A New Kernel for Parameterized Max-Bisection Above Tight Lower Bound

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Abstract. In this paper, we study kernelization of Parameterized Max-Bisection above Tight Lower Bound problem, which is to find a bisection (V_1, V_2) of G with at least $\lceil |E|/2 \rceil + k$ crossing edges for a given graph $G = (V, E)$. The current best vertex kernel result for the problem is of size $16k$. Based on analysis of the relation between maximum matching and vertices in Gallai-Edmonds decomposition of G, we divide graph G into a set of blocks, and each block in G is closely related to the number of crossing edges of bisection of G. By analyzing the number of crossing edges in all blocks, an improved vertex kernel of size $8k$ is presented.

1 Introduction

Given a graph $G = (V, E)$, for two subsets V_1, V_2 of V, if $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, and $||V_1| - |V_2|| \leq 1$, then (V_1, V_2) is called a *bisection* of G. An edge of G with one endpoint in V_1 and the other endpoint in V_2 is called a *crossing edge* of (V_1, V_2). The Maximum Bisection problem is to find a bisection (V_1, V_2) of G with maximum number of crossing edges. Jansen et al. [8] proved that the Maximum Bisection problem is NP-hard on planar graph. Díaz and Kamiński [1] proved that the Maximum Bisection is NP-hard on unit disk graphs.

Frieze and Jerrum [4] gave an approximation algorithm for the Maximum Bisection problem with ratio 0.651. Ye [13] presented an improved approximation algorithm with ratio 0.699. Halperin and Zwick [7] gave an approximation algorithm with ratio 0.701. Feige et al. [3] studied the Maximum Bisection problem on regular graphs, and presented an approximation algorithm with ratio 0.795. Karpiński et al. [9] studied approximation algorithms for Maximum Bisection problem on low degree regular graphs and planar graphs. For three regular graphs, an approximation algorithm of ratio 0.847 was presented in [9]. For four and five regular graphs, two approximation algorithms with ratios 0.805, 0.812 were presented in [9], respectively. For planar graph of a sublinear degree, a polynomial time approximation scheme was presented in [9]. Jansen et al. [8] studied Maximum Bisection problem on planar graphs, and gave the first polynomial time approximation scheme for the problem.

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For a given graph G, it is easy to find a bisection with $\lceil |E|/2 \rceil$ crossing edges by probabilistic method. In this paper, we study the following problem.

Parameterized Max-Bisection above Tight Lower Bound (PMBTLB): Given a graph $G = (V, E)$ and non-negative integer k, find a bisection of G with at least $\lceil |E|/2 \rceil + k$ crossing edges, or report that no such bisection exists.

Gutin and Yeo [5] gave a vertex kernel of size $O(k^2)$ for the PMBTLB problem. Based on the relation between edges in maximum matching and crossing edges, a parameterized algorithm of running time $O^*(16^k)$ was presented in [5]. Mnich and Zenklusen [11] presented a vertex kernel of size $16k$ for PMBTLB problem based on Gallai-Edmonds decomposition of the given graph.

In this paper, we further analyze the relation between maximum matching and vertices in Gallai-Edmonds decomposition for a given graph G. The vertices in Gallai-Edmonds decomposition are divided into several categories, which play important role in getting improved kernel. Based on the categories of vertices, we divide graph G into a set of blocks, where each block is closely related to the number of crossing edges of bisection of G. By analyzing the number of crossing edges in all blocks, a vertex kernel of size $8k$ is presented.

2 Preliminaries

For a given graph $G = (V, E)$, we use n, m to denote the number of vertices in V and the number of edges in E, respectively. Assume that all the graphs discussed in the paper are loopless undirected graph with possible parallel edges. For a graph $G = (V, E)$, if $|V|$ is a odd number, we can add an isolated vertex into G such that the number of crossing edges in each bisection of G is not changed. For simplicity, we assume that all the graphs in the paper have even number of vertices.

For two subsets $A, B \subseteq V$, let $E(A)$ be the set of edges in $G[A]$, and let $E(A, B)$ be the set of edges with one endpoint in A and the other endpoint in B. For two vertices u and v in G, for simplicity, let uv denote an edge between u and v, and let $E(u, v)$ denote the set of edges between u and v. For a vertex v in G, let $d(v)$ denote the degree of v in G. For a subset $X \subseteq V$ and a vertex v in X, let $d_X(v)$ denote the degree of v in induced subgraph $G[X]$, and let $\delta(G[X])$ be the number of connected components in $G[X]$. For a subgraph H of G, let $V(H)$ be the set of vertices contained in H. Let $N[V(H)]$ denote the set of neighbors of vertices in $V(H)$, where $V(H)$ is contained in $N[V(H)]$, and let $N(V(H)) = N[V(H)] - V(H)$.

Given a matching M in G, let $V(M)$ denote the set of vertices in M. If a vertex u in G is not contained in $V(M)$, then u is called an *unmatched vertex*. Matching M is called a *near-perfect matching* of G if there is exactly one unmatched vertex in G. For a connected graph G, and any vertex u in G, if the size of maximum matching in $G \backslash \{u\}$ is equal to the size of maximum matching in G, then G is called a *factor-critical* graph. A Gallai-Edmonds decomposition of graph G is a tuple (X, Y, Z), where X is the set of vertices in G

which are not covered by at least one maximum matching of G, Y is $N(X)$, and $Z = V(G)\backslash(X \cup Y)$. The Gallai-Edmonds decomposition of G can be obtained in polynomial time [10].

