Lambda) := Aut(\Lambda)$$

(thus $L(g)(\lambda) = Ad(\gamma)(\lambda) = \gamma\lambda\gamma^{-1}, \ \forall\gamma$ a lift of g)

(ii) An **affine realization defined over a field K $\supset$ Z** of an abstract Euclidean crystallographic group Γ is a homomorphism (necessarily injective)

$$\rho : \Gamma \to Aff(\Lambda \otimes_{\mathbb{Z}} K)$$

such that

[1] Λ acts by translations on $V_K := \Lambda \otimes_{\mathbb{Z}} K$, $\rho(\lambda)(v) = v + \lambda$,
[2] for any γ a lift of $g \in G$ we have:

$$V_K \ni v \mapsto \rho(\gamma)(v) = Ad(\gamma)v + u_\gamma = L(g)v + u_\gamma, \text{ for some } \ u_\gamma \in V_K.$$

(iii) More generally we can say that an affine realization of Γ is obtained via a lattice $\Lambda' \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ if there exists a homomorphism $\rho' : \Gamma \to Aff(\Lambda')$ such that $\rho = \rho' \otimes_{\mathbb{Z}} K$ (then necessarily $\Lambda \subset \Lambda'$).

Remark 3 In the previous formulae in (ii) [2] and in the following we used a shorthand notation, we extend the action of $L(g)$ on Λ to V_K naturally as $L(g) \otimes_{\mathbb{Z}} Id_K$. We shall also often write $g(v) := L(g)(v)$, and $\gamma(v) = Ad(\gamma)(v)$.

We note that for a crystallographic group Γ, realizing it via a lattice Λ' as in (iii) amounts to having all the u_γ inside the lattice Λ', in the formula appearing in (ii) [2].

Remark 4 Given a Euclidean crystallographic group Γ as above, the exact sequence $(*)$ is unique up to isomorphism, since Λ is the unique maximal normal abelian subgroup of Γ of finite index.

In fact, if Λ' has the same property, then their intersection $\Lambda^0 := \Lambda \cap \Lambda'$ is a normal subgroup of finite index, in particular $\Lambda^0 \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = V_{\mathbb{Q}}$ and any automorphism of Λ which is the identity on Λ^0 is the identity.

Since $\Lambda' \subset \ker(Ad : \Gamma \to \mathrm{GL}(\Lambda^0))$, $\Lambda' \subset \ker(Ad : \Gamma \to \mathrm{GL}(\Lambda)) = \Lambda$: by maximality $\Lambda' = \Lambda$.

Remark 5

(a) L makes Λ and V_K left G-modules, and to give an affine realization is equivalent to giving a 1-cocycle in $Z^1(\Gamma, V_K)$ such that $u_\lambda = \lambda \ \forall \lambda \in \Lambda$, since

$$\rho(g_1g_2) = \rho(g_1)\rho(g_2) \iff L(g_1)(L(g_2)v + u_{\gamma_2}) + u_{\gamma_1} = L(g_1g_2)v + u_{\gamma_1\gamma_2} \iff$$

$$\iff u_{\gamma_1\gamma_2} = u_{\gamma_1} + L(g_1)(u_{\gamma_2}) = u_{\gamma_1} + g_1(u_{\gamma_2}).$$

(b) Two such cocycles $(u_\gamma), (u'_\gamma)$, are cohomologous if and only if there exists a vector $w \in V_K$ such that:

$$u'_\gamma - u_\gamma = \gamma w - w, \ \forall \gamma \in \Gamma.$$

(c) Hence two such cocycles are cohomologous if and only if the respective affine realizations are conjugate by a translation in V_K, since

$$\rho(\gamma)(v+w)-w = \gamma(v+w)+u_\gamma - w = \gamma v + u_\gamma + (\gamma w - w) = \gamma v + u'_\gamma = \rho'(\gamma)(v).$$

Theorem 6 *Given an abstract Euclidean crystallographic group there is a unique class of affine realization, for each field $K \supset \mathbb{Z}$.*

There is moreover an effectively computable minimal number $d \in \mathbb{N}$ such that the realization is obtained via $\frac{1}{d}\Lambda$.

Proof $Ad : \Gamma \to GL(\Lambda)$ makes Λ a Γ-module, a trivial Λ module, hence also a G-module.

We have seen in Remark 5, (a), that an affine realization is given by a cocycle u_γ in $Z^1(\Gamma, V_K)$ such that $u_\lambda = \lambda, \ \forall \lambda \in \Lambda$; and moreover the class of the realization depends only on the cohomology class in $H^1(\Gamma, V_K)$.

Consider now the exact sequence of cohomology groups

$$H^1(G, V_K) \to H^1(\Gamma, V_K) \to H^1(\Lambda, V_K) = Hom(\Lambda, V_K) \to H^2(G, V_K).$$

Since G is a finite group and K is field of characteristic zero, $H^1(G, V_K) = H^2(G, V_K) = 0$ ([15], pp. 355–363): hence we get an isomorphism $H^1(\Gamma, V_K) \to H^1(\Lambda, V_K) = Hom(\Lambda, V_K)$.

We look for a cohomology class $[u_\gamma]$ such that its image in $H^1(\Lambda, V_K) = Hom(\Lambda, V_K)$ is the tautological map $\lambda \mapsto \lambda \in V_K$, composition of the identity of Λ with the inclusion $\Lambda \subset V_K$. By the above isomorphism such cohomology class exists, is unique and not equal to zero.

In particular, this applies for $K = \mathbb{Q}$, and since G is finite there is an integer D such that that $u_\gamma \in \frac{1}{D}\Lambda$.

Hence we get a cocycle in $H^1(\Gamma, \frac{1}{D}\Lambda)$ whose image in $H^1(\Gamma, (\frac{1}{D}\Lambda)/\Lambda)$ comes form a unique class in $H^1(G, (\frac{1}{D}\Lambda)/\Lambda)$. This last class has order dividing D, and we let δ be its order. Hence, setting $D = d\delta$ we obtain that $[u_\gamma]$ comes from $H^1(\Gamma, (\frac{1}{d}\Lambda))$. □

Remark 7 The above result provides a correct proof for a claim made in [4] and [5], thus answering the question posed in [12], Remark 23, p. 312. It also generalizes the result of lemma 18, p. 310 ibidem.

We realized later on that the first statement, about the unicity of the affine realization, was proved by Bieberbach in 1912 ([6, 7])

We have moreover

Proposition 8 *Suppose that Γ is an Euclidean crystallographic group, which by Theorem 6 is a subgroup of $Aff(V_{\mathbb{Q}})$. Consider a lattice $\Lambda' \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, such that $\Lambda \subset \Lambda'$, and let Γ' be the corresponding group of affine transformations of $V_{\mathbb{Q}}$,* *generated by* Γ *and* Λ'*. Then the group* Γ' *sits into a canonical exact sequence*

$$0 \to \Lambda' \to \Gamma' \to G \to 1,$$

and such an exact sequence splits if and only if the affine realization of Γ *is defined over* Λ'*.*

Proof The group Γ' is obtained from Γ by adding some translations, which lie in the kernel of $Ad : Aff(V_{\mathbb{Q}}) \to GL(V_{\mathbb{Q}})$, so the image $Ad(\Gamma')$ coincides with $G = Ad(\Gamma)$. Hence we get the exact sequence above.

Assume now that the affine realization is defined over Λ'. Then, for each $g \in G$, let $\gamma \in \Gamma \subset \Gamma'$ be any lift of g. From $\gamma(v) = L(g)v + u_\gamma$ for some $u_\gamma \in \Lambda'$, we obtain a second lift $\gamma' \in \Gamma'$ of g by setting $\gamma' = u_\gamma^{-1} \circ \gamma$, i.e. $\gamma'(v) = L(g)v$. Clearly the homomorphism $g \mapsto \gamma'$ yields a splitting.

Conversely, if the exact sequence splits, then we have a semidirect product of Λ' with a group G' isomorphic to G. Then there is a fixed point w for the action of G' on $V_{\mathbb{Q}}$, obtained as $w := \Sigma_{g \in G'} gv$, where v is any vector in Λ. Choosing w as the origin, we obtain an affine realization of Γ' defined over Λ', a fortiori the same holds for Γ. □

Proposition 9

(I) Let $\epsilon \in H^2(G, \Lambda)$ *be the extension class of* Γ*.*
Then there is an affine realization of Γ *defined over* Λ'
$\Leftrightarrow \epsilon \otimes_{\mathbb{Z}} \Lambda' = 0$
$\Leftrightarrow$ *there is a fixed point* $[w] \in \Lambda'/\Lambda$ *for the action of* G *on* $V_{\mathbb{Q}}/\Lambda$*.*

(II) G *acts freely on the real torus* $T := (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda = V_{\mathbb{R}}/\Lambda$ *if and only if* Γ *is* ***torsion free****: this means that the subset*

$$Tors(\Gamma) := \{\gamma | \exists m \in \mathbb{N}_+, \gamma^m = 1_\Gamma\}$$

consists only of the identity.

Proof

(I) An extension splits if and only if its cohomology class $\epsilon' \in H^2(G, \Lambda') = 0$ ([15], Theorem 6.15, p. 365), hence the first assertion follows from Proposition 8.

The second assertion follows as in the proof of Proposition 8.

(II) if g acts with a fixed point on T, there is a lift γ of g such that γ has a fixed point w in V. Then, choosing coordinates such that $w = 0$, the action of γ is linear, hence the order of γ equals the order of g.

Conversely, if γ has finite order m, the vector $w := \sum_1^m \gamma^i(0)$ is fixed by γ, hence $[w]$ is fixed by g.

□

Remark 10 Let $X = T/G$ be a Generalized Hyperelliptic Manifold. The action of $g \in G$ is induced by an affine transformation $x \mapsto \alpha x + b$ on the universal cover, hence it does not have a fixed point on $T = V/\Lambda$ if and only if there is no solution of the equation

$$g(x) \equiv x \pmod{\Lambda}$$

to be solved in $x \in V$, i.e. to the equation

$$\lambda \in \Lambda,\ (\alpha - \mathrm{Id})x = \lambda - b$$

in $(x, \lambda) \in V \times \Lambda$. This remark shows that if the action of G on X is free it is necessary that 1 be an eigenvalue of $\alpha = L(g)$ for all non trivial transformations $g \in G$.

3 Actions of a Finite Group on a Complex Torus T

Assume that we have the action of a finite group G' on a complex torus $T = V/\Lambda'$, where V is a complex vector space, and $\Lambda' \otimes_{\mathbb{Z}} \mathbb{R} \cong V$.

Since every holomorphic map between complex tori lifts to a complex affine map of the respective universal covers, we can attach to the group G' the group Γ of (complex) affine transformations of V which are lifts of transformations of the group G'.

Proposition 11 *Γ is an Euclidean crystallographic group, via the exact sequence*

$$0 \to \Lambda \to \Gamma \to G \to 1$$

where $\Lambda \supset \Lambda'$ is the lattice in V such that $\Lambda := Ker(Ad),\ Ad : \Gamma \to GL(\Lambda')$.

Defining G^0 to be the subgroup of G' consisting of all the translations in G', then $G^0 = \Lambda/\Lambda'$, and moreover $G \subset Aut(V/\Lambda)$ contains no translations.

Proof Λ is a subgroup of the group of translations in V, hence it is obviously Abelian, and maps isomorphically onto a lattice of V which contains Λ'. We shall identify this lattice with Λ.

Λ is normal, $G^0 = \Lambda/\Lambda'$ and G embeds in $GL(\Lambda') \subset GL(\Lambda)$.

□

Hence the datum of the action of a finite group G' on a complex torus T is equivalent to giving:

(1) a crystallographic group Γ
(2) a G- invariant sublattice Λ' of the maximal normal abelian subgroup Λ of finite index (equivalently, to give a normal such sublattice Λ'), so that we may set $G' := \Gamma/\Lambda'$
(3) a complex structure J on the real vector space $V_{\mathbb{R}}$ which makes the action of G complex linear.

While the data (1) and (2) are discrete data, the choice of the complex structure J on V may give rise to positive dimensional moduli spaces. We introduce however (see [12]) a further discrete invariant, called Hodge type.

Definition 12

(i) Given a faithful representation $G \to Aut(\Lambda)$, where Λ is a free abelian group of even rank $r = 2n$, a **G-Hodge decomposition** is a G-invariant decomposition

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}, \ H^{0,1} = \overline{H^{1,0}}.$$

(ii) Write $\Lambda \otimes \mathbb{C}$ as the sum of isotypical components

$$\Lambda \otimes \mathbb{C} = \oplus_{\chi \in \mathrm{Irr}(G)} U_\chi.$$

Write also $U_\chi = W_\chi \otimes M_\chi$, where W_χ is the irreducible representation corresponding to the character χ, and M_χ is a trivial representation whose dimension is denoted n_χ.

Write accordingly $V := H^{1,0} = \oplus_{\chi \in \mathrm{Irr}(G)} V_\chi$, where $V_\chi = W_\chi \otimes M_\chi^{1,0}$.

Then the **Hodge type** of the decomposition is the datum of the dimensions

$$\nu(\chi) := dim_{\mathbb{C}} M_\chi^{1,0}$$

corresponding to the Hodge summands for non real representations (observe in fact that one must have: $\nu(\chi) + \nu(\bar{\chi}) = \dim(M_\chi)$).

Definition 13 A crystallographic group Γ is said to be **even** if:

i) Λ is a free abelian group of even rank $r = 2n$
ii) considering the associated faithful representation $G \to Aut(\Lambda)$, for each real representation χ, $\mathbf{M}_\chi$ **has even dimension** (over $\mathbb{C}$).

Remark 14

(i) Given a group action on a complex torus, we obtain an Euclidean crystallographic group which is even, since $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ admits a G- invariant complex structure.
(ii) Given an even crystallographic group Γ, the G- Hodge decompositions of a fixed Hodge type (satisfying the necessary condition $\nu(\chi) + \nu(\bar{\chi}) = \dim(M_\chi)$) are parametrized by a (non empty) open set in a product of Grassmannians, as follows.

For a non real irreducible representation χ one may simply choose $M_\chi^{1,0}$ to be a complex subspace of dimension $\nu(\chi)$ of M_χ, and for $M_\chi = \overline{M_\chi}$, one simply chooses a complex subspace $M_\chi^{1,0}$ of middle dimension.

They must satisfy the open condition that (since $M_\chi^{0,1} := \overline{M_\chi^{1,0}}$)

$$M_\chi = M_\chi^{1,0} \oplus M_\chi^{0,1} \iff M_\chi = M_\chi^{1,0} \oplus \overline{M_{\bar\chi}^{1,0}}.$$

4 Proof of Theorem 1

1. Let $X = T/G$ be a GHM. Then X is Kähler, since, averaging (by the action of G) a Kähler metric on T with constant coefficients, we obtain a G-invariant one, which descends to the quotient manifold X.
2. Since the universal covering of X is the vector space V, which is contractible, and $X = V/\pi_1(X)$, where $\pi_1(X)$ acts freely, we obtain, setting $\pi_1(X) =: \Gamma$ that X is a $K(\Gamma, 1)$ manifold.
3. That Γ is an Euclidean crystallographic group follows from Proposition 11, that Γ is torsion free by (II) of Proposition 9, and Γ is even by (i) of Remark 14.
4. If X' is also a compact Kähler manifold which is a $K(\Gamma, 1)$, then X' admits a Galois unramified covering T' with group G such that T' is a compact Kähler manifold with the same integral cohomology algebra of a complex torus, hence T' is a complex torus, as shown in [8], see also [12]. Hence also $X' = T'/G$ is a GHM.
5. Moreover, given an Euclidean crystallographic group such that Γ is torsion free and even, $V_{\mathbb{R}}$ admits a complex structure by (i) and (ii) of Remark 14, and the action of Γ on V is free since Γ is torsion free. Hence for any such complex structure we obtain a quotient $X = V/\Gamma = (V/\Lambda)/G = T/G$ which is a GHM.
6. Observe that the family of GHM is stable by deformation in the large. Indeed, every deformation of a GHM $X = T/G$ yields a deformation of the covering torus T (observe that the covering $T \to X$ is associated to the unique surjection $\pi_1(X) \cong \Gamma \to G$) and that in [1, 9] and [10], was proven that a deformation in the large of a complex torus is a complex torus, so that the natural family of n-dimensional complex tori is a connected component $\mathcal{T}_n$ of the Teichmüller space $\mathcal{T}(T)$ (but not the only one).
7. The connected component $\mathcal{T}_n$ of the Teichmüller space of n-dimensional complex tori (see [9, 10] and [11]) is the open set $\mathcal{T}_n$ of the complex Grassmann Manifold $Gr(n, 2n)$, image of the open set of matrices

   $\mathcal{F} := \{\Omega \in \mathrm{Mat}(2n, n; \mathbb{C}) \mid i^n det(\Omega\overline{\Omega}) > 0\}$.

   Over $\mathcal{F}$ lies the following tautological family of complex tori: consider a fixed lattice $\Lambda := \mathbb{Z}^{2n}$, and associate to each matrix Ω as above the subspace V of $\mathbb{C}^{2n} \cong \Lambda \otimes \mathbb{C}$ given as

$$V := \Omega\mathbb{C}^n,$$

   so that $V \in Gr(n, 2n)$ and $\Lambda \otimes \mathbb{C} \cong V \oplus \bar{V}$.

To V we associate then the torus

$$T_V := V/p_V(\Lambda) = (\Lambda \otimes \mathbb{C})/(\Lambda \oplus \bar{V}),$$

$p_V : V \oplus \bar{V} \to V$ being the projection onto the first summand.

The crystallographic group Γ determines an action of $G \subset SL(2n, \mathbb{Z})$ on $\mathcal{F}$ and on $\mathcal{T}_n$, obtained by multiplying the matrix Ω with matrices $g \in G$ on the right.

Define then $\mathcal{T}_n^G$ as the locus of fixed points for the action of G. If $V \in \mathcal{T}_n^G$, then G acts as a group of biholomorphisms of T_V, and we associate then to such a V the GHM

$$X_V := T_V/G.$$

Since the induced family $\mathcal{X} \to \mathcal{T}_n^G$ is differentially trivial, we obtain a map $\psi : \mathcal{T}_n^G \to \mathcal{T}(X)$.

8. We see that $\mathcal{T}_n^G$ consists of a finite number of components, indexed by the Hodge type of the Hodge decomposition. Observe in fact that the Hodge type is invariant by deformation, so it distinguishes a finite number of connected components of $\mathcal{T}_n^G$. That these connected components are just a product of Grassmannians follows from (ii) of Remark 14.
9. Assume in greater generality that we have an unramified Galois cover $p : Y \to X$ such that the associated subgroup $\pi_1(Y) =: \Lambda$ is a characteristic subgroup of $\pi_1(X) =: \Gamma$, and denote by G the quotient group. Then, via pull back, the space $\mathcal{CS}(X)$ of complex structures J' on X is contained in the space $\mathcal{CS}(Y)$ of complex structures J on Y, and is actually equal to the fixed locus of G,

$$\mathcal{CS}(X) = \mathcal{CS}(Y)^G = \{J | g_*(J) = J,\ \forall g \in G\} = \{J | G \subset Bihol(Y_J)\}.$$

10. Since Λ is a characteristic subgroup, all diffeomorphisms of X lift to Y, and we have an exact sequence

$$1 \to G \to \mathcal{N}_Y(G) \to \mathcal{D}iff(X) \to 1,\ \mathcal{N}_Y(G) := \{\phi \in \mathcal{D}iff(Y) | \phi G = G\phi\},$$

since the diffeomorphisms in the normalizer $\mathcal{N}_Y(G)$ of G are the diffeomorphisms which descend to X.
11. If X, Y are classifying spaces, then $\mathcal{D}iff(X)^0$ is the subgroup acting trivially on $\pi_1(X) = \Gamma$, and similarly for Y.
12. In our case G acts non-trivially on the first homology, hence we get an inclusion

$$\mathcal{D}iff(X)^0 \subset \mathcal{D}iff(T)^0,$$

as the normalizer subgroup $\mathcal{N}_Y(G)^0$ of G.

13. We consider now Teichmüller space $\mathcal{T}(X) = \mathcal{CS}(X)/\mathcal{D}iff(X)^0$. Because of 9. and 12.,

$$\mathcal{T}(X) = \mathcal{CS}(X)/\mathcal{D}iff(X)^0 = \mathcal{CS}(Y)^G/\mathcal{N}_T(G)^0.$$

We get therefore a continuous map $j : \mathcal{T}(X) \to \mathcal{T}(T)^G$, where $\mathcal{T}(Y)^G$ is the image of $\mathcal{CS}(Y)^G$ inside $\mathcal{T}(Y)$.

14. We want to show that in our case j is a homeomorphism, at least when restricted to the inverse image of $\mathcal{T}_n$, which we denote by $\mathcal{T}(X)_{GH}$.

It suffices to observe that j and ψ are inverse to each other, and to show that they are local homeomorphisms. We use Remark 14 and Proposition 15 of [11]. In fact, locally for the torus T such that $X = T/G$, Teichmüller space $\mathcal{T}(T)$ is locally the Kuranishi space $Def(T)$, and in turn $Def(X)$ is the closed subspace $Def(X) = Def(Y)^G$ of fixed points for the action of G.

Locally there is a surjection $Def(X) \to \mathcal{T}(X)$. Composing it with j we get the composition of the inclusion $Def(X) \subset Def(T)$ with the local homeomorphism $Def(T) \cong \mathcal{T}(T)$: hence the composition is injective.

By 3) Remark 14 of [11] $Def(X) \to \mathcal{T}(X)$ is a local homeomorphism and we can also conclude that j is a homeomorphism with its image $\mathcal{T}(T)^G$.

4.1 Concluding Remark

We raise the following question: given a generalized Hyperelliptic Manifold X, classify the projective manifolds which are a deformation of X (see [13] for the special case of Bagnera de Franchis manifolds and [17] and [16] for results in dimension 3).

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The Monge-Ampère Energy Class $\mathcal{E}$

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In memory of Paolo de Bartolomeis

Abstract In this short note, based on a joint work with Tamas Darvas and Chinh Lu, we introduce and investigate pluripotential tools. In particular we give a characterization of the Monge-Ampère energy class $\mathcal{E}$ in terms of "envelopes" and we focus on some consequences.

1 Non-pluripolar Monge-Ampère Measure

We recall basic facts concerning pluripotential theory of big cohomology classes. We borrow notation and terminology from [5], and we also refer to this work for further details.

Let (X, ω) be a compact Kähler manifold of dimension n. We fix θ a smooth closed $(1, 1)$-form on X such that $\{\theta\}$ is *big*, i.e., there exists a function ψ such that $\theta + dd^c\psi \geq \varepsilon\omega$ for some small constant $\varepsilon > 0$. Here, d and d^c are real differential operators defined as $d := \partial + \bar{\partial}$, $d^c := \frac{i}{2\pi}\left(\bar{\partial} - \partial\right)$.

A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic if it is locally written as the sum of a plurisubharmonic function and a smooth function. φ is called θ-plurisubharmonic (θ-psh for short) if it is quasi-psh and $\theta + dd^c\varphi \geq 0$ in the sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the set of θ-psh functions which are not identically $-\infty$.

For any positive $(1, 1)$-current $T := \theta + dd^c\varphi$, or equivalently for any θ-psh function φ, we denote by $\nu(T, x)$, or $\nu(\varphi, x)$, its Lelong number at a point $x \in X$ defined as

$$\nu(T, x) = \nu(\varphi, x) := \sup\{\gamma \geq 0 \, : \, \varphi(z) \leq \gamma \log d(x, z) + C\}$$

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where z is a coordinate in an analytic neighborhood of x and d is a distance on it. The *Lelong number* of T along a prime divisor D is

$$\nu(T, D) := \inf\{\nu(T, x) \, : \, x \in D\}.$$

We refer the reader to [9] for a more extensive account on Lelong numbers.

A θ-psh function φ is said to have *analytic singularities* if there exists $c > 0$ such that locally on X,

$$\varphi = \frac{c}{2} \log \sum_{j=1}^{N} |f_j|^2 + u,$$

where u is smooth and $f_1, \ldots, f_N$ are local holomorphic functions.

The *ample locus* $\mathrm{Amp}(\{\theta\})$ of $\{\theta\}$ is the set of points $x \in X$ such that there exists a Kähler current $T \in \{\theta\}$ with analytic singularities and smooth in a neighbourhood of x. The ample locus $\mathrm{Amp}(\{\theta\})$ is a Zariski open subset, and it is nonempty [4].

If φ and φ' are two θ-psh functions on X, then φ' is said to be *less singular* than φ if they satisfy $\varphi \le \varphi' + C$ for some $C \in \mathbb{R}$. Furthermore, we say that φ' and φ are *comparable*, $\varphi' \sim \varphi$, (or equivalently that they have the same singularity type) if they satisfy $\varphi' - C \le \varphi \le \varphi' + C$ for some $C \in \mathbb{R}$.

A θ-psh function φ is said to have *minimal singularities* if it is less singular than any other θ-psh function. Such θ-psh functions with minimal singularities always exist: one can consider for example

$$V_\theta := \sup\{\varphi \;\; \theta\text{-psh}, \varphi \le 0 \text{ on } X\}.$$

More generally, if f is an upper-continuous function on X, we define the Monge-Ampère envelope of f in the class $\mathrm{PSH}(X, \theta)$ by

$$P_\theta(f) := \left(\sup\{u \in \mathrm{PSH}(X, \theta) \mid u \le f\}\right)^*,$$

where * denotes the upper semi-continuous regularization and we adopt the convention that $\sup \emptyset = -\infty$. Observe that $P_\theta(f)$ is a θ-psh function on X if and only if there exists some $u \in \mathrm{PSH}(X, \theta)$ lying below f. Note also that $V_\theta = P_\theta(0)$.

Let us also recall the envelope construction originally due to Ross–Witt Nyström [15]. Given ψ, φ, two θ-psh functions, we define:

$$P_{[\theta,\psi]}(\varphi) := \left\{ \lim_{C \to +\infty} P_\theta(\min(\psi + C, \varphi)) \right\}^*. \tag{1}$$

Given $T_1 := \theta_1 + dd^c\varphi_1, \ldots, T_p := \theta_p + dd^c\varphi_p$ positive $(1,1)$-currents, where θ_j are closed smooth $(1,1)$-forms, following the construction of Bedford-Taylor [1] in the local setting, it has been shown in [5] that the sequence of currents

$$\mathbf{1}_{\bigcap_j \{\varphi_j > V_{\theta_j} - k\}} (\theta_1 + dd^c \max(\varphi_1, V_{\theta_1} - k)) \wedge \ldots \wedge (\theta_p + dd^c \max(\varphi_p, V_{\theta_p} - k))$$

is non-decreasing in k and converges weakly to the so called *non-pluripolar product*

$$\langle T_1 \wedge \ldots \wedge T_p \rangle.$$

The resulting positive (p, p)-current does not charge pluripolar sets and it is *closed*. The particular case when $T_1 = \cdots = T_p$ will be important for us in the sequel. For a θ-psh function φ, the *non-pluripolar complex Monge-Ampère measure* of φ is

$$\theta_\varphi^n := \langle (\theta + dd^c \varphi)^n \rangle.$$

The volume of a big class $\{\theta\}$ is defined by

$$\mathrm{Vol}(\{\theta\}) := \int_{\mathrm{Amp}(\{\theta\})} \theta_{V_\theta}^n = \int_X \theta_{V_\theta}^n > 0.$$

Moreover, by [5, Theorem 1.16], in the above expression one can replace V_θ with any θ-psh function with minimal singularities. Let me recall that, thanks to [3, Theorem 4.8] a cohomology class $\alpha = \{\theta\}$ is big if and only if $\mathrm{Vol}(\{\theta\}) > 0$.

Remark 1 If $\alpha = \{D\}$ is the class of a divisor, then

$$\mathrm{Vol}(\alpha) = \mathrm{Vol}(\{D\}) = \lim_{k \to +\infty} \frac{n!}{k^n} h^0(X, kD)$$

[3, Theorem 1.2]. Moreover, when α is big and nef we have $\mathrm{Vol}(\alpha) = \alpha^n$ [3, Proposition 4.3].

A θ-psh function φ is said to have *full Monge-Ampère mass* if

$$\int_X \theta_\varphi^n = \mathrm{Vol}(\{\theta\}),$$

and we then write $\varphi \in \mathscr{E}(X, \theta)$, or equivalently that $T := \theta + dd^c\varphi \in \mathscr{E}(X, \{\theta\})$. Let us stress that since the non-pluripolar product does not charge pluripolar sets, for a general θ-psh function φ we only have $\mathrm{Vol}(\{\theta\}) \geq \int_X \theta_\varphi^n$.

Observe that by definition all potentials with minimal singularities belong to $\mathscr{E}$. Quasi-plurisubharmonic functions with full Monge-Ampère mass can be unbounded but their singularities are mild: for example, if $\varphi \in \mathscr{E}(X, \theta)$ then $\nu(\varphi, x) = 0$ for any $x \in \mathrm{Amp}(\{\theta\})$ [10, Proposition 2.9].

2 Main Results

Using techniques from [6, 7] we prove that whenever φ, ψ belong to $\mathscr{E}(X, \theta)$ then $P_\theta(\min(\varphi, \psi))$ also belongs to $\mathscr{E}(X, \theta)$, and viceversa:

Theorem 1 *Let $\{\theta\}$ be a big cohomology class and $\varphi \in \mathscr{E}(X, \theta)$. Then $\psi \in \mathscr{E}(X, \theta)$ if and only if $P_{[\theta,\varphi]}(\psi) = \psi$.*

We refer to [8, Theorem 1.2] for a proof.

Our main result clarifies the local/global singular behavior of potentials in $\mathcal{E}(X, \theta)$:

Theorem 2 *Assume that θ is a smooth closed $(1, 1)$-form such that $\{\theta\}$ is big. Let V_θ be the envelope of θ. Then we have the following:*

(i) for any $\varphi \in \mathcal{E}(X, \theta)$ we have

$$\nu(\varphi, x) = \nu(V_\theta, x), \ \forall x \in X,$$

where $\nu(\varphi, x)$ is the Lelong number of φ at x.

(ii) If $\{\eta\}$ is a big and nef class, then

$$\mathcal{E}(X, \eta) \cap PSH(X, \theta) \subset \mathcal{E}(X, \theta).$$

In particular, $\nu(\varphi, x) = 0$ for any $x \in X, \varphi \in \mathcal{E}(X, \eta)$.

In the particular case when $\{\theta\}$ is semi-positive and big, Theorem 2 answers affirmatively an open question in [12, Question 36], saying that potentials in $\mathcal{E}(X, \theta)$ have zero Lelong numbers. A very specific instance of this was verified in [2, Theorem 1.1], using techniques from algebraic geometry.

It is also worth mentioning that in the simpler case when both θ, η are Kähler forms, part (i) of the above Theorem was proved in [13, Corollary 1.8] while (ii) was observed in [10, Theorem B].

The proof we present below follows the one in [8] with some more details.

Proof We first argue (i). From Theorem 1 it follows that $P_{[\theta,\varphi]}(V_\theta) = V_\theta$. Take any $x \in X$. Then trivially $\nu(\varphi, x) \geq \nu(V_\theta, x)$. We will argue by contradiction. Assume that $\nu(\varphi, x) > \nu(V_\theta, x)$. Fix a holomorphic coordinate around x so that we identify x with $0 \in \mathbb{B} \subset \mathbb{C}^n$ where B is the unit ball in $\mathbb{C}^n$. By definition of the Lelong numbers we have

$$\varphi(z) \leq \gamma \log \|z\| + O(1),$$

where $\gamma = \nu(\varphi, x) > 0$. Let g be a smooth local potential for θ in $\mathbb{B}$ and observe that if $\psi \in \mathrm{PSH}(X, \theta)$ then $g + \psi$ is psh in $\mathbb{B}$. Furthermore, w.l.o.g. we can assume that $g+\varphi, g+V_\theta \leq 0$ in $\mathbb{B}$. By the very definition of the envelope we have the following inequality

$$V_\theta + g = P_{[\theta,\varphi]}(V_\theta) + g \leq \sup\{u \in \mathrm{PSH}(\mathbb{B}) \mid u \leq 0\,,\ u \leq \gamma \log \|z\| + O(1)\},$$

in $\mathbb{B}$. Indeed, by definition [15, Definition 4.3]

$$P_{[\theta,\varphi]}(V_\theta) := \sup\{\psi \in \mathrm{PSH}(X, \theta), \psi \leq V_\theta, \psi \sim \varphi\}.$$

Thus, any candidate ψ in the envelope is such that $\psi + g \leq V_\theta + g \leq 0$, $\psi + g$ is psh in $\mathbb{B}$ and, since ψ and φ have the same singularity type, we have that $\nu(\psi, x) = \nu(\varphi, x) = \gamma$ for any $x \in \mathbb{B}$. Moreover, since g is smooth, $\nu(\psi + g, x) = \nu(\psi, x)$. This implies that $\psi + g \leq \gamma \log \|z\| + O(1)$. Hence $\psi + g$ is a candidate in the envelope at the right-hand side.

The latter is the pluricomplex Green function $G_{\mathbb{B}}(z, 0)$ of $\mathbb{B}$ with a logarithmic pole at 0 of order γ. By Klimek [14, Proposition 6.1] we have that

$$G_{\mathbb{B}}(z, 0) \sim \gamma \log \|z\| + O(1).$$

But this contradicts with the assumption that $\nu(V_\theta, x) < \gamma$.

Now we turn to part (ii). Fix ω a Kähler form on X. We can suppose that $\theta, \eta \leq \tilde{\omega} := \eta + \omega$ and $\tilde{\omega}$ is Kähler. Assume that $\varphi \in \mathcal{E}(X, \eta) \cap \mathrm{PSH}(X, \theta)$. By Theorem 1 we get that $P_{[\eta,\varphi]}(V_\eta) = V_\eta$. This implies $P_{[\tilde{\omega},\varphi]}(V_\eta) = V_\eta$ since $P_{[\eta,\varphi]}(V_\eta) \leq P_{[\tilde{\omega},\varphi]}(V_\eta) \leq V_\eta$.

Furthermore, we claim that $V_\eta \in \mathcal{E}(X, \tilde{\omega})$, i.e., $\int_X \tilde{\omega}^n_{V_\eta} = \mathrm{Vol}(\tilde{\omega})$. Indeed, as η is nef, expanding the sum of Kähler classes $(\eta + (1+\varepsilon)\omega)^n$ gives

$$\mathrm{Vol}(\{\eta + (1+\varepsilon)\omega\})^n = \sum_{k=0}^{n} \binom{n}{k} \{\eta + \varepsilon\omega\}^k \cdot \{\omega\}^{n-k}.$$

It follows from the comments after [5, Definition 1.17] that the left-hand side converges to $\mathrm{Vol}(\tilde{\omega})$ while the right-hand side converges to $\sum_{k=0}^n \binom{n}{k} \{\eta\}^k \cdot \{\omega\}^{n-k}$, ultimately giving

$$\mathrm{Vol}(\{\tilde{\omega}\}) = \mathrm{Vol}(\{\eta + \omega\}) = \sum_{k=0}^{n} \binom{n}{k} \{\eta\}^k \cdot \{\omega\}^{n-k}.$$

On the other hand, by multilinearity of the non-pluripolar product we get

$$\int_X \tilde{\omega}^n_{V_\eta} = \int_X (\eta + \omega + dd^c V_\eta)^n = \sum_{k=0}^{n} \binom{n}{k} \int_X (\eta + dd^c V_\eta)^k \wedge \omega^{n-k},$$

and moreover $\{(\eta + dd^c V_\eta)^k\} = \{\eta\}^k$ for each $0 \leq k \leq n$ thanks to [5, Definition 1.17], proving the claim.

Given that $P_{[\tilde{\omega},\varphi]}(V_\eta) = V_\eta$ and $V_\eta \in \mathcal{E}(X, \tilde{\omega})$, we can use [6, Theorem 4] to conclude that $\varphi \in \mathcal{E}(X, \tilde{\omega})$. Because $\theta \leq \tilde{\omega}$ and $\varphi \in \mathrm{PSH}(X, \theta)$, we get $\varphi \in \mathcal{E}(X, \theta)$, as follows from [10, Theorem B].

Remark 2 Observe that Theorem 2 (ii) is in general false for classes $\{\eta\}$ that are big but not nef. Indeed, if $\{\eta\}$ is only big, it may happen that V_η has a non-zero Lelong number at some point, and then [13, Corollary 2.18] would give us that V_η does not have full mass with respect to any Kähler class $\{\theta\}$ satisfying $\eta \leq \theta$, contradicting $\mathcal{E}(X, \eta) \cap \mathrm{PSH}(X, \theta) \subset \mathcal{E}(X, \theta)$.

As a direct consequence we obtain the following additivity property of the set of full mass currents of big and nef cohomology classes.

Corollary 1 *Let* $\{\theta_1\}, \{\theta_2\}$ *be big and nef classes. Then for any* $\varphi_1 \in \mathrm{PSH}(X, \theta_1)$ *and* $\varphi_2 \in \mathrm{PSH}(X, \theta_2)$ *we have*

$$\varphi_1 + \varphi_2 \in \mathscr{E}(X, \theta_1 + \theta_2) \Longleftrightarrow \varphi_1 \in \mathscr{E}(X, \theta_1), \varphi_2 \in \mathscr{E}(X, \theta_2).$$

Proof The implication ($\Longrightarrow$) is proved in [10, Theorem B]. For the reverse implication, fix a Kähler form ω such that $\theta_j \leq \omega, j = 1, 2$. It follows from part (ii) of Theorem 2 that $\varphi_j \in \mathscr{E}(X, \omega)$, $\forall j = 1, 2$. By the convexity of $\mathscr{E}(X, \omega)$ proved in [13, Proposition 1.6] it follows that $\varphi_1 + \varphi_2$ belongs to $\mathscr{E}(X, 2\omega)$. Now, [10, Theorem B] gives that $\varphi_1 + \varphi_2 \in \mathscr{E}(X, \theta_1 + \theta_2)$, hence the result follows.
As observed in Remark 2 the assumption on $\{\theta_1\}$ and $\{\theta_2\}$ of being big and nef is crucial. In the following we give an example (see [10, Example 4.7] for more details) in which the conclusion of Corollary 1 does not hold since one of the two cohomology classes is merely big and not nef.

Example 1 Let $\pi : X \to \mathbb{P}^2$ be the blow up at one point p and set $E := \pi^{-1}(p)$. Consider $\alpha_1 = \pi^\star\{\omega_{FS}\} + \{E\}$ and $\alpha_2 = 2\pi^\star\{\omega_{FS}\} - \{E\}$. Thus $\alpha_1 + \alpha_2 = 3\pi^\star\{\omega_{FS}\}$. Since α_2 is a Kähler class there exists a Kähler form $\omega \in \mathscr{E}(X, \alpha_2)$. Let $T := \pi^\star\omega_{FS} + [E] \in \mathscr{E}(X, \alpha_1)$. We want to show that $T + \omega \notin \mathscr{E}(X, \alpha_1 + \alpha_2)$. Now, from the multilinearity of the non-pluripolar product we get

$$\int_X \langle (T + \omega)^2 \rangle = \int_X \langle (\pi^\star\omega_{FS} + [E] + \omega)^2 \rangle = \int_X \langle (\pi^\star\omega_{FS} + \omega)^2 \rangle = 8$$

Hence $\int_X \langle (T + \omega)^2 \rangle = 8 < 9 = (\alpha_1 + \alpha_2)^2 = \mathrm{Vol}(\alpha_1 + \alpha_2)$.

3 Comments

In this section we discuss some immediate consequences, of Corollary 1. Let us mentioning that the comments below are new with respect to the presentation in [8].

In the following we are going to make use of the Siu's decomposition for currents and of the Zariski decomposition for cohomology classes. We refer the interested reader to [4] for an account on the subject.

Consider α_i, $i = 1, \cdots n$ (different) big and nef cohomology classes, let $T_{\min,i} \in \alpha_i$ be currents with minimal singularities and $T_i \in \mathscr{E}(X, \alpha_i)$. Thanks to Corollary 1 we infer that both the $(1, 1)$-currents $T_{\min,1} + \cdots + T_{\min,n}$ and $T_1 + \cdots + T_n$ are with full Monge-Ampère mass in $\alpha_1 + \cdots \alpha_n$. Thus

$$\int_X \langle (T_1 + \cdots + T_n)^n \rangle = \int_X \langle (T_{\min,1} + \cdots + T_{\min,n})^n \rangle = \langle (\alpha_1 + \cdots + \alpha_n)^n \rangle.$$

Using the multilinearity of the non-pluripolar product we expand both sides and we then get

$$\int_X \langle T_1 \wedge \cdots \wedge T_n \rangle = \int_X \langle T_{\min,1} \wedge \cdots \wedge T_{\min,n} \rangle. \tag{2}$$

In particular if we chose $T_{k+1} = \omega_{k+1}, \cdots T_n = \omega_n$ where ω_i are Kähler forms we get that

$$\int_X \langle T_1 \wedge \cdots \wedge T_k \rangle \wedge \omega_{k+1} \wedge \cdots \wedge \omega_n = \int_X \langle T_{\min,1} \wedge \cdots \wedge T_{\min,k} \rangle \wedge \omega_{k+1} \wedge \cdots \wedge \omega_n$$

or equivalently that for any $k \in [0, n]$

$$\{\langle T_1 \wedge \cdots \wedge T_k \rangle\} = \{\langle T_{\min,1} \wedge \cdots \wedge T_{\min,k} \rangle\} = \langle \alpha_1 \cdots \alpha_k \rangle = \alpha_1 \cdots \alpha_k$$

where the last identity follows from the fact that the cohomology classes α_i are all big and nef. From this we can deduce the following:

Proposition 1 *Let α be a big and nef cohomology class and $T \in \mathscr{E}(X, \alpha)$, then*

$$T = \langle T \rangle.$$

Moreover, when α is merely big and $\dim_{\mathbb{C}} X = 2$ *we have that given $T \in \mathscr{E}(X, \alpha)$,*

$$\langle T \rangle = T_1$$

where T_1 is the positive part in the Siu's decomposition of T.

Proof First we prove that, given $T_{\min} \in \alpha$ then $T_{\min} = \langle T_{\min} \rangle$. Observe that $T_{\min} - \langle T_{\min} \rangle$ is a $(1, 1)$-currents that is supported on the non-Kähler locus $E_{nk}(\alpha) = Amp(\alpha)^c$. By Demailly [9, Corollary 2.14] we have

$$T_{\min} = \langle T_{\min} \rangle + \sum a_i E_i$$

where $E_i \subset E_{nk}(\alpha)$. But, by assumption α is big and nef, hence $0 = \nu(T_{\min}, E_i) = a_i$. It follows that $T_{\min} = \langle T_{\min} \rangle$. Now, observe that from the previous observation we obtain that $\{< T >\} = \{< T_{\min} >\} = \{T_{\min}\} = \{T\}$. So, $T - \langle T \rangle$ is a positive $(1, 1)$-current that is cohomologous to zero. Hence the conclusion. The last statement follows from the fact that when $\dim_{\mathbb{C}} X = 2$, the Zariski decomposition of α is

$$\alpha = \alpha_1 + \sum_{i=1}^{N} a_i \{D_i\}$$

where α_1 is a big and nef class and $a_i = \nu(T_{\min}, D_i) = \nu(T, D_i)$. Note that the last identity follows from [11, Theorem 3.1]. It thus follows that the Siu's decomposition of $T = T_1 + \sum_{i=1}^N a_i D_i$ corresponds to the Zariski decomposition at the level of cohomology. In particular, $\{T_1\} = \alpha_1$. Using the above arguments we get $T_1 = \langle T_1 \rangle$. On the other hand by definition we have $\langle T \rangle = \langle T_1 \rangle$, hence the conclusion.

Remark 3 The equality in (2) holds true without the "nef" assumption in complex dimension 2. More precisely, we claim that if X is a compact Kähler surface, given α, β big cohomology classes and $T \in \mathcal{E}(X, \alpha)$, $S \in \mathcal{E}(X, \beta)$, we have

$$\int_X \langle T \wedge S \rangle = \int_X \langle T_{\min} \wedge S_{\min} \rangle.$$

Indeed, let $\alpha = \alpha_1 + \sum_{i=1}^N a_i \{D_i\}$ and $\beta = \beta_1 + \sum_{j=1}^M b_j \{E_j\}$ be the Zariski decompositions of α and β respectively. Hence α_1, β_1 are nef classes and D_i, E_j are effective divisors. Moreover observe that $T_{\min} \in \alpha$ decomposes as $T_{\min} = T_{\min,1} + \sum_{i=1}^N a_i D_i$ where $T_{\min,1}$ has minimal singularities in α_1. Since $T = T_1 + \sum_{i=1}^N a_i D_i$, $S = S_1 + \sum_{j=1}^M b_j E_j$ and $\langle D_i \wedge S \rangle = 0$ for any i, using (2) we get

$$\int_X \langle T \wedge S \rangle = \int_X \langle T_1 \wedge S_1 \rangle = \int_X \langle T_{\min,1} \wedge S_{\min,1} \rangle = \int_X \langle T_{\min} \wedge S_{\min} \rangle.$$

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Quasi-Negative Holomorphic Sectional Curvature and Ampleness of the Canonical Class

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In memory of Paolo De Bartolomeis

Abstract This note is an extended version of a 50 min talk given at the INdAM Meeting "Complex and Symplectic Geometry", held in Cortona from June 12th to June 18th, 2016. What follows was the abstract of our talk.

Let X be a compact Kähler manifold with a Kähler metric whose holomorphic sectional curvature is strictly negative. Very recent results by Wu–Yau and Tosatti–Yang confirmed an old conjecture by S.-T. Yau which claimed that under this curvature assumption X should be projective and canonically polarized. We will explain how one can relax the assumption on the holomorphic sectional curvature to the weakest possible, i.e. non positive and strictly negative in at least one point, in order to have the same conclusions. We shall also try to motivate this generalization by arguments coming from birational geometry, such as the abundance conjecture.

The results presented here were originally contained in the joint work with Diverio and Trapani (Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle, 2016, ArXiv e-prints 1606.01381v3).

1 Introduction

Let (X, ω) be a Kähler manifold, $\Theta(T_X, \omega)$ be its Chern curvature and $R(T_X^{\mathbb{R}}, g_\omega)$ be the Riemann curvature tensor of the underling real Riemannian manifold with the induced Riemannian metric $g_\omega := \omega(\cdot, J\cdot)$, where J is the complex structure of X. Since ω is Kähler, the Riemann tensor can be identified with the Chern curvature tensor via the usual isomorphism $\xi\colon T_X \to T_X^{1,0}$, $\xi(v) = (v - iJv)/2$.

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The holomorphic bisectional curvature of ω in the directions given by two (holomorphic) tangent vectors $v, w \in T_{X,x} \setminus \{0\}$ is defined by

$$\mathrm{HBC}_\omega(x, [v], [w]) := \frac{1}{||v||_\omega^2 ||w||_\omega^2} \big\langle \Theta_x(T_X, \omega) \cdot v, w \big\rangle_\omega (v, \bar{w}).$$

In the above formula, $\Theta(T_X, \omega)$ firstly acts as an endomorphism of the holomorphic tangent space and then, once contracted with w using the hermitian product defined by ω, eats the pair $(v, \bar{w})$ as a $(1,1)$-form.

The holomorphic sectional curvature of ω in the direction given by one (holomorphic) tangent vector $v \in T_{X,x} \setminus \{0\}$ is defined by

$$\mathrm{HSC}_\omega(x, [v]) := \mathrm{HBC}_\omega(x, [v], [v]) = \frac{1}{||v||_\omega^4} \big\langle \Theta_x(T_X, \omega) \cdot v, v \big\rangle_\omega (v, \bar{v}).$$

It coincides with the Riemannian sectional curvature $K_{g_\omega}(v, Jv)$ relative to the 2-plane spanned by (v, Jv) in $T_{X,x}$.

Next, the (Chern-)Ricci tensor $\mathrm{Ric}(\omega)$ is the closed, real $(1,1)$-form defined up to a constant as the trace (in the endomorphism part) of the Chern curvature:

$$\mathrm{Ric}(\omega) := \frac{i}{2\pi} \operatorname{Tr}_{T_X} \Theta_x(T_X, \omega).$$

It is the Chern curvature $\frac{i}{2\pi} \Theta(K_X^{-1}, \omega^n)$ of the anti-canonical bundle K_X^{-1} of X equipped with the metric ω^n induced by ω.

Finally, the scalar curvature s_ω is the trace with respect to ω of the Ricci curvature. It is thus defined by the relation

$$\mathrm{Ric}(\omega) \wedge \omega^{n-1} = \frac{s_\omega}{n} \omega^n.$$

Both the Chern–Ricci and the scalar curvature correspond to the namesakes in Riemannian geometry.

It is well-known that both the Riemannian sectional curvature and the holomorphic sectional curvature completely determine the curvature tensor. A natural question is wether and how the signs of the different curvatures introduced above are correlated. The answer is summed up in the following diagram:

$$\begin{array}{ccccccc} K_{\omega_g} & \Longrightarrow & \mathrm{HBC}_\omega & \Longrightarrow & \mathrm{HSC}_\omega & \Longrightarrow & s_\omega \\ & & & \searrow\!\!\searrow & & \nearrow\!\!\nearrow & \\ & & & & \mathrm{Ric}(\omega) & & \end{array}$$

The arrows $\Rightarrow$ in the diagram mean that the positivity (resp. semi-positivity, negativity, semi-negativity) of the source curvature implies the positivity (resp.

semi-positivity, negativity, semi-negativity) of the target curvature. It is however a priori not clear if the sign of the holomorphic sectional curvature propagates and determines the signs of the Ricci curvature.

Nevertheless, it was conjectured by Yau that a compact Kähler manifold (X, ω) with negative holomorphic sectional curvature should always admit (a possibly different) Kähler metric ω' with negative Ricci curvature. If so, then K_X would be ample and, in particular, X would be projective. This conjecture has been proved only very recently in the projective case by Wu and Yau in [19], and extended shortly after to the Kähler case in [16] (see also Nomura's recent proof contained in [13] for a Kähler–Ricci flow approach to the problem). Before these major breakthroughs, only some cases were known under some extra conditions. For instance, this was proven in [9] supposing the abundance conjecture to hold true (which is the case indeed in dimension less than or equal to three).

From our point of view, one important motivation to study such a problem comes from the general conjectural picture in Kobayashi hyperbolicity theory of compact Kähler manifolds. Namely, it was conjectured in the 1970s by S. Kobayashi himself that a compact hyperbolic Kähler manifold should have positive canonical bundle, and thus by Yau's celebrated solution of the Calabi conjecture, would admit a Kähler metric with negative Ricci curvature (even more, a Kähler–Einstein metric of Einstein constant -1). Now, if a compact complex manifold admits a hermitian metric of negative holomorphic sectional curvature, then it is well-known that the manifold in question is Kobayashi hyperbolic (the converse does not hold in general, see [4, Theorem 8.2] for a very interesting class of projective examples). Thus, negativity of the holomorphic sectional curvature is a strong way to have hyperbolicity and the result of Wu–Yau and Tosatti–Yang can also be seen as a weak confirmation of the Kobayashi conjecture.

Now, what about compact Kähler manifolds with merely non positive holomorphic sectional curvature? They surely have nef canonical bundle thanks to [16] (in the projective case this is a well-known consequence of Mori's theorem, since they do not admit any rational curve, see next section for more details). Anyway, such a condition is not strong enough in order to obtain positivity of the canonical bundle, as flat complex tori immediately show. A less obvious but still easy counterexample is given by the product (with the product metric) of a flat torus and, say, a compact Riemann surface of genus greater than or equal to two endowed with its Poincaré metric. In this example, over each point there are some directions with strictly negative holomorphic sectional curvature but always some flat directions, too (we refer the reader to the recent paper [10] for some nice results about this merely non positive case).

So, if we look for the weakest condition, as long as the sign of holomorphic sectional curvature is concerned, for which one can hope to obtain the positivity of the canonical bundle, we are led to give the following (standard, indeed) definition.

Definition 1.1 The holomorphic sectional curvature is said to be *quasi-negative* if $\mathrm{HSC}_\omega \leq 0$ and moreover there exists at least one point $x \in X$ such that $\mathrm{HSC}_\omega(x,[v]) < 0$ for every $v \in T_{X,x} \setminus \{0\}$.

Now, why should we hope that such a condition would be sufficient? The reason comes from the birational geometry of complex Kähler manifolds, and in particular from the Minimal Model Program and the Abundance conjecture. Let us illustrate why.

We begin with the following elementary observation.

Proposition 1.2 *Let* (X,ω) *be a compact Kähler manifold such that* $\mathrm{HSC}_\omega \leq 0$, *and suppose there exists a direction* $[v] \in P(T_{X,x_0})$ *such that* $\mathrm{HSC}_\omega(x_0,[v]) < 0$, *for some* $x_0 \in X$. *Then,* $c_1(X) \in H^2(X,\mathbb{R})$ *cannot be zero.*

Sketch of the Proof A computation shows (see for instance [1], or [6, Section 2.1] for a more general computation) that, up to a positive constant multiple (which depends only on $\dim X$), we have for all $x \in X$

$$s_\omega(x) \simeq \int_{P(T_{X,x})} \mathrm{HSC}_\omega(x,[v])\, d\,\mathrm{Vol}_{FS}([v]),$$

where $d\,\mathrm{Vol}_{FS}$ is the Fubini–Study volume form on $P(T_{X,x})$ induced by ω. The hypotheses imply therefore that $s_\omega(x) \leq 0$ for all $x \in X$ and $s_\omega(x_0) < 0$. But then,

$$\begin{aligned} c_1(X)\cdot[\omega]^{n-1} &= \int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1} \\ &= \int_X \frac{s_\omega}{n}\,\omega^n < 0. \end{aligned}$$

□

As a direct consequence, if X is moreover projective and $\mathrm{Pic}(X)$ is infinite cyclic, then K_X must be ample. This gives back (and slightly generalize) a result of [17].

Now, let (X,ω) be a compact Kähler manifold with quasi-negative holomorphic sectional curvature. Then, Proposition 1.2 implies that X cannot have trivial first real Chern class. Moreover, as we saw, by [16] K_X is nef.

Suppose that not only K_X is nef, but moreover it is semi-ample, i.e. some tensor power of K_X is globally generated. This further hypothesis should be in principle removed since conjecturally guaranteed by the abundance conjecture for compact Kähler manifolds. It is in particular always verified if $\dim X \leq 3$ [2] (note moreover that if $\dim X \leq 4$ and X is projective, then since X does not contain any rational curves as we shall see later, the abundance conjecture reduces to the weaker nonvanishing conjecture [11, Theorem 1.5]).

Let $\phi\colon X \to B$ be the semiample Iitaka fibration associated to K_X. So, ϕ is a proper holomorphic mapping with connected fibers and such that some multiple of the canonical bundle is the pull-back of an ample line bundle on the normal projective variety B (see [12, Theorem 2.1.26] for more details). The dimension of B

is exactly the Kodaira dimension $\kappa(X)$ of X, that is the Kodaira–Iitaka dimension of K_X. Moreover, since K_X is supposed to be semi-ample, its Kodaira–Iitaka dimension coincides with its numerical dimension $\nu(K_X) \geq 0$, which is the largest integer ℓ such that $c_1(K_X)^\ell$ is non zero in real cohomology. Thus, since X has non trivial first real Chern class, we have that $\kappa(X) > 0$.

If $\kappa(X) = \nu(K_X) = \dim X$, then X is by definition of general type and, as we shall see, projective (see discussion right after formula (1)) and without rational curves [14, Corollary 2]. Then, Lemma 2.1 implies that in this case K_X is ample.

Next, suppose by contradiction that $1 \leq \kappa(X) \leq \dim X - 1$ so that if we call F the general fiber of ϕ, we have that F is a smooth compact Kähler manifold of positive dimension and different from X itself. Now, on the one hand, the short exact sequence of the fibration shows that $K_F \simeq K_X|_F$ and therefore it follows that $c_1(F)$ must be zero in real cohomology. On the other hand, the classical Griffiths' formulae for curvature of holomorphic vector bundles imply that the holomorphic sectional curvature decreases when passing to submanifolds, that is for every $x \in F \subset X$

$$\mathrm{HSC}_{\omega|_F}(x, [v]) \leq \mathrm{HSC}_\omega(x, [v]),$$

where $v \in T_{F,x}$ and, in the right hand side, v is seen as a tangent vector to X.

The quasi-negativity of the holomorphic sectional curvature implies, since F is a general fiber, that there exists a tangent vector to F along which the holomorphic sectional curvature of $\omega|_F$ is strictly negative. Thus, Proposition 1.2 implies that F cannot have trivial first real Chern class, which is absurd.

As a consequence, me may indeed hope to extend Wu–Yau–Tosatti–Yang theorem to the optimal, quasi-negative case. This is precisely the main contribution of [7].

Theorem 1.3 ([7, Theorem 1.2]) *Let (X, ω) be a connected compact Kähler manifold. Suppose that the holomorphic sectional curvature of ω is quasi-negative. Then, K_X is ample. In particular, X is projective.*

This answers a question raised in [16] (see also [10, Remark 7.2] for a related discussion). A particular case of this theorem was already proved in [17] (see also the comment right after Proposition 1.2) under the additional assumption that the Picard group of X is infinite cyclic, and in [10] under the additional assumption that X is a projective surface. Let us finally note that Wu and Yau have subsequently given a slightly different proof of our Theorem 1.3 in [18].

The rest of this note will be devoted to give a proof of Theorem 1.3, which somehow simplifies the one contained in our original paper. It is the outcome of exchanges with H. Guenancia, who is warmly acknowledged.

2 Reduction to the Key Inequality

Let (X, ω) be a n-dimensional compact Kähler manifold such that $\mathrm{HSC}_\omega \leq 0$. Then, it is classically known (see for instance [14, Corollary 2]) that X cannot contain any (possibly singular) rational curve, i.e. it does not admit any non constant map $\mathbb{P}^1 \to X$. Now, if X is projective, Mori's theorem immediately gives us that K_X must be nef. If X is merely supposed to be Kähler, then the nefness of K_X still holds true and is a direct consequence of the non positivity of the holomorphic sectional curvature, but this is the much more recent result [16, Theorem 1.1].

Now, suppose that one can show under the quasi-negativity assumption of the holomorphic sectional curvature that

$$c_1(K_X)^n > 0. \tag{1}$$

Then, by [5, Theorem 0.5], we deduce that K_X is big. In particular, carrying a big line bundle, X is Moishezon. Since X is Kähler and Moishezon, by Moishezon's theorem X is projective. But then, the following lemma implies that K_X is ample.

Lemma 2.1 (Exercise 8, page 219 of [3]) *Let X be a smooth projective variety of general type which contains no rational curves. Then, K_X is ample.*

Proof Since there are no rational curves on X, Mori's theorem implies as above that K_X is nef. Since K_X is big and nef, the Base Point Free theorem tells us that K_X is semi-ample. If K_X were not ample, then the morphism defined by (some multiple of) K_X would be birational but not an isomorphism. In particular, there would exist an irreducible curve $C \subset X$ contracted by this morphism. Therefore, $K_X \cdot C = 0$. Now, take any very ample divisor H. For any $\varepsilon > 0$ rational and small enough, $K_X - \varepsilon H$ remains big and thus some large positive multiple, say $m(K_X - \varepsilon H)$, of $K_X - \varepsilon H$ is linearly equivalent to an effective divisor D. Set $\Delta = \varepsilon' D$, where $\varepsilon' > 0$ is a rational number. We have:

$$\begin{aligned}(K_X + \Delta) \cdot C &= \varepsilon' D \cdot C \\ &= \varepsilon' m(K_X - \varepsilon H) \cdot C \\ &= -\varepsilon\varepsilon' m H \cdot C < 0.\end{aligned}$$

Finally, if ε' is small enough, then (X, Δ) is a klt pair. Thus, the (logarithmic version of the) Cone Theorem would give the existence of an extremal ray generated by the class of a rational curve in X, contradiction. □

Remark 2.2 The same conclusion can be directly obtained by means of [15, Theorem 1.1]. This theorem states, among other things, that the non-ample locus of the canonical divisor of a smooth projective variety of general type is uniruled. In particular, if there are no rational curves, the non-ample locus must be empty and thus K_X is ample.

It is thus sufficient to prove inequality (1). Since K_X is nef, for any $\varepsilon > 0$, the cohomology class $[\varepsilon\omega - \mathrm{Ric}(\omega)] = \varepsilon[\omega] + c_1(K_X)$ is a Kähler class. By [19, Proposition 8], for every $\varepsilon > 0$, there exists a smooth function u_ε which solves the following Monge–Ampère equation:

$$\begin{cases} \left(\varepsilon\omega - \mathrm{Ric}(\omega) + i\partial\bar\partial u_\varepsilon\right)^n = e^{u_\varepsilon}\omega^n, \\ \omega_\varepsilon := \varepsilon\omega - \mathrm{Ric}(\omega) + i\partial\bar\partial u_\varepsilon > 0. \end{cases} \tag{2}$$

Moreover, again by [19, Proposition 8], there exists a constant $C > 0$ which only depends on ω and $n = \dim X$, such that

$$\sup_X u_\varepsilon < C.$$

Now,

$$\begin{aligned} \int_X e^{u_\varepsilon}\,\omega^n &= \int_X \omega_\varepsilon^n \\ &= \left(\varepsilon[\omega] + c_1(K_X)\right)^n \\ &= c_1(K_X)^n + \sum_{j=0}^{n-1}\binom{n}{j}\varepsilon^{n-j}[\omega]^{n-j}\cdot c_1(K_X)^j. \end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\int_X e^{u_\varepsilon}\,\omega^n = c_1(K_X)^n,$$

and what we have to show is that

$$\lim_{\varepsilon\to 0^+}\int_X e^{u_\varepsilon}\,\omega^n > 0. \tag{3}$$

The next section will be entirely devoted to the proof of inequality (3), which in the sequel will be referred to as "key inequality".

3 Proof of the Key Inequality

The first observation is that the functions u_ε are all ω'-plurisubharmonic for some fixed Kähler form ω' and $\varepsilon > 0$ small enough. For, let $\ell > 0$ be such that $\ell\omega - \mathrm{Ric}(\omega)$ is positive and call $\omega' = \ell\omega - \mathrm{Ric}(\omega)$. Thus, for all $0 < \varepsilon < \ell$, one has

$$0 < \varepsilon\omega - \mathrm{Ric}(\omega) + i\partial\bar\partial u_\varepsilon < \ell\omega - \mathrm{Ric}(\omega) + i\partial\bar\partial u_\varepsilon = \omega' + i\partial\bar\partial u_\varepsilon.$$

Therefore, by [8, Proposition 2.6], either $\{u_\varepsilon\}$ converges uniformly to $-\infty$ on X or it is relatively compact in $L^1(X)$. Suppose for a moment that we are in the second case. Then, there exists a subsequence $\{u_{\varepsilon_k}\}$ converging in $L^1(X)$ and moreover the limit coincides *a.e.* with a uniquely determined ω'-plurisubharmonic function u. Up to pass to a further subsequence, we can also suppose that u_{ε_k} converges pointwise *a.e.* to u. But then, $e^{u_{\varepsilon_k}} \to e^u$ pointwise *a.e.* on X. On the other hand, we have $e^{u_{\varepsilon_k}} \le e^C$ so that, by dominated convergence, we also have $L^1(X)$-convergence and

$$\lim_{k\to\infty} \int_X e^{u_{\varepsilon_k}}\, \omega^n = \int_X e^u \omega^n > 0.$$

The upshot is that what we need to prove is that $\{u_\varepsilon\}$ does not converge uniformly to $-\infty$ on X. From now on, we shall suppose by contradiction that

$$\sup_X u_\varepsilon \to -\infty.$$

Now, as in [19], consider the smooth positive function S_ε on X defined by

$$\omega \wedge \omega_\varepsilon^{n-1} = \frac{S_\varepsilon}{n}\, \omega_\varepsilon^n.$$

Now, define T_ε to be $\log S_\varepsilon$. In other words, T_ε is the logarithm of the trace of ω with respect to ω_ε.

Lemma 3.1 *The function T_ε satisfies the following inequality:*

$$T_\varepsilon > -\frac{u_\varepsilon}{n}.$$

In particular, if $\{u_\varepsilon\}$ converges uniformly to $-\infty$ on X, then T_ε converges uniformly to $+\infty$ on X.

Proof Let $0 < \lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of ω_ε with respect to ω, so that $0 < 1/\lambda_n \le \cdots \le 1/\lambda_1$ are the eigenvalues of ω with respect to ω_ε. Then,

$$e^{T_\varepsilon} = \operatorname{tr}_{\omega_\varepsilon} \omega = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} > \frac{1}{\lambda_1}.$$

Thus, $e^{-T_\varepsilon} < \lambda_1$ so that $e^{-nT_\varepsilon} < (\lambda_1)^n \le \lambda_1 \cdots \lambda_n$. But, $e^{u_\varepsilon}\omega^n = \omega_\varepsilon^n = \lambda_1 \cdots \lambda_n\, \omega^n$, and so we get $e^{-nT_\varepsilon} < e^{u_\varepsilon}$, or, in other words,

$$T_\varepsilon > -\frac{u_\varepsilon}{n}.$$

□

Next, since we do not dispose of a negative constant uniform upper bound for HSC_ω, we are naturally led to consider the following continuous function on X:

$$\kappa : X \to \mathbb{R}$$
$$x \mapsto - \max_{v \in T_{X,x} \setminus \{0\}} \mathrm{HSC}_\omega(x, [v]).$$

The quasi-negativity of the holomorphic sectional curvature of Theorem 1.3 translates in $\kappa \geq 0$ and $\kappa(x_0) > 0$ for some $x_0 \in X$.

By [19, Proposition 9], for every $\varepsilon > 0$ we have the following crucial inequality which makes the holomorphic sectional curvature enter into the picture:

$$\Delta_{\omega_\varepsilon} T_\varepsilon(x) \geq \left(\frac{n+1}{2n} \kappa(x) + \frac{\varepsilon}{n} \right) e^{T_\varepsilon(x)} - 1. \tag{4}$$

Set $M(x) = \frac{n+1}{2n} \kappa(x)$. By plain minoration of the right hand side, we obtain that the T_ε's satisfy the following differential inequality:

$$\Delta_{\omega_\varepsilon} T_\varepsilon(x) \geq M(x)\, e^{T_\varepsilon(x)} - 1. \tag{5}$$

For each $\varepsilon > 0$, integrate (5) over X using the volume form associated to ω_ε, to get

$$0 = \int_X \Delta_{\omega_\varepsilon} T_\varepsilon \, \omega_\varepsilon^n \geq \int_X \left(M e^{T_\varepsilon} - 1 \right) \omega_\varepsilon^n.$$

We obtain therefore the following integral inequality:

$$\int_X M e^{T_\varepsilon} e^{u_\varepsilon} \, \omega^n \leq \int_X e^{u_\varepsilon} \, \omega^n,$$

and setting $v_\varepsilon = u_\varepsilon - \sup_X u_\varepsilon$ one has

$$\int_X M e^{T_\varepsilon} e^{v_\varepsilon} \, \omega^n \leq \int_X e^{v_\varepsilon} \, \omega^n.$$

Next, if we define $C_\varepsilon := \inf_X e^{-u_\varepsilon/n}$, we have that $e^{T_\varepsilon} > C_\varepsilon$, and

$$C_\varepsilon \int_X M e^{v_\varepsilon} \, \omega^n \leq \int_X e^{v_\varepsilon} \, \omega^n. \tag{6}$$

Moreover, recall that we are assuming by contradiction that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$.

Now, the same reasoning made at the beginning of this section tells us that there exists a subsequence $\{v_{\varepsilon_k}\}$ of $\{v_\varepsilon\}$ converging in $L^1(X)$ and moreover the limit coincides *a.e.* with a uniquely determined ω'-plurisubharmonic function v.

Indeed, the case where $\{v_\varepsilon\}$ converges uniformly to $-\infty$ is not possible here since the supremum of the v_ε's is fixed and equal to 0. Again, up to pass to a further subsequence, we can also suppose that v_{ε_k} converges pointwise *a.e.* to v. But then, $e^{v_{\varepsilon_k}} \to e^v$ pointwise *a.e.* on X. On the other hand, we have $e^{v_{\varepsilon_k}} \le 1$ so that, by dominated convergence, we also have $L^1(X)$-convergence and therefore

$$\lim_{k\to\infty} \int_X e^{v_{\varepsilon_k}}\, \omega^n = \int_X e^v \omega^n > 0,$$

and

$$\lim_{k\to\infty} \int_X M e^{v_{\varepsilon_k}}\, \omega^n = \int_X M e^v \omega^n > 0,$$

since M is non negative and strictly positive in at least one point, while the set of points where $v = -\infty$ has zero measure.

