# Chapter 9
# Homomorphisms from Functional Equations in Probability

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*"Niedaleko jabłko spada od jabłoni"*
*"The apple never falls far from the tree"*

**Abstract** We showcase the significance to probability theory of homomorphisms and their simplifying rôle by reference to the Goldie functional equation (*GFE*), an equation at the heart of regular variation theory (RV) encoding asymptotic flows, but with an apparent lack of symmetry. Like the Gołąb–Schinzel equation (*GS*), of which it is a disguised equivalent, it and its Pexiderized form can be transmuted into homomorphy under a 'generalized circle product' due to Popa, conformally with the *Pompeiu equation*. This not only forges a specific direct connection to Beurling's Tauberian Theorem, but also generally both helps simplify classical RV-analysis, lending it a flow-type intuition as a guide, and elevates it to unfamiliar contexts. This is illustrated by a new approach to the one-dimensional random walks with stable laws.

We review some new literature, offer some new insights and, in Sections 9.4 and 9.5, some new contributions; possible generalizations are indicated in Section 9.6.

**Keywords** Random walks • Stable laws • Goldie equation • Gołąb–Schinzel equation • Regular variation • Circle groups • Hypergroups

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## 9.1 Introduction

The leitmotif of this article is the critical rôle of homomorphisms underlying several of the functional equations arising in probability theory. When homomorphy is patently present in a functional equation, then it surely guides the work of extending classical theorems to a wider context. As for the converse: if absent, one seeks out any latent structures capable of expressing homomorphy, and so of bearing the fruits of unity and clarity—through closeness to a paradigm, as in the introductory motto. We offer several examples, both old and new.

Generally speaking, functional equations, more properly their (continuous) solutions, play a significant rôle in the asymptotic analysis needed to elicit the characterization of various laws in probability theory (see [58] for the origins of such a programme). Below we meet familiar examples of functional equations in such situations.

The classical context of $\mathbb{R}$ generalizes naturally to the metric-group frameworks of harmonic analysis: a general locally compact group $G$, alternatively a linear space—indeed a Hilbert space $H$. A remarkable instance of generalization is to be seen in the characterization of infinitely divisible laws, which on $\mathbb{R}$ goes back to Lévy and Khintchine; here the most basic is the *Cauchy functional equation (CFE)* in the general form of a *homomorphy equation* between groups:

$$\chi(xy) = \chi(x)\chi(y)\,, \tag{CFE}$$

its (continuous) solutions termed *characters*, and the symmetric bi-homomorphy variant:

$$\Psi(xy, z) = \Psi(x, z)\Psi(y, z) \qquad \text{with } \Psi(x, y) = \Psi(y, x)\,.$$

In the bi-additive case $\Psi : G^2 \to \mathbb{R}$, putting

$$\psi(x) := \Psi(x, x)$$

yields the important associated *quadratic form* $\psi : G \to \mathbb{R}$, which may be equivalently defined (as in [74, Section 6, (6.1)], or with more explicit details as in [50, L. 5.2.4]) by the *Apollonius* or *quadratic functional equation*:

$$\psi(xy) + \psi(xy^{-1}) = 2(\psi(x) + \psi(y))\,;$$

see also [2, Section 11.1; cf. Chapter 8, the related d'Alembert equation], [84, Chapter 13], and [86, Section 2.2], the latter in connection with the Chebyshev 'polynomial hypergroup'—for which see [22], Section 9.6 (and presently below). Their continuous solutions are critical in establishing the characterization of a Gaussian measure $\mu$ [27] either on a locally compact abelian $G$, or in Hilbert space $H$, along the following lines.

The first of the three equations above introduces duality considerations into a locally compact abelian $G$, employing the group $\hat{G}$ of continuous unitary characters $\chi : G \to \mathbb{T}$, with $\mathbb{T}$ the unit circle group in $\mathbb{C}$, and draws on the Pontryagin structure theorem for $G$. That and the third equation, with $\hat{G}$ replacing $G$, yields a functional characterization of a Gaussian measure $\mu$ via its Fourier transform $\hat{\mu}$: for some $g \in G$,

$$\hat{\mu}(\chi) = \chi(g)\exp(-\psi(\chi)) \qquad (\chi \in \hat{G}).$$

For details see [74, IV Theorem 6.1], or [50, Section 5.2], [85, Section 3.2]; for an example see Section 9.3.3 below. A similar result holds in Hilbert space, which is of course self-dual, so $H$ replaces both $G$ and $\hat{G}$ above—see [74, VI Theorem 4.9].

