Outline of a Generalization of Kinetic Theory to Study Opinion Dynamics

Stefania Monica and Federico Bergenti

Dipartimento di Scienze Matematiche, Fisiche e Informatiche
Università degli Studi di Parma
Parco Area delle Scienze 53/A, 43124 Parma, Italy
{stefania.monica,federico.bergenti}@unipr.it *

Abstract. This paper presents an analytic framework to study the dynamics of the opinion in multi-agent systems. In the proposed framework, each agent is associated with an attribute which represents its opinion, and the opinion of an agent changes because of interactions with other agents, without supervised coordination. Each interaction involves only two agents, and it corresponds to an exchange of messages. The framework assumes that time is modeled as a sequence of discrete steps, which do not necessarily have the same duration, and that at each step two random agents interact. Various sociological phenomena can be incorporated in the proposed framework, and the framework allows studying macroscopic properties of a system starting from microscopic models of such phenomena, obtaining analytic results. In detail, the proposed framework is inspired by the kinetic theory of gas mixtures, which relies on the use of balance equations that can be properly adopted to study opinion dynamics in a multi-agent system. After a short introduction on the kinetic theory of gas mixtures, this paper shows how the main ideas behind it can be generalized to study the dynamics of the opinion in multi-agent systems starting from a detailed description of microscopic interactions among agents.

1 Introduction

In this paper, we outline the presentation of an analytical framework to study collective behaviors in multi-agent systems without supervised coordination. According to the proposed framework, observable macroscopic properties of a system can be analytically derived, under proper assumptions, from the description of the effects of microscopic interactions among agents. The term *interaction* is frequently used throughout the paper to denote a message exchange among two agents, and each interaction corresponds to a single time step. Hence, time is modeled as a sequence of discrete steps, which may not have the same duration, and each step corresponds to a single interaction among a randomly chosen pair of agents. We assume that each agent is associated with a scalar attribute, and

* The work presented in this paper is partially supported by Gruppo Nazionale per il Calcolo Scientifico (GNCS).

S. Omatu et al. (eds.), *Distributed Computing and Artificial Intelligence, 14th International Conference*, Advances in Intelligent Systems and Computing 620, DOI 10.1007/978-3-319-62410-5_37

since the scenarios considered in this paper refer to the study of opinion dynamics, we assume that such an attribute represents an agent's opinion. Most of the existing agent-based models used to study opinion dynamics are based on simulations and, hence, the validity of obtained results depends on the specific type of system that is simulated, and on the actual values of the parameters of simulations. At the opposite, the framework outlined in this paper adopts an analytic point of view, thus leading to results which are valid regardless of the details of simulations and of the number of agents in the studied multi-agent system, provided that the hypotheses used to derive the analytic results remain valid. Notably, the ideas behind the proposed framework are not limited to the study of opinion dynamics and they could be reworked to describe other attributes and other collective behaviours.

In detail, the analytic framework described in this paper to study opinion dynamics is inspired by the classic models that physics uses to study gases. This idea is not new, and other models used to study phenomena like opinion dynamics are derived from physics. For instance, it is worth mentioning models based on statistical mechanics [1] and on Brownian motion [2]. Besides these, other models of opinion dynamics are closely related to the proposed framework (see, e.g., [3]) because they are also inspired from the kinetic theory of gases, a branch of physics according to which macroscopic properties of gases can be explained starting from the details of microscopic interactions among molecules. The similarities between the kinetic theory of gases and the study of opinion dynamics are evident because the study of opinion dynamics typically starts from the description of microscopic interactions among agents and aims at deriving observable properties of the system concerning, e.g., the temporal evolution of the average opinion [4, 5]. Due to such similarities, a simple parallelism between the molecules in gases and the agents in a multi-agent system can be drawn. This parallelism supports the derivation of kinetic-inspired analytical models to study opinion dynamics in multi-agent systems. Observe that classic kinetic theory of gases assumes that all molecules are equal. Hence, when generalizing the ideas of classic kinetic theory to study opinion dynamics, it is not possible to account for agents with different characteristics. In order to describe multi-agent systems in which agents may have different characteristics, the framework outlined in this paper extends those inspired by classic kinetic theory of gases. More precisely, instead of considering classic kinetic theory of gases, the discussed framework starts from kinetic theory of gas mixtures, which accounts for gases composed of different types of molecules. This allows modelling multi-agent systems composed of agents with different characteristics, thus adopting a parallelism between different types of molecules and different classes of agents.