Plugging this information into inequality (6) we obtain the desired contradiction since the left hand side blows up while the right hand side converges to some fixed positive number.

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Surjective Holomorphic Maps onto Oka Manifolds

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Abstract Let X be a connected Oka manifold, and let S be a Stein manifold with $\dim S \geq \dim X$. We show that every continuous map $S \to X$ is homotopic to a surjective strongly dominating holomorphic map $S \to X$. We also find strongly dominating algebraic morphisms from the affine n-space onto any compact n-dimensional algebraically subelliptic manifold. Motivated by these results, we propose a new holomorphic flexibility property of complex manifolds, the *basic Oka property with surjectivity*, which could potentially provide another characterization of the class of Oka manifolds.

1 Introduction

A complex manifold X is said to be an *Oka manifold* if every holomorphic map $U \to X$ from an open convex set U in a complex Euclidean space $\mathbb{C}^N$ can be approximated uniformly on compacts in U by holomorphic maps $\mathbb{C}^N \to X$. This *convex approximation property* (CAP) of X, which was first introduced in [12], implies that maps from any Stein manifold S to X satisfy the parametric Oka principle with approximation and interpolation (see [13, Theorem 5.4.4]; it suffices to verify CAP for the integer $N = \dim S + \dim X$). In particular, every continuous map $f : S \to X$ from a Stein manifold S to an Oka manifold X is homotopic to a holomorphic map $F : S \to X$, and F can chosen to approximate f on a compact $\mathscr{O}(S)$-convex set $K \subset S$ provided that f is holomorphic on a neighborhood of K. For the theory of Oka manifolds, we refer to the monograph [13] and the surveys [14, 15, 22]; for the theory of Stein manifolds, see [18, 19].

In this note, we construct *surjective* holomorphic maps from Stein manifolds to Oka manifolds, and surjective algebraic morphisms of affine algebraic manifolds to certain compact algebraic manifolds. We say that a (necessarily surjective)

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holomorphic map $F: S \to X$ is *strongly dominating* if for every point $x \in X$ there exists a point $p \in S$ such that $F(p) = x$ and $dF_p: T_pS \to T_xX$ is surjective. Equivalently, $F(S \setminus \mathrm{br}_F) = X$ where $\mathrm{br}_F \subset S$ is the branch locus of F.

Theorem 1.1 *Let X be a connected Oka manifold. If S is a Stein manifold and* $\dim S \ge \dim X$ *then every continuous map $f: S \to X$ is homotopic to a strongly dominating (surjective) holomorphic map $F: S \to X$. In particular, there exists a strongly dominating holomorphic map $F: \mathbb{C}^n \to X$ for $n = \dim X$.*

Theorem 1.1 answers a question that arose in author's discussion with Jörg Winkelmann (see the Acknowledgement). The result also holds, with the same proof, if S is a reduced Stein space. A similar result in the algebraic category is given by Theorem 1.6.

A version of Theorem 1.1, with $X \subset \mathbb{C}^n$ a non-autonomous basin of a sequence of attracting automorphisms with uniform bounds, is due to Fornæss and Wold [9, Theorem 1.4]. With the exception of surjectivity, the results in the cited theorem had been know earlier for maps of Stein manifolds to Oka manifolds; see [13, Theorem 7.9.1 and Corollary 7.9.3, pp. 324–325] for the existence of embeddings, while the existence of maps with dense images follows immediately from the fact that Oka manifolds enjoy the Oka property with interpolation [13, Theorem 5.4.4].

According to the standard terminology, a holomorphic map $F: \mathbb{C}^n \to X$ is said to be *dominating* at the point $x_0 = F(0) \in X$ if the differential $dF_0: T_0\mathbb{C}^n \to T_{x_0}X$ is surjective; if such F exists then X is *dominable at* x_0. A complex manifold which is dominable at every point is called *strongly dominable*. Every Oka manifold is strongly dominable, but the converse is not known. For a discussion of this subject, see e.g. [16]. Theorem 1.1 furnishes a map $F : \mathbb{C}^n \to X$ such that the family of maps $\{F \circ \phi_a\}_{a \in \mathbb{C}^n}$, where $\phi_a: \mathbb{C}^n \to \mathbb{C}^n$ is the translation $z \mapsto z + a$, dominates at every point of X.

On the other hand, we do not know whether every Oka manifold X is the image of a *locally biholomorphic* map $\mathbb{C}^n \to X$ with $n = \dim X$. A closely related problem is to decide whether locally biholomorphic self-maps of $\mathbb{C}^n$ for $n > 1$ satisfy the Runge approximation theorem; see [13, Problem 8.11.3 and Theorem 8.12.4].

Theorem 1.1 is proved in Sect. 3. The proof is based on an approximation result for holomorphic maps from Stein manifolds to Oka manifolds which we formulate in Sect. 2 (see Theorem 2.1). The approximation takes place on a locally finite sequence of compact sets in a Stein manifold S which are separated by the level sets of a strongly plurisubharmonic exhaustion function and satisfy certain holomorphic convexity conditions. Although Theorem 2.1 follows easily from the proof of the Oka principle with approximation (see [13, Chapter 5]), this formulation is useful in certain situations like the one considered here, and hence we feel it worthwhile to record it.

Theorem 1.1 is motivated in part by results to the effect that certain complex manifolds S are *universal sources*, in the sense that they admit a surjective holomorphic map $S \to X$ onto every complex manifold of the same dimension. This holds for the polydisc and the ball in $\mathbb{C}^n$ (see Fornæss and Stout [7, 8]; in this case, the map can be chosen locally biholomorphic and finitely sheeted), and also for any bounded domain with $\mathscr{C}^2$ boundary in $\mathbb{C}^n$ (see Løw [24]). Further results, with emphasis on the case $X = \mathbb{C}^n$, were obtained by Chen and Wang [4]. In these results, the source manifold is assumed to be Kobayashi hyperbolic. This condition cannot be substantially weakened since a holomorphic map is distance decreasing with respect to the Kobayashi pseudometrics on the respective manifolds. In particular, a manifold with vanishing Kobayashi pseudometric (such as $\mathbb{C}^n$) does not admit any nonconstant holomorphic map to a hyperbolic manifold. Furthermore, the existence of a nondegenerate holomorphic map $\mathbb{C}^n \to X$ to a connected compact complex manifold X of dimension n implies that X is not of general type (see Kodaira [21] and Kobayashi and Ochiai [20]). By an extension of the Kobayashi-Ochiai argument, Campana proved that such X is *special* [2, Corollary 8.11]. Special manifolds are important in Campana's structure theory of compact Kähler manifolds. Recently, Diverio and Trapani [5] and Wu and Yau [28, 29] proved that a compact connected complex manifold X, which admits a Kähler metric whose holomorphic sectional curvature is everywhere nonpositive and is strictly negative at least at one point, has positive canonical bundle K_X. (See also Tosatti and Yang [26] and Nomura [25].) Hence, such X is projective and of general type, and therefore it does not admit any nondegenerate holomorphic map $\mathbb{C}^n \to X$ with $n = \dim X$.

These observations justify the hypothesis in Theorem 1.1 that X be an Oka manifold.

Let us recall a related but weaker holomorphic flexibility property introduced by Gromov [17]. A complex manifold X is said to enjoy the *basic Oka property*, BOP, if every continuous map $S \to X$ from a Stein manifold S is homotopic to a holomorphic map. The only difference with respect to the class of Oka manifolds is that BOP does not include any approximation or interpolation conditions. Thus, every Oka manifold satisfies BOP, but the converse fails e.g. for contractible hyperbolic manifolds (such as bounded convex domains in $\mathbb{C}^n$). The basic Oka property was studied by Winkelmann [27] for maps between Riemann surfaces, and by Campana and Winkelmann [3] for more general complex manifolds. (Their use of the term *homotopy principle* is equivalent to BOP.) In particular, they proved in [3, Main Theorem] that a projective manifold satisfying BOP is special in the sense of [2]. (The converse is an open problem.) We thus have

$$\text{Oka} \Longrightarrow \text{BOP} \Longrightarrow \text{special},$$

where the second implication holds for compact projective manifolds (and is expected to be true for all compact Kähler manifolds).

Concerning the relationship between Oka manifolds and manifolds with BOP, one has the feeling that these two classes are essentially the same after eliminating the obvious counterexamples provided by contractible hyperbolic manifolds; the latter may be used as building blocks in manifolds with BOP, but not in Oka manifolds. With this in mind, we propose the following new Oka property.

Definition 1.2 A connected complex manifold X satisfies the *basic Oka property with surjectivity*, abbreviated BOPS, if every continuous map $f : S \to X$ from a

Stein manifold S with $\dim S \geq \dim X$ is homotopic to a *surjective* holomorphic map $F: S \to X$.

Theorem 1.1 says that Oka $\Rightarrow$ BOPS. Applying the BOPS axiom to a constant map $\mathbb{C}^n \to x_0 \in X$ gives the following observation.

Proposition 1.3 *A connected complex manifold X satisfying BOPS admits a surjective holomorphic map $\mathbb{C}^n \to X$ with $n = \dim X$. In particular, the Kobayashi pseudometric of a complex manifold satisfying BOPS vanishes identically.*

Since the BOPS axiom eliminates the obvious counterexamples to the (false) implication BOP $\Rightarrow$ Oka, the following seems a reasonable question.

Problem 1.4

(a) Assuming that a complex manifold X satisfies BOPS, does it follow that X is an Oka manifold? That is, do we have the implication BOPS $\Rightarrow$ Oka?
(b) Do the properties BOP and BOPS coincide in the class of compact (or compact Kähler, or compact projective) manifolds?

Let us mention another question related to Theorem 1.1. Let $\mathbb{B}^n$ denote the open ball in $\mathbb{C}^n$. It is an open problem whether $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is an Oka manifold when $n > 1$.

Problem 1.5 Let $n > 1$. Does there exist a surjective holomorphic map $\mathbb{C}^n \to \mathbb{C}^n \setminus \overline{\mathbb{B}}^n$?

In this connection, we mention that Dixon and Esterle (see [6, Theorem 8.13, p. 182]) constructed for every $\epsilon > 0$ a finitely sheeted holomorphic map $f: \mathbb{C}^2 \to \mathbb{C}^2$ whose image avoids the closed unit ball $\overline{\mathbb{B}}^2$ but contains the complement of the ball of radius $1 + \epsilon$: $\mathbb{C}^2 \setminus (1 + \epsilon)\overline{\mathbb{B}}^2 \subset f(\mathbb{C}^2) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}^2$.

Theorem 1.1 shows that a negative answer to Problem 1.5 would imply that $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ fails to be Oka. Since $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is a union of Fatou-Bieberbach domains (obtained for example as attracting basins of holomorphic automorphisms of $\mathbb{C}^n$ which map the ball $\mathbb{B}^n$ into itself), this would provide an example of a strongly dominable manifold which is not Oka.

The above example is also connected to the open problem whether every Oka manifold is elliptic or subelliptic, the latter being the main known geometric conditions implying all versions of the Oka property (see Gromov [17], Forstnerič [10], and [13, Definition 5.5.11 (d) and Corollary 5.5.12]). The following implications hold for any complex manifold:

$$\text{homogeneous} \Longrightarrow \text{elliptic} \Longrightarrow \text{subelliptic} \Longrightarrow \text{Oka} \Longrightarrow \text{strongly dominable.}$$

It was shown by Andrist et al. [1] that $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is not subelliptic when $n \geq 3$. Since $\mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ is strongly dominable, at least one of the two right-most implications cannot be reversed. Therefore, the question whether $\mathbb{C}^3 \setminus \overline{\mathbb{B}}^3$ is an Oka manifold is of particular interest.

It is natural to look for an analogue of Theorem 1.1 in the algebraic category. At this time, we do not have a good notion of an algebraic Oka manifold. However, a useful geometric condition on an algebraic manifold X, which gives the approximation of certain holomorphic maps $S \to X$ from affine algebraic manifolds S by algebraic morphisms $S \to X$, is *algebraic subellipticity*; see [13, Definition 5.5.11 (e)] or Sect. 4 below. (We emphasize that all algebraic maps in this paper are understood to be morphisms, i.e., without singularities.) In Sect. 4 we prove the following result in this direction.

Theorem 1.6 *Assume that X is a compact algebraically subelliptic manifold and S is an affine algebraic manifold such that* $\dim S \geq \dim X$. *Then, every algebraic map $S \to X$ is homotopic (through algebraic maps) to a surjective strongly dominating algebraic map $S \to X$. In particular, X admits a surjective strongly dominating algebraic map $F\colon \mathbb{C}^n \to X$ with $n =$* $\dim X$.

The proof of Theorem 1.6 is based on Theorem 4.1 which is taken from [11]. It says in particular that, given an affine algebraic manifold S and an algebraically subelliptic manifold X, a holomorphic map $S \to X$ that is homotopic to an algebraic map through a family of holomorphic maps can be approximated by algebraic maps $S \to X$.

Example 1.7 Let X be an algebraic manifold of dimension n which is covered by Zariski open sets that are biregularly isomorphic to $\mathbb{C}^n$. Such manifolds are said to be of Class $\mathcal{A}_0$ (see [13, Definition 6.4.5]). Then X is algebraically subelliptic (see [13, Proposition 6.4.6]). Furthermore, the total space Y of any blow-up $Y \to X$ along a closed algebraic submanifold of X is also algebraically subelliptic according to Lárusson and Truong [23, Corollary 2]. If X (and hence Y) is compact, then Theorem 1.6 furnishes a strongly dominating morphisms $\mathbb{C}^n \to Y$. This holds for example if Y is obtained by blowing up a projective space or a Grassmanian along a compact submanifold.

In dimension 2, Theorem 1.6 says in particular that every rational smooth compact algebraic surface is a regular image of $\mathbb{C}^2$.

It was shown by Lárusson and Truong [23, Proposition 6] that every algebraically subelliptic manifold (not necessarily compact) is strongly algebraically dominable. We do not know whether the analogue of Theorem 1.6 holds if X is a *noncompact* algebraically subelliptic manifold.

2 Approximation of Maps from a Stein Manifold to an Oka Manifold on a Sequence of Stein Compacts

We denote by $\mathscr{O}(S)$ the algebra of all holomorphic functions on a complex manifold S, endowed with the compact-open topology. Recall that a compact set K in S is said to be $\mathscr{O}(S)$*-convex* if $K = \widehat{K}_{\mathscr{O}(S)}$, where the holomorphic hull of K is defined by

$$\widehat{K}_{\mathscr{O}(S)} = \big\{p \in S : |f(p)| \leq \sup_K |f| \ \ \forall f \in \mathscr{O}(S)\big\}.$$

In this section, we prove the following approximation result. In the next section, we will apply it to prove Theorem 1.1.

Theorem 2.1 *Let S be a reduced Stein space and $(K_j)_{j=1}^{\infty}$ be a sequence of compact pairwise disjoint subsets of S satisfying the following properties:*

(a) *Every compact set in S intersects at most finitely many of the sets K_j.*
(b) *The union $\cup_{j=1}^{k} K_j$ is $\mathscr{O}(S)$-convex for each $k \in \mathbb{N}$.*
(c) *Set $K = \cup_{j=1}^{\infty} K_j$. There exist a strongly plurisubharmonic exhaustion function $\rho: S \to \mathbb{R}_+ = [0, +\infty)$ and an increasing sequence $0 < a_1 < a_2 < \dots$ with $\lim_{j\to\infty} a_j = +\infty$ such that for every $j \in \mathbb{N}$ we have $K \cap \{\rho = a_j\} = \emptyset$ and*

$$\text{the compact set } M_j := \{\rho \le a_j\} \cup (K \cap \{\rho \le a_{j+1}\}) \text{ is } \mathscr{O}(S)\text{-convex.} \tag{1}$$

Let X be an Oka manifold, and let $f: S \to X$ be a continuous map which is holomorphic on a neighborhood of the set $K = \cup_{j=1}^{\infty} K_j$. Let dist *be a distance function on X inducing the manifold topology. Given a sequence $\epsilon_j > 0$ $(j \in \mathbb{N})$, there exists a holomorphic map $F: S \to X$, homotopic to f by a family of maps $F_t: S \to X$ $(t \in [0,1])$ that are holomorphic on a neighborhood of K, such that*

$$\sup_{p \in K_j} \mathrm{dist}(f(p), F(p)) < \epsilon_j \quad \textit{for all } j = 1, 2, \dots. \tag{2}$$

Furthermore, given a discrete sequence of points $(p_j)_{j\in\mathbb{N}} \subset K$ and integers $k_j \in \mathbb{N}$, we can choose F to agree with f to order k_j at p_j.

Proof We may assume that dist is a complete metric on X and that $\sum_j \epsilon_j < \infty$. Let $(a_j)_{j\in\mathbb{N}}$ be the sequence of real numbers in condition (c). Set

$$S_j := \{p \in S : \rho(p) \le a_j\}, \quad A_j := \{p \in S : a_j \le \rho(p) \le a_{j+1}\}, \quad j \in \mathbb{N}.$$

Note that S_j is compact $\mathscr{O}(S)$-convex, and we have

$$S_{j+1} = S_j \cup A_j \ \text{ and } \ M_j = S_j \cup (K \cap A_j) \ \text{ for every } j = 1, 2, \dots.$$

(Recall that $K = \cup_{j=1}^{\infty} K_j$.) For consistency of notation we also set

$$S_0 = \emptyset, \quad M_0 := K \cap S_1, \quad F_0 = f.$$

By hypothesis (c), we have that $K \cap bS_j = \emptyset$ for all $j \in \mathbb{N}$. Furthermore, condition (a) in the theorem implies that each set S_j contains at most finitely many of the sets K_i. Set

$$\eta_j := \min\{\epsilon_i : K_i \subset S_j\} > 0, \quad j = 1, 2, \dots. \tag{3}$$

To prove the theorem, we shall construct sequences of continuous maps $F_j: S \to X$, homotopies $F_{j,t}: S \to X$ $(t \in [0,1])$, and numbers $b_j, c_j > 0$ satisfying the following conditions for every $j \in \mathbb{N}$:

(i$_j$) $a_j < b_j < c_j < a_{j+1}$ and $K \cap A_j \subset \{c_j < \rho < a_{j+1}\}$.
(ii$_j$) F_j is holomorphic on $\{\rho < b_j\}$ and $F_j = F_{j-1}$ on $\{\rho \geq c_j\}$.
(iii$_j$) $\mathrm{dist}(F_j(p), F_{j-1}(p)) < 2^{-j}\eta_j$ for every $p \in M_{j-1}$.
(iv$_j$) $F_{j,0} = F_{j-1}$ and $F_{j,1} = F_j$.
(v$_j$) For every $t \in [0,1]$ the map $F_{j,t}$ is holomorphic on a neighborhood of M_{j-1} and $F_{j,t} = F_{j-1}$ holds on $\{\rho \geq c_j\}$.
(vi$_j$) $\mathrm{dist}(F_{j,t}(p), F_{j-1}(p)) < 2^{-j}\eta_j$ for every $p \in M_{j-1}$ and $t \in [0,1]$.

We could also add a suitable condition on F_j to ensure jet interpolation along a discrete sequence $(p_j) \subset K$ (see the last sentence in the theorem). Since this interpolation is a trivial addition in what follows, we shall delete it to simplify the exposition.

A sequence of maps and homotopies satisfying these properties can be constructed recursively by using [13, Theorem 5.4.4] at every step; we offer some details.

Assume that maps $F_0, F_1, \ldots, F_j$ and homotopies $F_{1,t}, \ldots, F_{j,t}$ with these properties have been found for some $j \in \mathbb{N}$. (Recall that $F_0 = f$.) In view of property (ii$_j$) the map F_j is holomorphic on the set $\{\rho < b_j\}$, and we have $F_j = F_{j-1} = \cdots = F_0$ on $\{\rho \geq c_j\}$. Since $K \cap A_j \subset \{c_j < \rho < a_{j+1}\}$ by property (i$_j$), it follows that F_j is holomorphic on a neighborhood of the set M_j (1). Since M_j is $\mathscr{O}(S)$-convex, we can apply [13, Theorem 5.4.4] to find a number $c_{j+1} > a_{j+1}$ close to a_{j+1}, a holomorphic map $F_{j+1}\colon \{\rho < c_{j+1}\} \to X$ satisfying property (iii$_{j+1}$), and a homotopy of maps $F_{j+1,t}\colon \{\rho < c_j\} \to X$ ($t \in [0,1]$) satisfying properties (iv$_{j+1}$) and (vi$_{j+1}$). It remains to extend this homotopy to all of S such that condition (v$_{j+1}$) holds as well. This is accomplished by using a cut-off function in the parameter of the homotopy. Explicitly, pick a number b_{j+1} such that $a_{j+1} < b_{j+1} < c_{j+1}$, and let $\chi\colon S \to [0,1]$ be a continuous function which equals 1 on the set $\{\rho \leq b_{j+1}\}$ and has support contained in $\{\rho < c_{j+1}\}$. The homotopy of continuous maps

$$(p,t) \longmapsto F_{j+1,\chi(p)t}(p) \in X, \quad p \in S,\ t \in [0,1]$$

then agrees with the homotopy $F_{j+1,t}$ on the set $\{\rho \leq b_{j+1}\}$ (since $\chi = 1$ there), and it agrees with the map F_j (and hence with $F_0 = f$) on $\{\rho \geq c_{j+1}\}$ since χ vanishes there. This established the condition (v$_{j+1}$) and completes the induction step.

In view of (iii$_j$) and the definition of the numbers η_j (3), the sequence $F_j\colon S \to X$ converges uniformly on compacts in S to a holomorphic map $F = \lim_{j\to\infty} F_j\colon S \to X$ satisfying the estimates (2). Furthermore, conditions (iv$_{j+1}$)–(vi$_{j+1}$) imply that the sequence of homotopies $F_{j,t}\colon S \to X$ ($j \in \mathbb{N}$) can be assembled into a homotopy $F_t\colon S \to X$ ($t \in [0,1]$) connecting $F_0 = f$ to the final holomorphic map $F_1 = F$ such that F_t is holomorphic on a neighborhood of the set K for every $t \in [0,1]$ and every map F_t in the homotopy satisfies the estimates (2). This assembling is accomplished by writing $[0,1) = \cup_{j=1}^{\infty} I_j$, where $I_j = [1 - 2^{-j+1}, 1 - 2^{-j}]$, and placing the homotopy $(F_{j,t})_{t\in[0,1]}$ onto the subinterval $I_j \subset [0,1]$ by suitably reparametrizing the t-variable. $\square$

3 Construction of Surjective Holomorphic Maps to Oka Manifolds

Proof of Theorem 1.1 Let X be a complex manifold of dimension n. Choose a countable family of compact sets $L'_j \subset L_j \subset X$ ($j \in \mathbb{N}$) satisfying the following conditions:

(i) $L'_j \subset \mathring{L}_j$ for every $j \in \mathbb{N}$.
(ii) $\cup_{j=1}^{\infty} L'_j = X$.
(iii) For every $j \in \mathbb{N}$ there are an open set $V_j \subset X$ containing L_j and a biholomorphic map $\psi_j\colon V_j \to \psi_j(V_j) \subset \mathbb{C}^n$ such that $\psi_j(L_j) = \overline{\mathbb{B}}^n$ is the closed unit ball in $\mathbb{C}^n$.

A compact set $L_j \subset X$ satisfying condition (iii) will be called a (closed) *ball* in X. If the manifold X is compact, then we can cover it by a finite family of such balls.

Let S be a Stein manifold of dimension $m = \dim S \geq n$. Choose a smooth strongly plurisubharmonic exhaustion function $\rho\colon S \to \mathbb{R}_+ = [0, +\infty)$. Pick an increasing sequence of real numbers $a_j > 0$ with $\lim_{j\to\infty} a_j = +\infty$. For each $j \in \mathbb{N}$ we choose a small $\mathscr{O}(S)$-convex ball K_j in S such that

$$K_j \subset \{p \in S\colon a_j < \rho(p) < a_{j+1}\} \tag{4}$$

and

$$\text{the compact set } M_j := K_j \cup \{\rho \leq a_j\} \text{ is } \mathscr{O}(S)\text{-convex.} \tag{5}$$

The last condition can be achieved by taking the balls K_j small enough; here is an explanation. By the assumption, there are an open set $U_j \subset S$ containing K_j and a biholomorphic coordinate map $\phi_j\colon U_j \to \phi_j(U_j) \subset \mathbb{C}^m$ such that $\phi_j(K_j) = \overline{\mathbb{B}}^m \subset \mathbb{C}^m$. In view of (4) we may assume that $\overline{U}_j \cap \{\rho \leq a_j\} = \emptyset$. Let $p_j := \phi_j^{-1}(0) \in K_j$ be the center of K_j. The compact set $\{\rho \leq a_j\} \cup \{p_j\}$ is clearly $\mathscr{O}(S)$-convex, and hence it has a basis of compact $\mathscr{O}(S)$-convex neighborhoods. In particular, there is a compact neighborhood $T \subset U_j$ of the point p_j such that $T \cup \{\rho \leq a_j\}$ is $\mathscr{O}(S)$-convex. Choose a number $0 < r_j < 1$ small enough such that $r_j\overline{\mathbb{B}}^m \subset \phi_j(T)$. The ball $K'_j := \phi_j^{-1}(r_j\overline{\mathbb{B}}^m)$ is then contained in T and is $\mathscr{O}(T)$-convex. Hence, the set $K'_j \cup \{\rho \leq a_j\}$ is $\mathscr{O}(S)$-convex. Replacing K_j by K'_j and rescaling the coordinate map ϕ_j accordingly so that it takes this set onto $\overline{\mathbb{B}}^m$, condition (5) is satisfied.

Denote by $\pi\colon \mathbb{C}^m \to \mathbb{C}^n$ the coordinate projection $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_n)$. (Recall that $m \geq n$.) Then $\pi(\overline{\mathbb{B}}^m) = \overline{\mathbb{B}}^n$. Let $U_j \supset K_j$ and $\phi_j\colon U_j \to \phi_j(U_j) \subset \mathbb{C}^m$ be as above, and let ψ_j be as in (iii). There is an open neighborhood $U'_j \subset S$ of K_j, with $U'_j \subset U_j$, such that the map

$$f_j = \psi_j^{-1} \circ \pi \circ \phi_j\colon U'_j \to V_j \subset X$$

is a well defined holomorphic submersion satisfying $f_j(K_j) = L_j$ for every $j \in \mathbb{N}$.

Since the set L'_j is contained in the interior of L_j and f_j is a submersion, there is a compact set K'_j contained in the interior of K_j such that $L'_j \subset f_j(\mathring{K}'_j)$. By Rouché's theorem we can choose $\epsilon_j > 0$ small enough such that for every holomorphic map $F: K_j \to X$ defined on a neighborhood of K_j we have that

$$\sup_{p \in K_j} \operatorname{dist}(f_j(p), F(p)) < \epsilon_j \implies L'_j \subset F(K'_j). \tag{6}$$

Let $f: S \to X$ be a continuous map. By a homotopic deformation of f, supported on a contractible neighborhood of the ball $K_j \subset S$ for each j, we can arrange that $f = f_j$ on a neighborhood of K_j for each $j \in \mathbb{N}$. The homotopy is kept fixed outside a somewhat bigger neighborhood of each K_j in S, and these neighborhoods are chosen to have pairwise disjoint closures. We denote the new map by the same letter f.

Theorem 2.1, applied to the map f and the sequences K_j and ϵ_j, furnishes a holomorphic map $F: S \to X$ that is homotopic to f and satisfies the estimate

$$\sup_{p \in K_j} \operatorname{dist}(f(p), F(p)) < \epsilon_j, \quad j = 1, 2, \ldots$$

(see (2)). By the choice of ϵ_j (6) it follows that $L'_j \subset F(K'_j)$ for each $j \in \mathbb{N}$, and hence

$$F(S) = \cup_{j=1}^{\infty} F(K'_j) = \cup_{j=1}^{\infty} L'_j = X.$$

Furthermore, if the numbers $\epsilon_j > 0$ are chosen small enough, then F has maximal rank equal to $\dim X$ at every point of K'_j (since this holds for the map f on the bigger set K_j), and hence F is strongly dominating. □

Remark 3.1 The same proof applies if S is a reduced Stein space with $\dim S \geq \dim X$. In this case, we just pick the balls K_j (4) in the regular locus of S.

4 Surjective Algebraic Maps to Compact Algebraically Subelliptic Manifolds

In this section we prove Theorem 1.6. We begin by recalling the relevant notions.

An algebraic manifold X is said to be *algebraically subelliptic* if it admits a finite family of algebraic sprays $s_j: E_j \to X$ ($j = 1, \ldots, k$), defined on total spaces E_j of algebraic vector bundles $\pi_j: E_j \to X$, which is *dominating* in the sense that for each point $x \in X$ the vector subspaces $(ds_j)_{0_x}(E_{j,x}) \subset T_x X$ span the tangent space $T_x X$:

$$(ds_1)_{0_x}(E_{1,x}) + \cdots + (ds_k)_{0_x}(E_{k,x}) = T_x X \quad \forall x \in X.$$

See [11, Definition 2.1] or [13, Definition 5.5.11 (e)] for the details. Here, X could be a projective (or quasi-projective) algebraic manifold, although the same theory applies to more general algebraic manifolds. By an *algebraic map*, we always mean an algebraic morphism without singularities.

The following result is [11, Theorem 3.1]; see also [13, Theorem 7.10.1]. As pointed out there, this is a version of the *h-Runge approximation theorem* in the algebraic category. For the analytic case of this result, see Gromov [17] and [13, Theorem 6.6.1].

Theorem 4.1 *Assume that S is an affine algebraic manifold and X is an algebraically subelliptic manifold. Given an algebraic map $f\colon S \to X$, a compact $\mathscr{O}(S)$-convex subset K of S, an open set $U \subset S$ containing K, and a homotopy $f_t\colon U \to X$ of holomorphic maps ($t \in [0,1]$) with $f_0 = f|_U$, there exists for every $\epsilon > 0$ an algebraic map $F\colon S \times \mathbb{C} \to X$ such that*

$$F(\cdot\,,0) = f \quad \textit{and} \quad \sup_{p\in K,\, t\in[0,1]} \operatorname{dist}(F(p,t), f_t(p)) < \epsilon.$$

Proof of Theorem 1.6 The proof uses Theorem 4.1 and is similar to that of Theorem 1.1. The main difference is that the initial map $f\colon S \to X$ must be algebraic. For the sake of simplicity, we present the details only in the special case when $S = \mathbb{C}^n$ with $n = \dim X$.

Fix a point $x_0 \in X$ and let $f\colon \mathbb{C}^n \to X$ be the constant map $f(z) = x_0 \in X$.

Since X is compact, there is finite family of pairs of compact sets $L'_j \subset L_j \subset X$ ($j = 1,\ldots,\ell$) satisfying properties (i)–(iii) stated at the beginning of proof of Theorem 1.1 (see Sect. 3). In particular, each set L_j is a ball in a suitable local coordinate, and we have that $\cup_{j=1}^{\ell} L'_j = X$.

Let $n = \dim X$. Choose pairwise disjoint closed balls $K_1,\ldots,K_\ell$ in $\mathbb{C}^n$ whose union $K := \cup_{j=1}^{\ell} K_j$ is polynomially convex. Let $p_j \in K_j$ denote the center of K_j. For each $j = 1,\ldots,\ell$ there are an open ball $U_j \subset \mathbb{C}^n$ containing K_j and a biholomorphic map $g_j\colon U_j \to g_j(U_j) \subset X$ such that $g_j(K_j) = L_j$. We may assume that the sets $U_1,\ldots,U_\ell$ are pairwise disjoint. By using a contraction of K_j and L_j to their respective centers, and after shrinking the neighborhoods $U_j \supset K_j$ if necessary, we can find homotopies of holomorphic maps $f_{j,t}\colon U_j \to X$ ($t \in [0,1]$, $j = 1,\ldots,\ell$) such that

$$f_{j,0} = f|_{U_j} \quad \text{and} \quad f_{j,1} = g_j \quad \text{for all } j = 1,\ldots,\ell.$$

Set $U = \cup_{j=1}^{\ell} U_j$ and denote by $f_t\colon U \to X$ the holomorphic map whose restriction to U_j agrees with $f_{j,t}$ for each $j = 1,\ldots,\ell$. Then $f_0 = f|_U$ is the constant map $U \to x_0$.

Applying Theorem 4.1 to the source manifold $S = \mathbb{C}^n$, the constant (algebraic) map $f\colon S \to x_0 \in X$, and the homotopy $\{f_t\}_{t\in[0,1]}$ furnishes an algebraic map $F\colon \mathbb{C}^n \to X$ whose restriction to K_j approximates the map g_j for each $j = 1,\ldots,\ell$. Assuming that the approximation is close enough, we see as in the proof of Theorem 1.1 that $F(\mathbb{C}^n) = X$ and that F can be chosen strongly dominating. $\square$

Remark 4.2 A major source of examples of algebraically elliptic manifolds are the *algebraically flexible* manifolds; see e.g. [22, Definition 12]. An algebraic manifold X is said to be algebraically flexible of it admits finitely many algebraic vector fields $V_1, \dots, V_N$ with complete algebraic flows $\phi_{j,t}$ ($t \in \mathbb{C}$, $j = 1, \dots, N$), such that the vectors $V_1(x), \dots, V_N(x)$ span the tangent space T_xX at every point $x \in X$. Note that every $(\phi_{j,t})_{t\in\mathbb{C}}$ is a unipotent 1-parameter group of algebraic automorphisms of X. The composition of the flows $\phi_{1,t_1} \circ \cdots \circ \phi_{N,t_N}$ is a dominating algebraic spray $X \times \mathbb{C}^N \to X$, and hence such X is algebraically elliptic.

For a survey of this subject, we refer to Kutzschebauch's paper [22].

Remark 4.3 The argument in the proof of Theorem 1.6 also applies in the holomorphic case and gives a simple proof of Theorem 1.1 when the manifold X is compact and the initial map $f: S \to X$ is assumed to be holomorphic. I wish to thank Tuyen Truong for this observation.

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Stabilized Symplectic Embeddings

Richard Hind

Abstract We survey some symplectic embedding results focussing on the case when both domain and range are products of 4-dimensional ellipsoids or polydisks with Euclidean space. The stabilized problems have additional flexibility but some 4-dimensional obstructions persist.

1 Introduction

The symplectic embedding problem is among the easiest to state in symplectic topology. Nevertheless it provides a model situation to search for boundaries between symplectic rigidity and flexibility, and to test the power of symplectic invariants.

Problem. Let U and V be open subsets of $\mathbb{R}^{2n}$. Find the infimum of $\lambda > 0$ such that there exists a symplectic embedding $f : U \hookrightarrow \lambda V$.

Saying that f is symplectic means that $f^*\omega = \omega$, where $\omega = \sum dx_i \wedge dy_i$ is the standard symplectic form on $\mathbb{R}^{2n}$.

The only classical obstruction to symplectic embeddings is volume. Note that $\frac{1}{n!}\omega^n$ is the standard volume form and so symplectic embeddings are volume preserving. Hence if there exists a symplectic embedding $U \hookrightarrow \lambda V$ then necessarily $\mathrm{vol}(U) \leq \lambda^{2n}\mathrm{vol}(V)$.

We can say that an embedding problem is flexible if this estimate is sharp. We know of rather few nontrivial examples of flexible embedding problems. On the other hand rigidity for symplectic embeddings was discovered only in 1985 by Gromov in his seminal work on the subject.

Define a ball of capacity c by

$$B^{2n}(c) = \{\sum_{i=1}^{n} \pi(x_i^2 + y_i^2) < c\}$$

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and a cylinder of capacity c by

$$Z^{2n}(c) = \{\pi(x_1^2 + y_1^2) < c\}.$$

Nonsqueezing Theorem **(Gromov, [8])** $B^{2n}(a) \hookrightarrow Z^{2n}(c)$ *if and only if* $a \leq c$.

There are now several proofs of the Nonsqueezing Theorem. Gromov's original proof applied his theory of pseudoholomorphic curves and the subsequent developments we describe here follow basically the same scheme.

It is well known in the field that pseudoholomorphic curves can be an especially useful tool when we work in dimension 4, due to positivity of intersection. This gives a topological criterion for the curves to be embedded. As a consequence, much more is known about symplectic embeddings in dimension 4. In particular Hutchings has developed a powerful set of embedding obstructions coming from his Embedded Contact Homology, ECH, see [15] for example.

In the current article we describe some first steps in extending 4-dimensional theorems to higher dimension. We will simply take a 4-dimensional problem and stabilize it by adding Euclidean factors to both the domain and range. In some cases the 4-dimensional rigidity generalizes directly to the stabilized case, but in others we will see that there is significant additional flexibility.

In Sect. 2 we describe the situation for ellipsoid embeddings and in Sect. 3 the polydisk situation. For general U and V however, the embedding problem remains broadly open, even in dimension 4.

2 Embedding Ellipsoids

For given $a_i \in (0, \infty]$ we define a symplectic ellipsoid by

$$E(a_1, \dots, a_n) = \{\sum_{i=1}^{n} \frac{\pi}{a_i}(x_i^2 + y_i^2) < 1\}.$$

Then we have $E(c, \dots, c) = B^{2n}(c)$ and $E(c, \infty, \dots, \infty) = Z^{2n}(c)$.

In this section we discuss the case when our domain U is an ellipsoid and the range V is either a ball or a stabilized ball. In dimension 4 there are also solutions when the range is a cube (due to Frenkel–Müller, [7]) or certain polydisks (due to Cristofaro-Gardiner–Frenkel–Schlenk, [4]) but in these cases only conjectures for the corresponding stabilized problems, see [10], Conjecture 1.19, and [4], Conjecture 1.4.

The solution to the problem of 4-dimensional ellipsoid embeddings into a ball can be expressed in the function

$$e_2(x) = \inf\{c > 0 | E(1, x) \hookrightarrow B^4(c)\}.$$

By rescaling and reordering the factors we may assume $x \geq 1$. The description of the function e_2 is due to McDuff and Schlenk and reveals both the beauty and intricate nature of the symplectic embedding problem.

To give their solution we need to fix some notation. First define the sequence $\{g_k\}_{k=0}^{\infty}$ where $g_0 = 1$ and g_k for $k \geq 1$ is the kth odd index Fibonacci number. Hence g_k is the sequence beginning $1, 1, 2, 5, 13, 34, \ldots$. Then we can define sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ by $a_k = (\frac{g_{k+1}}{g_k})^2$ and $b_k = \frac{g_{k+2}}{g_k}$. These are increasing sequences with $a_0 < b_0 < a_1 < b_1 < \ldots$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \tau^4$, where τ is the golden ratio.

Theorem 2.1 (McDuff–Schlenk, [17], Theorem 1.1.2) *For $1 \leq x < \tau^4$ the function $e_2(x)$ is linear on the intervals $[a_k, b_k]$ and constant on intervals $[b_k, a_{k+1}]$ with $e_2(a_k) = \sqrt{a_k}$.*

If $x \geq 8\frac{1}{36}$ then $e_2(x) = \sqrt{x}$.

In other words, the first part of the graph is an infinite staircase with ever shorter steps converging to τ^4. Meanwhile if $x \geq 8\frac{1}{36}$ the embedding problem is flexible. For brevity we have not tried to describe the graph over the interval $(\tau^4, 8\frac{1}{36})$. Here there are nine 'exotic' additional steps. It turns out that the ECH capacities give a sharp obstructions in all cases, see [16].

In McDuff and Schlenk's proof, the existence of holomorphic curves is used both to obstruct embeddings and construct the optimal embeddings. Thus the embeddings themselves are completely non-explicit. Elementary embedding constructions were the subject of the earlier book [20] of Schlenk. In terms of concrete embeddings in dimension 4 the methods Schlenk describes have not generally been improved upon (although see [18] for the volume filling embeddings $E(1, k^2) \hookrightarrow B^4(k)$ when $k \in \mathbb{N}$). These 'folded' embeddings are sufficient to read off the asymptotic behavior $\lim_{x\to\infty} \frac{e_2(x)}{\sqrt{x}} = 1$, although for $x > 2$ (that is, when the inclusion map is not optimal) they never reproduce the embeddings established in Theorem 2.1.

Given an $n \geq 3$ the stabilized embedding function is given by

$$e_n(x) = \inf\{c > 0 | E(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}\}.$$

By taking product embeddings we see immediately that $e_n(x) \leq e_2(x)$ for all x, n and it would be natural to guess that we always have equality. This notion was disproved in a remarkable paper of Guth [9] where an explicit construction demonstrated the extra flexibility present in higher dimension. This construction was improved by the author in [10] using folding methods as in [20] to obtain the following.

Theorem 2.2 (Hind, [10], Pelayo–Ngọc, [19]) $E(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}$ *whenever $c > \frac{3x}{x+1}$.*

The paper [10] dealt only with compact subsets. To obtain embeddings of all of $E(1, x) \times \mathbb{R}^{2(n-2)}$ we use the technique from Theorem 4.3 in [19].

The graph of $\frac{3x}{x+1}$ intersects $\sqrt{x}$ precisely at $x = \tau^4$, where it also coincides with $e_2(x)$. It follows that $e_n(x) < e_2(x)$ for all $x > \tau^4$ and $n \geq 3$.

As a possible first step in extending the ECH capacities to higher dimension, Cristofaro-Gardiner established the following in collaboration with the author, showing that when $x \le \tau^4$ symplectic rigidity persists in the stabilized case.

Theorem 2.3 (Cristofaro-Gardiner–Hind, [3]) *If $x \le \tau^4$ then $e_n(x) = e_2(x)$.*

While the holomorphic curves giving the 4-dimensional ECH obstructions remain useful in higher dimensions when $x < \tau^4$, the graph of $e_n(x)$ when $x > \tau^4$ remains mysterious. Kerman in collaboration with the author has shown that the folding construction is sharp at least asymptotically.

Theorem 2.4 (Hind–Kerman, [11, 12]) *For all $n \ge 3$, $\lim_{x\to\infty} e_n(x) = 3$.*

At the other end of the scale there is a sequence of points converging to τ^4 from above (called 'ghost stairs') at which the folded embedding from Theorem 2.2 is again sharp. To describe these points let $\{h_k\}_{k=1}^{\infty}$ be the even index Fibonacci numbers, that is, the sequence beginning $1, 3, 8, 21 \ldots$. Then let $x_k = \frac{h_{2k+3}}{h_{2k+1}}$ for $k \ge 0$.

Theorem 2.5 (Cristofaro-Gardiner–Hind–McDuff, [5]) *$e_n(x_k) = \frac{3x_k}{x_k+1}$ for all $k \ge 0$, $n \ge 3$.*

These are labelled ghost stairs because they give a staircase of obstructions which originally appeared in the paper [17] of McDuff and Schlenk, but in dimension 4 the obstructions are not sharp and so do not appear in the graph of e_2. In higher dimension we do not know if the x_k are the tips of a staircase in the graph of e_n, or alternatively if $e_n(x) = \frac{3x}{x+1}$ for all $n \ge 3$ and $x \ge \tau^4$.

3 Embedding Polydisks

For given $a_i \in (0, \infty]$ we define a symplectic polydisk by

$$P(a_1, \ldots, a_n) = \{\pi(x_i^2 + y_i^2) < a_i \text{ for all } i\}.$$

In dimension 4 a solution to the embedding problem for polydisks into a ball amounts to describing the function

$$p_2(x) = \inf\{c > 0 | P(1, x) \hookrightarrow B^4(c)\}$$

for $x \ge 1$. The techniques from [17] do not apply to this case, and indeed the only embedding constructions available come from Schlenk's book [20]. The known theorems show that at least for small x folding cannot be improved.

Theorem 3.1 (Hind–Lisi, [13], Hutchings, [15], Christianson–Nelson, [2]) *If $1 \le x \le 2$ then $p_2(x) = 1 + x$.*

If $2 \le x \le \frac{\sqrt{7}-1}{\sqrt{7}-2}$ then $p_2(x) = 2 + \frac{x}{2}$.

To be precise, Hind–Lisi established the case $x = 2$, Hutchings dealt with all $x \leq 12/5$ and Christianson–Nelson completed the theorem as stated.

Although the 4-dimensional case remains incomplete, we can nevertheless write down the stabilized function

$$p_n(x) = \inf\{c > 0 | P(1, x) \times \mathbb{R}^{2(n-2)} \hookrightarrow B^4(c) \times \mathbb{R}^{2(n-2)}\}.$$

Surprisingly, we can say a lot about $p_3(x)$.

Theorem 3.2 *For $x \geq 2$ we have $p_3(x) = 3$.*

We conclude with an outline of a proof of this assuming some familiarity with pseudoholomorphic curves, and in particular finite energy curves, see [6] and [1]. Detailed results about Lagrangian submanifolds will appear in a joint paper with Opshtein, [14].

First note that it suffices to prove that $p_3(2) = 3$. Indeed, Theorem 2.4 implies that $\lim_{x\to\infty} p_n(x) = 3$ and the p_n are clearly nondecreasing.

Arguing by contradiction, suppose that there exists a symplectic embedding $P(1, 2, S) \hookrightarrow B^4(c) \times \mathbb{R}^2$ for an S extremely large and $c < 3$. We will identify $P(1, 2, S)$ with its image under this embedding.

Now fix a large d, such that $3d - 1 > dc$ but still with $d << S$ and an $\epsilon > 0$ such that $d\epsilon$ is small. Then we define an open subset $U = U(1, 2, S, \epsilon) \subset P(1, 2, S)$ by

$$\{1 - \epsilon < \pi(x_1^2 + y_1^2) < 1, 2 - (3d - 1)\epsilon < \pi(x_2^2 + y_2^2) < 2, \frac{S}{2} < \pi(x_3^2 + y_3^2) < S\}.$$

We can think of U as a tubular neighborhood of a Lagrangian torus, say

$$L = \{\pi(x_1^2 + y_1^2) = 1 - \frac{\epsilon}{2}, \pi(x_2^2 + y_2^2) = 2 - \frac{3d - 1}{2}\epsilon, \pi(x_3^2 + y_3^2) = \frac{3S}{4}\}.$$

In fact U admits a symplectic embedding into T^*T^3 taking L to the zero-section.

For $\delta_2, \delta_3 < \delta_1 << \epsilon$ very small, there is another symplectic embedding $E = E(\epsilon - \delta_1, (3d - 1)(\epsilon - \delta_2), (3d - 1)(\epsilon - \delta_3)) \hookrightarrow U$ which extends to the closure of the ellipsoid. Composing the two we get

$$E \hookrightarrow U \hookrightarrow B^4(c) \times \mathbb{R}^2 \subset \mathbb{C}P^2(c) \times \mathbb{R}^2$$

where the last inclusion is a standard compactification of the ball factor, so we are adding an $L_\infty = l_\infty \times \mathbb{R}^2$ with l_∞ the line at infinity in $\mathbb{C}P^2$.

To study holomorphic curves we must fix an almost-complex structure on $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus E$ with a cylindrical end on ∂E as in [1], Section 3 (identifying E with its image as usual). To control the projection of holomorphic curves to the $\mathbb{R}^2$ factor we also assume that our almost-complex structures are equal to a standard product structure on a fixed region $\{x_3^2 + y_3^2 > R\}$. We can then study finite energy curves asymptotic to Reeb orbits on ∂E. For definitions and basic properties see [1], Section 6. The Reeb orbits here are closed loops in ∂E tangent to $\ker \omega|_{\partial E}$, and for suitable δ_i these will be exactly covers of the $\gamma_i = \partial E \cap \{x_j = y_j = 0 \,\text{for}\, j \neq i\}$. Denote by γ_i^r the r-fold cover.

The analysis in [11] and [3] implies that for a generic almost-complex structure and sequence of $d \to \infty$ there exist finite energy planes asymptotic to γ_1^{3d-1} and intersecting L_∞ exactly d times, counting with multiplicity. We may assume our d lies in this sequence, so the planes exist and have area $dc - (3d-1)(\epsilon - \delta_1)$. (For the area formula note that if we compactify our planes by adding a $(3d-1)$ times cover of the (x_1, y_1) plane inside E, then the curves will project to spheres of degree d in $\mathbb{C}P^2(c)$.)

Our goal is to take a limit of such finite energy planes as we perform a neck stretching (as in [1], Section 3.4) along a smoothing Σ of ∂U. Reeb orbits on Σ appear in 2-dimensional families indexed by $(k, l, m) \in \mathbb{Z}^3 \setminus \{0\}$ describing the homology class when we project the orbit in T^*T^3 to the zero-section. That is, k gives the winding about $\{0\}$ in the (x_1, y_1)-plane and so on.

The compactness theorem in [1] describes the limit as a holomorphic building, that is, a collection of finite energy holomorphic curves in completions of $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$ and $U \setminus E$ with matching asymptotic limits along Σ. (We should really also include curves mapping to the symplectization $\mathbb{R} \times \Sigma$ in our discussion, however these do not affect the argument.) An analysis as in [14] implies that the limit consists of finite energy planes in $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$ with negative ends asymptotic to Reeb orbits on Σ and a single finite energy curve in $U \setminus E$ with a number of positive ends on Σ but a single negative end asymptotic to γ_1^{3d-1} on ∂E. A potential limiting building with degree $d = 2$ is illustrated in Fig. 1.

Identifying U with a subset of T^*T^3 we see that the symplectic form is exact and moreover has a primitive whose integral over a Reeb orbit of Σ in the class (k, l, m) is given (in an arbitrarily large range, and up to a small correction due to the smoothing) by

$$\mathcal{A}(k, l, m) = \frac{\epsilon}{2}|k| + \frac{(3d-1)\epsilon}{2}|l| + \frac{S}{4}|m|.$$

Now as $d << S$ and our curves have area of order dc, we see that the limiting component in U cannot have any positive ends asymptotic to Reeb orbits with $m \neq 0$.

Given this we investigate the planes in $(\mathbb{C}P^2(c) \times \mathbb{R}^2) \setminus U$. If the plane has intersection number g with L_∞ and is asymptotic to a Reeb orbit in the class $(k, l, 0)$ then the deformation index given by

$$\text{index} = 6g - 2(k + l).$$

Again following [14] we can show that for a generic almost-complex structure all limiting planes have index either 0 or 2 (the index is necessarily even). The idea behind this is that genericity of our almost-complex structure can be used to exclude curves of negative index. Thus the planes have nonnegative index, but if their index exceeds the dimension of the asymptotic family of Reeb orbits then other curves in

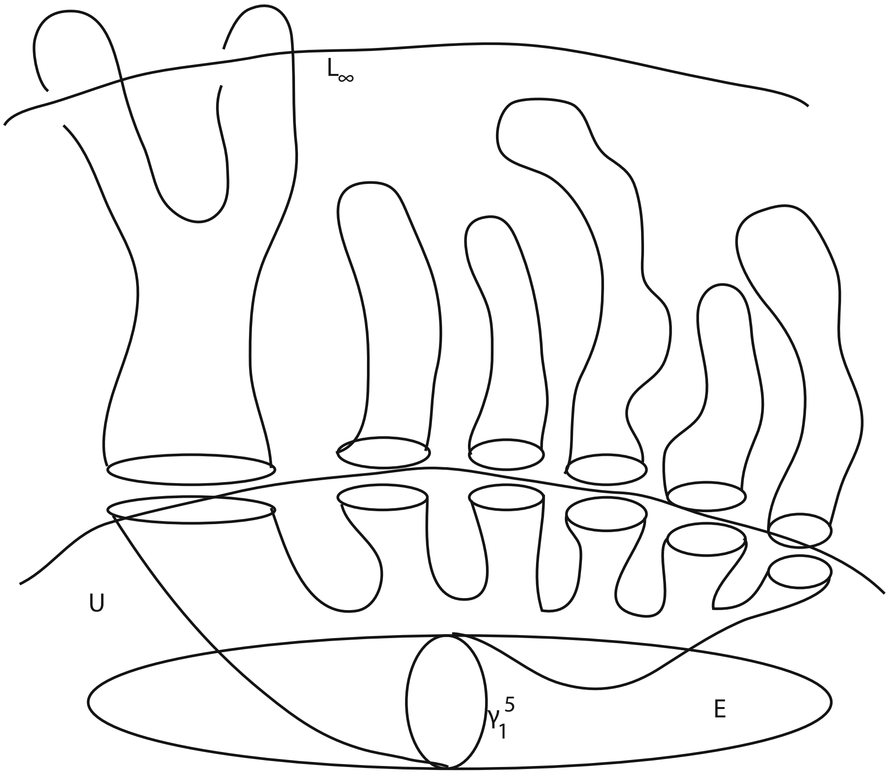

Fig. 1 A limiting building with $d = 2$

the building will be forced to have negative index. Index 0 planes are rigid but for those of index 2 the asymptotic limit will vary in the moduli space.

We also observe here the consequence that if our plane is asymptotic to an orbit of class $(k, 0, 0)$ with $k < 0$ then in fact we must have $k = -1$, the index must be 2, and the intersection number $g = 0$.

The symplectic area of such a plane is given by the area of a disk with boundary on L up to a correction of order ϵ,

$$\text{area} = cg - (k + 2l) + O(\epsilon) = (c - 3)g + (3g - k - l) - l + O(\epsilon).$$

Increasing c if necessary the area of our planes can be bounded above 0 by a constant independent of ϵ. On the other hand the first term in this formula is nonpositive, and by our index calculation the second term is either 0 or 1. Therefore we must have $l \leq 0$. But the asymptotic limits of our finite energy planes bound a cycle in U, and hence the sum of their homology classes is 0. Hence, all planes are asymptotic to orbits in classes $(k, 0, 0)$, and by the matching conditions for curves in a holomorphic building all positive asymptotic limits of our curve in U are also of this type.

Stokes' theorem now gives the area of our curve in U as

$$\frac{\epsilon}{2}\sum |k_i| - (3d-1)(\epsilon - \delta_1)$$

where the sum is over the covering degrees of the positive limits. Again since these limits bound a cycle we have $\sum_{k_i>0}|k_i| = \sum_{k_i<0}|k_i|$ and so

$$\sum_{k_i<0} |k_i| \geq (3d-1)(1-\frac{\delta_1}{\epsilon}).$$

We can take δ_1 arbitrarily small, so $\sum_{k_i<0}|k_i| \geq 3d-1$ and by the observation above our limiting building must contain $3d-1$ planes asymptotic to orbits of class $(-1,0,0)$ and each having area $1+O(\epsilon)$. The total area of the limit is equal to the symplectic area of our initial planes. Therefore we get

$$dc - (3d-1)(\epsilon - \delta_1) \geq (3d-1)(1+O(\epsilon)).$$

As $dc < 3d-1$, when ϵ is sufficiently small this gives a contradiction.

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On the Obstruction of the Deformation Theory in the DGLA of Graded Derivations

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Abstract In a recent paper, the authors studied the deformation theory in the DGLA of graded derivations $\mathcal{D}^*(M)$ of differential forms on M. They proved the existence of canonical solutions e_Φ of Maurer-Cartan equation depending on a vector valued differential form Φ and gave a classification of these canonical solutions by their type: a canonical solution e_Φ is of finite type r if $\Phi^r[\Phi,\Phi]_{\mathcal{FN}} = 0$ and $r = \min\left\{j \in \mathbb{N} : \Phi^j[\Phi,\Phi]_{\mathcal{FN}} = 0\right\}$, where $[\cdot,\cdot]_{\mathcal{FN}}$ is the Frölicher-Nijenhuis bracket. In this paper it is shown that the deformation theory in the DGLA of graded derivations is not obstructed, but it is level-wise obstructed.

1 Introduction

In the papers [1, 2] the authors studied deformations of Levi-flat hypersurfaces in complex manifolds and the deformations of Levi-flat structures in smooth manifolds. As starting point, they developed a theory of deformations of integrable distributions of codimension 1 in smooth manifolds, by defining a DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ associated to a codimension 1 foliation on a co-oriented manifold L as a subalgebra of the algebra $(\Lambda^*(L), \delta, \{\cdot,\cdot\})$ of differential forms on L. Its definition depends on the choice of a DGLA defining couple (γ, X), where γ is a 1-differential form on L and X is a vector field on L such that $\gamma(X) = 1$, but the cohomology classes of the underlying differential vector space structure do not depend on this choice. The deformations are given by forms in $\mathcal{Z}^1(L)$ verifying the Maurer-Cartan equation and the moduli space takes in account the diffeomorphic deformations. The tangent cone at the origin of the moduli space is the collection of

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the cohomology classes in $H^1(\mathcal{Z}^*(L), \delta)$ of the tangent vectors at 0 to the curves in the set of solutions of the Maurer-Cartan equation.

In [5] Kodaira and Spencer developed a theory of deformations of the multifoliate structures which are more general than the foliations (see Remark 14 of [2] for a discussion). They defined a DGLA structure $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ on the graded algebra of graded derivations introduced by Frölicher and Nijenhuis in [4] and the deformations of the multifoliate structures are related to the solutions of the Maurer-Cartan equation in this algebra. The graded derivations of $\mathcal{D}^*(M)$ are given by means of vector valued differential forms on M: every $D \in \mathcal{D}^k(M)$ has a unique decomposition $D = \mathcal{L}(D) + \mathcal{I}(D)$, where $\mathcal{L}(D) = \mathcal{L}_\Phi$ is a derivation of type d_* and $\mathcal{I}(D) = \mathcal{I}_\Psi$ is a derivation of type i_* with $\Phi \in \Lambda^k M \otimes TM$, $\Psi \in \Lambda^{k+1} M \otimes TM$ (see the first paragraph for precise definitions).

In [3], the authors studied the deformation theory of integrable distributions of arbitrary codimension on smooth manifolds, by solving the Maurer-Cartan equation in the DGLA of graded derivations. The canonical solutions of the Maurer-Cartan equation in this DGLA are obtained by means of deformations of the d-operator depending on a vector valued differential 1-form Φ. For every $\Phi \in \Lambda^1 M \otimes TM$ such that $Id_{TM} + \Phi$ is invertible, there exists a unique canonical solution e_Φ of the Maurer-Cartan equation such that $\mathcal{L}(e_\Phi) = \mathcal{L}_\Phi$. The canonical solutions of the Maurer-Cartan equation are classified on their type: a canonical solution of the Maurer-Cartan equation associated to an endomorphism Φ is of finite type r if there exists $r \in \mathbb{N}$ such that $\Phi^r[\Phi, \Phi]_{\mathcal{FN}} = 0$ and r is minimal with this property, where $[\cdot,\cdot]_{\mathcal{FN}}$ is the Frölicher-Nijenhuis bracket.

As a general fact, if v is a solution of the Maurer-Cartan equation in a DGLA $(V^*, d, [\cdot,\cdot])$, the deformations of v are given by the solutions of the Maurer-Cartan equation for the derivation $d_v = d + [v,\cdot]$ (see Lemma 1 of [2]).

Denote $\mathfrak{MC}(M)$ (respectively $\mathfrak{MC}_k(M)$) the canonical solutions (respectively the canonical solutions of type k) of the Maurer-Cartan equation in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$. We say that the deformation theory in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ is not obstructed (respectively k-not obstructed) if for every $\Phi \in \Lambda^1 M \otimes TM$ such that $Id_{TM} + \Phi$ is invertible, and every $\Psi \in \Lambda^1 M \otimes TM$ there exists a smooth $\mathfrak{MC}(M)$-valued curve (respectively a $\mathfrak{MC}_k(M)$-valued curve) γ through e_Φ such that $\mathcal{L}(\gamma'(e_\Phi)) = \mathcal{L}_\Psi$.

In this paper we study the obstructedness of the deformation theory in the DGLA of graded derivations $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$. We show that the deformation theory in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ is not obstructed but it is level-wise obstructed.

The results of this paper were obtained during the spring of 2015, when the first author visited the Mathematics Institute of Jussieu and spring 2016 when the second author visited the Fibonacci Laboratory Pisa. Unfortunately, Paolo de Bartolomeis passed away on November 29th, 2016, while we were planning to write this paper. The second author would like to express his gratitude to the hosting institutions.

2 The *DGLA* of Graded Derivations

In this paragraph we recall some definitions and properties of the DGLA of graded derivations from [4], [5] (see also [6]).

Notation 1 *Let M be a smooth manifold. We denote by $\Lambda^* M$ the algebra of differential forms on M, by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M and by $\Lambda^* M \otimes TM$ the algebra of TM-valued differential forms on M, where TM is the tangent bundle to M. In the sequel, we will identify $\Lambda^1 M \otimes TM$ with the algebra $End(TM)$ of endomorphisms of TM by their canonical isomorphism: for $\sigma \in \Lambda^1 M$, $X, Y \in \mathfrak{X}(M)$, $(\sigma \otimes X)(Y) = \sigma(Y) X$.*

Definition 1 A differential graded Lie algebra (DGLA) is a triple $(V^*, d, [\cdot,\cdot])$ such that:

(1) $V^* = \oplus_{i \in \mathbb{N}} V^i$, where $\left(V^i\right)_{i \in \mathbb{N}}$ is a family of $\mathbb{C}$-vector spaces and $d : V^* \to V^*$ is a graded homomorphism such that $d^2 = 0$. An element $a \in V^k$ is said to be homogeneous of degree $k = \deg a$.

(2) $[\cdot,\cdot] : V^* \times V^* \to V^*$defines a structure of graded Lie algebra i.e. for homogeneous elements we have

$$[a, b] = -(-1)^{\deg a \deg b} [b, a]$$

and

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg a \deg b} [b, [a, c]]$$

(3) d is compatible with the graded Lie algebra structure i.e.

$$d[a, b] = [da, b] + (-1)^{\deg a} [a, db].$$

Definition 2 Let $(V^*, d, [\cdot,\cdot])$ be a DGLA and $v \in V^1$. We say that v verifies the Maurer-Cartan equation in $(V^*, d, [\cdot,\cdot])$ if

$$dv + \frac{1}{2}[v, v] = 0.$$

Definition 3 Let $A = \oplus_{k \in \mathbb{Z}} A_k$ be a graded algebra. A linear mapping $D : A \to A$ is called a graded derivation of degree $p = |D|$ if $D : A_k \to A_{k+p}$ and $D(ab) = D(a) b + (-1)^{p \deg a} a D(b)$.

Definition 4 Let M be a smooth manifold. We denote by $\mathcal{D}^*(M)$ the graded algebra of graded derivations of $\Lambda^* M$.

Definition 5 Let P, Q be homogeneous elements of degree $|P|$, $|Q|$ of $\mathcal{D}^*(M)$. We define

$$[P, Q] = PQ - (-1)^{|P||Q|} QP,$$

$$\daleth P = [d, P].$$

Lemma 1 *Let M be a smooth manifold. Then $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ is a DGLA.*

Definition 6 Let $\alpha \in \Lambda^* M$ and $X \in \mathfrak{X}(M)$.

(1) Let $\sigma \in \Lambda^* M$. We define $\mathcal{L}_{\alpha\otimes X}, \mathcal{I}_{\alpha\otimes X}$ by

$$\mathcal{L}_{\alpha\otimes X}\sigma = \alpha \wedge \mathcal{L}_X \sigma + (-1)^{|\alpha|} d\alpha \wedge \iota_X \sigma, \tag{1}$$

$$\mathcal{I}_{\alpha\otimes X}\sigma = \alpha \wedge \iota_X \sigma, \tag{2}$$

where $\mathcal{L}_X$ is the Lie derivative and ι_X the contraction by X.
For $\Phi \in \Lambda^* M \otimes TM$ we define $\mathcal{L}_\Phi \sigma$, $\mathcal{I}_\Phi \sigma$ as linear extensions of (1), (2).

(2) Let $\sigma \in \Lambda^* M$ and $Y \in \mathfrak{X}(M)$. We define

$$\mathcal{L}_{\alpha\otimes X}(\sigma \otimes Y) = \mathcal{L}_{\alpha\otimes X}\sigma \otimes Y, \tag{3}$$

$$\mathcal{I}_{\alpha\otimes X}\sigma = \mathcal{I}_{\alpha\otimes X}\sigma \otimes Y \tag{4}$$

For $\Phi, \Psi \in \Lambda^* M \otimes TM$ we define $\mathcal{L}_\Phi \Psi$, $\mathcal{I}_\Phi \Psi$ as linear extensions of (3), (4).

Lemma 2 *For every $\Phi \in \Lambda^k M \otimes TM$, $\mathcal{L}_\Phi, \mathcal{I}_\Phi \in \mathcal{D}^*(M)$, $|\mathcal{L}_\Phi| = k$, $|\mathcal{I}_\Phi| = k - 1$.*

Notation 2

$$\mathcal{L}(M) = \{\mathcal{L}_\Phi : \ \Phi \in \Lambda^* M \otimes TM\}, \ \mathcal{I}(M) = \{\mathcal{I}_\Phi : \ \Phi \in \Lambda^* M \otimes TM\}.$$

In [4] the graded derivations of $\mathcal{L}(M)$ (respectively of $\mathcal{I}(M)$) are called of type d_* (respectively of type ι_*).

Remark 1 The mapping $\mathcal{L} : \Lambda^* M \otimes TM \to \mathcal{D}^*(M)$ defined by $\mathcal{L}(\Phi) = \mathcal{L}_\Phi$ is an injective morphism of graded Lie algebras.

Lemma 3

(1) For every $D \in \mathcal{D}^k(M)$ there exist unique forms $\Phi \in \Lambda^k M \otimes TM$, $\Psi \in \Lambda^{k+1} M \otimes TM$ such that

$$D = \mathcal{L}_\Phi + \mathcal{I}_\Psi,$$

so

$$\mathcal{D}^*(M) = \mathcal{L}(M) \oplus \mathcal{I}(M).$$

We denote $\mathcal{L}_\Phi = \mathcal{L}(D)$ and $\mathcal{I}_\Psi = \mathcal{I}(D)$

(2) For every $\Phi \in \Lambda^* M \otimes TM$

$$\daleth (-1)^{|\Phi|} \mathcal{I}_\Phi = [\mathcal{I}_\Phi, d] = \mathcal{L}_\Phi.$$

(3)

$$\mathcal{L}(M) = \ker \daleth.$$

Remark 2 $d \in \mathcal{D}^1(M)$ and

$$d = \mathcal{L}_{Id_{T(M)}} = -\daleth \mathcal{I}_{Id_{T(M)}}.$$

By Lemma 3 and the Jacobi identity, for every $\Phi \in \Lambda^p M \otimes TM$, $\Psi \in \Lambda^q M \otimes TM$ we have

$$\begin{aligned} \daleth([\mathcal{L}_\Phi, \mathcal{L}_\Psi]) &= [d, [\mathcal{L}_\Phi, \mathcal{L}_\Psi]] = [[d, \mathcal{L}_\Phi], \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, [d, \mathcal{L}_\Psi]] \\ &= [\daleth \mathcal{L}_\Phi, \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, \daleth \mathcal{L}_\Phi] = 0, \end{aligned}$$

so there exists a unique form $[\Phi, \Psi] \in \Lambda^{p+q} M \otimes TM$ such that

$$[\mathcal{L}_\Phi, \mathcal{L}_\Psi] = \mathcal{L}_{[\Phi,\Psi]}. \tag{5}$$

This gives the following

Definition 7 Let $\Phi, \Psi \in \Lambda^* M \otimes TM$. The Frölicher-Nijenhuis bracket of Φ and Ψ is the unique form $[\Phi, \Psi]_{\mathcal{FN}} \in \Lambda^* M \otimes TM$ verifying (5).

Proposition 1 *Let* $\Phi, \Psi \in \Lambda^* M \otimes TM$*. Then*

$$[\mathcal{L}_\Phi, \mathcal{I}_\Psi] = \mathcal{I}_{[\Phi,\Psi]_{\mathcal{FN}}} - (-1)^{|\Phi|(|\Psi|+1)} \mathcal{L}_{\mathcal{I}_\Psi \Phi}$$

$$[\mathcal{I}_\Phi, \mathcal{I}_\Psi] = \mathcal{I}_{\mathcal{I}_\Phi \Psi - (-1)^{|\Phi|(|\Psi|+1)} \mathcal{I}_\Psi \Phi}.$$

3 Canonical Solutions of Finite Type of Maurer-Cartan Equation

In this paragraph we recall some definitions and results from [3].