Lemma 1 ([2,10]). *For a given graph G, a Gallai-Edmonds decomposition (X, Y, Z) of G has the following properties:*

1. *the components of the subgraph induced by X are factor-critical,*
2. *the subgraph induced by Z has a perfect matching,*
3. *if M is any maximum matching of G, it contains a near-perfect matching of each component of $G[X]$, a perfect matching of each component of $G[Z]$, and matches all vertices of Y with vertices in distinct components of $G[X]$,*
4. *the size of the maximum matching is $\frac{1}{2}(|V| - \delta(G[X]) + |Y|)$.*

For two subsets A, B of V, if $A \cap B = \emptyset$ and $|A| = |B|$, then (A, B) is called a *basic block* of graph G. Let $\mathcal{C} = \{C_1, \ldots, C_h\}$ be the set of basic blocks of G, where $C_i = (A_i, B_i)$. For a basic block $C_i = (A_i, B_i)$, let $V(C_i)$ denote the set of vertices in $A_i \cup B_i$. Given two basic blocks $C_i, C_j \in \mathcal{C}$, for simplicity, let $E(C_i, C_j) = E(V(C_i), V(C_j))$. For all basic blocks in $\mathcal{C}$, if $V(C_i) \cap V(C_j) = \emptyset$ $(i \neq j)$ and $\bigcup_{i=1}^{h} V(C_i) = V$, then $\mathcal{C}$ is called a *block cluster* of G. For a basic block $C \in \mathcal{C}$, we use $\mathcal{C} - C$ to denote $\mathcal{C}\backslash\{C\}$.

Based on the block cluster $\mathcal{C}$ and V, a bisection (V_1, V_2) of G can be constructed in the following way: for each basic block $C_i = (A_i, B_i)$ in $\mathcal{C}$, put all vertices in A_i into V_1 and V_2 with probability 1/2, 1/2, respectively; if A_i is put into V_1, then B_i will be put into V_2, and if A_i is put into V_2, then B_i will be put into V_1.

Let $r_1 = \sum_{i=1}^{h} |E(A_i, B_i)|$, $r_2 = \sum_{i=1}^{h-1} \sum_{j=i+1}^{h} |E(C_i, C_j)|$, and $r_3 = \sum_{i=1}^{h} (|E(A_i)| + |E(B_i)|)$. For a basic block $C_i = (A_i, B_i)$ in $\mathcal{C}$, let $r(C_i) = |E(A_i, B_i)| - |E(A_i)| - |E(B_i)|$. Let $r(\mathcal{C}) = \sum_{i=1}^{h} r(C_i)$.

Lemma 2. *For any block cluster $\mathcal{C}$ of graph G, there exists a bisection (V_1', V_2') of G obtained from $\mathcal{C}$ such that $|E(V_1', V_2')|$ is at least $\lceil m/2 \rceil + r(\mathcal{C})/2$.*

Proof. For any two basic blocks C_i, C_j $(i \neq j)$ in $\mathcal{C}$, we now analyze the expected number of crossing edges from $E(C_i, C_j)$ for bisection (V_1, V_2). Assume that A_i is in V_1, and B_i is in V_2. In the process of constructing (V_1, V_2), A_j is put into V_1 and V_2 with probability 1/2, 1/2, respectively. Therefore, the expected number of crossing edges from $E(C_i, C_j)$ is $|E(C_i, C_j)|/2$. Moreover, for a basic block C_i in $\mathcal{C}$, if $E(A_i, B_i) \neq \emptyset$, then the edges in $E(A_i, B_i)$ are all crossing edges, and edges in $E(A_i) \cup E(B_i)$ are not crossing edges. Therefore, the expected number of crossing edges in (V_1, V_2) is $r_1 + r_2/2 = (2r_1 + r_2)/2 = (r(\mathcal{C}) + r_3 + r_1 + r_2)/2$. Since $r_1 + r_2 + r_3 = m$, $(r(\mathcal{C}) + r_3 + r_1 + r_2)/2 = m/2 + r(\mathcal{C})/2$. Therefore, there must exist a bisection (V_1', V_2') of G with $|E(V_1', V_2')| \geq \lceil m/2 \rceil + r(\mathcal{C})/2$. □

Lemma 3. *For a given instance (G, k) of PMBTLB problem and any block cluster $\mathcal{C}$ of G, if $r(\mathcal{C}) \geq 2k$, then G has a a bisection of size at least $\lceil m/2 \rceil + k$ based a standard derandomization as given by Ries and Zenklusen [12].*

3 Kernelization for PMBTLB Problem

For a given instance (G, k) of PMBTLB, assume that (X, Y, Z) is a Gallai-Edmonds decomposition of G. Let MM be a maximum matching of G. Based on the degree of vertices in X and the maximum matching MM, we divide X into following subsets:

$X_0 = \{v|v \in X, d(v) = 0\}$,
$X_1 = \{v|v \in X, d_X(v) = 0, v \in V(MM)\}$,
$X_2 = \{v|v \in X, d_X(v) = 0, v \notin V(MM)\}$,
$X_3 = \{v|v \in X, d_X(v) \geq 1, \exists u \in Y, uv \in MM\}$,
$X_4 = \{v|v \in X, \exists u \in X, uv \in MM\}$,
$X_5 = \{v|v \in X, d_X(v) \geq 1, v \notin V(MM)\}$.