Noteworthy is that the last formula speaks *entirely* in the language of homomorphy.

Indeed, also the Fourier transformation taking $\mu$ to its 'characteristic function' (which uniquely determines the measure):

$$\hat{\mu}(\chi) := \int_G \chi(g)d\mu(g), \tag{†}$$

is itself both an additive and multiplicative homomorphism (on the measures on $G$, which form a semigroup under convolution).

A further ubiquitous functional equation is the *Gołąb–Schinzel* equation [43], cf. [30]:

$$\eta(v + u\eta(v)) = \eta(u)\eta(v) \qquad (u, v \in \mathbb{R}), \tag{GS}$$

whose continuous solutions that are *positive on* $\mathbb{R}_+$ (briefly: *positive*) satisfy for some $\rho \geq 0$

$$\eta(t) \equiv \eta_\rho(t) := 1 + \rho t \qquad (t \in \mathbb{R}_+).$$

For a new approach to the proof see Section 9.5. We write $\eta \in GS$ to mean that $\eta$ satisfies (*GS*). Equation (*GS*) is the focus for much of the text below, for good reason: indeed, for three reasons.

The classical theory of regular variation, RV for short, introduced by Karamata, studies for $f : \mathbb{R}_+ \to \mathbb{R}_+ := (0, \infty)$ the limit function

$$\kappa(t) = \kappa_f(t) := \lim_{x\to\infty} \frac{f(tx)}{f(x)} \qquad (t \in \mathbb{A}),$$

or *Karamata kernel*, with domain $\mathbb{A} \subseteq \mathbb{R}_+$; if $\mathbb{A} = \mathbb{R}_+$, $f$ is called *regularly varying*. This is the *multiplicative* formulation, thematic here and of practical significance; for the additive variant, more convenient in theoretical considerations (for instance, in Section 9.3.1), see Section 9.7(1). The standard text for RV is [21], BGT below.

(There is also an associated notion of regularly varying measures: see [53], or [79], and Section 9.7(4) below.) In his seminal text on probability Feller laid claim to RV as an important tool: the opening second paragraph of [42, VIII.8], motivating the significance of RV to probability theory, highlights the *quantifier weakening* aspect (visited below) of being prepared to work on the premise of good limiting behaviour (as above) but initially on only a *dense* set $\mathbb{A}$ (cf. Section 9.2.1).

Above, if $\mathbb{A} = \mathbb{R}_+$ and if $\kappa \equiv 1$, then $f$ is called *slowly varying*. In general, however, as $\kappa$ satisfies the multiplicative Cauchy equation:

$$\kappa(st) = \kappa(s)\kappa(t)\,,$$

a regularly varying function that is measurable/Baire (i.e. with the Baire property) has a natural characterization as the product of a power function with a slowly varying factor.

For the purposes of extending the Wiener Tauberian theorem (Section 9.2.6 below) to encompass the Borel summability method (cf. [16, Section 1]), Beurling introduced what we now know as *Beurling slow variation*, BSV, employing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\frac{\varphi(x + t\varphi(x))}{\varphi(x)} \to 1,$$

with $\varphi(x) = o(x)$ as $x \to \infty$. This includes the case, significant to the Borel and Valiron summability method, of

$$\varphi(x) := \sqrt{x}.$$

Such functions are called *self-neglecting,* $\varphi \in SN$, provided a further technical condition holds, that the convergence is locally uniform in $t$. Conditions implying self-neglect are studied in [16], where for $\varphi \in SN$ a more comprehensive theory of $\varphi$-regular variation is established by studying limit functions

$$g(t) := \lim_{x\to\infty} \frac{f(x + t\varphi(x))}{f(x)} \qquad (t \in \mathbb{A})\,.$$