Definition 8 Let $\Phi \in \Lambda^1 M \otimes TM$.

(a) Let $\sigma \in \Lambda^p M$. We define $\Phi\sigma \in \Lambda^p M$ by $\Phi\sigma = \sigma$ if $p = 0$ and

$$(\Phi\sigma)(V_1, \cdots, V_p) = \sigma(\Phi V_1, \cdots, \Phi V_p) \ \textit{if}\ p \geqslant 1,\ V_1, \cdots, V_p \in \mathfrak{X}(M).$$

(b) Let $\Psi \in \Lambda^p M \otimes TM$. We define $\Phi\Psi \in \Lambda^p M \otimes TM$ by $\Phi\Psi = \Psi$ if $p = 0$ and

$$\Phi\Psi\left(V_1, \ldots, V_p\right) = \Phi\left(\Psi\left(V_1, \ldots, V_p\right)\right), \; V_1, \ldots, V_p \in \mathfrak{X}(M) \; \textit{if } p \geqslant 1.$$

Lemma 4 *Let* $\Phi \in \Lambda^1 M \otimes TM$, $\Psi \in \Lambda^2 M \otimes TM$. *Then*

$$\mathcal{I}_\Psi \Phi = \Phi\Psi.$$

Notation 3 *Let* $\Phi \in \Lambda^1 M \otimes TM$ *such that* $R_\Phi = Id_{TM} + \Phi$ *is invertible. Set*

$$d_\Phi = R_\Phi d R_\Phi^{-1},$$

$$e_\Phi = d_\Phi - d$$

and

$$b(\Phi) = -\frac{1}{2} R_\Phi^{-1} [\Phi, \Phi]_{\mathcal{FN}}.$$

Theorem 1 *Let* $\Phi \in \Lambda^1 M \otimes TM$ *such that* $R_\Phi = Id_{T(M)} + \Phi$ *is invertible. Then:*

(i) e_Φ *is a solution of the Maurer-Cartan equation in* $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$.
(ii)

$$e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{b(\Phi)}.$$

Theorem 2 *Let* D *be a solution of the Maurer-Cartan equation in* $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$. *Let* $\Phi \in \Lambda^1 M \otimes TM$ *such that* $\mathcal{L}(D) = \mathcal{L}_\Phi$. *Suppose that* $R_\Phi = Id_{T(M)} + \Phi$ *is invertible. Then* $D = e_\Phi$.

Notation 4 *We denote by* $\aleph : \mathcal{D}^*(M) \to \mathcal{D}^*(M)$ *the mapping defined by*

$$\aleph(D) = (-1)^{|D|} \mathcal{I}(D)$$

Remark 3

$$Id = \daleth\aleph + \aleph\daleth.$$

Theorem 3 *Let* $\Phi \in \Lambda^1 M \otimes TM$ *such that* $Id_{T(M)} + \Phi$ *is invertible and* $\left(Id_{T(M)} + \Phi\right)^{-1} = \sum_{h=0}^{\infty} (-1)^h \Phi^h$. *Let* e_Φ *be the canonical solution of Maurer-Cartan equation associated to* Φ. *Then*

(a) $e_\Phi = \sum_{k=1}^{\infty} \gamma_k$, *where* $\gamma_k \in \mathcal{D}^1(M)$ *are defined by induction as*

$$\gamma_1 = \mathcal{L}_\Phi, \; \gamma_k = -(-1)^k \frac{1}{2} \sum_{(p,q)\in\mathbb{N}^*,\; p+q=k} \aleph\left(\left[\gamma_p, \gamma_q\right]\right), \; k \geqslant 2.$$

(b)

$$\gamma_k = (-1)^{k+1}\frac{1}{2}\mathcal{I}_{\Phi^{k-2}[\Phi,\Phi]_{\mathcal{FN}}},\ k \geqslant 2.$$

(c)

$$\mathcal{I}_{b(\Phi)} = \sum_{k=2}^{\infty}\gamma_k.$$

Proposition 2

(1) Let $\Phi \in \Lambda^1 M \otimes TM$ and $e_\Phi = \mathcal{L}_\Phi$. Then e_Φ is a solution of the Maurer-Cartan equation in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ if and only if $[\Phi,\Phi]_{\mathcal{FN}} = 0$.

(2) Let $\Phi \in \Lambda^1 M \otimes TM$ and $e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{\frac{1}{2}\sum_{k=0}^{r-1}(-1)^{k+1}\Phi^k[\Phi,\Phi]_{\mathcal{FN}}}$, $r \geqslant 1$. Then e_Φ is a solution of Maurer Cartan equation in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ if and only if $\Phi^r[\Phi,\Phi]_{\mathcal{FN}} = 0$.

Definition 9 Let $\Phi \in \Lambda^1 M \otimes TM$ having the property that there exists $s \in \mathbb{N}$ such that $\Phi^s[\Phi,\Phi]_{\mathcal{FN}} = 0$. We say that the canonical solution of Maurer-Cartan equation $e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{\frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\Phi^k[\Phi,\Phi]_{\mathcal{FN}}}$ associated to Φ is of finite type. We say that e_Φ is of finite type r if $r = \min\{s \in \mathbb{N} :\ \Phi^s[\Phi,\Phi]_{\mathcal{FN}} = 0\}$.

4 Unobstructedness of Deformation Theory in the DGLA of Graded Derivations

Notation 5

1. *We denote $\mathfrak{MC}(M) = \left\{D \in \mathcal{D}^1(M) : \daleth D + \frac{1}{2}[D,D] = 0\right\}$.*
2. *Let $\mathcal{A}$ be a subset of $\mathfrak{MC}(M)$ and $D \in \mathcal{A}$. An $\mathcal{A}$-valued smooth curve through D is a smooth map $\gamma :]-\varepsilon,\varepsilon[\to \mathcal{A}$ such that $\gamma(0) = D$. Then $\gamma'(0) \in \mathcal{D}^1(M)$ is a tangent vector at D to the curve γ. We denote by $T_D(\mathcal{A})$ the collection of tangent vectors at D of $\mathcal{A}$-valued smooth curves.*

Remark 4 Let $\Phi \in \Lambda^1 M \otimes TM$ such that $R_\Phi = Id_{T(M)} + \Phi$ is invertible and let $\widehat{\sigma}$ be a smooth $\mathfrak{MC}(M)$-valued curve through e_Φ. Set $\mathcal{L}(\widehat{\sigma}(t)) = \mathcal{L}_{\sigma(t)}$. Since $\mathcal{L}(\widehat{\sigma}(0)) = \mathcal{L}_\Phi$ and R_Φ is invertible, $R_{\sigma(t)}$ is invertible for every t small enough and by Theorem 1 it follows that $\widehat{\sigma}(t) = e_{\sigma(t)}$, with σ a $\Lambda^1 M \otimes TM$-valued curve such that $\sigma(0) = \Phi$.

Definition 10 We say that the deformation theory in $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ is not obstructed at e_Φ if for every $\Psi \in \Lambda^1 M \otimes TM$ there exists $\alpha \in T_{e_\Phi}(\mathfrak{MC}(M))$ such that $\mathcal{L}(\alpha) = \mathcal{L}_\Psi$.

Theorem 4 *Let $\Phi \in \Lambda^1 M \otimes TM$ such that $R_\Phi = Id_{T(M)} + \Phi$ is invertible. Then*

$$T_{e_\Phi}(\mathfrak{MC}(M)) = \left\{\mathcal{L}_\Psi + \mathcal{I}_{\omega(\Psi)} : \Psi \in \Lambda^1 M \otimes TM\right\}.$$

where

$$\omega(\Psi) = -\frac{1}{2} R_\Phi^{-1} \Psi R_\Phi^{-1} + R_\Phi^{-1} [\Phi, \Psi]_{\mathcal{FN}}.$$

Proof Let $t \mapsto e_{\sigma(t)}$ a $\mathfrak{MC}(M)$-valued curve through e_Φ, with $\sigma(t) = \Phi + t\Psi + o(t)$, $\Psi \in \Lambda^1 M \otimes TM$. We have

$$e_{\sigma(t)} = \mathcal{L}_{\sigma(t)} + \mathcal{I}_{b(\sigma(t))} = \mathcal{L}_{\sigma(t)} + \mathcal{I}_{-\frac{1}{2} R_{\sigma(t)}^{-1} [\sigma(t), \sigma(t)]_{\mathcal{FN}}}$$

so

$$\frac{de_{\sigma(t)}}{dt}_{|t=0} = \mathcal{L}_\Psi + \mathcal{I}_{-\frac{1}{2}\frac{d}{dt}\left(R_{\sigma(t)}^{-1}[\sigma(t),\sigma(t)]_{\mathcal{FN}}\right)_{|t=0}}$$

Since

$$\frac{d}{dt}\left(R_{\sigma(t)} R_{\sigma(t)}^{-1}\right)_{|t=0} = \frac{dR_{\sigma(t)}}{dt}_{|t=0} R_{\sigma(0)}^{-1} + R_{\sigma(0)} \frac{dR_{\sigma(t)}^{-1}}{dt}_{|t=0} = 0,$$

it follows that

$$\frac{dR_{\sigma(t)}^{-1}}{dt}_{|t=0} = -R_{\sigma(0)}^{-1} \frac{dR_{\sigma(t)}}{dt}_{|t=0} R_{\sigma(0)}^{-1} = -R_\Phi^{-1} \Psi R_\Phi^{-1}.$$

But

$$[\sigma(t), \sigma(t)]_{\mathcal{FN}} = [\Phi + t\Psi + o(t), \Phi + t\Psi + o(t)]_{\mathcal{FN}} = [\Phi, \Phi]_{\mathcal{FN}} + 2t [\Phi, \Psi]_{\mathcal{FN}} + o(t)$$

so

$$\begin{aligned} R_{\sigma(t)}^{-1} [\sigma(t), \sigma(t)]_{\mathcal{FN}} &= \left(R_\Phi^{-1} - t R_\Phi^{-1} \Psi R_\Phi^{-1} + o(t)\right)\left([\Phi, \Phi]_{\mathcal{FN}} + 2t [\Phi, \Psi]_{\mathcal{FN}} + o(t)\right) \\ &= R_\Phi^{-1} [\Phi, \Phi]_{\mathcal{FN}} + t\left(-R_\Phi^{-1} \Psi R_\Phi^{-1} [\Phi, \Phi]_{\mathcal{FN}} + 2R_\Phi^{-1} [\Phi, \Psi]_{\mathcal{FN}}\right) + o(t) \end{aligned}$$

and

$$\frac{de_{\sigma(t)}}{dt}_{|t=0} = \mathcal{L}_\Psi + \mathcal{I}_{-\frac{1}{2} R_\Phi^{-1} \Psi R_\Phi^{-1} + R_\Phi^{-1} [\Phi, \Psi]_{\mathcal{FN}}} = \mathcal{L}_\Psi + \mathcal{I}_{\omega(\Psi)}.$$

Conversely, let $\Psi \in \Lambda^1 M \otimes TM$ and $\sigma(t) = \Phi + t\Psi$. Then $Id_{T(M)} + \sigma(t)$ is invertible for $|t|$ small enough and the tangent vector at the origin to the $\mathfrak{MC}(M)$-valued curve $t \mapsto e_{\sigma(t)}$ is

$$\frac{de_{\sigma(t)}}{dt}_{|t=0} = \frac{d}{dt}\left(\mathcal{L}_{\sigma(t)} + \mathcal{I}_{-\frac{1}{2}R^{-1}_{\sigma(t)}[\sigma(t),\sigma(t)]_{\mathcal{FN}}}\right)_{|t=0}$$
$$= \mathcal{L}_\Psi + \mathcal{I}_{\omega(\Psi)}.$$

□

Notation 6 *Let $k \in \mathbb{N}$. We denote*

$$\mathfrak{MC}_k(M) = \{D \in \mathfrak{MC}(M) : D \textit{ canonical solution of type } k\}.$$

Theorem 5 *Let $\Phi \in \Lambda^1 M \otimes TM$ such that $R_\Phi = Id_{T(M)} + \Phi$ is invertible.*

(1) Suppose $e_\Phi \in \mathfrak{MC}_0(M)$. Then

$$T_{e_\Phi}(\mathfrak{MC}_0(M)) \subset \left\{\mathcal{L}_\Psi : \Psi \in \Lambda^1 M \otimes TM,\ [\Psi, \Phi] = 0\right\}.$$

(2) Suppose $e_\Phi \in \mathfrak{MC}_k(M)$, $k \geqslant 1$. For $j = 1, \cdots, k$ set

$$\chi_j^\Phi(\Psi) = [\Phi,\Phi]_{\mathcal{FN}}\left(\sum_{p=1}^{j} \Phi^{p-1}\Psi\Phi^{j-p} + 2\Phi^j[\Phi,\Psi]_{\mathcal{FN}}\right).$$

Then

$$T_{e_\Phi}(\mathfrak{MC}_k(M)) \subset \left\{\mathcal{L}_\Psi + \mathcal{I}_{-\frac{1}{2}\sum_{j=0}^{k-1}\chi_j^\Phi(\Psi)} : \Psi \in \Lambda^1 M \otimes TM,\ \chi_\Phi^k(\Psi) = 0\right\}.$$

Proof

(1) Let $t \mapsto e_{\sigma(t)}$ a $\mathfrak{MC}_0(M)$-valued curve through e_Φ, with $\sigma(t) = \Phi + t\Psi + o(t)$, $\Psi \in \Lambda^1 M \otimes TM$ and $[\sigma(t), \sigma(t)] = o(t)$. Since

$$[\sigma(t), \sigma(t)_{\mathcal{FN}}] = [\Phi + t\Psi + o(t), \Phi + t\Psi + o(t)]_{\mathcal{FN}}$$
$$= [\Phi,\Phi]_{\mathcal{FN}} + 2t[\Psi,\Phi]_{\mathcal{FN}} + o(t), \tag{6}$$

we obtain $[\Psi, \Phi] = 0$. Moreover

$$\frac{de_{\sigma(t)}}{dt}_{|t=0} = \mathcal{L}_\Psi.$$

(2) Let $t \mapsto e_{\sigma(t)}$ a smooth $\mathfrak{MC}_k(M)$-valued curve through e_Φ, with $\sigma(t) = \Phi + t\Psi + o(t)$, $\Psi \in \Lambda^1 M \otimes TM$ and $\sigma(t)^k[\sigma(t), \sigma(t)] = 0$ for every t. We have

$$e_{\sigma(t)} = \mathcal{L}_{\sigma(t)} + \mathcal{I}_{-\frac{1}{2}\sum_{j=0}^{k-1}\sigma(t)^j[\sigma(t),\sigma(t)]}$$

and

$$\begin{aligned}\sigma(t)^j[\sigma(t), \sigma(t)] &= \sigma^j(t)[\sigma(t), \sigma(t)] \\ &= (\Phi + t\Psi + o(t))^j([\Phi, \Phi]_{\mathcal{FN}} + 2t[\Phi, \Psi]_{\mathcal{FN}} + o(t)) \\ &= \left(\Phi^j + t\sum_{i=1}^{j}\Phi^{i-1}\Psi\Phi^{j-i}\right)([\Phi, \Phi]_{\mathcal{FN}} + 2t[\Phi, \Psi]_{\mathcal{FN}}) + o(t) \\ &= \Phi^j[\Phi, \Phi]_{\mathcal{FN}} + t\chi_j^\Phi(\Psi) + o(t).\end{aligned}$$

Since $\sigma(t)^k[\sigma(t), \sigma(t)] = 0$ for every t it follows that $\chi_k^\Phi(\Psi) = 0$ and

$$\frac{de_{\sigma(t)}}{dt}_{|t=0} = \mathcal{L}_\Psi + \mathcal{I}_{-\frac{1}{2}\sum_{j=0}^{k-1}\chi_j^\Phi(\Psi)}.$$

□

From Theorems 4 and 5 we obtain

Corollary 1 *The deformation theory in* $(\mathcal{D}^*(M), \daleth, [\cdot,\cdot])$ *is not obstructed, but it is level-wise obstructed.*

Remark 5 Since $[Id_{TM}, \Phi] = 0$ for every $\Phi \in \Lambda^1 M \otimes TM$, it follows that $e_{Id_{TM}} = 0 \in \mathfrak{MC}_0(M)$.

Corollary 2 $T_0(\mathfrak{MC}_0(M)) = \{\mathcal{L}_\Psi : \Psi \in \Lambda^1 M \otimes TM,\ [\Psi, \Psi] = 0\}$.

Proof By Theorem 5, $T_0(\mathfrak{MC}_0(M)) \subset \{\mathcal{L}_\Psi : \Psi \in \Lambda^1 M \otimes TM,\ [\Psi, \Psi] = 0\}$.

Conversely, if $\Psi \in \Lambda^1 M \otimes TM$ and $[\Psi, \Psi] = 0$, set $\sigma(t) = Id_{TM} + t\Psi$ and $\widehat{\sigma}(t) = \mathcal{L}_{\sigma(t)}$. Then

$$[\sigma(t), \sigma(t)] = [Id_{TM} + t\Psi, Id_{TM} + t\Psi] = 0.$$

so $\widehat{\sigma}(t) = \mathcal{L}_{\sigma(t)}$ is a $\mathfrak{MC}_0(M)$-valued curve and $\widehat{\sigma}'(0) = \mathcal{L}_\Psi$. □

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Cohomologies on Hypercomplex Manifolds

Mehdi Lejmi and Patrick Weber

Abstract We review some cohomological aspects of complex and hypercomplex manifolds and underline the differences between both realms. Furthermore, we try to highlight the similarities between compact complex surfaces on one hand and compact hypercomplex manifolds of real dimension 8 with holonomy of the Obata connection in $\mathrm{SL}(2, \mathbb{H})$ on the other hand.

1 Introduction

We describe a recipe that allows one to adapt some cohomological results from complex manifolds to hypercomplex manifolds. A hypercomplex manifold is a complex manifold together with a second complex structure that anticommutes with the first one. To extract cohomological information out of a hypercomplex manifold, we may thus start with the double complex of the underlying complex manifold, twist this data by the second complex structure and see what information we get about the hypercomplex manifold in question. This approach turns out to be surprisingly successful if we want to adapt results from complex geometry to hypercomplex geometry and the resulting cohomology groups have the additional advantage of being easily computable.

We would like to anticipate that this way of proceeding also suffers from some drawbacks and that there is an alternative approach available in the literature. If a manifold admits two anticommuting complex structures I and J, then $K = IJ$ is another almost-complex structure, anticommuting with both I and J. This then leads to a whole 2-sphere worth of almost-complex structures

$$S^2 = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$$

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and it has been shown that all of these almost-complex structures are integrable as soon as I and J are (see for example [18]). From this point of view, all these complex structures should be treated equally on a hypercomplex manifold and singling out a preferred complex structure, as we do with I, is not very natural. Unfortunately, the cohomology groups based upon the "averaged complex structures" often tend to be quite cumbersome to work with and less explicit to compute. For further information, we refer the interested reader to [11, 29, 34, 36].

In the present note we summarise some results from the recent preprints [15] and [22]. We would like to thank the organisers of the INdAM meeting *Complex and Symplectic Geometry* for the great conference held in June 2016 in Cortona, Italy.

1.1 History and Examples

While complex manifolds have been around for a long time, the study of hypercomplex manifolds only became prominent in the eighties with publications such as [8, 29]. Probably the most well-known class of hypercomplex manifolds are hyperkähler manifolds. However, the realm of hypercomplex manifolds is much broader than the one of hyperkähler manifolds. To cite but a few hypercomplex non-hyperkähler manifolds, note that some nilmanifolds, that is quotients of a nilpotent Lie group by a cocompact lattice, admit hypercomplex structures [5]. Furthermore, Dominic Joyce constructed many left-invariant hypercomplex structures on Lie groups [16] and similar ones have been analysed by physicists interested in string theory [32] in the context of $N = 4$ supersymmetry. In more recent years, various authors constructed inhomogeneous hypercomplex structures: see for example [9] for hypercomplex structures on Stiefel manifolds as well as [7, 27].

A complete classification of compact hypercomplex manifolds of real dimension 4, called *quaternionic curves*, has been established by Boyer [8]. These are either 4-tori or K3 surfaces, both of whom are hyperkähler, or else quaternionic Hopf surfaces [19] which, even if non-hyperkähler, remain locally conformally hyperkähler. On the other hand, the situation becomes much more complicated for compact hypercomplex manifolds of real dimension 8, called *hypercomplex surfaces*. While compact complex surfaces are nowadays well understood thanks to the work of Kodaira [20], a similar classification for compact hypercomplex surfaces is still missing. In the sequel of this note, we will hence focus on hypercomplex manifolds of real dimension 8, the first "unsolved dimension".

2 Cohomological Properties of Complex and Hypercomplex Manifolds

In this Section we first briefly review some well-known cohomological aspects of complex manifolds and then show how these can be adapted to hypercomplex manifolds. For the cohomological properties of complex manifolds we refer the reader to [1] and the references therein whilst the hypercomplex cohomologies appear in [11, 15, 22, 29, 34, 36] to cite but a few of them.

2.1 Cohomologies on Complex Manifolds

An almost complex manifold (X, I) is a smooth manifold X of real dimension $2n$ together with an endomorphism of the tangent bundle $I : TX \to TX$ that satisfies $I^2 = -\mathrm{Id}_{TX}$. This *almost complex structure* I can be used to decompose the bundle of complex-valued one-forms $\Omega^1(X) \otimes \mathbb{C}$ into the subbundle $\Omega_I^{1,0}(X)$ and the subbundle $\Omega_I^{0,1}(X)$, with I acting on the sections of $\Omega_I^{1,0}(X)$ by i and on those of $\Omega_I^{0,1}(X)$ by $-i$. We get the following decomposition

$$\Omega^k(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_I^{p,q}(X).$$

We denote by $\Lambda_I^{p,q}(X)$ the sections of $\Omega_I^{p,q}(X)$ and define the *Dolbeault operators*

$$\partial = \pi^{p+1,q} \circ d : \Lambda_I^{p,q}(X) \to \Lambda_I^{p+1,q}(X), \quad \bar{\partial} = \pi^{p,q+1} \circ d : \Lambda_I^{p,q}(X) \to \Lambda_I^{p,q+1}(X),$$

where d is the exterior derivative and $\pi^{p,q}$ is the projection onto $\Lambda_I^{p,q}(X)$. Clearly, $df = \left(\partial + \bar{\partial}\right) f$ for any function f. However, a priori, the same is not true for higher degree forms as explained in Fig. 1:

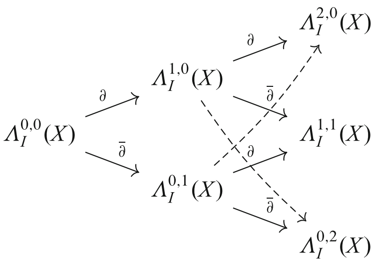

Fig. 1 In general, the two dashed maps $N_I = \pi^{0,2} \circ d : \Lambda_I^{1,0}(X) \to \Lambda_I^{0,2}(X)$ and $N_I^* = \pi^{2,0} \circ d : \Lambda_I^{0,1}(X) \to \Lambda_I^{2,0}(X)$ do not need to vanish. If they do, then the almost complex structure I is called integrable and $d = \partial + \bar{\partial}$ not only on functions but also on forms of higher degree

An almost complex manifold (X, I) is integrable if and only if

$$\partial^2\alpha = \bar{\partial}^2\alpha = \left(\partial\bar{\partial} + \bar{\partial}\partial\right)\alpha = 0 \quad \text{for all } \alpha \in \Lambda_I^{p,q}(X). \tag{1}$$

On any complex manifold (X, I), there is a double complex $(\Lambda_I^{p,q}(X), \partial, \bar{\partial})$ with two anti-commuting differentials.

2.2 Cohomologies on Hypercomplex Manifolds

An *almost hypercomplex manifold* (M, I, J, K) is a smooth manifold M of real dimension $4n$ equipped with three almost-complex structures I, J, K satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_{TM}.$$

If all three almost-complex structures are integrable, then (M, I, J, K) is called a *hypercomplex manifold*. We would like to mimic the above characterisation of integrability (1) in terms of differential operators. To this end, we will keep the decomposition of complexified differential forms with respect to the almost-complex structure I. As the almost-complex structures I and J anticommute, we deduce that J interchanges $\Lambda_I^{1,0}(M)$ with $\Lambda_I^{0,1}(M)$. This action then extends to an action $J : \Lambda_I^{p,q}(M) \to \Lambda_I^{q,p}(M)$:

$$J(\varphi)(X_1, \ldots, X_p, Y_1, \ldots, Y_q) = (-1)^{p+q}(\varphi)(JX_1, \ldots, JX_p, JY_1, \ldots, JY_q).$$

On any almost hypercomplex manifold, the *twisted Dolbeault operator* ∂_J is defined by the commutative diagram

$$\begin{array}{ccc} \Lambda_I^{p,q}(M) & \xrightarrow{\partial_J} & \Lambda_I^{p+1,q}(M) \\ {\scriptstyle J}\downarrow & & \uparrow{\scriptstyle J^{-1}} \\ \Lambda_I^{q,p}(M) & \xrightarrow{\bar{\partial}} & \Lambda_I^{q,p+1}(M) \end{array}$$

Both ∂ and ∂_J increase the first index in the bidegree as illustrated in Fig. 2.

One checks that $\partial^2\alpha = 0 = \partial_J^2\alpha$ for all $\alpha \in \Lambda_I^{p,0}(M)$ if and only if the Nijenhuis tensor N_I of the almost complex structure I vanishes, that is if and only if the almost complex structure I is integrable. Moreover, a direct computation shows that $(\partial\partial_J + \partial_J\partial)\alpha = 0$ for all $\alpha \in \Lambda_I^{p,0}(M)$ if and only if the Nijenhuis tensor N_J of the almost complex structure J vanishes. We deduce the following result [29, 34]: An almost

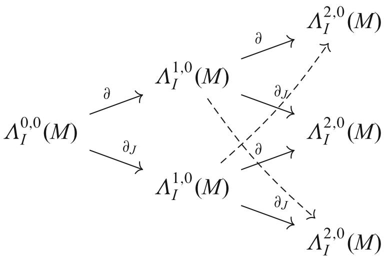

Fig. 2 On general almost hypercomplex manifolds, the two dashed maps $J^{-1} \circ N_I : \Lambda_I^{1,0}(M) \to \Lambda_I^{2,0}(M)$ and $N_I^* \circ J : \Lambda_I^{1,0}(M) \to \Lambda_I^{2,0}(M)$ do not need to vanish. If they do, then the almost complex structure I is called integrable

hypercomplex manifold (M, I, J, K) is integrable if and only if

$$\partial^2 \alpha = \partial_J^2 \alpha = (\partial\partial_J + \partial_J \partial)\,\alpha = 0 \quad \text{for all } \alpha \in \Lambda_I^{p,0}(M).$$

On any hypercomplex manifold (M, I, J, K), there is always a cochain complex $(\Lambda_I^{p,0}(M), \partial, \partial_J)$ with two anti-commuting differentials. This naturally leads to a definition of cohomology groups on hypercomplex manifolds.

2.3 *Complex and Quaternionic Cohomology Groups*

As soon as one is facing a cochain complex with two differential operators that anticommute, one may think about defining the following cohomology groups: the Dolbeault cohomology groups, the Bott–Chern cohomology groups and the Aeppli cohomology groups. Table 1 below gives precise definitions of these groups for both the double complex $(\Lambda_I^{p,q}(X), \partial, \bar\partial)$ on a complex manifold (X, I) and the single complex $(\Lambda_I^{p,0}(M), \partial, \partial_J)$ on a hypercomplex manifold (M, I, J, K).

Table 1 Some cohomology groups on compact complex manifolds (X, I) (left) and their analogues on compact hypercomplex manifolds (M, I, J, K) (right)

Complex Dolbeault cohomology groups	*Quaternionic Dolbeault cohomology groups*
$H_{\partial}^{p,q}(X) = \frac{\{\varphi \in \Lambda_I^{p,q}(X) \mid \partial\varphi = 0\}}{\partial \Lambda_I^{p-1,q}(X)} = \frac{\text{Ker } \partial}{\text{Im } \partial}$	$H_{\partial}^{p,0}(M) = \frac{\{\varphi \in \Lambda_I^{p,0}(M) \mid \partial\varphi = 0\}}{\partial \Lambda_I^{p-1,0}(M)} = \frac{\text{Ker } \partial}{\text{Im } \partial}$
$H_{\bar\partial}^{p,q}(X) = \frac{\{\varphi \in \Lambda_I^{p,q}(X) \mid \bar\partial\varphi = 0\}}{\bar\partial \Lambda_I^{p,q-1}(X)} = \frac{\text{Ker } \bar\partial}{\text{Im } \bar\partial}$	$H_{\partial_J}^{p,0}(M) = \frac{\{\varphi \in \Lambda_I^{p,0}(M) \mid \partial_J\varphi = 0\}}{\partial_J \Lambda_I^{p-1,0}(M)} = \frac{\text{Ker } \partial_J}{\text{Im } \partial_J}$
Complex Bott–Chern cohomology groups	*Quaternionic Bott–Chern cohomology groups*
$H_{BC}^{p,q}(X) = \frac{\{\varphi \in \Lambda_I^{p,q}(X) \mid \partial\varphi = 0 = \bar\partial\varphi\}}{\partial\bar\partial \Lambda_I^{p-1,q-1}(X)} = \frac{\text{Ker } \partial \cap \text{Ker } \bar\partial}{\text{Im } \partial\bar\partial}$	$H_{BC}^{p,0}(M) = \frac{\{\varphi \in \Lambda_I^{p,0}(M) \mid \partial\varphi = 0 = \partial_J\varphi\}}{\partial\partial_J \Lambda_I^{p-2,0}(M)} = \frac{\text{Ker } \partial \cap \text{Ker } \partial_J}{\text{Im } \partial\partial_J}$
Complex Aeppli cohomology groups	*Quaternionic Aeppli cohomology groups*
$H_{AE}^{p,q}(X) = \frac{\{\varphi \in \Lambda_I^{p,q}(X) \mid \partial\bar\partial\varphi = 0\}}{\partial \Lambda_I^{p-1,q}(X) + \bar\partial \Lambda_I^{p,q-1}(X)} = \frac{\text{Ker } \partial\bar\partial}{\text{Im } \partial + \text{Im } \bar\partial}$	$H_{AE}^{p,0}(M) = \frac{\{\varphi \in \Lambda_I^{p,0}(M) \mid \partial\varphi = 0 = \partial_J\varphi\}}{\partial\partial_J \Lambda_I^{p-2,0}(M)} = \frac{\text{Ker } \partial \cap \text{Ker } \partial_J}{\text{Im } \partial\partial_J}$

On compact hypercomplex manifolds, the groups $H^{p,0}_{\partial}(M)$, $H^{p,0}_{\partial_J}(M)$, $H^{p,0}_{BC}(M)$ and $H^{p,0}_{AE}(M)$ are finite-dimensional complex vector spaces [15].

2.4 Conjugation Symmetry

On a complex manifold (X, I), conjugation defines a map

$$\Lambda^{p,q}_I(X) \to \Lambda^{q,p}_I(X) : \alpha \mapsto \bar{\alpha}.$$

As this map passes to cohomology, we deduce that $H^{p,q}_{\partial}(X) \cong H^{q,p}_{\bar{\partial}}(X)$. Furthermore, this also implies that

$$H^{p,q}_{BC}(X) \cong H^{q,p}_{BC}(X) \quad \text{and} \quad H^{p,q}_{AE}(X) \cong H^{q,p}_{AE}(X).$$

On a hypercomplex manifold (M, I, J, K), conjugation followed by the action of J similarly defines a map

$$\Lambda^{p,0}_I(M) \to \Lambda^{p,0}_I(M) : \alpha \mapsto J(\bar{\alpha}).$$

Once more, this map descends to cohomology and leads to the isomorphism

$$H^{p,0}_{\partial}(M) \cong H^{p,0}_{\partial_J}(M)$$

but we do not get any isomorphisms for $H^{p,0}_{BC}(M)$ or $H^{p,0}_{AE}(M)$.

2.5 The $\partial\partial_J$-Lemma

On a compact complex manifold (X, I) and on a compact hypercomplex manifold (M, I, J, K), the identity map induces the following maps:

$$\begin{array}{ccccc} & & H^{p,q}_{BC}(X) & & \\ & \swarrow & & \searrow & \\ H^{p,q}_{\partial}(X) & & & & H^{p,q}_{\bar{\partial}}(X) \\ & \searrow & & \swarrow & \\ & & H^{p,q}_{AE}(X) & & \end{array} \qquad \begin{array}{ccccc} & & H^{p,0}_{BC}(M) & & \\ & \swarrow & & \searrow & \\ H^{p,0}_{\partial}(M) & & & & H^{p,0}_{\partial_J}(M) \\ & \searrow & & \swarrow & \\ & & H^{p,0}_{AE}(M) & & \end{array}$$

In general, these maps have no reason to be either injective or surjective. We say that the $\partial\bar{\partial}$-Lemma holds if the map $H^{p,q}_{BC}(X) \to H^{p,q}_{\bar{\partial}}(X)$ is injective and similarly that the $\partial\partial_J$-Lemma is satisfied if the map $H^{p,0}_{BC}(M) \to H^{p,0}_{\partial_J}(M)$ is injective. In other words, the $\partial\bar{\partial}$-Lemma holds if every ∂-closed $\bar{\partial}$-exact (p,q)-form is $\partial\bar{\partial}$-exact while the $\partial\partial_J$-Lemma holds if every ∂-closed, ∂_J-exact $(p,0)$-form is $\partial\partial_J$-exact. As it turns out, this actually implies that all of the maps in the above diagram become isomorphisms [12].

2.6 A Quaternionic Frölicher-Type Inequality

We deduce that, on a compact complex manifold (X, I), the Bott–Chern and Aeppli cohomology groups may differ from the Dolbeault and deRham cohomology groups (if the $\partial\bar{\partial}$-Lemma does not hold). The following result by Angella–Tomassini quantifies this difference:

Theorem 1 ([2]) *Let (X, I) be a compact complex manifold of real dimension $2n$. Then*

$$\sum_{p+q=k} \left(\dim H^{p,q}_{BC}(X) + \dim H^{p,q}_{AE}(X)\right) \geqslant 2 \dim H^k_{dR}(X) \tag{2}$$

for any $0 \leqslant k \leqslant n$ where

$$H^k_{dR}(X) = \frac{\{\varphi \in \Lambda^k(X) \mid d\varphi = 0\}}{d\Lambda^k(X)} = \frac{Ker\, d}{Im\, d}$$

denotes deRham cohomology. Moreover, the $\partial\bar{\partial}$-Lemma holds if and only if we have equality for all $0 \leqslant k \leqslant n$.

A similar result can be established for quaternionic cohomologies on compact hypercomplex manifolds:

Theorem 2 ([22]) *Let (M, I, J, K) be a compact hypercomplex manifold of real dimension $4n$. Then*

$$\dim H^{p,0}_{BC}(M) + \dim H^{p,0}_{AE}(M) \geqslant 2 \dim E^{p,0}_2(M) \tag{3}$$

for any $0 \leqslant p \leqslant 2n$ where the space $E^{p,0}_2(M)$ is defined by

$$E^{p,0}_2(M) = \frac{\{\varphi \in \Lambda^{p,0}_I(M) \mid \partial\varphi = 0 \text{ and } \partial_J\varphi + \partial\alpha_1 = 0\}}{\{\varphi \in \Lambda^{p,0}_I(M) \mid \varphi = \partial\beta_1 + \partial_J\beta_2 \text{ and } \partial\beta_2 = 0\}}.$$

Moreover, the $\partial\partial_J$-Lemma holds if and only if we have equality for all $0 \leqslant p \leqslant 2n$.

While these results look very similar, the conclusions we draw differ. More precisely, recall that the Betti numbers appearing in the right-hand-side of (2) are topological invariants. As the dimensions of the cohomology groups are upper semi-continuous, Angella and Tomassini deduce from Theorem 1 that, on compact complex manifolds, the $\partial\bar{\partial}$-Lemma is stable by small complex deformations [2, 35, 37]. However, the same reasoning fails on compact hypercomplex manifolds, because the term $\dim E_2^{p,0}(M)$ appearing in the right-hand-side of (3) in Theorem 2 has no reason to be a topological invariant. Indeed, it can be shown that the $\partial\partial_J$-Lemma is not stable by small hypercomplex deformations as illustrated in the Example in Sect. 4.5.

Finally, Theorems 1 and 2 also allow us to quantify how far away a complex manifold is from being "cohomologically Kähler" and similarly how far away a hypercomplex manifold is from being "cohomologically HKT" (see Sect. 3). Define the *non-Kähler-ness degrees* [3] on complex manifolds

$$\Delta^k(X) = \sum_{p+q=k} \left(\dim H^{p,q}_{BC}(X) + \dim H^{p,q}_{AE}(X)\right) - 2\dim H^k_{dR}(X)$$

and the *non-HKT-ness degrees* [22] on hypercomplex manifolds

$$\Delta^p(M) = \dim H^{p,0}_{BC}(M) + \dim H^{p,0}_{AE}(M) - 2\dim E_2^{p,0}(M).$$

3 Metric Structures

Every complex manifold (X, I) admits a Hermitian metric, that is a Riemannian metric g such that

$$g(\cdot,\cdot) = g(I\cdot, I\cdot).$$

We can build out of this the Hermitian form $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$ and various special metrics can be characterised via conditions on ω. Similarly, any hypercomplex manifold (M, I, J, K) admits a quaternionic Hermitian metric, that is a Riemannian metric g which satisfies

$$g(\cdot,\cdot) = g(I\cdot, I\cdot) = g(J\cdot, J\cdot) = g(K\cdot, K\cdot).$$

This leads to three (not necessarily closed) differential forms $\omega_I(\cdot,\cdot) = g(I\cdot,\cdot)$, $\omega_J(\cdot,\cdot) = g(J\cdot,\cdot)$ and $\omega_K(\cdot,\cdot) = g(K\cdot,\cdot)$ that can be assembled to build the fundamental form

$$\Omega = \omega_J + \sqrt{-1}\omega_K$$

which is of type $(2, 0)$ with respect to the complex structure I. Once more, various special metrics can be characterised by imposing conditions on the form Ω. If, for instance, the form Ω is d-closed then (M, I, J, K, Ω) is called a *hyperkähler* manifold whereas if Ω is ∂-closed, then (M, I, J, K, Ω) is called *hyperkähler with*

Table 2 Correspondence between metric structures on complex and hypercomplex manifolds

Complex	Condition	Hypercomplex	Condition
Gauduchon	$\partial\bar{\partial}\omega^{n-1} = 0$	Quaternionic Gauduchon	$\partial\partial_J\Omega^{n-1} = 0$
Strongly Gauduchon	$\partial\omega^{n-1} \in \operatorname{Im}\bar{\partial}$	Quaternionic strongly Gauduchon	$\partial\Omega^{n-1} \in \operatorname{Im}\partial_J$
Balanced	$d\omega^{n-1} = 0$	Quaternionic balanced	$\partial\Omega^{n-1} = 0$
Kähler	$d\omega = 0$	Hyperkähler with torsion (HKT)	$\partial\Omega = 0$
		Hyperkähler	$d\Omega = 0$

torsion, or *HKT* for short (see [14] for a nice introduction). Table 2 summarises some special metrics on hypercomplex manifolds together with their associated conditions on Ω as well as their complex counterparts. We point out that HKT metrics, just as Kähler metrics in the complex setup, admit a local potential [4].

A first important difference between complex and hypercomplex manifolds is the existence of a preferred metric. Indeed, a complex manifold always admits a Gauduchon metric and this metric is unique in its conformal class up to a constant. On the other hand, to recover existence of a quaternionic Gauduchon metric on hypercomplex manifolds, we will impose an additional holonomy constraint as described in the next Section.

4 $SL(n, \mathbb{H})$-manifolds

There is a particular class of hypercomplex manifolds, called $SL(n, \mathbb{H})$-manifolds, that shares more properties of complex manifolds than general hypercomplex manifolds do. The key reason for this is that the canonical bundle of an $SL(n, \mathbb{H})$-manifold is holomorphically trivial and this leads to a version of Hodge theory when HKT [34] and to a version of Serre duality on the bundle $\Omega_I^{*,0}(M)$.

4.1 The Obata Connection

Another important difference between complex and hypercomplex geometry is the existence of a special connection. A complex manifold generally admits infinitely many torsion-free connections which preserve the complex structure [17]. On the other hand, any hypercomplex manifold admits a unique torsion-free connection ∇ such that

$$\nabla I = \nabla J = \nabla K = 0.$$

This connection is called the Obata connection [26]. In general, the Obata connection does not preserve the metric, except when the manifold is hyperkähler. Given

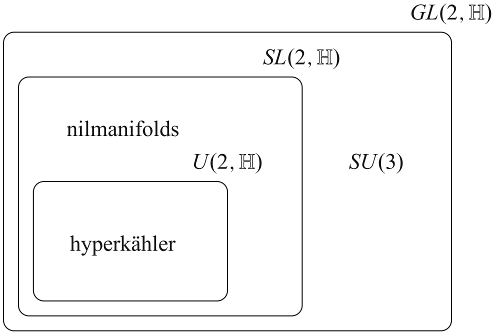

Fig. 3 This figure shows the possible holonomy groups of compact hypercomplex manifolds in real dimension 8. Left-invariant structures on Lie groups are conjectured to have holonomy equal to $GL(2,\mathbb{H})$ just as it has been proven for $SU(3)$

any torsion-free affine connection, the holonomy group introduced by Élie Cartan measures the failure of the parallel translation associated to a connection to be holonomic. Merkulov and Schwachhöfer classified the groups which can possibly arise as irreducible holonomy groups of torsion-free connections [24]. As illustrated in Fig. 3, in the case of the Obata connection, there are three possible choices: $GL(n,\mathbb{H})$, $SL(n,\mathbb{H})$ and $U(n,\mathbb{H})$. Indeed, as the Obata connection preserves all three complex structures, its holonomy is necessarily contained in the quaternionic general linear group $GL(n,\mathbb{H})$. It turns out that, for all nilmanifolds, the holonomy is contained in the commutator subgroup $SL(n,\mathbb{H})$ [6]. Finally, a hyperkähler manifold is characterised by the fact that the holonomy of the Obata connection is equal to the compact symplectic group $Sp(n) = U(n,\mathbb{H})$, i.e. the hyperunitary group. For the homogeneous hypercomplex structure on $SU(3)$ constructed by Joyce, the holonomy is equal to $GL(2,\mathbb{H})$ [31].

4.2 Hodge Theory

One important aspect of $SL(n,\mathbb{H})$-manifolds is that, if the metric is HKT, then it is possible to establish a version of Hodge theory [34]. Indeed, any $SL(n,\mathbb{H})$-manifold has holomorphically trivial canonical bundle. The nowhere degenerate real holomorphic section Φ (that is a nowhere degenerate section Φ such that $J\bar{\Phi} = \Phi$ and $\partial\bar{\Phi} = 0$) which trivialises $\Omega_I^{2n,0}(M)$ may then be used to define a Hodge star operator on a $SL(n,\mathbb{H})$-manifold (M, I, J, K, Ω)

$$\star_\Phi : \Lambda_I^{p,0}(M) \to \Lambda_I^{2n-p,0}(M)$$

via

$$\alpha \wedge (\star_\Phi \beta) \wedge \bar{\Phi} = h(\alpha,\beta)\frac{\Omega^n \wedge \bar{\Phi}}{n!},$$

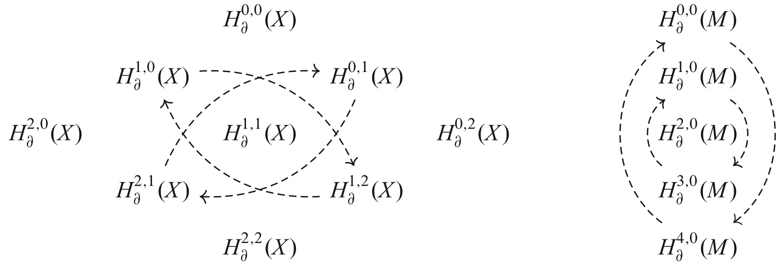

Fig. 4 Serre duality on compact complex surfaces (*left*) and $SL(2,\mathbb{H})$-symmetry on compact hypercomplex surfaces (*right*)

where h is the $\mathbb{C}$-bilinear extension with respect to I of the quaternionic Hermitian metric g associated to Ω. On compact manifolds, this leads to the adjoints

$$\partial^{*_\Phi} = - \star_\Phi \partial \star_\Phi \quad \text{and} \quad \partial_J^{\star_\Phi} = - \star_\Phi \partial_J \star_\Phi$$

and thus to the Laplacians

$$\Delta_\partial = \partial\partial^{*_\Phi} + \partial^{*_\Phi}\partial \quad \text{and} \quad \Delta_{\partial_J} = \partial_J\partial_J^{*_\Phi} + \partial_J^{*_\Phi}\partial_J.$$

On $SL(2,\mathbb{H})$-manifolds, the Hodge $\star_\Phi$ acts as an involution on $\Lambda_I^{2,0}(M)$ and hence we may decompose $(2,0)$-forms into $\star_\Phi$-self-dual ones and $\star_\Phi$-anti-self-dual ones. As $\star_\Phi$ commutes with Δ_∂, this splitting descends to cohomology. We conclude that, on a compact $SL(2,\mathbb{H})$-manifold, the space $H_\partial^{2,0}(M)$ can be decomposed as a direct sum of ∂-closed $\star_\Phi$-self-dual and ∂-closed $\star_\Phi$-anti-self-dual forms.

4.3 Serre Duality and $SL(n,\mathbb{H})$-symmetry

Besides the conjugation symmetry, compact complex manifolds also satisfy Serre duality coming from the pairing on $H_\partial^{p,q}(X) \times H_\partial^{n-p,n-q}(X)$ given by

$$([\alpha],[\beta]) \mapsto \int_X \alpha \wedge \beta.$$

On compact $SL(n,\mathbb{H})$ manifolds, an analogue of this exists and can be formulated as follows (see also Fig. 4). Consider the pairing $H_\partial^{p,0}(M) \times H_\partial^{2n-p,0}(M)$ given by

$$([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \bar{\Phi}.$$

Note that, for this to be well-defined we really need $\partial\bar{\Phi} = 0$.

Furthermore, Serre duality and $SL(n, \mathbb{H})$-symmetry also provide links between Bott–Chern and Aeppli cohomologies. Indeed, using the above pairings, it can be shown that Serre duality on compact complex manifolds of real dimension $2n$ implies that $H^{p,q}_{BC}(X) \cong H^{n-p,n-q}_{AE}(X)$ and similarly $SL(n, \mathbb{H})$-symmetry on compact $SL(n, \mathbb{H})$-manifolds implies that $H^{p,0}_{BC}(M) \cong H^{2n-p,0}_{AE}(M)$.

4.4 $SL(2, \mathbb{H})$-manifolds

We saw that compact $SL(2, \mathbb{H})$-manifolds share many properties of compact complex surfaces, most notably a version of Hodge theory when it is HKT and similar symmetries. Hence it should not surprise that many results valid on compact complex surfaces can be adapted to results on $SL(2, \mathbb{H})$-manifolds. To illustrate this link, we display in Table 3 some results which show that HKT metrics play a similar role on $SL(2, \mathbb{H})$-manifolds than Kähler metrics do on complex surfaces.

4.5 Computations

On a compact hypercomplex nilmanifold (M, I, J, K) of real dimension 8, if one assume that the Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(X)$ with respect to I can be computed using left-invariants forms then the quaternionic Dolbeault cohomology groups $H^{p,0}_{\partial}(M)$ and $H^{p,0}_{\partial_J}(M)$, the quaternionic Bott–Chern cohomology groups $H^{p,0}_{BC}(M)$ as well as the quaternionic Aeppli cohomology groups $H^{p,0}_{AE}(M)$ can be computed using only left-invariant forms [22]. Hence we may explicitly calculate these cohomologies for the following example based upon the central extension of the quaternionic Lie algebra $\mathbb{R} \times H_7$. We consider a path of hypercomplex structures as done in [13, 15, 22]. We end up with an $SL(2, \mathbb{H})$-manifold carrying a family $t \in (0, 1)$ of hypercomplex structures which is HKT for $t = \frac{1}{2}$ but not HKT for all

Table 3 Results valid on compact complex surfaces (left) and the corresponding results on compact $SL(2, \mathbb{H})$-manifolds (right)

Compact complex surfaces	Compact $SL(2, \mathbb{H})$-manifolds
Kähler if and only if $\dim H^1_{dR}(X)$ even [10, 21, 25, 30]	HKT if and only if $\dim H^{1,0}_{\partial}(M)$ even [15]
Kähler if and only if strongly Gauduchon [28]	HKT if and only if quaternionic strongly Gauduchon [22]
Kähler if and only if the second non-Kähler-ness degree vanishes [3, 23, 33]	HKT if and only if the second non-HKT-ness degree vanishes [22]

other values of t. The structure equations of the Lie algebra are:

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = 0, \\ de^6 = e^1 \wedge e^2 + e^3 \wedge e^4, \\ de^7 = e^1 \wedge e^3 + e^4 \wedge e^2, \\ de^8 = e^1 \wedge e^4 + e^2 \wedge e^3. \end{cases}$$

Consider the family of hypercomplex structures (I_t, J_t, K_t) parametrised by $t \in (0, 1)$:

$$\begin{array}{llll} I_t e^1 = \frac{t-1}{t} e^2, & I_t e^3 = e^4, & I_t e^5 = \frac{1}{t} e^6, & I_t e^7 = e^8, \\ J_t e^1 = \frac{t-1}{t} e^3, & J_t e^2 = -e^4, & J_t e^5 = \frac{1}{t} e^7, & J_t e^6 = -e^8. \end{array}$$

A basis of left-invariant $(1, 0)$-forms is given by:

$$\varphi^1 = e^1 - i\tfrac{t-1}{t} e^2, \ \varphi^2 = e^3 - ie^4, \ \varphi^3 = e^5 - i\tfrac{1}{t} e^6, \ \varphi^4 = e^7 - ie^8.$$

The structure equations become:

$$d\varphi^1 = 0, \ d\varphi^2 = 0, \ d\varphi^3 = \tfrac{1}{2(1-t)} \varphi^{1\bar{1}} - \tfrac{1}{2t} \varphi^{2\bar{2}}, \ d\varphi^4 = \tfrac{2t-1}{2t-2} \varphi^{12} - \tfrac{1}{2t-2} \varphi^{\bar{1}2}.$$

If $t = \frac{1}{2}$, then $d\varphi^i \subseteq \Lambda_I^{1,1}(M)$ and the complex structure is abelian whereas otherwise it is not. In terms of the differentials ∂ and ∂_J, the structure equations can be written as:

$$\begin{array}{llll} \partial\varphi^1 = 0, & \partial\varphi^2 = 0, & \partial\varphi^3 = 0, & \partial\varphi^4 = \frac{2t-1}{2(t-1)} \varphi^{12}, \\ \partial_J\varphi^1 = 0, & \partial_J\varphi^2 = 0, & \partial_J\varphi^3 = -\frac{2t-1}{2(t-1)} \varphi^{12}, & \partial_J\varphi^4 = 0. \end{array}$$

We conclude: if $t \neq \frac{1}{2}$, then we get Table 4.

Whereas if $t = \frac{1}{2}$ then both $\partial\varphi^4 = 0$ and $\partial_J\varphi^3 = 0$ which leads to Table 5.

Table 4 Dimensions of the quaternionic cohomology groups when $t = \frac{1}{2}$

$(p, 0)$	$h_\partial^{p,0}$	$h_{\partial_J}^{p,0}$	$h_{BC}^{p,0}$	$h_{AE}^{p,0}$
$(1, 0)$	3	3	2	4
$(2, 0)$	4	4	5	5
$(3, 0)$	3	3	4	2

Table 5 Dimensions of the quaternionic cohomology groups when $t \neq \frac{1}{2}$

$(p, 0)$	$h_\partial^{p,0}$	$h_{\partial_J}^{p,0}$	$h_{BC}^{p,0}$	$h_{AE}^{p,0}$
$(1, 0)$	4	4	4	4
$(2, 0)$	6	6	6	6
$(3, 0)$	4	4	4	4

We deduce that, just as the HKT property, the $\partial\partial_J$-Lemma is not stable by small hypercomplex deformations [13]. This differs from the complex setup where the $\partial\bar{\partial}$-Lemma is stable by small complex deformations [2, 35, 37].

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The Teichmüller Stack

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Abstract This paper is a comprehensive introduction to the results of Meersseman (The Teichmüller and Riemann Moduli Stacks, Available via arxiv. http://arxiv.org/abs/1311.4170, 2015). It grew as an expanded version of a talk given at INdAM Meeting Complex and Symplectic Geometry, held at Cortona in June 12–18, 2016. It deals with the construction of the Teichmüller space of a smooth compact manifold M (that is the space of isomorphism classes of complex structures on M) in arbitrary dimension. The main problem is that, whenever we leave the world of surfaces, the Teichmüller space is no more a complex manifold or an analytic space but an analytic Artin stack. We explain how to construct explicitly an atlas for this stack using ideas coming from foliation theory. Throughout the article, we use the case of $\mathbb{S}^3 \times \mathbb{S}^1$ as a recurrent example.

1 Introduction

Let M be a compact C^∞ connected oriented manifold. Assume that M is even-dimensional and admits complex structures. We are interested in the Teichmüller space $\mathcal{T}(M)$. To define it, we start with the *moduli space* $\mathcal{M}(M)$ of complex structures on M. We can formally define it as the set of complex manifolds diffeomorphic to M up to biholomorphisms. In short,

$$\mathcal{M}(M) = \{X \text{ complex manifold } \mid X \simeq_{so} M\}/\sim \tag{1}$$

where $X \simeq_{so} M$ means that there exists a C^∞-diffeomorphism from X to M preserving the orientations and where $X \sim Y$ if they are biholomorphic.

Thanks to Newlander-Nirenberg Theorem [9], a structure of a complex manifold on M is equivalent to an integrable complex operator J on M, that is a C^∞ bundle

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operator J on the tangent bundle TM such that

$$J^2 = -Id \qquad \text{and} \qquad [T^{0,1}, T^{0,1}] \subset T^{0,1} \tag{2}$$

for

$$T^{0,1} = \{v + iJv \mid v \in TM \otimes \mathbb{C}\} \tag{3}$$

the subbundle of the complexified tangent bundle $TM \otimes \mathbb{C}$ formed by the eigenvectors of J with eigenvalue $-i$. Here of course, J has been linearly extended to the complexified tangent bundle. We may thus rewrite (1) as

$$\mathcal{M}(M) = \{J \text{ o.p. integrable complex operator on } M\}/\sim \tag{4}$$

where o.p. means orientation preserving, i.e. the orientation induced on M by J coincides with that of M.

Now, it is easy to check that (M, J) and (M, J') are biholomorphic if and only if there exists a diffeomorphism f of M whose differential df satisfies

$$J' = (df)^{-1} \circ J \circ df \tag{5}$$

In other words, denoting $J \cdot f$ the right hand side of (5), we see that (5) defines an action of the diffeomorphism group Diff(M) onto $\mathcal{I}(M)$, the set of o.p. integrable complex operators appearing in (4). Since our operators are o.p., this is even in fact an action of Diff$^+(M)$, the group of diffeomorphisms of M that preserve the orientation.

So we end with

$$\mathcal{M}(M) = \mathcal{I}(M)/\text{Diff}^+(M) \tag{6}$$

and we are in position to define the *Teichmüller space* of M as

$$\mathcal{T}(M) = \mathcal{I}(M)/\text{Diff}^0(M) \tag{7}$$

where Diff$^0(M)$ is the group of diffeomorphisms C^∞-isotopic to the identity, that is the connected component of the identity in Diff$^+(M)$.

It is important to notice that (6) and (7) define *topological* spaces and not only sets as (1). Indeed, we endow $\mathcal{I}(M)$ and Diff(M) with the topology of uniform convergence of sequences of operators/functions and all their derivatives (the C^∞-topology) and we endow (6) and (7) with the quotient topology.

In fact, more can be said. Replacing C^∞ operators, resp. C^∞ functions with Sobolev L^2_l operators, resp. L^2_{l+1} functions (for l big), then $\mathcal{I}(M)$ is a Banach complex analytic space in the sense of [3]. Also Diff$^0(M)$ is a complex Hilbert manifold and acts by holomorphic transformations on $\mathcal{I}(M)$.

Remark 1 The Teichmüller space may have several/countably many connected components (defined as the quotient of a connected component of the space of operators $\mathcal{I}(M)$ by the $\mathrm{Diff}^0(M)$ action). We will always consider a single connected component of it.

For M a smooth surface, this definition of Teichmüller space coincides with the usual one. Then $\mathcal{T}(M)$ is a complex manifold and enjoys wonderful properties such as the existence of several nice metrics or of explicit interesting compactifications. The situation is completely different in the higher dimensional case. It is known since quite a long time, at least since the first works of Kodaira-Spencer at the end of the fifties, that the Teichmüller space is not even a complex analytic space in general. In order to put an analytic structure in some sense on the space $\mathcal{T}(M)$, one has to consider it as an analytic stack. And in order to make this stack structure concrete and useful, one has to give an explicit atlas of it.

This was done in [7]. The crucial idea is to understand the action of $\mathrm{Diff}^0(M)$ onto $\mathcal{I}$ as defining a foliation (in a generalized sense) on $\mathcal{I}$. Hence $\mathcal{T}(M)$ is the leaf space of this foliation, so as a stack, an atlas can be obtained as a (generalized) holonomy groupoid for this foliation. The aim of this paper is to serve as a comprehensive survey of [7], putting emphasis on the main ideas, on examples and on applications. Only Sect. 6 contains new results: we briefly report on work in progress by C. Fromenteau.

2 Examples

Example 1 To begin with, let us consider the Teichmüller space of $\mathbb{S}^1 \times \mathbb{S}^1$. By Riemann's uniformization theorem, every Riemann surface diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ is a compact complex torus, that is the quotient of $\mathbb{C}$ by a lattice $\mathbb{Z}v \oplus \mathbb{Z}w$ with (v, w) a direct $\mathbb{R}$-basis.[1] Indeed, its universal covering cannot be $\mathbb{P}^1$ for topological reasons, and the case of $\mathbb{D}$ is discarded because its automorphism group does not contain any $\mathbb{Z}^2$ subgroup acting freely.

A classical computation shows that the lattice can be assumed to be of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$ with τ belonging to the upper half-plane $\mathbb{H}$. Moreover, two such lattices give rise to biholomorphic tori if and only they are related through the formula

$$\tau' = \frac{a\tau + b}{c\tau + d} =: A \cdot \tau \qquad \text{with} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \tag{8}$$

Hence, the moduli space $\mathcal{M}(\mathbb{S}^1 \times \mathbb{S}^1)$ is the complex orbifold $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, the action being defined through (8).

[1] It should be noted that, in the definition of Teichmüller space, we consider *any* complex structure, not only projective ones. Hence, in the case of $\mathbb{S}^1 \times \mathbb{S}^1$, we *have to prove* that every such structure is projective.

However, the Teichmüller space $\mathcal{T}(\mathbb{S}^1\times\mathbb{S}^1)$ is just $\mathbb{H}$. This is because a non-trivial element A of $\mathrm{SL}_2(\mathbb{Z})$ sends the lattice associated to τ isomorphically onto the lattice associated to τ' (we use the same notations as in (8)), but does not send 1 to 1 and τ to τ'. Since 1 and τ in $\mathbb{C}$ descends as two loops on $\mathbb{S}^1\times\mathbb{S}^1$ which generates a basis of the first homology group with values in $\mathbb{Z}$, and since 1 and τ' defines the same basis, this means that the biholomorphism induced by A does not act trivially in homology. So it cannot be isotopic to the identity and τ differs from $A\cdot\tau$ in the Teichmüller space.

The Teichmüller space of Example 1 has the wonderful property of being a complex manifold. Moreover, this complex structure is natural in the sense that it is "compatible" with deformation theory. More precisely, every deformation of complex tori, that is every smooth morphism $\mathcal{X}\to B$ with fibers diffeomorphic to $\mathbb{S}^1\times\mathbb{S}^1$ defines a mapping from the parameter space B to $\mathcal{T}(\mathbb{S}^1\times\mathbb{S}^1)$. The point here is that this mapping is *holomorphic*.

The fundamental question is to know if this is a general property of Teichmüller spaces or something specific to dimension one. Surely a Teichmüller space has to be a complex object so we ask

Is it possible to endow any Teichmüller space with the structure of an analytic space?

There are many cases for which this is true at least locally. For example, in [2], Catanese shows this is ok locally for Kähler manifolds with trivial or torsion canonical bundle and for minimal surfaces of general type with no automorphisms or rigidified (i.e. with no automorphism smoothly isotopic to the identity) with ample canonical bundle.

Recall that an analytic space is a Hausdorff topological space locally modelled onto the zero set of holomorphic functions and that the Hausdorffness requirement does not follow from the local models. There exist non-Hausdorff analytic spaces, as well as non-Hausdorff manifolds, that is objects having the right local model but not separated as topological spaces. More important for us, this is exactly what happens for the Teichmüller space of irreducible Hyperkähler manifolds[2] [11]. So we should allow non-Hausdorff analytic spaces to expect a positive answer to our question.

Nevertheless, this is indeed not enough. The situation is even worse than that, as shown by the following example.

Example 2 Our guiding example is that of $\mathbb{S}^3\times\mathbb{S}^1$. It was proved by Kodaira in [5] that any complex surface homeomorphic to $\mathbb{S}^3\times\mathbb{S}^1$ is a *Hopf surface*, that is the quotient of $\mathbb{C}^2\setminus\{(0,0)\}$ by the group generated by a holomorphic contraction of $\mathbb{C}^2$. Besides, every such contraction can be either linearized and then diagonalized with eigenvalues of modulus strictly less than 1, or reduced to the following resonant

[2]Here this is defined as the space of isomorphism classes of Hyperkähler complex structures on a fixed smooth manifold, and not of arbitrary complex structures.

normal form

$$(z,w) \longmapsto g_{\lambda,p}(z,w) := (\lambda z + w^p, \lambda^p w) \qquad \text{for } p \in \mathbb{N}^*,\ 0 < |\lambda| < 1 \tag{9}$$

Note that $p = 1$ corresponds to the linear but non diagonalizable case, whereas $p > 1$ corresponds to non-linear cases. As a consequence, the classification of Hopf surfaces up to biholomorphism is as follows. Let X_g be such a surface defined by the contracting biholomorphism g of $\mathbb{C}^2$. Let λ_1 and λ_2 be the eigenvalues of the linear part of g at $(0,0)$ and assume that $|\lambda_2| \leq |\lambda_1|$. Then,

1. If there is no resonance, that is if λ_2^p is different from λ_1 for all $p \in \mathbb{N}^*$, then X_g is biholomorphic to $X_{\left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)}$.
2. If there is a resonance of order p, then X_g is biholomorphic either to $X_{\left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)}$ or to $X_{g_{\lambda,p}}$.

All these models are pairwise non biholomorphic. As a consequence, one can show that $\mathcal{T}(\mathbb{S}^3 \times \mathbb{S}^1)$, as a topological space, is as follows[3] [7]. Let

$$A \in \mathrm{GL}_2^c(\mathbb{C}) \xrightarrow{\ \pi\ } (\det A, \operatorname{Tr} A) \in \mathbb{C}^2 \tag{10}$$

where the superscript c means that we only consider contracting matrices. The image of the map π in (10), say $\mathcal{D}$, is a bounded domain in $\mathbb{C}^2$ that is exactly the Teichmüller space of linear diagonal Hopf surfaces.

To add the linear but non diagonalizable case, one has to add a non-separated copy $\mathcal{C}$ of the curve

$$\pi_*\{A \in \mathrm{GL}_2^c(\mathbb{C}) \mid 4 \det A = (\operatorname{Tr} A)^2\} \tag{11}$$

in $\mathcal{D}$. This is because a point in this curve corresponds to two non biholomorphic Hopf surfaces. Hence we distinguish $(\lambda^2, 2\lambda) \in \mathcal{D}$ which encodes $X_{\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)}$ and the same point in the added curve $\mathcal{C}$ which encodes $X_{\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)}$. The augmented domain, say $\mathcal{D}_{\mathcal{C}}$, is the Teichmüller space of linear Hopf surfaces and has the topology of the conjugacy classes of contracting matrices. In particular, a sufficiently small neighborhood of a point $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$ of $\mathcal{C}$ in $\mathcal{D}_{\mathcal{C}}$ does not contain the corresponding point $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$ of $\mathcal{D}$, whereas every neighborhood of $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right) \in \mathcal{D}$ contains $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right) \in \mathcal{C}$.

But we are not done, since the resonant non linear Hopf surfaces are missing. To include them, for each $p > 1$, we add a non-separated copy $\mathcal{C}_p$ of the curve

$$\{(\lambda^{p+1}, \lambda + \lambda^p) \mid 0 < |\lambda| < 1\} \tag{12}$$

[3] Recall that we just consider one connected component of the Teichmüller space. Here, the Teichmüller space has several connected components, cf. the discussion in [7].

to encode the contractions $g_{p,\lambda}$. Thus, a point $(\lambda^{p+1}, \lambda + \lambda^p)$ in $\mathcal{D}$ encodes $X_{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^p \end{pmatrix}}$ whereas its double in $\mathcal{C}_p$ encodes $X_{g_{\lambda,p}}$.

We therefore obtain finally a non-Hausdorff space

$$\mathcal{D}_{(\mathcal{C})} = \mathcal{D}_{\mathcal{C}} \sqcup \mathcal{C}_2 \sqcup \dots \tag{13}$$

which looks like a bounded domain in $\mathbb{C}^2$ plus a countable number of pairwise disjoint analytic curves and is endowed with the previously explained topology. The space is not Hausdorff along these curves, since a point in such a curve and the corresponding point in $\mathcal{D}_{\mathcal{C}}$ cannot be separated. However, two such points do not play symmetric roles. Every open neighborhood of $(\lambda^{p+1}, \lambda+\lambda^p)$ in $\mathcal{D}_{\mathcal{C}}$ contains the corresponding point of $\mathcal{C}_p$, whereas a sufficiently small neighborhood of $(\lambda^{p+1}, \lambda + \lambda^p)$ in $\mathcal{C}_p$ does not see the corresponding point of $\mathcal{D}_{\mathcal{C}}$.

We note the following important consequence, which is known since Kodaira-Spencer works on deformations in the sixties, but which is still frequently overlooked.

Proposition 1 *The Teichmüller space $\mathcal{T}(\mathbb{S}^3 \times \mathbb{S}^1)$ cannot be endowed with the structure neither of an analytic space nor of a non-Hausdorff analytic space.*

Proof An analytic space, even non-Hausdorff, is locally Hausdorff since it is locally modelled onto the zero set of some holomorphic functions in $\mathbb{C}^n$. This contradicts (13) and the subsequent discussion. □

3 Artin Analytic Stacks

As shown by Proposition 1, the Teichmüller space does not always admit the structure of an analytic space, even locally. Hence, to see it as an analytic object, one needs to use the more general notion of analytic stacks.

Let $\mathcal{A}$ be the category of (complex) analytic spaces and morphisms. We consider the euclidean topology on the analytic spaces. Especially, a covering is just an open covering for the euclidean topology. Fix some smooth manifold M as in Sect. 1. By a M-deformation, we understand a smooth morphism $\mathcal{X} \to A$ over an analytic space all of whose fibers are complex manifolds diffeomorphic to M.

We consider the contravariant functor $\underline{\mathcal{M}}(M)$ from $\mathcal{A}$ to the category of groupoids[4] such that

1. For A an analytic space, $\underline{\mathcal{M}}(M)(A)$ is the groupoid formed by M-deformations over A (objects) and isomorphisms of M-deformations (morphisms).
2. For f a morphism from A to B, $\underline{\mathcal{M}}(M)(f)$ is the natural pull back mapping from $\underline{\mathcal{M}}(M)(B)$ to $\underline{\mathcal{M}}(M)(A)$.

[4]Recall that a groupoid is a category all of whose morphisms are invertible.