We now give the process to construct a block cluster $\mathcal{C}$ of graph G, as given in Fig. 1. Assume that $\mathcal{C} = \{C_1, \ldots, C_h\}$ is the block cluster of G obtained by algorithm BBDA1 in Fig. 1.

BBDA1(G, MM)
Input: a graph $G = (V, E)$, and a maximum matching MM in G.
Output: a block cluster $\mathcal{C}$ of G.
1. $\mathcal{C} = \emptyset$;
2. **for** each edge uv in MM **do**
2.1 let $A = \{u\}$, $B = \{v\}$;
2.2 construct a basic block $C = (A, B)$, and add it into $\mathcal{C}$;
3. let $V' = V \backslash V(MM)$;
4. **if** V' is not empty **then**
4.1 randomly choose $|V'|/2$ vertices to put into A, and put the remaining vertices into B;
4.2 construct a basic block $C = (A, B)$, and add it into $\mathcal{C}$;
5. return $\mathcal{C}$.

Fig. 1. Algorithm for constructing block cluster $\mathcal{C}$

Lemma 4 ([6]). *If M is a matching in a graph G, then G has a bisection of size at least $\lceil m/2 \rceil + \lfloor |M|/2 \rfloor$, which can be found in $O(m + n)$ time.*

By Lemma 4, we can get that the size of matching M is less than $2k$, otherwise a bisection with at least $\lceil m/2 \rceil + k$ crossing edges can be found in polynomial time. For a maximum matching MM of G, if $V' = V \backslash V(MM)$ is empty, then G is a graph with perfect matching. Since the size of matching MM is less than $2k$, the number of vertices in G is bounded by $4k$.

In the following, assume that V' is not empty. Since $V' = V \backslash V(MM)$, V' is an independent set. Assume that C_h is the basic block constructed by step 4 of algorithm BBDA1. According to the construction process of $\mathcal{C}$, for each basic block C_i in $\mathcal{C} - C_h$, $r(C_i) \geq 1$. Especially, $r(C_h) = 0$. We now construct a new block cluster based on $\mathcal{C}$. The general idea is to move vertices of $V(C_h)$ to the basic blocks in $\mathcal{C} - C_h$ to get new basic blocks. In the construction process, if no vertex in added into a basic block $C = (A, B)$, then the value $r(C)$ is not changed. Since vertices in $V(C_h)$ form an independent set, after removing some vertices of C_h, the vertices in the remaining basic block C_h still form an independent set, and $r(C_h)$ is still zero. In the following, we give the process to get a new block cluster $\mathcal{C}' = \{C'_1, \ldots, C'_h\}$ based on $\mathcal{C}$, which is given in Fig. 2.

BBDA2((X, Y, Z), $\mathcal{C}$)
Input: a Gallai-Edmonds decomposition (X, Y, Z) of G, and a block cluster $\mathcal{C}$ of G returned by algorithm BBDA1.
Output: a new block cluster $\mathcal{C}'$ of G and a vertex set S.
1. let $\mathcal{C}_Y = \{C_i, \ldots, C_j\}$ be the subset of $\mathcal{C}$ such that for each C_l in $\mathcal{C}_Y$, $V(C_l)$ contains one vertex from Y;
2. $\mathcal{C}' = \emptyset$; $C'_h = C_h$; $\mathcal{C}'_Y = \mathcal{C}_Y$; $S = X_2$;
3. **for** each C_l in $\mathcal{C}'_Y$ **do**
3.1 assume A_l of C_l contains one vertex from Y, denoted by u_l;
3.2 **for** each vertex v in S **do**
3.3 **if** there exists a vertex w in S with $|E(v, u_l)| > |E(w, u_l)|$ **then**
3.4 $B_l = B_l \cup \{v\}$; $A_l = A_l \cup \{w\}$; $S = S \backslash \{v, w\}$; $C'_h = C'_h \backslash \{v, w\}$;
3.5 $C_l = (A_l, B_l)$;
4. $\mathcal{C}' = (\mathcal{C} - \mathcal{C}_Y - C_h) \cup \mathcal{C}'_Y \cup C'_h$;
5. return $\mathcal{C}'$ and S.

Fig. 2. Algorithm for constructing block cluster $\mathcal{C}'$

Let $\mathcal{C}'$ be the block cluster returned by algorithm BBDA2. We now analyze the difference between $r(\mathcal{C})$ and $r(\mathcal{C}')$. For two vertices v, w in X_2 that are added into C_l in step 3 of algorithm BBDA2, $r(C_l)$ is increased by at least one. Assume that S is returned by algorithm BBDA2. For any two vertices w, v in S, it is easy to see that for each vertex u in G, $|E(w, u)| = |E(v, u)|$. Since all vertices in $|X_2 \backslash S|$ are moved to $\mathcal{C}'_Y$ in algorithm BBDA2, $r(\mathcal{C}'_Y) - r(\mathcal{C}_Y)$ is at least $|X_2 \backslash S|/2$, and $r(\mathcal{C}') - r(\mathcal{C})$ is at least $|X_2 \backslash S|/2$. In algorithm BBDA1, each edge in MM is chosen to construct a basic block. Therefore, the value $r(\mathcal{C})$ is at least $|MM|$. Since $\mathcal{C}'$ is constructed based on $\mathcal{C}$, we have

$$r(\mathcal{C}') \geq |MM| + |X_2 \backslash S|/2. \tag{1}$$

Since the vertices in X_0, X_2 and X_5 are not in $V(MM)$, in algorithm BBDA1, $V(C_h) = X_0 \cup X_2 \cup X_5$. In algorithm BBDA2, the vertices in $X_2 \backslash S$ are moved from C_h to $\mathcal{C}'_Y$. Therefore, $V(C'_h) = X_0 \cup S \cup X_5$.

