German-Speaking Traditions in Mathematics Education Research

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Abstract This paper describes and analyzes developments that have taken place in German mathematics education research during the last 40 years. Which developments and ideas were characteristic for the discussion and how was Germany influenced by and how did it interact with the international community? The themes range from subject matter didactics to large-scale studies.

Keywords Subject-matter didactics · Design science · Modelling · *Allgemeinbildung* · Theory · Classroom studies · Educational research · Large-scale studies

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Introduction

First, what do we mean by *traditions* in this paper? As most readers know, in 1976, ICME-3 took place in Germany in the city of Karlsruhe, and ICME returned to Germany exactly 40 years later. Thus, it is quite natural to ask which developments have taken place in German mathematics education research during these 40 years, which developments and ideas were characteristic, which people proved to be influential, and how was Germany influenced by and how did it interact with the international community. Thus, the present paper will be confined to this period. However, since there was a great period of educational thinking in the 19th century, it will also digress a bit into the era of W. v. Humboldt from around 1800.

"German speaking" encompasses more than just Germany. Austria and Switzerland belong to the family, and the former German Democratic Republic (GDR) has its own traditions that are still influential. In preparing this event, the authors discussed these problems seriously. We felt that we should limit ourselves and confine the paper to Germany, with small references to Austria and the former GDR.

The paper splits German mathematics education research into eight sub-themes ranging from subject-matter didactics to large-scale studies without any claims that these sub-themes exhaust the whole field.

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Subject-Matter Didactics (German: *Stoffdidaktik*)

In the development of the didactics of mathematics as a professional field in Germany, subject-related approaches played an important role. Felix Klein created a model that has been referred to for a long time. A general goal was to develop approaches for representing mathematical concepts and knowledge in a way that corresponded to the cognitive abilities and personal experiences of the students while simultaneously simplifying the material without disturbing the mathematical substance. A fundamental claim was that such simplifications should be "intellectually honest" and "upwardly compatible" (Kirsch, 1977). Concepts and explanations should be taught to students with sufficient mathematical rigor in a manner that connects with and expands their knowledge of the subject. For this reason, subject-matter didactics placed value on constructing viable and robust mental representations (*Grundvorstellungen*) to capture mathematical concepts and procedures as they are represented in the mental realm. In the 80s, views of the nature of learning as well as objects and methods of research in mathematics education changed and the perspective was widened and opened towards new directions and gave more attention to the learners' perspective. This shift of view issued new challenges to subject-related considerations that have been enhanced by the recent discussion about professional mathematical knowledge for teaching.

The session started with an overview lecture on the main issues of subject-matter didactics given by Lisa Hefendehl-Hebeker and Rudolf vom Hofe entitled, "Subject-matter didactics: Overview of origin, main issues, theory, methods, and fields of application." Subsequent presentations concentrated on two paramount concepts of subject-matter didactics that can serve as guiding orientations in a local and global sense to present mathematical knowledge corresponding to the overarching goals.

The concept of *Grundvorstellungen*, which can be roughly translated as "basic mental models," describes relationships between mathematical content and the phenomenon of individual concept formation. For example, the actions of distributing and measuring provide basic mental models for the operation of division within the domain of natural numbers (partitive and quotitive basic model). Sebastian Wartha and Axel Schulz unfolded this concept in the context of natural numbers and fractions: "Numbers, fractions, operations and representations: *Grundvorstellungen* in primary school."

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The tension between *clarity* and *rigour* in calculus has been a main theme in the German tradition of subject-matter didactics and still is an actual problem field, especially in upper secondary school teaching. Blum and Kirsch (1991) suggested more intuitive approaches (at least for basic courses) with the original naïve ideas of function and limit and sequential steps of exactitude, which could be achieved according to the capacity of the learners. In reference to this discussion, Andreas Büchter and Hans Humenberger gave a presentation entitled, “Clarity and rigour in calculus courses.”

Design Science

Within the German-speaking tradition, considering mathematics education as a design science primarily draws on the work of Wittmann. He underlined the role of substantial learning environments while elaborating on how mathematics education can be established as a scientific field in its own right. From their very nature, substantial learning environments contain substantial mathematical content, even beyond school level, and also offer rich mathematical activities for (pre-service) teachers on a higher level. Exploring the epistemological structure reflected in such learning environments or reflecting didactical principles while testing the learning environments in practice adds to a deeper understanding of both the mathematics involved and students’ learning processes.