This functor satisfies several properties, in particular

1. **Descent.** Given an open covering (A_α) of A, and M-deformations $\mathcal{X}_\alpha$ over A_α and a cocycle of isomorphisms over the intersections, there exists a unique $\mathcal{X}$ over A obtained by gluing all the $\mathcal{X}_\alpha$.
2. **Sheaf.** Given an open covering (A_α) of A, M-deformations $\mathcal{X}$ and $\mathcal{X}'$ over A, a collection of isomorphisms f_α over A_α between the M-deformations that coincide over the intersections glue into a unique morphism f.

This is an example of *a stack over the site*[5] $\mathcal{A}$, cf. [4]. This is the stack version of the moduli space (6). To obtain the stack version of (7), we have to modify slightly the construction. A M-deformation over A is smoothly a M bundle over A with structural group $\text{Diff}^+(M)$. The functor $\underline{\mathcal{T}}(M)$ associates to each A the set of isomorphism classes of M-deformations over A whose smooth bundle structure has structural group $\text{Diff}^0(M)$. Let us call *reduced* such a M-deformation.

Let us make a break. We propose to replace the Teichmüller space $\mathcal{T}(M)$ with a complicated contravariant functor $\underline{\mathcal{T}}(M)$ which describes all the M-deformations over analytic spaces with structural group $\text{Diff}^0(M)$. At first sight, there is no reason to do this. This is not even clear that we are dealing with something similar to the Teichmüller space.

The point here is that in many cases a stack can also be described by a single groupoid, called *atlas* or *presentation* of the stack. An atlas is far from being unique and we can look for a "nice" one. In our case we can choose as an atlas a groupoid that looks much more like the Teichmüller space and which is in fact an enriched version of it. To see this less theoretically, let us revisit the example of complex tori of dimension one.

Example 3 We let again M to be $\mathbb{S}^1 \times \mathbb{S}^1$ with a fixed orientation. Then $\underline{\mathcal{T}}(M)$ describes all reduced deformations with complex tori as fibers and their morphisms. We claim that an atlas for this stack can be obtained as follows. Consider the universal family $\mathcal{X} \to \mathbb{H}$ where $\mathcal{X}$ is the quotient of $\mathbb{C} \times \mathbb{H}$ by the free and proper holomorphic action

$$(z, \tau, p, q) \in \mathbb{C} \times \mathbb{H} \times \mathbb{Z} \times \mathbb{Z} \longmapsto (z + p + q\tau, \tau) \tag{14}$$

This is a reduced $\mathbb{S}^1 \times \mathbb{S}^1$-deformation. The fiber over τ is the complex torus $\mathbb{E}_\tau$ of lattice $(1, \tau)$. We rewrite this family as the following groupoid

$$\mathcal{X} \rightrightarrows \mathbb{H} \tag{15}$$

with both arrows equal to π. This must be thought of as follows. The set of objects is $\mathbb{H}$, that is the Teichmüller space $\mathcal{T}(\mathbb{S}^1 \times \mathbb{S}^1)$. The set of morphisms is $\mathcal{X}$ and the two maps are the projection on the source and the target of the morphism. In (15), since

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[5] In fact, a stack is a 2-functor but we will not go into that.

both equals π, every morphism has same source and target τ. The set of morphisms above τ is $\mathbb{E}_\tau$ and represents the group of translations of the complex torus $\mathbb{E}_\tau$. Here, we set the projection of $(0,\tau) \in \mathbb{C}\times\mathbb{H}$ in $\mathcal{X}$ to be the zero translation. Hence our groupoid is just $\mathcal{T}(\mathbb{S}^1\times\mathbb{S}^1)$ with its translation group above each complex torus τ.

We will not prove our claim, this would force us to give many and many definitions, but we can give a heuristic interpretation of it. The key point is that if you are given an analytic space A, then every torus deformation above A can be recovered from (15). From the one hand, above a sufficiently small open set A_α of A, such a deformation is completely and uniquely up to isomorphism characterized by a holomorphic map from A_α to $\mathbb{H}$.[6] From the other hand, gluings of these families over A_α are completely characterized by maps from $A_\alpha \cap A_\beta$ to $\mathcal{X}$ since such gluings are translations along the fibers of the deformation.

Note that (15) is a complex Lie groupoid: its sets of objects and morphisms are complex manifolds and the source and target maps are holomorphic submersions.

In the general case, an atlas for $\underline{\mathcal{T}}(M)$ is given by the action groupoid

$$\mathrm{Diff}^0(M)\times\mathcal{I}(M) \rightrightarrows \mathcal{I}(M) \tag{16}$$

with source map s and target map t defined as

$$s(f,J) = J \qquad \text{and} \qquad t(f,J) = J\cdot f \tag{17}$$

with $J\cdot f$ defined as the right hand side of (5). Since there is a dictionary between a stack and an atlas for it, (16) explains why we say that $\underline{\mathcal{T}}(M)$ is the stack version of $\mathcal{T}(M)$.

Compared with (15), this is no longer a complex Lie groupoid but an infinite dimensional object. And it does not help us to understand the Teichmüller space since it is just a rewriting of (7). But recall that an atlas is not unique. So introducing $\underline{\mathcal{T}}(M)$ is interesting only if we can find a nice and useful atlas of it. By nice, we think of a complex Lie groupoid, but we will see it is too much too expect. We need a singular version of a complex Lie groupoid. So we define

Definition 1 A stack over the site $\mathcal{A}$ is an *Artin analytic stack* if it admits an atlas $A_1 \rightrightarrows A_0$ with A_0 and A_1 being finite dimensional complex analytic spaces and the source and target maps being smooth morphisms. Such a groupoid is called a *singular Lie groupoid*.

The main result of [7] is to prove that, under a mild uniform condition on the automorphism group of M when considered as a complex manifold, $\underline{\mathcal{T}}(M)$ is an Artin analytic stack and to construct an explicit atlas with the properties of Definition 1. This is the best possible answer to the question of Sect. 2. The Teichmüller space $\mathcal{T}(M)$ is not an analytic space but its stack version $\underline{\mathcal{T}}(M)$ is Artin analytic.

[6] This is the meaning of universal family in Kodaira-Spencer deformation theory.

4 Foliations and Holonomy Groupoid

To understand the stack structure of $\underline{\mathcal{T}}(M)$, we need to make a diversion through foliation theory. Stacks and groupoids are also useful to understand the structure of the leaf space of a foliation.

So assume we start with a smooth foliation $\mathcal{F}$ of a smooth manifold M. Then $\mathcal{F}$ is defined through charts with values in $\mathbb{R}^p \times \mathbb{R}^{n-p}$ such that the changes of charts are of the type

$$(x,t) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \longmapsto (g(x,t), h(t)) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \tag{18}$$

that is send plaques $\{t = Cst\}$ onto plaques. Gluing the plaques following the changes of charts gives the leaves, that is disjoint immersed submanifolds that form a partition of M. Transverse local sections to $\mathcal{F}$ are given by $\{x = Cst\}$.

Example 4 An easy but yet interesting example is that of a linear foliation of a torus. Consider the trivial foliation of $\mathbb{R}^2$ given by parallel straight lines making a fixed angle α with the horizontal. It descends on the torus $\mathbb{R}^2/\mathbb{Z}^2$ as a foliation by curves with leaves wrapping around the torus. Its properties depend on the arithmetic type of α.

1. If $\alpha = p/q$ is rational then the leaves of the foliation are closed curves diffeomorphic to $\mathbb{S}^1$ that make p turns in the vertical direction and q in the horizontal one. The foliation has only compact leaves.
2. If α is irrational then the leaves are diffeomorphic to $\mathbb{R}$ and are dense in the torus. The foliation is minimal.

If we look for the leaf space, we can fix a meridian T on our torus. It is a global transverse to the foliation. Each leaf L will cut the circle T in at least one point. More precisely, the intersection $L \cap T$ is an orbit of the rotation of angle α and the leaf space is given by the quotient of T by the group G generated by this rotation.

1. If $\alpha = p/q$ is rational then G is isomorphic to $\mathbb{Z}_q$ and T/G identifies with $\mathbb{S}^1$.
2. If α is irrational then G is isomorphic to $\mathbb{Z}$ and T/G is not Hausdorff.

Let us see now how stacks and groupoids can help us in defining the leaf space.

In Example 4, identifying T with $\mathbb{S}^1 \subset \mathbb{C}$ and G with the group generated by the rotation $z \mapsto r_\alpha(z) := \exp 2i\pi\alpha z$, we may encode this as the Lie groupoid $\langle r_\alpha \rangle \times \mathbb{S}^1 \rightrightarrows \mathbb{S}^1$ with source and target maps given by

$$s(g,z) = z \qquad \text{and} \qquad t(g,z) = g(z) \tag{19}$$

This is an example of an action Lie groupoid: the source map is the projection onto the second factor and the target map is given by the action, cf. (17). Of course, just doing this is not enough. We have to think of this groupoid as defining a stack over the category of smooth manifolds. And as such, it is not the only groupoid defining this stack. Hence we have to think as the leaf space not as $G \times T \rightrightarrows T$ but as an equivalence class of groupoids.

There exists a general construction to encode the leaf space in a Lie groupoid: the étale holonomy groupoid. Roughly speaking, one starts with a foliated atlas of $\mathcal{F}$ and define as objects of the groupoid the disjoint union of a complete sets of local transverse sections. Then the set of morphisms encodes the holonomy morphisms. This is quite technical to do and we refer to [8, §5.2] for more details.

As in Example 4, this groupoid is an atlas for a stack over the category of smooth manifolds. And this stack has lots of different atlases, all equivalent. We will not give the precise definition of the equivalence relation needed here.[7] Let us just say that this equivalence class is an enriched version of the topological quotient. It does not remember neither M nor $\mathcal{F}$ but encodes the topological leaf space and moreover the smoothness of the initial construction (since it remains in the realm of Lie groupoids) and all the holonomy data. This is the best definition of a leaf space. In the case of Example 4, if α is rational, then $G \times T \rightrightarrows T$ is equivalent to the trivial groupoid $\mathbb{S}^1 \rightrightarrows \mathbb{S}^1$, that is the stack is just the manifold $\mathbb{S}^1$. However, if α and α' are irrational, then the corresponding Lie groupoids $G \times T \rightrightarrows T$ are equivalent if and only if $\alpha' = A \cdot \alpha$ as in (8), see [10]. Stacks allow to distinguish the leaf spaces in the irrational case.

Of course everything works in the analytic context. If the foliation is holomorphic, then the étale holonomy groupoid is a complex Lie groupoid and defines a stack over the category of complex manifolds. If we look at regular foliations (i.e. leaves are manifolds) on a singular space, then everything works except that the étale holonomy groupoid is now a singular Lie groupoid and the stack is Artin analytic in the sense of Definition 1.

5 The Teichmüller Stack

We are now in position to give the main results of [7] and the main ingredients of the proof. As before, we denote by M a compact connected oriented even-dimensional smooth manifold, by $\mathcal{T}(M)$ its Teichmüller space.

Definition 2 We call *Teichmüller stack* the stack $\underline{\mathcal{T}}(M)$ defined in Sect. 3.

We have

Theorem 1 (cf. [7]) *Assume that there exists a constant $a \in \mathbb{N}$ such that, for all $J \in \mathcal{T}(M)$, the dimension of the automorphism group of the corresponding complex manifold X_J is bounded by a.*

Then $\underline{\mathcal{T}}(M)$ is an Artin analytic stack.

[7] It is called Morita equivalence and basically is an adaptation of equivalence of category in the world of smooth manifolds.

Before explaining the main ideas of the proof, some important remarks have to be done.

1. The proof is constructive and geometric. It builds a concrete singular Lie groupoid as atlas for $\underline{\mathcal{T}}(M)$ which comes from the existence of a geometric structure (a foliated structure) of $\mathcal{I}(M)$. This is perhaps the most interesting aspect of the result.
2. The hypothesis is used to control that the constructed atlas is finite-dimensional. In any case this is a mild restriction since we may easily stratify $\mathcal{T}(M)$ by strata satisfying the hypotheses for a given a. Classical results of Grauert ensure that this gives a nice analytic stratification, see [7] for more details.

The crucial idea is to understand that the action of $\text{Diff}^0(M)$ defines a"generalized foliation" on $\mathcal{I}(M)$ so that the construction of a "generalized étale holonomy groupoid" can be carried out. To do this, many technical problems have to be overcome, the most serious one being the presence of non-trivial automorphisms (remark that the isotropy groups of the actions are constituted by automorphisms).
1st case: no automorphisms.

Here we assume that, for all $J \in \mathcal{T}(M)$, we have

$$\text{Aut}\ (X_J) \cap \text{Diff}^0(M) = \{Id\} \tag{20}$$

Hypothesis (20) exactly means that the $\text{Diff}^0(M)$ action is free. It is thus natural to expect that it defines a foliation in some sense. Now, this is a reformulation of Kuranishi's theorem of existence of a versal space [6].

Theorem 2 *Let $X_0 = (M, J_0)$ be a compact complex manifold whose underlying smooth structure is M. Assume (20). Then there exists a finite-dimensional analytic space K_0 such that the space $\mathcal{I}(M)$ is locally isomorphic to $K_0 \times \mathit{Diff}^0(M)$ in a neighborhood of J_0.*

Of course this local isomorphism preserves locally the action, so this gives really a foliated chart for the action. The plaques are open neighborhoods of the identity in the Fréchet manifold $\text{Diff}^0(M)$ and a transverse local section is given by the analytic space K_0. So this is an infinite dimensional but of finite codimension foliation of an infinite-dimensional analytic space, cf. Sect. 1, just before Remark 1.

In this situation, we can carry out the construction of the étale holonomy groupoid. Only slight adaptations have to be done. Note that the finite codimension of the foliation ensures the finite dimensionality of the holonomy groupoid.
2nd case: general case. Kuranishi's theorem takes now the following more general form.

Theorem 3 *Let $X_0 = (M, J_0)$ be a compact complex manifold whose underlying smooth structure is M. Let $\mathit{Aut}^0(X_0)$ be the connected component of the identity in the automorphism group of X_0. Then*

1. *A neighborhood of the identity in the quotient space $(\mathit{Diff}^0(M)/\mathit{Aut}^0(X_0))$ is a Fréchet manifold.*

2. *There exists a finite-dimensional analytic space* K_0 *such that the space* $\mathcal{I}(M)$ *is locally isomorphic to* $K_0 \times (\mathit{Diff}^0(M)/\mathit{Aut}^0(X_0))$ *in a neighborhood of* J_0.

As in the previous case, this local isomorphism preserves locally the action, but this time this does not give a foliated chart for the action. The problem is that the plaques are now modelled onto $(\text{Diff}^0(M)/\text{Aut}^0(X_0))$, i.e. depends on the automorphism group of the base manifold X_0. Hence plaques of different charts cannot be glued. There is no leaf to be constructed from the plaques.

Now we can reformulate Theorem 3 as giving a local isomorphism at J_0 between $\mathcal{I}(M)$ and the product

$$\text{Diff}^0(M) \times [K_0/\text{Aut}^0(X_0)] \tag{21}$$

where the brackets mean that we consider the right hand side as an Artin analytic stack. In other words, $\text{Aut}^0(X_0)$ acts[8] on K_0 and we consider its quotient as a stack. In foliated terms, we force the plaques to be open neighborhoods of the identity in the Fréchet manifold $\text{Diff}^0(M)$. This is possible subject to the condition that we let the transverse sections to be analytic stacks rather than analytic spaces.

Then (21) can be interpreted as a foliated chart in a generalized sense and the gluings will respect this foliated structure. In [7], we call it a *foliation transversely modelled on a translation groupoid* or in short a *TG foliation*.

The next step consists in showing how to adapt the machinery of holonomy étale groupoid to the world of TG foliations. This forms the technical core of [7]. The gap with the classical theory is important and lots of work is needed. We will not get into that, since we attain our assigned goal: give a comprehensive introduction to the results and objects of [7].

Remark 2 In the realm of Sect. 4, we get an étale holonomy groupoid. This is more than a singular Lie groupoid since the source and target maps are not only smooth morphisms but also étale morphisms. In the general case, the holonomy groupoid associated to a TG foliation has no more this property.

6 The Teichmüller Stack of Hopf Surfaces

In this last section, we shall briefly report on work in progress by C. Fromenteau on the Teichmüller stack of $\mathbb{S}^3 \times \mathbb{S}^1$. We explain in Example 2 the quite complicated topology of the Teichmüller *space* of $\mathbb{S}^3 \times \mathbb{S}^1$ as well as the different normal forms for the associated complex structures. It is known that the automorphism group of a Hopf surface has dimension 2, 3 or 4 depending on the normal form. Hence the hypothesis of Theorem 1 is fulfilled and $\underline{\mathcal{T}}(\mathbb{S}^3 \times \mathbb{S}^1)$ is an Artin analytic stack. The construction of the holonomy groupoid referred to in Sect. 5 does not give a

[8] In fact, this is not exactly an action, see [7] for more details.

useful atlas in this case. In particular its set of objects has countably many connected components. To really work with $\underline{\mathcal{T}}(\mathbb{S}^3 \times \mathbb{S}^1)$ we need another atlas.

C. Fromenteau gave a much nicer atlas that can be described as follows. Let G be the Lie group biholomorphic to $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ as a complex manifold but with the following product rule

$$(A, t) * (B, s) = (AB, t + s \det A) \tag{22}$$

Let M be the product $\mathrm{GL}_2^c(\mathbb{C}) \times \mathbb{C}$. Then one may define

1. a holomorphic action $\cdot$ of G onto M.
2. a holomorphic injection $\imath$ of M into G

such that the Lie groupoid $(G \times M)/\mathbb{Z} \rightrightarrows M$ is an atlas of $\underline{\mathcal{T}}(\mathbb{S}^3 \times \mathbb{S}^1)$. Here the $\mathbb{Z}$-action is defined as

$$(p, g, m) \in \mathbb{Z} \times G \times M \longmapsto (\imath(m)^p g, m) \tag{23}$$

and the source and target maps are the projections of the maps[9]

$$(g, m) \in G \times M \longmapsto m \qquad \text{and} \qquad (g, m) \mapsto m \cdot g \tag{24}$$

One interest of this atlas is for cohomological computations. For example, $\underline{\mathcal{T}}(\mathbb{S}^3 \times \mathbb{S}^1)$ has well defined de Rham cohomology groups, cf. [1]. This cohomology is different from the cohomology of the topological space $\mathcal{T}(\mathbb{S}^3 \times \mathbb{S}^1)$ since it takes also into account the cohomology of the automorphism groups of Hopf surfaces.

In any case, these cohomology groups are very difficult to compute in general. Having such an atlas makes the calculations possible. Roughly speaking, they are just the equivariant cohomology groups of the action $\cdot$ of G onto M. Here, with more work, one can compute them and show that the generators in dimension 2 can be realized as non-trivial holomorphic bundles above $\mathbb{P}^1$ with fiber a Hopf surface (which must be thought of as isotrivial but not trivial $\mathbb{S}^1 \times \mathbb{S}^3$-deformation above $\mathbb{P}^1$). The generator of the first cohomology group (which is equal to $\mathbb{C}$) cannot be realized as a *holomorphic* $\mathbb{S}^1 \times \mathbb{S}^3$-deformation above a compact Riemann surface.

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[9]The action groupoid $G \times M \rightrightarrows M$ with source and target maps defined in (24) is an atlas for the stack of reduced $\mathbb{S}^3 \times \mathbb{S}^1$-deformations admitting a covering $\mathbb{C}^2 \setminus \{(0, 0)\}$-deformation plus a choice of a base point in the covering family. Together with $\imath$, this forms a gerbe with band $\mathbb{Z}$.

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Embedding of LCK Manifolds with Potential into Hopf Manifolds Using Riesz-Schauder Theorem

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Abstract A locally conformally Kähler (LCK) manifold with potential is a complex manifold with a cover which admits a positive automorphic Kähler potential. A compact LCK manifold with potential can be embedded into a Hopf manifold, if its dimension is at least 3. We give a functional-analytic proof of this result based on Riesz-Schauder theorem and Montel theorem. We provide an alternative argument for compact complex surfaces, deducing the embedding theorem from the Spherical Shell Conjecture.

1 Introduction

A **locally conformally Kähler (LCK)** manifold is a Hermitian manifold (M, J, g) such that the fundamental two-form $\omega(X, Y) = g(X, JY)$ satisfies the equation

$$d\omega = \theta \wedge \omega$$

for a **closed** one-form θ, see [9].

The one-form θ is called the **Lee form**, and it produces a twisted cohomology associated to the operator $d_\theta := d - \theta\wedge$.

An equivalent definition requires the existence of a covering $\Gamma \to \tilde{M} \to M$ endowed with a Kähler metric $\tilde{g}$ with respect to which the deck group Γ acts by

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holomorphic homotheties. This gives rise to a character $\chi : \Gamma \to \mathbb{R}^{>0}$ which associates to a homothety $\gamma \in \Gamma$ its scale factor c_γ. A differential form η on $\tilde{M}$ is then called **automorphic** if $\gamma^*\eta = \chi(\gamma)\eta$. Clearly, $\tilde{\omega}$ is automorphic.

An LCK manifold is called **with potential** if there exists a Kähler covering with Kähler form having positive, automorphic *global* potential ψ: $\tilde{\omega} = dd^c\psi$. This is equivalent to the existence of a function φ on M such that $\omega = d_\theta d^c_\theta(\varphi)$, see [19]. Note that φ *is not* a potential on M.

Important examples are the Hopf surfaces and, more generally, the **linear Hopf manifolds** $(C^N \setminus \{0\})/\langle A\rangle$, where $\langle A\rangle$ is the cyclic group generated by a linear operator $A \in \mathrm{GL}(n, \mathbb{C})$ with all eigenvalues $|\alpha_i| < 1$, [20].

On the other hand, there exist compact complex manifolds which admit LCK metrics, but no LCK metric with potential: such are blow-ups of LCK manifolds [25] and the LCK Inoue surfaces, [21] (see also [2]).

We proved in [17] (see also [19]) that if (M, ω, θ) is a compact LCK manifold with potential, there exists another LCK structure (ω', θ'), close to (ω, θ) in the $\mathscr{C}^\infty$-topology, such that the corresponding $\tilde{\omega}'$ has a **proper** potential, this being equivalent with the monodromy of the covering, $\mathrm{im}(\chi)$, being isomorphic with $\mathbb{Z}$.

Vaisman manifolds are LCK manifolds whose Kähler coverings are Riemannian cones over Sasakian manifolds (see [9], and [4] for the classification of Vaisman compact surfaces). All Vaisman manifolds are LCK with potential (represented by the squared Kähler norm of the pull-back of the Lee form). The converse is not true, as the example of non-diagonal Hopf manifolds shows, see [18]. Still, the covering of an LCK manifold with potential is very close to being a cone:

Theorem 1.1 ([18]) *Let M be an LCK manifold with proper potential,* $\dim_{\mathbb{C}} M \geq 3$*, and $\widetilde{M}$ its Kähler $\mathbb{Z}$-covering. Then the metric completion $\widetilde{M}_c$ admits a structure of a complex variety, actually Stein, compatible with the complex structure on $\widetilde{M} \subset \widetilde{M}_c$. Moreover, $\tilde{M}_c \setminus \tilde{M}$ is just one point.*

Remark 1.2 The same result seems to be true for $\dim_{\mathbb{C}} M = 2$. In Sect. 5, we deduce it from classification of surfaces and the spherical shell conjecture on surfaces of Kodaira class VII (Conjecture 5.5).

The main property of an LCK manifold with potential is the following Kodaira type embedding result:

Theorem 1.3 ([18]) *A compact LCK manifold with proper potential, of complex dimension at least* 3*, can be holomorphically embedded in a linear Hopf manifold.*

Remark 1.4 The hypothesis $\dim_{\mathbb{C}} M > 2$ in Theorem 1.1 is essential in order to apply a result in [1, 22] (see Theorem 3.2 below) from which we deduce that the completion $\tilde{M}_c$ is Stein. Once we know that the completion is Stein, Theorem 1.3 follows without further assumptions on the dimension.

The aim of this note is to give new proofs of the above two theorems, based on applications of Montel and Riesz-Schauder theorems. This will require several notions of functional analysis (see, e.g. [14]) that we recall for the reader's convenience. In the last section we comment on the possible validity of the result for complex surfaces.

2 Preliminaries of Functional Analysis

2.1 Normal Families of Functions

Definition 2.1 Let M be a complex manifold, and $\mathscr{F}$ a family of holomorphic functions $f_i \in H^0(\mathscr{O}_M)$. $\mathscr{F}$ is called **a normal family** if for each compact $K \subset M$ there exists $C_K > 0$ such that for each $f \in \mathscr{F}$, $\sup_K |f| \leq C_K$.

Lemma 2.2 *Let M be a complex Hermitian manifold, $\mathscr{F} \subset H^0(\mathscr{O}_M)$ a normal family, and $K \subset M$ a compact subset. Then there exists a number $A_K > 0$ such that* $\sup_K |f'| \leq A_K$.

Proof By contradiction, suppose there exists $x \in K$, $v \in T_xM$, and a sequence $f_i \in \mathscr{F}$ such that $\lim_i |D_v f_i| = \infty$. We choose a disk $\Delta \overset{j}{\hookrightarrow} M$ with compact closure in M, tangent to v in x, such that $j(0) = x$. Let $w = tv$ have norm 1. Then $\sup_\Delta |f_i| < C_\Delta$ by the normal family condition. By Schwarz lemma (see [12]), this implies $|D_w f_i| < C_\Delta$. However, $t^{-1} \lim_i |D_w f_i| = \lim_i |D_v f_i| = \infty$, yielding a contradiction. ■

2.2 Topologies on Spaces of Functions

Definition 2.3 Let $C(M)$ be the space of functions on a topological space. The **topology of uniform convergence on compacts** (also known as **compact-open topology**, usually denoted as $\mathscr{C}^0$) is the topology on $C(M)$ whose base of open sets is given by

$$U(X, C) := \{f \in C(M) \mid \sup_K |f| < C\},$$

for all compacts $K \subset M$ and $C > 0$.

A sequence $\{f_i\}$ of functions converges to f if it converges to f uniformly on all compacts.

Remark 2.4 In a similar way one defines the $\mathscr{C}^0$-topology on the space of sections of a bundle.

Definition 2.5 Let B be a vector bundle on a smooth manifold M, and $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ a connection. Define the $\mathscr{C}^1$-topology on the space of sections of B (denoted, as usual, by the same letter B) as one where a sub-base of open sets is given by $\mathscr{C}^0$-open sets on B and $\nabla^{-1}(W)$, where W is an open set in $\mathscr{C}^0$-topology in $B \otimes \Lambda^1 M$.

Remark 2.6 A sequence $\{f_i\}$ converges in the $\mathscr{C}^1$-topology if it converges uniformly on all compacts, and the first derivatives $\{f_i'\}$ also converge uniformly on all compacts. This can be seen as an equivalent definition of the $\mathscr{C}^1$-topology.

2.3 Montel Theorem for Normal Families

Theorem 2.7 (Montel) *Let M be a complex manifold and $\mathscr{F} \subset H^0(\mathscr{O}_M)$ a normal family of functions. Denote by $\bar{\mathscr{F}}$ its closure in the $\mathscr{C}^0$-topology. Then $\bar{\mathscr{F}}$ is compact and contained in $H^0(\mathscr{O}_M)$.*

Proof Let $\{f_i\}$ be a sequence of functions in $\mathscr{F}$. By Tychonoff's theorem, for each compact K, there exists a subsequence of $\{f_i\}$ which converges pointwise on a dense countable subset $Z \subset K$. Taking a diagonal subsequence, we find a subsequence $\{f_{p_i}\} \subset \{f_i\}$ which converges pointwise on a dense countable subset $Z \subset M$. Since $|f_i'|$ is uniformly bounded on compacts, the limit $f := \lim_i f_i$ is Lipschitz on all compact subsets of M. It is thus continuous, because a pointwise limit of Lipschitz functions is again Lipschitz.

Then, since $|f_i'|$ is uniformly bounded on compacts, we can assume that f_i' also converges pointwise in Z, and $f := \lim_i f_i$ is differentiable. Since a limit of complex-linear operators is complex linear, Df is complex linear, and f is holomorphic. This implies that $\bar{\mathscr{F}} \cap H^0(\mathscr{O}_M)$ is compact. ■

2.4 The Banach Space of Holomorphic Functions

We begin by proving:

Theorem 2.8 *Let M be a complex manifold, and $H^0_b(\mathscr{O}_M)$ the space of all bounded holomorphic functions, equipped with the sup-norm $|f|_{\sup} := \sup_M |f|$. Then $H^0_b(\mathscr{O}_M)$ is a Banach space.*

Proof Let $\{f_i\} \in H^0_b(\mathscr{O}_M)$ be a Cauchy sequence in the sup-norm. Then $\{f_i\}$ converges to a continuous function f in the sup-topology.

Since $\{f_i\}$ is a normal family, it has a subsequence which converges in $\mathscr{C}^0$-topology to $\tilde{f} \in H^0(\mathscr{O}_M)$, by Montel's Theorem (Theorem 2.7). However, the $\mathscr{C}^0$-topology is weaker than the sup-topology, hence $\tilde{f} = f$. Therefore, f is holomorphic. ■

2.4.1 Compact Operators

Recall that a subset of a topological space is called **precompact** if its closure is compact.

Definition 2.9 Let V, W be topological vector spaces, and let $\phi : V \longrightarrow W$ be a continuous linear operator. It is called **compact** if the image of any bounded set is precompact.

Remark 2.10 Note that the notion of **bounded set** makes sense in all topological vector spaces V. Indeed, a set $K \subset V$ is called **bounded** if for any open set $U \ni 0$, there exists a number $\lambda_U \in \mathbb{R}^{>0}$ such that $\lambda_U K \subset U$.

Claim 2.11 Let $V = H^0(\mathscr{O}_M)$ be the space of holomorphic functions on a complex manifold M with $\mathscr{C}^0$-topology. Then any bounded subset of V is precompact. In this case, the identity map is a compact operator.

Proof This is a restatement of Montel's theorem (Theorem 2.7). ■

Remark 2.12 By Riesz theorem, a closed ball in a normed vector space V is never compact, unless V is finite-dimensional. This means that $(H^0(\mathscr{O}_M), \mathscr{C}^0)$ does not admit a norm. A topological vector space where any bounded subset is precompact is called **Montel space.**

2.4.2 Holomorphic Contractions

Definition 2.13 A **contraction** of a manifold M to a point $x \in M$ is a continuous map $\phi : M \longrightarrow M$ such that for any compact subset $K \subset M$ and any open set $U \ni x$, there exists $N > 0$ such that for all $n > N$, the map ϕ^n maps K to U.

Theorem 2.14 *Let X be a complex variety, and let $\gamma : X \longrightarrow X$ be a holomorphic contraction such that $\gamma(X)$ is precompact. Consider the Banach space $V = H_b^0(\mathscr{O}_X)$ with the sup-metric. Then $\gamma^* : V \longrightarrow V$ is compact, and its operator norm $\|\gamma^*\| := \sup_{|v|\leq 1} |\gamma^*(v)|$ is strictly less than 1.*

Proof Let $B_C := \{v \in V \mid |v|_{\sup} \leq C\}$. Then

$$|\gamma^* f|_{\sup} = \sup_{x \in \overline{\gamma(X)}} |f(x)|.$$

Therefore, for any sequence $\{f_i\}$ converging in the $\mathscr{C}^0$-topology, the sequence $\{\gamma^* f_i\}$ converges in the sup-topology. However, B_C is precompact in the $\mathscr{C}^0$-topology, because it is a normal family. Then $\gamma^* B_C$ is precompact in the sup-topology.

Since $\sup_X |\gamma^* f| = \sup_{\gamma(X)} |f| \leq \sup_X |f|$, one has $\|\gamma^*\| \leq 1$. If this inequality is not strict, for some sequence $f_i \in B_1$ one has $\lim_i \sup_{x\in\gamma(X)} |f_i(x)| = 1$. Since B_1 is a normal family, f_i has a subsequence converging in $\mathscr{C}^0$-topology to f. Then $\gamma(f_i)$ converges to $\gamma(f)$ in sup-topology, giving

$$\lim_i \sup_{x\in\gamma(X)} |f_i(x)| = \sup_{x\in\gamma(X)} |f(x)| = 1.$$

Since, by the maximum principle, a holomorphic functions has no strict maxima, this means that $|f(x)| > 1$ somewhere on X. Then f cannot be the $\mathscr{C}^0$-limit of $f_i \in B_1$. ■

2.4.3 The Riesz-Schauder Theorem

The following result is a Banach analogue of the usual spectral theorem for compact operators on Hilbert spaces. It will be the central piece in our argument.

Theorem 2.15 (Riesz-Schauder [7]) *Let $F: V \longrightarrow V$ be a compact operator on a Banach space. Then for each non-zero $\mu \in \mathbb{C}$, there exists a sufficiently large number $N \in \mathbb{Z}$ such that for each $n > N$ one has*

$$V = \ker(F - \mu\,\mathrm{Id})^n \oplus \overline{\mathrm{im}(F - \mu\,\mathrm{Id})^n},$$

where $\overline{\mathrm{im}(F - \mu\,\mathrm{Id})^n}$ is the closure of the image of $(F - \mu\,\mathrm{Id})^n$. Moreover, $\ker(F - \mu\,\mathrm{Id})^n$ is finite-dimensional and independent on n.

3 Proof of Theorem 1.1

Recall that $\widetilde{M}_c$ denotes the metric completion of the $\mathbb{Z}$-cover $\tilde{M}$ of M.

Claim 3.1 The complement $\widetilde{M}_c \backslash \widetilde{M}$ is just one point, called **the origin**.

Proof Indeed, let $z_i = \gamma^{n_i}(x_i)$ be a sequence of points in $\widetilde{M}$, with each x_i in the fundamental domain $\phi^{-1}([1,\lambda])$ of the $\Gamma = \mathbb{Z}$-action. Clearly, the distance between two fundamental domains $M_n := \gamma^n \phi^{-1}([1,\lambda]) = \phi^{-1}([\lambda^n, \lambda^{n+1}])$ and $M_{n+k+2} = \gamma^{n+k+2}\phi^{-1}([1,\lambda])$ is written as

$$d(M_n, M_{n+k+2}) = \sum_{i=0}^{k} \lambda^{n+i} v, \tag{1}$$

where v is the distance between M_0 and M_2. Then, z_i may converge only if $\lim_i n_i = -\infty$ or if all n_i, except finitely many, belong to the set $(p, p+1)$ for some

p. The second case is irrelevant, because each M_i is compact, and in the first case, $\{z_i\}$ is always a Cauchy sequence, as follows from (1). All such $\{z_i\}$ are therefore equivalent, hence converge to the same point in the metric completion. ■

Recall now the following result in complex analysis:

Theorem 3.2 ([1, 22]) *Let S be a compact strictly pseudoconvex CR manifold, $\dim_{\mathbb{R}} S \geq 5$, and let $H^0(\mathscr{O}_S)_b$ the ring of bounded CR holomorphic functions. Then S is the boundary of a Stein manifold M with isolated singularities, such that $H^0(\mathscr{O}_S)_b = H^0(\mathscr{O}_M)_b$, where $H^0(\mathscr{O}_M)_b$ denotes the ring of bounded holomorphic functions. Moreover, M is defined uniquely, $M = \operatorname{Spec}(H^0(\mathscr{O}_S)_b)$.*

The proof of Theorem 1.1 now goes as follows:

Step 1: Applying Rossi-Andreotti-Siu Theorem 3.2 to $\phi^{-1}([a,\infty[)$, we obtain a Stein variety $\widetilde{M}_a$ containing $\phi^{-1}([a,\infty[)$. Since $\widetilde{M}_a$ contains $\phi^{-1}([a_1,\infty[)$ for any $a_1 > a$, and the Rossi-Andreotti-Siu variety is unique, one has $\widetilde{M}_a = \widetilde{M}_{a_1}$. This implies that $\widetilde{M}_a =: \widetilde{M}_c$ is independent from the choice of $a \in \mathbb{R}^{>0}$.

It remains to identify $\widetilde{M}_c$ with the metric completion of $\widetilde{M}$. By Claim 3.1, this is equivalent to the complement $\widetilde{M}_c \backslash \widetilde{M}$ being a singleton.

Step 2: The monodromy group $\Gamma = \mathbb{Z}$ acts on $\widetilde{M}_c$ by holomorphic automorphisms. Indeed, any holomorphic function (hence, any holomorphic map) can be extended from $\widetilde{M}$ to $\widetilde{M}_c$ uniquely.

Step 3: Denote by γ the generator of Γ which decreases the metric by $\lambda < 1$, and let $\widetilde{M}_c^a$ be the Stein variety associated with $\phi^{-1}(]0,a]) \subset \widetilde{M}$ as above. Since $\gamma(\widetilde{M}_c^a) = \widetilde{M}_c^{\lambda a}$, for any holomorphic function f on $\widetilde{M}_c$, one has

$$\sup_{z\in\widetilde{M}_c^a} |f(\gamma^n(z))| = \sup_{z\in\widetilde{M}_c^{\lambda^n a}} |f(z)| \leq \sup_{z\in\widetilde{M}_c^a} |f(z)|.$$

Therefore, $\{f(\gamma^n(z))\}$ is a normal family.

Step 4: Let f_{lim} be any limit point of the sequence $\{f(\gamma^n(z))\}$. Since the sequence $t_i := \sup_{z\in\widetilde{M}_c^{\lambda^i a}} |f(z)|$ is non-increasing, it converges, and $\sup_{z\in\widetilde{M}_c^a} f_{\mathsf{lim}} = \lim t_i$. Similarly, $\sup_{z\in\widetilde{M}_c^{\lambda a}} f_{\mathsf{lim}} = \lim t_i$. By the strong maximum principle, [10], a non-constant holomorphic function on a complex manifold with boundary cannot have local maxima (even non-strict) outside of the boundary. Since $\widetilde{M}_c^{\lambda a}$ does not intersect the boundary of $\widetilde{M}_c^a$, the function f_{lim} must be constant.

Step 5: Consider now the complement $V := \widetilde{M}_c \backslash \widetilde{M}$, and suppose it has two distinct points x and y. Let f be a holomorphic function which satisfy $f(x) \neq f(y)$. Replacing f by an exponent of μf if necessarily, we may assume that $|f(x)| < |f(y)|$. Since γ fixes Z, which is compact, for any limit f_{lim} of the sequence $\{f(\gamma^n(z))\}$, supremum f_{lim} on Z is not equal to infimum of f_{lim} on Z. This is impossible, hence $f = \mathsf{const}$ on V, and V is one point.

This finishes the proof of Theorem 1.1. ■

4 Proof of the Embedding Theorem

Theorem 1.3 is implied by Theorem 4.2. To see this, we need the following:

Definition 4.1 Let γ be an endomorphism of a vector space V. A vector $v \in V$ is called ***γ*-finite** if the subspace $\langle v, \gamma(v), \gamma^2(v), \dots \rangle$ is finite-dimensional.

Theorem 4.2 *Let M be an LCK manifold with potential,* $\dim_{\mathbb{C}} M > 2$, *and $\widetilde{M}$ its Kähler $\mathbb{Z}$-covering. Consider the metric completion $\widetilde{M}_c$ with its complex structure and a contraction $\gamma : \widetilde{M}_c \longrightarrow \widetilde{M}_c$ generating the $\mathbb{Z}$-action. Let $H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ be the space of functions which are γ^*-finite. Then $H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ is dense in the* sup*-topology on each compact subset of $\widetilde{M}_c$.*

4.1 Theorem 4.2 Implies Theorem 1.3

Step 1: Let $W \subset H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ be an m-dimensional γ^*-invariant subspace W with basis $\{w_1, \dots, w_m\}$. Then the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Psi} & \mathbb{C}^m \\ \gamma \big\downarrow & & \big\downarrow \gamma^* \\ M & \xrightarrow{\Psi} & \mathbb{C}^m \end{array}$$

where $\Psi(x) = (w_1(x), w_2(x), \dots, w_m(x))$.

Suppose that the map Ψ associated with a given $W \subset H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ is injective. Then the quotient map gives an embedding $\Psi : \widetilde{M}/\mathbb{Z} \longrightarrow (\mathbb{C}^m \backslash 0)/\gamma^*$; all eigenvalues of γ^* are < 1 because its operator norm is < 1, by Theorem 2.14.

Step 2: To find an appropriate $W \subset H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$, choose a holomorphic embedding $\Psi_1 : \widetilde{M}_c \hookrightarrow \mathbb{C}^n$, which exists because $\widetilde{M}_c$ is Stein. Let $\tilde{w}_1, \dots, \tilde{w}_n$ be the coordinate functions of Ψ_1. Theorem 4.2 allows one to approximate $\tilde{w}_i$ by $w_i \in H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ in $\mathscr{C}^0$-topology. Choosing w_i sufficiently close to $\tilde{w}_i$ in a compact fundamental domain of the $\mathbb{Z}$-action, we obtain that $x \mapsto (w_1(x), w_2(x), \dots, w_n(x))$ is injective in a compact fundamental domain of $\mathbb{Z}$.

Finally, take $W \subset H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ generated by the γ^* from $w_1, \dots, w_n$, and apply Step 1. ■

4.2 Proof of Theorem 4.2

The core of our argument is an application of Riesz-Schauder theorem. First we prove:

Proposition 4.3 *Fix a precompact subset* $\widetilde{M}_c^a := \phi^{-1}([0,a[)$, *where* $\phi : \widetilde{M}_c : \longrightarrow \mathbb{R}^{>0}$ *is the Kähler potential. Let* A *be the ring of bounded holomorphic functions on* $\widetilde{M}_c^a$, *and* $\mathfrak{m}$ *the maximal ideal of the origin point. Clearly,* γ^* *preserves* $\mathfrak{m}$ *and all its powers. Let* $P_k(t)$ *be the minimal polynomial of* $\gamma^*\Big|_{A/\mathfrak{m}^k}$. *Then* $\operatorname{im}(P_k(\gamma^*)) \subset \mathfrak{m}^k(A)$, *and* $\ker P_k(\gamma^*)$ *generates* $A/\mathfrak{m}^k$.

Proof Since $P_k(t)$ is a minimal polynomial of γ^* on $A/\mathfrak{m}^k$, the endomorphism $P_k(\gamma^*)$ acts trivially on $A/\mathfrak{m}^k$, by Cayley-Hamilton theorem, hence it maps A to $\mathfrak{m}^k$.

From Riesz-Schauder theorem applied to the Banach space A and $F = P_k(\gamma^*) - P_k(0)$, with $\mu = -P_k(0)$, it follows that $A = \ker(P_k(\gamma^*) \oplus \overline{\operatorname{im}(P_k(\gamma^*))^n}$. Since $P_k(\gamma^*)$ acts trivially on $A/\mathfrak{m}^k$, its image lies in $\mathfrak{m}^k$. This gives a surjection of $\ker P_k(\gamma^*)$ onto $A/\mathfrak{m}^k$. ■

This implies:

Proposition 4.4 *Let* $H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}} \subset H^0(\mathscr{O}_{\widetilde{M}_c})$ *be the set of* γ^**-finite functions and* $\mathfrak{m}$ *the maximal ideal of the origin in* $\widetilde{M}_c$. *Then* $H^0(\mathscr{O}_{\widetilde{M}_c})_{\mathsf{fin}}$ *is dense in* $\mathfrak{m}$*-adic topology.*[1]

Proof A subspace $V \subset A$ is dense in $\mathfrak{m}$-adic topology in $A \Leftrightarrow$ the quotient $V/v \cap \mathfrak{m}^k$ surjects to $A/\mathfrak{m}^k$. This is proven in Proposition 4.3 for the ring of bounded holomorphic functions on $\widetilde{M}_c^a$. However, any such function can be extended to γ^*-finite function on $\widetilde{M}_c$ using the γ^*-action. ■

To finish the proof, observe that Theorem 4.2 is implied by the following:

Claim 4.5 Let X be a connected complex variety, A the ring of bounded holomorphic functions on X, $x \in X$ a point, $\mathfrak{m} \subset A$ its maximal ideal, and $R : A \longrightarrow \hat{A}$ the natural map from A to its $\mathfrak{m}$-adic completion. Then R is continuous in $\mathscr{C}^0$-topology and induces homeomorphism of any bounded set to its image.

Proof Continuity is clear because the $\mathscr{C}^0$-topology on holomorphic functions is equivalent to $\mathscr{C}^1$-topology, $\mathscr{C}^2$-topology and so on, by Montel Theorem (Theorem 2.7). Therefore, taking successive derivatives in a point is continuous in $\mathscr{C}^0$-topology. However, R takes a function and replaces it by its Taylor series.

To see that R is a homeomorphism, notice that any bounded, closed subset of A is compact, hence its image under a continuous map is also closed. Then R induces a homeomorphism on all bounded sets. To see that the preimage of a converging sequence is converging, notice that any such sequence is bounded in A by another application of Schwarz lemma. ■

[1]Recall that for a ring A with a proper ideal $\mathfrak{m}$, the $\mathfrak{m}$-adic topology on A is given by the base of open sets formed by $\mathfrak{m}^k$ and their translates.

5 Kato Conjecture and Non-Kähler Surfaces

All surfaces in this section are assumed to be compact.

Definition 5.1 A complex surface M with $b_1(M) = 1$ and Kodaira dimension $-\infty$ is called **Kodaira class VII surface**. If it is also minimal, it is called **class VII$_0$ surface**.

Remark 5.2 Class VII surfaces are obviously non-Kähler. Indeed, their b_1 is odd.

The main open question in the classification of non-Kähler surfaces is the following conjecture, called **spherical shell conjecture**, or **Kato conjecture**. To state it, recall first:

Definition 5.3 Let $S \subset M$ be a real submanifold in a complex surface, diffeomorphic to S^3. We call S **spherical shell** if $M\backslash S$ is connected, and S has a neighbourhood which is biholomorphic to an annulus in $\mathbb{C}^2$. A class VII$_0$ surface which contains a spherical shell is called a **Kato surface**.

Remark 5.4 From [13], we know that any Kato surface contains exactly $b_2(M)$ distinct rational curves (the converse was proven in [8]).

Conjecture 5.5 (Spherical Shell Conjecture) Any class VII$_0$ surface with $b_2 > 0$ is a Kato surface.

Theorem 5.6 *Assume that the spherical shell conjecture is true. Then Theorems 1.1 and 1.3 are true in dimension 2.*

Proof The only part of the proof missing for dimension 2 is Rossi-Andreotti-Siu Theorem (3.2), see also Remark 1.4. We used it to prove the following result (which is stated here as a conjecture, because we don't know how to prove it for class VII$_0$ non-Kato surfaces).

Conjecture 5.7 Let M be an LCK complex surface with proper potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-cover. Then the metric completion of $\tilde{M}$, realized by adding just one point, is a Stein variety.

Remark 5.8 For $\dim M \geq 3$, this is Theorem 1.1

Conjecture 5.7 follows from the spherical shell conjecture and the classification of surfaces.

First of all, notice that an LCK surface M with an LCK potential cannot contain rational curves. Indeed, if M contains rational curves, by homotopy lifting, $\tilde{M}$ would also contain rational curves, but $\tilde{M}$ is embedded to a Stein variety. This implies that M cannot be a Kato surface, and that M is minimal.

If the spherical shell conjecture is true, class VII surfaces which are not Kato have $b_2 = 0$. However, class VII surfaces with $b_2 = 0$ were classified by Bogomolov [5, 6], Li et al. [16], Li and Yau [15], Teleman [23], and Teleman [24]. From these works it follows that any class VII surface with $b_2 = 0$ is biholomorphic to a Hopf surface or to an Inoue surface.

Inoue surfaces don't have LCK potential for topological reasons [21]. The Hopf surfaces are quotients of $\mathbb{C}^2 \setminus 0$ by an action of $\mathbb{Z}$, hence their 1-point completions are affine, and hence Stein.

The only non-Kähler minimal surfaces which are not of class VII are non-Kähler elliptic surfaces [3]. These surfaces are obtained as follows. Let X be a 1-dimensional compact complex orbifold, and L an ample line bundle on X. Consider the space $\tilde{M}$ of all non-zero vectors in the total space of L^*, and let $\mathbb{Z}$ act on $\tilde{M}$ as $v \mapsto \alpha v$, where $\alpha \in \mathbb{C}$ is a fixed complex number, $|\alpha| > 1$. Any non-Kähler elliptic surface is isomorphic to $\tilde{M}/\mathbb{Z}$ for appropriate α, X and L. However, the sections of $L^{\otimes n}$ define holomorphic functions on $\tilde{M} \subset \mathrm{Tot}(L^*)$, identifying $\tilde{M}$ and the corresponding cone over X. As such, M is Vaisman [4], in particular LCK with potential. The completion of this cone is $\tilde{M}_c$, and it is affine [11, §8], and hence Stein. This finishes the proof of Theorem 5.6. ■

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Orbits of Real Forms, Matsuki Duality and *CR*-cohomology

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Abstract We discuss the relationship between groups of CR cohomology of some compact homogeneous CR manifolds and the corresponding Dolbeault cohomology groups of their canonical embeddings.

This paper gives an overview on some topics concerning compact homogeneous *CR* manifolds, that have been developed in the last few years. The algebraic structure of compact Lie group was employed in [3] to show that a large class of compact *CR* manifolds can be viewed as the total spaces of fiber bundles over complex flag manifolds, generalizing the classical Hopf fibration for odd dimensional spheres and the Boothby-Wang fibration for homogeneous compact contact manifolds (see [6]). If a compact group $\mathbf{K}_0$ acts as a transitive group of *CR* diffeomorphisms of a *CR* manifold M_0, which is $\mathfrak{n}$-reductive in the sense of [3], one can construct a homogeneous space $M_- = \mathbf{K}/\mathbf{V}$ of the complexification $\mathbf{K}$ of $\mathbf{K}_0$ such that the map $M_0 \to M_-$ associated to the inclusion $\mathbf{K}_0 \hookrightarrow \mathbf{K}$ is a generic *CR* embedding. The manifold M_- is algebraic over $\mathbb{C}$ and a tubular neighborhood of M_0. For instance, if M_0 is the sphere S^{2n-1} in $\mathbb{C}^n$ and $\mathbf{K}_0 = \mathbf{SU}(n)$, the embedding $M_0 \hookrightarrow M = \mathbb{CP}^n$ is useful to compute the maximal group of *CR* automorphisms of M_0 (see [10, 11]), while the embedding $M_0 \hookrightarrow M_- = \mathbb{C}^n \setminus \{0\}$ better reflects the topology and the *CR cohomology* of M_0. Thus, for some aspects of *CR* geometry, we can consider M_- to be the *best* complex realization of M_0. This is the essential contents of the PhD thesis of the first Author [18]: his aim was to show that, in a range which depends on the pseudoconcavity of M_0, the groups of tangential Cauchy-Riemann cohomology of M_0 are isomorphic to the corresponding Dolbeault cohomology groups of M_-. The

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class of compact homogeneous *CR* manifolds to which this theory applies includes the intersections of Matsuki dual orbits in complex flag manifolds (cf. [21]).

The paper is organized as follows. In the first section we discuss some basic facts on compact homogeneous *CR* manifolds, including Matsuki duality and the notion of $\mathfrak{n}$-reductivity. The second section describes $\mathbf{K}_0$-covariant fibrations $M_- \to M_0$ and, under a special assumption on the partial complex structure of M_0 (that we call HNR), results on the cohomology, which in part are contained in [18] and have been further developed in [19]. In a final section we discuss a simple example of $\mathfrak{n}$-reductive compact *CR* manifolds which are intersections of Matsuki dual orbits.

1 Compact Homogeneous *CR* Manifolds

A *compact homogeneous CR manifold* is a *CR* manifold M on which a compact Lie group $\mathbf{K}_0$ acts transitively as a group of *CR* diffeomorphisms. Its *CR* structure is uniquely determined by the datum, for the choice of a *base point* p_0 of M, of the *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$, where $\mathfrak{k}_0 = \text{Lie}(\mathbf{K}_0)$ and $\mathfrak{v} = d\pi^{-1}(T^{0,1}_{p_0}M)$, for the complexification $d\pi$ of the differential of the canonical projection $\pi : \mathbf{K}_0 \ni x \to x \cdot p_0 \in M$. We recall that, by the formal integrability of the partial complex structure of M, the subspace $\mathfrak{v}$ is in fact a Lie subalgebra of the complexification $\mathfrak{k}$ of $\mathfrak{k}_0$. These pairs where introduced in [22] to discuss homogeneous *CR* manifolds and the compact case was especially investigated in [3].

1.1 Lie Algebras of Compact Lie Groups

Compact Lie algebras are characterized in [8] by

Proposition 1.1 *For a real Lie algebra $\mathfrak{g}_u$ the following are equivalent:*

(1) $\mathfrak{g}_u$ *is the Lie algebra of a compact Lie group;*
(2) *the analytic Lie subgroup* $\text{Int}(\mathfrak{g}_u)$ *of* $\mathbf{GL}_{\mathbb{R}}(\mathfrak{g}_u)$ *with Lie algebra* $\text{ad}(\mathfrak{g}_u)$ *is compact;*
(3) *on* $\mathfrak{g}_u$ *a symmetric bilinear form can be defined which is invariant and positive definite;*
(4) $\mathfrak{g}_u$ *is reductive, i.e. its adjoint representation is semi-simple and, for every* $X \in \mathfrak{g}_u$ *the endomorphism* $\text{ad}(X)$ *is semisimple with purely imaginary eigenvalues;*
(5) $\mathfrak{g}_u$ *is reductive with a negative semidefinite Killing form.* □

1.2 Complex Flag Manifolds

Let $\mathbf{G}_u$ be a compact Lie group, with Lie algebra $\mathfrak{g}_u$. The negative of the trace form of a faithful representation of $\mathfrak{g}_u$ yields an invariant scalar product b on $\mathfrak{g}_u$, that we use to identify $\mathfrak{g}_u$ with its dual $\mathfrak{g}_u^*$. In particular, the coadjoint orbits of $\mathbf{G}_u$ are canonically isomorphic to the adjoint orbits in $\mathfrak{g}_u$. We follow [16, §5.2]. Fix an element $H_0 \in \mathfrak{g}_u$ with $\mathrm{ad}(H_0)$ of maximal rank. Then the commutant

$$\mathfrak{t}_u = \{H \in \mathfrak{g}_u \mid [H_0, H] = 0\}$$

is a maximal torus of $\mathfrak{g}_u$. Denote by $\mathbf{T}_{H_0} = \{x \in \mathbf{G}_u \mid \mathrm{Ad}(x)(H_0) = H_0\}$ the corresponding torus of $\mathbf{G}_u$. The Weyl group W_{H_0} is the quotient of the normalizer $N_{\mathbf{G}_u}(\mathfrak{t}_u) = \{x \in \mathbf{G}_u \mid \mathrm{Ad}(x)(\mathfrak{t}_u) \subset \mathfrak{t}_u\} = \{x \in \mathbf{G}_u \mid [\mathrm{Ad}(x)(H_0), H_0] = 0\}$ with respect to $\mathbf{T}_{H_0}$.

We set $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_u$, $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}_u$ and denote by $\mathcal{R}$ the set of nonzero $\alpha \in \mathfrak{h}_{\mathbb{R}}^*$ such that $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha X,\ \forall H \in \mathfrak{h}_{\mathbb{R}}\} \neq \{0\}$. We choose a system of simple roots $\alpha_1 \dots, \alpha_\ell$ in $\mathcal{R}$ with $\alpha_j(iH_0) > 0$ for $j = 1, \dots, \ell$, and denote by

$$C(H_0) = \{X \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(X) > 0,\ 1 \leq i \leq \ell\}$$

the corresponding positive Weyl chamber.

Lemma 1.2 *Every adjoint orbit M in $\mathfrak{g}_u$ intersects $iC(H_0)$ in exactly one point.*

Proof Let $f(X) = b(X, H_0)$. Since M is compact, f has stationary points on M. A stationary point X_0 is characterized by

$$df(X_0) = 0 \iff 0 = b([X, X_0], H_0) = b(X, [X_0, H_0]),\ \forall X \in \mathfrak{g}_u \iff X_0 \in \mathfrak{t}_u.$$

This shows that the intersection $M \cap \mathfrak{t}_u$ is not empty and all its points are critical for f. On the other hand, if $\mathrm{Ad}(x)(X_0) \in \mathfrak{t}_u$, we can find $x' \in \mathbf{G}_u$ with $\mathrm{Ad}(x')(X_0) = \mathrm{Ad}(x)(X_0)$ and $\mathrm{Ad}(x)(\mathfrak{t}_u) = \mathfrak{t}_u$, so that the Weyl group is transitive on $M \cap \mathfrak{t}_u$. □

Proposition 1.3 *Let $\mathbf{G}_u$ be a connected compact Lie group. Then:*

(1) *For each $\Upsilon \in \mathfrak{g}_u$ the stabilizer $\mathbf{E}(\Upsilon) = \{x \in \mathbf{G}_u \mid \mathrm{Ad}(x)(\Upsilon) = \Upsilon\}$ of Υ contains a maximal torus of $\mathbf{G}_u$.*
(2) *If $\mathbf{T}_0$ is any fixed maximal torus in $\mathbf{G}_u$, then there are finitely many subgroups $\mathbf{A}_0$ with $\mathbf{T}_0 \subseteq \mathbf{A}_0 \subseteq \mathbf{G}_u$.*
(3) *There is a unique maximal orbit M whose stabilizer at each point is a maximal torus of $\mathfrak{g}_u$.* □

Definition 1.1 A *flag manifolds* of $\mathbf{G}_u$ is an orbit of its adjoint action on $\mathfrak{g}_u$.

Theorem 1.4 *On a flag manifold M of $\mathbf{G}_u$ it is possible to define a complex structure and a $\mathbf{G}_u$-invariant Kähler structure.*

If $\mathbf{T}$ is a maximal torus of $\mathbf{G}_u$, the Weyl group $\mathcal{W}(\mathbf{T}, \mathbf{G}_u)$ act transitively on the $\mathbf{G}_u$-invariant complex structures of M.

Proof Fix a point p_0 of M, corresponding to $\Upsilon_0 \in \mathfrak{g}_u$. The stabilizer $\mathbf{E}_{\mathbf{G}_u}(\Upsilon_0)$ contains a maximal torus $\mathbf{T}$ and therefore there are finitely many parabolic subalgebras $\mathfrak{q}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_u$ with $\mathfrak{q} \cap \mathfrak{g}_u = \mathfrak{stab}_{\mathfrak{g}_u}(\Upsilon_0)$ (the Lie algebra of

$\mathbf{E}_{\mathbf{G}_u}(\Upsilon_0)$). Each $\mathfrak{q}$ corresponds to a different $\mathbf{G}_u$-invariant complex structure on M, and the Weyl group $\mathcal{W}(\mathbf{T}, \mathbf{G}_u)$ act transitively on the $\mathfrak{q}$'s. □

Let $\mathbf{G}$, with Lie algebra $\mathfrak{g}$, be the complexification of $\mathbf{G}_u$. The analytic subgroup $\mathbf{Q}$ corresponding to the subalgebra $\mathfrak{q}$ in the proof of Theorem 1.4 is parabolic and the complex structure of M is also defined by its representation $M \simeq \mathbf{G}/\mathbf{Q}$ as a complex homogeneous space. Vice versa, if $\mathbf{Q}$ is parabolic in $\mathbf{G}$, the homogeneous space $\mathbf{G}/\mathbf{Q}$ is $\mathbf{G}_u$-diffeomorphic to a flag manifold of its compact form $\mathbf{G}_u$.

1.3 Matsuki's Dual Orbits

Let $\mathbf{G}_0$ be any real form and $\mathbf{G}_u$ a compact form of a semisimple complex Lie group $\mathbf{G}$, with $\mathbf{K}_0 = \mathbf{G}_0 \cap \mathbf{G}_u$ a maximal compact subgroup of $\mathbf{G}_0$ and σ, τ the commuting conjugations on $\mathbf{G}$, and on its Lie algebra $\mathfrak{g}$, with respect to $\mathbf{G}_0$ and $\mathbf{G}_u$, respectively. The composition $\theta = \sigma \circ \tau$ is the complexification of a Cartan involution of $\mathbf{G}_0$ (and of its Lie algebra $\mathfrak{g}_0$) commuting with σ and τ. The complexification $\mathbf{K}$ of $\mathbf{K}_0$ is the subgroup $\mathbf{G}^\theta$ of the elements $x \in \mathbf{G}$ which are fixed by θ.

Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$ and consider the actions of the Lie groups $\mathbf{G}_0$, $\mathbf{K}$, $\mathbf{K}_0$ on the complex flag manifold $M = \mathbf{G}/\mathbf{Q}$.

Notation For $p \in M = \mathbf{G}/\mathbf{Q}$, let us set

$$M_+(p) = \mathbf{G}_0 \cdot p \simeq \mathbf{G}_0/\mathbf{E}_0, \qquad \text{with } \mathbf{E}_0 = \{x \in \mathbf{G}_0 \mid x \cdot p = p\}, \tag{1}$$

$$M_-(p) = \mathbf{K} \cdot p \simeq \mathbf{K}/\mathbf{V}, \qquad \text{with } \mathbf{V} = \{z \in \mathbf{K} \mid z \cdot p = p\}, \tag{2}$$

$$M_0(p) = \mathbf{K}_0 \cdot p = \mathbf{K}_0/\mathbf{V}_0, \qquad \text{with } \mathbf{V}_0 = \{x \in \mathbf{K}_0 \mid x \cdot p = p\}. \tag{3}$$

We note that $M_+(p)$ are real, $M_-(p)$ complex and $M_0(p)$ compact submanifolds of M, and denote by $\mathcal{M}_+$, $\mathcal{M}_-$, $\mathcal{M}_0$ the sets of orbits of $\mathbf{G}_0$, $\mathbf{K}$, $\mathbf{K}_0$ in M, respectively. We know (see [20, 21, 26]) that

Theorem 1.5 *There are finitely many orbits in $\mathcal{M}_+$ and in $\mathcal{M}_-$.*

There is a one-to-one correspondence $\mathcal{M}_+ \leftrightarrow \mathcal{M}_-$ such that $M_+(p_+)$ and $M_-(p_-)$ are related if and only if $M_+(p_+) \cap M_-(p_-) = M_0(p_0) \in \mathcal{M}_0$ (Matsuki duality).

The proof of Theorem 1.5 is done by considering the elements of $\mathcal{M}_\pm$ in the Grassmannian of $\dim(\mathfrak{q})$-subspaces of $\mathfrak{g}$. On each orbit we can pick a $\mathfrak{q}'$ containing a θ-stable Cartan subalgebra of $\mathfrak{g}$. The fact that $\mathcal{M}_\pm$ is finite is then a consequence of the fact that there are only finitely many conjugacy classes of Cartan subalgebras with respect to the action of either $\mathbf{G}_0$ or $\mathbf{K}$ (see e.g. [17, Prop. 6.64]). The last part of the statement is a consequence of the following

Lemma 1.6 *Let $\mathfrak{b}$ and $\mathfrak{b}'$ be Borel subalgebras containing θ-stable Cartan subalgebras of $\mathfrak{g}_0$. If $\mathfrak{b}$ and $\mathfrak{b}'$ are either $\mathbf{G}_0$- or $\mathbf{K}$-conjugate, then they are $\mathbf{K}_0$-conjugate.*

Proof Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$, where $\mathfrak{h}$ and $\mathfrak{h}'$ are the complexifications of θ-stable Cartan subalgebras $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ of $\mathfrak{g}_0$ and $\mathfrak{n}$, $\mathfrak{n}'$ the nilradicals of $\mathfrak{b}$, $\mathfrak{b}'$, respectively.

Assume that $\mathfrak{b} = \mathrm{Ad}(g_0)(\mathfrak{b}')$, for some $g_0 \in \mathbf{G}_0$. The proof in the case where $\mathfrak{b} = \mathrm{Ad}(k)(\mathfrak{b}')$ for some $k \in \mathbf{K}$ is similar, and will be omitted.

Both $\mathrm{Ad}(g_0)(\mathfrak{h}'_0)$ and $\mathfrak{h}_0$ are Cartan subalgebras of $\mathfrak{b} \cap \mathfrak{g}_0$, which is a solvable Lie subalgebra of $\mathfrak{g}_0$. Hence $\mathrm{Ad}(g_0)(\mathfrak{h}'_0) = \mathrm{Ad}(\exp(X_0))(\mathfrak{h}_0)$ for some X_0 in the nilradical of $\mathfrak{b} \cap \mathfrak{g}_0$. For $g_1 = \exp(-X_0) \cdot g_0$, we have $\mathrm{Ad}(g_1)(\mathfrak{h}'_0) = \mathfrak{h}_0$ and $\mathrm{Ad}(g_1)(\mathfrak{b}') = \mathfrak{b}$.

To show that $g_1 \in \mathbf{K}_0$, we use the Cartan decomposition $\mathbf{G}_0 = \exp(\mathfrak{p}_0) \cdot \mathbf{K}_0$, where $\mathfrak{p}_0 = \{ Y \in \mathfrak{g}_0 \mid \theta(Y) = -Y\}$, to write $g_1 = \exp(Y_0) \cdot k_0$, with $Y_0 \in \mathfrak{p}_0$ and $k_0 \in \mathbf{K}_0$. From

$$\mathrm{Ad}(g_1)(\mathfrak{h}'_0) = \mathfrak{h}_0 = \theta(\mathfrak{h}_0) = \theta(\mathrm{Ad}(g_1)(\theta(\mathfrak{h}'_0)) = \mathrm{Ad}(\theta(g_1))(\mathfrak{h}'_0)$$

we obtain that

$$\mathrm{Ad}(\exp(Y_0))(\mathrm{Ad}(k_0)(\mathfrak{h}'_0)) = \mathrm{Ad}(\exp(-Y_0))(\mathrm{Ad}(k_0)(\mathfrak{h}'_0)),$$

i.e. $y_0 = \mathrm{Ad}(\exp(2Y_0))$ normalizes the θ-stable Cartan subalgebra $\mathrm{Ad}(k_0)(\mathfrak{h}'_0)$. This implies that $y_0 \in \mathbf{K}_0$ and thus that $Y_0 = 0$, yielding $g_1 = k_0 \in \mathbf{K}_0$. □

From Lemma 1.6 we immediately obtain the statement on the Matsuki duality in the case where $\mathbf{Q}$ is a Borel subgroup $\mathbf{B}$. The general case follows by considering the natural fibration $\mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{Q}$ for a Borel $\mathbf{B} \subset \mathbf{Q}$. □

1.4 CR Manifolds

Let M_0 be a smooth manifold of real dimension m, countable at infinity. A *formally integrable partial complex structure* of type (n, k) (with $m = 2n + k$) on M_0 is a pair (HM_0, J), consisting of a rank $2n$ vector subbundle HM_0 of its tangent bundle TM_0 and a smooth fiber-preserving vector bundle isomorphism $J : HM_0 \to HM_0$ with $J^2 = -I$, satisfying the integrability conditions

$$\begin{cases} [X, Y] - [JX, JY] \in \Gamma(M_0, HM_0), \\ [X, JY] + [JX, Y] = J([X, Y] - [JX, JY]), \end{cases} \qquad \forall X, Y \in \Gamma(M_0, HM_0).$$

This is equivalent to the fact that the subbundles $T^{1,0}M_0 = \{x - iJX \mid X \in HM_0\}$, $T^{0,1}M_0 = \{x + iJX \mid X \in HM_0\}$ of the complexification $\mathbb{C} \otimes HM_0$ of the structure bundle HM_0 are formally *Frobenius integrable*, i.e. that

$$[\Gamma(M_0, T^{1,0}M_0), \Gamma(M_0, T^{1,0}M_0)] \subset \Gamma(M_0, T^{1,0}M_0),$$

or, equivalently, that

$$[\Gamma(M_0, T^{0,1}M_0), \Gamma(M_0, T^{0,1}M_0)] \subset \Gamma(M_0, T^{0,1}M_0).$$

These complex subbundles are the eigenspaces of J corresponding to the eigenvalues $\pm i$ and $T^{1,0}M_0 = \overline{T^{0,1}M_0}$, $T^{1,0}M_0 \cap T^{0,1}M_0 = \{0\}$, $T^{1,0}M_0 \oplus T^{0,1}M_0 = \mathbb{C} \otimes HM_0$.

An (abstract) *CR manifold* M_0 of type (n, k) is a smooth paracompact real manifold M_0 on which a formally integrable partial complex structure (HM_0, J), of type (n, k), has been fixed. The integers n and k are its *CR* dimension and codimension, respectively. Complex manifolds have $k = 0$, while for $n = 0$ we say that M_0 is *totally real.*

A smooth map $f : M_0 \to N_0$ between *CR* manifolds is *CR* iff its differential df maps HM_0 to HN_0 and commutes with the partial complex structures.