Lemma 5. *For any basic block $C'_l = (A'_l, B'_l)$ in $\mathcal{C}'$, where $V(C'_l)$ contains one vertex u of Y and $u \in N(S)$, assume that $u \in A'_l$. Then, S cannot be connected to any vertex in B'_l, and the number of basic blocks in $\mathcal{C}'$ containing one vertex in $N(S)$ is $|N(S)|$.*

For any connected component H in $G[X]$ with at least three vertices, there is exactly one vertex v in $V(H)$ such that v is in either X_3 or X_5, and other vertices in $V(H)\backslash\{v\}$ are in X_4.

Lemma 6. $|X_5| < 2k$.

Proof. If X_5 is empty, then this lemma is correct. Let $\mathcal{H} = \{H_1, \ldots, H_l\}$ be the set of connected components in $G[X]$, each of which has size at least three and contains one vertex in X_5. For any H_i $(1 \le i \le l)$ in $\mathcal{H}$, there exists a perfect matching in $G[V(H_i)\backslash\{v\}]$, and the number of edges from $E(H_i)$ in MM is $(|V(H_i)| - 1)/2$. By above discussion, the number of edges in MM is less than $2k$. Therefore, $\sum_{i=1}^{l}(|V(H_i)| - 1)/2 < 2k$. Thus, $|X_5| < 2k$. □

In the following, we will construct two new block clusters $\mathcal{C}''$ and $\mathcal{C}'''$ based on $\mathcal{C}'$ by adding vertices in X_0, X_5 and S into basic blocks of $\mathcal{C}' - C'_h$.

Case 1. $|X_0| \ge |S|$. Under this case, the general idea to construct $\mathcal{C}''$ is to use the vertices in S and X_0 firstly. When all vertices in S are added into basic blocks in $\mathcal{C}' - C'_h$ to get new basic blocks in $\mathcal{C}''$, we consider the vertices X_5 and the remaining vertices in X_0 to construct basic blocks in $\mathcal{C}'''$ based on $\mathcal{C}''$.

By Lemma 5, find a basic block $C'_i = (A'_i, B'_i)$ in $\mathcal{C}'$ such that $V(C'_i) \cap N(S) = \{u\}$, and assume that u is contained in A'_i. A new basic block $C''_i = (A''_i, B''_i)$ of $\mathcal{C}''$ can be constructed from C'_i by the following steps: $B''_i = B'_i \cup S$; arbitrarily choose $|S|$ vertices from X_0, denoted by P; $A''_i = A'_i \cup P$. Let $C''_h = C'_h\backslash(S \cup P)$, $\mathcal{C}'' = (\mathcal{C}' - C'_i - C'_h) \cup C''_i \cup C''_h$. Since no vertex in S is connected to any vertex in B'_i and no vertex in P is connected to any vertex in $V(C'_i)$, $r(C''_i) - r(C'_i)$ is at least $|S|$. Thus,

$$r(\mathcal{C}'') - r(\mathcal{C}') \ge |S|. \quad (2)$$

Let $\mathcal{H}$ be the set of connected components in $G[X]$ such that for each connected component H in $\mathcal{H}$, $V(H)$ contains one vertex of X_5, and $N(V(H)) \cap Y \ne \emptyset$. For each connected component H in $\mathcal{H}$, assume that E' is the set of edges in H that are contained in MM. Let $\mathcal{C}_H$ be a subset of $\mathcal{C}''$, where $\mathcal{C}_H$ can be constructed by the edges in E'. Since H is factor-critical, there exists a vertex v in $V(H)$ such that v is an unmatched vertex. If v is not connected to any vertex in Y, then find a vertex u in $V(H)$ that is connected to some vertices in Y, and find a perfect matching M' in $G[V(H)\backslash\{u\}]$. Construct a set $\mathcal{F}$ of basic blocks by the edges in M', and let $\mathcal{C}'' = (\mathcal{C}'' - \mathcal{C}_H) \cup \mathcal{F}$. After dealing with all connected components in $\mathcal{H}$ by the above process, a new maximum matching MM' can be obtained, and for each connected component H in $G[X]$ satisfying that $V(H)$ has at least three vertices and $N(H) \cap Y \ne \emptyset$, if v is an unmatched vertex in H, then v is connected to at least one vertex in Y.

The vertices in X_5 are divided into the following two types. Let X_5^y be a subset of X_5 such that each vertex v in X_5^y is connected to at least one vertex in Y, and let X_5^i be a subset of X_5 such that each vertex u in X_5^i is not connected to any vertex in Y.