It also emerged in [16, Section 10.3] (a matter followed through in [18]) that an even more satisfactory development may be had by going beyond BSV to obtain the even more comprehensive notion of *Beurling regular variation*, BRV, which encompasses both the Karamata theory and the related Bojanić-Karamata/de Haan theory (cf. BGT Chapter 3). BRV is built around functions $\varphi$ that are *self-equivarying*, as in [71]; for these functions a limit value more general than the ‘1’ above is permitted, so that

$$\frac{\varphi(x + t\varphi(x))}{\varphi(x)} \to \eta^{\varphi}(t) \qquad (t \in \mathbb{A})\,, \tag{$SE_{\mathbb{A}}$}$$

here with $\mathbb{A} = \mathbb{R}_+$ (but see Proposition 1 in Section 9.2.3), and this convergence demands a side-condition of local uniformity as in ($SN$) above (and $\varphi(x) = O(x)$). For $\eta \equiv 1$, these specialize to the *self-neglecting* functions of Beurling, as above. The key result from [71] is that the limit function $\eta^{\varphi}(t)$ satisfies ($GS$), and this is the *first reason* for interest in ($GS$) in RV.

The *second reason* is that ($GS$) may be 'converted' very simply into an expression of homomorphism and so throws much light on an alternative form of the equation occurring in RV, 'disguised from birth' in [11] (cf. BGT Lemma 3.2.1), now known as the *Goldie equation*. The latter contains a further *auxiliary function* $\psi$ and takes the form:

$$K(x + y) - K(y) = \psi(y)K(x) \qquad (x, y \in \mathbb{R}_+)\,. \tag{$GFE$}$$

In the functional equations literature this is a special case of the *Levi-Civita* functional equation, albeit a *conditioned* one, as the quantifiers are bounded: quantifying over $\mathbb{R}_+$—cf. [84, Section 5.4]. However, tracing the direct connection of ($GFE$) to ($GS$), and so to homomorphy, brings untold benefits: see Section 9.2.4, as already mentioned.

The 'algebraicization' needed to release these benefits originates with a largely forgotten contribution, due first to Popa [76] and later Javor [57], based on the binary operation, generated from an arbitrary $\eta : \mathbb{R} \to \mathbb{R}$:

$$u \circ_\eta v := u + v\eta(u)\,,$$

for which see Section 9.2.3 below (cf. [28]). This may be traced back to the 'circle product' of ring theory:

$$x \circ y := x + y + xy\,;$$

indeed, $\circ_\eta$ reduces to just that for

$$\eta(x) = 1 + x.$$

(For historical background see [72, Section 2.1].) This binary operation re-expresses ($GS$) as homomorphy:

$$\eta(u \circ_\eta v) = \eta(u)\eta(v)\,,$$

where the right-hand side may be interpreted in various group structures (e.g. the *Pompeiu equation* of [84, Example 3.24], where the original circle product $\circ$ appears on both sides).

The *third reason* can now be declared as the benefit of homomorphy: homomorphism into $(\mathbb{R}_+, \times)$ lessens the burdens of proof in the Beurling theory of regular variation: the algebra becomes virtually identical to that of the $\mathbb{R}_+$ classical theory, leaving only the analysis of local uniformity to be undertaken (cf. Theorem 6 below).

We therefore advocate a more systematic use of the tool of homomorphy, as a unifier and clarifier.

The bulk of the material below falls naturally into two parts: first Sections 9.2 and 9.3, and then Sections 9.4 and 9.5, as follows.

In the first part, Section 9.2 discusses (*GFE*), indicating its relation to (*GS*), and considers the Popa operation $\circ_\eta$. We then describe the connection with the *Beurling Tauberian Theorem*, a proper extension of the celebrated *Wiener Tauberian Theorem*. In passing, we indicate briefly how to solve (*GFE*) using integration, which also permits a side glance at the rôle of flows—a natural consequence of the presence of a *group action*. In Section 9.3 we pass beyond Karamata kernels to the *Beurling kernels* of BRV, and as an application sketch how (*GFE*) helps to deduce very directly the form of *stable laws* associated with one-dimensional *random walks* (i.e. walks on the additive group $\mathbb{R}$—see [9] for an a very informative survey of the theory and application of random walks). The starting point is their *characteristic functional equation* (*ChFE*), which is briefly deduced ab initio and then reduced after some work to (*GFE*)—see Section 9.3.3 below. We also indicate further literature.