If $f : M_0 \to N_0$ is a *CR* map and a smooth immersion (resp. embedding) such that $df^{-1}(HN_0) = HM_0$, then we say that f is a *CR* immersion (resp. embedding). Let M_0 be of type (n_{M_0}, k_{M_0}) and N_0 of type (n_{N_0}, k_{N_0}). A CR immersion (or embedding) $f : M_0 \to N_0$ is *generic* if $n_{M_0} + k_{M_0} = n_{N_0} + k_{N_0}$.

The *characteristic bundle* H^0M_0 of a *CR* manifold M_0, of type (n, k), is the annihilator bundle of its structure bundle HM_0. It is a rank k linear subbundle of the real cotangent bundle T^*M_0. Its elements parametrize the *scalar Levi forms* of M_0: if $\xi \in H^0_{p_0}M_0$ and $X \in H_{p_0}M_0$, then

$$\mathrm{L}_\xi(X) = d\tilde{\xi}(X, JX) = \xi([J\tilde{X}, \tilde{X}]),$$

where $\tilde{\xi} \in \Gamma(M_0.H^0M_0)$ extends ξ and $\tilde{X} \in \Gamma(M_0, HM_0)$ extends X, is a quadratic form on $H_{p_0}M$, which is Hermitian with respect to the complex structure defined by J. This L_ξ is the *scalar Levi form at the characteristic* ξ.

If $\tilde{Z} = \tilde{X} + iJ\tilde{X} \in \Gamma(M_0, T^{0,1}M_0)$, then $\mathrm{L}_\xi(X) = \frac{1}{2}\xi(i[\tilde{Z}^*, \tilde{Z}])$ and therefore we can as well consider the scalar Levi forms as defined on $T^{0,1}_{p_0}M_0$.

Let $\lambda(\alpha) = (\lambda^+(\xi), \lambda^-(\xi))$ be the signature of the Hermitian form L_ξ. Then $\nu_\xi = \min\{\lambda^+(\xi), \lambda^-(\xi)\}$ is its *Witt index.*

We say that M_0 is *strongly q-pseudoconcave at a point* p_0 if the Witt index of L_ξ is greater or equal to q for all nonzero $\xi \in H^0_{p_0}M_0$.

For the relevant definitions of the tangential Cauchy-Riemann complex and the relationship of its groups with q-pseudoconcavity we refer e.g. to [1, 2, 5, 9, 13–15, 25].

1.5 Homogeneous CR Manifolds

Let $\mathbf{G}_0$ be a real Lie group with Lie algebra $\mathfrak{g}_0$, and denote by $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_0$ its complexification. A $\mathbf{G}_0$-*homogeneous* CR *manifold* is a $\mathbf{G}_0$-homogeneous *smooth* manifold endowed with a $\mathbf{G}_0$-invariant CR structure.

Let M_0 be a $\mathbf{G}_0$-homogeneous CR manifold. Fix a point $p_0 \in M_0$ and denote by $\mathbf{E}_0$ its stabilizer in $\mathbf{G}_0$. The natural projection

$$\pi : \mathbf{G}_0 \longrightarrow \mathbf{G}_0 / \mathbf{E}_0 \simeq M_0,$$

makes $\mathbf{G}_0$ the total space of a principal $\mathbf{E}_0$-bundle with base M_0. Denote by $\mathcal{Z}(\mathbf{G}_0)$ the space of smooth sections of the pullback to $\mathbf{G}_0$ of $T^{0,1}M$. It is the set of complex valued vector fields Z on $\mathbf{G}_0$ such that $d\pi^{\mathbb{C}}(Z_g) \in T^{0,1}_{\pi(g)}M_0$, for all $g \in \mathbf{G}_0$.

Since $T^{0,1}M$ is formally integrable, $\mathcal{Z}(\mathbf{G}_0)$ is formally integrable, i.e.

$$[\mathcal{Z}(\mathbf{G}_0), \mathcal{Z}(\mathbf{G}_0)] \subset \mathcal{Z}(\mathbf{G}_0).$$

Being invariant by left translations, $\mathcal{Z}(\mathbf{G}_0)$ is generated, as a left $C^\infty(\mathbf{G}_0)$-module, by its left invariant vector fields. Hence

$$\mathfrak{e} = (d\pi^{\mathbb{C}})^{-1}(T^{0,1}_{x_0}M) \subset \mathfrak{g} = T^{\mathbb{C}}_e \mathbf{G}_0 \tag{4}$$

is an $\mathrm{Ad}(\mathbf{E}_0)$-invariant complex Lie subalgebra of $\mathfrak{g}$. We have:

Lemma 1.7 *Denote by $\mathfrak{e}_0$ the Lie algebra of the isotropy subgroup $\mathbf{E}_0$. Then (4) establishes a one-to-one correspondence between the $\mathbf{G}_0$-homogeneous CR structures on $M_0 = \mathbf{G}_0 / \mathbf{E}_0$ and the $\mathrm{Ad}_{\mathfrak{g}}(\mathbf{E}_0)$-invariant complex Lie subalgebras $\mathfrak{e}$ of $\mathfrak{g}$ such that $\mathfrak{e} \cap \mathfrak{g}_0 = \mathfrak{e}_0$.* □

The pair $(\mathfrak{g}_0, \mathfrak{e})$ completely determines the homogeneous *CR* structure of M_0 and is called *the CR-algebra* of M_0 at p_0 (see [22]).

Let M_0, N_0 be $\mathbf{G}_0$-homogeneous CR manifolds and $\phi : M_0 \to N_0$ a $\mathbf{G}_0$-equivariant smooth map. Fix $p_0 \in M_0$ and let $(\mathfrak{g}_0, \mathfrak{e})$ and $(\mathfrak{g}_0, \mathfrak{f})$ be the CR algebras associated to M_0 at p_0 and to N_0 at $\phi(p_0)$, respectively. Then $\mathfrak{e} \cap \bar{\mathfrak{e}} \subset \mathfrak{f} \cap \bar{\mathfrak{f}}$, and ϕ is *CR* if and only if $\mathfrak{e} \subset \mathfrak{f}$ and is *a CR-submersion* if and only if $\mathfrak{f} = \mathfrak{e} + \mathfrak{f} \cap \bar{\mathfrak{f}}$.

The fibers of a $\mathbf{G}_0$-equivariant *CR* submersion are homogeneous *CR* manifolds: if $\mathbf{F}_0$ is the stabilizer of $\phi(p_0) \in N_0$, with Lie algebra $\mathfrak{f}_0 = \mathfrak{f} \cap \mathfrak{g}_0$, then $\Phi_0 = \phi^{-1}(\phi(p_0)) \simeq \mathbf{F}_0/\mathbf{E}_0$ has $(\mathfrak{f}_0, \mathfrak{e} \cap \bar{\mathfrak{f}})$ as associated *CR* algebra at p_0.

Corollary 1.8 *A CR-submersion $\phi : M_0 \to N_0$ has :*

(1) *totally real fibers if and only if $\mathfrak{e} \cap \bar{\mathfrak{f}} = \bar{\mathfrak{e}} \cap \mathfrak{f} = \mathfrak{e} \cap \bar{\mathfrak{e}}$;*
(2) *complex fibers if and only if $\mathfrak{e} \cap \bar{\mathfrak{f}} + \bar{\mathfrak{e}} \cap \mathfrak{f} = \mathfrak{f} \cap \bar{\mathfrak{f}}$.* □

1.6 $\mathfrak{n}$-Reductive Compact CR Manifolds

Let κ be a reductive complex Lie algebra and

$$\kappa = \mathfrak{z} \oplus \mathfrak{s}$$

its decomposition into the sum of its center $\mathfrak{z} = \{X \in \kappa \mid [X, Y] = 0, \ \forall Y \in \kappa\}$ and its semisimple ideal $\mathfrak{s} = [\kappa, \kappa]$. We fix a faithful linear representation of κ in which the elements of $\mathfrak{z}$ correspond to semisimple matrices. We call semisimple and nilpotent the elements of κ which are associated to semisimple and nilpotent matrices, respectively. Each $X \in \kappa$ admits a unique Jordan-Chevalley decomposition

$$X = X_s + X_n, \quad \text{with } X_s,\ X_n \in \kappa \text{ and } X_s \text{ semisimple}, X_n \text{ nilpotent}. \tag{5}$$

A real or complex Lie subalgebra $\mathfrak{v}$ of κ is called *splittable* if, for each $X \in \mathfrak{v}$, both X_s and X_n belong to $\mathfrak{v}$.

Let $\mathfrak{v}$ be a Lie subalgebra of κ and $\text{rad}(\mathfrak{v})$ its radical (i.e. its maximal solvable ideal). Denote by

$$\mathfrak{v}_n = \{X \in \text{rad}(\mathfrak{v}) \mid \text{ad}(X) \text{ is nilpotent}\} \tag{6}$$

its nilradical (see [12, p. 58]). It is the maximal nilpotent ideal of $\mathfrak{v}$. We have (see [7, Ch. VII, §5, Prop. 7]):

Proposition 1.9 *Every splittable Lie subalgebra* $\mathfrak{v}$ *is a direct sum*

$$\mathfrak{v} = \mathfrak{v}_r + \mathfrak{v}_n, \tag{7}$$

of its nilradical $\mathfrak{v}_n$ *and of a reductive subalgebra* $\mathfrak{v}_r$*, which is uniquely determined modulo conjugation by elementary automorphisms of* $\mathfrak{v}$. □

We assume in the following that κ is the complexification of a compact Lie algebra κ_0. Since compact Lie algebras are reductive, and the complexification of a reductive real Lie algebra is reductive, κ is complex reductive. Conjugation in κ will be taken with respect to the real form κ_0.

Proposition 1.10 *For any complex Lie subalgebra* $\mathfrak{v}$ *of* κ*, the intersection* $\mathfrak{v} \cap \bar{\mathfrak{v}}$ *is reductive and splittable. In particular* $\mathfrak{v} \cap \bar{\mathfrak{v}} \cap \mathfrak{v}_n = \{0\}$*. A splittable* $\mathfrak{v}$ *admits a Levi-Chevalley decomposition with a reductive Levi factor containing* $\mathfrak{v} \cap \bar{\mathfrak{v}}$. □

Let M_0 be a $\mathbf{K}_0$-homogeneous *CR* manifold, $p_0 \in M_0$ and $(\kappa_0, \mathfrak{v})$ its *CR* algebra at p_0.

Definition 1.1 We say that M_0 and its CR-algebra $(\kappa_0, \mathfrak{v})$ are $\mathfrak{n}$-*reductive* if

$$\mathfrak{v} = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus \mathfrak{v}_n, \tag{8}$$

i.e. if $\mathfrak{v}_r = \mathfrak{v} \cap \bar{\mathfrak{v}}$ is a reductive Levi factor in $\mathfrak{v}$.

If $(\kappa_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive, then $\mathfrak{v}$ is splittable. Indeed the elements of $\mathfrak{v}_n$ are nilpotent and those of $\mathfrak{v} \cap \kappa_0$ semisimple. Having a set of generators that are either nilpotent or semisimple, $\mathfrak{v}$ is splittable (see [7, Ch. VII, §5, Thm. 1]).

Let us consider the situation of Sect. 1.3 and keep the notation therein. The submanifolds $M_0(p)$ are examples of compact $\mathbf{K}_0$-homogeneous *CR* submanifold of the complex flag manifolds M. In [3, §6] it was shown that

Proposition 1.11 *The inclusion $M_0(p) \hookrightarrow M_-(p)$ is a generic CR embedding. A necessary and sufficient condition for $M_0(p)$ to be $\mathfrak{n}$-reductive is that*

$$M_0(p) = M_+(p) \cap M_-(p).$$

2 Mostow Fibration and Applications to Cohomology

We use the notation of Sect. 1.6. Let $\mathbf{V}$ be the analytic subgroup of $\mathbf{K}$ with Lie algebra $\mathfrak{v}$ and $\mathbf{V}_0$ the isotropy subgroup at $p_0 \in M_0$, having Lie algebra $\mathfrak{v}_0 = \mathfrak{v} \cap \mathfrak{k}_0$.

Proposition 2.1 ([3, Theorem 26]) *If M_0 is $\mathfrak{n}$-reductive, then $\mathbf{V}$ is an algebraic subgroup of $\mathbf{K}$ and the natural map*

$$M_0 = \mathbf{K}_0/\mathbf{V}_0 \longrightarrow M_- = \mathbf{K}/\mathbf{V}$$

is a generic CR embedding. □

When, as in Sect. 1.3, M_0 is the intersection of two Matsuki dual orbits in a complex flag manifold, M_- has a compactification $\tilde{M}_-$ which is a complex projective variety. Thus, in principle, we can study its Dolbeault cohomology by algebraic geometric techniques. On the other hand, by Mostow's decomposition (see [23, 24]) we know that M_- is a $\mathbf{K}_0$-equivariant fiber bundle over M_0, whose fibers are totally real Euclidean subspaces. We can exploit this fact for constructing an exhaustion function on M_- whose level sets can be used to relate the tangential *CR* cohomology of M_0 to the Dolbeault cohomology of M_-.

2.1 Mostow Fibration of M_-

The isotropy $\mathbf{V}_0$ is a maximal compact subgroup of $\mathbf{V}$ and, putting together the Levi-Chevalley decomposition of $\mathbf{V}$ and the Cartan decomposition of $\mathbf{V}_r$, we have a diffeomorphism

$$\mathbf{V}_0 \times \mathfrak{v}_0 \times \mathfrak{v}_n \ni (x, Y, Z) \longrightarrow x \cdot \exp(iY) \cdot \exp(Z) \in \mathbf{V}. \tag{9}$$

Then by [24, Theorem A] *we can find a closed Euclidean subspace F of $\mathbf{K}$ such that* $\mathrm{ad}(y)(F) = F$, *for all $y \in \mathbf{V}_0$, and*

$$\mathbf{K}_0 \times F \times \mathfrak{v}_0 \times \mathfrak{v}_n \ni (x, f, Y, Z) \longrightarrow x \cdot f \cdot \exp(iY) \cdot \exp(Z) \in \mathbf{K} \tag{10}$$

is a diffeomorphism.

Let b be an $\mathrm{Ad}(\mathbf{K}_0)$-invariant scalar product on $\mathfrak{k}_0$ and set

$$\mathfrak{m}_0 = ((\mathfrak{v} + \overline{\mathfrak{v}}) \cap \mathfrak{k}_0)^{\perp}, \quad \mathfrak{m} = \mathbb{C} \otimes \mathfrak{m}_0.$$

Since[1] $\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n = \mathfrak{v}_n \oplus ([\mathfrak{v}_n \oplus \bar{\mathfrak{v}}_n] \cap \kappa_0)$, we obtain the decomposition

$$\kappa = \kappa_0 \oplus i\,\mathfrak{m}_0 \oplus (i\,\mathfrak{v}_0 \oplus \mathfrak{v}_n),$$

which suggests that $\exp(i\mathfrak{m}_0)$ could be a reasonable candidate for the fiber F of the Mostow fibration. Let us consider the smooth map

$$\mathbf{K}_0 \times \mathfrak{m}_0 \times \mathfrak{v}_0 \times v_n \ni (x, T, Y, Z) \longrightarrow x \cdot \exp(iT) \cdot \exp(iY) \cdot \exp(Z) \in \mathbf{K}. \qquad (11)$$

Proposition 2.2 *The map* (11) *is onto and we can find* $r > 0$ *such that its restriction to* $\{b(T, T) < r^2\}$ *is a diffeomorphism with the image.* □

Let $\mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{m}_0$ be the quotient of $\mathbf{K}_0 \times \mathfrak{m}_0$ by $(x_1, T_1) \sim (x_2, T_2)$ iff $x_2 = x_1 \cdot y$ and $T_1 = \mathrm{Ad}(y)(T_2)$ and $\pi : \mathbf{K} \ni z \to z \cdot \mathbf{V} \in M_-$ the canonical projection. By passing to the quotients, (11) yields a smooth map

$$\mathbf{K}_0 \times_{\mathbf{V}_0} \mathfrak{m}_0 \ni [x, T] \longrightarrow \pi(x \cdot \exp(iT)) \in M_-, \qquad (12)$$

which is surjective and, when restricted to $\{b(T, T) < r^2\}$, defines a tubular neighborhood U_r of M_0 in M_-. The function $\phi([x, T]) = b(T, T)/(r^2 - b(T, T))$ is then an exhaustion function of the tubular neighborhood U_r.

2.2 A Local Result

We can use Proposition 2.2 to precise, in this special case, the size of the tubular neighborhoods of [13, Theorem 2.1]:

Proposition 2.3 *Assume that* M_0 *is q-pseudoconcave, of type* (n, k)*. Then we can find* $r_0 > 0$ *such that* U_r *is q-pseudoconcave and* $(n - q)$*-pseudoconvex and the natural restriction maps*

$$H^{p,j}_{\bar\partial}(U_r) \longrightarrow H^{p,j}_{\bar\partial_{M_0}}(M_0)$$

are isomorphisms of finite dimensional vector spaces for $0 < r < r_0$*, for all* $0 \le p \le n + k$ *and either* $j < q$ *or* $j > n - q$*.* □

[1] In fact, if $\bar{Z} \in \bar{\mathfrak{v}}_n$, then $\bar{Z} = -Z + (Z + \bar{Z})$, with $Z = \bar{\bar{Z}} \in \mathfrak{v}_n$ and $Z + \bar{Z} \in \kappa_0$. The sum is direct because $\mathfrak{v}_n \cap \kappa_0 = \{0\}$.

2.3 Mostow Fibration of M_- in the HNR Case

In [19] the Authors show that (11) and (12) are not, in general, global diffeomorphisms. To single out a large class of compact homogeneous $\mathfrak{n}$-reductive *CR* manifolds having a *nice* Mostow fibration, they introduce the following notion.[2]

Definition 2.1 The *CR* algebra $(\mathfrak{k}_0, \mathfrak{v})$ is HNR if $\mathfrak{v} = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus \mathfrak{v}_n$ and

$$\mathfrak{q} = \{Z \in \mathfrak{k} \mid [Z, \mathfrak{v}_n] \subset \mathfrak{v}_n\}$$

is parabolic.

Note that, when $(\mathfrak{k}_0, \mathfrak{v})$ is $\mathfrak{n}$-reductive, it is always possible to find a parabolic $\mathfrak{q}$ in $\mathfrak{k}$ with $\mathfrak{v} \subset \mathfrak{q}$, $\mathfrak{q} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \oplus \mathfrak{q}_n$, and $\mathfrak{v}_n \subset \mathfrak{q}_n$. Then $(\mathfrak{k}_0, \mathfrak{v}_r \oplus \mathfrak{q}_n)$ is HNR and describes a *stronger* $\mathbf{K}_0$-homogeneous *CR* structure on the same manifold M_0.

Proposition 2.4 *If its CR algebra* $(\mathfrak{k}_0, \mathfrak{v})$ *is* HNR, *then* (11) *and* (12) *are diffeomorphisms.*

In this case M_- admits *a Mostow fibration with Hermitian fiber.*

2.4 Application to Cohomology

In the HNR case we can use $\phi([x, T]) = b(T, T)$ as an exhaustion function on M_-. Assume that M_0 has type (n, k). It is shown in [19] that, for $T \in \mathfrak{m}_0$ and $\xi = b(T, \cdot) \in H^0_{p_0} M_0$, if the Levi form L_ξ has signature $(\lambda^+(\xi), \lambda^-(\xi))$, then the complex Hessian of ϕ at the point $\pi(x \cdot \exp(iT))$ has signature $(\lambda^+(\xi) + k, \lambda^-(\xi))$. Then we get

Proposition 2.5 *Assume that* M_0 *is q-pseudoconcave and has a CR algebra which is* HNR. *Then* M_- *is q-pseudoconcave and* $(n - q)$*-pseudoconvex and the natural restriction maps*

$$H^{p,j}_{\bar\partial}(U_r) \longrightarrow H^{p,j}_{\bar\partial_{M_0}}(M_0)$$

are isomorphisms of finite dimensional vector spaces for $0 < r < r_0$, *for all* $0 \le p \le n + k$ *and either* $j < q$ *or* $j > n - q$.

Proof The statement follows from [4] and the computation of the signature of the exhaustion function ϕ. □

[2]Actually they consider a slightly less restrictive condition, which is related to a notion of weak *CR*-degeneracy for homogeneous *CR* manifolds that was introduced in [22].

3 Example: General Orbits of SU(p,q) in the Grassmannian

Let us consider the orbits of the real form $\mathbf{G}_0 = \mathbf{SU}(p,q)$ of $\mathbf{G} = \mathbf{SL}_{p+q}(\mathbb{C})$ in $\mathcal{Gr}_m(\mathbb{C}^{p+q})$. We assume that $p \leq q$. Let $\hbar$ be the Hermitian symmetric form of signature (p,q) in $\mathbb{C}^{p+q}$ employed to define $\mathbf{SU}(p,q)$.

The orbits of $\mathbf{G}_0$ are classified by the signature of the restriction of $\hbar$ to their m-planes: to a pair of nonnegative integers a, b with

$$a+b \leq m, \quad p_0 = \max\{0, m-q\} \leq a \leq p, \quad q_0 = \max\{0, m-p\} \leq b \leq q, \qquad (*)$$

correspond the orbit $M_+(a,b)$ consisting of m-planes ℓ for which $\ker(\hbar|_\ell)$ has signature (a,b).

To fix a maximal compact subgroup of $\mathbf{SU}(p,q)$ we choose a couple of $\hbar$-orthogonal subspaces $W_+ \simeq \mathbb{C}^p$, $W_- \simeq \mathbb{C}^q$ of $\mathbb{C}^{p+q}$, with $\hbar > 0$ on W_+, $\hbar < 0$ on W_-, $\mathbb{C}^{p+q} = W_+ \oplus W_-$. Then $\mathbf{K}_0 \simeq \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$, and $\mathbf{K} = \mathbb{C} \otimes \mathbf{K}_0 \simeq \mathbf{S}(\mathbf{GL}_p(\mathbb{C}) \times \mathbf{GL}_q(\mathbb{C}))$, with the first factor operating on W_+ and the second on W_-. The orbits of $\mathbf{K}$ in $\mathcal{Gr}_m(\mathbb{C}^{p+q})$ are characterized by the dimension of $\ell \cap W_+$ and $\ell \cap W_-$. Let us set, for a, b satisfying $(*)$,

$$\begin{aligned}
&M_+(a,b) = \{\ell \in \mathcal{Gr}_m(\mathbb{C}^{p+q}) \mid \hbar|_\ell \text{ has signature } (a,b)\},\\
&M_-(a,b) = \{\ell \in \mathcal{Gr}_m(\mathbb{C}^{p+q}) \mid \dim_{\mathbb{C}}(\ell \cap W_+) = a, \ \dim_{\mathbb{C}}(\ell \cap W_-) = b\},\\
&M_0(a,b) = M_+(a,b) \cap M_-(a,b).
\end{aligned}$$

To describe the *CR* structure of $M_0(a,b)$ it suffices to compute the Lie algebra of the stabilizer in $\mathbf{K}$ of any of its points. Let $e_1, \dots, e_p$ be an orthonormal basis of W_+ and $e_{p+1}, \dots, e_{p+q}$ an orthonormal basis of W_-. Set $c = m-a-b$ and $n_1 = a$, $n_2 = c$, $n_3 = p-a-c$, $n_4 = b$, $n_5 = c$, $n_6 = q-b-c$. Let us choose the base point $p_0 = \langle e_1, \dots, e_a, (e_{a+1}+e_{p+b+1}), \dots, (e_{a+c}+e_{p+b+c}), e_{p+1}, \dots, e_{p+b}\rangle$. Then

$$\mathfrak{v} = \left\{ \begin{pmatrix} Z_{1,1} & 0 & Z_{1,3} & 0 & 0 & 0 \\ 0 & Z_{2,2} & Z_{2,3} & 0 & 0 & 0 \\ 0 & 0 & Z_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_{4,4} & 0 & Z_{4,6} \\ 0 & 0 & 0 & 0 & Z_{2,2} & Z_{5,6} \\ 0 & 0 & 0 & 0 & 0 & Z_{6,6} \end{pmatrix} \middle| \, Z_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap \mathfrak{sl}_{p+q}(\mathbb{C}).$$

We see that $(\kappa_0, \mathfrak{v})$ is HNR and $M_0(a,b)$ has *CR* dimension

$$n = n_1 n_3 + n_2 n_3 + n_4 n_6 + n_2 n_6.$$

We have

$$\mathfrak{m} = \left\{ \left. \begin{pmatrix} 0 & W_{1,2} & 0 & 0 & 0 & \\ W_{2,1} & W_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{4,5} & 0 \\ 0 & 0 & 0 & W_{5,4} & -W_{2,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| W_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap \mathfrak{sl}_{p+q}(\mathbb{C})$$

and hence the *CR* codimension of $M_0(a, b)$ is

$$k = n_1 n_2 + n_2 n_4 + n_2^2 = n_2(n_1 + n_2 + n_4).$$

We observe that for $c = 0$ the $M_+(a, b)$ are the open orbits, while the minimal orbit of $\mathbf{G}_0$ is $M_+(p_0, q_0)$. Then $M_0(a, b)$ is μ-pseudoconcave with

$$\mu = \min\{p - a - c, q - b - c\},$$

and $M_-(a, b)$ is μ-pseudoconcave and $(n - \mu)$-pseudoconvex.

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Generalized Geometry of Norden and Para Norden Manifolds

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Abstract We study complex structures $\widehat{J}$ on the generalized tangent bundle of a smooth manifold M compatible with the standard symplectic structure. In particular we describe the class of such generalized complex structures defined by a pseudo Riemannian metric g and a g-symmetric operator H such that $H^2 = \mu I$, $\mu \in \mathbb{R}$. These structures include the case of complex Norden manifolds for $\mu = -1$ and the case of Para Norden manifolds for $\mu = 1$ (Nannicini, J Geom Phys 99:244–255, 2016; Nannicini, On a class of pseudo calibrated generalized complex structures: from Norden to para Norden through statistical manifolds, preprint, 2016). We describe integrability conditions of $\widehat{J}$ with respect to a linear connection ∇ and we give examples of geometric structures that naturally give rise to integrable generalized complex structures. We define the concept of generalized $\bar{\partial}$-*operator* of (M, H, g, ∇), and we describe certain holomorphic sections. We survey several results appearing in a series of author's previous papers, (Nannicini, J Geom Phys 56:903–916, 2006; Nannicini, J Geom Phys 60:1781–1791, 2010; Nannicini, Differ Geom Appl 31:230–238, 2013; Adv Geom 16(2):165–173, 2016; Nannicini, Adv Geom 16(2):165–173, 2016; Nannicini, J Geom Phys 99:244–255, 2016; Nannicini, On a class of pseudo calibrated generalized complex structures: from Norden to para Norden through statistical manifolds, preprint, 2016), with special attention to recent results on the generalized geometry of Norden and Para Norden manifolds (Nannicini, J Geom Phys 99:244–255, 2016; Nannicini, Balkan J Geom Appl 22:51–69, 2017).

1 Introduction

In 1990 Courant introduced the concept of *Dirac structure* to unify Poisson and Symplectic Geometry [3]. The complex analogue of Dirac structure is the concept of *Generalized Complex Structure*, introduced by Hitchin in 2003, [10], and further investigated by Gualtieri, [8], in order to unify Symplectic and Complex Geometry.

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Now there is a vaste literature on this topic, also because of its relation to *Mirror Symmetry*.

Let M be a smooth manifold, let $T(M)$ be the tangent bundle and let $T^\star(M)$ be the cotangent bundle of M. $E = T(M) \oplus T^\star(M)$ is called the *generalized tangent bundle* of M, it is the main object of *Generalized Geometry*. Hitchin considered complex structures of the generalized tangent bundle compatible with the standard metric of neutral signature on E. In this paper we describe a class of complex structures on the generalized tangent bundle compatible with the standard symplectic structure of E, that is a class of *pseudo calibrated generalized complex structures*. The concept we consider, also investigated in [4], differs from Hitchin's definition, one of the purpose of this study is to complement the existing literature on generalized complex structures highlighting similarities and differences with Hitchin's theory. We survey several results on this subject appearing in a series of author's previous papers, [15–20], with special attention to recent results on the generalized geometry of Norden and Para Norden manifolds [19, 20]. The paper is organized as follows. In Sects. 2 and 3 we introduce preliminary material of the generalized tangent bundle and of generalized complex structures; in Sect. 4 we state integrability conditions and in Sect. 5 we give examples of integrable structures; Sect. 6 is devoted to the study of complex Lie algebroids naturally associated to integrable pseudo calibrated generalized complex structures; in Sect. 7 we define the concept of generalized $\overline{\partial}$*-operator* of (M, H, g, ∇) and in Sect. 8 we describe some generalized holomorphic sections.

2 Geometry of the Generalized Tangent Bundle

In this section we recall the main geometrical properties of the generalized tangent bundle.

Let M be a smooth manifold of real dimension n and let $E = T(M) \oplus T^\star(M)$ be the *generalized tangent bundle* of M. Smooth sections of E are elements $X + \xi \in C^\infty(E)$ where $X \in C^\infty(T(M))$ is a vector field and $\xi \in C^\infty(T^\star(M))$ is a 1-form. E is equipped with a natural *symplectic structure*, $(\ ,\ )$, defined on two elements $X + \xi,\ Y + \eta \in C^\infty(E)$ by:

$$(X + \xi, Y + \eta) = -\frac{1}{2}(\xi(Y) - \eta(X)).$$

E is equipped with a natural *indefinite metric*, $<\ ,\ >$, defined by:

$$< X + \xi, Y + \eta >= -\frac{1}{2}(\xi(Y) + \eta(X));$$

$<\ ,\ >$ is non degenerate and of signature (n, n).

On $C^\infty(E)$ the *Courant bracket* is defined naturally by:

$$[X+\xi, Y+\eta]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}d(\xi(Y) - \eta(X))$$

where $\mathcal{L}$ means Lie derivative, d is the differential operator and $[\, ,\,]$ denotes the Lie bracket of vector fields on M. Moreover a linear connection on M, ∇, defines a bracket on $C^\infty(E)$, $[\, ,\,]_\nabla$, as follows:

$$[X+\xi, Y+\eta]_\nabla = [X,Y] + \nabla_X\eta - \nabla_Y\xi.$$

A direct computation gives the following.

Lemma 1 ([15]) *For all $X, Y \in C^\infty(T(M))$, for all $\xi, \eta \in C^\infty(T^\star(M))$ and for all $f \in C^\infty(M)$ we have:*

1. $[X+\xi, Y+\eta]_\nabla = -[Y+\eta, X+\xi]_\nabla$
2. $[f(X+\xi), Y+\eta]_\nabla = f[X+\xi, Y+\eta]_\nabla - Y(f)(X+\xi)$
3. *Jacobi's identity holds for $[\, ,\,]_\nabla$ if and only if ∇ has zero curvature.*

The following proposition gives a geometrical interpretation of the introduced bracket.

Proposition 2 ([16]) *Let ∇ be a linear connection on M, there is a bundle morphism:*

$$\Phi^\nabla : T(M) \oplus T^\star(M) \to T(T^\star(M))$$

which is an isomorphism on the fibres and such that

1. $(\Phi^\nabla)^\star(\Omega) = -2(\, ,\,)$ *if and only if ∇ has zero torsion*
2. $(\Phi^\nabla)([\, ,\,]_\nabla) = [\Phi^\nabla, \Phi^\nabla]$ *if and only if ∇ has zero curvature.*

where Ω is the canonical symplectic form on $T^\star(M)$ defined by the Liouville 1-form.

3 Pseudo Calibrated Generalized Complex Structures

In this section we define the concept of pseudo calibrated generalized complex structure, rather than the usual terminology from generalized complex geometry we consider the following.

Definition 3 A *generalized complex structure* on M is an endomorphism $\widehat{J} : E \to E$ such that $\widehat{J}^2 = -I$.

Definition 4 A generalized complex structure $\widehat{J}$ is called *pseudo calibrated* if it is $(\, ,\,)$-invariant and if the bilinear symmetric form defined by $(\, ,\widehat{J})$ on $T(M)$ is non degenerate. Moreover $\widehat{J}$ is called *calibrated* if it is pseudo calibrated and $(\, ,\widehat{J})$ is positive definite.

From the definition we get the following block matrix form of a generalized pseudo calibrated complex structure:

$$\widehat{J} = \begin{pmatrix} H & -(I+H^2)g^{-1} \\ g & -H^\star \end{pmatrix}$$

where $g : T(M) \to T^\star(M)$ is identified to the bemolle musical isomorphism of a pseudo Riemannian metric g on M, $H : T(M) \to T(M)$ is a g-symmetric operator, $H^\star : T^\star(M) \to T^\star(M)$ is the dual operator of H defined by $H^\star(\xi)(X) = \xi(H(X))$.

$\widehat{J}$ is calibrated if and only if g is a Riemannian metric, namely:

$$(g(X))(Y) = g(X, Y) = 2(X, \widehat{J}Y).$$

Remark 5 Let (M, g) be a pseudo Riemannian manifold and let H be a g-symmetric operator, we have:

$$\widehat{J}_H = \begin{pmatrix} H & -(I+H^2)g^{-1} \\ g & -H^\star \end{pmatrix} = \begin{pmatrix} I & Hg^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} I & -Hg^{-1} \\ 0 & I \end{pmatrix}$$

in particular, if H and K are two g-symmetric operators, we get:

$$\widehat{J}_K = \begin{pmatrix} I & (K-H)g^{-1} \\ 0 & I \end{pmatrix} \widehat{J}_H \begin{pmatrix} I & (H-K)g^{-1} \\ 0 & I \end{pmatrix}.$$

In the following we will consider g-symmetric operators $H : T(M) \to T(M)$ such that $H^2 = \mu I$ were $\mu \in \mathbb{R}$ and I denotes identity; if we pose $\lambda = -1 - \mu$ then we have:

$$\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^\star \end{pmatrix}.$$

Remark 6 If $\lambda = 0$ then (M, H, g) is a *Norden manifold*, (see Sect. 5.2). If $\lambda = -1$, and $ImH = KerH$, then (M, H, g) is an *almost tangent manifold* [2]. If $\lambda = -2$ then (M, H, g) is a *Para Norden manifold*, (see Sect. 5.3).

4 Integrability

In this section we define integrability of generalized complex structures with respect to a linear connection and we state integrability conditions for the class under consideration.

4.1 Linear Pseudo Calibrated Generalized Complex Structures

In order to give an interpretation of integrability, see Proposition 35, we prove the following result of linear algebra.

Proposition 7 *Let V be a real vector space and let $V^\star$ be the dual, a complex subspace $L \subset (V \oplus V^\star) \otimes \mathbb{C}$ is the holomorphic space of a pseudo calibrated generalized complex structure on V if and only if*

$$L = L(F, \epsilon) = \{v + \xi \in F \oplus (V \otimes \mathbb{C})^\star | \xi_{|F} = i_v(\epsilon)\}$$

where F is a complex subspace of $V \otimes \mathbb{C}$ such that $F + \bar{F} = V \otimes \mathbb{C}$, $\epsilon \in F^\star \otimes F^\star$ is symmetric complex bilinear and $Im(\epsilon_{|\Delta})$ is non degenerate, were $\Delta \in V$ is the real part of $F \cap \bar{F}$.

Proof Let (,) be the natural symplectic structure of $V \oplus V^\star$ and let J be a complex structure on $V \oplus V^\star$ (,)-invariant. Let $L = \{v - iJv | v \in V\}$, then L is a Lagrangian subspace of $(V \oplus V^\star) \otimes \mathbb{C}$. Let $p : (V \oplus V^\star) \otimes \mathbb{C} \to V \otimes \mathbb{C}$ and $p^\star : (V \oplus V^\star) \otimes \mathbb{C} \to V^\star \otimes \mathbb{C}$ be the canonical projections, we pose $F = p(L)$ and $\epsilon(p(x)) = p^\star(x)$ for all $x \in (V \oplus V^\star) \otimes \mathbb{C}$. We can easily prove that ϵ is well defined and symmetric. Moreover $L \cap \bar{L} = \{0\}$ implies that $Im(\epsilon_{|\Delta})$ is non degenerate. The converse it is easily seen. □

4.2 ∇-Integrability

Lemma 8 ([16]) *Let $\widehat{J} : E \to E$ be a generalized complex structure on M and let*

$$N^\nabla(\widehat{J}) : C^\infty(E) \times C^\infty(E) \to C^\infty(E)$$

defined for all $\sigma, \tau \in C^\infty(E)$ by:

$$N^\nabla(\widehat{J})(\sigma, \tau) = [\widehat{J}\sigma, \widehat{J}\tau]_\nabla - \widehat{J}[\widehat{J}\sigma, \tau]_\nabla - \widehat{J}[\sigma, \widehat{J}\tau]_\nabla - [\sigma, \tau]_\nabla.$$

$N^\nabla(\widehat{J})$ is a skew symmetric tensor called the **Nijenhuis tensor** *of $\widehat{J}$ with respect to ∇.*

Let $E^{\mathbb{C}} = (T(M) \oplus T^\star(M)) \otimes \mathbb{C}$ be the complexified generalized tangent bundle. The splitting in $\pm i$ eigenspaces of $\widehat{J}$ is denoted by: $E^{\mathbb{C}} = E^{1,0}_{\widehat{J}} \oplus E^{0,1}_{\widehat{J}}$ with $E^{0,1}_{\widehat{J}} = \overline{E^{1,0}_{\widehat{J}}}$. Let $P_+ : E^{\mathbb{C}} \to E^{1,0}_{\widehat{J}}$ and $P_- : E^{\mathbb{C}} \to E^{0,1}_{\widehat{J}}$ be the projection operators: $P_\pm = \frac{1}{2}(I \mp i\widehat{J})$. The following holds.

Lemma 9 ([19]) *For all* $\sigma, \tau \in C^\infty(E^{\mathbb{C}})$ we have:

$$P_{\mp}[P_{\pm}(\sigma), P_{\pm}(\tau)]_\nabla = -\frac{1}{4}P_{\mp}(N^\nabla(\widehat{J})(\sigma, \tau)).$$

Corollary 10 *For any linear connection* ∇ *on* M *we have that* $E^{1,0}_{\widehat{J}}$ *and* $E^{0,1}_{\widehat{J}}$ *are* $[\,,\,]_\nabla$*-involutive if and only if* $N^\nabla(\widehat{J}) = 0$.

Definition 11 Let $\widehat{J} : E \to E$ be a generalized complex structure on M, $\widehat{J}$ is called ∇*-integrable* if $N^\nabla(\widehat{J}) = 0$.
Let (M, g) be a pseudo Riemannian manifold, let ∇ be a linear connection on M, the *torsion* of ∇, T^∇, and the *exterior differential* associated to ∇ acting on g, $(d^\nabla g)$, are defined on $X, Y \in C^\infty(T(M))$ respectively by: $T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ and $(d^\nabla g)(X, Y) = (\nabla_X g)(Y) - (\nabla_Y g)(X) + g(T^\nabla(X, Y))$.

Let H be a g-symmetric operator on $T(M)$ and let $N(H)$ be the Nijenhuis tensor of H, defined on $X, Y \in C^\infty(T(M))$ by:

$$N(H)(X, Y) = [HX, HY] - H[HX, Y] - H[X, HY] + H^2[X, Y].$$

Let us suppose $H^2 = \mu I$, $\mu \in \mathbb{R}$, and let $\lambda = -1 - \mu$. Let $\widehat{J}$ be the pseudo calibrated generalized complex structure defined by g and H:

$$\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^\star \end{pmatrix}.$$

∇-integrability of $\widehat{J}$ is described as follows.

Theorem 12 ([20]) *For* $\lambda(\lambda + 1) \neq 0$ *the pseudo calibrated generalized complex structure* $\widehat{J}$ *is* ∇*-integrable if and only if the following conditions hold:*

$$\begin{cases} N(H) = 0 \\ \nabla H = 0 \\ d^\nabla g = 0. \end{cases}$$

For $\lambda = 0$ *the pseudo calibrated generalized complex structure* $\widehat{J} = \begin{pmatrix} H & 0 \\ g & -H^\star \end{pmatrix}$ *is* ∇*-integrable if and only if for all* $X, Y \in C^\infty(T(M))$ *the following conditions hold:*

$$\begin{cases} N(H) = 0 \\ (\nabla_{HX} H) - H(\nabla_X H) = 0 \\ (d^\nabla g)(HX, Y) + (d^\nabla g)(X, HY) - g((\nabla_X H)(Y) - (\nabla_Y H)(X)) = 0. \end{cases}$$

For $\lambda = -1$ *the pseudo calibrated generalized complex structure* $\widehat{J} = \begin{pmatrix} H & -g^{-1} \\ g & -H^{\star} \end{pmatrix}$ *is* ∇*-integrable if and only if for all* $X, Y \in C^{\infty}(T(M))$ *the following conditions hold:*

$$\begin{cases} N(H) = 0 \\ (\nabla_{HX} H) - H(\nabla_X H) = 0 \\ (\nabla_X H)(Y) - (\nabla_Y H)(X) = 0 \\ d^{\nabla} g = 0. \end{cases}$$

Corollary 13 *If* $H = 0$ *the pseudo calibrated generalized complex structure* $\widehat{J} = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$ *is* ∇*-integrable if and only if the following condition holds:*

$$d^{\nabla} g = 0.$$

5 Examples

In this section we will describe examples of geometrical structures on the manifold M that define integrable generalized pseudo calibrated complex structures.

5.1 Statistical Manifolds

The first class of examples can be found in the context of *Statistical manifolds*. They were introduced in [1, 21], are manifolds of probability distributions, used in Information Geometry and related to Codazzi tensors and Affine Geometry.

Definition 14 Let (M, g) be a pseudo Riemannian manifold and let ∇ be a linear connection on M. (M, g, ∇) is called *quasi statistical manifold* if

$$d^{\nabla} g = 0.$$

If ∇ is torsion free then (M, g, ∇) is called *statistical manifold.*

From Corollary 13 we get immediately the following.

Proposition 15 *Let* (M, g) *be a pseudo Riemannian manifold and let* ∇ *be a linear connection on M, the generalized complex structure on M defined by g:*

$$\widehat{J} = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix},$$

is ∇*-integrable if and only if* (M, g, ∇) *is a quasi statistical manifold.*

5.2 Norden Manifolds

The second class of examples can be found in the context of *Norden manifolds*. They were introduced by Norden in [22] and then studied also as *almost complex manifolds with B-metric* and *anti Kählerian manifolds*. They have applications in mathematics and in theoretical physics.

Definition 16 Let (M, H) be an almost complex manifold and let g be a pseudo Riemannian metric on M such that H is a g-symmetric operator, g is called *Norden metric* and (M, H, g) is called *Norden manifold*.

Definition 17 Let (M, H, g) be a Norden manifold, if H is integrable then (M, H, g) is called *complex Norden manifold*.

Definition 18 Let (M, H, g) be a Norden manifold and let ∇ be the Levi Civita connection of g, if $\nabla H = 0$ then (M, H, g) is called *Kähler Norden manifold*.

Remark 19 For a Kähler Norden manifold (M, H, g) the structure H is integrable. The following result is well known.

Theorem 20 ([7]) *Let (M, H, g) be a complex Norden manifold, there exists a unique linear connection D with torsion T on M such that:*

$$\begin{cases} (D_X g)(Y, Z) = 0 \\ T(HX, Y) + T(X, HY) = 0 \\ g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0. \end{cases}$$

for all X, Y, Z on M. D is called the **natural canonical connection**.

Remark 21 D is defined by: $D_X Y = \nabla_X Y - \frac{1}{2} H(\nabla_X H)Y$, where ∇ is the Levi Civita connection of g and satisfies the condition $DH = 0$.

Corollary 22 *For a Kähler Norden manifold (M, H, g) the natural canonical connection is the Levi Civita connection.*

Proposition 23 ([19]) *Let (M, H, g) be a complex Norden manifold and let D be the natural canonical connection on M, the generalized complex structure on M defined by H and g:*

$$\widehat{J} = \begin{pmatrix} H & 0 \\ g & -H^\star \end{pmatrix}$$

is D-integrable.

5.3 Para Norden Manifolds

The third class of examples lies in the context of *Para Norden manifolds*, [24].

Definition 24 An *almost paracomplex Norden manifold* (M, H, g) is a real $2n$-dimensional smooth manifold with a pseudo Riemannian metric g and a $(1, 1)$ tensor field such that $H^2 = I$, the two eigenbundles, T^+M, T^-M, associated to the two eigenvalues, $+1$ and -1 of H respectively have the same rank and H is a g-symmetric operator.

Definition 25 A *paraholomorphic Norden manifold*, or *para Kähler Norden manifold*, is an almost paracomplex Norden manifold (M, H, g) such that $\nabla H = 0$, where ∇ is the Levi Civita connection of g.

Proposition 26 ([20]) *Let (M, H, g) be a paraholomorphic Norden manifold and let ∇ be the Levi Civita connection of g, the generalized complex structure on M defined by H and g:*

$$\widehat{J} = \begin{pmatrix} H & -2g^{-1} \\ g & -H^\star \end{pmatrix}$$

is ∇-integrable.

6 Complex Lie Algebroids

Lie algebroids are generalization of Lie algebras and tangent vector bundles, they were introduced by Pradines in [23].

6.1 Preliminaries

Definition 27 A *complex Lie algebroid* is a complex vector bundle L over a smooth real manifold M such that: a Lie bracket $[\ ,]$ is defined on $C^\infty(L)$, a smooth bundle map $\rho : L \to T(M) \otimes \mathbb{C}$, called *anchor*, is defined and, for all $\sigma, \tau \in C^\infty(L)$, for all $f \in C^\infty(M)$ the following conditions hold:

1. $\rho([\sigma, \tau]) = [\rho(\sigma), \rho(\tau)]$
2. $[f\sigma, \tau] = f([\sigma, \tau]) - (\rho(\tau)(f))\sigma.$

Let L and its dual vector bundle $L^\star$ be Lie algebroids; on sections of $\wedge L$, respectively $\wedge L^\star$, the *Schouten bracket* is defined by:

$$[\ ,\ ]_L : C^\infty(\wedge^p L) \times C^\infty(\wedge^q L) \to C^\infty(\wedge^{p+q-1} L)$$

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]_L =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j]_L \wedge X_1 \wedge \overset{\widehat{i}}{\ldots} \wedge X_p \wedge Y_1 \wedge \overset{\widehat{j}}{\ldots} \wedge Y_q$$

for $f \in C^\infty(M)$, $X \in C^\infty(L)$:

$$[X,f]_L = -[f,X]_L = \rho(X)(f);$$

$$[\,,\,]_{L^\star} : C^\infty(\wedge^p L^\star) \times C^\infty(\wedge^q L^\star) \to C^\infty(\wedge^{p+q-1} L^\star)$$

$$[X_1^\star \wedge \ldots \wedge X_p^\star, Y_1^\star \wedge \ldots \wedge Y_q^\star]_{L^\star} =$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[X_i^\star, Y_j^\star]_{L^\star} \wedge X_1^\star \wedge ..\widehat{i}.. \wedge X_p^\star \wedge Y_1^\star \wedge ..\widehat{j}.. \wedge Y_q^\star$$

for $f \in C^\infty(M)$, $X \in C^\infty(L^\star)$:

$$[X,f]_{L^\star} = -[f,X]_{L^\star} = \rho(X)(f).$$

The *exterior derivative* d associated with the Lie algebroid structure of L is defined by:

$$d : C^\infty(\wedge^p L^\star) \to C^\infty(\wedge^{p+1} L^\star)$$

$$(d\alpha)(\sigma_0, \ldots, \sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\alpha(\sigma_0..\widehat{i}..\sigma_p) + \sum_{i<j}(-1)^{i+j}\alpha([\sigma_i, \sigma_j]_L, \sigma_0, ..\widehat{i}..\widehat{j}..\sigma_p)$$

for $\alpha \in C^\infty(\wedge^p L^\star)$, $\sigma_0, \ldots, \sigma_p \in C^\infty(L)$.

The *exterior derivative* $d_\star$ associated with the Lie algebroid structure of $L^\star$ is defined by:

$$d_\star : C^\infty(\wedge^p L) \to C^\infty(\wedge^{p+1} L)$$

$$(d_\star\alpha)(\sigma_0, \ldots, \sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\alpha(\sigma_0..\widehat{i}..\sigma_p) + \sum_{i<j}(-1)^{i+j}\alpha([\sigma_i, \sigma_j]_{L^\star}, \sigma_0, ..\widehat{i}..\widehat{j}..\sigma_p)$$

for $\alpha \in C^\infty(\wedge^p L)$, $\sigma_0, \ldots, \sigma_p \in C^\infty(L^\star)$.

Definition 28 A *complex Lie bialgebroid* is a pair of complex dual Lie algebroids $(L, L^\star)$ such that the differential $d_\star$ is a derivation of $(C^\infty(\wedge L^\star), [\,,\,]_L)$, that is for all $\sigma, \tau \in C^\infty(L)$ the following compatibility condition is satisfied:

$$d_\star[\sigma, \tau]_L = [d_\star, \tau]_L + [\sigma, d_\star\tau]_L.$$

6.2 Lie Algebroids in Generalized Geometry

Let (M, H, g) be a smooth pseudo Riemannian manifold with a g-symmetric operator H on $T(M)$ such that $H^2 = \mu I$, and let $\widehat{J}$ be the generalized pseudo calibrated complex structure on M defined by g and H.

Lemma 29 *For $\lambda \neq 0$ the map: $\psi : T(M) \otimes \mathbb{C} \to T(M) \otimes \mathbb{C}$ defined by: $\psi(Z) = Z + iHZ$ is an isomorphism and the following holds*

$$\psi(Z - iHZ) - ig(\psi(Z)) = -\lambda Z - ig(Z + iHZ) = -i(-\lambda iZ + g(Z + iHZ)).$$

Proof ψ is injective if and only if i is not an eigenvalue of H. □
A direct computation gives:

$$E^{1,0}_{\widehat{J}} = \{Z - iHZ + g(W + iHW - iZ) + i(-\lambda)W|Z, W \in C^\infty(T(M) \otimes \mathbb{C})\}.$$

Thus we get:

Corollary 30 *If $\lambda \neq 0$ then:*

$$E^{1,0}_{\widehat{J}} = \{-\lambda Z - ig(Z + iHZ)|Z \in C^\infty(T(M) \otimes \mathbb{C})\}$$

$$E^{0,1}_{\widehat{J}} = \{-\lambda Z + ig(Z - iHZ)|Z \in C^\infty(T(M) \otimes \mathbb{C})\}.$$

Computing Jacobiator on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ we get the following result:

Theorem 31 ([20]) *Let ∇ be a linear connection on M, let $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^\star \end{pmatrix}$ with $\lambda \neq 0$, if $\widehat{J}$ is ∇-integrable then $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are complex Lie algebroids.*
In the case $\lambda = 0$ we have the following:

Proposition 32 ([19]) *Let (M, H, g, D) be a complex Norden manifold with the natural canonical connection D, Jacobi identity holds on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ if and only if for all $X, Y, Z \in C^\infty(T(M))$ the curvature operator of D, R^D, $R^D(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$, satisfies the following conditions:*

$$R^D(HX, HY) - HR^D(HX, Y) - HR^D(X, HY) - R^D(X, Y) = 0$$

$$(R^D(HX, Y) - R^D(X, HY))Z + (R^D(HZ, X) + R^D(Z, HX))Y+$$

$$+(R^D(Y, HZ) - R^D(HY, Z))X = 0.$$

In particular, from the properties of the curvature operator of the Norden metric of a Kähler Norden manifold, [12], we get the following:

Theorem 33 ([19]) *Let (M, H, g) be a Kähler Norden manifold then $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are complex Lie algebroids.*

Remark 34 In Hitchin's context integrability of a generalized complex structure always implies that the $\pm i$-eigenbundles are complex Lie algebroids with respect to the Courant bracket [8].
Extending Proposition 7 pointwise to manifolds, we can hope to express integrability of a pseudo calibrated complex structure on M in terms of its holomorphic bundle, as in [8]. In the case of $H = 0$ we have the following:

Proposition 35 *Let (M, g) be a pseudo Riemannian manifold and let ∇ be a linear connection on M. Let $\widehat{J} = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$, then $\nabla = T(M)$ and, denoted $L = E^{1,0}_{\widehat{J}}$, we have $\epsilon = -ig$. In particular $\widehat{J}$ is ∇-integrable if and only if*

$$d^{\nabla}\epsilon = 0.$$

Proof In this case is $E^{1,0}_{\widehat{J}} = \{Z - ig(Z) | Z \in C^{\infty}(T(M) \otimes \mathbb{C})\}$. In particular, denoted by $p : (T(M) \oplus (T(M))^{\star} \otimes \mathbb{C} \to T(M) \otimes \mathbb{C}$ the canonical projection we have $F = p(E^{1,0}_{\widehat{J}}) = T(M) \otimes \mathbb{C}$, then $F + \bar{F} = T(M) \otimes \mathbb{C}$, $F \cap \bar{F} = F$ and $\Delta = T(M)$. Moreover, from the expression of $E^{1,0}_{\widehat{J}}$ we get immediately $\epsilon = -ig$.

The last statement follows from Corollary 13. The condition $d^{\nabla}\epsilon = 0$ can be interpreted as a sort of Maurer-Cartan equation. □

7 Generalized $\bar{\partial}$-Operator

In this section we define the concept of generalized $\bar{\partial}$-operator associated to a generalized complex structure and to a linear connection on M.

Lemma 36 ([20]) *Let (M, H, g) be a pseudo Riemannian manifold with H g-symmetric operator of $T(M)$ such that $H^2 = (-1 - \lambda)I$. Let $\widehat{J}$ be the generalized complex structure on M defined by g and H, the natural symplectic structure on E defines an isomorphism*

$$\phi : E^{0,1}_{\widehat{J}} \to (E^{1,0}_{\widehat{J}})^{\star}$$

by: $\phi(\sigma)(\tau) = (\sigma, \tau)$ for all $\sigma \in E^{0,1}_{\widehat{J}}$ and for all $\tau \in E^{1,0}_{\widehat{J}}$.

Let (M, H, g) be as in the previous lemma and let ∇ be a linear connection on M, the isomorphism ϕ between $E^{0,1}_{\widehat{J}}$ and the dual bundle of $E^{1,0}_{\widehat{J}}$, $(E^{1,0}_{\widehat{J}})^{\star}$, allow us to define the $\bar{\partial}_{\widehat{J}}$-*operator* associated to the complex structure $\widehat{J}$ and to the connection ∇. Precisely the ***generalized*** $\bar{\partial}_{\widehat{J}}$-*operator*, or **generalized** $\bar{\partial}$-*operator of* (M, H, g, ∇), is defined as in the following [18–20].

Let $f \in C^\infty(M)$ and let $df \in C^\infty(T^\star(M)) \hookrightarrow C^\infty(T(M) \oplus T^\star(M))$, we define

$$\bar{\partial}_{\widehat{J}} f = 2(df)^{0,1}.$$

Moreover we define $\bar{\partial}_{\widehat{J}} : C^\infty(E^{0,1}_{\widehat{J}}) \to C^\infty(\wedge^2(E^{0,1}_{\widehat{J}}))$ as

$$\bar{\partial}_{\widehat{J}} : C^\infty((E^{1,0}_{\widehat{J}})^\star) \to C^\infty(\wedge^2(E^{1,0}_{\widehat{J}})^\star)$$

$$(\bar{\partial}_{\widehat{J}}\alpha)(\sigma, \tau) = \rho(\sigma)\alpha(\tau) - \rho(\tau)\alpha(\sigma) - \alpha([\sigma, \tau]_\nabla)$$

where $\alpha \in C^\infty((E^{1,0}_{\widehat{J}})^\star), \sigma, \tau \in C^\infty(E^{1,0}_{\widehat{J}})$ and $\rho : C^\infty(E^{1,0}_{\widehat{J}}) \to C^\infty(T(M) \otimes \mathbb{C})$ is the anchor.

In general we define $\bar{\partial}_{\widehat{J}} : C^\infty(\wedge^p(E^{1,0}_{\widehat{J}})^\star) \to C^\infty(\wedge^{p+1}(E^{1,0}_{\widehat{J}})^\star)$ as

$$(\bar{\partial}_{\widehat{J}}\alpha)(\sigma_0, \ldots, \sigma_p) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i)\alpha(\sigma_0, ..\overset{\widehat{i}}{.}.., \sigma_p) + \sum_{i<j}(-1)^{i+j}\alpha([\sigma_i, \sigma_j]_\nabla, \sigma_0, ..\overset{\widehat{i}\,\widehat{j}}{.}.., \sigma_p)$$

where $\alpha \in C^\infty(\wedge^p(E^{1,0}_{\widehat{J}})^\star), \sigma_0, \ldots, \sigma_p \in C^\infty(E^{1,0}_{\widehat{J}})$.

Also we define $\partial_{\widehat{J}} : C^\infty(\wedge^p(E^{1,0}_{\widehat{J}})) \to C^\infty(\wedge^{p+1}(E^{1,0}_{\widehat{J}}))$ as

$$(\partial_{\widehat{J}}\alpha)(\sigma_0^\star, \ldots, \sigma_p^\star) =$$

$$= \sum_{i=0}^{p}(-1)^i \rho(\sigma_i^\star)\alpha(\sigma_0^\star, ..\overset{\widehat{i}}{.}.., \sigma_p^\star) + \sum_{i<j}(-1)^{i+j}\alpha([\sigma_i^\star, \sigma_j^\star]_\nabla, \sigma_0^\star, ..\overset{\widehat{i}\,\widehat{j}}{.}.., \sigma_p^\star)$$

where $\alpha \in C^\infty(\wedge^p(E^{1,0}_{\widehat{J}})), \sigma_0^\star, \ldots, \sigma_p^\star \in C^\infty(E^{1,0}_{\widehat{J}})^\star$.

Proposition 37 *If $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are complex Lie algebroids then $(\bar{\partial}_{\widehat{J}})^2 = 0$ and $(\partial_{\widehat{J}})^2 = 0$.*

Proof It follows from the fact that Jacobi identity holds on $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$. □

It turns out that $\bar{\partial}_{\widehat{J}}$ is the exterior derivative d of the Lie algebroid $L = E^{1,0}_{\widehat{J}}$ and $\partial_{\widehat{J}}$ is the exterior derivative $d_{L^\star}$ of the Lie algebroid $L^\star = (E^{1,0}_{\widehat{J}})^\star$.

It is well known that a Lie algebroid is equivalent to a Gerstenhaber algebra, [13], in particular we get:

Remark 38 $(C^\infty(\wedge^\bullet(E^{1,0}_{\widehat{J}})), \wedge, \bar{\partial}_{\widehat{J}}, [\,,\,]_\nabla)$ is a *differential Gerstenhaber algebra*, where $\wedge$ denotes Schouten bracket.

8 Generalized Holomorphic Sections

In this section we define the concept of generalized holomorphic section and we describe some of them.

Definition 39 Let $\alpha \in C^\infty(\wedge^p(E^{1,0}_{\widehat{J}})^\star)$, α is called *generalized holomorphic section* if

$$\bar{\partial}_{\widehat{J}}\alpha = 0.$$

Proposition 40 ([20]) *Let (M, H, g) be a pseudo Riemannian manifold with H g-symmetric operator of $T(M)$ such that $H^2 = (-1-\lambda)I, \lambda \neq 0$. Let $\widehat{J}$ be the generalized complex structure on M defined by g and H and let ∇ be a linear connection on M such that $\widehat{J}$ is ∇-integrable. Let $W \in C^\infty(T(M))$ and let $\sigma = -\lambda W + ig(W - iHW) \in E^{0,1}_{\widehat{J}}$, then $\bar{\partial}_{\widehat{J}}\sigma = 0$ if and only if $g(W)$ is a Lagrangian submanifold of $T^\star(M)$ with respect to the standard symplectic structure.*

Examples of generalized holomorphic sections can be obtained in the context of *Hessian Geometry*. The concept of Hessian manifold was inspired by the Bergman metric on bounded domains in $\mathbb{C}^n$ and now is a very interesting topic, related to many fields in mathematics and theoretical physics: Kähler and symplectic geometry, affine differential geometry, special manifolds, string theory and mirror symmetry [5, 6, 9, 14, 25, 26].

Definition 41 Let (M, g) be a pseudo Riemannian manifold, g is called of *Hessian type* if there exists $u \in C^\infty(M)$ such that $g = Hess(u) = \nabla^2 u$, where ∇ is the Levi Civita connection of g. (M, g) is called *Hessian pseudo Riemannian manifold* if g is of Hessian type.

Proposition 42 ([20]) *Let (M, H, g) be a Hessian pseudo Riemannian manifold with H g-symmetric operator of $T(M)$ such that $H^2 = (-1-\lambda)I, \lambda \neq 0$. Let ∇ be the Levi Civita connection of g, assume that $\widehat{J} = \begin{pmatrix} H & \lambda g^{-1} \\ g & -H^\star \end{pmatrix}$ is ∇-integrable. Let $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ be local frames for $T(M)$, if the curvature of ∇, R^∇, vanishes, then for all $k = 1, \ldots, n$ the local section of $(E^{0,1}_{\widehat{J}})$*

$$\sigma_k = -\lambda \frac{\partial}{\partial x_k} + ig(\frac{\partial}{\partial x_k} - iH\frac{\partial}{\partial x_k})$$

is $\bar{\partial}_{\widehat{J}}$-closed.

In the case of Norden manifolds we have the following.

Proposition 43 ([19]) *Let (M, H, g) be a Kähler Norden manifold, let $\widehat{J} = \begin{pmatrix} H & 0 \\ g & -H^\star \end{pmatrix}$ be the generalized complex structure on M defined by g and H and let $W \in C^\infty(T(M))$ be an infinitesimal automorphism of H. $g(W - iHW)$ and*

$W + iHW + ig(W)$ are generalized holomorphic sections of $E^{0,1}_{\widehat{J}}$ if and only if $g(W)$ is a Lagrangian submanifold of $T^\star(M)$ with respect to the standard symplectic structure.

Finally, in the case of $H = 0$, as an application of the theory of generalized holomorphic sections, we can prove the following:

Proposition 44 ([18]) *Let (M, g, ∇) be an affine Hessian manifold then the pair of complex dual Lie algebroids $(E^{1,0}_{\widehat{J}}, (E^{0,1}_{\widehat{J}})^\star)$ is a Lie bialgebroid if and only if ∇ is the Levi Civita connection of g.*

Remark 45 The generalized $\bar{\partial}_{\hat{J}}$-operator introduced in this paper,

$$\bar{\partial}_{\hat{J}} : C^\infty(E^{0,1}_{\hat{J}}) \to C^\infty(\wedge^2(E^{0,1}_{\hat{J}}))$$

and the $\bar{\partial}_J$-operator for Hitchin's generalized complex structures,

$$\bar{\partial}_J : C^\infty(E^{0,1}_J) \to C^\infty(\wedge^2(E^{0,1}_J))$$

are defined formally in the same way [8, 11]. Here we use $[,]_\nabla$, restricted to sections of $E^{0,1}_{\hat{J}}$, instead of the Courant bracket, restricted to sections of $E^{0,1}_J$, and the standard symplectic form instead of the natural indefinite metric on $T(M) \oplus T^\star(M)$, in the identifications $E^{0,1}_{\hat{J}} \cong (E^{0,1}_{\hat{J}})^\star$ and $E^{0,1}_J \cong (E^{0,1}_J)^\star$ respectively. However Proposition 44 shows different behaviour of the two operators regarding Lie bialgebroid structure of $(E^{1,0}_{\hat{J}}, (E^{1,0}_{\hat{J}})^\star)$ and $(E^{1,0}_J, (E^{1,0}_J)^\star)$, since a generalized complex structure in Hitchin's sense always induces a Lie bialgebroid structure.

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Spectral and Eigenfunction Asymptotics in Toeplitz Quantization

Roberto Paoletti

Abstract Toeplitz operators on quantized compact symplectic manifolds were introduced by Guillemin and Boutet de Monvel, who studied their spectral asymptotics in analogy with the theory developed by Duistermaat, Guillemin, and Hörmander for pseudodifferential operators. In this survey, we review some recent results concerning eigenfunction asymptotics in this context, largely based on the microlocal description of Szegö kernels by Boutet de Monvel and Sjöstrand, and its revisitation and generalization to the almost complex symplectic category by Shiffman and Zelditch. For simplicity, the exposition is restricted to the complex projective setting.

1 Introduction

In this paper we shall review some recent results on the asymptotic distribution of eigenvalues and concentration of eigensections in the context of Toeplitz quantization of compact symplectic manifolds. We shall restrict our presentation to the Kähler setting for simplicity, but with the appropriate conceptual background [4, 19] these results may be formulated in the general symplectic almost complex framework. The approach stems from the techniques developed in [2, 23], and [19], which in turn are based on the microlocal description of the Szegö kernel as a Fourier integral operator in [5].

The geometric setting is as follows: M is a compact Kähler manifold, ω a Kähler form on it, and (A, h) is a positive (holomorphic) line bundle on M, with the property that the unique compatible connection on it has curvature form $\Theta = -2i\,\omega$. We consider the unit circle bundle $X \subseteq A^\vee$, where $A^\vee$ is the dual line bundle to A, and remark that X is the boundary of a strictly pseudoconvex domain in view of the positivity of (A, h). So X is a CR manifold and we may introduce its Hardy space $H(X)$, with the orthogonal projector $\Pi : L^2(X) \to H(X)$; the latter operator is the celebrated *Szegö projector*, and its distributional kernel is the *Szegö kernel*.

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What follows revolves about the fundamental fact, due to Boutet de Monvel and Sjöstraend, that Π has a rather explicit description as a Fourier integral operator with complex phase [4, 5, 19].

An important object built into this picture is the closed symplectic cone in $T^*X \setminus \{0\}$ (the complement of the zero section in the cotangent bundle of X) sprayed by the connection 1-form $\alpha \in \Omega^1(X)$:

$$\Sigma =: \{(x, r\alpha_x) \,:\, x \in X,\, r > 0\}\,.$$

Analytically Σ, or more precisely the diagonal in $\Sigma \times \Sigma$, is the wave front (that is, the locus of the microlocal singularities) of Π. As a consequence, the asymptotics in point are controlled by the geometry of various Hamiltonian flows naturally defined on Σ.

A Toeplitz operator (in the sense of [4]) is a (generally unbounded) operator on $L^2(X)$ of the form $T = \Pi \circ Q \circ \Pi$, where Q is a pseudodifferential operator on X. The degree of T is by definition the degree of Q, and its symbol $\sigma_T : \Sigma \to \mathbb{C}$ is the restriction to Σ of the symbol of Q. Thus if Q and Q' are pseudodifferential operators on X of the same degree, whose principal symbols agree on Σ, then they define the same Toeplitz operator up to operators of lesser degree. A Toeplitz operator T is elliptic if it is defined by a pseudodifferential operator Q which is elliptic in a conic neighborhood of Σ. If T is self-adjoint, there is no loss in assuming that so is Q; in particular, σ_T is real-valued.

The reduced symbol of T is the $\mathcal{C}^\infty$ function ς_T on X, defined by $\varsigma_T(x) =: \sigma_T(x, \alpha_x)$ $(x \in X)$.

Let us consider the following basic question. If T is an elliptic and self-adjoint operator, what can be said about the asymptotic distribution of its eigenvalues and and eigensections? For pseudodifferential operators, this problem was studied extensively in [11] and [7] (see also the discussion in [8]). For Toeplitz operators, the spectral asymptotics of T were studied in [4], using the theory of Fourier-Hermite distributions and their symbolic calculus based on symplectic spinors. In a series of papers, I have attempted to revisit this theme by applying directly the microlocal theory of [4] and its developments in [19]. The emphasis is on the local aspect, that is, on the asymptotic concentration of eigenfunctions; global consequences can then be drawn by integration. What follows is a condensed survey of some of these results.