Lemma 7. *Given a vertex $v \in X_5^y$, for any basic block $C_l'' = (A_l'', B_l'')$ in $\mathcal{C}''$, where $V(C_l'')$ contains one vertex u of Y and $u \in N(v)$, assume that $u \in A_l''$. Then, v cannot be connected to any vertex in B_l''.*

Let $X_0' = X_0 \backslash P$. We now construct basic blocks of $\mathcal{C}'''$ by vertices in X_0' and X_5^y. Assume that $|X_0'| \geq |X_5^y|$. For a vertex v in X_5^y, by Lemma 7, there exists a basic block $C_i'' = (A_i'', B_i'')$ in $\mathcal{C}''$ containing u such that $u \in N(v)$. Without loss of generality, assume that u is contained in A_i''. A new basic block $C_i''' = (A_i''', B_i''')$ of $\mathcal{C}'''$ can be constructed from C_i'' by the following process: $B_i''' = B_i'' \cup \{v\}$; arbitrarily choose a vertex w in X_0', let $A_i''' = A_i'' \cup \{w\}$. Let $C_h''' = C_h'' \backslash \{v, w\}$, $\mathcal{C}''' = (\mathcal{C}'' - C_i'' - C_h'') \cup \{C_i'''\} \cup \{C_h'''\}$. Since v is not connected to any vertex in B_i'' and w is not connected to any vertex in $V(C_i'')$, it is easy to see that $r(C_i''') - r(C_i'') \geq 1$. By considering all vertices in X_5^y, we have

$$r(\mathcal{C}''') - r(\mathcal{C}'') \geq |X_5^y|. \tag{3}$$

For the case when $|X_0'| < |X_5^y|$, a similar construction process can be obtained as above. By considering all vertices in X_0', we can get that

$$r(\mathcal{C}''') - r(\mathcal{C}'') \geq |X_0'|. \tag{4}$$

Lemma 8. *Based on MM', $|X_5^i| \leq \delta(G \backslash X_0)$.*

Proof. We prove this lemma by discussing the types of connected components in $G \backslash X_0$. For a connected component H in $G \backslash X_0$, if $V(H)$ contains no vertex of X, then no vertex in $V(H)$ is contained in X_5^i. If all vertices in $V(H)$ are from X, then H is a factor-critical connected component, and no vertex in $V(H)$ is connected to Y. Under this case, one vertex from $V(H)$ is contained in X_5^i. Assume that $V(H) \backslash X$ is not empty. By the construction process of MM', if a vertex v in $N(Y) \cap V(H)$ is contained in X_5, then v is in X_5^y, and no vertex in $V(H)$ is in X_5^i. Therefore, $|X_5^i| \leq \delta(G \backslash X_0)$. □

Rule 1. For a given instance (G, k) of PMBTLB, if $|X_0| + \delta(G \backslash X_0) > \frac{n}{2}$, then arbitrarily delete $|X_0| + \delta(G \backslash X_0) - \frac{n}{2}$ vertices from X_0.

Lemma 9. *Rule 1 is safe.*

Proof. Assume that $|X_0| + \delta(G \backslash X_0) > \frac{n}{2}$, and assume that (V_1, V_2) is a maximum bisection of G. If $X_0 = \emptyset$, then the number of connected components in $G \backslash X_0$ is at most $n/2$, because each connected component in $G \backslash X_0$ has at least two vertices. Assume that the number of connected components in $G \backslash X_0$ is at least one, otherwise, the PMBTLB problem can be trivially solved. Assume that $\mathcal{H} = \{H_1, \ldots, H_l\}$ is the set of connected components in $G \backslash X_0$. We now prove that all vertices in X_0 cannot be contained totally in V_1 or V_2. Assume that all vertices of X_0 are contained in V_1. Since $|X_0| + \delta(G \backslash X_0) > \frac{n}{2}$, there must exist a connected component H_i in $\mathcal{H}$ such that all vertices in $V(H_i)$ are contained in V_2. Choose a vertex v in $V(H_i)$, and find a vertex u of X_0 in V_1. Let $V_1' = (V_1 \backslash \{u\}) \cup \{v\}$, $V_2' = (V_2 \backslash \{v\}) \cup \{u\}$. Then, $|E(V_1', V_2')| - |E(V_1, V_2)| \geq 1$, contradicting the fact that

(V_1, V_2) is a maximum bisection of G. Therefore, $V_1 \cap X_0 \neq \emptyset$ and $V_2 \cap X_0 \neq \emptyset$. Assume that X_0' and X_0'' are two subsets of X_0 such that X_0' is contained in V_1, and X_0'' is contained in V_2. Without loss of generality, assume that $|X_0'| \leq |X_0''|$. Let R be any subset of X_0'' of size $|X_0'|$. Let $V_1'' = V_1 - X_0'$, $V_2'' = V_2 - R$. Then, $|V_1''| = |V_2''|$. Denote the new graph with bisection (V_1'', V_2'') by G'. It is easy to see that $|E(V_1, V_2)|$ is bounded by $\lceil m/2 \rceil + k$ if and only if $|E(V_1'', V_2'')|$ is bounded by $\lceil m/2 \rceil + k$. Repeat the above process until $|X_0| + \delta(G \backslash X_0) \leq \frac{n}{2}$. □

For a given instance (G, k) of PMBTLB, by applying Rule 1 on G exhaustively, we can get the following result.

Lemma 10. *For a given instance (G, k) of PMBTLB, if $|X_0| \geq |S|$, then the number of vertices in G is bounded by $8k$.*

Proof. Based on block cluster $\mathcal{C}'''$, we prove this lemma by analyzing the sizes of X_0' and X_5^y.