The main objective of design science has been developing feasible designs for conceptual and practical innovations, involving the teachers (and educators as well) actively in any design process, for example, designing teaching concepts and learning units, tasks, examples and materials for different lessons, curricula, assessments, and programs for teacher education. In this sense, the development of substantial learning environments can be seen from a twofold perspective: First, designing such learning environments should be based on substantial mathematics, meaning that students can be immersed in mathematical processes such as mathematizing, exploring, reasoning, and communicating. Second, investigating substantial learning environments should be the essential starting point of mathematics education research. In collective teaching experiments, the research focus lies on the induced learning processes and children’s thinking as well as on the mathematical communication in the classroom. By working together with teachers in schools, the researchers reflect the effects of the designed substantial learning environments. However, researchers are not the only ones who analyze empirical data: Teachers also collect and reflect on their own empirical data and use it to improve their teaching. Bringing these two intentions together allows bridging of theory and practice in mathematical research.

From a broader perspective, the design science approach has played a distinctive role within prominent European traditions concerned with designing and evaluating learning material and processes (such as Realistic Mathematics Education in the Netherlands or the theory of didactical situations in France, for example). On the one hand, different conceptualizations for designing learning environments for students (or teachers) have developed in light of the didactical traditions of each country. On the other hand, these conceptualizations reflect the different theories involved that connect design and research, and balance theory and practice effectively. Nowadays, the variety of approaches used by researchers and teachers to work together collaboratively to promote mathematical learning and to develop substantial learning environments indicate the progress of design science and give insights into different ways of connecting design and empirical research. The following presentations were given: Marcus Nührenbörger and Bettina Rösken-Winter, "Mathematics education as a 'design science': Where did we start?" Susanne Prediger and Paul Cobb (USA); "Trends and developments: German trends in design science and design research at the system level"; Michael Link (Switzerland), Ralph Schwarzkopf, Anna S. Steinweg, and Chun Ip Fung (China), "Designing and researching substantial learning environments: Four examples of design experiments"; Erich Ch. Wittmann, "Design science revisited: Where are we now?"

Modelling

German work on modelling in mathematical education started in the 80s. In his talk about "Mathematical modelling in German-speaking countries: Introduction and overview," Gilbert Greefrath outlined the German discussion of mathematical modelling by presenting definitions, pedagogical aims, typical modelling cycles, and key examples of the German debate on mathematical modelling. In addition, he gave an overview of central pragmatic and specific approaches and addressed current development in research, educational standards, modelling competencies, comparative studies, and final exams. He also discussed the role of technology in mathematical modelling (see Greefrath & Vorhölter, 2016).

Afterwards, four important aspects of the German modelling discussion of the last decades were deepened, subdivided into two parts: Cognitive and empirical approaches and promoting modelling competencies.

In the first part, Rita Borromeo Ferri took a cognitive approach in her presentation on "Classification of modelling cycles: An insight into cognitive processes." In this presentation, she gave a classification of modelling cycles that focused on how these give a better insight into the cognitive processes of learners when solving modelling problems (Borromeo Ferri, 2006). In a short overview, she showed how this knowledge has been used for empirical and theoretical research in the German modelling debate. Focusing on "Quantitative research on modelling: Examples from German-speaking countries," Dominik Leiss and Stanislaw Schukajlow-Wasjutinski gave an overview of some empirical approaches in the German modelling discussion. They presented current research projects on mathematical modelling that are meeting the challenge of going beyond case studies to increase the external validity of their results. In the presented studies, a wide spectrum of quantitative research methods ranging from correlative analyses to mediation analyses were used. They reported on findings from studies conducted in German-speaking countries on the role of quantitative research methods while searching for the "best" learning environment for teaching modelling in a regular classroom (see Schukajlow et al., 2012).