The second part, comprising Sections 9.4 and 9.5, contains new contributions as supporting material: a new theorem about (*ChFE*) and novel approaches to solutions of (*GS*) that are positive (on $\mathbb{R}_+$). The latter functions play a significant rôle in RV, so direct proofs are of interest.

We complete the circle of ideas in Section 9.6, ending as we began: with the theme of homomorphy—noting how the characteristic functions of random walks on some other groups give rise to an *integrated functional equation* (IFE)—for background here see [78], inspired by the work of Choquet and Deny [32]. However, the more natural setting for these is that of a *hypergroup* structure (sketchily reproduced below) with binary operation $\star$ and involution, within which these particular IFEs again reduce to a homomorphy:

$$K(x \star y) = K(x)K(y)\,. \tag{$\star$}$$

In brief, cf. [22], or [86, 87]: a *hypergroup* has as underlying domain a topological space $X$ (possibly a topological group). The topology may be discrete. Upon this space is imposed (axiomatically) both a measure-theoretic and a group-like structure: first, the points $x$ of $X$ are identified with probability measures $\delta_x$ degenerate at the points of $X$; then a binary operation $\star$ is introduced on these (later extended to a wider domain of measures), and is interpreted much as convolution, so as to yield a probability measure with compact support (continuously mapped to the hyperspace $\mathscr{K}(X)$ of (nonempty) compact subsets of $X$, the latter equipped with the topology inherited from the *Vietoris* topology [41, 2.7.20] on the (nonempty) closed subsets, known also as the *Michael* topology, in view of the contribution [67]); and lastly, an involution operation is provided on the point-masses.

This allows a very broad algebraicization of random 'dynamics', generated by $\star$, within which measures describe the location of 'random points' of $X$. Sometimes the hypergroup is not much more than a group, as when

$$\delta_x \star \delta_y := \delta_{xy}.$$

But often the introduction of $\star$ calls for some quite intriguing ingenuity—as the two examples of Section 9.6 show.

We close in Section 9.7 with complements, including in Section 9.7(4) indications of some generalizations.

## 9.2 From Beurling via Goldie to Gołąb–Schinzel

We begin with a discussion of Equation (*GFE*) introduced in Section 9.1.

### *9.2.1 The Goldie Equation*

In RV Equation (*GFE*) emerges from asymptotic analysis (see Section 9.3.1) and is initially valid on a *subset* of $\mathbb{R}$ (as the domain of convergence of a limit operation), so it is natural to formalize this phenomenon by *weakening the quantifiers*, as indicated in Section 9.1, allowing the free variables to range over a set $\mathbb{A}$ smaller than $\mathbb{R}$, which typically will be a subgroup that is dense. (There is an implicit appeal to Kronecker's density theorem here and the presence of two incommensurable elements in $\mathbb{A}$.) The functional equation in the result below, denoted by ($G_{\mathbb{A}}$), is thus a second form of the Goldie functional equation. As we see in Theorem 1 below, the two coincide in the principal case of interest—compare the insightful Footnote 3 of [26]. The notation $H_\gamma$ below (originating in [26]) is from BGT Sections 3.1.7 and 3.2.1, implying

$$H_0(t) \equiv t.$$

Equation ($G_{\mathbb{A}}$) below when $\mathbb{A} = \mathbb{R}$ is a special case of a generalized Pexider equation studied by Aczél [1]. In Theorem 1 (*CEE*) is the *Cauchy exponential equation*. Versions of the specific result here, taken from [17, Theorem 1] (where the proof—based on the *Cauchy nucleus* of $K$ [63, Section 18.5]—may be consulted), also appear elsewhere in the literature.