(1) $|X_0'| \geq |X_5^y|$. By Lemma 3 and inequalities (1), (2) and (3), we can get that $r(\mathcal{C}''') = |MM| + \frac{|X_2 \backslash S|}{2} + |S| + |X_5^y| < 2k$. Based on Gallai-Edmonds decomposition (X, Y, Z) and MM, we know that $|MM| = (|Z| + |Y| + |X \backslash (X_0 \cup X_2 \cup X_5)|)/2$. Then, $|Z| + |Y| + |X \backslash (X_0 \cup S \cup X_5)| + 2|S| + 2|X_5^y| < 4k$. Since $X_5 = X_5^y \cup X_5^i$, and S, X_5^i, X_5^y, X_0 are disjoint, we can get that

$$|Z| + |Y| + |X| - |X_0| - |X_5^i| + |S| + |X_5^y| < 4k. \quad (5)$$

By Lemmas 8 and 9, $|X_0| + |X_5^i| \leq \frac{n}{2}$. Then, we can get that

$$\begin{aligned} |V| = |Z| + |Y| + |X| &= |Z| + |Y| + |X \backslash (X_0 \cup X_{5i})| + |X_0| + |X_5^i| \\ &= \underbrace{|Z| + |Y| + |X| - |X_0| - |X_5^i|}_{<4k \text{ by } (5)} + |X_0| + |X_5^i| \\ &< 4k + |V|/2. \end{aligned}$$

Therefore, $|V| < 8k$.

(2) $|X_0'| < |X_5^y|$. By Lemma 3 and inequalities (1), (2) and (4), we can get that $r(\mathcal{C}''') = |MM| + \frac{|X_2 \backslash S|}{2} + |S| + |X_0'| < 2k$. Since $P \subseteq X_0$, $X_0' = X_0 \backslash P$ and $|S| = |P|$, we can get that $r(\mathcal{C}''') = |MM| + \frac{|X_2 \backslash S|}{2} + |X_0| < 2k$. Similarly, $|MM| = (|Z| + |Y| + |X \backslash (X_0 \cup X_2 \cup X_5)|)/2$. Then, $|Z| + |Y| + |X \backslash (X_0 \cup S \cup X_5)| + 2|X_0| < 4k$. Since S, X_5, X_0 are disjoint and $|X_0| \geq |S|$, we can get that

$$|Z| + |Y| + |X| - |X_5| < 4k. \quad (6)$$

By Lemma 6, $|X_5| < 2k$. Then, we can get that

$$\begin{aligned} |V| = |Z| + |Y| + |X| &= |Z| + |Y| + |X \backslash X_5| + |X_5| \\ &= \underbrace{|Z| + |Y| + |X| - |X_5|}_{<4k \text{ by } (6)} + |X_5| \\ &< 4k + 2k = 6k. \end{aligned}$$

Therefore, if $|X_0| \geq |S|$, the number of vertices in G is bounded by $8k$. □

Case 2. $|X_0| < |S|$. Under this case, the general idea to construct $\mathcal{C}''$ is to use the vertices in S and X_0 firstly. When all vertices in X_0 are added into basic blocks in $\mathcal{C}' - C'_h$ to get new basic blocks in $\mathcal{C}''$, we consider vertices in X_5 and the remaining vertices in S to construct basic blocks in $\mathcal{C}'''$ based on $\mathcal{C}''$.

By Lemma 5, find a basic block $C'_i = (A'_i, B'_i)$ in $\mathcal{C}'$ such that $V(C'_i) \cap N(S) = \{u\}$, and assume that u is contained in A'_i. A new basic block $C''_i = (A''_i, B''_i)$ of $\mathcal{C}''$ can be constructed from C'_i by the following steps: $A''_i = A'_i \cup X_0$; arbitrarily choose $|X_0|$ vertices from S, denoted by Q; $B''_i = B'_i \cup Q$. Let $C''_h = C'_h \backslash (X_0 \cup Q)$, $\mathcal{C}'' = (\mathcal{C}' - C'_i - C'_h) \cup \{C''_i\} \cup \{C''_h\}$. Since no vertex in Q is connected to any vertex in B'_i and no vertex in X_0 is connected to any vertex in $V(C'_i)$, $r(C''_i) - r(C'_i)$ is at least $|X_0|$. Thus,

$$r(\mathcal{C}'') - r(\mathcal{C}') \geq |X_0|. \tag{7}$$

Since $|X_0| < |S|$, S is not empty. For a connected component H in $G[X]$, and for any three vertices w, v and u, where $w \in S$, $v \in V(H)$ and $u \in N(S)$, if $N(V(H)) = N(S)$ and $|E(w, u)| = |E(v, u)|$, then H is called a *special component* in $G[X]$.

Let $\mathcal{H}$ be the set of connected components in $G[X]$ such that for each connected component H in $\mathcal{H}$, $V(H)$ contains one vertex of X_5 and H is not a special component. For each connected component H in $\mathcal{H}$, assume that E' is the set of edges in H that are contained in MM. Let $\mathcal{C}_H$ be a subset of $\mathcal{C}''$, where $\mathcal{C}_H$ can be constructed by the edges in E'. Since H is factor-critical, there exists a vertex v in $V(H)$ such that v is an unmatched vertex. Based on $N(S)$ and $N(v)$, we give following five conditions: (1) $N(v) \cap N(S) = \emptyset$; (2) $N(v) \backslash N(S) \neq \emptyset$; (3) $N(v) \subset N(S)$; (4) $N(v) = N(S)$, and for any two vertices $w \in S$ and $u \in N(S)$, $|E(v, u)| > |E(w, u)|$; (5) $N(v) = N(S)$, and for any two vertices $w \in S$ and $u \in N(S)$, $|E(w, u)| < |E(v, v)|$.