In the second part, the promotion of modelling competencies in German schools was addressed in two different ways. Katja Maaß focused in her talk on "Mathematical modelling in professional development: Traditions in Germany on professional development courses," addressing topics such as differentiation and assessment when modelling. Based on expert interviews and a desktop analysis, she outlined the most important milestones, thereby showcasing important steps which might be useful for other countries as well (see Maaß & Mischo, 2011). In addition, Katrin Vorhölter gave an overview on "Implementing mathematical modelling in schools" by presenting several projects of the last two decades aiming at the implementation of modelling in Germany. Two kinds of implementation were distinguished: One the one hand, teaching units for promoting modelling competencies during mathematics lessons were presented whose implementations are often accompanied by research. On the other hand, so-called modelling days and weeks, highly requested by teachers but not often systematically researched, were introduced (see Greefrath & Vorhölter, 2016).

Finally, Gloria Stillman gave an "International perspective on the German modelling debate." She pointed out that the German debate has strong historical roots but also shows a healthy vibrancy where new people are continually coming into the field and the field is expanding and broadening in views and its research base.

Allgemeinbildung and Mathematical Literacy

In Germany, the idea that mathematics should be a constitutive component for the cultivation of human beings and, thus, an indispensable part of *Allgemeinbildung* dates back to Wilhelm von Humboldt in the beginning of the 19th century. This constituted a tradition of pedagogical thinking that is still influential in modern times.

In his talk, "Mathematics and *Allgemeinbildung* in the time of W. v. Humboldt," Hans Niels Jahnke showed that during Humboldt's time, the German view on mathematics put an emphasis on its cultural meaning. Humboldt's opinions on mathematics were dominated by a pronounced anti-utilitarianism, a preference for pure mathematics, an affinity between mathematics and aesthetics, and a high esteem of rigorous thinking. The education of the individual should not be regulated by demands from outside and future professional life. Rather, the aims of education should be defined in terms of an individual's needs for self-development. The strong emphasis of the reformers around Humboldt on theoretical thinking and pure science was in their eyes not a denial of the demands of practical life, but the best way to meet these demands. According to them, theoretical thinking is a necessary condition for change. To educate young people in theoretical thinking is the best way to make them 'apt for the future' The second part of the talk showed how the basic ideas of this approach can be identified in modern papers on *Allgemeinbildung* and mathematics by H. Winter.

In his reaction entitled "*Bildung, Paideia,* and some undergraduate programs manifesting them," Michael F. Fried discussed how similar notions are enshrined in the idea of paideia and the classical concept of the liberal arts. He showed that such ideas also work in modern times by hinting at the examples of prominent colleges in North America.

In his talk on *"Allgemeinbildung,* mathematical literacy, and competence orientation," Rolf Biehler gave a sketch of the discussion on *Allgemeinbildung* and mathematical literacy in Germany from the late 60s to today. In terms of mathematics, *Allgemeinbildung* was related to those components of mathematics that are considered to be relevant to the general public. In the 70s, educational goals for *Allgemeinbildung* were condensed in different visions of, for example, a 'scientifically educated human being,' a 'reflected citizen,' an 'emancipated individual being able to critique society,' and a person 'well educated for the needs of the economic system.' Among others, these ideas led to the first approaches of critical mathematics education (Christine Keitel and colleagues, see Damerow et al., 1974). In 1995, Hans-Werner Heymann, "Why teach mathematics," related the discussion of *Allgemeinbildung* in the educational sciences to mathematics education, developing a system of justifications about why mathematics should be taught (Heymann, 2003; see also Biehler, Heymann, & Winkelmann, 1995). Contrary to Heymann's intentions, the public reception of this book focused narrowly on one aspect, namely, that seven years of mathematics would be enough if mathematics education were only devoted to immediate everyday applications. Due to bad results in TIMSS and PISA starting in the late 90s, a new discussion on educational goals in mathematics arose. PISA's conception of mathematical literacy was extended by ideas from the German debate (Humboldt, Freudenthal, Winter, and Heymann) and a new notion of *Mathematische Grundbildung* emerged (Neubrand, 2003) that very much influenced the new national standards in mathematics in Germany (2003, 2012). Last but not least, the challenge of mathematics education given the heterogeneity of students was thematized with some advanced and basic examples stemming from statistical literacy education (http://www.procivicstat.org).

In his reaction, Mogens Niss from Roskilde University, Denmark, related German development to the international development on competence orientation (featuring the KOM project), including the various conceptualizations in the various PISA frameworks (Niss & Højgaard, 2011).