**Theorem 1 ([26, (2.2)], BGT Lemma 3.2.1; cf. [3], [84, Proposition 5.8])** *For $\psi$ with $\psi(0) = 1$, if $K \not\equiv 0$ satisfies*

$$K(u + v) = \psi(v)K(u) + K(v) \qquad (u, v \in \mathbb{A}), \tag{$G_{\mathbb{A}}$}$$

*with $\mathbb{A}$ a dense subgroup, then:*

(i) *the following is an additive subgroup on which $K$ is additive:*

$$\mathbb{A}_\psi := \{u \in \mathbb{A} : \ \psi(u) = 1\};$$

(ii) *if $\mathbb{A}_\psi \neq \mathbb{A}$ and $K \not\equiv 0$, there is a constant $\kappa \neq 0$ with*

$$K(t) \equiv \kappa(\psi(t) - 1) \qquad (t \in \mathbb{A}), \tag{*}$$

*and $\psi$ satisfies*

$$\psi(u+v) = \psi(v)\psi(u) \qquad (u, v \in \mathbb{A})\,. \tag{CEE}$$

(iii) *So for $\mathbb{A} = \mathbb{R}$ and $\psi$ locally bounded at* $0$ *with $\psi \neq 1$ except at* $0$ :

$$\psi(x) \equiv e^{-\gamma x}\,,$$

*for some constant $\gamma \neq 0$, and so $K(t) \equiv cH_\gamma(t)$ for some constant c, where*

$$H_\gamma(t) := (1 - e^{-\gamma t})/\gamma\,.$$

For the needs of Section 9.5 below, we note briefly that the proof rests on symmetry in the equation:

$$\begin{aligned}\psi(v)K(u) + K(v) &= K(u+v) = K(v+u)\\ &= \psi(u)K(v) + K(u)\,.\end{aligned}$$

So, for $u$, $v$ not in $\{x : \psi(x) = 1\}$, an additive subgroup,

$$K(u)[\psi(v) - 1] = K(v)[\psi(u) - 1],$$

$$\frac{K(u)}{\psi(u) - 1} = \frac{K(v)}{\psi(v) - 1} = \text{const.} = \kappa,$$

as in BGT Lemma 3.2.1. If $K(\cdot)$ is to satisfy (*GFE*), $\psi(\cdot)$ needs to satisfy (*CEE*).

### *9.2.2 The Disguised GS*

By Theorem 1, assuming its local boundedness, the *auxiliary* function $\psi$ of (*GFE*) is exponential; with this in mind, we can trace the connection to (*GS*) as follows.

Recall from Section 9.1 that a function is *positive* if it takes positive values on $\mathbb{R}_+$.

Recall also that the positive (and likewise, ultimately, the continuous) solutions of (*GS*) take the form

$$\eta \equiv \eta_\rho(x) := 1 + \rho x\,,$$

with $\rho > 0$, for $x > \rho^* := -\rho^{-1}$—see Section 9.5. Writing (*GS*) in the form

$$\eta(a + \eta(a)b) = \eta(a)\eta(b)\,,$$

put

$$A := \eta(a) > 0, \quad B := \eta(b) > 0,$$

and take $f := \eta_\rho^{-1}$ (which exists to the right of $\rho^*$); then $a = f(A), b = f(B)$. Applying $f$ to (*GS*) yields

$$a + Ab = f(AB) : \qquad f(A) + Af(B) = f(AB)\,.$$

Apply the logarithmic transformation: $u = \log A$, $v = \log B$, set $K(x) := f(e^x)$; then

$$f(e^u) + e^u f(e^v) = f(e^{u+v}) : \qquad K(u) + e^u K(v) = K(u+v)\,.$$

The reverse direction can be effected for non-trivial (i.e. invertible) solutions $K$ of this last equation – see [20, §7].