Note that only some results of kinetic theory of gas mixtures are directly applicable to the study of opinion dynamics because the details of collisions among molecules intrinsically differ from those of interactions among agents. This is evident when comparing the interaction rules used to model collisions among molecules with the various interaction rules that can be considered in the study of opinion dynamics to account for various sociological phenomena occurring in the interactions among agents. The following list enumerates, using the nomenclature proposed by sociologists, some of the most common sociological phenomena that can be accommodated in the analytic framework outlined in this paper by means of specific interaction rules [6–8]:

- *Compromise:* the tendency of agents to move their opinions towards those of agents they interact with, trying to reach consensus [9];
- *Diffusion:* the phenomenon according to which the opinion of each agent can be influenced by the social context [10];
- *Homophily:* the process according to which agents interact only with those with similar opinions [11];
- *Negative Influence:* the idea according to which agents evaluate their peers, and they only interact with those with positive scores [12];
- *Opinion Noise:* the process according to which a random additive variable may lead to arbitrary opinion changes with small probability [13]; and
- *Striving for Uniqueness:* the process based on the idea that agents want to distinguish from others and, hence, they decide to change their opinions if too many agents share the same opinion [14].

This paper is organized as follows. Section 2 outlines the basic nomenclature of the kinetic theory of gas mixtures. Section 3 presents the proposed framework using the nomenclature of kinetic theory of gas mixtures, and briefly discusses analytic results obtained considering only compromise. Finally, Section 4 concludes the paper and outlines ongoing and future work.

2 Overview of Kinetic Theory of Gas Mixtures

This section outlines the main ideas of the kinetic theory of gas mixtures, a branch of physics which was introduced to to allow modelling gases composed of different kinds of molecules, typically called *species*. Let $M \geq 1$ be the number of species in the considered gas. Then, M distribution functions $\{f_s(\underline{x}, \underline{v}, t)\}_{s=1}^{M}$, representing the density of molecules of species s at position $\underline{x} \in \mathbb{R}^3$ with velocity $\underline{v} \in \mathbb{R}^3$ at time $t \geq 0$, need to be considered, and a proper balance equation for each density function must be defined. Typically, the balance equation for distribution functions $\{f_s(\underline{x}, \underline{v}, t)\}_{s=1}^{M}$ is the Boltzmann equation and it is normally expressed as

$$\frac{\partial f_s}{\partial t} = \sum_{r=1}^{M} \mathcal{Q}_{sr}(f_s, f_r)(\underline{x}, \underline{v}, t) \qquad 1 \leq s \leq M. \tag{1}$$

The right-hand side of (1) represents the *collisional operator* relative to species s, which is meant to account for the details of interactions among agents. According to (1), such a collisional operator is written as the sum of the collisional operators $\mathcal{Q}_{sr}(f_s, f_r)$ relative to the collisions among the molecules of species s with the molecules of any species $r \in \{1, \ldots, M\}$. Observe that the collisional operator relative to species s does not only depend on the distribution function f_s, but it also depends on the distribution functions related to other species $\{f_r\}_{r=1, r\neq s}^{M}$.

Using the outlined analytic framework, proper macroscopic properties of the considered gas mixture can be studied, as follows. First, observe that the number of molecules of species s per unit volume at time t with position $\underline{x}$ is obtained by integrating the distribution function relative to species s with respect to $\underline{v}$

$$n_s(\underline{x}, t) = \int_{\mathbb{R}^3} f_s(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v} \qquad 1 \leq s \leq M. \tag{2}$$

Then, according to the considered probabilistic approach, the average velocity of molecules of species s can be computed as

$$\underline{u}_s(\underline{x}, t) = \frac{1}{n_s(\underline{x}, t)} \int_{\mathbb{R}^3} \underline{v}\, f_s(\underline{x}, \underline{v}, t)\, \mathrm{d}_3\underline{v} \qquad 1 \leq s \leq M. \tag{3}$$

Finally, the temperature $T_s(\underline{x}, t)$ of molecules of species $s \in \{1, \ldots, M\}$ is computed according to the following formula

$$\frac{3}{2} n_s(\underline{x}, t) k T_s(\underline{x}, t) = m_s \int_{\mathbb{R}^3} |(\underline{v} - \underline{u}_s(\underline{x}, t))|^2 f_s(\underline{v}, \underline{x}, t)\, \mathrm{d}_3\underline{v} \qquad 1 \leq s \leq M \tag{4}$$

where parameters $\{m_s\}_{s=1}^{M}$ represent the mass of molecules of species s.