Theory Traditions in German-Speaking Countries

In the 70s and 80s, teacher education was established at universities, and scientific media and a scientific society in mathematics education were founded in the German-speaking countries. This raised the issue of how to develop mathematics education a scientific discipline. At about the same time, the question of how far the didactics of mathematics already had developed as a scientific discipline was intensively discussed. Referring to Kuhn and Masterman, Burscheid (1983) used a four-stage model to identify the developmental stage of the scientific discipline. Critical reactions from Steiner (1983) and Fischer (1983) required more focus on the needs of mathematics education itself. Since the development of any scientific discipline is deeply intertwined with its theoretical work, there was a need to clarify what kinds of theories were adequate for the discipline. This was done by Jahnke (1978) and Bigalke (1984), who proposed the Sneed and Stegmüller's concept a suitable theory concept for the field:

> A theory in mathematics education is a structured entity shaped by propositions, values, and norms about learning mathematics. It consists of a kernel that encompasses the unimpeachable foundations and norms of the theory and an empirical component that contains all possible expansions of the kernel and all intended applications that arise from the kernel and its expansions. (Bigalke, 1984, p. 152, translated, ABB)

In 1984, Hans-Georg Steiner inaugurated a series of five international conferences on Theories of Mathematics Education (TME), pursuing a scientific program that aimed at founding and developing the didactics of mathematics as a scientific discipline on the international level. His program addressed three partly overlapping areas:

> (1) Identification and elaboration of basic problems in the orientation, foundation, methodology, and organization of mathematics education as a discipline; (2) the development of a comprehensive approach to mathematics education in its totality when viewed as an interactive system comprising research, development, and practice; and (3) self-referent research and meta-research related to mathematics education that provides information about the state of the art—the situation, problems, and needs of the discipline—while respecting national and regional differences. (Steiner, 1987, p. 46)

The spirit of TME has been renewed today by the more bottom-up meta-theoretical approach of the networking of theories exploring how research with multiple theories can be conducted (specifically when they have emerged within specific educational systems), where the limits are, and how far new insights can be gained. Addressing networking strategies, this approach takes up the principle of complementarity, which Steiner (1987) worked out in the TME program (ibid., p. 48), being open for the theoretical diversity of the field.

In the 1990s, the research field in German-speaking countries began to investigate various methodologies based on a growing diversity in theory use. As examples, two theory traditions were presented in the session at ICME-13. Building on views of Peirce and Wittgenstein, Dörfler (2016) outlined a semiotic perspective on mathematics as an activity of diagrammatic reasoning and related to it as sign games and their techniques deeply involving rules for acting. Regina Bruder & Schmitt (2016) complemented this more home-grown theoretical view with the theory of learning activity that was developed based on activity theory by Hans Joachim Lompscher to inform the practice of teaching and learning in a school in the GDR. Bruder took up Lompscher's work and adapted it to the needs of teaching and learning mathematics. By applying the two theoretical views, mathematics as diagrammatic reasoning and learning activitiy, to the same data set in her presentation, Bikner-Ahsbahs (2016) readdressed Steiner's concern about complementarity by analyzing the data on the basis of the networking of theories; she wrote:

> Both approaches may enrich each other to inform practice (see TME program): coming from the learning activity we may zoom into (see Jungwirth 2009 cited by Prediger et al. 2009, p. 1532) diagram use, and coming from diagram use we may zoom out (ibid., p. 1532) to embed the diagram use into the whole course of the learning activity. (Bikner-Ahsbahs, 2016, p. 41)