### 9.2.3 Popa (Circle) Operation: Basics

The operation

$$x \circ_\eta y := x + y\eta(x)\,,$$

with $\eta : \mathbb{R} \to \mathbb{R}$ arbitrary, was introduced in 1965 for the study of Equation (*GS*) by Popa [76], and later Javor [57] (in the broader context of $\eta : \mathbb{E} \to \mathbb{F}$, with $\mathbb{E}$ a vector space over a commutative field $\mathbb{F}$), who observed that this equation is *equivalent to the operation* $\circ_\eta$ *being associative* on $\mathbb{R}$, and that then $\circ_\eta$ confers a group structure on $\mathbb{G}_\eta := \{g \in \mathbb{R} : \eta(g) \neq 0\}$—see [76, Proposition 2], [57, Lemma 1.2]. We term this a *Popa circle group*, or *Popa group* for short, as the case

$$\eta_1(x) = 1 + x$$

(i.e. for $\rho = 1$ above, so a translation) yields precisely the circle group of the ring $\mathbb{R}$, as noted in Section 9.1.

The operation $\circ_\eta$ turns $\eta$ into a homomorphism from

$$\mathbb{G}_\eta^+ = \{g \in \mathbb{G}_\eta : \eta(g) > 0\}$$

to $(\mathbb{R}_+, \times)$. For $\eta = \eta^\varphi$, arising from $\varphi \in SE$ as in ($SE_{\mathbb{A}}$) with natural domain $\mathbb{A} = \mathbb{R}_+$, one may in fact extend the definition of $\eta^\varphi$ from $\mathbb{R}_+$ to $\mathbb{G}_\eta^+$ preserving homomorphy, as we see presently (Proposition 1). Below, when

$$\eta(t) = 1 + \rho t,$$

we use the variants $(\mathbb{G}_\eta, \circ_\eta)$ and $(\mathbb{G}_\rho, \circ_\rho)$ interchangeably and call $\rho^* := -\rho^{-1}$ the *Popa centre* of $\mathbb{G}_\rho$. Other notation associated with $\mathbb{G}_\eta$ includes $1_\eta$ for the neutral element, and $t_\eta^{-1}$ for the inverse of $t$, and obvious variants of these.

**Proposition 1 (Non-zero Uniform Involutive Extension, [18, L.1])** *For $\varphi \in SE$, $\circ = \circ_\rho$ with $\rho = \rho_\varphi > 0$, put*

$$\eta^\varphi(t_\circ^{-1}) = \eta^\varphi(-t/\eta^\varphi(t)) := 1/\eta^\varphi(t) \qquad (t > 0)\,;$$

*then ($SE_{\mathbb{A}}$) holds for $\mathbb{A} = \mathbb{G}_+^\rho = (\rho^*, \infty)$. Moreover, this is a maximal non-vanishing extension: for each $s < \rho^*$, assuming $\varphi(x + s\varphi(x)) > 0$ is defined for all large $x$,*

$$\lim_{x\to\infty} \eta_x^\varphi(s) = \lim_{x\to\infty} \varphi(x + s\varphi(x))/\varphi(x) = 0 = \eta(\rho^*)\,.$$

Here we see the critical rôle of the Popa origin $\rho^* = -\rho^{-1}$ : the domain of the limit operation

$$\lim_{x\to\infty} \eta_x^\varphi(s),$$

used to extend $\eta^\varphi$, is $\mathbb{G}_+^\rho$. So the argument $s$ here has to take values to the right of the Popa origin. As $\rho \to 0+$ the Popa centre recedes to $-\infty$ and this extension falls into line with the natural extension to $\mathbb{R}_-$ (taken for granted) in the Karamata theory: see BGT (2.11.2).

With this much isomorphy in place (in fact conjugacy with $\mathbb{R}$), it is natural to seek further group structures in order to allow (*GFE*), as a statement about $K$, to assert homomorphism between Popa groups:

$$K(x \circ_\eta y) = K(y) \circ_\sigma K(x) \text{ for some } \sigma \in GS\,, \qquad (GBE)$$

with the side-condition

$$\sigma(K(y)) \equiv \psi(y).$$

We term the above the *Goldie–Beurling equation* (*GBE*), acknowledging the Beurling connection via $\eta$; it is a natural extension of the *Pompeiu equation* to which it reduces when $\eta \equiv \sigma \equiv \eta_1$ [84, Example 3.24], and so links with results not only of Aczél, but also of Chudziak [33–35], and Jabłońska [55], concerned with the equation

$$f(x \circ_g y) = f(x) \circ f(y) \qquad (ChE)$$

with $f : \mathbb{R} \to (S, \circ)$ for $(S, \circ)$ some group or semigroup, and $g : \mathbb{R} \to \mathbb{R}$ continuous, or locally bounded above.