3 A Kinetic Framework for Opinion Dynamics

As discussed in the Introduction, it is possible to generalize the kinetic approach described in Section 2 to investigate opinion dynamics in multi-agent systems. According to described assumptions, we aim at studying interesting macroscopic properties of a system made of interacting agents, starting from the analysis of the effects of single interactions. While in kinetic theory the molecules of gases are associated with physical parameters, e.g., their positions and velocities, here we assume that each agent is associated with a single scalar parameter v which represents its opinion, and we assume that v varies continuously in a proper closed interval, denoted as $I \subseteq \mathbb{R}$. It is worth mentioning that, in the literature on opinion dynamics, opinion has been modeled either as a discrete [15] or as a continuous [16] variable. Discrete models are typically used to address situations where a finite number of possible options are available, e.g., in political elections. Continuous models, instead, are typically used to study opinions related to a single topic, varying from strongly disagree to completely agree.

In order to account for the peculiarities of the proposed framework, we call the groups of agents sharing the same characteristics *classes*, and not species, and we assume that the agents of the considered multi-agent systems are divided into $M \geq 1$ classes. It is possible to define M distribution functions $\{f_s(v, t)\}_{s=1}^{M}$, each of which represents the number of agents of a single class s with opinion in $(v, v + \mathrm{d}v)$ at time t. In agreement with the related equation of kinetic theory of gas mixtures, the number of agents of class s at time t can be written as

$$n_s(t) = \int_I f_s(v, t) \mathrm{d}v \qquad 1 \leq s \leq M. \tag{5}$$

The total number of agents is obtained by adding the number of agents of each class, as follows

$$n(t) = \sum_{s=1}^{M} n_s(t). \tag{6}$$

Moreover, following the same approach used in kinetic theory of gas mixtures, we can compute the average opinion of agents of class s as

$$u_s(t) = \frac{1}{n_s} \int_I f_s(v,t) v \mathrm{d}v \qquad 1 \leq s \leq M. \tag{7}$$

Observe that (7) is the analogous of (3), which rules the average velocity of each species in a gas mixture. The definition of the average opinions of all classes allows computing $u(t)$, i.e., the global average opinion of the system at time t

$$u(t) = \frac{1}{n(t)} \sum_{s=1}^{M} n_s(t) u_s(t). \tag{8}$$

Observe that the average opinion of the system is obtained by a properly weighed sum of the average opinions of all the classes. Finally, the standard deviation of the opinion relative to agents of class s can be computed as

$$\sigma_s^2(t) = \frac{1}{n(t)} \int_I (v - u_s)^2 f_s(v,t) \mathrm{d}v \qquad 1 \leq s \leq M. \tag{9}$$

Observe that (9) is related to (4), which is used to compute the temperature of each species in a gas mixture.

In the proposed framework, we assume that each distribution function $f_s(v,t)$ evolves according to a balance equation, which is the analogous of the Boltzmann equation (1)

$$\frac{\partial f_s}{\partial t}(v,t) = \sum_{r=1}^{M} \mathcal{Q}_{sr}(f_s, f_r) \qquad 1 \leq s \leq M. \tag{10}$$

From (10) it is evident that the collisional operator relative to class s corresponds to the sum of the contributions $\mathcal{Q}_{sr}(f_s, f_r)$ of the collisions between an agent of class s with an agent of class $r \in \{1, \ldots, M\}$.

As usual in the studies of the kinetic theory of gas mixtures, the Boltzmann equation is often treated in its *weak form* to try to obtain interesting analytic results. Actually, the study of the weak form of the Boltzmann equation is useful to derive proper differential equations involving the temporal evolution of interesting macroscopic parameters of the system, namely the average opinion of each class and the variance of the opinion of each class. The weak form of (10) is obtained by multiplying it by a so called *test function* $\phi(v)$ and by integrating with respect to v. Hence, the weak form of the Boltzmann equation for a given test function can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I f_s(v,t) \phi(v) \mathrm{d}v = \sum_{r=1}^{M} \int_I \mathcal{Q}_{sr}(f_s, f_r) \phi(v) \mathrm{d}v \qquad 1 \leq s \leq M. \tag{11}$$

Proper choices of the test function allow obtaining analytic results regarding:

1. The number of agents of each class, using test function $\phi(v) = 1$;
2. The average opinion of each class, using test function $\phi(v) = v$; and
3. The variance of the opinion of each class, using test function $\phi(v) = (v - u_s)^2$.