As a significant variant, one can explore similar questions and adapt the present approach to the setting of Berezin-Toeplitz geometric quantization [1, 6]. Let $H(X)_k \subseteq H(X)$ be the k-th equivariant piece for the standard S^1-action on the circle bundle X. Rather than considering arbitrary self-adjoint Toeplitz operators, in Berezin-Toeplitz quantization one restricts to S^1-invariant ones. If T is an S^1-invariant self-adjoint Toeplitz operator, it induces by restriction operators T_k on each $H(X)_k$. In this setting, $1/k$ plays the heuristic role of Planck's constant $\hbar$, and letting $k \to +\infty$ corresponds to considering the semiclassical limit of the model. A natural issue is then the study of the local asymptotics for $k \to +\infty$ of spectral projectors pertaining to a shrinking spectral band of T_k, in the spirit of the classical Gutzwiller trace formula. I will not report on this theme in this short survey, but refer to [3, 14, 18].

2 A Local Weyl Law

Let T be a first order self-adjoint Toeplitz operator on X with strictly positive symbol. Then the eigenvalues of T may be arranged in a non-decreasing sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_j \uparrow +\infty$. Let (e_j) be a complete orthonormal system for $H(X)$ associated to the λ_j's. The *local spectral function* of T is the $\mathcal{C}^\infty$ function on X given by

$$\mathcal{T}(\lambda, x) =: \sum_{\lambda_j \leq \lambda} |e_j(x)|^2 \qquad (x \in X).$$

Thus, in the asymptotic regime $\lambda \to +\infty$, $\mathcal{T}(\lambda, x)$ measures the asymptotic concentration at x of the eigenfunctions of T pertaining to eigenvalues $\leq \lambda$. For the following, see [12].

Theorem 2.1 *Let T be a self-adjoint first order elliptic Toeplitz operator on X with $\sigma_T > 0$. Then as $\lambda \to +\infty$ we have*

$$\mathcal{T}(\lambda, x) = \frac{\pi}{d+1} \cdot \left(\frac{\lambda}{\pi\, \varsigma_T(x)}\right)^{\mathrm{d}+1} + O\left(\lambda^{\mathrm{d}}\right),$$

uniformly in $x \in X$.

A global integration leads to the Weyl law of [4] for the counting function

$$N_T(\lambda) =: \sharp\left\{j \in \mathbb{N} \,:\, \lambda_j \leq \lambda\right\} \qquad (\lambda \in \mathbb{R}). \tag{1}$$

Namely, we get (see [4], Theorem 13.1):

Corollary 2.1 *Let $\Sigma_1 \subseteq \Sigma$ be the locus where $\sigma_T < 1$. In the situation of Theorem 2.1, as $\lambda \to +\infty$ we have*

$$N_T(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{\mathrm{d}+1} \mathrm{vol}(\Sigma_1) + O\left(\lambda^{\mathrm{d}}\right).$$

3 A Local Trace Formula: The Holomorphic Case

The symbol ς_T generates a Hamiltonian flow on the conic symplectic manifold Σ. In the presence of periodic trajectories, the asymptotic distribution of the eigenvalues λ_j exhibits an interesting structure along some arithmetic progressions related to the periods of the flow. This leads to singularities of a certain distributional trace related to T. This general theme was studied in [4], and in [7] in the case of pseudodifferential operators, and has led to a large body of work (see, for instance, [9, 10, 20, 22]). Here we shall consider the related problem of eigenfunction concentration, with an emphasis on scaling asymptotics along certain special loci defined by the dynamics. One important feature is that it is the periodic trajectories on Σ (equivalently, on X) that control the asymptotics; even when the flow is a lifting of an underlying flow on M, the periods on X are generally only a proper subset of the periods on M. In particular $0 \in \mathbb{R}$ may be viewed as a 'degenerate' period, and the study of the corresponding singularity is related to the topic of the previous section.

Let us first consider the first order Toeplitz operator associated to a holomorphic flow. This is a rather special object, but nonetheless of great significance in geometry and harmonic analysis, since it naturally arises in the context of Hamiltonian group actions on polarized complex manifolds. We shall call a Hamiltonian $f \in \mathcal{C}^\infty(M)$ *compatible* (with the Kähler structure) if its Hamiltonian flow $\phi_\tau^M : M \to M$ with respect to $2\,\omega$ is holomorphic. The Hamiltonian vector field υ_f of f has a natural lift to a contact vector field on X, given by $\widetilde{\upsilon}_f = \upsilon_f^\sharp - f\,(\partial/\partial\theta)$; here $\upsilon_f^\sharp$ is the horizontal lift of υ_f, and $\partial/\partial\theta$ is the infinitesimal generator of the S^1-action.

Given that f is compatible, the contact flow ϕ_τ^X generated by $\widetilde{\upsilon}_f$ preserves the Hardy space, and the restriction of $i\,\widetilde{\upsilon}_f$ to $H(X)$ may be viewed as a Toeplitz operator T_f in the sense of [4]. If we assume that f is positive, the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ of T_f drift to $+\infty$ and it makes sense to consider the tempered distribution $\sum_j e^{i\tau\lambda_j}$ on $\mathbb{R}$. The latter may be viewed as the distributional trace of the 'Toeplitz wave group' $U_H(\tau)$ generated by T_f (see [4] or [13] for more detailed definitions).

It is a basic theme in the study of spectral asymptotics that the singularities of the distributional traces associated to elliptic operators are concentrated on the set of periods of the Hamiltonian flow of the principal symbol [4, 7]. In this specific case, it is the periods of the contact flow on X that control the singularities of the trace.

In order to study the microlocal singularities (wave front) of a distribution on a manifold at a given point, one first multiplies it by a function with small compact support near the point, and then takes the Fourier transform of the product along various directions. In the case at hand, this brings into the picture the smoothing operator

$$S_{\chi\, e^{-i\lambda(\cdot)}} =: \int_{-\infty}^{+\infty} \chi(\tau)\, e^{-i\lambda\tau}\, U_H(\tau)\, d\tau; \tag{2}$$

here χ is compactly supported near a given $\tau_0 \in \mathbb{R}$, and we are interested in the asymptotics for $\lambda \to +\infty$ (the asymptotics for $\lambda \to -\infty$ are trivial). In spectral terms, and with the previous notation, the latter operator has distributional kernel

$$S_{\chi\, e^{-i\lambda(\cdot)}}(x, y) =: \sum_j \widehat{\chi}(\lambda - \lambda_j)\, e_j(x)\, \overline{e_j(y)}; \tag{3}$$

heuristically, it may thus be interpreted as a slight smoothing of the spectral projection kernel of some interval which is traveling to the right as $\lambda \to +\infty$.

Assume then that χ has small support near a given period τ_0. Then $S_{\chi\, e^{-i\lambda(\cdot)}}(x, x)$ concentrates asymptotically near the fixed locus of $\phi^X_{\tau_0}$, and exhibits a rather explicit exponential decay along normal directions to it. Namely, if x_0 belongs to the fixed locus Fix $\left(\phi^X_{\tau_0}\right)$ and $\mathbf{u}$ is a normal vector to it at x_0, by Theorem 1.1 of [13] we have for $\lambda \to +\infty$ an asymptotic expansion of the form:

$$S_{\chi\, e^{-i\lambda(\cdot)}}\left(x_0 + \frac{\mathbf{u}}{\sqrt{\lambda}}, x_0 + \frac{\mathbf{u}}{\sqrt{\lambda}}\right) \tag{4}$$
$$\sim \frac{2\pi\, e^{-i\lambda\tau_0}}{f(m_0)^{\mathrm{d}+1}} \left(\frac{\lambda}{\pi}\right)^{\mathrm{d}} e^{f(m_0)^{-1}\cdot\psi_2\left(d_{m_0}\phi^M_{-\tau_0}(\mathbf{u}),\mathbf{u}\right)}\, \chi(\tau_0)\cdot \left[1 + \sum_{j=1}^{+\infty} \lambda^{-j/2} G_j(x_0, \mathbf{u})\right].$$

Here the G_j's are polynomials in $\mathbf{u}$, of the same parity as j, and ψ_2 is (in the right coordinates) the universal exponent controlling off-diagonal equivariant Szegö kernel asymptotics [19]:

$$\psi_2(\mathbf{u}, \mathbf{w}) = -i\,\omega(\mathbf{u}, \mathbf{v}) - \frac{1}{2}\,\|\mathbf{u} - \mathbf{w}\|^2;$$

ω is the symplectic structure and $\|\cdot\|$ the Euclidean norm on point of M below x_0. Computing the trace of $S_{\chi\, e^{-i\lambda(\cdot)}}$ from (4) then amounts to performing a Gaussian integral along normal directions; we obtain a contribution from each connected component of the fixed locus, involving Poincaré type data along it (Corollary 1.1 of [19]). The asymptotics are trivial for $\lambda \to -\infty$.

The previous local rescaled asymptotics may be globally integrated to obtain information about the trace of (2); we then recover the asymptotics of the Fourier transform of the distributional trace [4] as a sum of contributions coming from each component of the fixed locus, and depending on the Poincaré map. More explicitly, we obtain

$$\left\langle \sum_j e^{i\lambda_j\tau}, \chi\, e^{-i\lambda(\cdot)} \right\rangle = \sum_{j=1}^{N_{\tau_0}} I_j(\chi, \lambda), \tag{5}$$

where each I_j admits an asymptotic expansion

$$I_j(\chi, \lambda) \sim 2\pi\, e^{-i\lambda\tau_0} \left(\frac{\lambda}{\pi}\right)^{\mathrm{f}_j} \frac{\chi(\tau_0)}{c(\tau_0, j)} \cdot \left(\int_{M_{\tau_0 j}} \frac{1}{f(m)^{\mathrm{f}_j+1}}\, dV_{M_{\tau_0 j}}(m)\right)$$
$$\cdot \left(1 + \sum_{j=1}^{+\infty} \lambda^{-j} p_j\right). \tag{6}$$

Here $c(\tau_0, j)$ is the determinant of the complex automorphism of the normal bundle to the fixed locus induced by $\mathrm{id}_{N_{m_0}} - \mathrm{d}_{m_0}\phi^M_{-\tau_0}$, and is constant along each connected component.

4 A Local Trace Formula for General Hamiltonian Flows

It is natural to seek an extension of the previous local asymptotics to general quantized Hamiltonian flows. However, the Toeplitz quantization of a general Hamiltonian flow is a subtle issue, since the same procedure used in the compatible case leads to operators that do no preserve the Hardy space.

To clarify this point, let $f \in \mathcal{C}^\infty(M)$ be an arbitrary positive Hamiltonian, and let υ_f be its Hamiltonian vector field, ϕ^M_τ its Hamiltonian flow. As mentioned, there is associated to f a contact flow ϕ^X_τ on (X, α), which lifts ϕ^M_τ and is generated by the contact vector field $\widetilde{\upsilon}_f$. However, unless f is compatible one generally has $\left(\phi^X_\tau\right)^* \left(H(X)\right) \neq H(X)$; thus pull-back induces a unitary action on $L^2(X)$, but not on $H(X)$. If on the other hand $\left(\phi^X_\tau\right)^*$ is composed with the Szegö projector $\Pi : L^2(X) \to H(X)$, we obtain a 1-parameter family of generally *non-unitary* operators on $H(X)$.

An intrinsic procedure to obtain a 1-parameter family of unitary operators from a general Hamiltonian flow has been developed by Zelditch [21], who proved the following: there is a natural family of Toeplitz operators $T_f(\tau)$ associated to f, such that the composition $U_f(\tau) =: T_f(\tau) \circ \Phi^X_\tau$ is a one-parameter family of (essentially) unitary operators on $H(X)$. One thus regards $\tau \mapsto U_f(\tau)$ as the *quantization* of the flow ϕ^M_τ.

One is thus led to consider the analogue of the problem considered in the previous section, with the Toeplitz correction introduced by Zelditch; more generally, we may consider one-parameter families of endomorphisms of $H(X)$ of the form $U(\tau) = R_\tau \circ \left(\phi^X_\tau\right)_\tau$, where R_τ is a smoothly varying zeroth order Toeplitz operator. The first step is to extend the notion of trace to this more general setting.

To this end, let us introduce the diagonal map $\Delta : \mathbb{R} \times X \to \mathbb{R} \times X \to \mathbb{R} \times X \times X$ and the projection $p : \mathbb{R} \times X \to X$. We can view the Schwartz kernel of U as a distribution on $\mathbb{R} \times X \times X$. The distributional trace of the family $\tau \mapsto U(\tau)$ is then defined functorially by setting

$$\mathrm{trace}(U) = p_* \left(\Delta^*(U)\right).$$

For example, in the case considered by Zelditch, let us assume that the eigenvalues of $U_f(\tau)$ can be written as functions $\tau \mapsto g_j(\tau) \in S^1$. Then - perhaps up a smoothing contribution - $\mathrm{trace}(U) = \sum_j g_j(\tau)$.

Again, we can view the diagonal asymptotics of the smoothing kernel $S_{\chi\, e^{-i\lambda(\cdot)}}$ as local contributions to the Fourier transform of $\chi \cdot \mathrm{trace}(U)$, with compactly supported near an isolated period τ_0 of ϕ^X. Under certain cleanness conditions on the fixed locus of the period τ_0, there are scaling asymptotics for $S_{\chi\, e^{-i\lambda(\cdot)}}$ similar to those of the integrable case. The simple universal exponent ψ_2 in (4) is replaced however by a considerably more intricate quadratic form on the normal bundle to the fixed locus, depending on Poincaré type data.

More precisely, suppose that $f > 0$ and that τ_0 is a *very clean* (see [15] for precise definitions) isolated period of the contact flow ϕ^X. If $x_0 \in \mathrm{Fix}\left(\phi^X_{\tau_0}\right)$ and $\mathbf{n}$ is a normal vector to $\mathrm{Fix}\left(\phi^X_{\tau_0}\right)$ with $\|\mathbf{n}\| \leq C\,\lambda^{1/9}$, the following asymptotic expansion holds for $\lambda \to +\infty$ [15]:

$$
\begin{aligned}
&S_{\chi\, e^{-i\lambda(\cdot)}}\left(x_0 + \frac{\mathbf{n}}{\sqrt{\lambda}}, x_0 + \frac{\mathbf{n}}{\sqrt{\lambda}}\right) \qquad (7)\\
&\sim \rho_{\tau_0}(x_0)\,\frac{2\pi\, e^{-i\lambda\tau_0}}{f(m_0)^{\mathrm{d}+1}} \cdot \frac{2^{\mathrm{d}}}{\sqrt{\det(Q_A)}} \cdot e^{f(m_0)^{-1}\Psi_2^A(\mathbf{n})}\,\chi(\tau_0)\\
&\cdot \left(\frac{\lambda}{\pi}\right)^{\mathrm{d}}\left[1 + \sum_{j\geq 1}\lambda^{-j/2}G_j(x_0,\mathbf{n})\right],
\end{aligned}
$$

where ρ_τ is the symbol of R_τ, and the G_j's are polynomials in $\mathbf{n}$, depending smoothly on x_0. Furthermore, if A is the matrix representing the Poincaré map of the period τ_0 in a 'good' system of local coordinates on M, then $Q_A = I + A^t A$. Finally, the quadratic form Ψ_2^A is defined in terms of A and polar decomposition, and still describes an exponential decay along normal directions. The previous expansion reduces to the one described in Sect. 3 when f is integrable.

5 Directional Local Trace Formulae for Commuting Hamiltonian Flows

A natural generalization of the previous discussion concerns commuting Hamiltonian flows. Suppose given Poisson commuting Hamiltonian's $f_1, \ldots, f_r \in \mathcal{C}^\infty(M)$; for the sake of simplicity, let us assume that each f_j is compatible. Then the Hamiltonian and contact flows of the f_j's commute; collectively, they give rise to a Hamiltonian action $\phi^M : \mathbb{R}^r \times M \to M$, with moment map $\Phi = (f_1, \ldots, f_r) : M \to \mathbb{R}^r$, and to a contact action $\phi^X : \mathbb{R}^r \times X \to X$ lifting ϕ^M.

Let υ_k be the Hamiltonian vector field associated to f_k, and $\widetilde{\upsilon}_k$ be its contact lift to X ($k = 1, \ldots, r$). On the quantum side, we have commuting Toeplitz operators $\mathfrak{T}_k$, given by the restriction of $i\,\widetilde{\upsilon}_k$ to $H(X)$. We can find a complete orthonormal system (e_j) of $H(X)$ consisting of simultaneous eigenvectors of the $\mathfrak{T}_k$'s, associated to *joint eigenvalues* $\Lambda_j =: (\lambda_{1j}, \ldots, \lambda_{rj})^{\mathrm{t}} \in \mathbb{R}^r$; in other words,

$$
\mathfrak{T}_k(e_j) = \lambda_{kj}\, e_j.
$$

If $\mathbf{0} \notin \Phi(M)$ then, as in the case $r = 1$, the Λ_j's drift to infinity, and we may consider the distributional trace

$$\mathrm{tr}(\mathfrak{U}) =: \sum_j e^{i\langle \Lambda_j, \cdot \rangle}; \tag{8}$$

the latter is a well-defined temperate distribution on $\mathbb{R}^r$, whose singularities encapsulate asymptotic information on the distribution of the Λ_j's. The singular support of $\mathrm{tr}(\mathfrak{U})$ is contained in the set of periods of ϕ^X,

$$\mathrm{Per}(\phi^X) =: \left\{\mathbf{s} \in \mathbb{R}^r \,:\, \exists x \in X \text{ such that } \phi^X_{\mathbf{s}}(x) = x\right\}. \tag{9}$$

Let us fix a period $\mathbf{s}_0 \in \mathrm{Per}(\phi^X)$. To analyze the singularity of $\mathrm{tr}(\mathfrak{U})$ at $\mathbf{s}_0$, we shall fix a bump function $\chi_{\mathbf{s}_0}$ supported in a small neighborhood of $\mathbf{s}_0$, and consider the compactly supported distribution $\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U})$. A convenient choice is $\chi_{\mathbf{s}_0}(\cdot) =: \chi(\cdot - \mathbf{s}_0)$, where $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^r)$ is a bump function vanishing for $\|\mathbf{s}\| \geq \epsilon$. In order to microlocalize the analysis, let us also fix a covector $\beta \in (\mathbb{R}^r)^\vee$ of unit length, and measure the singularity of $\mathrm{tr}(\mathfrak{U})$ at $\mathbf{s}_0$ in the direction β. This implies considering the asymptotics of the Fourier transform

$$\mathcal{F}\big(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U})\big)(\lambda\,\beta) = \left\langle \mathrm{tr}(\mathfrak{U}),\, \chi_{\mathbf{s}_0}\, e^{-i\lambda\,\langle \beta, \cdot \rangle} \right\rangle \tag{10}$$

for $\lambda \to \infty$. We then obtain for (10):

$$\begin{aligned}\mathcal{F}\big(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U})\big)(\lambda\,\beta) &= \sum_j \left\langle e^{i\langle \Lambda_j, \cdot\rangle},\, \chi_{\mathbf{s}_0}\, e^{-i\lambda\,\langle \beta, \cdot \rangle} \right\rangle \\ &= e^{-i\,\lambda\langle \beta, \mathbf{s}_0\rangle} \sum_j e^{i\,\langle \Lambda_j, \mathbf{s}_0\rangle}\, \widehat{\chi}(\lambda\,\beta - \Lambda_j).\end{aligned} \tag{11}$$

This is the genuine trace of the smoothing operator

$$\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0) =: \int_{\mathbb{R}^r} \chi_{\mathbf{s}_0}(\mathbf{s})\, e^{-i\lambda\cdot\langle \beta, \mathbf{s}\rangle}\, \mathfrak{U}(\mathbf{s})\, \mathrm{d}\mathbf{s}. \tag{12}$$

Thus, if $\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0, \cdot, \cdot) \in \mathcal{C}^\infty(X \times X)$ is the Schwartz kernel of $\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0)$, we have

$$\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0, x, y) = \sum_j e^{-i\,\langle \lambda\beta - \Lambda_j, \mathbf{s}_0\rangle}\, \widehat{\chi}(\lambda\,\beta - \Lambda_j)\, e_j(x) \cdot \overline{e_j(y)}, \tag{13}$$

and

$$\mathcal{F}\big(\chi_{\mathbf{s}_0} \cdot \mathrm{tr}(\mathfrak{U})\big)(\lambda\,\beta) = \int_X \mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0, x, x)\, \mathrm{dV}_X(x). \tag{14}$$

Let us assume that the moment map Φ be transverse to the ray $\mathbb{R}_+\beta$, and let X_β be the inverse image in X of the same ray under the composition $\Phi \circ \pi$. Also, let $X_\beta(\mathbf{s}_0)$ be the locus of points in X_β that are fixed by $\phi^X_{\mathbf{s}_0}$. Then $X_\beta(\mathbf{s}_0)$ is a submanifold, and $\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0, x, x)$ asymptotically concentrates in a shrinking neighborhood of $X_\beta(\mathbf{s}_0)$ when $\lambda \to +\infty$. Furthermore, there is an asymptotic expansion in rescaled displacements away from $X_\beta(\mathbf{s}_0)$, generalizing (4).

To explain this, we need to introduce some notation. Suppose $x \in X_\beta(\mathbf{s}_0)$, and let $m_x =: \pi(x)$ be its projection in M. Consider the evaluation map $\mathrm{val}_{m_x} : \xi \in \mathfrak{g} \cong \mathbb{R}^r \mapsto \xi_M(m_x) \in T_{m_x}M$ from the Lie algebra to the tangent space at m_x (thus ξ_M is the vector field associated to ξ). The previous transversality assumption implies that val_{m_x} is injective on the subspace $\ker\big(\Phi(m_x)\big)$. Thus we may relate the metric $\langle\cdot,\cdot\rangle_0$ on $\ker\big(\Phi(m_x)\big)$ coming from the standard Euclidean scalar product on $\mathbb{R}^r$ on the one hand, and the one coming from the Riemannian scalar product on $T_{m_x}M$ on the other. We shall call $\mathcal{D}(m_x)$ the corresponding *distortion function* (see [17] for a precise definition).

Then the normal space to $X_\beta(\mathbf{s}_0)$ at x splits as a direct sum

$$N_x\big(X_\beta(\mathbf{s}_0)\big) \cong \ker\big(d_{m_x}\phi^M_{\mathbf{s}_0} - \mathrm{id}_{T_{m_x}M}\big)^\perp \oplus^\perp \big[J_{m_x} \circ \mathrm{val}_{m_x}\big(\ker\Phi(m_x)\big)\big];$$

here $T_{m_x}M$ and its subspaces are viewed as vector subspaces of T_xX, by identifying $T_{m_x}M$ with the horizontal tangent space $H_x \subset T_xX$. We shall accordingly write $\mathbf{v} \in N_x\big(X_\beta(\mathbf{s}_0)\big)$, with $\|\mathbf{v}\| \le D'\,\lambda^{\delta-1/2}$ for some $D' > 0$ (we may take $D' = D + \epsilon'$ for any $\epsilon' > 0$). In turn, in view of the previous direct sum decomposition we can also write any $\mathbf{v} \in N_x\big(X_\beta(\mathbf{s}_0)\big)$ as a sum $\mathbf{v} = \mathbf{w} + \mathbf{n}$, for unique

$$\mathbf{w} \in \ker\big(d_{m_x}\phi^M_{\mathbf{s}} - \mathrm{id}_{T_{m_x}M}\big)^\perp \quad \text{and} \quad \mathbf{n} = J_{m_x}\big(\xi_M(m_x)\big), \tag{15}$$

with $\xi \in \ker\Phi(m)$, of norm $O\big(\lambda^{\delta-1/2}\big)$.

Working in a system of Heisenberg local coordinates at x, a rescaled displacement from $X_\beta(\mathbf{s}_0)$ may be represented in the form

$$y_\lambda =: x + \mathbf{v}/\sqrt{\lambda},$$

we have [17]

$$\begin{aligned} &\mathcal{S}_\chi(\lambda\,\beta, \mathbf{s}_0, y_\lambda, y_\lambda) \\ &\sim \frac{2^{\frac{r+1}{2}}\,\pi}{\|\Phi(m_x)\|} \cdot \left(\frac{\lambda}{\pi\,\|\Phi(m_x)\|}\right)^{d+\frac{1-r}{2}} \frac{e^{-i\lambda\,\langle\beta,\mathbf{s}_0\rangle}}{\mathcal{D}(m)}\, e^{\left[\psi_2(A\mathbf{w},\mathbf{w}) - 2\,\|\mathbf{n}\|^2\right]/\|\Phi(m_x)\|} \\ &\quad \cdot \sum_{\ell \ge 0} \lambda^{-\ell/2}\,\mathcal{R}_\ell(x; \mathbf{n}, \mathbf{w}), \end{aligned} \tag{16}$$

$\mathcal{R}_\ell(x; \cdot, \cdot)$ being a polynomial of degree $\leq 3\ell$, and parity $(-1)^\ell$; also $\mathcal{R}_0 = \chi(\mathbf{0})$. Finally, A is the matrix representing $d_x\phi^X_{-\mathbf{s}_0} : T_{m_x}M \to T_{m_x}M$.

Inserting (16) in (14) we obtain a global expansion involving Poincaré type data along each component of $X_\beta(\mathbf{s}_0)$, generalizing (5) and (6).

6 Equivariant Spectral Projectors

In the presence of symmetries, one is naturally led to study the local asymptotics associated to spectral projectors pertaining to a given fixed isotype. Globally, this yields an equivariant Weyl law.

Suppose given an e-dimensional connected compact Lie group, and a holomorphic and Hamiltonian action $\mu^M : G \times M \to M$, with moment map $\Phi : M \to \mathfrak{g}^\vee$, such that $\mathbf{0} \in \mathfrak{g}^\vee$ is a regular value. Let us also suppose that μ^M lifts to an contact action $\mu^X : G \times X \to X$. Then there is an induced unitary representation of G on the Hardy space $H(X)$.

Let T be a G-invariant self-adjoint elliptic first order Toeplitz operator, with everywhere positive symbol $\varsigma_T > 0$. For any irreducible representation ϖ of G, we may consider the restriction of T to the associated isotypical subspace, $T^{(\varpi)} : H(X)^{(\varpi)} \to H(X)^{(\varpi)}$, with eigenvalues $\lambda_1^{(\varpi)} \leq \lambda_2^{(\varpi)} \leq \cdots$, and corresponding orthonormal eigenvectors $(e_j^{(\varpi)})$. The equivariant analogue of (3) is

$$S^{(\varpi)}_{\chi\cdot e^{-i\lambda(\cdot)}}(x, y) =: \sum_j \widehat{\chi}\big(\lambda - \lambda_j^{(\varpi)}\big)\ e_j^{(\varpi)}(x) \cdot \overline{e_j^{(\varpi)}(y)}, \tag{17}$$

having trace

$$\text{trace}\left(S^{(\varpi)}_{\chi\cdot e^{-i\lambda(\cdot)}}\right) = \sum_j \widehat{\chi}\left(\lambda - \lambda_j^{(\varpi)}\right),$$

Since $\mathbf{0} \in \mathfrak{g}^\vee$ is a regular value of Φ,

$$X' =: (\Phi \circ \pi)^{-1}(\mathbf{0}) \subseteq X \tag{18}$$

is connected and G-invariant submanifold of X, of real codimension e.

Then $S^{(\varpi)}_{\chi\cdot e^{-i\lambda(\cdot)}}(x, x)$ is rapidly decreasing away from X', and asymptotically concentrated in a shrinking neighborhood of X' as $k \to +\infty$. Along X', in addition, we have an asymptotic expansion in rescaled Heisenberg local coordinates [16]. The general form of the expansion requires introducing quite a bit of notation, so we shall refer to *ibidem* for a general and detailed statement. However, for *normal* rescaled displacements from X' the expansion takes a simpler form.

Suppose $x \in X'$, and let $\mathbf{w}$ be a normal vector to X' at x. Then as $\lambda \to +\infty$ we have

$$S^{(\varpi)}_{\chi\cdot e^{-i\lambda(\cdot)}}\left(x+\frac{\mathbf{w}}{\sqrt{\lambda}},x+\frac{\mathbf{w}}{\sqrt{\lambda}}\right)$$
$$\sim 2\pi \cdot A^T_{\varpi}(x)\,a_{\Phi,\varpi}(m_x)\left(\frac{\lambda}{\pi}\right)^{d-e/2}\cdot\exp\left(-\frac{2}{\varsigma_T(x)}\,\|\mathbf{w}\|^2\right)$$
$$\cdot\left[1+\sum_{j\geq 1}\lambda^{-j/2}F_j(x,\mathbf{w})\right],$$

for polynomials F_j of the same parity as j. The coefficients in front are local invariants depending on the local geometry of the action and on the specific representation ϖ, as well as on the reduced symbol ς_T.

By integrating the previous expansion and applying a classical Tauberian argument, one obtains an estimate on the ϖ-equivariant counting function

$$N_T^{(\varpi)}(\lambda) =: \sharp\left\{j:\lambda_j^{(\varpi)}\leq\lambda\right\}. \tag{19}$$

Namely, as $\lambda \to +\infty$

$$N_T^{(\varpi)}(\lambda) = \frac{\pi}{d-e+1}\cdot\dim(V_{\varpi})\,a_{\text{gen}}(\Phi,\varpi)\,\Gamma(\Phi,\varsigma_T)\left(\frac{\lambda}{\pi}\right)^{d-e+1}+O\left(\lambda^{d-e}\right),$$

where $a_{\text{gen}}(\Phi,\varpi)$ is the generic value of $a_{\Phi,\varpi}$.

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On Bi-Hermitian Surfaces

M. Pontecorvo

Abstract We present an overview of results giving a satisfactory classification of compact bi-Hermitian surfaces $(S, J_{\pm})$. That is to say compact complex surfaces (S, J_{+}) admitting a Hermitian metric g and a different complex structure J_{-} which is also g-Hermitian.

1 Introduction

An almost complex structure on a smooth manifold M is an endomorphism of the tangent bundle $J : TM \to TM$ such that $J^2 = -id$. this implies that M is even dimensional with a preferred orientation. By the Newlander-Nirenberg theorem J induces local holomorphic coordinates with respect to which J becomes multiplication by $\sqrt{-1}$ on the holomorphic tangent bundle $T^{1,0}M$ if and only if the Nijenhuis tensor of J vanishes identically. J is said to be a *complex structure* in this case.

Fixed a Riemannian metric g on M we say that J is *orthogonal* if it acts as an isometry $g(J\cdot, J\cdot) = g(\cdot, \cdot)$; because such a J is orthogonal with respect to any representative in the conformal class $c = \{e^f g \mid f \in C^\infty(M)\}$ the pair (g, J) is called a Hermitian metric on M and (c, J) a *conformal Hermitian* structure. If we further fix an orientation on M we can consider the *twistor space* of (M, g, or) as the bundle of all almost complex structures which are *compatible* that is to say its sheaf of sections are locally defined almost complex structures which are g-orthogonal,

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inducing the given orientation:

$$Z = \{J \in \mathrm{End}^+(TM) : J^2 = -id., g(J\cdot, J\cdot) = g(\cdot,\cdot)\}.$$

We denote by $t : Z \to M$ the resulting fibration whose fiber is the Hermitian symmetric space $F := SO(2n)/U(n)$. The Levi-Civita connection of g defines a horizontal distribution $\mathscr{H} \subset TZ$ and the tangent space at a point $z \in$Z can be written as a direct sum $T_zZ = T_zF \oplus \mathscr{H}_z$. The fiber bundle Z inherits in this way a tautological almost complex structure $\mathbb{J} := J_F \oplus J_t$ which at each point $z \in Z$ is the direct sum of the standard complex structure on the vertical space F plus the complex structure defined by z itself on the horizontal space $\mathscr{H}_z \cong T_{t(z)}M$.

We see that the twistor space has real dimension $2n+n(2n-1)-n^2 = n(n+1)$ and the almost complex manifold $(Z, \mathbb{J})$ enjoys the following natural properties which show that this is a reasonable context for studying orthogonal complex structures in a conformal setting; recall that the Weyl tensor is the conformally invariant part of the curvature, see [8] and [1] for example.

1. $\mathbb{J}$ only depends on the conformal class c of the metric g of M.
2. Let $U \subset M$ be an open set and let J be a g-orthogonal almost complex structure on U. Then J is an integrable complex structure if and only if the image $J(U)$ of J viewed as a local section of the twistor fibration is a $\mathbb{J}$-holomorphic map.
3. When M is four-dimensional, $(Z, \mathbb{J})$ is a complex 3-manifold if and only if the self-dual Weyl tensor of the metric g, hence conformal class, vanishes identically: $W_+ = 0$ at every point $p \in M$. We say g is anti-self-dual, abbreviated by ASD.
4. When M is $2n$-dimensional with $n \geq 3$, the twistor space $(Z, \mathbb{J})$ is a complex $\frac{n}{2}(n+1)$-manifold if and only if the Weyl tensor W of g identically vanishes—i.e. g is conformally flat.

Using property 2. we can think of a Hermitian structure J on $(M, g, or.)$ as a smooth section of the twistor fibration whose image $J(M)$ is an almost complex submanifold of $(Z, \mathbb{J})$. When the twistor space happens to be a complex manifold there will be plenty of such sections, locally. This is equivalent to say that any conformally-flat—or in four-dimensions any anti-self-dual—metric admits an abundance of locally defined Hermitian complex structures. Also notice that if J is compatible, the same holds for the conjugate complex structure $-J$ if and only if the real dimension is a multiple of four. In what follows however we shall consider $-J$ equivalent to J and not a new complex structure.

Conversely, it was proved by Muskarov [40] that, in general, the Nijenhuis tensor of an almost complex manifold (X, J) of real dimension $2n$ vanishes identically if X admits many complex hypersurfaces or holomorphic functions, locally; this shows that enough locally defined orthogonal and positively oriented complex structures make the twistor space into a complex manifold and therefore imply the vanishing of the conformal Weyl tensor, or in four dimension of its self-dual part which is a much weaker condition.

We already see from the above natural properties that four-dimension is special, this is because the curvature operator $\mathscr{R}$ of a Riemannian metric—thought as an endomorphism of 2-forms $\mathscr{R} : \Lambda^2 \to \Lambda^2$—decomposes in general into three irreducible components which can be identified with the scalar curvature s, the traceless Ricci tensor Ric_0 and the Weyl tensor W of g. It is only in dimension four however that the bundle Λ^2 splits as $\Lambda^2_+ \oplus \Lambda^2_-$ into self-dual and anti-self-dual forms under the action of the Hodge-star operator. This splitting corresponds to the splitting of Lie groups $SO(4) = SU(2) \times_{\mathbb{Z}_2} SU(2)$ and produces a decomposition of the Weyl tensor $W = W_+ + W_-$. A Riemannian metric is said to be *anti-self-dual* if it satisfies $W_- = 0$ and this is a conformal property of the metric as it only concerns the Weyl tensor W and the Hodge star operator acting on forms of middle dimension.

From now on we specialize to real dimension four and therefore to *complex surfaces*. It was noticed in this context by Boyer [12, Prop.1] that the integrability condition of a compatible almost complex structure on $(M^4, g, or.)$ forces the vanishing of one out of three complex components of the positive Weyl tensor W_+. In a similar vein, Salamon proved the following precise local result which lies at the heart of four-dimensional bi-Hermitian geometry:

Theorem 1.1 ([44, 45]) *Each point $p \in (M^4, c, or.)$ belongs to a neighbourhood in which there are zero, one, two or infinitely many distinct (pairs $\pm J$ of) compatible complex structures.*

The last alternative forces W_+ to vanish identically and therefore implies a result of Boyer that a hyperhermitian four-manifold is actually anti-self-dual: $W_+ = 0$, see [13].

A global version of the above statement is the following:

Theorem 1.2 ([43]) *A compact $(M^4, c, or.)$ can admit none, one, two compatible complex structures or else it is hyperhermitian.*

Recall that in any dimension multiple of four, hyperkähler or more in general hyperhermitian structures correspond to two integrable complex structures I and J satisfying the usual quaternionic condition $IJ = -JI =: K$ and therefore generate a family of complex structures $aI + bJ + cK$ parametrized by the unit 2-sphere. There always is a compatible Riemannian metric in this case, in fact a unique conformal structure. Hyperhermitian compact surfaces have been classified by Boyer [13] and in this work we will be concerned with strictly bi-Hermitian structures by which we mean that $(M^4, g, or.)$ admits exactly two (distinct) compatible complex structures J_+ and J_-—meaning that there is a point $p \in M$ where $J_+(p) \neq \pm J_-(p)$. Notice that four-dimension is again special because $J_\pm$ constitute the maximal number of compatible complex structures which do not necessarily belong to an infinite family. This observation provides motivation to study the following:

Question 1 Classify all compact complex surfaces (S, J) which admit a bi-Hermitian structure $(M, c, J_\pm)$ such that S is diffeomorphic to M^4 and $J = J_+$ as endomorphism of the tangent bundle.

1.1 Generalized Kähler Four-Manifolds

Hitchin successfully introduced the study of generalized geometry [32] which was developed by Gualtieri [31] who considered generalized Kähler structures as a pair of commuting complex structures $(\mathscr{J}_1, \mathscr{J}_2)$ on the direct sum $TM \oplus T^*M$ of the tangent and cotangent bundle of an even-dimensional smooth manifold M^{2m} satisfying some compatibility conditions which he showed to be equivalent to the existence of a Riemannian metric g on M and two orthogonal complex structures $J_\pm$ satisfying the following equations already considered in Physics by Gates, Hull and Rocek in the context of $N = (2, 2)$-supersymmetric σ-models [26]

$$d^c_+\omega_+ + d^c_-\omega_- = 0 \quad \text{and} \quad dd^c_\pm\omega_\pm = 0 \tag{1}$$

where of course $\omega_\pm(\cdot,\cdot) := g(J_\pm\cdot,\cdot)$ denote the fundamental $(1, 1)$-forms of the two Hermitian metrics $(g, J_\pm)$ while the operator $d^c := -JdJ$ so that $d^c = i(\bar\partial - \partial)$ and $dd^c = 2i\partial\bar\partial$.

In particular, we have a bi-Hermitian metric—or *ambi-Hermitian* if $J_\pm$ induce different orientations—of special type and we will now follow [5] to present the generalized Kähler condition for compact conformal surfaces. First of all notice that the second equation requires the existence of Hermitian metrics on M whose fundamental $(1, 1)$-form is $\partial\bar\partial$-closed which may well be impossible in high dimension. However, complex dimension-2 is special let's see why: the fundamental $(1, 1)$-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ of a Hermitian metric (g, J) is always non-degenerate; it is symplectic—ω is closed—if and only if the metric is Kähler. As the real dimension is four, taking the wedge product defines an isomorphism $\omega : \Lambda^1 \to \Lambda^3$ so that there always exist a unique 1-form θ—called the *Lee form* solving the equation

$$d\omega = \theta \wedge \omega.$$

Furthermore, $\theta = 0$ if and only if the metric is Kähler while $\theta = df$ is exact if and only if the conformally related metric $e^f g$ is Kähler. We say in this case that (g, J) is a globally conformal Kähler metric.

We now examine the two generalized Kähler equations (1) in complex dimension 2: first of all $d^c\omega = -JdJ\omega = -Jd\omega = -J\theta \wedge \omega$ because $\omega \in \Lambda^{1,1}$ is J-invariant. On the other hand we always have $\theta \wedge J\theta \wedge \omega = *||\theta||^2$ which shows that

$$d^c\omega = - * \theta \tag{2}$$

and therefore

$$dd^c\omega = 0 \quad \text{is equivalent to} \quad \delta\theta = 0 \quad \text{– i.e. the Lee form is coclosed.} \tag{3}$$

It is now time to recall the following result of Gauduchon which is a milestone in conformal Hermitian geometry, we only state it in dimension 2 for simplicity and it will be used a number of times in what follows; it shows that metrics with coclosed Lee form always exist on compact complex surfaces—while in higher dimension there always exist metrics with $\partial\bar{\partial}\omega^{n-1} = 0$.

Theorem 1.3 ([27]) *In any conformal class of Hermitian metrics on a compact complex surface there is a unique—up to homothetis—representative for which the Lee form is coclosed. Furthermore, the Lee form is coexact if and only if the first Betti number is even.*

The above unique—up to a multiplicative constant—metric is usually called the *Gauduchon metric* of the Hermitian conformal class (c, J).

The generalized Kähler equation (1) in real dimension four then requires the existence of a bi-Hermitian structure $(M, c, J_\pm)$ for which the J_+-Gauduchon metric is also J_- Gauduchon and with respect to this metric $d^c_+\omega_+ + d^c_-\omega_- = 0$; when the two complex structures induce the same orientation—so that they have the same $*$-operator—we then get from (1.1) the condition $*(\theta_+ + \theta_-) = 0$: showing that the two Lee forms are opposite [5]. In order to put this equation in our conformal Hermitian setting we recall

Proposition 1.4 ([6, 12]) *For the Lee forms $\theta_\pm$ of a bi-Hermitian surface $(M, c, J_\pm)$ the following holds:*

1. *The differential of the sum of the Lee forms is anti-self-dual: $d(\theta_+ + \theta_-)_{SD} = 0$*
2. *The pointwise norm satisfies $\|\theta_+\| + 2\delta\theta_+ = \|\theta_-\| + 2\delta\theta_-$ If furthermore M is compact then,*
3. *The sum of the two Lee forms is closed: $d(\theta_+ + \theta_-) = 0$*
4. *The two Lee forms have the same L^2-norm: $\|\theta_+\|_{L^2} = \|\theta_-\|_{L^2}$*

Proof The proof of Salamon's theorem (1.1) shows that there exist severe constrains for the integrability of a compatible almost complex structure J on a conformal oriented four-manifold $(M, c, or.)$ which where first described by Boyer in [12, Lemma 1] as follows: we can think of the self-dual Weyl tensor W_+ as the conformally invariant part of the curvature operator $\mathscr{R} : \Lambda^2 \to \Lambda^2$ acting on self-dual 2-forms as $W_+ : \Lambda^2_+ \to \Lambda^2_+$. At each point $x \in M$, W_+ is a symmetric 3×3 matrix of zero trace, let $\lambda_- \leq \lambda_0 \leq \lambda_+$ denote its three eigenvalues: $\lambda_- + \lambda_0 + \lambda_+ = 0$.

Then, if θ is the Lee form of a compatible *integrable* almost complex structure the following alternative must hold for the self-dual part of its differential $(d\theta)_{SD}$: at any point where (at least) two eigenvalues coincide $(d\theta)_{SD} = 0$; at all other points, $(d\theta)_{SD}$ is a unit vector in the (1-dimensional) eigenspace of λ_0 which completely determines $\pm J$.

On the open dense set $U := M \setminus T$ where $J_+ \neq \pm J_-$ we then have $(d\theta_-)_{SD} = -(d\theta_+)_{SD}$. This proves the first statement which implies the third because the exact 2-form $d(\theta_+ + \theta_-)$ being anti-self-dual is actually harmonic. The second statement follows from [6, equation (3)] and implies the fourth by Stoke's theorem. □

This leads to the following conformal invariant formulation of the generalized Kähler condition:

Remark 1.5 Let $(M, c, J_{\pm})$ be a compact bi-Hermitian surface with the usual assumption that $J_{\pm}$ induce the same orientation. The bi-Hermitian structure comes from a generalized Kähler structure if and only if, for one and hence any metric in the conformal class, the sum of the Lee forms $\theta_{+} + \theta_{-}$ is an exact 1-form.
Indeed, if $(M, c, J_{\pm})$ is generalized Kähler we have $\theta_{+} + \theta_{-} = 0$ for the common Gauduchon metric g. For $\tilde{g} = e^{f} g$ any other metric in the conformal class we easily see $\tilde{\theta}_{\pm} = \theta_{\pm} + df$ so that $\tilde{\theta}_{+} + \tilde{\theta}_{-} = 2df$ is exact. Conversely, let g be the J_{+}-Gauduchon metric and consider the orthogonal decomposition of Lee forms into their g-harmonic, g-coexact and exact parts: $\theta_{+} = \theta_{+}^{h} + \delta\alpha_{+}$ and $\theta_{-} = \theta_{-}^{h} + \delta\alpha_{-} + da$. The sum can be exact only for $\theta_{+}^{h} = -\theta_{-}^{h}$ and $\delta\alpha+ = -\delta\alpha_{-}$. However, we know from (1.4) that the two Lee forms always have the same global L^2-norm and we conclude that $da = 0$, showing that the sum of the two Lee forms vanishes for the J_{+}-Gauduchon metric which is therefore J_{-}-Gauduchon too.

1.2 Locally Conformal Kähler Surfaces

Let (g, J) be a Kähler metric, the fundamental $(1, 1)$-form of a conformally related metric is $\tilde{\omega} = e^{f}\omega$ and $d\tilde{\omega} = d(e^{f}) \wedge \omega$ because ω is closed. This shows that a Hermitian conformal structure is locally conformal to a Kähler metric, abbreviated by l.c.K. , if and only if the fundamental form of any metric in the conformal class satisfies two conditions:

$$d\omega = \theta \wedge \omega \quad \text{with} \quad d\theta = 0 \tag{4}$$

we already know that the first equation always holds on complex surfaces so that in this case the l.c.K. condition is equivalent to the Lee form being closed. In higher dimensions the situation is opposite as the first equation is not satisfied in general, but it implies the second one.

We described above the interplay between bi-Hermitian surfaces and generalized Kähler geometry in four-dimension. Our aim in this section is to outline some connections among self-dual four-manifolds, l.c.K. and bi-Hermitian surfaces in the compact case.

The story started with Boyer who showed in [12] that anti-self dual Hermitian metrics are necessarily l.c.K. in the compact case. The theory of anti-self dual bi-Hermitian metrics was developed in [43] and later on in [22] mainly for the purpose of showing some examples and constructions. The argument in the proof of Prop. 1.4 shows that a bi-Hermitian structure $(M, c, J_{\pm})$ is anti-self-dual if and only if one and hence both Hermitian structures $(g, J_{\pm})$ are l.c.K. Notice that this is a conformal statement. The twistor construction of [22] then produced new l.c.K. surfaces and soon after Brunella showed that indeed all Kato surfaces admit l.c.K. metrics [15] .

Finally, in the last section of this work we show how this circle of ideas was closed up by Apostolov-Bailey-Dloussky who proved that l.c.K. metrics can be used to produce new bi-Hermitian structures, never anti-self-dual [7].

Degree of Holomorphic Line Bundles

One of the main application of the fundamental result of Gauduchon 1.3 is that it allows to define the notion of degree of holomorphic line bundles $\mathscr{L}$ which is a powerful tool to prove vanishing theorems for the group of holomorphic sections $H^0(X, \mathscr{L})$. On a compact Hermitian conformal n-manifold (X, c, J) the theorem produces a unique—up to multiplicative constant—metric g with $\partial\bar{\partial}$-closed form ω^{n-1} and then one sets

$$\deg_g(\mathscr{L}) := \frac{1}{2\pi}\int_M \rho \wedge \omega^{n-1}$$

where ρ is the curvature 2-form of the Chern connection of $\mathscr{L}$, the unique holomorphic connection which is compatible with a Hermitian metric on $\mathscr{L}$. When (g, J) is Kähler the Chern connection is the Levi Civita connection and $\deg_g(\mathscr{L})$ is the cup product of the first Chern class of $\mathscr{L}$ with the closed form ω^{n-1}; otherwise the Chern connection has torsion and its curvature ρ is a $(1, 1)$-form which is well defined modulo $\partial\bar{\partial}$-exact forms; this shows that $\deg_g(\mathscr{L})$ is well posed exactly when g is a Gauduchon metric. The degree measures the g-volume with sign of a virtual meromorphic section; in particular $deg(\mathscr{L}) > 0$ whenever $H^0(S, \mathscr{L}) \neq 0$ and $\mathscr{L} \neq \mathscr{O}$. Notice that the *sign* of the degree is an invariant of the Hermitian conformal structure (c, J).

In the special case $\mathscr{L}$ is a flat holomorphic line bundle of real type so that $\mathscr{L} = \mathscr{L}_a$ for some $a \in H^1_{dR}(M, \mathbb{R})$, then the curvature of the Chern connection is the 2-form $\rho = -d^c a$ and the degree is the global L^2-inner product of the closed 1-form a with the coclosed Lee form θ of the Gauduchon metric, see [4]:

$$\begin{aligned} 2\pi \deg(\mathscr{L}_a) &:= \textstyle\int_M \rho \wedge \omega = -\int_M d^c a \wedge \omega = -\int_M g(d^c a, \omega) vol = -\langle a, *d^c * \omega\rangle = \\ &\quad -\langle a, -J * d\omega\rangle = -\langle a, \theta\rangle = -\langle a^h, \theta^h\rangle. \end{aligned} \tag{5}$$

To conclude this introductory section we observe that the degree of a Lee bundle $\mathscr{L}_\theta$—i.e. the holomorphic flat line bundle of a l.c.K. metric with Lee form θ—is always strictly negative because from Eq. (5) it follows that $\deg(\mathscr{L}_\theta) = -||\theta_h||^2 < 0$ otherwise g would be conformally Kähler and $b_1(S)$ even. In particular then, $H^0(S, \mathscr{L}_\theta) = 0$ and the same holds for any positive power.

2 Obstructions

The aim of this section is to present some necessary conditions for the existence of bi-Hermitian structures $(M, c, J_\pm)$ on compact complex surfaces S—i.e. the complex structure J of S satisfies $J = J_+$. The first observation is that the Kodaira dimension of such an S cannot be positive because the surface admits effective (twisted) anticanonical divisors or else its canonical bundle must be trivial.

To see this we start with a geometric construction of a numerical anticanonical divisor (NAC divisor in the terminology of Dloussky) by which we mean a holomorphic section of the anticanonical bundle K^{-1}, twisted by a line bundle of zero Chern class $F \in \mathrm{Pic}_0(S)$.

Think of the complex structures $\pm J_\pm$ on $(M, [g], or.)$ as defining four sections of its almost complex twistor space $(Z, \mathbb{J})$, denoted by $S_+ := J_+(M)$, $\bar{S}_+ := -J_+(M)$ and so on. By the integrability of $J_\pm$ they form two pairs of disjoint almost complex hypersurfaces in $(Z, \mathbb{J})$, which we denote by $X_\pm := S_\pm \sqcup \bar{S}_\pm$. The relevant geometric observation is that the Poincaré dual of the union of these four irreducible components represents the first Chern class $c_1(Z)$.

To this end, recall that M is a four-manifold while Z has real dimension 6 and the fibers $SO(4)/U(2)$ of the twistor fibration $t : Z \to M$ are pseudo-holomorphically imbebbed $\mathbb{CP}_1$'s. By Leray-Hirsch theorem the first Chern class $c_1(Z)$ turns out to be represented by 4 times the cohomology class of a twistor line and $X_+ \cup X_- = S_\pm \cup \bar{S}_\pm$ intersects each twistor line in precisely 4 points. In fact the almost complex hypersurfaces X_+ and X_- are homologous in Z and we can compute the first Chern class $c_1(S) = c_1(M, J_+)$ by adjunction formula in the twistor space as follows; denoting by * the Poincaré dual we have:

$$c_1(S) = [c_1(Z) - X_+^*]_{|S}.$$

We consider the intersection of the almost complex hypersurfaces S_+ and X_- as an almost complex submanifold of S_+ (with (positive) multiplicities) and therefore an effective or zero divisor in the bi-Hermitian surface S. As the restriction $t_{|S_+} : (S_+, \mathbb{J}) \to (S, J_+)$ is a biholomorphism the image $\mathbf{T} := t(S_+ \cap X_-)$ is an effective, or zero, divisor which satisfies the following *fundamental equation* which provides the necessary condition for the existence of bi-Hermitian metrics on compact four-manifolds.

$$\mathbf{T} = K^{-1} \otimes F \text{ for some holomorphic line bundle of real type } F \in \mathrm{Pic}_0(S) \qquad (6)$$

where K^{-1} denotes the anticanonical line bundle of $S = (M, J_+)$. We will call the flat line bundle F the *fundamental line bundle* of the bi-Hermitian structure which is necessarily of real type: $F \in \mathrm{Pic}^0_{\mathbb{R}}(S)$ and refer to $\mathbf{T}$ as the *fundamental divisor*. Notice that F is a divisor if and only if S admits anticanonical divisors. In what follows we will use tensor product notation for line bundles and additive notation for divisors; for example, if the anticanonical bundle K^{-1} happens to have a meromorphic section we will denote by $-K$ the corresponding divisor.

We see from the above geometric picture that the support T of the divisor $\mathbf{T} \geq 0$ is the set where the two complex structures J_+ and J_- commute

$$T := \mathrm{supp}(\mathbf{T}) = \{p \in M s.t. J_+(p) = J_-(p)\} \sqcup \{p \in M s.t. J_+(p) = -J_-(p)\} \qquad (7)$$

Remark 2.1 A few interesting properties of this set are the following [23]:

1. $T \subset M$ is a compact complex curve in each surface $(M, J_\pm)$ which often turns out to be the maximal curve. In any case, T is the closure of all smooth 2-dimensional surfaces which are simultaneously $J_\pm$-holomorphic curves.
2. Knowing the set T one can reconstruct the divisor $\mathbf{T}$ completely by taking the unique linear combination of the irreducible components of T such that $c_1(\mathbf{T}) = c_1(S)$
3. $\mathbf{T}$ can be the trivial divisor—i.e. $T = \emptyset$—in which case $(M, c, J_\pm)$ is said to be strongly bi-Hermitian. Otherwise, from the classification of surfaces it can easily be checked that the number of connected components of a non-trivial NAC divisor is either 1 or 2, in other words $b_0(T) \leq 2$, see Proposition 2.10.

We now pass to a short discussion of blown-up surfaces aiming at restricting our presentation to the minimal case—i.e. there is no smooth rational curve $C \subset S$ such that $C^2 = -1$. The first observation is that the blow-up formula for the canonical bundle states that if $b : \tilde{S} \to S$ is the blow-up at a point $p \in S$ with exceptional divisor E then

$$K_{\tilde{S}} = b^* K_S \otimes E$$

It follows that $\tilde{S}$ has a NAC divisor if and only if it is obtained by blowing up one point on a NAC divisor of S. Furthermore, the exceptional divisor E belongs to the new NAC divisor if and only if $\mathbf{T}$ has multiplicity at least two at p.

Therefore, we can conclude that if a compact complex surface S satisfies the necessary condition of having a divisor $\mathbf{T} \geq 0$ with $\mathbf{T} = K^{-1} \otimes F$ then the same holds for its minimal model; furthermore, the number of connected components does not change under blow-ups.

Conversely, the results of [22] and [16] show that the blow-up of a bi-Hermitian surface is again bi-Hermitian and it will therefore be sufficient to answer the following

Question 2 Which *minimal* compact complex surfaces admit a conformal Riemannian metric with exactly two orthogonal complex structures?

A key role in the study of bi-Hermitian surfaces is played by the sum of the two Lee forms. We have already seen that in the compact case $\theta_+ + \theta_-$ is a closed 1-form; in fact, by the following result of Apostolov-Gauduchon-Grantcharov, it is always associated to the fundamental line bundle $F \in \mathrm{Pic}^0(S)$

Proposition 2.2 ([6]) *On any compact bi-Hermitian surface $(M, c, J_\pm)$ the flat bundle F in the fundamental equation (2) is associated to the de Rham class $[\theta_- + \theta_+] \in H^1_{dR}(M)$ via the natural map $H^1_{dR}(M) \cong H^1(M, \mathbb{R}) \overset{exp}{\to} H^1(M, \mathbb{R}^{>0}) \hookrightarrow H^1(M, \mathbb{C}^*) \to \mathrm{Pic}^0(S)$. In particular, F is the trivial line bundle if and only if $(c, J_\pm)$ is generalized Kähler.*

It is well known that a compact complex surface has even first Betti number b_1 if and only if it admits a Kähler metric. Although bi-Hermitian metrics are conformally Kähler only when they are actually conformally hyperkähler, it is convenient to distinguish between the two cases of b_1 even/odd in the present context too, mainly because of Gauduchon's Theorem 1.3.

2.1 Kählerian Case

In case the first Betti number is even, the main observation is that the generalized Kähler condition always holds.

Proposition 2.3 ([6, Lemma 4]) *When $(M, c, J_{\pm})$ is compact bi-Hermitian with even first Betti number the two Lee forms always satisfy the generalized Kähler condition that*

$$\theta_+ + \theta_- \quad \textit{is exact.} \tag{8}$$

Proof Let g be the J_+-Gauduchon metric, as $b_1(M)$ is even, we have $\theta_+ = \delta\alpha$. On the other hand $d(\theta_+ + \theta_-) = 0$ so that the orthogonal decomposition of θ_- into harmonic, coexact and coclosed part is of the form $\theta_- = \theta_-^h - \delta\alpha + df$. Because the two Lee forms have the same global L^2-norm we conclude that $\theta_-^h = 0 = df$ so that $\theta_+ + \theta_- = 0$ and g is J_--Gauduchon too. □

The fundamental equation (2) then reduces to

$$\mathbf{T} = K^{-1} \tag{9}$$

and the necessary condition to admit bi-Hermitian metrics is then $H^0(S, K^{-1}) \neq 0$.

This shows that $Kod(S) \leq 0$ for any bi-Hermitian compact surface S with $b_1(S)$ even and equality holds if and only if the canonical bundle is trivial—i.e. S can only be a torus or a $K3$ surface, but cannot be a finite quotient; for example it cannot be an Enriques surface. When $c_1(S) \neq 0$ there is an effective anticanonical divisor $\mathbf{T} = -K$: these surfaces must be ruled, of arbitrary genus and the complete list can be found in [9]; notice that the anticanonical divisor is always connected and this agrees with the result of [6, Prop. 4].

2.2 Non-Kähler Case

We now look for obstructions to bi-Hermitian metrics on a compact complex surface S with $b_1(S)$-odd. In this case, Apostolov [2] proved that the Kodaira dimension must be negative and this is shown by the following degree computation which will be followed by a discussion on surfaces in Kodaira class VII.

Proposition 2.4 ([2]) *Let $(M, c, J_\pm)$ be a compact bi-Hermitian surface with $b_1(M)$ odd. Then, with respect to either complex structures $J_\pm$, the degree of the fundamental line bundle satisfies*

$$\deg(F) \leq 0$$

with equality if and only if the metric is generalized Kähler.

Proof As F corresponds to the closed 1-form $\theta_+ + \theta_-$ its degree with respect to the J_+-Gauduchon metric g with Lee form θ_+ is given by the following formula in which $\langle,\rangle$ denotes the global L^2-inner product, see (5), $\deg(F) = -\langle\theta_+, \theta_+ + \theta_-\rangle = -\|\theta_+\|^2 - \langle\theta_+, +\theta_-\rangle = -\frac{1}{2}\langle\theta_+ + \theta_-, \theta_+ + \theta_-\rangle$ because $\|\theta_+\| = \|\theta_-\|$ holds for the global norm (1.4). The same computation holds for the other complex structure J_- as well. □

Corollary 2.5 ([2]) *The canonical bundle of a compact bi-Hermitian surface S of odd first Betti number has negative degree, in particular $Kod(S) = -\infty$ and S is said to be in class VII of Kodaira classification.*

Proof As the degree computes the signed volume of the divisor of a virtual meromorphic section and $\mathbf{T}$ is an effective or zero divisor we have from the fundamental equation (2): $0 \leq vol(\mathbf{T}) = deg(F) - deg(K)$ so that $deg(K) \leq deg(F) - vol(\mathbf{T}) \leq 0$; furthermore, the last equality holds if and only if $\mathbf{T} = 0$ is empty and F is the trivial line bundle. The last two equalities are however in contradiction because in the $b_1(S)$-odd case $F = \mathcal{O}$ implies that $\mathbf{T}$ has two non-empty connected components by [6, Prop. 4]. This shows the first statement which by definition implies $S \in$ VII for odd b_1. □

2.2.1 Class-VII$_0$ Surfaces

With the aim at answering Question 2, we recall that a minimal compact complex surface with invariants b_1 odd and $Kod = -\infty$ is said to be in class VII$_0$ of Kodaira classification; it was shown by Kodaira that in fact $b_1(S) = 1$ always holds in this class. The only other compact complex surfaces with $b_1 = 1$ are secondary Kodaira surfaces; namely, finite quotients of surfaces with trivial canonical bundle and in particular $Kod = 0$.

For a general compact complex surface S, the number of positive eigenvalues of its intersection form satisfies $b_2^+(S) = 1 + 2h^0(S, K)$ if S is Kähler and $b_2^+(S) = 2h^0(S, K)$ otherwise. It follows that the topological signature $\sigma(S) = -b_2(S)$ for $S \in$ VII$_0$ and it agrees with the Chern number $c_1^2(S) = 2\chi(S) + 3\sigma(S) = -b_2(S)$.

Although there is no complete classification of surfaces in this class, we now present a few results which are particularly useful for discussing bi-Hermitian surfaces. We start from the case of vanishing first Chern class where a classification has been known for some time.

2.2.2 Surfaces in Class VII and $b_2(S) = 0$

A Theorem of Bogomolov [10], later clarified by Teleman [46] and Yau et al. [39] asserts that there exist only two types of surfaces $S \in \mathrm{VII}_0$ with $b_2(S) = 0$.

The first class is given by Hopf surfaces which are defined to be compact quotients of $\mathbb{C}^2 \setminus \{0\}$; they have fundamental group isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/\mathbb{Z}_k)$ for $k \geq 1$ and are diffeomorphic to $S^1 \times S^3$, up to finite coverings. The easiest example is when the action is generated by the contraction $(z, w) \mapsto \frac{1}{2}(z, w)$. The complete list of contractions was described by Kodaira and shows that a Hopf surface can be elliptic (admits an infinite number of elliptic curves) or has exactly two elliptic curves when the action is diagonal (up to a finite covering)—i.e. of the form $(z, w) \mapsto (az, bw)$—in which case we say that S is a *diagonal Hopf surface*. Otherwise S has just one elliptic curve E and $-K$ is a positive multiple of E. In any case a Hopf surface can only have elliptic curves which are always disjoint as $b_2(S) = 0$. The anticanonical divisor is always effective, consisting of two elliptic curves in the diagonal case and of one curve with multiplicity in the other case.

It follows that every Hopf surface satisfies the necessary condition (2) for existence of bi-Hermitian metrics: admits effective numerical anticanonical divisors $-K + mE$ the only condition is for the elliptic curve E to represent a *real* divisor, when $m > 0$. It will be shown later that there is a surprising abundance of bi-Hermitian Hopf surfaces with all possible values of $b_0(T) = 0, 1, 2$.

By Bogomolov theorem the only other case $S \in \mathrm{VII}$ with $b_2(S) = 0$ is that S is a so called Inoue-Bombieri surface [11, 35] which can be characterized as discrete quotients of $\mathscr{H} \times \mathbb{C}$, the product of the upper-half plane with a complex line. Inoue-Bombieri surfaces however cannot admit bi-Hermitian metrics by Corollary 2.5 because the degree of their canonical bundle is always positive by [47].

2.2.3 Kato Surfaces

A minimal non-Kähler surface of Kodaira dimension $-\infty$ and positive second Betti number is said to belong to class VII_0^+. All known examples are so called Kato surfaces which were introduced in [36] and by definition are compact complex surfaces S admitting a global spherical shell: there is a holomorphic embedding $\phi\colon U \to S$, where $U \subset \mathbb{C}^2 \setminus \{0\}$ is a neighborhood of the unit sphere S^3, such that $S \setminus \phi(U)$ is connected.

The following statement summarizes some of the main results about Kato surfaces

Theorem 2.6 ([17, 20, 21, 36, 41]) *For a surface S in class VII_0^+ the following conditions are equivalent and they imply that S is diffeomorphic to $(S^1 \times S^3)\#\overline{\mathbb{CP}}_2$.*

1. *S is a Kato surface*
2. *S contains $b_2(S) =: b$ rational curves*
3. *S admits an effective divisor $D = G - mK$ with $c_1(G) = 0$ and $0 \leq m \in \mathbb{Z}$*

As already mentioned, a non-trivial divisor of the form $D = G - mK$ with $c_1(G) = 0$ is called a NAC (numerically anticanonical) divisor in the terminology of Dloussky [17] because its Chern class is a negative multiple of the canonical class. It is known that D is automatically effective on $S \in \mathrm{VII}_0^+$. The smallest $m \geq 0$ for which a NAC divisor exists is called the *index* of the Kato surface S. The equivalence between 1. and 3. is a powerful result which we call Enoki-Dloussky theorem. The case $m = 0$ is due to Enoki who also showed that S must be a so called Enoki surface. The case $m \geq 1$ is due to Dloussky. The equivalence between 1. and 2. is due to Dloussky-Oljeklaus-Toma and it was known from the work of Nakamura [41] that Kato surfaces admit deformations which are blown-up Hopf surfaces, in fact this holds more in general for any surface with a cycle or rational curves.

There are recent important results about Kato surfaces going in different directions; concerning their Hermitian geometry Brunella proved the following strong property

Theorem 2.7 ([15]) *Every Kato surface admits l.c.K. metrics.*

But Kato surfaces are important mainly because they are the only known examples of surfaces in class-VII_+^0. A strong conjecture of Nakamura [41, 8.5] says that there should be no other examples in this class. We only point out here that there is recent important progress in this direction by Teleman [48].

Our main interest in Kato surfaces comes from the following result

Proposition 2.8 ([2, 17]) *A minimal compact bi-Hermitian surface S with b_1-odd is either a Hopf surface when $b_2 = 0$, or else is a Kato surface of index* 1 *when $b := b_2 > 0$. In particular, S is diffeomorphic to a finite quotient of $S^1 \times S^3$ or to $(S^1 \times S^3)\#b\overline{\mathbb{CP}}_2$.*

Proof Suppose $b_2(S) = 0$ and that S is not a Hopf surface, then by Bogomolov theorem S must be an Inoue-Bombieri surface which however is impossible by Corollary 2.5, as already mentioned. The case $b = b_2(S) > 0$ is handled by Theorem 2.6 (3) because the fundamental divisor $\mathbf{T} = F - K$ satisfies $\mathbf{T}^2 = c_1^2(S) = 2\chi(S) + 3\sigma(S) = 2b - 3b = -b$ and therefore is a non-trivial NAC divisor on S. □

Our main task at this point is to understand which Kato surfaces can possibly admit bi-Hermitian metrics—i.e. satisfy the fundamental equation (2). To see this we now present a description of these surfaces based on the intersection properties of their rational curves $D_1, \ldots, D_b$. It is useful to consider the Dloussky number $\mathrm{Dl}(S) := -\sum_{i=1}^b D_i^2$ which is the negative sum of the self-intersection numbers of the rational curves. As S is minimal and the intersection form is negative definite the lower bound is certainly $\mathrm{Dl}(S) \geq 2b$. On the other hand a rational curve of self-intersection $-(k+2)$ must come together with a chain of -2-curves, of length $(k-1)$; for these reasons $2b \leq Dl(S) \leq 3b$ always holds.

We will use the terminology of the Japanese school and examine different Kato surfaces according to their Dloussky number and configuration of rational curves.

$\mathrm{Dl}(S) = 2b$. This is the case of so called Enoki surfaces [21]: the union of all rational curves is a cycle C which necessarily satisfies $C^2 = 0$ so that C is a divisor with $c_1(C) = 0$ and Enoki surfaces are the only class-VII surfaces with this property. Enoki surfaces fill up a complex b-dimensional open set in the moduli space of Kato surfaces; generically C is the maximal curve of the Enoki surface, however there is a 1-dimensional subfamily of so called *parabolic Inoue surfaces* which also admit a smooth elliptic curve E with $E\dot{C} = 0$ and the sum $E + C = -K$ is both the maximal curve and the anticanonical divisor. We conclude that an Enoki surface with bi-Hermitian metrics is necessarily parabolic Inoue; furthermore, for the fundamental divisor we have either $\mathbf{T} = E + C$ or else $\mathbf{T} = E$ because these are the only NAC divisors of index 1 with $\deg(G) \leq 0$ on parabolic Inoue surfaces.

$\mathrm{Dl}(S) = 3b$. Kato surfaces in this class are rigid and come in two types, they are characterized by the fact that every rational curve D_i belongs to a cycle of negative self-intersection. The number of cycles can be one or two. *Half-Inoue surfaces* have only one cycle and they are double covered by *hyperbolic Inoue surfaces* which have two cycles whose union is the maximal curve as well as the anticanonical divisor $-K = C_1 + C_2$. We see from this discussion that every bi-Hermitian metric (if any) on hyperbolic Inoue surfaces is generalized Kähler: $\mathbf{T} = -K$. By contrast, half-Inoue surfaces admit no bi-Hermitian metrics at all because the cycle C is the only NAC divisor and $C = G - K$ holds with G a non-trivial flat line bundle of order 2 which has zero degree without being trivial, in particular cannot be of real type.