Javor's observation regarding associativity has interesting corollaries. (Recall that *positive* means positive on $\mathbb{R}_+$.)

**Lemma$_{\mathbf{com}}$ ([72])** *If* (*GBE*) *holds for some injective $K$, $\sigma$ with $\circ_\sigma$ commutative, and $\eta : \mathbb{R}_+ \to \mathbb{R}$, then*

$$\eta(u) \equiv 1 + \rho u,$$

*for some constant* $\rho$.

*Proof* Here

$$K(u + v\eta(u)) = K(u) \circ_\sigma K(v) = K(v) \circ_\sigma K(u) = K(v + u\eta(v)),$$

as $\circ_\sigma$ is commutative. By injectivity, for all $u, v \geq 0$,

$$u + v\eta(u) = v + u\eta(v) : \quad u(1 - \eta(v)) = v(1 - \eta(u)),$$

so, as in Theorem 1,

$$(\eta(u) - 1)/u \equiv \rho = \text{const.},$$

for $u > 0$; taking $v = 1$ above,

$$\eta(u) \equiv 1 + \rho u,$$

for all $u \geq 0$. □

**Lemma$_{\mathbf{assoc}}$ ([72])** *If* (*GBE*) *holds for some injective* $K$, $\sigma$ *with* $\circ_\sigma$ *associative, and a positive continuous* $\eta : \mathbb{R} \to \mathbb{R}$, *then*

$$\eta(u) = 1 + \rho u \quad (u \geq 0),$$

*for some constant* $\rho$.

*Proof* This follows, e.g., from Javor's observation above connecting associativity with (*GS*) [57, p. 235].

## *9.2.4 Creating Homomorphisms*

In this section we demonstrate how to convert two functional equations into expressions of homomorphy. The immediate use this serves is to enable the solutions to be 'read back' from those of the Cauchy functional equation (*CFE*), as in Theorem 5 below. This process is captured in the following routine result concerning (*GBE*). For

$$\circ_\eta = \circ_0 \quad \text{and} \quad \circ_\sigma = \circ_\infty,$$

the equation reduces to the exponential format of (*CFE*) ([63, Section 13.1]; cf. [54]). The critical case for Beurling regular variation is for $\rho \in (0, \infty)$, with positive continuous solutions described in the table below; the four corner formulas correspond to classical variants of (*CFE*). The proof, which we omit, proceeds by a straightforward reduction to a classical variant of (*CFE*) by an appropriate shift and rescaling.

**Proposition 2 ([72, Proposition A]; cf. [33])** *For*

$$\circ_\eta = \circ_r, \quad \circ_\sigma = \circ_s,$$

*and $K$ Baire/measurable satisfying* (*GBE*), *there is $\gamma \in \mathbb{R}$ so that $K(t)$ is given by:*

| *Popa parameter* | $s = 0$ | $s \in (0, \infty)$ | $s = \infty$ |
|---|---|---|---|
| $r = 0$ | $\gamma t$ | $(e^{\gamma t} - 1)/s$ | $e^{\gamma t}$ |
| $r \in (0, \infty)$ | $\gamma \log(1 + rt)$ | $[(1 + rt)^\gamma - 1]/s$ | $(1 + rt)^\gamma$ |
| $r = \infty$ | $\gamma \log t$ | $(t^\gamma - 1)/s$ | $t^\gamma$ |

Below and elsewhere a function $K$ is *non-trivial* if $K \not\equiv 0$ and $K \not\equiv 1$.