We now introduce how to get a new maximum matching based on the above five conditions. Since $V(H)$ is not a special component, if v does not satisfy any condition from conditions (1)–(5), then a vertex u in H can be found such that u satisfies one of the above conditions. Then, find a perfect matching M' in $G[V(H) \backslash \{u\}]$, and construct a set $\mathcal{F}$ of basic blocks by the edges in M'. Let $\mathcal{C}'' = (\mathcal{C}'' - \mathcal{C}_H) \cup \mathcal{F}$. After dealing with all connected components in $\mathcal{H}$ by above process, a new maximum matching MM' can be obtained.

The vertices in X_5 are divided into the following three types. Let X_5^1 be a subset of X_5 such that for each vertex v in X_5^1, the connected component in $G[X]$ containing v is a special component. Let X_5^2 be a subset of X_5 such that for each vertex v in X_5^2, v satisfies one of conditions (1), (2), and (4). Let X_5^3 be a subset of X_5 such that for each vertex v in X_5^3, v satisfies one of conditions (3) and (5).

Lemma 11. *For any vertex v in X_5^2 and any vertex w in S, there exists a vertex u in $N(v)$ with $|E(v, u)| > |E(w, u)|$ such that a basic block $C''_i = (A''_i, B''_i)$ in $\mathcal{C}''$ can be found with $V(C''_i) \cap Y = \{u\}$. Assume that $A''_i \cap Y = \{u\}$. Then, no vertex in B''_i is connected to $\{v, w\}$.*

Lemma 12. *For any vertex v in X_5^3 and any vertex w in S, there exists a vertex u in $N(w)$ with $|E(w,u)| > |E(v,u)|$ such that a basic block $C_i'' = (A_i'', B_i'')$ in $\mathcal{C}''$ can be found with $V(C_i'') \cap Y = \{u\}$. Assume that $A_i'' \cap Y = \{u\}$. Then, no vertex in B_i'' is connected to $\{v, w\}$.*

Let $S' = S \backslash Q$. We now construct basic blocks of $\mathcal{C}'''$ by vertices in S', X_5^2 and X_5^3. Assume that $|S'| \geq |X_5^2| + |X_5^3|$. For a vertex v in X_5^2 and any vertex w in S', by Lemma 11, there exists a vertex u in $N(v)$ with $|E(v,u)| > |E(w,u)|$ such that a basic block $C_i'' = (A_i'', B_i'')$ in $\mathcal{C}''$ can be found with $V(C_i'') \cap Y = \{u\}$. Without loss of generality, assume that u is contained in A_i''. A new basic block $C_i''' = (A_i''', B_i''')$ can be constructed from C_i'' by the following process: $B_i''' = B'' \cup \{v\}$ and $A_i''' = A'' \cup \{w\}$. Let $C_h''' = C_h'' \backslash \{v, w\}$, $\mathcal{C}''' = (\mathcal{C}'' - C_i'' - C_h'') \cup \{C_i'''\} \cup \{C_h'''\}$. By Lemma 11, no vertex in B_i'' is connected to $\{v, w\}$. It is easy to see that $r(C_i''') - r(C_i'') \geq 1$.

For a vertex v in X_5^3 and any vertex w in S', by Lemma 12, there exists a vertex u in $N(w)$ with $|E(w,u)| > |E(v,u)|$ such that a basic block $C_j'' = (A_j'', B_j'')$ in $\mathcal{C}''$ can be found with $V(C_j'') \cap Y = \{u\}$. Without loss of generality, assume that u is contained in A_j''. A new basic block C_j''' can be constructed from C_j'' by the following process: $B_j''' = B'' \cup \{w\}$ and $A_j''' = A'' \cup \{v\}$. Let $C_h''' = C_h'' \backslash \{v, w\}$, $\mathcal{C}''' = (\mathcal{C}'' - C_j'' - C_h'') \cup \{C_j'''\} \cup \{C_h'''\}$. By Lemma 12, no vertex in B_j'' is connected to $\{v, w\}$. It is easy to see that $r(C_j''') - r(C_j'') \geq 1$.

By considering all vertices in $X_5^2 \cup X_5^3$, we can get that

$$r(\mathcal{C}''') - r(\mathcal{C}'') \geq |X_5^2| + |X_5^3|. \tag{8}$$

For the case when $|S'| < |X_5^2| + |X_5^3|$, a similar construction process can be obtained as above. By considering all vertices in S', we can get that

$$r(\mathcal{C}''') - r(\mathcal{C}'') \geq |S'|. \tag{9}$$

Rule 2. For a given instance (G, k) of PMBTLB, if $|S| + |X_5^1| > \frac{n}{2}$, then arbitrarily delete $|S| + |X_5^1| - \frac{n}{2}$ vertices from S.

Lemma 13. *Rule 2 is safe.*

Proof. Assume that $|S| + |X_5^1| > \frac{n}{2}$, and assume that (V_1, V_2) is a maximum bisection of G. Since $|S| > |X_0|$, S is not empty. We prove this lemma by discussing whether X_5^1 is empty or not.