It remains to discuss the case of Kato surfaces with $2b < \mathrm{Dl}(S) < 3b$ which are usually called *intermediate.* The configuration of rational curves is rather different from the previously described Kato surfaces because the union of the rational curves is always connected with precisely one cycle and a positive number of trees attached to it—i.e. an arboreal cycle. In this situation, unlike the other cases, a numerical anticanonical divisor is never reduced, it exists if and only if $\mathrm{index}(S) = 1$ and is supported on the maximal curve which is connected.

We described in [24] precise conditions for an intermediate Kato surface to have index 1 showing in particular that the number of irreducible components in the trees cannot exceed the number of cycle components. As a finer condition on the complex structure we showed that the following three sets of intermediate Kato surfaces of index 1 are mutually disjoint: surfaces with holomorphic vector fields, surfaces with anticanonical divisor and surfaces with bi-Hermitian metrics.

We end the section with the following

Remark 2.9 It follows from the above presentation that the number of connected components of a NAC divisor on a compact complex surface can be at most 2. We certainly have $b_0(T) = 0$ for a hyperhermitian structure because there are no points where $J_{\pm}$ commute and more in general a bi-Hermitian structure with $T = \emptyset$ is said to be strongly bi-Hermitian; such structure can only exist when $c_1(S) = 0$. The opposite case is characterized as follows:

Proposition 2.10 ([23, Prop.4.7]) *Let T be the set where the two complex structures of a compact bi-Hermitian surface $(M, c, J_{\pm})$ are dependent (2) then $b_0(T) \leq 2$. Furthermore, equality holds if and only if $b_1(M) = 1$ and the metric is generalized Kähler; in this case the complex structure can only be that of a Hopf surface, a parabolic Inoue surface or a hyperbolic Inoue surfaces.*

Proof The fact that $\mathbf{T}$ is disconnected when b_1 is odd and $(g, J_{\pm})$ is generalized Kähler was proved in [6, Prop.4]. Conversely, when $b_2(S)$ is even we know from (9) that the fundamental divisor $\mathbf{T} = -K$ is anticanonical and by Enriques classification $T = \emptyset$ if and only if S is either a K3 surface or a torus. In all other cases S is ruled and it follows that any anticanonical divisor is connected. In the b_1-odd case, by a result of Nakamura [43], [22, Lemma 3.3] if $S \in$VII has disconnected numerical anticanonical divisor then the complex structure is that of a diagonal Hopf surface or else of a parabolic or hyperbolic Inoue surface. □

Now that we have seen several necessary conditions for existence of bi-Hermitian structures on a given compact complex surface we would like to present some positive results and constructions. It will turn out that the previously described conditions are indeed (almost) sufficient.

3 Existence of Bi-Hermitian Surfaces

The first non-ASD examples of bi-Hermitian structures on four-manifolds are due to Kobak [37] and to a Hamiltonian flow construction which produced strongly bi-Hermitian metrics on surfaces which can admit hyperhermitian structures, see [6, Prop.3].

New impetus came after the introduction of generalized Kähler geometry [31]: Hitchin produced new examples on Del Pezzo surfaces [33, 34] using the positivity of the curvature of the anticanonical bundle coupled with a Hamiltonian flow. The result was then generalized by Gualtieri [30]—using similar techniques—and by Goto who introduced a new theory of K-deformations [28] to show that a compact Kähler surface has bi-Hermitian structures if and only if it admits a holomorphic anticanonical section. Notice that these bi-Hermitian metrics are never conformally Kähler and that Goto's result proves that the necessary condition of Prop. 9 is actually sufficient when the first Betti number is even.

Let us now examine the non-Kähler situation which is more involved, in the sense that the fundamental line bundle F may be non-trivial; nevertheless a lot of progress has been made.

3.1 Surfaces of Non-Kähler Type

Recall that the necessary condition for a compact complex surface S to admit bi-Hermitian metrics is $H^0(S, K^{-1} \otimes F) \neq 0$ with F a holomorphic flat line bundle which is real of negative degree. We have seen that $F = \mathscr{O}$ always in the b_2-even case and we now examine surfaces with $b_2(S)$ odd with an effective or trivial divisor $\mathbf{T} = F - K$ which we may call a"real-negative" NAC divisor.

The best possible result would be that conversely every such $\mathbf{T}$ is the fundamental divisor of a bi-Hermitian metric on S. The aim of the present section is to show that this can actually be proved in all situations except for *intermediate* Kato surfaces in which case we can only show it up to logarithmic deformations. Let us proceed with a case by case presentation based on the number of connected components of the set $T = \operatorname{supp} \mathbf{T}$ which we recall can be at most 2.

Generalized Kähler structures: $b_0(T) = 2$. The fundamental equation (2) reduces to $\mathbf{T} = -K$ in this case and the divisor is disconnected. All types of surfaces satisfying the necessary condition, see Prop. 2.10, admit such metrics because the standard conformally flat metric on Hopf surfaces with reality condition $|a| = |b|$ is bi-Hermitian [43] as well as LeBrun's hyperbolic ansatz metrics on parabolic Inoue surfaces of "real type" [38]. The picture was completed by a twistor construction on all hyperbolic Inoue surfaces in [22].

These bi-Hermitian metrics are actually ASD and l.c.K. which always implies generalized Kähler [43, Prop.3.11]; as a matter of fact these where the first examples of l.c.K. metrics on Kato surfaces. It is important for the subsequent discussion to compute the Lee class of such metrics. We do this is the following Proposition whose proof strongly uses twistor methods and can be found in [25].

Proposition 3.1 *Let $(M, c, J_\pm)$ be any ASD bi-Hermitian structure on a parabolic Inoue surface S—as constructed in [38] and [22], for example. Then S is a real parabolic Inoue surface in the sense that in the complex 1-dimensional Kuranishi family S_z of parabolic Inoue surfaces over the punctured unit disc, $S = S_t$ with $t \in \mathbb{R}$. Furthermore the Lee class $\theta \in H^1_{dR}(M)$ of the metric corresponds to the holomorphic flat line bundle C^* under the injection $H^1_{dR}(M) \hookrightarrow H^1(M, \mathbb{C}) \overset{exp}{\to} H^1(M, \mathbb{C}^*)$ where $C \subset S$ is the cycle of rational curves.*

Notice that all known examples of bi-Hermitian metrics with $b_1(S)$-odd and $b_0(T) = 2$ are actually ASD. The remaining cases $b_0(T) = 0, 1$ present an interesting interplay between bi-Hermitian and l.c.K. metrics. The following strong result of Apostolov-Bailey-Dloussky will be used to construct all the remaining new examples; its proof can be seen as a twisted version of the Hamiltonian construction in [30, Thm.6.2].

Theorem 3.2 ([7, Theorem 1.1]) *Let $S = (M, J)$ be a compact surface in class VII which admits a l.c.K. metric and denote by $\mathscr{L}_\theta$ the Lee bundle of this metric. Suppose S also admits a NAC divisor $\mathbf{T} = F - K$—i.e. the line bundle $F \otimes K^{-1}$ admits a non-trivial holomorphic section—satisfying $F = \mathscr{L}_\theta$ then, M has bi-Hermitian conformal structures $(c, J_\pm)$ such that the following holds:*

1. $J_+ = J$
2. $\mathbf{T} = F - K$ *is the fundamental divisor of the bi-Hermitian structure* $(M, c, J_\pm)$ *so that* $T = \operatorname{supp}(\mathbf{T})$ *is the subset where* $J_\pm$ *agree up to sign.*

Notice that this theorem does not apply in the generalized Kähler situation because $F = \mathscr{O}$ cannot be the Lee bundle of a l.c.K. metric. We are then led to consider surfaces with "real-negative" NAC divisor $\mathbf{T} = F - K$, $F \neq \mathscr{O}$.

Hopf surfaces. When S is a Hopf surface, $b_2(S) = 0$ and every holomorphic line bundle is flat; furthermore, the anticanonical bundle is always effective, its divisor is either the sum of two (reduced) elliptic curves or else is one elliptic curve with multiplicity. In any case the point is that Hopf surfaces admit l.c.K. metrics with Lee class in every possible de Rham class, in the sense that given any $F \in \mathrm{Pic}^0_{\mathbb{R}}(S)$ with $\deg(F) < 0$ there is a l.c.K. metric whose Lee bundle is F.

The Apostolov-Bailey-Dloussky theorem is therefore very powerful for Hopf surfaces and produces bi-Hermitian metrics for every $\mathbf{T}$ which is either empty or connected. Indeed $\mathbf{T}$ is empty, or equivalently the metric is strongly bi-Hermitian if and only if $K = F$ is of real type [3]. The other case is when $\mathbf{T} = F - K$ is connected and therefore supported on a unique elliptic curve E which may or may not be reduced. As we can freely choose the Lee form class the only condition is on the canonical bundle for which we have $-K = E_1 + E_2$ in the diagonal case, and therefore $-F$ must be one of the two elliptic curves; or else S is of non-diagonal type in which case admits a unique elliptic curve E and a unique $m \in \mathbb{N}$ with $-K = mE$. The real line bundle F must then be chosen to satisfy $F = -nE$ with $n < m$.

This concludes the discussion about Hopf surfaces and notice that they admit a surprising abundance of bi-Hermitian metrics with all possible values of $b_0(T) = 0, 1, 2$.

3.2 Bi-Hermitian Kato Surfaces with $b_0(T) = 1$

It only remains to consider the case of Kato surfaces with bi-Hermitian metrics of non-generalized Kähler type which was tackled in [25]. From the fundamental equation (2) and the discussion of last section, S can only be a parabolic Inoue surface or an intermediate Kato surface. In the first case we can use the Lee class of the generalized Kähler metric computed in Prop. 3.1 to get the following precise existence result:

Theorem 3.3 ([25]) *A Kato surface with* $Dl(S) \in \{2b, 3b\}$ *admits bi-Hermitian metrics with* $b_0(\mathbf{T}) = 1$ *if and only if S is a parabolic Inoue surface of real type.*

Proof We know from the Dloussky number and the discussion of the previous section that the only candidates are parabolic or hyperbolic Inoue surfaces. In both cases the anticanonical bundle is effective and disconnected: $-K = E+C$. It follows from the fundamental equation (2) that F must be effective and real and this implies that S is parabolic Inoue of real type with $F = -C$. Viceversa, we have shown that such a surface S admits generalized Kähler metrics which are actually l.c.K. with Lee bundle $-C$; we can then conclude by applying Theorem 3.2. □

We are now left with the intermediate Kato case for which we don't have precise information about the Lee classes. The existence result of Brunella 2.7 gives Lee classes of the form $t\theta$ for values of t in a neighbourhood of $+\infty$. For this reason our existence result for bi-Hermitian metrics—necessarily of type $b_0(T) = 1$—is of an asymptotic nature based on deformation theory of Kato surfaces as developed by Dloussky, Oeljeklaus and Toma [18, 19, 42]. Recall that a bi-Hermitian intermediate Kato surface must have index 1, as a weak converse we have:

Theorem 3.4 ([25]) *Every intermediate Kato surface S of index* 1 *admits a logarithmic deformation S_λ which is bi-Hermitian with $b_0(T) = 1$.*

Idea of Proof Let S be an intermediate Kato surface and let $D := D_1 + \cdots + D_b$ be the union of all rational curves in S which is in fact its maximal curve; by a *logarithmic deformation* of S we mean a deformation of the pair (S, D). It is known that the Kuranishi family is always smooth of complex dimension $3b - Dl(S)$ and recall that $2b <$Dl$(S) < 3b$. Every element in the family is a Kato surface because it contains b rational curves which in fact have the same configuration of the irreducible components of D in the original surface S. In particular the index of S is invariant under logarithmic deformations. Following [19] we identify $\mathrm{Pic}^0(S)$ with $\mathbb{C}^*$ by fixing a particular line bundle L and writing every element as L^ν for some $\nu \in \mathbb{C}^*$ with the property that $\nu = 1$ corresponds to the trivial line bundle $\mathscr{O}$. Suppose S has index 1, by [19] if u is the parameter of the logarithmic deformation of S there is a unique $\nu = \nu(u)$ such that $H^0(S_u, K^{-1} \otimes L^\nu \neq 0$. Conversely, if S is of intermediate type, given any $\nu \in \mathbb{C}^*$, there is a logarithmic deformation S_u such that $H^0(S_u, K^{-1} \otimes L^\nu) \neq 0$.

The geometric idea for existence of bi-Hermitian metrics on S_u is the following, see also [25] for more details and a different approach. Start with a hyperbolic Inoue surface S_0, its logarithmic deformation is trivial and therefore in any neighborhood of its deformation as a Kato surface—i.e. preserving the rational curves, but not their configuration—there are Kato surfaces with arbitrary configuration of rational curves, in particular Kato surfaces of index 1. By the result of Brunella, S_0 has l.c.K. metrics with Lee bundle L^ν for every real value of ν such that $\nu > M_0$, for some big constant M_0. By the stability of l.c.K. metrics on Kato surfaces, see [4, 14, 25, 29], we can therefore find a Kato surface S_u of index 1 and $H^0(S_u, K^{-1} \otimes L^\nu \neq 0)$ which has l.c.K. metrics with Lee bundle L^ν and apply Theorem 3.2 to produce bi-Hermitian metrics with $\mathbf{T}$ supported on the maximal curve of S_u, which is always connected. □

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Kähler-Einstein Metrics on ℚ-Smoothable Fano Varieties, Their Moduli and Some Applications

Cristiano Spotti

Abstract We survey recent results on the existence of Kähler-Einstein metrics on certain smoothable Fano varieties, focusing on the importance of such metrics in the construction of compact algebraic moduli spaces of K-polystable Fano varieties. Moreover, we give some applications and we discuss some natural problems which deserve future investigations.

1 Introduction

Let X be a smooth Fano manifold, i.e., a compact n-dimensional complex manifold with positive first Chern class or, equivalently, with ample anticanonical bundle K_X^{-1}.

In this survey we discuss the *moduli problem* of this important class of complex varieties, showing its deep connections with the theory of *canonical metrics* on complex manifolds. More precisely, we focus on the so-called *Kähler-Einstein* (KE) metrics. This correspondence can be thought as an higher dimensional generalization of the relations between the theory of compact complex curves and their natural algebraic degenerations to nodal curves (Deligne-Mumford moduli compactification), and the theory of metrics with constant negative Gauss curvature and formation of hyperbolic cusps. However, crucially, in our higher dimensional KE Fano situation the value of the constant scalar curvature is *positive*, fact that imposes, as we will see, important constraints on the possible degenerations of such spaces.

Beside a pioneering work of Mabuchi and Mukai in a special complex two dimensional case [50] (based on fundamental works on geometric limits of Kähler-Einstein manifolds in real dimension four by, among others, Anderson [3] and Tian [68]), the precise picture on "geometric compactified" moduli spaces for Fano manifolds remained unclear. In particular, it is important to note that the "space" of all Fano manifolds is non-Hausdorff and Fano varieties may have continuous

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families of automorphisms. Thus some care has to be considered in studying such moduli problem.

However, the recent advances on the equivalence between existence of KE metrics and the purely algebro-geometric notion of K-stability on Fano manifolds [16], combined with the results [24] by Donaldson and Sun on geometric limits of non-collapsing KE metrics (based on the so-called Cheeger-Colding-Tian theory of limit spaces), made possible to study in more detail *moduli spaces of KE (or K-stable) Fano manifolds and their degenerations* showing that, if we restrict our attention to such special Fano varieties, their moduli theory becomes much well behaved. In particular, we now have a complete explicit picture in complex dimension two [57], and more general abstract results [48, 49, 56, 62] in higher dimension regarding existence of weak KE metrics on singular Fanos and on the structure of the (compactified) moduli spaces.

Finally, for completeness, we should mention here in the introduction that the relation between canonical metrics and compact moduli spaces of varieties is also fundamental in the higher dimensional case of varieties with negative first Chern class. Contrary to the Fano case, we have that *all* such smooth manifold admit KE metrics with negative "cosmological constant" by the works of Aubin [6] and Yau [72]. If we consider singular varieties, K-stability is equivalent to "KSBA stability" [52, 53] (a condition on the singularities of canonical polarized varieties, generalizing Deligne-Mumford stability for curves, used to construct compact separated moduli spaces, e.g., [1, 40, 41]). Moreover, it has been proved in [9] that certain singular KE metrics [9] always exist precisely on this type of singular varieties. We will briefly explain these relations in more detail at the end of Sect. 4.

In conclusion, we can say that, at least for (anti)canonical polarized varieties, K-stability, with its relation with KE metrics, provides a *unified* way to construct nice (compact) moduli spaces of algebraic varieties, and thus KE/K-moduli spaces are important objects to be further studied in the near future.

2 An Overview of Fano KE/K-Moduli Problem

Let X be an n-dimensional smooth Fano manifold and let $\chi(K_X^{-k})$ be the Euler characteristic of power of the anticanonical line bundle K_X^{-1}. By Kodaira's vanishing, such Euler characteristic is equal to $h^0(K_X^{-k})$ and, moreover, it coincides with the Hilbert polynomial associated to the anticanonical polarization.

Thus, for a given polynomial h, we can define the following *moduli set*:

$$M_h := \{X^n \text{ Fano mfd with } \chi(K_X^{-k}) = h(k)\}/\text{bi-holo}.$$

Being a "parameter space" for certain algebraic manifolds, we would like this set to admit a *natural* algebraic/complex analytic structure of complex variety: i.e., if $\pi : \mathscr{X} \to B$ is a flat family where $\pi^{-1}(b) = X_b$ is a Fano manifold, the natural map $B \to M_h$ should be *holomorphic* with respect to the analytic structure on M_h.

However, for dimension $n \geq 3$ such structure cannot exist for trivial reasons, known as "jumps of complex structures": there exist flat families of smooth Fanos $\pi : \mathscr{X} \to \Delta$ over the complex disc, such that $X_t \cong X_s$ for any $t, s \neq 0$, but $X_0 \ncong X_t$, for $t \neq 0$. Thus $[X_0] \in \overline{[X_t]}$ (here the square bracket denotes the isomorphism class). Hence, $[X_t] \in M_h$ would be a non-closed point, condition which is incompatible with the existence of a natural structure of complex analytic variety on M_h (in particularly inducing a Hausdorff topology). A well-known concrete example of this phenomenon is given by deformations of Mukai-Umemura Fano threefold (see [70], where the relations with KE metrics is discussed).

Thus the only hope to find a moduli space of Fanos which indeed admits a nice classical analytic structure is to *restrict the class of Fanos to consider*. Of course this non-Hausdorff issue is typical in many moduli problems (e.g., moduli of vector bundles). An answer for solving this problem is usually found in restricting the attention to "stable" vector bundles or, thanks to the Hitchin-Kobayashi correspondence, to bundles which admit Hermitian-Einstein metrics.

This suggests that also *in the case of varieties* we should look to certain "stable" varieties or, somehow equivalently, varieties which carry special Riemannian metrics. However, understanding the "right" stability condition to consider in the case of varieties turned out to be a highly non-trivial task, and many scholars worked in the last 30 years to better understand the relations between special metrics and algebraic stabilities, guided by the so-called *Yau-Tian-Donaldson conjecture* (YTD for short): given a polarized complex manifold (X, L), the existence of a Kähler metric with constant scalar curvature (cscK) in $2\pi c_1(L)$ should be equivalent to certain purely algebraic notion of stability of (X, L) (for a gentle introduction on this topic, focused on a moduli perspective, one can read [67]). In particular, not all polarized manifolds carry canonical metrics, contrary to the Calabi-Yau or negative first Chern class case. Classical obstructions to the existence of such metrics are given by the reductivity of the automorphism group [51] and the vanishing of the so-called Futaki invariant [31].

We are not going to describe the huge literature in the subject here, but instead we focus on our Fano case of anti-canonical polarized manifolds, where the natural differential geometric notion for a canonical metric in $2\pi c_1(K_X^{-1})$ reduces to the so-called *Kähler-Einstein* (KE) condition. Recall that a KE metric on a Fano manifold is a Kähler metric $\omega \in 2\pi c_1(K_X^{-1})$ which satisfies the Einstein geometric PDE, necessarily with "positive cosmological constant" (here normalized to 1):

$$Ric(\omega) \left(= i\bar{\partial}\partial \log(\omega^n)\right) = \omega.$$

Thanks to the Kähler condition, such Einstein equation, in general obstructed, reduces to a complex Monge-Ampère equation on a potential function and thus it can be studied using techniques coming from pluri-potential theory.

In [70] Tian introduced the notion of *K-stability*, extending the notion of Futaki invariant, stability condition later further generalized and made completely algebraic by Donaldson [21]. K-(poly)stability (be aware that sometimes people call K-stable something that for us is K-polystable!) is a "geometric invariant theory (GIT)-like" notion of stability for varieties in which one promotes an abstract version of the Hilbert-Mumford criterion as a definition for stability.

A *test-configuration* for X (the analogous to a one-parameter subgroup in standard GIT) is the datum of a $\mathbb{C}^*$-equivariant relative polarized normal flat family of schemes over $\mathbb{C}$:

$$\mathbb{C}^* \curvearrowright ((\mathscr{X}, \mathscr{L}) \to \mathbb{C}),$$

such that over 1 (hence away from zero), we have an isomorphism $(X_1, \mathscr{L}_{|X_1}) \cong (X, K_X^{-r})$. Clearly $\mathbb{C}^*$ acts naturally on the d_k-dimensional vector space $H^0(X_0, \mathscr{L}^k_{|X_0})$ with weight w_k on its top exterior power. Note that X_0 can be highly singular.

The *Donaldson-Futaki invariant* for the test configuration $(\mathscr{X}, \mathscr{L})$ for X (the analogous of weight in GIT) is the k^{-1} coefficient in the Riemann-Roch expansion:

$$\frac{w_k}{kd_k} = C + DF(X, (\mathscr{X}, \mathscr{L}))k^{-1} + \mathscr{O}(k^{-2}).$$

X is called *K-stable* (rep. semistable) if and only if $DF(X, (\mathscr{X}, \mathscr{L})) > 0$ (resp. ≥ 0) for all test-configurations, and *K-polystable* if and only if it is K-semistable and $DF = 0$ iff $\mathscr{X} \cong X \times \mathbb{C}$.

Note that to define K-stability we haven't taken a specific embedding of X in some *fixed* projective space and considered only embedded test configurations: the definition require a-priori to test stability for all possible equivariant degenerations inside any $\mathbb{P}^N$ where X embeds, letting $N \to \infty$. For this reason testing K-stability from the actual definition is very challenging, even if we can reduce to the so-called *special test configurations* [47]. However, some criteria related to the so-called log-canonical-threshold, or to the very recent notion of Ding stability are available (e.g.,[30]). More abstractly, K-stability may be thought as a GIT like notion on the stack of Fano varieties where the DF-invariant is actually realized as the weight of a stacky line bundle, called CM line bundle [58]. As we will see, this point of view is quite important for the moduli discussion.

We are now ready to state the fundamental theorem relating KE metrics with K-stability.

Theorem 2.1 (YTD Conjecture for Fano Manifolds) *Let X be an n-dimensional smooth Fano manifold. Then*

$$\textit{there exists a KE metric in } 2\pi c_1(K_X^{-1}) \iff X \textit{ is K-polystable.}$$

The direction "⇒" has been proved in various degrees of generality by Tian [70], Donaldson [22], Stoppa [64] and finally by Berman [8]. The other direction is the content of the recent breakthrough of Chen, Donaldson and Sun [16]. The proof uses a combination of analytic, geometric and algebraic techniques, in particular related to the notion of Gromov-Hausdorff (GH) convergence (notion that, as we will see, is deeply relevant also for the moduli problem). More recently different proofs have been found: via Aubin's continuity path [18], via Kähler-Ricci flow [17] or, for the case of finite automorphisms groups, via calculus of variation techniques [10].

But now let us go back to the moduli discussion. We can define the "differential geometric" KE moduli space of equivalence classes, up to *biholomorphic isometries*, of KE Fano manifolds (with fixed Hilbert polynomial h):

$$\mathscr{E}M_h := \{(X, \omega) \,|\, \omega \, \text{KE}\}/\sim .$$

Similarly, we can consider the subset of M_h defined by the algebro geometric condition of K-polystability:

$$\mathscr{K}M_h := \{[X] \,|\, X \, \text{K-ps}\} \subsetneq M_h.$$

Thus we have the following *Hitchin-Kobayashi map for varieties*:

$$\phi_h : \mathscr{E}M_h \longrightarrow \mathscr{K}M_h,$$

naturally given by forgetting the metric structure, i.e., $\phi_h([(X, \omega)]) := [X]$. This map is:

- well-defined, by "$\Rightarrow$" in Theorem 2.1.
- surjective, by "$\Leftarrow$" in Theorem 2.1.
- injective, by Bando-Mabuchi uniqueness [7].

Thanks to the canonical metric structure induced by the KE metric, we can now put a natural topology on the differential geometric moduli space $\mathscr{E}M_h$. Such topology is essentially induced by the *Gromov-Hausdorff (GH) distance* between compact metric spaces: given two compact metric spaces, say (S, d_S) and (T, d_T), one defines

$$d_{GH}(S, T) := \inf_{S,T \hookrightarrow U} \inf\{C > 0 \,|\, S \subseteq N_C(T) \;\&\; T \subseteq N_C(S)\},$$

where $N_C(S)$ denotes the distance C neighborhood of S isometrically embedded in a metric space U. The above defines a metric structure, in particular a Hausdorff topology, on the space of isomorphisms classes of compact metric spaces. In practice, one usually estimates the GH distance (which is sufficient for studying convergence) via maps $f : S \to T$ which are ϵ-dense and ϵ-isometries. See [12] for an introduction to such notion of convergence.

One of the immediate advantage of the GH topology is that, by its very definition, it gives a possible precise way to study *degenerations* of Riemannian manifolds to *singular* spaces.

Moreover, in our KE Fano case, we have to following remarkable precompactness property: any sequence of complex n-dimensional KE Fano manifolds (X_i, ω_i) subconverges in the GH sense to a compact length metric space S_∞ of real Hausdorff dimension equal to $2n$. This follows by Gromov's theorem on convergence of Riemannian manifolds with Ricci uniformly bounded below and diameter bounded above (condition that in our case is implied by the positivity of the Ricci tensor, thanks to Myers'theorem) and by the volume non-collapsing condition, i.e., the volume of balls of radius r is uniformly bounded below by Cr^{2n}.

The metric space limit S_∞ can be considered as a very weak limit. However, since we are considering limits of spaces which admit many additional structures (a Riemannian metric and a complex structure) it is natural to expect that in a suitable sense such structures are preserved in the limit. The first important result is the "Riemannian regularity" provided by Cheeger-Colding theory [14] which shows that S_∞ is actually an incomplete smooth Einstein space off a set of Hausdorff codimension 4, and also gives some geometric stratification of the singular set based on the local behavior of the metric structure (via splittings of metric tangent cones). The second regularity result is the recent theorem of Donaldson and Sun [24] which, in addition, shows that S_∞ admits a natural structure of *normal algebraic Fano variety*. Such structure is constructed by realizing the GH convergence as convergence of algebraic cycles in a sufficiently big projective space, via *uniform* Tian's L^2-orthonormal embeddings by plurianticanonical sections. Thus S_∞ is homeomorphic to a Fano limit cycle X_0. Note that the above also gives a refinement of the GH topology, which now "remembers" the complex structure (otherwise there is some ambiguity related to complex conjugations [61]). More technically (see Sect. 4 for details), X_0 turned out to be a $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano variety, i.e., a *normal* variety with $\mathbb{Q}$-Cartier anticanonical divisor and Kawamata-log-terminal (klt) singularities admitting nice smoothings and a weak KE metric.

It is important to remember that in complex dimension two the above convergence results were already known by the works mentioned in the introduction of Anderson, Tian and others. In this situation, GH limits must have isolated orbifold singularities (which is precisely the klt condition in dimension two), i.e., quotients of $\mathbb{C}^2$ by finite subgroups of $U(2)$ acting freely on the 3-sphere. Moreover, the KE metric is orbifold smooth, i.e., it extends to a smooth metric on the local orbifold covers.

Since, by the result of Berman [8], it is known that the direction KE implies K-polystability holds also for singular varieties, one can naturally define the *extended Hitchin-Kobayashi map*

$$\tilde{\phi}_h : \overline{\mathscr{E}M}_h^{GH} \longrightarrow \overline{\mathscr{K}M}_h,$$

where $\overline{\mathscr{E}M}_h^{GH}$ is a compact Hausdorff topological space with respect to the refined GH topology obtained by adding all GH limits, and $\overline{\mathscr{K}M}_h$ denotes the set of $\mathbb{Q}$-Gorenstein smoothable K-polystable $\mathbb{Q}$-Fano ($\mathbb{Q}$-smoothable for short) varieties up to isomorphism.

With all of this in mind, it is natural to ask the following foundational questions:

1. *YTD for $\mathbb{Q}$-smoothable Fanos*: does any $X \in \overline{\mathscr{K}M}_h$ admit a weak KE metric? (i.e., is $\tilde{\phi}_h$ surjective?)
2. *Existence of K-moduli*: does $\overline{\mathscr{K}M}_h$ admit a natural algebraic structure such that $\tilde{\phi}_h$ is an homeomorphism with respect to the GH topology and the euclidean topology of the algebraic space?

3. Can we find *explicit examples* of such compactifications?

We start by discussing the dimension two case, i.e., the case of *del Pezzo surfaces*. In this situation the answer to all such questions is complete. Note that in dimension one, if one does not consider weighted/cone angle case, the moduli problem for Fanos is clearly trivial, being $\mathbb{P}^1$ with the Fubini-Study metric the only such space.

3 KE/K-Moduli of del Pezzo Surfaces

In complex dimension two Fano manifolds are traditionally called *del Pezzo surfaces*. Such varieties are also completely classified: they are given by $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-up of the plane in up to eight points in "very general" position. Let us denote with $d = c_1^2(X) \leq 9$ their *degree* (which also uniquely determines the Hilbert polynomial, as a consequence of Riemann-Roch arguments). The problem of understanding which *smooth* del Pezzo surface admits a KE metric was addressed in the seminal paper of Tian [68]. The answer is: they all admit such metrics, beside the well-known obstructed cases of the blow-up of the plane in one or two points.

Since we are interested in moduli problem, we restrict to the case $d \leq 4$, i.e., to the case when there are non-trivial complex deformations. Together with recovering Tian's theorem in the smooth case and, somehow, providing a conceptual explanation why all smooth del Pezzo of degree $d \leq 4$ admit KE metrics, the following theorem computes via explicit algebro-geometric techniques the GH compactifications of such KE moduli spaces, classifying the geometric limits (which are KE del Pezzo orbifolds, by the result recalled in the previous section).

Theorem 3.1 ([57]) *For any positive integer degree $d \leq 4$, there exists an* explicit *compact algebraic space $\overline{M}_d^{ALG}$ (moduli space of certain degree d del Pezzo orbifolds) such that the Hitchin-Kobayashi map*

$$\tilde{\phi}_d : \overline{\mathscr{E}M}_d^{GH} \longrightarrow \overline{M}_d^{ALG} \left(\cong \overline{\mathscr{K}M}_d \right)$$

is a homeomorphism*, and $\mathscr{E}M_d$ is identified with a Zariski dense subset of $\overline{M}_d^{ALG}$.*

As we said in the introduction, the degree $d = 4$ case was previously understood by Mabuchi and Mukai [50].

An important differential geometric application, generalizing Tian's results in the smooth setting, is the following corollary, answering a conjecture of Cheltsov and Kosta [15]. Since KE del Pezzo orbifolds with orbifold groups contained in $SU(2)$ (i.e., with *canonical singularities*) are classified and they always admit $\mathbb{Q}$-Gorenstein smoothings, from the above explicit KE/K-moduli compactification we have:

Corollary 3.2 ([57]) *KE del Pezzo orbifolds with orbifold groups at the singularities contained in $SU(2)$ are classified.*

For example, KE del Pezzo orbifolds of degree three with such singularities are precisely given by *all* cubic surfaces in $\mathbb{P}^3$ with only nodal (i.e., A_1) singularities plus the toric cubic $\{xyz = t^3\} \cong \mathbb{P}^2/\mathbb{Z}_3$, since in this case the GH compactification agrees with the classical GIT quotient of cubic surfaces. Partial results where previously known (e.g., [15, 20, 60]).

We now indicate the main passages in the proof of the above Theorem 3.1 for the interesting $d = 2$ case.

Let X_∞ be the GH limit of smooth degree 2 del Pezzo surfaces (i.e., GH limit of KE double covers of $\mathbb{P}^2$ branched at a smooth quartics).

- Step 1: we first improve our understanding of the singularity of X_∞, combining Bishop-Gromov monotonicity formula (which shows that the order of the orbifold group at the singularity can be at most 6) with the Kollár and Shepherd-Barron classification of two dimensional $\mathbb{Q}$-Gorenstein smoothable quotient singularities [42].
- Step 2: using classification results of singular del Pezzo surface, we show that X_∞ has to be given by the following hypersurfaces (of degree four and eight, respectively) in weighted projective spaces:

  1. $X_\infty \cong \{f_4 = t^2\} \subseteq \mathbb{P}(1, 1, 1, 2)$;
  2. $X_\infty \cong \{f_8 = z^2 + t^2\} \subseteq \mathbb{P}(1, 1, 4, 4)$;

- Step 3: in both two cases there is a natural action of two groups on the parameter spaces: more precisely, in the first case $SL(3, \mathbb{C})$ acts on the space of quartics $\mathbb{P}(Sym^4(\mathbb{C}^3))$ and in the second case $SL(2, \mathbb{C})$ acts on $\mathbb{P}(Sym^8(\mathbb{C}^2))$. This gives two GIT quotients, with natural linearizations.
- Step 4: since X_∞ is K-polystable by Berman [8], a comparison of stabilities using the CM line bundle, shows that X_∞ has to be also stable with respect to the above classical notions of GIT stability. Next we can blow-up the first quotient semistable stack at the point corresponding to the double conic to get a space mapping to a categorical quotient $\overline{M}_2^{ALG}$: i.e., we can define

$$\mathscr{M}_2 := [\mathbb{P}(Sym^4(\mathbb{C}^3))^{ss}/SL(3, \mathbb{C})] \cup_{\{[q^2=t^2]\}} [\mathbb{P}(Sym^8(\mathbb{C}^2))^{ss}/SL(2, \mathbb{C})] \to \overline{M}_2^{ALG}.$$

- Step 5: since there exists at least one smooth degree two KE del Pezzo surface [71], we can define a natural continuous map (the Hitchin-Kobayashi map) from $\overline{\mathscr{E}M}_2^{GH}$ to $\overline{M}_2^{ALG}$. Finally a standard open-closed topological argument, combined with the fact that $\overline{M}_2^{ALG}$ is an Hausdorff space of del Pezzo orbifolds, implies the statement.

Note that in general there are non-canonical singularities in the limits, e.g., the limit toric variety $X_\infty = \{x^4y^4 = z^2 + t^2\} \cong (\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_4$ has in particular two singularities of type $\frac{1}{4}(1, 1)$. This is related to the existence of *torsion* Calabi-Yau ALE metric bubbles from limits of Einstein spaces [65]: loosing speaking, ALE bubbles are spaces that metrically model the formation of singularities in this non-collapsing setting, and thus they are somehow the equivalent of hyperbolic collars in the (locally collapsing) curve case near the formation of a node.

Let us also note that, since an Einstein deformation of a smooth KE del Pezzo surface has to be KE [44], such moduli spaces, quotienting with respect to the natural involution given by conjugating the complex structure, *are explicit compactification of connected component of (real) Einstein moduli spaces* on the real oriented manifolds $\mathbb{CP}^2 \sharp k \overline{\mathbb{CP}^2}$ with $5 \leq k \leq 8$. See, for example, the general discussion on Einstein moduli by Koiso [39]. The topological types of our spaces could be easily understood.

In degree one the construction of the algebraic compactification is more involved, since bi-meromorphic contractions are used (which also cause the a-priori loss of projectivity for the moduli space). However, in all cases the algebraic compactifications obtained show that such moduli spaces admit "more structure": namely, they are *KE moduli Artin stack* (essentially étale covered by affine GIT quotients parametrizing del Pezzo orbifold deformations, see definitions 3.13–14 in [57]). This is related to the Alper's notion of *Good Moduli Spaces* for an Artin stack [2].

Finally, the fact that our explicit moduli agrees with the K-compactification (that is, all $\mathbb{Q}$-smoothable K-polystable del Pezzo surfaces appear in their boundary) is a consequence of more general results which we are going to discuss in the next section.

4 KE Metrics on $\mathbb{Q}$-Smoothable Fano Varieties

By what we explained in Sect. 2, in order to study the boundary of the moduli problem is natural to consider $\mathbb{Q}$-Gorenstein smoothings of a K-polystable $\mathbb{Q}$-Fano ($\mathbb{Q}$-smoothings for short) varieties, i.e., flat families $\mathscr{X} \to \Delta$ over the complex disc where:

- X_0 is a normal (K-polystable) Fano variety with $\mathbb{Q}$-Cartier canonical divisor (i.e., some power is a line bundle) satisfying $K_{\hat{X}_0/X_0} =_{\mathbb{Q}} \sum_i a_i E_i$, with $a_i > -1$, for any log-resolution $\hat{X}_0$. Equivalently, from a more differential geometric view-point, for any $p \in X_0$, $\int_{U \cap X_0^{reg}} s^{\frac{1}{m}} \wedge \bar{s}^{\frac{1}{m}} < \infty$, where s is a local trivialization over a small neighborhood U of the m-th power of the canonical bundle $K_{X_0}^m$ near p.
- $K_{\mathscr{X}/\Delta}$ is $\mathbb{Q}$-Cartier.
- X_t is smooth.

We remark that *not* $\mathbb{Q}$-*Gorenstein* smoothings of $\mathbb{Q}$-Fano varieties exist, but such deformations are not relevant for KE/K-moduli problems. We can now state the main theorem of this section, which in particular give an answer to question one in Sect. 2.

Theorem 4.1 ([62]) *Let $\mathscr{X} \to \Delta$ be a $\mathbb{Q}$-smoothing of a K-polystable variety X_0. Then X_0 admits a weak KE metric ω_0. Moreover, if Aut(X_0) is finite, X_t admit smooth KE metrics ω_t for t sufficiently small, and $(X_t, \omega_t) \to (X_0, \omega_0)$ in the GH topology.*

Thanks to the above correspondence between metric limits and flat families, as a corollary we have the following algebraic separatedness statement.

Corollary 4.2 ([62]) *If two $\mathbb{Q}$-smoothings of K-stable $\mathbb{Q}$-Fanos (with finite automorphisms) agree away from the singular fiber, then the central (singular) fibers are isomorphic.*

Before giving a quick survey of the main ideas in the proof, it is useful to discuss some properties of weak KE metrics, which are the natural pluri-potential theoretic generalization on singular varieties of smooth KE metrics ([26], or [19] for a recent survey).

Near any point $p \in X_0$ such weak KE metrics are given by the restriction of the $i\partial\bar{\partial}$ of a *continuous* potential for an embedding in $\mathbb{C}^N$ of the analytic germ of the singularity. As two dimensional orbifold singularities show, such regularity for the potential is essentially optimal. Regarding more geometric considerations, we have that the regular part X_0^{reg} is a *smooth* incomplete KE space, and its metric completion $\overline{X_0^{reg}}$ topologically agrees with X_0.

The actual "asymptotic behavior" at the singularities of these weak KE metrics is quite delicate, and more complicated with respect to the two dimensional orbifold case. Recently we have seen important results which put some light on the aspect of the metric near the singular locus, at least when we consider singular KE spaces arising as limits of smooth ones (note that orbifold singularities appear only exactly in complex codimension two, by the famous Schlessinger's rigidity of quotient singularities). From a metric measure theoretic perspective, it is known that the metric "looks the same" at all sufficiently small scales near a singularity (uniqueness of metric Calabi-Yau tangent cone [25]). But, as first observed by Hein and Naber [35], phenomena of local jumping of complex structures can happen when "zooming" to find such metric tangent cone. For example, it is expected that in complex dimension 3, metric tangent cones at the isolated singularities of type A_k (i.e., locally analytical of type $x_1^2 + x_2^2 + x_3^2 + x_4^{k+1} = 0$) for $k \geq 3$ should all be isometric to the flat cone $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ (singular along a line). See [23] for a discussion.

Roughly speaking (but the situation is slightly more subtle in reality), these jumping phenomena have their origins in the fact that typical complex links of klt singularities are Fano varieties, but the existence of a Calabi-Yau cone metric model implies that such links have to be KE! So there is indication that some notion of *stability for singularities* is required (related to Sasaki-Einstein stability, in the simplest cases). Some recent works, such as [45, 46], are trying to understand this picture from an algebraic perspective. However, in certain situations (e.g., for the A_1 case, where a CY cone metric can be found via Calabi's ansatz [13] and where the corresponding smoothing bubble was explicitly found by Stenzel [63]) it is expected, and very recently proved in the CY case by Hein and Sun [36], that the weak KE metrics are *polynomially asymptotic* to the CY cone models in a suitable local holomorphic gauge.

Let us now briefly describe the strategy in the prove of Theorem 4.1. The very rough idea consists in constructing the weak KE metric on the singular fiber via a

GH limit of certain conically singular KE metrics on the smooth nearby fibers, thus running in *families* the so-called "Donaldson's cone angle path", i.e., the continuity path used by [16] for proving the YTD conjecture in the Fano case.

- Step 1: by a Bertini's type argument we can take a divisor in $\mathscr{D} \in |-\lambda K_{\mathscr{X}/\Delta}|$, for λ big enough, which gives a smooth pair (X_t, D_t) when restricted at $t \neq 0$ and a klt pair $(X_0, (1-\beta)D_0)$ on the singular fiber for $\beta \leq 1$. Thus we want to consider the following *two parameters family* of PDEs:

$$Ric(\omega_t + i\partial\bar{\partial}\phi_{t,\beta}) = (1-(1-\beta)\lambda)(\omega_t + i\partial\bar{\partial}\phi_{t,\beta}) + 2\pi(1-\beta)\delta_{D_t},$$

  where ω_t is the restriction of a Fubini-Study metric from an embedding of the family and δ_{D_t} the current of integration along D_t. At least when $t \neq 0$ a solution of this equation is a KE metric with cone angle equal to β along D_t (e.g. [38] or [33]).
- Step 2: via a log-canonical-threshold argument one shows that the above equation has a positive KE solution for all t, if β is sufficiently small.
- Step 3: using some pluri-potential techniques (e.g., Berndsson's positivity of direct images) one can find, for fixed β, an a-priori bound of type $||\varphi_{t,\beta}||_{L^\infty} \leq C(\beta)$, for $t \neq 0$. Taking the universal embedding in $\mathbb{P}^N$ provided by the conical generalization of Donaldson and Sun convergence theorem [16] III, one sees that the conical metrics on the smooth fibers GH converge to the weak conical KE metric on the central fiber.
- Step 4: the above convergence is used to prove that the function

$$\beta_t := \sup\{\beta \in (1-\lambda^{-1}, 1] \,|\, \exists\, \omega_{t,\beta} \text{ KE on } X_t\},$$

  is a lower semi-continuous function in the euclidean topology of the disc. This is connected with some properties of the automorphism groups.
- Step 5: the above semi-continuity result, combined with a gap argument for some natural energy functional (Aubin's energy), gives that the set of cone angles βs for which a weak conical KE metric exists on X_0 is open. The closeness follows again by taking limits from the smooth nearby fibers.

Thus the KE metric on X_0 is constructed as a kind of "diagonal" GH limit of cone angle KE metrics $\omega_{t,\beta(t)}$ with $\beta(t) \to 1$ as $t \to 0$. In particular, it is a weak KE metric thanks to the regularity theory for GH limits.

4.1 Algebraic Structure on Fano KE/K-Moduli

Theorem 4.1 above shows that the YTD conjecture also holds in the case of $\mathbb{Q}$-smoothable Fano varieties and it provides a natural correspondence between flat limits and GH convergence, at least in the case of finite automorphisms groups. The next step is then related to the construction of a natural algebraic structure on the differential geometric KE moduli space or, equivalently, showing that our question two asked at the end of the second section admits a positive answer.

The rough idea for constructing such algebraic moduli space of $\mathbb{Q}$-smoothable K-polystable Fano varieties is the following. One knows that, being KE, such varieties have linear reductive automorphism groups. Thus one can consider a Luna's slice type argument applied to the Hilbert scheme in the uniform $\mathbb{P}^N$ embedding where all GH limits of smooth KE spaces (say with fixed Hilbert polynomial h) live. Here one shows that, étale locally, K-polystability is completely captured by some "local GIT" stability on a small enough (affine) slice. The actual argument is similar to Steps 4–5 in the construction of explicit moduli space of del Pezzo surfaces. Thus this expected "local GIT picture" (e.g. [61]), generalizing the one in the smooth case obtained by Broennle [11] and Székelyhidi [66], provides the natural, compatible with the GH topology, *algebraic atlas for a compact moduli space*. The above has been fully proved by Li et al. [48] and, independently, by Odaka [56] (using Theorem 4.1 recalled above and some preliminary propositions in the first version of [48]), generalizing [54]. In conclusion we have:

Theorem 4.3 ([48, 56]) *In any dimension, $\overline{\mathscr{K}M_h}$ admits a natural algebraic structure (given étale locally by affine GIT quotients) such that the map $\tilde{\phi}_h$ is a homeomorphism.*

As in the del Pezzo case, this moduli space carries more structure: in particular, $\overline{\mathscr{K}M_h}$ is a categorical quotient of a KE/K-moduli stacks as we have previously discussed. Moreover, further analyzing the structure of such moduli spaces (in particular studying openness of K-semistability), the authors in [48] showed that $\overline{\mathscr{K}M_h}$ is "dominated" by a good Artin moduli stack $\overline{\mathscr{K}\mathscr{M}_h}$ of *K-semistable* $\mathbb{Q}$-smoothable Fano varieties, with a unique K-polystable point in the K-semistable equivalence classes. However, as we have seen, we stress that at present the existence/construction of such algebraic moduli spaces of $\mathbb{Q}$-smoothable K-polystable Fano varieties depends crucially on transcendental complex analytic techniques related to KE metrics. Some discussion on the potential dependence on N (the dimension of the projective space were all GH limits live) of the algebraic structure on the compactified moduli space can be found in the original papers. Moreover, it will be very important to find a purely algebraic way to form such moduli spaces and, furthermore, to remove the smoothability hypothesis used in the present construction. We expect new birational geometric techniques to be relevant for this progress.

Finally, we mention that such KE/K compact moduli spaces of Fano varieties are the analogous of the KSBA compactification of moduli spaces of manifolds with negative first Chern class [41], and thus a special instance of the more general theme: relations between special Kähler metrics and moduli of polarized varieties. In [52, 53] Odaka showed that K-stability is equivalent to the KSBA conditions on the singularities (semi-log-canonical) of a variety (satisfying the conditions G1 and S2) with ample canonical divisor required to form compact moduli spaces. Moreover, Berman and Guenancia showed in [9] that precisely on varieties with such type of singularities is possible to construct weak KE metrics of negative scalar curvature.

Here the metric can be *complete* near the non-klt locus, and locally collapsing (this is precisely the higher dimensional analogous of the hyperbolic cusps in the complement of a node in a DM stable curve). Thus, even if indirectly, one recovers the equivalence between K-stability and (negative) KE metric. However, the complete metric convergence picture is not fully understood, due to these collapsing phenomena. It is known that if X_0, the central fiber of a smoothing, has only simple normal crossing singularities, then the KE metrics in the nearby fiber naturally converge to complete KE metrics on the irreducible components of X_0 (e.g., [59, 69]). Some properties of (special) collapsing regions has been recently studied by Zhang in [73] inspired by the SYZ picture in collapsings of Calabi-Yau manifolds. Related to the last point, we should mention that canonical metrics should be relevant also (at least) in the study of compactified moduli space of polarized Calabi-Yau manifolds: the non-collapsing case is well understood (e.g., [74]), however the full GH collapsing to lower dimensional spaces (e.g., [32]) remains quite mysterious (but see conjectures of Gross and Wilson, and Kontsevich and Soibelman, e.g., [43], where collapsing to certain spaces of real dimension at most equal to half of the original dimension is expected), and possibly related to certain moduli of tropical varieties [55]. For relations with algebraic geometry, fixing the polarization in the study of degenerations of Calabi-Yau manifolds is going to be essential, as the purely trascendental collapsing of K3 surfaces to real three dimensional spaces in [28] suggests.

5 Some Applications and Future Perspectives

In this last section we describe some possible applications of the previously discussed results and, moreover, we will mention some natural problems to be considered in the near future.

The first application, more differential geometric in nature, consists in using singular KE metrics to construct examples of *smooth Kähler metrics of constant scalar curvature* (cscK), a notorious difficult problem, via certain geometric transitions. Next we discuss some properties related to the study of the "geometry" of KE/K-moduli spaces or stacks. Finally, we briefly mention the problem of understanding *explicit* examples of KE/K-moduli spaces.

5.1 Generalized cscK Conifold Transitions

Through this section let $X_0 \hookrightarrow \mathscr{X} \to \Delta$ be a $\mathbb{Q}$-smoothing of a K-stable Fano variety (with discrete automorphism group). By Theorem 4.1, X_0 and its sufficiently small deformations X_t are KE, and moreover the family is continuous in the GH topology. Now let us take a *resolution* $\hat{X}_0$ of the singular variety X_0.

Question 5.1 Can we find a family g_ϵ of "canonical" Kähler metrics on a resolution $\hat{X}_0$ which also degenerate, as $\epsilon \to 0$, to the singular KE space X_0?

The natural notion of best metric to consider on the resolution $\hat{X}_0$ is given by the more general notion of cscK metrics. In a loose sense, we can think of KE metrics on the smoothings to be a family of metrics where the underlying symplectic structure is fixed while the complex structures changes and becomes degenerate. For the metrics on the resolution the relations is the opposite one: the complex structure is now fixed but the symplectic structures vary (the ϵ parameter being related to let the Kähler classes of the metrics approaching a special point in the boundary of the Kähler cone on $\hat{X}_0$). We call such paths of canonical metrics connecting in the GH sense smooth complex manifolds, in general not diffeomorphic, through a singular variety *generalized cscK conifold transitions*. Such terminology originates from a similar geometric situation considered in Physics for Calabi-Yau threefolds.

The construction of these geometric transitions is expected to be hard in general. However, in the case when X_0 has only isolated singularities of some special type, one can hope to show existence of cscK metrics on some resolutions via gluing techniques, similar to the strategy used in [4] in the case of (orbifold) smooth metrics. For example, if the singularities of X_0 are locally analytically modeled on the blow down of the zero section of the canonical bundle of a KE Fano manifold (in general not orbifold) and the KE metric on X_0 is asymptotic near the singularities to the conical CY cone metric given by the Calabi's ansatz [13], we can prove the following:

Theorem 5.2 ([5]) *Under the above hypothesis, X_0 has a natural crepant resolution $\hat{X}_0$ admitting a family of cscK metrics of positive scalar curvature converging to the KE metric on X_0 in the GH topology, and thus X_0 is the degenerate variety of a generalized cscK conifold transition.*

The above theorem is a special case of more general results in [5] (combined with Theorem 4.1), where X_0 is not assumed to be Fano (e.g., it could have a KE metric of zero or negative Einstein constant), nor smoothable, and the singularities belong to a bigger class. We crucially remark that having the needed asymptotic behavior of the weak KE metric near the singularities is in general a major problem. However, as we have previously recalled, for the smoothable Ricci-flat case (but modifications of the arguments should also work for KE metrics with different sign of the Einstein constant) Hein and Sun have recently shown [36] that the required asymptotic decay property of the weak KE metric for isolated singularities of the type considered in the above theorem.

5.2 Geometry of KE/K-Moduli Stacks

As we have explained in the previous sections, inside the non-separated, non-proper moduli stack $\mathscr{M}_h$ of Fano varieties, in general not of finite type even if we restrict the attention to $\mathbb{Q}$-smoothable Fanos, we can find a nice subspace of K-semistable

$\mathbb{Q}$-smoothable Fano varieties $\overline{\mathscr{K}\mathscr{M}}_h$ mapping canonically to a compact algebraic space $\overline{\mathscr{K}M}_h$, the coarse moduli variety of smoothable K-polystable or KE objects. Thus we ended up in a set-up of good moduli space for an Artin stack [2], here described by a kind of generalized GIT quotient space with respect to a more abstract stability notion (K-stability) given by the CM line bundle on the moduli stacks $\mathscr{M}_h$ (see [34] and [37] for more information on this view point, which we think will be relevant in the future). The natural next step in this moduli theory is to understand further the geometry of such spaces. For example, even if the CM line bundle λ_{CM} is in general not ample [27], it is expected -thanks to its relation with Weil-Petersson geometry [29]- that, once descended to the coarse algebraic space, it becomes a natural $\mathbb{Q}$-polarization for $\overline{\mathscr{K}M}_h$, which then will be a *projective variety* (the quasi-projectivity of the part parameterizing smooth KE/K-polystable Fanos has been recently shown in [49]).

Furthermore, it would be interesting to study properties of "subvarieties" of the moduli stacks (i.e., families of K-stable Fano varieties), canonical bundles or sheaves on such moduli spaces, and cohomological properties of them, similar to the ones studied for curves and varieties with ample canonical class.

As a toy example of such possible investigations, we will now compute the CM-volume of a simple curve (i.e., a family over a one dimensional space) in the KE/K-moduli space of degree 3 del Pezzos, i.e., cubic surfaces. We first need an easy lemma:

Lemma 5.3 *Let $\gamma : \mathscr{X} \to \mathscr{C}_g$ be a curve of degree d del Pezzo orbifolds with generically smooth fibers for which $K^{-1}_{\mathscr{X}/\mathscr{C}}$ makes sense. Then*

$$c_1(\lambda_{CM}(\mathscr{X} \to \mathscr{C}_g)) = 6d(1-g) - c_1^3(\mathscr{X}).$$

In fact, by the definition of the CM line bundle for the relative anticanonical polarization and by Grothendieck-Riemann-Roch, we have $c_1(\lambda_{CM}(\mathscr{C})) = -\gamma_*(c_1^3(K^{-1}_{\mathscr{X}/\mathscr{C}}))$. Hence, $c_1(\lambda_{CM}(\mathscr{C})) = -\gamma_*\left(c_1^3(K^{-1}_{\mathscr{X}}) + 3c_1^2(K^{-1}_{\mathscr{X}})c_1(\gamma^*K_{\mathscr{C}})\right)$ which is indeed equal to $-c_1^3(\mathscr{X}) - 6d(g-1)$, as claimed.

Thus, for example, if $d = 3$ and $\mathscr{C} = \mathbb{P}^1$, we have $c_1(\lambda_{CM}) = 18 - c_1^3(\mathscr{X})$. Moreover note that, by the positivity of the CM line bundle, if $\mathscr{X} \to \mathscr{C}_g$ is a "K-polystable curve" then $c_1^3(\mathscr{X}) \leq 6d(1-g)$. Similar Chern numbers inequalities can be founded in higher dimension too.

Now, if we take a *generic pencil* of cubic surfaces $tc_1 + sc_2 = 0$, by genericity we may assume that the generic member in the associated *Lefschetz's fibration* $\mathscr{X} \to \mathbb{P}^1$ is smooth and the singular fibers have only one nodal A_1-singularity. Thus we have the following "intersection number computation".

Proposition 5.4 *The degree of the CM line bundle on the base of a generic Lefschetz's fibration of (K-stable by Theorem 3.1) cubic surfaces is equal to*

$$c_1(\lambda_{CM}(\mathscr{X} \to \mathbb{P}^1)) = 8.$$

For this, thanks to the previous lemma, it is sufficient to compute $c_1^3(\mathscr{X})$, where $\mathscr{X} = Bl_{\Sigma_g}\mathbb{P}^3$ with $\Sigma_g = c_1 \cap c_2$ surface of genus $g = 10$, by adjunction. Since $-K_{\mathscr{X}} = 4H - E$, where H is the pull-back of the hyperplane bundle of $\mathbb{P}^3$ and E is the exceptional divisor ($\mathbb{P}^1$ bundle over Σ_g), we have that $c_1^3(\mathscr{X}) = 64H^3 - 48H^2E + 12HE^2 - E^3$. But $H^2E = 0$, $HE^2 = 9E.f = -9$ (where f is a fiber of the $\mathbb{P}^1$-bundle), and $-E^3 = N_{\Sigma_g} = -K_{\mathbb{P}^3}.C + 2g - 2 = 54$. Hence $c_1^3(\mathscr{X}) = 10$, which implies the result.

We expect similar computations to be relevant in the study of properties of the Picard group of K-moduli stacks $\overline{\mathscr{K}\mathscr{M}}_h$.

5.3 Examples of Fano KE/K-Moduli

Beside the complex dimensional two case, (where the KE/K-moduli picture is complete, at least for the components corresponding to compactifications of smooth surfaces), in higher dimension we are completely lacking of explicit examples. There are several reasons to look for such examples. For us the two more important ones are:

- they will provide a complete understanding of which Fano manifolds in a given family admit KE metrics.
- they may provide hints to study recurrent properties of K/KE-moduli spaces.

We expect that the techniques developed in the proofs of the theorems presented and discussed in this survey note (e.g., stability comparisons, local moduli picture, properties of singularities, etc.) will be essential in the future studies. Natural situations to investigate are given by Fano threefolds, log settings, “special” Fanos, non-smoothable KE del Pezzo orbifolds. It is natural to believe that explicit K-moduli compactifications could be found by birational modifications of standard GIT quotients, as we have shown for the two dimensional del Pezzo case.

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Cohomological Aspects on Complex and Symplectic Manifolds

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Abstract We discuss how quantitative cohomological informations could provide qualitative properties on complex and symplectic manifolds. In particular we focus on the Bott-Chern and the Aeppli cohomology groups in both cases, since they represent useful tools in studying non Kähler geometry. We give an overview on the comparisons among the dimensions of the cohomology groups that can be defined and we show how we reach the $\partial\overline{\partial}$-lemma in complex geometry and the Hard-Lefschetz condition in symplectic geometry. For more details we refer to Angella and Tardini (Proc Am Math Soc 145(1):273–285, 2017) and Tardini and Tomassini (Int J Math 27(12), 1650103 (20 pp.), 2016).

1 Introduction

In this note we discuss the informations that we can obtain on both complex and symplectic (not necessarily Kähler) manifolds studying the space of forms endowed with suitable differential operators; in particular, we focus on how quantitative cohomological properties could provide qualitative informations on the manifold. Recall that a smooth Kähler manifold is a complex manifold endowed with a Hermitian metric whose fundamental 2-form is d-closed. For dimensional reasons every Riemann surface is Kähler but in higher dimension this is not true in general. In complex dimension two, Kählerness can be topologically characterized in terms of the first Betti number (see [20, 25, 27]) but a similar result does not hold in dimension greater than two. Nevertheless there are many topological obstructions to the existence of a Kähler metric on a manifold, for example the odd Betti numbers are even and the even Betti numbers are positive. These results follow from the strong requests on the involved geometric structures and their deep relations. It seems therefore natural to ask what happens if we weaken those structures and/or their relations. In particular we could weaken the complex condition looking at non integrable almost-complex structures or we could look at complex manifolds with a weaker metric condition (e.g., balanced metrics [24], SKT metrics [13], etc.). On the

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other side we could ignore the (almost-)complex structure focusing the attention on the existence of a non-degenerate d-closed 2-form (i.e., a symplectic form) moving therefore to symplectic geometry. In any case, an important global tool in studying smooth manifold is furnished by cohomology, more precisely cohomology groups that are invariant for the considered geometric structures.

In complex non-Kähler geometry it turns out that the classical de Rham and Dolbeault cohomology groups do not suffice in studying a complex manifold (see e.g., [2]), indeed many informations are contained in the Bott-Chern and Aeppli cohomologies, defined, on a complex manifold X, respectively as

$$H^{\bullet,\bullet}_{BC}(X) := \frac{Ker\ \partial \cap Ker\ \overline{\partial}}{Im\ \partial\overline{\partial}}\ , \qquad H^{\bullet,\bullet}_{A}(X) := \frac{Ker\ \partial\overline{\partial}}{Im\ \partial + Im\ \overline{\partial}}.$$

They represent a bridge between a topological invariant (the de Rham cohomology) and a complex invariant (the Dolbeault cohomology). In general we have the following picture:

$$\begin{array}{ccccc} & & H^{\bullet,\bullet}_{BC}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H^{\bullet,\bullet}_{\partial}(X) & & H^{\bullet}_{dR}(X;\mathbb{C}) & & H^{\bullet,\bullet}_{\overline{\partial}}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{\bullet,\bullet}_{A}(X)\ , & & \end{array}$$

where the maps are the ones induced by the identity. Generally such maps are neither injective nor surjective but when the map $H^{\bullet,\bullet}_{BC}(X) \longrightarrow H^{\bullet,\bullet}_{A}(X)$ is injective, the manifold X is said to satisfy the $\partial\overline{\partial}$-*lemma*. Every Kähler manifold satisfies the $\partial\overline{\partial}$-lemma but the converse is not true. In this paper we will compare the dimensions of these cohomology groups recalling some results contained in [7] and [6]; in particular we will focus on how just knowing the dimensions of the Bott-Chern (and dually Aeppli) cohomology groups we can understand whether the $\partial\overline{\partial}$-lemma holds. More precisely,

Theorem 1 (See Theorems 3 and 5, Remark 2) *Let X be a compact complex manifold. Then, the following facts are equivalent:*

1. *the* $\partial\overline{\partial}$*-lemma holds on* X*;*
2. $\Delta^k := \sum_{p+q=k}\left(dim_{\mathbb{C}}H^{p,q}_{BC}(X) + dim_{\mathbb{C}}H^{p,q}_{A}(X)\right) - 2\,b_k \;=\; 0$ *, for any* $k \in \mathbb{Z}$*;*
3. $\sum_{p+q=k}\left(dim_{\mathbb{C}}H^{p,q}_{BC}(X) - dim_{\mathbb{C}}H^{p,q}_{A}(X)\right) \;=\; 0$*, for any* $k \in \mathbb{Z}$*.*

Moreover, if X has complex dimension 2*, then X is Kähler if and only if* $\Delta^2 = 0$*.*

In a similar fashion on a compact symplectic manifold (X, ω) it is possible to consider the symplectic Bott-Chern and Aeppli cohomology groups, as defined by Tseng and Yau in [31] by using the operators d and its symplectic-adjoint d^{Λ}. They are the appropriate cohomology groups in order to study symplectic Hodge theory. In the present work, similarly to the complex case, we will consider some comparisons among the dimensions of these cohomology groups collecting some results contained in [6, 8] and [28]. It turns out that the symplectic Bott-Chern cohomology $H^{\bullet}_{d+d^{\Lambda}}(X)$ (and dually Aeppli cohomology $H^{\bullet}_{dd^{\Lambda}}(X)$) suffices to characterize the dd^{Λ}-lemma, indeed we have the following

Theorem 2 (See [8, 15, 16, 22, 23, 35], Theorem 8) *Let (X, ω) be a compact symplectic manifold of dimension $2n$. Then, the following facts are equivalent:*

1. *the* hard-Lefschetz condition *(HLC for short) holds, i.e., the maps*

$$[\omega]^k : H^{n-k}_{dR}(X) \longrightarrow H^{n+k}_{dR}(X), \qquad 0 \le k \le n$$

   are all isomorphisms;
2. *the* dd^{Λ}-lemma *holds, i.e., the natural maps induced by the identity* $H^{\bullet}_{d+d^{\Lambda}}(X) \longrightarrow H^{\bullet}_{dR}(X)$ *are injective;*
3. $\Delta^k := \dim H^k_{d+d^{\Lambda}}(X) + \dim H^k_{dd^{\Lambda}}(X) - 2\, b_k \ = \ 0$ *, for any* $k \in \mathbb{Z}$.

Moreover, if X has dimension 4, then X satisfies HLC if and only if $\Delta^2 = 0$.

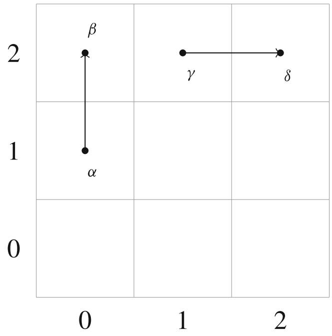
2 Complex Cohomologies

We begin this section with some preliminaries and fixing some notations. Let X be a compact complex manifold of complex dimension n. With $A^{p,q}(X)$ we denote the space of complex (p, q)-forms on X. As a consequence of the integrability of the complex structure the triple $\left(A^{\bullet,\bullet}(X),\ \partial,\ \overline{\partial}\right)$ represents a double complex, indeed the following relations hold: $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

The complex *de Rham*, *Dolbeault* and *conjugate Dolbeault* cohomology groups of X have been widely studied and they are defined, respectively, as

$$H^{\bullet}_{dR}(X; \mathbb{C}) := \frac{Ker\ d}{Im\ d}\,, \qquad H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{Ker\ \overline{\partial}}{Im\ \overline{\partial}}\,, \qquad H^{\bullet,\bullet}_{\partial}(X) := \frac{Ker\ \partial}{Im\ \partial}\,.$$

Roughly speaking, if we draw a double complex as follows, for the Dolbeault cohomology we are looking at vertical arrows, since the operator $\overline{\partial}$ changes the second degree of a (p, q)-form, and for its conjugate we are looking at horizontal arrows, since the operator ∂ changes the first degree of a (p, q)-form. For a more detailed explanation of the interpretation of a double complex as a sum of indecomposable objects as zig-zag, dots and squares we refer to [3].

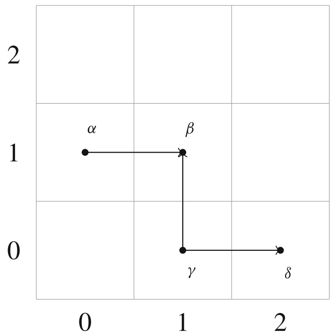


For instance, in the above picture we mean that $\overline{\partial}\alpha = \beta$, $\partial\alpha = \partial\beta = 0$, $\partial\gamma = \delta$ and $\overline{\partial}\gamma = \overline{\partial}\delta = 0$. So α and β are representatives of two non-trivial classes in $H^{\bullet,\bullet}_{\partial}(X)$ and β represents the trivial class in $H^{0,2}_{\overline{\partial}}(X)$. Similarly goes for δ and γ. Notice that we can not have two consecutive vertical (resp. horizontal) arrows because $\overline{\partial}^2 = 0$ (resp. $\partial^2 = 0$).

Nevertheless there is no natural map between the de Rham (a topological invariant) and Dolbeault (a holomorphic invariant) cohomologies, in this sense a bridge between them is furnished by the *Bott-Chern* [14] and the *Aeppli* [1] cohomology groups defined by

$$H^{\bullet,\bullet}_{BC}(X) := \frac{Ker\ \partial \cap Ker\ \overline{\partial}}{Im\ \partial\overline{\partial}}\,, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{Ker\ \partial\overline{\partial}}{Im\ \partial + Im\ \overline{\partial}}.$$

The same definitions can be stated, more generally, for a double complex $(B^{\bullet,\bullet}, \partial, \overline{\partial})$ of vector spaces. In this way we are taking into accounts the corners in the double complex of forms. For example looking at this picture



the forms α and γ are representatives of two non-trivial classes in $H^{\bullet,\bullet}_{A}(X)$ and β, δ in $H^{\bullet,\bullet}_{BC}(X)$. Namely, ingoing corners contribute to the Bott-Chern cohomology and outgoing corners to the Aeppli cohomology.

As regards the algebraic structure, a very easy computation shows that the product induced by the wedge product on forms induces a structure of algebra for the Bott-Chern cohomology of a complex manifold $H^{\bullet,\bullet}_{BC}(X)$ and a structure of $H^{\bullet,\bullet}_{BC}(X)$-module for the Aeppli cohomology $H^{\bullet,\bullet}_{A}(X)$.