Sociological Perspectives on Classroom Interaction

The specific aspects of sociological perspectives on classroom interaction, as a focus within the German-speaking traditions in mathematics education research, rest on a fundamental sociological orientation in mathematics lessons. This orientation has its origin in the works of Heinrich Bauersfeld and his colleagues at the Institut für Didaktik der Mathematik (IDM) at Bielefeld (Bauersfeld, 1980; Krummheuer & Voigt, 1991). These early studies unfolded the power of sociological description by reconstructing social processes regarding the negotiation of meaning and the social constitution of shared knowledge through collective argumentation in the daily practice of mathematic lessons. The "social" in these interactionist studies of mathematics classroom micro-culture was firmly located in the interpersonal space of those who interact. This space was considered a contingent sphere in which mathematical meaning emerges as the product of processes of negotiation. With respect to the sociological reference theories (primarily) of symbolic interactionism and ethnomethodology, a microsociology of mathematics lessons was created and elaborated. This theoretical approach to the mathematics classroom was based on three assumptions: (1) The mathematics that students learn and the conditions of the learning process are partly open to a process of negotiation of meaning in which the learners and the teacher(s) interactively exchange their definitions of the learning situation. (2) A process of collective argumentation concerning the mathematical content (concepts, terms, procedures, algorithms, etc.) is a constitutive social condition of the possibility of learning of this content. Participation in this process, albeit in different forms, is necessary for success in school mathematics. (3) Increased autonomous participation in such collective argumentation is the indication of successful learning in the mathematics classroom. The results of the empirically based development of a theory of learning in mathematics classrooms show that interaction in mathematics classes occurs in patterns of interaction in which the mathematical content is relevant. The patterns that support learning of mathematics are formats of collective argumentation. By increasing their autonomous participation in formats of argumentation, the learners take part in a process of development towards a full participation in school mathematics practice (Krummheuer, 2007).

The focus on a sociological theory of learning mathematics has been taken up and complemented by other sociological perspectives that have aimed at reconstructing the conditions and the structure surrounding the construction of performance and success in mathematics lessons. From these perspectives, schools and classrooms are not only considered places in which the learning of mathematics occurs, but also as institutional loci in which further societal functions of schooling, such as cultural reproduction and allocation, need to be pursued in parallel. At stake in mathematical activities in the classroom is not only the development of students' knowledge and skills, but also the creation of hierarchies of achievement in mathematics, of differential access to valued forms of mathematics, and of familiarization with work ethic (Gellert, 2008). From such a point of view, issues such as the distribution of knowledge, access, and students' resources are crucial ingredients to the forms the interaction in the mathematics classrooms may take.

During the session on sociological perspectives on classroom interaction, Götz Krummheuer's introductory talk, "Interpretative classroom research: Origins, insights, developments," summarized the development of a theory of learning mathematics. Two reactions to the presentation were prepared by Núria Planas (Spain) and Michelle Stephan (USA). The sociological zoom was then expanded by Uwe Gellert's presentation of "Classroom research as part of the social-political agenda" and of studies of German scholars concerning this matter. A prepared reaction by Eva Jablonka (UK) finalized the program.

Educational Research on Learning and Teaching of Mathematics

Educational research aims at generating knowledge on teaching and learning mathematics. To achieve this goal in the complex research domain, many empirical studies triangulate data, methods, investigators, and theory. This is reflected in the following strategies:

(1) a narrow focus on distinct phenomena concerning learning and teaching mathematics,
(2) an interdisciplinary perspective that integrates different background theories, and
(3) a mixed-method approach that combines different methodological practices.

As early as the 80s, mathematics education was already actively using and developing such research strategies, e.g., Ursula Viet's investigation of the cognitive development of fifth- and sixth-grade students in arithmetic and geometry. The benefits and limitations of such approaches for mathematics education were discussed by considering recent research projects from the last two decades.

In the first part of the presentation, Timo Leuders focused on two interdisciplinary research projects: Between 2000 and 2006, the priority program Educational Quality of Schools initiated more than 30 interdisciplinary cooperations to analyze domain-specific and cross-curricular learning. One of these projects was a multi-step research project by Regina Bruder (mathematics education) and Bernhard Schmitz (educational psychology) that investigated the problem-solving and self-regulatory behavior of students, also connecting the two perspectives theoretically. In another project within the program, Alexander Renkl (educational psychology) and Kristina Reiss (mathematics education) investigated how students' proof competence can be fostered by learning with worked-out examples and self-explanation prompts. For both projects in the presentation, the researchers reported in video interviews on the experiences, advantages, and challenges of their interdisciplinary approach.