(a) $X_5^1 \neq \emptyset$. Assume that $\mathcal{H} = \{H_1, \ldots, H_l\}$ is the set of connected components in $G[X]$ containing one vertex of X_5^1, and assume that $S = \{v_1, \ldots, v_j\}$. We now prove that all vertices in S cannot be contained totally in V_1 or V_2. Assume that all vertices of S are contained in V_1. Since $|S| + |X_5^1| > \frac{n}{2}$, there must exist a connected component H_i in $\mathcal{H}$ such that all vertices in $V(H)$ are contained in V_2. Choose a vertex v in H_i, and find a vertex u of S in V_1. Let $V_1' = (V_1 - \{u\}) \cup \{v\}$, $V_2' = (V_2 - \{v\}) \cup \{u\}$. Then, $|E(V_1', V_2')| - |E(V_1, V_2)| \geq 1$, contradicting the fact that (V_1, V_2) is a maximum bisection of G. Therefore, $V_1 \cap S \neq \emptyset$ and $V_2 \cap S \neq \emptyset$.

(b) $X_5^1 = \emptyset$. Since $|S| > \frac{n}{2}$, $V_1 \cap S \neq \emptyset$ and $V_2 \cap S \neq \emptyset$.

Assume that S' and S'' are two subsets of S such that S' is contained in V_1, and S'' is contained in V_2. Without loss of generality, assume that $|S'| \leq |S''|$. Let R be any subset of S'' of size $|S'|$. Let $V_1'' = V_1 - S'$, $V_2'' = V_2 - R$. Then, $|V_1''| = |V_2''|$. Denote the new graph with bisection (V_1'', V_2'') by G'. It is easy to see that $|E(V_1, V_2)|$ is bounded by $\lceil m/2 \rceil + k$ if and only if $|E(V_1'', V_2'')|$ is bounded by $\lceil m'/2 \rceil + k$, where m' is the number of edges in G'. Repeat the above process until $|S| + |X_5^1| \leq \frac{n}{2}$. □

For a given instance (G, k) of PMBTLB, by applying Rule 2 on G exhaustively, we can get following result.

Lemma 14. *For a given instance (G, k) of PMBTLB, if $|S| > |X_0|$, then the number of vertices in G is bounded by $8k$.*

Proof. Based on block cluster $\mathcal{C}'''$, we prove this lemma by analyzing the sizes of S', X_5^2 and X_5^3.

(1) $|S'| \geq |X_5^2| + |X_5^3|$. By Lemma 3 and inequalities (1), (7) and (8), we can get that $r(\mathcal{C}''') = |MM| + \frac{|X_2 \setminus S|}{2} + |X_0| + |X_5^2| + |X_5^3| < 2k$. Since $|MM| = (|Z| + |Y| + |X \setminus (X_0 \cup X_2 \cup X_5)|)/2$, we can get that $|Z| + |Y| + |X \setminus (X_0 \cup S \cup X_5)| + 2(|X_0| + |X_5^2| + |X_5^3|) < 4k$. Since $X_5 = X_5^1 \cup X_5^2 \cup X_5^3$, and S, X_5^1, X_5^2, X_5^3, X_0 are disjoint, we have

$$|Z| + |Y| + |X| - |S| - |X_5^1| + |X_0| + |X_5^2| + |X_5^3| < 4k. \tag{10}$$

By Lemma 13, $|S| + |X_5^1| < n/2$. Then, we can get that

$$\begin{aligned} |V| = |Z| + |Y| + |X| &= |Z| + |Y| + |X \setminus (S \cup X_5^1)| + |S| + |X_5^1| \\ &= \underbrace{|Z| + |Y| + |X| - |S| - |X_5^1|}_{<4k \text{ by } (10)} + |S| + |X_5^1| \\ &< 4k + |V|/2. \end{aligned}$$

Therefore, $|V| < 8k$.

(2) $|S| < |X_5^2| + |X_5^3|$. By Lemma 3 and inequalities (1), (7) and (9), we have $r(\mathcal{C}''') = |MM| + \frac{|X_2 \setminus S|}{2} + |X_0| + |S'| < 2k$. Since $Q \subseteq S$, $S' = S \setminus Q$ and $|X_0| = |Q|$, we can get that $r(\mathcal{C}''') = |MM| + \frac{|X_2 \setminus S|}{2} + |S| < 2k$. Similarly, $|MM| = (|Z| + |Y| + |X \setminus (X_0 \cup X_2 \cup X_5)|)/2$. Then, $|Z| + |Y| + |X \setminus (X_0 \cup S \cup X_5)| + 2|S| < 4k$. Since S, X_5, X_0 are disjoint and $|X_0| < |S|$, we can get that

$$|Z| + |Y| + |X| - |X_5| < 4k. \tag{11}$$

By Lemma 6, $|X_5| < 2k$. Then, we can get that

$$\begin{aligned} |V| = |Z| + |Y| + |X| &= |Z| + |Y| + |X \setminus X_5| + |X_5| \\ &= \underbrace{|Z| + |Y| + |X| - |X_5|}_{<4k \text{ by } (11)} + |X_5| \\ &< 4k + 2k = 6k. \end{aligned}$$

Therefore, if $|X_0| < |S|$, the number of vertices in G is bounded by 8k. □

For a given instance (G, k) of PMBTLB, our kernelization algorithm is to apply Rule 1 and Rule 2 on G exhaustively. By Lemmas 10 and 14, we can get the following result.

Theorem 1. *The PMBTLB problem admits a vertex kernel of size* $8k$.

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