In [26], see also [21], Hodge theory for the Bott-Chern and the Aeppli cohomologies is developed. In particular, once fixed a Hermitian metric g on X the Bott-Chern and the Aeppli cohomology groups of X are, respectively, isomorphic to the kernel of the following 4th-order elliptic self-adjoint differential operators

$$\Delta^g_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial$$

and

$$\Delta^g_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^*\left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right)\left(\overline{\partial}\partial^*\right)^* .$$

Therefore these cohomologies are finite-dimensional vector spaces. Moreover, differently from the Poincaré and Serre duality for the Dolbeault cohomology, the Hermitian duality does not preserve these cohomologies; more precisely when a Hermitian metric is fixed on X, the $\mathbb{C}$-anti-linear Hodge-$*$-operator induces an (un-natural) isomorphism between the Bott-Chern cohomology and the Aeppli cohomology, namely

$$* : H^{p,q}_{BC}(X) \longrightarrow H^{n-p,n-q}_{A}(X)$$

is an isomorphism for any $p, q \in \mathbb{Z}$; this means that we do not have symmetry with respect to the center in the Bott-Chern (and Aeppli) diamond. Therefore, we have the following equalities: $\dim_{\mathbb{C}} H^{p,q}_{BC}(X) = \dim_{\mathbb{C}} H^{q,p}_{BC}(X) = \dim_{\mathbb{C}} H^{n-q,n-p}_{A}(X) = \dim_{\mathbb{C}} H^{n-p,n-q}_{A}(X)$, where the first one and the last one are due to the fact that the conjugation preserves the Bott-Chern and the Aeppli cohomologies respectively (giving a symmetry in the Bott-Chern diamond with respect to the central column).

Remark 1 Notice that, in general, the isomorphism $H^{\bullet,\bullet}_{BC}(X) \simeq Ker\Delta^g_{BC}$ is of vector spaces not algebras, indeed the wedge product of harmonic forms is not necessarily harmonic. The study of Hermitian metrics whose space of Bott-Chern harmonic forms has a structure of algebra has been developed in [9] and in [29] in terms of geometric formality.

By definition, the identity induces natural maps of (bi-)graded vector spaces between the Bott-Chern, Dolbeault, de Rham, and Aeppli cohomologies:

$$\begin{array}{ccccc} & & H^{\bullet,\bullet}_{BC}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H^{\bullet,\bullet}_{\partial}(X) & & H^{\bullet}_{dR}(X;\mathbb{C}) & & H^{\bullet,\bullet}_{\overline{\partial}}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{\bullet,\bullet}_{A}(X)\ . & & \end{array}$$

Recall that a compact complex manifold is said to satisfy the $\partial\overline{\partial}$-*Lemma* if the natural map $H^{\bullet,\bullet}_{BC}(X) \longrightarrow H^{\bullet,\bullet}_{A}(X)$ is injective. This is equivalent to any of the above maps being an isomorphism, see [18, Lemma 5.15]. Since any compact Kähler manifold satisfies the $\partial\overline{\partial}$-lemma the Bott-Chern and Aeppli cohomologies could provide more informations on a compact complex manifold which does not admit any Kähler metric. For this reason, from now on, we will implicitly assume that our manifolds are not Kähler.

2.1 Inequalities on Compact Complex Manifolds

In this section we are mainly interested in discussing quantitative cohomological informations on complex manifolds with the final aim of understanding which integers can appear as dimensions of cohomology groups of complex manifolds. In the compact Kähler case the Hodge decomposition Theorem states that the Dolbeault cohomology groups give a decomposition of the de Rham cohomology, inducing at the level of cohomology the decomposition of complex forms in (p,q)-forms. This is no longer true if we drop the Kähler assumption. Frölicher in [19] constructs a spectral sequence whose first page is isomorphic to the Dolbeault cohomology and converging to the de Rham cohomology proving, consequently, that on any compact complex manifold X there is a topological lower bound for the Hodge numbers (the dimensions of the Dolbeault cohomology groups) in terms of the Betti numbers (the dimensions of the de Rham cohomology groups), namely for any $k \in \mathbb{Z}$,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \ \geq \ b_k\ .$$

A Frölicher type inequality has been proven by Angella and Tomassini in [7] taking into consideration the Bott-Chern and the Aeppli cohomology groups. For clearness we report here the complete statement.

Theorem 3 ([7, Theorem A, Theorem B]) *Let X be a compact complex manifold. Then, for any $k \in \mathbb{Z}$,*

$$\Delta^k(X) \;:=\; \sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) - 2\, b_k \;\geq\; 0\,.$$

Moreover, X satisfies the $\partial\overline{\partial}$-Lemma if and only if, for any $k \in \mathbb{Z}$, there holds $\Delta^k(X) = 0$.

It provides a lower bound for the dimension of the Bott-Chern and Aeppli cohomologies in terms of the Betti numbers (the proof actually shows a lower bound also in terms of the Hodge numbers), and it yields also a quantitative characterization of the $\partial\overline{\partial}$-Lemma. The proof of this Theorem is essentially algebraic and it is based on Varouchas exact sequences [32]. The idea relies on the fact that the Dolbeault cohomology is computed by looking at vertical arrows in a double complex and its conjugate by looking at horizontal arrows. Nevertheless the Bott-Chern and the Aeppli cohomologies compute the number of ingoing and outgoing corners therefore, by combinatoric arguments, one gets that the dimensions of the Bott-Chern and Aeppli cohomology groups are greater or equal than the sum of Hodge numbers and their conjugates, which are greater or equal than the Betti numbers by Frölicher. As a corollary one gets also the stability of the $\partial\overline{\partial}$-lemma under small deformations of the complex structure (see also [33] and [34] for different proofs). In [8] a generalization to double complexes is developed, with applications to compact symplectic manifolds.

Remark 2 Consider the special case when X is a compact complex surface, i.e., $\dim_{\mathbb{C}} X = 2$. By duality the non-negative numbers Δ^1 and Δ^2 give all the informations. Since Kählerness can be topologically characterized in terms of the parity of the first Betti number b_1, the Kähler condition is then equivalent to the $\partial\overline{\partial}$-lemma holding on X, leading to the equivalence: X is Kähler if and only if $\Delta^1 = \Delta^2 = 0$.

Nevertheless we can be even more precise, indeed, it is proven in [30] that Δ^1 vanishes on any compact complex surface (see [10] for explicit examples). This is not true in higher dimension. Therefore the number Δ^2 measure the non-Kählerness of a compact surface:

$$\text{Kähler} \qquad \Longleftrightarrow \qquad \Delta^2 = 0.$$

In general, on surfaces Teleman in [30] proves that there are only two options for Δ^2: it is either 0 or 2. For a generalization in higher dimension see [11].

We have seen above that the Bott-Chern and the Aeppli numbers dominate the Hodge numbers and then, by Frölicher the Betti numbers. In joint work with Angella in [6] (see also [3]) we prove that they are also dominated by Hodge numbers.

Theorem 4 ([6, Theorem 2.1, Remark 2.2]) *Let X be a compact complex manifold of complex dimension n. Then, for any $k \in \mathbb{Z}$,*

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{A}(X)$$

$$\leq \min\{k+1, (2n-k)+1\} \cdot \left(\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) + \sum_{p+q=k+1} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \right)$$

$$\leq (n+1) \cdot \left(\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) + \sum_{p+q=k+1} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \right) ,$$

and

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X)$$

$$\leq \min\{k+1, (2n-k)+1\} \cdot \left(\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) + \sum_{p+q=k-1} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \right)$$

$$\leq (n+1) \cdot \left(\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) + \sum_{p+q=k-1} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \right) .$$

Proof The proof is essentially algebraic and, for example, the idea behind the first inequality is obtained by thinking that the outgoing corners in a zig-zag contribute to the Aeppli cohomology and the extremal points of a zig-zag to the Dolbeault cohomology and/or its conjugate. Therefore, for any outgoing corners we have two extremal points and the number of outgoing corners depends on the length of the zig-zag. For a detailed proof we refer to [6] (see also [3]). □

A similar result holds in case of double complexes under some additional hypothesis of boundedness, leading to a similar result in symplectic geometry.

2.2 A Characterization of the $\partial\overline{\partial}$-Lemma

By the above inequalities we then get that the difference $\sum_{p+q=k} \big(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) - \dim_{\mathbb{C}} H^{p,q}_{A}(X)\big)$ is bounded from both above and below by the Hodge numbers. In [6] together with Angella we prove that there is also a characterization of the $\partial\overline{\partial}$-Lemma in terms of this quantity.

Theorem 5 ([6, Theorem 3.1]) *A compact complex manifold X satisfies the $\partial\overline{\partial}$-Lemma if and only if, for any $k \in \mathbb{Z}$, there holds*

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) - \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \;=\; 0 \;.$$

Proof The first implication is trivial. For the other one notice that, roughly speaking, the vanishing of the numbers $\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) - \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right)$ means that the number of ingoing corners is equal to the number of outgoing corners on any diagonal of the same total degree; since in degree 0 we do not have ingoing corners then we do not have any arrows in the picture of the double complex and therefore the $\partial\overline{\partial}$-lemma holds on X. Nevertheless the precise proof of Theorem 5 is based on Varouchas exact sequences [32] but it is not algebraic, indeed conjugation is needed; a similar result cannot be expected in the symplectic case. □

Remark 3 This result means that on a compact complex manifold a non canonical isomorphism between the Bott-Chern and the Aeppli cohomology forces all the natural maps in the cohomology diagram to be isomorphisms and so these cohomologies are not providing additional informations on the manifold. By the Schweitzer duality between the Bott-Chern and the Aeppli cohomology [26, §2.c], the above condition can be written just in terms of the Bott-Chern cohomology as follows: for any $k \in \mathbb{Z}$, there holds

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X) \;=\; \sum_{p+q=2n-k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X) \;,$$

namely there is a symmetry in the Bott-Chern numbers. The study of this property was initially motivated by the development of Sullivan theory of formality in the context of Bott-Chern cohomology (see [9] and [29] for results in this direction).

Notice that there exist special classes of complex manifolds where the dimensions of the Bott-Chern (and by duality Aeppli) cohomology groups can be computed explicitly by means of suitable sub-complexes of the complex of forms (see [4]) making this result concrete in studying the $\partial\overline{\partial}$-lemma.

3 Symplectic Cohomologies

We consider now the symplectic case and we show that similar results hold in this setting. Let (X, ω) be a compact symplectic manifold of dimension $2n$, then Tseng and Yau in [31] define a symplectic version of the Bott-Chern and the Aeppli cohomology groups. Denoting with $A^\bullet(X)$ the space of differential forms on X, the *symplectic-$\star$-Hodge operator* (see [15]) $\star : A^\bullet(X) \longrightarrow A^{2n-\bullet}(X)$ is defined as follows: given $\beta \in A^k(X)$, for any $\alpha \in A^k(X)$ there holds $\alpha \wedge \star\beta = (\omega^{-1})^k(\alpha, \beta)\, \omega^n$, where on simple elements $(\omega^{-1})^k(\alpha^1 \wedge \ldots \wedge \alpha^k, \beta^1 \wedge \ldots \wedge \beta^k) := \det\left(\omega^{-1}(\alpha^i, \beta^j)\right)_{i,j}$.

The *Brylinski co-differential* is defined as

$$d^{\Lambda} := [d, \Lambda] = d\Lambda - \Lambda d = (-1)^{k+1} \star d \star \ ,$$

where $\Lambda : A^{\bullet}(X) \longrightarrow A^{\bullet-2}(X)$ is the adjoint of the Lefschetz operator $L = \omega \wedge - : A^{\bullet}(X) \longrightarrow A^{\bullet+2}(X)$. By definition $d^{\Lambda} : A^{\bullet}(X) \longrightarrow A^{\bullet-1}(X)$ and the following relations hold: $\left(d^{\Lambda}\right)^2 = 0$ and $dd^{\Lambda} + d^{\Lambda}d = 0$.

Notice that the operator $dd^{\Lambda} + d^{\Lambda}d$ is not the analogue of the de-Rham Laplacian in the classical Riemannian Hodge theory because it is not elliptic (it is always zero!) and we should think at d^{Λ} as the analogue of the operator d^c in complex geometry (actually they are deeply related once fixed a compatible triple, see [31] for more details).

Then, for $k \in \mathbb{Z}$, (see [31]) the d^{Λ}*-cohomology groups* are

$$H^k_{d^{\Lambda}}(X) := \frac{Ker(d^{\Lambda}) \cap A^k(X)}{Im\ d^{\Lambda} \cap A^k(X)},$$

the *symplectic Bott-Chern cohomology groups* are

$$H^k_{d+d^{\Lambda}}(X) := \frac{Ker(d + d^{\Lambda}) \cap A^k(X)}{Im\ dd^{\Lambda} \cap A^k(X)}$$

and the *symplectic Aeppli cohomology groups* are

$$H^k_{dd^{\Lambda}}(X) := \frac{Ker(dd^{\Lambda}) \cap A^k(X)}{(Im\ d + Im\ d^{\Lambda}) \cap A^k(X)}.$$

By construction they are invariant under symplectomorphisms and so they are good symplectic cohomologies encoding global invariants. For similar definitions in the locally conformal symplectic setting see [12].

Moreover these cohomology groups have been introduced because in symplectic geometry the de Rham cohomology is not the appropriate one when talking about Hodge theory.

Consider a compatible triple (ω, J, g) on X, namely

- J is a ω*-compatible almost-complex structure*, i.e.,
  - ω is positive on the J-complex lines, $\omega(\cdot, J\cdot) > 0$;
  - ω is J-invariant, $\omega(J\cdot, J\cdot\cdot) = \omega(\cdot, \cdot\cdot)$;
- g is the *corresponding Riemannian metric* on X defined by $g(\cdot, \cdot\cdot) := \omega(\cdot, J\cdot\cdot)$.

Denoting with $*$ the standard *Hodge-operator* with respect to the Riemannian metric g, there are canonical isomorphisms (see [31])

$$\mathscr{H}^k_{d^{\Lambda}}(X) := \ker \Delta_{d^{\Lambda}} \simeq H^k_{d^{\Lambda}}(X)\,,$$

where $\Delta_{d^\Lambda} := d^{\Lambda *} d^\Lambda + d^\Lambda d^{\Lambda *}$ is a second-order elliptic self-adjoint differential operator and

$$\mathscr{H}^k_{d+d^\Lambda}(X) := \ker \Delta_{d+d^\Lambda} \simeq H^k_{d+d^\Lambda}(X), \qquad \mathscr{H}^k_{dd^\Lambda}(X) := \ker \Delta_{dd^\Lambda} \simeq H^k_{dd^\Lambda}(X).$$

where Δ_{d+d^Λ}, Δ_{dd^Λ} are fourth-order elliptic self-adjoint differential operators defined by

$$\Delta_{d+d^\Lambda} := (dd^\Lambda)(dd^\Lambda)^* + (dd^\Lambda)^*(dd^\Lambda) + d^* d^\Lambda d^{\Lambda *} d + d^{\Lambda *} dd^* d^\Lambda + d^* d + d^{\Lambda *} d^\Lambda,$$

$$\Delta_{dd^\Lambda} := (dd^\Lambda)(dd^\Lambda)^* + (dd^\Lambda)^*(dd^\Lambda) + dd^{\Lambda *} d^\Lambda d^* + d^\Lambda d^* dd^{\Lambda *} + dd^* + d^\Lambda d^{\Lambda *}.$$

In particular, the symplectic cohomology groups are finite-dimensional vector spaces on a compact symplectic manifold. For $\sharp \in \{d^\Lambda, d + d^\Lambda, dd^\Lambda\}$ we set $h^\bullet_\sharp =: h^\bullet_\sharp(X) := \dim H^\bullet_\sharp(X) < \infty$ when the manifold X is understood.

Similarly to the classical Hodge theory the differential forms closed both for the operators d and d^Λ were called by Brylinski *symplectic harmonic* [15]. The existence of a symplectic harmonic form in each de Rham cohomology class does not occur in general. As regards uniqueness there is no hope, indeed on any symplectic manifold (X, ω) if $\alpha \in A^1(X)$ is symplectic-harmonic then $\alpha + df$ is still symplectic-harmonic, for any smooth function f on X, because $d^\Lambda(\alpha + df) = d^\Lambda df = d\Lambda df = 0$ for degree reasons.

In particular, the following facts are equivalent on a compact symplectic manifold (X^{2n}, ω) (cf. [15, 16, 22, 23, 35])

- the *hard-Lefschetz condition* (HLC for short) holds, i.e., the maps

$$L^k : H^{n-k}_{dR}(X) \longrightarrow H^{n+k}_{dR}(X), \qquad 0 \le k \le n$$

  are all isomorphisms;
- the *Brylinski conjecture*, i.e., the existence of a symplectic harmonic form in each de Rham cohomology class;
- the *dd^Λ-lemma*, i.e., every d^Λ-closed, d-exact form is also dd^Λ-exact;
- the natural maps induced by the identity $H^\bullet_{d+d^\Lambda}(X) \longrightarrow H^\bullet_{dR}(X)$ are injective;
- the natural maps induced by the identity $H^\bullet_{d+d^\Lambda}(X) \longrightarrow H^\bullet_{dR}(X)$ are surjective;
- the natural maps induced by the identity in the following diagram are isomorphisms

$$\begin{array}{ccccc} & & H^{\bullet}_{d+d^{\Lambda}}(X) & & \\ & \swarrow & & \searrow & \\ H^{\bullet}_{dR}(X) & & & & H^{\bullet}_{d^{\Lambda}}(X). \\ & \searrow & & \swarrow & \\ & & H^{\bullet}_{dd^{\Lambda}}(X) & & \end{array}$$

In this sense $H^{\bullet}_{d+d^{\Lambda}}(X)$ and $H^{\bullet}_{dd^{\Lambda}}(X)$ represent more appropriate cohomologies talking about existence and uniqueness of harmonic representatives on symplectic manifolds.

Nevertheless, in general, on a symplectic manifold of dimension $2n$ the following maps are all isomorphisms (see [31, Prop. 3.24])

$$\begin{array}{ccc} \mathscr{H}^{k}_{d+d^{\Lambda}}(X) & \xrightarrow{\ *\ } & \mathscr{H}^{2n-k}_{dd^{\Lambda}}(X) \\ \downarrow{\scriptstyle L^{n-k}} & & \downarrow{\scriptstyle \Lambda^{n-k}} \\ \mathscr{H}^{2n-k}_{d+d^{\Lambda}}(X) & \xrightarrow{\ *\ } & \mathscr{H}^{k}_{dd^{\Lambda}}(X)\,, \end{array}$$

in particular, it follows that $h^{k}_{d+d^{\Lambda}} = h^{2n-k}_{d+d^{\Lambda}} = h^{k}_{dd^{\Lambda}} = h^{2n-k}_{dd^{\Lambda}}$ for all $k = 0, \ldots, 2n$.

Remark 4 Note that, as proved in [6] (see Theorem 5), on a compact complex manifold the equality between the dimensions of the Bott-Chern cohomology groups and the Aeppli cohomology groups characterizes the $\partial\overline{\partial}$-lemma; nevertheless, the "analogous" condition on a compact symplectic manifold X, namely $h^{\bullet}_{d+d^{\Lambda}}(X) = h^{\bullet}_{dd^{\Lambda}}(X)$, is always verified.

3.1 Inequalities on Compact Symplectic Manifolds

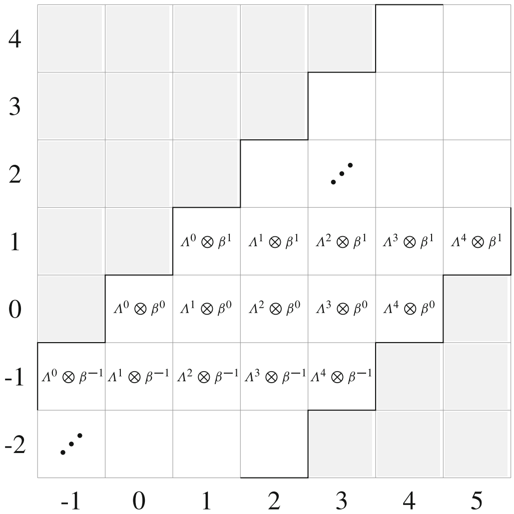
The proof of Theorem 4 is essentially algebraic and it can be generalized to double complexes with some hypothesis of boundedness. For the general statement we refer to [6], here we consider the application to the symplectic cohomologies. Let X be a compact manifold of dimension $2n$ endowed with a symplectic structure ω. As in [15, 16], we define the double complex associated to $(A^{\bullet}(X), d, d^{\Lambda})$ as

$$\left(B^{\bullet_1,\bullet_2} := \wedge^{\bullet_1-\bullet_2}X \otimes \beta^{\bullet_2},\ d \otimes \mathrm{id},\ d^{\Lambda} \otimes \beta\right)\ ,$$

where β is a generator of the infinite cyclic commutative group $\beta^{\mathbb{Z}}$. Note that, for any $q \in \mathbb{Z}$, we have

$$B^{p,q} = \{0\}, \qquad p \notin \{q, \ldots, q+2n\} ,$$

hence there exists a diagonal strip of width $2n+1$ such that the double complex $B^{\bullet,\bullet}$ has support in this strip. In the picture below we have an example for $2n = 4$.



The Bott-Chern and Aeppli cohomologies of $B^{\bullet,\bullet}$ are related to the symplectic cohomologies of X, $H^{\bullet}_{d+d^{\Lambda}}(X)$, $H^{\bullet}_{dd^{\Lambda}}(X)$, more precisely,

$$H^{\bullet_1,\bullet_2}_{BC}(B^{\bullet,\bullet}) = H^{\bullet_1-\bullet_2}_{d+d^{\Lambda}}(X) \otimes \beta^{\bullet_2}, \qquad H^{\bullet_1,\bullet_2}_{A}(B^{\bullet,\bullet}) = H^{\bullet_1-\bullet_2}_{dd^{\Lambda}}(X) \otimes \beta^{\bullet_2} .$$

The conjugate-Dolbeault and Dolbeault cohomologies of $B^{\bullet,\bullet}$ are both related to the de Rham cohomology of X. With the same idea of the proof of Theorem 4 we can prove the following

Theorem 6 ([6, Theorem 6.2]) *Let X be a compact differentiable manifold of dimension $2n$ endowed with a symplectic structure ω. Then, for any $k \in \mathbb{Z}/2\mathbb{Z}$,*

$$\sum_{h=k \bmod 2} \dim_{\mathbb{R}} H^{h}_{d+d^{\Lambda}}(X) \leq 2(2n+1) \cdot \sum_{h\in\mathbb{Z}} \dim_{\mathbb{R}} H^{h}_{dR}(X;\mathbb{R}) ,$$

and

$$\sum_{h=k \bmod 2} \dim_{\mathbb{R}} H^{h}_{dd^{\Lambda}}(X) \leq 2(2n+1) \cdot \sum_{h\in\mathbb{Z}} \dim_{\mathbb{R}} H^{h}_{dR}(X;\mathbb{R}) .$$

3.2 A Characterization of the Hard Lefschetz Condition

In [8] Angella and Tomassini, starting from a purely algebraic point of view, introduce on a compact symplectic manifold (X^{2n}, ω) the following non-negative integers

$$\Delta^k := h^k_{d+d^\Lambda} + h^k_{dd^\Lambda} - 2b_k \geq 0, \qquad k \in \mathbb{Z},$$

proving that, similarly to the complex case, their vanishing characterizes the dd^Λ-lemma which is equivalent to the validity of the Hard-Lefschetz condition. In this sense these numbers measure the HLC-degree of a symplectic manifold, as their analogue in the complex case do (cf. [7]).

Now, as already observed by Chan and Suen in [17], using the equality $\dim H^\bullet_{d+d^\Lambda}(X) = \dim H^\bullet_{dd^\Lambda}(X)$ proved in [31], we get

$$\Delta^k = 2(h^k_{d+d^\Lambda} - b_k), \qquad k \in \mathbb{Z};$$

therefore we can simplify them as in [28], considering just the difference between the dimensions of the Bott-Chern and the de Rham cohomology groups. We define

$$\tilde{\Delta}^k := h^k_{d+d^\Lambda} - b_k, \qquad k \in \mathbb{Z}.$$

Notice that a similar simplification can not be done in the complex case (cf. [26]). We put in evidence that, by duality, $\tilde{\Delta}^k = \tilde{\Delta}^{2n-k}$, $k = 0, \dots 2n$, so for a compact symplectic manifold (X, ω) of dimension $2n$ we will refer to $\tilde{\Delta}^k$, $k = 0 \dots n$, as the *non-HLC-degrees* of X. Note that $\tilde{\Delta}^0 = 0$.

As a consequence of the positivity of Δ^k, for any k, we have that for all $k = 1, \dots, n$

$$b_k \leq h^k_{d+d^\Lambda}$$

on a compact symplectic $2n$-dimensional manifold.

Moreover the equalities

$$b_k = h^k_{d+d^\Lambda}, \qquad \forall k = 1, \dots, n,$$

hold on a compact symplectic $2n$-dimensional manifold if and only if it satisfies the Hard-Lefschetz condition; namely the equality $b_\bullet = h^\bullet_{d+d^\Lambda}$ ensures the bijectivity of the natural maps $H^\bullet_{d+d^\Lambda}(X) \longrightarrow H^\bullet_{dR}(X)$, and hence the dd^Λ-lemma.

This considerations can be inserted in the more general setting of generalized complex manifolds, see [17] for more details.

Similarly to the complex case where Δ^2 characterizes the Kählerianity of a compact complex surface, if $2n = 4$ we want to show that the only degree which characterizes the Hard Lefschetz Condition is $\tilde{\Delta}^2$. Notice that, differently to the complex case, in any dimension we have the following

Theorem 7 ([28, Theorem 4.3]) *Let (X^{2n}, ω) be a compact symplectic manifold, then the natural map induced by the identity*

$$H^1_{d+d^\Lambda}(X) \longrightarrow H^1_{dR}(X)$$

is an isomorphism. In particular,

$$\tilde{\Delta}^1 = 0.$$

Proof For the sake of completeness we briefly recall here the proof. For the surjectivity, if α is a d-closed 1-form, then it is also d^Λ-closed, indeed

$$d^\Lambda \alpha = [d, \Lambda]\,\alpha = -\Lambda d\alpha = 0.$$

We need to prove the injectivity. Let $a = [\alpha] \in H^1_{d+d^\Lambda}(X)$ be such that $a = 0$ in $H^1_{dR}(X)$, namely $\alpha = df$ for some smooth function f on X. Considering the Hodge decomposition of f with respect to the d^Λ-cohomology (cf. [31]) we get $f = c + d^\Lambda\beta$ with c constant and β differential 1-form. Hence

$$\alpha = df = d(c + d^\Lambda\beta) = dd^\Lambda\beta,$$

i.e., $[\alpha] = 0 \in H^1_{d+d^\Lambda}(X)$.

As a consequence, $b_1 = h^1_{d+d^\Lambda}$, implying $\tilde{\Delta}^1 = h^1_{d+d^\Lambda} - b_1 = 0$ and concluding the proof. □

The analog result for the complex Bott-Chern cohomology is not true, see e.g., [4, Remark 3.6]. The previous Theorem lead us to the following quantitative characterization of the Hard Lefschetz condition in dimension 4.

Theorem 8 ([28, Theorem 4.5]) *Let (X^4, ω) be a compact symplectic 4-manifold, then it satisfies*

$$HLC \iff \tilde{\Delta}^2 = 0 \iff b_2 = h^2_{d+d^\Lambda}.$$

Therefore in 4-dimensions it is possible to study the Hard Lefschetz condition by studying the dependence of the space $H^2_{d+d^\Lambda}(X)$ on the symplectic structure.

Remark 5 As shown in [30] on a compact complex surface $\Delta^2 \in \{0, 2\}$; in [28] with Tomassini we provide an explicit example of a compact symplectic 4-manifold with $\Delta^2 \notin \{0, 2\}$, or equivalently $\tilde{\Delta}^2 \notin \{0, 1\}$, showing hence a different behavior in the symplectic case. More precisely we compute the non-HLC degree $\tilde{\Delta}^2$ when X is a compact 4-dimensional manifold diffeomorphic to a solvmanifold $\Gamma\backslash G$ (i.e., the compact quotient of a connected simply-connected solvable Lie group G by a discrete cocompact subgroup Γ) admitting a left-invariant symplectic structure; for a partial computation cfr. [5, Table 2].

In detail, if $X = \Gamma\backslash G$ is a compact solvmanifold of dimension 4 with ω left-invariant symplectic structure, then, according to $\mathfrak{g} = Lie(G)$, we have the following cases

a) if $\mathfrak{g} = \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$, then $\tilde{\Delta}^2 = 1$;
b) if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{3,4}^{-1}$, then $\tilde{\Delta}^2 = 0$;
c) if $\mathfrak{g} = \mathfrak{g}_{4,1}$, then $\tilde{\Delta}^2 = 2$.

See [28] for the computations.

Notice that, by applying Theorem 6, with an easy computation we obtain the following (quite large) inequalities for a general compact symplectic 4-manifold (X^4, ω),

$$b_2 \leq h^2_{d+d^\Lambda} \leq 10\, b_2 + 20\, b_1 + 18$$

and

$$0 \leq \tilde{\Delta}^2 \leq 9\, b_2 + 20\, b_1 + 18.$$

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Towards the Classification of Class VII Surfaces

Andrei Teleman

Abstract This article follows the ideas of my talk "A duality theorem for instanton moduli spaces on class VII surfaces" given at the workshop "*INdAM Meeting: Complex and Symplectic Geometry*" which took place in June 2016 in Cortona; it gives a survey on my recent results concerning the existence of a cycle of curves on class VII surfaces with small b_2. The main problem in the classification of class VII surfaces is the existence of holomorphic curves. My approach uses a combination of techniques coming from complex geometry and gauge theory. The main object used in the proofs is a moduli space $\mathcal{M}$ of polystable holomorphic bundles on the considered surface. This moduli space is identified with an instanton moduli space via the Kobayashi-Hitchin correspondence. The existence (non-existence) of curves on the base surface is related to geometric properties of the corresponding moduli space.

1 Class VII Surfaces: The Classification Problem and Conjectures

A class VII surface [1] is a (compact, connected) complex surface X with $b_1(X) = 1$, $\mathrm{kod}(X) = -\infty$. These surfaces are not classified yet.

The classification problem for class VII surfaces with $b_2 = 0$ is solved by the following theorem :

Theorem 1.1 *Any surface* $X \in \mathrm{VII}_0$ *is biholomorphic to either a Hopf surface or to an Inoue surface.*

The proof of this theorem [25] uses differential geometric methods, and the renowned Kobayashi-Hitchin correspondence on Gauduchon surfaces [2, 13, 19, 20] relating Hermite-Einstein connections to polystable holomorphic bundles. Taking into account Theorem 1.1, we will focus on the subclass

$$\mathrm{VII}^{\min}_{>0} := \{X \in \mathrm{VII} |\ b_2(X) > 0,\ \ \text{X is minimal}\},$$

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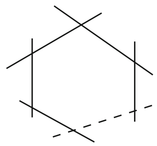
which has been studied with classical complex geometric methods by several authors [4–9, 12, 15–17, 21–24]. However a complete, general classification theorem for these surfaces is not available yet.

An important progress on this classification problem was made by Kato, who introduced the class of surfaces admitting a global spherical shell (a GSS). These surfaces are called GSS surfaces. Kato gave an explicit construction method for all GSS surfaces, and Dloussky refined this construction by proving an interesting relationship between surfaces with a GSS and contracting holomorphic germs. From the results of Kato and Dloussky it follows that any *known* surface $X \in \mathrm{VII}^{\min}_{>0}$ is a GSS surface. That is why the GSS surfaces in the class $\mathrm{VII}^{\min}_{>0}$ (the known surfaces in this class) are called Kato surfaces. The classification problem for class VII surfaces will be entirely solved if the following renowned GSS conjecture, stated by Nakamura [22], is proved:

Conjecture 1 Any surface $X \in \mathrm{VII}^{\min}_{>0}$ has a GSS, hence it is a Kato surface.
Kato surfaces have several remarkable properties, for example:

Proposition 1.2 *Any Kato surface X has exactly $b_2(X)$ rational curves, some of which forming one cycle, or two disjoint cycles of rational curves.*
Here, by a *cycle of rational curves* we mean an effective divisor $D \subset X$ which is either a rational curve with a simple singularity, or is a sum of $k \geq 2$ smooth rational curves whose intersecting numbers are given by the diagram:



Example 1.1 A surface $X \in \mathrm{VII}^{\min}_{>0}$ is called

(1) an *Enoki surface*, if it contains a non-empty, homologically trivial, effective divisor. Such a surface contains a homologically trivial cycle C of $b_2(X)$ rational curves, and is biholomorphic to a compactification of a holomorphic affine line bundle over an elliptic curve (see [12]);
(2) a *parabolic Inoue* surface, if it is an Enoki surface which contains an elliptic curve E. One has $E^2 = -b_2(X)$, and X is biholomorphic to a compactification of a holomorphic line bundle over E, such that E corresponds to the zero-section of this line bundle;
(3) a *half Inoue surface*, if it contains a cycle of $b_2(X)$ rational curves C with $C^2 = -b_2(X)$;
(4) an *Inoue-Hirzebruch* surface, if it contains two cycles of rational curves. The two cycles of an Inoue-Hirzebruch surface are disjoint, and contain together $b_2(X)$ curves.

By the results of Kato and Dloussky all these surfaces are examples of Kato surfaces. There also exist Kato surfaces containing a single cycle of rational curves, and trees (of rational curves) intersecting the cycle. Such surfaces are called *intermediate Kato surfaces*.

Property 1.2 is very important, because, by an important result of Nakamura, we know that:

Proposition 1.3 *Any class VII surface admitting an elliptic curve, or a cycle of rational curves, is the special fibre of a family of blown up primary Hopf surfaces.* Taking into account Proposition 1.3, we agree to call *cycle* (on a class VII surface) any effective divisor which is either an elliptic curve, or a cycle of rational curves. Combining Propositions 1.2, 1.3, we see that *Kato surfaces are just degenerations of blown up primary Hopf surfaces*.

An important step forward towards a proof of Conjecture 1 has been made in [9]:

Theorem 1.4 *Any surface $X \in \mathrm{VII}^{\min}_{>0}$ with $b_2(X)$ rational curves is a Kato surface.* This result answers positively a conjecture stated by Kato. This shows that the GSS conjecture (Conjecture 1) is equivalent to

Conjecture 2 Any surface $X \in \mathrm{VII}^{\min}_{>0}$ has $b_2(X)$ rational curves.

The advantage of this statement, compared with the GSS conjecture, is that it concerns the existence of *compact* subspaces in the surface. Note that, if true, Conjecture 2 will solve the classification problem for class VII surfaces up to biholomorphisms. For the coarser classification up to deformation equivalence, the relevant conjecture is:

Conjecture 3 Any surface $X \in \mathrm{VII}^{\min}_{>0}$ admits a cycle of curves.

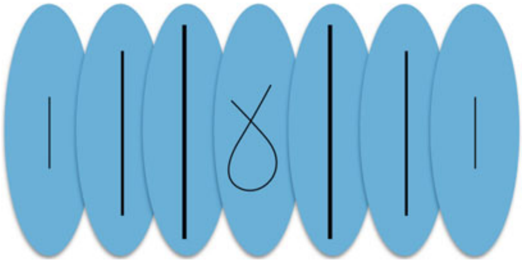
If true, this conjecture would imply, by Nakamura's Proposition 1.3, that any surface $X \in \mathrm{VII}^{\min}_{>0}$ is a degeneration of a family of blown up Hopf surfaces. The goal of our programme, developed in [26] and [29], is to prove Conjecture 3 at least for class VII surfaces with small b_2.

Supposing that Conjecture 3 is proved for any b_2 (using our programme, or a different method), one can wonder if the more ambitious Conjecture 2 will follow. If a surface $X \in \mathrm{VII}^{\min}_{>0}$ has a cycle of curves, then it will belong to the "known component" of the moduli stack of class VII surfaces; this component contains both Kato surfaces and blown up primary Hopf surfaces, the latter being generic. A natural question is

Can one obtain b_2 curves on X as "limits" of the exceptional curves in generic, non-minimal, small deformations of X?

Unfortunately the existence of curves cannot be obtained by "passing to the limit" in a family, because of a major, fundamental difficulty, specific to non-Kählerian geometry: *the explosion of area*.

Example 1.2 Denote by D the standard disk, and by $D^\bullet$ the punctured disk. There exists a holomorphic family $\mathcal{X} \to D$ of class VII surfaces, such that for any $z \in D^\bullet$, the fibre X_z is a blown up primary Hopf surface with $b_2 = 1$, and X_0 is an Enoki surface with a single irreducible curve. Therefore, for $z \neq 0$ the surface X_z contains an exceptional curve E_z with $E_z^2 = -1$, whereas the only irreducible curve of X_0 is a singular rational curve C_0 with $C_0^2 = 0$. As $z \to 0$, the area of E_z tends to infinity.



2 A Moduli Space of Polystable Holomorphic Bundles

In a series of articles [26–32], we have considered the existence of curves on class VII surfaces with small b_2 using a new method which makes use of notions and techniques coming from gauge theory [11].

Let X be a class VII surface with $b_2(X) > 0$. Suppose for simplicity $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}$. By the coefficients formula it follows that $H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $b := b_2(X)$. The condition $\mathrm{kod}(X) = -\infty$ implies $p_g(X) = 0$, so $b_+(X) = 0$, i.e. the intersection form

$$q_X : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$$

of X is negative definite. By Donaldson's first theorem [10], it follows that q_X is standard, i.e. there exists a basis $(e_1,\ldots,e_b)$ of $H^2(X,\mathbb{Z})$ such that

$$e_i \cdot e_j = -\delta_{ij} \text{ for } 1 \le i,j \le b.$$

Taking into account that $c_1(\mathcal{K}_X)$ is a characteristic element in the unimodular lattice $(H^2(X,\mathbb{Z}), q_X)$ [14, Definition 9.28], it follows that, replacing some of the classes e_i by their opposite, we can also assume that

$$c_1(\mathcal{K}_X) = \sum_{i=1}^{b} e_i.$$

(see [29] for details). The obtained basis will be unique up to order, and will be called the standard basis of $H^2(X,\mathbb{Z})$. Let $\mathfrak{I}$ be the index set $\{1,\ldots,b\}$, and for any $I \subset \mathfrak{I}$, put

$$e_I := \sum_{i\in I} e_i,\ \bar{I} := \mathfrak{I} \setminus I.$$

We denote by $\mathfrak{P}$ the quotient of the power set $\mathcal{P}(\mathfrak{I})$ of $\mathfrak{I}$ by the involution $I \mapsto \bar{I}$. This set can be identified with the set of unordered 2-term partitions of $\mathfrak{I}$.

Let E be a $\mathcal{C}^\infty$ rank 2-bundle on X with $c_2(E) = 0$, $\det(E) = K_X$, where K_X is the underlying differentiable line bundle of the canonical line bundle $\mathcal{K}_X$ of X. By the classification theorem for vector bundles on 4-dimensional CW complexes, it follows that such a bundle exists, and is unique up to isomorphism. Using the standard basis of $H^2(X,\mathbb{Z})$ one can prove easily that a differentiable line bundle L on X is isomorphic to a line sub-bundle of E if and only if there exists $I \subset \mathfrak{I}$ such that $c_1(L) = e_I$. Therefore $\mathfrak{P}$ can be identified with the set of isomorphism classes of (unordered) decompositions of E as direct sum of line bundles.

Let us fix a Gauduchon metric g on X [13], and let $\mathcal{M}^{\rm pst} = \mathcal{M}^{\rm pst}_{\mathcal{K}}(E)$ be the moduli space of polystable holomorphic structures $\mathcal{E}$ on E inducing the holomorphic structure $\mathcal{K}_X$ on $\det(E)$. By the Kobayashi-Hitchin correspondence this moduli space can be identified with a moduli space of projectively ASD unitary connections, so it comes with a natural Hausdorff topology [11, 26, 29]. The open subspace $\mathcal{M}^{\rm st} \subset \mathcal{M}^{\rm pst}$ corresponding to isomorphism classes of stable holomorphic structures has a natural complex space structure. The complement $\mathcal{R} := \mathcal{M}^{\rm pst} \setminus \mathcal{M}^{\rm st}$ corresponds to isomorphism classes of split poly-stable holomorphic structures in the moduli space, and will be called the subspace of reductions.

The moduli space $\mathcal{M}^{\rm pst}$ has the following important properties:

1. *Compactness*. $\mathcal{M}^{\rm pst}$ is compact. The proof of this property uses the Kobayashi-Hitchin correspondence and the Uhlenbeck compactness theorem. The lower strata in the Donaldson-Uhlenbeck compactification are automatically empty when $b \leq 3$ for topological reasons. For $b \geq 4$ one can prove that these strata are also empty with complex geometric methods [29]. This argument is due to N. Buchdahl.
2. *The subspace of reductions decomposes as a disjoint union of circles*. More precisely $\mathcal{R}$ decomposes as a disjoint union

$$\mathcal{R} = \coprod_{\lambda\in\mathfrak{P}} \mathcal{C}_\lambda$$

   of circles, called the circles of reductions in the moduli space. If $\lambda = \{I, \bar{I}\} \in \mathfrak{P}$, then $\mathcal{C}_\lambda$ is the subspace of $\mathcal{M}^{\rm pst}$ consisting of isomorphism classes of split poly-stable bundles $\mathcal{L} \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee)$ with $c_1(\mathcal{L}) \in \{e_I, e_{\bar{I}}\}$.
3. *Symmetry*. $\mathcal{M}^{\rm pst}$ comes with a natural involution denoted $\otimes l_0$, and given by the formula

$$[\mathcal{E}] \mapsto [\mathcal{E} \otimes \mathcal{L}_0],$$

   where $l_0 := [\mathcal{L}_0]$ is the nontrivial square root of $[\mathcal{O}_X]$ in the complex group $\mathrm{Pic}^0(X) \simeq \mathbb{C}^*$. This involution leaves invariant and acts freely on $\mathcal{R}$, and it has *finitely many* fixed points, which belong to $\mathcal{M}^{\rm st}$. Theses fixed points will be called twisted reductions.
4. *Regularity*. Suppose that X is minimal, is not an Enoki surface (see Sect. 1), and that the Gauduchon metric g has been chosen such that $\deg_g(\mathcal{K}_X) < 0$. Such

Gauduchon metrics exist [27, 29]. The proof of this important existence result is obtained by showing that, on a minimal class VII surface with positive b_2, the Bott-Chern class $c_1^{\mathrm{BC}}(\mathcal{K}_X)$ is not pseudo-effective [27, Proposition 4.5]. The proof makes use of the Buchdahl-Lamari description of the Gauduchon cone of a non-Kählerian complex surface [3, 18].

With these choices $\mathcal{M}^{\mathrm{st}}$ is a smooth complex manifold of dimension b, and all the reductions in the moduli space are regular, i.e. the second cohomology of the deformation complex of the associated reducible PU(2)-instanton vanishes. This result yields the following explicit description of the moduli space $\mathcal{M}^{\mathrm{pst}}$ around a circle of reductions:

Denote by $C_{\mathbb{P}^{b-1}_{\mathbb{C}}}$ the topological cone over $\mathbb{P}^{b-1}_{\mathbb{C}}$, and by $*$ its vertex. For any $\lambda \in \mathfrak{P}$ there exists a continuous map $h_\lambda : \mathcal{C}_\lambda \times C_{\mathbb{P}^{b-1}_{\mathbb{C}}} \to \mathcal{M}^{\mathrm{pst}}$ which maps homeomorphically $\mathcal{C}_\lambda \times C_{\mathbb{P}^{b-1}}$ onto a compact neighbourhood $\mathcal{N}_\lambda$ of $\mathcal{C}_\lambda$ and induces the obvious identification $\mathcal{C}_\lambda \times \{*\} \to \mathcal{C}_\lambda$. Such a neighbourhood will be called a tubular neighbourhood of $\mathcal{C}_\lambda$. It is easy to see that any circle $\mathcal{C}_\lambda$ has a fundamental system of tubular neighbourhoods. This result will be called the model theorem for the moduli space around the circles of reductions.

Let X be any class VII surface. Using Serre duality and Riemann-Roch theorems it follows that $h^1(\mathcal{K}_X) = 1$, so $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{K}_X) = H^1(\mathcal{K}_X) \simeq \mathbb{C}$. Therefore there exists an essentially unique non-trivial extension of the form

$$0 \to \mathcal{K}_X \xrightarrow{i} \mathcal{A} \xrightarrow{p} \mathcal{O}_X \to 0.$$

This extension will be called *the canonical extension of* X, and the isomorphy class $[\mathcal{A}]$ of its central term will be denoted by a. Our method to prove Conjecture 3 for surfaces with small b_2 starts with the following simple proposition (see [29]):

Proposition 2.1 *If $\mathcal{A}$ admits a holomorphic line subbundle $\mathcal{M} \neq i(\mathcal{K}_X)$, then X has a cycle of curves.*

Proof Let $\mathcal{M} \neq i(\mathcal{K}_X)$ be a line subbundle of $\mathcal{A}$, and let $j : \mathcal{M} \hookrightarrow \mathcal{A}$ be the embedding defined by the inclusion. The composition $p \circ j$ is non-zero. Indeed, if this composition vanishes, it will follow $\mathcal{M} \subset i(\mathcal{K}_X)$, so $\mathcal{M} = i(\mathcal{K}_X)$ because j is a bundle embedding (fibrewise injective). On the other hand $p \circ j$ is not an isomorphism, because, by definition, the canonical extension is non-split. Therefore $\mathrm{im}(p \circ j) = \mathcal{O}_X(-D)$ where $D > 0$ is the vanishing divisor of $p \circ j$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_X & \xrightarrow{i} & \mathcal{A} & \xrightarrow{p} & \mathcal{O}_X & \longrightarrow & 0 \\ & & & & {\scriptstyle j}\big\uparrow & & & & \\ & & & & \mathcal{M} & & & & \end{array}$$

Restricting the diagram to D, and taking into account that j is a bundle embedding, we obtain, for any $x \in D$, a monomorphism $\mathcal{M}(x) \to \mathcal{A}(x)$ whose image is $\mathcal{K}_X(x)$. Therefore the restriction of j to D induces an isomorphism

$$\mathcal{M}_D = \mathcal{O}(-D)_D \xrightarrow{\simeq} (\mathcal{K}_X)_D.$$

This gives a trivialisation of the dualising sheaf $\omega_D := \mathcal{K}_X(D)_D \simeq \mathcal{O}_D$. If D is reduced and irreducible, this implies already that D is a cycle. In the general case it follows that D contains a cycle (see [29] for details). ■

Corollary 2.2 *Suppose* $X \in \mathrm{VII}^{\min}_{>0}$, *and let* g *be a Gauduchon metric on* X *such that* $\deg_g(\mathcal{K}_X) < 0$. *If* $\mathcal{A}$ *is not stable, then* X *has a cycle of curves.*

Proof If $\mathcal{A}$ is not stable, there will exist a destabilising short exact sequence

$$0 \to \mathcal{M} \hookrightarrow \mathcal{A} \to \mathcal{K}_X \otimes \mathcal{M}^\vee \otimes \mathcal{I}_Z \to 0,$$

where $\mathcal{M}$ is a rank 1, locally free subsheaf of $\mathcal{A}$. By decomposing $c_1(\mathcal{M})$ with respect to the standard basis of $H^2(X, \mathbb{Z})$, and taking into account that $c_2(\mathcal{A}) = 0$, one obtains easily $Z = \emptyset$, i.e. $\mathcal{M}$ is a line subbundle of $\mathcal{A}$. This line subbundle cannot coincide with $i(\mathcal{K}_X)$ because, since $\deg_g(\mathcal{K}_X) < 0$, $i(\mathcal{K}_X)$ does not destabilise $\mathcal{A}$. The result follows now from Proposition 2.1. ■

In fact one can prove [27] that, for a given surface $X \in \mathrm{VII}^{\min}_{>0}$, the following holds: either there exist Gauduchon metrics on X for which $\mathcal{A}$ is stable, or X belongs to a special class of Kato surfaces.

Corollary 2.3 *Suppose* $X \in \mathrm{VII}^{\min}_{>0}$. *If* $[\mathcal{A}]$ *coincides with a twisted reduction, then* X *has a cycle of curves.*

Proof If $[\mathcal{A}]$ coincides with a twisted reduction, then $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{L}_0$, so $\mathcal{A}$ contains a line subbundle isomorphic with $\mathcal{K} \otimes \mathcal{L}_0$, which cannot coincide with $i(\mathcal{K})$, because $\mathcal{K} \otimes \mathcal{L}_0$ and $\mathcal{K}$ are not isomorphic. ■

Corollary 2.4 *If* $\mathcal{A}$ *is the central term of an extension*

$$0 \to \mathcal{K} \otimes \mathcal{L}^\vee \xrightarrow{j} \mathcal{A} \to \mathcal{L} \to 0 \tag{1}$$

with $c_1(\mathcal{L}) \neq 0$, *than* X *has a cycle of curves.*

Proof If $c_1(\mathcal{L}) \neq 0$, then $c_1(\mathcal{K} \otimes \mathcal{L}^\vee) \neq c_1(\mathcal{K}_X)$, so $j(\mathcal{K} \otimes \mathcal{L}^\vee)$ cannot coincide with $i(\mathcal{K})$, because $\mathcal{K} \otimes \mathcal{L}^\vee$ and $\mathcal{K}$ are not isomorphic. ■

Reading Corollaries 2.2–2.4 one can wonder what can be said about the bundle $\mathcal{A}$ on the known minimal class VII surfaces (see Sect. 1). It is easy to prove that:

Remark 2.5 If X is an Enoki surface, then $\mathcal{A}$ can be written as an extension of $\mathcal{K}(C)$ by $\mathcal{O}(-C)$, where C is the cycle of rational curves of X. This extension is split when X contains an elliptic curve (i.e. when X is a parabolic Inoue surface). If X is an Inoue-Hirzebruch surface, then $\mathcal{A}$ decomposes as $\mathcal{O}(-C_1) \oplus \mathcal{O}(-C_2)$, where C_1, C_2 are the two cycles of X. If X is a half Inoue surface, then $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{L}_0$, so $[\mathcal{A}]$ is a twisted reduction.

Our strategy for proving Conjecture 3 is based on the following idea: we prove that one of the hypotheses of Corollaries 2.2–2.4 holds. More precisely, we fix a Gauduchon metric on X with $\deg_g(\mathcal{K}_X) < 0$, and we prove that one of the following holds:

- $\mathcal{A}$ is non-stable, or
- $\mathcal{A}$ can be written as an extension of the form (1) with $c_1(\mathcal{L}) \neq 0$, or
- $[\mathcal{A}]$ is a twisted reduction.

For surfaces with $b_2 \leq 3$ this follows from the theorem below, whose proof will be sketched at the end of the next section.

Theorem 2.6 *Let $X \in \mathrm{VII}^{\min}_{>0}$ with $1 \leq b_2(X) \leq 3$, and let g be a Gauduchon metric on X with $\deg_g(\mathcal{K}_X) < 0$. Suppose that $\mathcal{A}$ is stable, that it cannot be written as an extension of the form (1) with $c_1(\mathcal{L}) \neq 0$, and that it does not coincide with a twisted reduction. Then, for a generic perturbation of g, there exists a compact complex subspace $Y \subset \mathcal{M}^{\mathrm{st}}$ of pure positive dimension such that*

(i) *$a \in Y$,*
(ii) *a has a Zariski open neighbourhood $Y' \subset Y$ such that $Y' \setminus \{a\}$ consist only of non-filtrable bundles.*

By the main result of [30], the existence of such a subspace leads to a contradiction. This results holds in full generality (for surfaces with arbitrary positive second Betti number). Taking into account Corollaries 2.2 – 2.4, we obtain the following result, which proves Conjecture 3 for class VII surfaces with small b_2:

Corollary 2.7 *Let X be a minimal class VII surface with $1 \leq b_2(X) \leq 3$. Then X has a cycle.*

The proof of Theorem 2.6 is based on a structure theorem for the moduli space $\mathcal{M}^{\mathrm{st}}$. In the next section we will state this theorem, and explain the idea of proof.

3 A Structure Theorem for $\mathcal{M}^{\mathrm{st}}$

We recall that a torsion-free coherent sheaf of rank 2 is called filtrable if it contains a coherent subsheaf of rank 1. A locally free sheaf $\mathcal{E}$ of rank 2 on a complex surface is filtrable if and only if it fits in a short exact sequence

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \otimes \mathcal{I}_Z \to 0,$$

where $\mathcal{L}$, $\mathcal{M}$ are locally free sheaves of rank 1, Z is a 0-dimensional locally complete intersection in X, and $\mathcal{I}_Z$ is its ideal sheaf. If X is a class VII surface, one can prove (see [29]):

Remark 3.1 Let X be a class VII surface, and let

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \otimes \mathcal{I}_Z \to 0,$$

be a short exact sequence on X, where $\mathcal{L}$, $\mathcal{M}$ are locally free sheaves of rank 1, Z is a 0-dimensional locally complete intersection in X, and $\mathcal{E}$ satisfies the conditions $\det(\mathcal{E}) \simeq \mathcal{K}_X$, $c_2(X) = 0$. Then $Z = \emptyset$, and there exists $I \subset \mathfrak{I}$ such that $c_1(\mathcal{L}) = e_I$. An extension of the form

$$0 \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{E} \to \mathcal{L} \to 0 \tag{2}$$

with $c_1(\mathcal{L}) = e_I$ will be called extension of type I. Remark 3.1 shows that the filtrable locus $\Phi \subset \mathcal{M}^{\mathrm{st}}$ (i.e. the subspace of points corresponding to filtrable stable bundles) decomposes as a union

$$\Phi = \bigcup_{I \in \mathcal{P}(\mathfrak{I})} \mathcal{M}_I^{\mathrm{st}}, \tag{3}$$

where $\mathcal{M}_I^{\mathrm{st}}$ is the subset of $\mathcal{M}^{\mathrm{st}}$ consisting of isomorphism classes of stable bundles which can be written as central terms of type I extensions. The subsets $\mathcal{M}_I^{\mathrm{st}}$ are not disjoint in general, because a bundle of rank 2 might be written as a line bundle extension in different ways. The incidence relation between these sets is determined by the structure of the curves of X.

Let $X \in \mathrm{VII}_{>0}^{\min}$ which is not an Enoki surface, let g be a Gauduchon metric on X, and put $\mathfrak{k} := \frac{1}{2}\deg_g(\mathcal{K}_X)$. Assume again for simplicity that $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}$. The sets $\mathcal{M}_I^{\mathrm{st}}$ can be described as follows:

A. The case $I \neq \emptyset$. Denote by p, q the projections of $\mathrm{Pic}(X) \times X$ on the two factors, and let $\mathscr{L}$ be a Poincaré line bundle on $\mathrm{Pic}(X) \times X$. For $y \in \mathrm{Pic}(X)$ denote by $\mathcal{L}_y$ the line bundle on X given by the restriction of $\mathscr{L}$ to the fibre $\{y\} \times X$. For $c \in H^2(X, \mathbb{Z})$ denote by $\mathrm{Pic}^c(X)_{>\mathfrak{k}}$ the open subset of $\mathrm{Pic}^c(X)$ defined by the inequality $\deg_g(y) > \mathfrak{k}$. For $I \neq \emptyset$ the restriction of the sheaf $R^1p_*(\mathscr{L}^{\vee\otimes 2} \otimes q^*(\mathcal{K}_X))$ to the punctured disk $\mathrm{Pic}^{e_I}(X)_{>\mathfrak{k}}$ is locally free of rank $|I|$, and its fibre at $y \in \mathrm{Pic}^{e_I}(X)_{>\mathfrak{k}}$ can be identified with $H^1(\mathcal{L}_y^{\vee\otimes 2} \otimes \mathcal{K}_X)$. Denote by $\Pi_{>\mathfrak{k}}^{e_I}$ the projectivisation of the bundle

$$R^1p_*(\mathscr{L}^{\vee\otimes 2} \otimes q^*(\mathcal{K}_X))\big|_{\mathrm{Pic}^{e_I}(X)_{>\mathfrak{k}}}.$$

For $\eta \in H^1(\mathcal{L}_y^{\vee\otimes 2} \otimes \mathcal{K}_X) \setminus \{0\}$ we denote by $\mathcal{E}_\eta$ the central term of an extension of $\mathcal{L}_y$ by $\mathcal{K} \otimes \mathcal{L}_y^\vee$ with extension invariant η. The subset

$$\{\Pi_{>\mathfrak{k}}^{e_I}\}^{\mathrm{st}} := \{[\eta] \in \Pi_{>\mathfrak{k}}^{e_I} |\ \mathcal{E}_\eta \text{ is stable}\}$$

is open in $\Pi_{>\mathfrak{k}}^{e_I}$. With these definitions it is easy to see that $\mathcal{M}_I^{\mathrm{st}}$ is just the image of the holomorphic map

$$\phi^{e_I} : \{\Pi_{>\mathfrak{k}}^{e_I}\}^{\mathrm{st}} \to \mathcal{M}^{\mathrm{st}},\ [\eta] \mapsto [\mathcal{E}_\eta].$$

$\{\Pi_{>\mathfrak{k}}^{e_I}\}^{\mathrm{st}}$ is a smooth complex manifold of dimension $|I|$, but it is non-compact, so its image $\mathcal{M}_I^{\mathrm{st}}$ via ϕ^{e_I} might be a very complicated subset of $\mathcal{M}^{\mathrm{st}}$. In particular it is not clear at all whether the closure of $\mathcal{M}_I^{\rm st}$ coincides with its Zariski closure, or whether this Zariski closure is still $|I|$-dimensional. On the other hand it is easy to see that, for sufficiently small $\varepsilon > 0$, the following holds:

(1) The set $\{\Pi^{e_I}_{>\mathfrak{k}}\}^{\rm st}$ contains the restriction $\Pi^{e_I}_{(\mathfrak{k},\mathfrak{k}+\varepsilon)}$ of the bundle $\Pi^{e_I}_{>\mathfrak{k}}$ to the annulus $\mathrm{Pic}^{e_I}(X)_{(\mathfrak{k},\mathfrak{k}+\varepsilon)} := \{ y \in \mathrm{Pic}^{e_I}(X) |\ \mathfrak{k} < \deg_g(y) < \mathfrak{k}+\varepsilon\}$.
(2) The restriction of ϕ^{e_I} to $\Pi^{e_I}_{(\mathfrak{k},\mathfrak{k}+\varepsilon)}$ is a holomorphic embedding.

Therefore $\mathcal{M}_I^{\rm st}$ contains the $|I|$-dimensional submanifold $\phi^{e_I}(\Pi^{e_I}_{(\mathfrak{k},\mathfrak{k}+\varepsilon)})$. Note that the closure of this submanifold in $\mathcal{M}^{\rm pst}$ contains the circle of reductions $\mathcal{C}_{I,\bar{I}}$.

B. The case $I = \emptyset$. The only non-trivial extensions of type $\emptyset$ have the form

$$0 \to \mathcal{K} \to \mathcal{A} \to \mathcal{O} \to 0,$$

$$0 \to \mathcal{K}\otimes\mathcal{L}_0 \to \mathcal{A}' := \mathcal{A}\otimes\mathcal{L}_0 \to \mathcal{L}_0 \to 0.$$

Therefore, putting $a' := [\mathcal{A}']$, we have

Remark 3.2 $\mathcal{M}_\emptyset^{\rm st}$ is either empty (when $\mathcal{A}$ is not stable), or is the singleton $\{a\}$ (when $\mathcal{A}$ is stable and $\mathcal{A}\simeq\mathcal{A}'$), or coincides with the two-point set $\{a, a'\}$.

Our structure theorem for the moduli space $\mathcal{M}^{\rm st}$ states

Theorem 3.3 *Let X be a minimal class VII surface with* $1 \le b_2(X) \le 3$*, and let g be a Gauduchon metric on X with* $\deg_g(\mathcal{K}_X) < 0$*. For a generic perturbation of g, the moduli space* $\mathcal{M}^{\rm st}$ *decomposes as*

$$\mathcal{M}^{\rm st} = \left(\bigcup_{\emptyset\ne I\subset\mathfrak{I}} \mathcal{M}_I^{\rm st}\right)\bigcup\left(\bigcup_{0\le d\le b} Y_d\right), \tag{4}$$

where $Y_d \subset \mathcal{M}^{\rm st}$ *is a compact analytic subset of pure dimension d, such that:*

(1) Y_0 *is a set of twisted reductions in the moduli space, hence it is finite.*
(2) For $0 < d \le b$*, the intersection* $Y_d \bigcap \left(\bigcup_{\emptyset\ne I\subset\mathfrak{I}} \mathcal{M}_I^{\rm st}\right)$ *is Zariski closed in* Y_d*.*

The cases $b_2(X) \in \{1, 2\}$ are implicitly treated in [26] and [29]. For the new case $b_2(X) = 3$ we will explain the main steps of the proof (see [32] for details). As explained above, we can assume that X is not an Enoki surface. The proof is obtained in the following steps:

Step 1. All circles of reductions $\mathcal{C}_\lambda$, $\lambda \in \mathfrak{P}$ belong to the same connected component $\mathcal{M}_0^{\rm pst}$ of $\mathcal{M}^{\rm pst}$. The proof makes use of:

(1) The model theorem for the moduli space $\mathcal{M}^{\rm pst}$ around the circles of reductions (see Sect. 2), and the explicit computation of the restriction of the Donaldson cohomology classes to the boundary $\partial\mathcal{N}_\lambda$ of a tubular neighbourhood of a circle of reductions (see [28, 31]).

(2) A cobordism argument.

Since $\mathcal{M}_0^{\rm pst}$ contains all reductions, its complement $\mathcal{M}^{\rm pst} \setminus \mathcal{M}_0^{\rm pst}$ (which is a union of connected components) will be a smooth, compact threefold. Put $Y_3 := \mathcal{M}^{\rm pst} \setminus \mathcal{M}_0^{\rm pst}$.

Step 2. Using the result given by Step 1, we see that $\mathcal{M}_0^{\rm st} := \mathcal{M}_0^{\rm pst} \setminus \mathcal{R}$ is a connected component of $\mathcal{M}^{\rm st}$ which contains all the sets $\mathcal{M}_I^{\rm st}$ with $\emptyset \neq I \subset \mathfrak{I}$. One has:

(1) $\{\Pi^{e_{\mathfrak{I}}}_{>\mathfrak{k}}\}^{\rm st} = \Pi^{e_{\mathfrak{I}}}_{>\mathfrak{k}}$, and the map $\phi^{e_{\mathfrak{I}}} : \Pi^{e_{\mathfrak{I}}}_{>\mathfrak{k}} \to \mathcal{M}_0^{\rm st}$ is an open embedding whose image is $\mathcal{M}_{\mathfrak{I}}^{\rm st}$.
(2) The complement $\mathcal{Z} := \mathcal{M}_0^{\rm st} \setminus \mathcal{M}_{\mathfrak{I}}^{\rm st}$ is a pure 2-dimensional analytic subset of $\mathcal{M}_0^{\rm st}$, so it is a hypersurface of this threefold.

Denote by $\mathcal{P}^k(\mathfrak{I}) \subset \mathcal{P}(\mathfrak{I})$ the set of subsets of $\mathfrak{I}$ of cardinality k.

Step 3. For every $I \in \mathcal{P}^2(\mathfrak{I})$, the subset $\mathcal{M}_I^{\rm st}$ is contained in an irreducible component $\mathcal{Z}_I$ of $\mathcal{Z}$. This irreducible component coincides with the closure, and also with the Zariski closure, of $\mathcal{M}_I^{\rm st}$ in $\mathcal{M}^{\rm st}$. Let Y_2 be the union of the irreducible components of $\mathcal{Z}$ other than $\mathcal{Z}_{\{1,2\}}$, $\mathcal{Z}_{\{2,3\}}$, $\mathcal{Z}_{\{1,3\}}$.

Step 4. There exists a bijection $\mathcal{P}^2(\mathfrak{I}) \to \mathcal{P}^1(\mathfrak{I})$, denoted $I \mapsto I'$ such that for any $I \in \mathcal{P}^2(\mathfrak{I})$ the hypersurface $\mathcal{Z}_I$ contains $\mathcal{M}_{I'}^{\rm st}$. Moreover, for any $I \in \mathcal{P}^2(\mathfrak{I})$ the following holds:

(1) The closure $\mathcal{Z}_{I'}$ of $\mathcal{M}_{I'}^{\rm st}$ in $\mathcal{M}^{\rm st}$ is an irreducible, pure 1-dimensional analytic subset which is contained in $\mathcal{Z}_I$.
(2) $\mathcal{Z}_I$ decomposes as a union

$$\mathcal{Z}_I = \mathcal{M}_I^{\rm st} \cup \mathcal{Z}_{I'} \cup \Sigma_{\bar{I}'} \cup Y_1^I,$$

where $\Sigma_{\bar{I}'}$ is an irreducible 1-dimensional analytic subset of $\mathcal{Z}_{\bar{I}'}$, and Y_1^I is a pure 1-dimensional analytic subset of $\mathcal{Z}_I$, which does not contain $\mathcal{Z}_{I'}$ or $\Sigma_{\bar{I}'}$ as an irreducible component.

Put

$$Y_1 := \bigcup_{I \in \mathcal{P}^2(\mathfrak{I})} Y_1^I.$$

Step 5. For any $J \in \mathcal{P}^1(\mathfrak{I})$ the complement of $\mathcal{M}_J^{\rm st}$ in $\mathcal{Z}_J$ is either empty, or the singleton formed by a twisted reduction. Denote by Y_0 the subset of twisted reductions in $\mathcal{M}^{\rm st}$.

Step 6. Replacing the metric g by a generic perturbation of it, the analytic subsets Y_1, Y_2 obtained above will be compact.

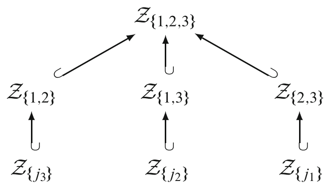
Step 7. For any compact analytic subset $Y \subset \mathcal{M}^{\rm st}$, the intersection

$$Y \bigcap \left(\bigcup_{\emptyset \neq I \subset \mathfrak{I}} \mathcal{M}_I^{\rm st} \right)$$

is Zariski closed in Y.

The subspaces Y_d $(0 \leq d \leq 3)$ obtained in this way will fulfil the conditions claimed in the conclusion of the theorem. ■

Putting $j_i := (\mathfrak{I} \setminus \{ i\})'$ for any $i \in \mathfrak{I}$, we see that the proof of Theorem 3.3 gives the following incidence relations between the closures $\mathcal{Z}_I := \bar{\mathcal{M}}_I^{\mathrm{st}}$:



We can now come back to the proof of Theorem 2.6:

Proof (of Theorem 2.6) If $\mathcal{A}$ is stable, then its isomorphism class a is a point of $\mathcal{M}^{\mathrm{st}}$. Suppose that $\mathcal{A}$ cannot be written as an extension of the form (1) with $c_1(\mathcal{L}) \neq 0$, and does not coincide with a twisted reduction. Taking into account the decomposition (4) given by Theorem 3.3, there exists $d > 0$ such that $a \in Y_d$. Put $Y := Y_d$, and

$$Y' := \begin{cases} Y \setminus \left(\bigcup_{\emptyset \neq I \subset \mathfrak{I}} \mathcal{M}_I^{\mathrm{st}} \right) & \text{if } a' = a \\ Y \setminus \left(\bigcup_{\emptyset \neq I \subset \mathfrak{I}} \mathcal{M}_I^{\mathrm{st}} \cup \{a'\} \right) & \text{if } a' \neq a \end{cases} .$$

Using Theorem 3.3 (3.3) it follows that Y' is a Zariski open neighbourhood of a in Y. On the other hand formula (3) and Remark 3.2 show that Y' does not intersect the filtrable locus $\Phi \subset \mathcal{M}^{\mathrm{st}}$. ■

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Erratum to: On Bi-Hermitian Surfaces

M. Pontecorvo

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The author found an error in the original version of the chapter. This error affected also his contribution to the book and hence the author wanted to publish the below disclaimer by this erratum.

Disclaimer

We found a gap in the proof of Theorem 3.4, as a result we do not know if the statement is correct.

The problem with the proof is that when S_u is a small deformation of a hyperbolic Inoue surface S_0 the inequality $H^0(S_u, K^{-1} \bigotimes L^v) \neq 0$ certainly holds but *deg* $L^v > 0$ by [18, 4.21, 4.22] and therefore Theorem 3.2 does not apply.

The updated online version of this chapter can be found at
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