In order to exemplify the analytic results that can be obtained using the outlined framework (see, e.g., previous works [6–8,17–20]), we briefly summarize major results obtained in the study of a very simple model of opinion dynamics which, among the variety of sociological phenomena cited in the introduction, considers only compromise. Let us remark that compromise is the key ingredient of many models of opinion dynamics and it is based on the idea that the opinions of two agents get closer after an interactions. Mathematically, denoting as v and w the pre-interaction opinions of two agents, and as v' and w' their post-interaction opinions, compromise corresponds to the following inequality

$$|v' - w'| < |v - w|. \tag{12}$$

In detail, compromise is modelled using the following interaction rules. Let us denote as s and r the generic classes of two interacting agents whose pre-interaction opinions are v and w, respectively. The post-interaction opinions v' and w' of the two agents are computed as

$$\begin{cases} v' = v - \gamma_{sr}(v - w) \\ w' = w - \gamma_{rs}(w - v) \end{cases} \tag{13}$$

where $\{\gamma_{sr}\}_{s,r=1}^{M}$ are the characteristic parameters of the model. The value of a single γ_{sr} quantifies the propensity of an agent of class s to change its opinion in favor of the opinion of an agent of class r. We remark that in order to reproduce compromise, namely to satisfy (12), the values of γ_{sr} must be chosen in the interval $\mathcal{I}_\gamma = (0, 1)$. As a matter of fact, using (13) it is possible to observe that (12) is equivalent to

$$|1 - (\gamma_{sr} + \gamma_{rs})| < 1 \tag{14}$$

which is always satisfied if all the parameters $\{\gamma_{sr}\}_{s,r=1}^{M}$ are defined in $\mathcal{I}_\gamma$. Using interaction rules (13), the weak form of the Boltzmann equation for each class $s \in \{1, \ldots, M\}$ can be explicitly written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)\phi(v)\mathrm{d}v = \beta \sum_{r=1}^{M} \int_{I^2} f_s(v,t) f_r(w,t)(\phi(v^*) - \phi(v))\mathrm{d}v\mathrm{d}w \tag{15}$$

where on the left-hand side we used the fact that for every test function

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} f_s(v,t)\phi(v)\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_I f_s(v,t)\phi(v)\mathrm{d}v \qquad 1 \le s \le M. \tag{16}$$

Under these assumptions, considering the system of differential equations (11) with $\phi(v) = 1$, it can be proved that the number of agents of each class is conserved and, according to (6), the total number of agents is also conserved [7]. Moreover, from the analysis of the weak form of the Boltzmann equation (11) with $\phi(v) = v$, it is possible to show that the asymptotic average opinions of all classes are equal [21].

4 Conclusions and Future Work

This paper briefly outlines an analytic framework to study the dynamics of the opinion in multi-agent systems. In the studied scenarios, each agent is associated with a scalar parameter which represents its opinion and which may vary continuously in a closed interval. The opinion of each agent is modified by interactions with other agents. Each interaction corresponds to a message exchange between two randomly chosen agents, and it is assumed that each agent can freely interact with any other agent, without supervised coordination. In detail, the analytic framework proposed in this paper is inspired by the kinetic theory of gas mixtures, and it allows deriving analytic results which only rely on the choice of the details used to describe the effects of interactions among agents. Notably, the outlined framework draws a strong parallelism between the kinetic theory of gas mixtures and the study of opinion dynamics in multi-agent systems, even though it is worth remarking that analytic results regarding the dynamics of the opinion are different from those of the kinetic theory of gas mixtures because they are obtained using different forms of interactions.