The second part of the presentation about flexible mixed-methods approaches by Andreas Schulz started with a complementary perspective on qualitative and quantitative research. He showed that both make use of inductive as well as deductive reasoning and that both can complement and compensate for their strengths and weaknesses within a mixed-methods design. This was illustrated by a video and audio presentation and discussion of two such approaches: Kathleen Philipp made use of a sequential mixed-methods design. She analyzed students' strategies during solving several mathematical problems and developed a competence model about experimental thinking in mathematics. This laid the groundwork for an intervention study that confirmed that experimental competences in mathematics can be fostered effectively. Susanne Prediger and Lena Wessel implemented an integrated/parallel mixed-methods design. They fostered students' understanding of fractions and scaffolded the learning processes by fostering students' abilities to talk about fractions and their meaning. The effectivity of the randomized control study was evaluated by both statistical analyses and qualitative analyses of the teaching-learning processes.

In the international commentary that followed, Kaye Stacey confirmed and illustrated the need for a flexible combination of quantitative and qualitative research to generate both meaningful and reliable evidence for the understanding of learning and teaching in the field of mathematics education. This lead to an engaged discussion with the international audience about the potential and challenges of interdisciplinary research and mixed-methods approaches in mathematics education.

Large-Scale Studies

Large-scale studies assess mathematical competence using large samples. They often compare mathematical competence between groups of individuals within or between countries. The development of sophisticated statistical methods in recent years has encouraged collaborations between researchers from mathematics education on the one hand and from statistics or psychology on the other. This development has also allowed the empirical verification of theoretical models of mathematical competence and competence development.

In Germany, international large-scale studies did not receive much attention before 1995, when Germany took part in the Third International Mathematics and Science Studies (TIMSS) for the first time. The results showed that German lower and upper secondary school students' mathematical performance did not meet the expectations of teachers, educators, and the public. German students performed below the international average and showed acceptable results only for routine problems (Baumert, Bos, & Lehmann, 2000). The results of PISA 2000 (Baumert et al., 2001) were again disappointing and became known as the "PISA shock." The consequences of these studies were intensive debates among educators and stakeholders and the launch of educational programs to improve mathematics instruction at school. Another consequence was the agreement to use large-scale assessments on a regular basis to monitor the outcome of school education.

Assessing students' mathematical competences requires models of what mathematical competence actually is. Initial models were predominantly based on theoretical and normative considerations, but rarely on empirical evidence. In a recursive process, Reiss and colleagues (e.g., Reiss, Heinze, Kessler, Rudolph-Albert, & Renkl, 2007; Reiss, Roppelt, Haag, Pant, & Köller, 2012) developed a model for primary mathematics education that took into account theoretical and normative perspectives and was continuously refined based on empirical evidence. The model suggests five levels of mathematical competence reaching from technical background knowledge and routine procedures to complex mathematical modelling.

To monitor the outcome of educational quality on a regular basis, new institutions have been founded in Germany, such as the Institute for Educational Quality Improvement (IQB, Berlin) and the Center for International Student Assessment (ZIB, Munich). However, the idea of system monitoring is not specific to Germany. Other countries founded similar institutions and developed similar models of mathematical competence to assess students' competences on a regular basis. In Austria, for example, the Federal Institute for Educational Research, Innovation, and Development of the Austrian School System (BIFIE) is responsible for assessments. These assessments are based on a model of mathematical competence that describes mathematical competence in three dimensions (process domain, content domain, and level of complexity). This model is not only used for assessment purposes but is also the basis for developing curricula for the mathematics classroom.

Large-scale studies allow monitoring of the outcome of mathematics education on the system level. The broad empirical data these studies collect have been used to empirically validate theoretical models of mathematical competence and have contributed to a more realistic view of what students are capable of learning at school.

Final Remark

Looking back at the eight themes above, the reader will realize the profound changes that have taken place in German-speaking mathematics education research during the last 40 years. The development comes near to a sort of revolution—not very typical for Germany. The only themes that could have appeared in the program of the Karlsruhe Congress in 1976 are subject-matter didactics and, with qualifications, design science and *Allgemeinbildung*. All other topics, especially modelling, theory traditions, classroom studies, and empirical research represent for Germany completely new fields of activity. Today, they define the stage on which German mathematics educators have to act. Nevertheless, the more traditional fields that are nearer to mathematics, subject matter analysis and elementarization, are still alive and will continue to be areas of intense work so that the common ground of mathematics and education will not be lost.

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