Various interaction rules can be accommodated in the proposed framework and they can be used to account for some of the most commonly studied sociological phenomena involved in opinion dynamics. For the sake of brevity, this paper only focused on one of such phenomena, namely compromise, which represents the tendency of individuals to move their opinions towards those of the peers they interact with. Further work on the kinetic study of opinion dynamics involves treating other sociological phenomena, such as diffusion and negative influence, which have not yet been studied using the framework of gas mixtures. Another interesting problem that we plan to study concerns the modelling of locality of interactions, namely the fact that only agents close to each other are allowed to interact in some interesting models of opinion dynamics.

References

1. D. Stauffer and T. J. P. Penna, "Crossover in the Cont-Bouchaud percolation model for market fluctuations," *Physica A: Statistical Mechanics and Its Applications*, vol. 256, pp. 284–290, 1998.
2. F. Schweitzer and J. Holyst, "Modelling collective opinion formation by means of active brownian particles," *European Physical Journal B*, pp. 723–732, 2000.
3. G. Toscani, "Kinetic models of opinion formation," *Communications in Mathematical Sciences*, vol. 4, pp. 481–496, 2006.
4. W. Weidlich, *Sociodynamics: A systematic approach to mathematical modelling in the social sciences.* Amsterdam: Harwood Academic Publisher, 2000.
5. L. Pareschi and G. Toscani, *Interacting Multiagent Systems: Kinetic Equations and Montecarlo Methods.* Oxford: Oxford University Press, 2013.
6. S. Monica and F. Bergenti, "A study of consensus formation using kinetic theory," in *Proceedings of the* 13$^{\text{th}}$ *International Conference on Distributed Computing and Artificial Intelligence* (DCAI 2016), Sevilla, Spain, June 2016, pp. 213–221.

7. F. Bergenti and S. Monica, "Analytic study of opinion dynamics in multi-agent systems with two classes of agents," in *Proceedings of* 17^{th} *Workshop "Dagli Oggetti agli Agenti"* (WOA 2016), Catania, Italy, July 2016.
8. S. Monica and F. Bergenti, "Opinion dynamics in multi-agent systems: Selected analytic models and verifying simulations," *Computational and Mathematical Organization Theory*, accepted September 2016.
9. R. P. Abelson, *Contributions to Mathematical Psycology.* New York: Frederiksen N., Gulliksen H. eds., 1964.
10. E. Bonabeau, "Agent-based modeling: Methods and techniques for simulating human systems." *Proc. Natl. Acad. Sci*, pp. 7280–7287, 2002.
11. A. Nowak, J. Szamrej, and B. Latan, "From private attitude to public opinion: A dynamic theory of social impact," *Psycol. Rev.*, vol. 97, pp. 362–376, 1990.
12. M. Mäs and A. Flache, "Differentiation without distancing. Explaining bi-polarization of opinions without negative influence," *PLOS One*, vol. 8, 2013.
13. S. Galam, Y. Gefen, and Y. Shapir, "Sociophysics: A new approach of sociological collective behavior," *Journal of Mathematical Sociology*, 2009.
14. M. Mäs, A. Flache, and D. Helbing, "Individualisazion as driving force of clustering phenomena in humans," *PLOS One*, vol. 6, no. 10, 2010.
15. E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione, "Noisy continuous opinion dynamics," *Journal of Statistical Mechanics*, vol. 9, pp. 1–13, 1982.
16. R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of Artificial Societies and Social Simulations*, vol. 5, no. 3, 2002.
17. S. Monica and F. Bergenti, "Simulations of opinion formation in multi-agent systems using kinetic theory," in *Proceedings of* 16^{th} *Workshop "Dagli Oggetti agli Agenti"* (WOA 2015), Napoli, Italy, June 2015.
18. S. Monica and F. Bergenti, "A kinetic study of opinion dynamics in multi-agent systems," in *Atti del Convegno* AI*IA 2015, Ferrara, Italy, September 2015.
19. S. Monica and F. Bergenti, "Kinetic description of opinion evolution in multi-agent systems: Analytic model and simulations," in *Proceedings of the* 18^{th} *International Conference on Principles and Practice of Multi-Agent Systems* (PRIMA 2015), Bertinoro, Italy, October 2015, pp. 483–491.
20. S. Monica and F. Bergenti, "An analytic study of opinion dynamics in multi-agent systems with additive random noise," in *Atti del Convegno* AI*IA 2016, Genova, Italy, December 2016.
21. S. Monica and F. Bergenti, "An analytic study of opinion dynamics in multi-agent systems," *Computer and Mathematics with Applications*, accepted March 2017.