**Theorem 2 (Conversion to Homomorphy, [72, Theorem 1])** *For $\eta \in GS$ in the setting above,* (*GBE*) *holds for positive $\psi$ in the side-condition and a non-trivial $K$ iff*

(i) *$K$ is injective;*
(ii) *$\sigma =: \psi K^{-1} \in GS$, equivalently, either $\psi \equiv 1$, or, for some $s > 0$,*

$$K(u) \equiv (\psi(u) - 1)/s \text{ and } \psi(0) = 1, \text{ so } K(0) = 0\,;$$

(iii)

$$K(x \circ_\eta y) = K(x) \circ_\sigma K(y)\,. \tag{Hom-1}$$

*Then*

(iv) *for some constants $c, \gamma$,*

$$K(x) \equiv c \cdot [(1 + \rho x)^\gamma - 1]/\rho\gamma, \text{ or } \qquad K(t) \equiv \gamma \log(1 + \rho t)$$
$$(\rho = \rho_\eta > 0),$$
$$\text{or} \qquad K(x) \equiv c \cdot (e^{\gamma x} - 1)/\gamma \qquad (\rho_\eta = 0).$$

A related functional equation replaces one instance of $K$ on the right of (*GBE*) by a further unknown function $\kappa$ multiplying $\psi$, yielding a 'Pexiderized' generalization[1]

$$K(x + y\eta(x)) - K(y) = \psi(y)\kappa(x) \qquad (x, y \in \mathbb{R})\,, \tag{GBE-P}$$

[1]Acknowledging the connection, the qualifier $P$ in (*GBE-P*) is for 'Pexiderized' Goldie–Beurling equation—referring to Pexider's equation: $f(xy) = g(x) + h(y)$ and its generalizations—cf. [29, 30], and the recent [54].

considered also in [36]. Passage to this more general form enables the inclusion of (*GS*) as the case

$$K \equiv \psi \equiv \eta$$

with $\kappa \equiv \eta - 1$.

To apply the earlier argument here, an extension of the Popa binary operation suggests itself; put

$$u \circ v = u \circ_{\alpha\beta} v := \alpha(u) + v\beta(u)\,,$$

with $\alpha, \beta$ continuous and $\alpha$ invertible; this seems reminiscent of [3].

**Proposition 3 ([72, Proposition A])** *The operation $\circ$ is a group operation on $\mathbb{A} \subseteq \mathbb{R}$ with $0 \in \mathbb{A}$ iff $\mathbb{A}$ is closed under $\circ$ and for some constants $b, c$ with $bc = 0$*

$$\alpha(x) \equiv x + b \text{ and } \beta(x) \equiv 1 + cx.$$

*That is:*

$$\alpha(x) \equiv x \text{ and } \beta(x) \equiv 1 + cx, \text{ OR } \alpha(x) \equiv x + b \text{ and } \beta(x) \equiv 1\,.$$

*So this is either a Popa group with*

$$x \circ y = x \circ_c y := x + y(1 + cx),$$

*or the b-shifted additive reals with the operation*

$$x +_b y := x + y + b.$$

*Remark* For the $b$-shifted additive reals, the neutral element is $e := -b$ and $x^{-1} = -x - 2b$.

Applying Proposition 3, we deduce the circumstances when (*GBE-P*) may be transformed to a homomorphism. Here we see that

$$K(x) \equiv (\psi(y) - 1)/s$$

only in the cases (i) and (iii), but not in (ii)—compare Theorem 2. Note that in all cases $\kappa$ is a homomorphism between Popa groups.

**Theorem 2′ (Conversion to Homomorphy, [72, Theorem 1′])** *If (GBE-P) is solved by $K$ for $\psi$ positive, $\kappa$ positive and invertible, $\eta(x) \equiv 1 + \rho x$ (with $\rho \geq 0$), then in the equation below $\circ$ is a group operation and $K^{-1}$ is a homomorphism under $\circ$:*

$$K^{-1}(u \circ_\sigma v) = K^{-1}(u) \circ K^{-1}(v) \qquad (u, v \in \mathbb{R})\,, \tag{Hom-2}$$

*iff* $\sigma := \psi K^{-1} \in GS$ *and one of the following three conditions holds:*

(i) $\rho = 0$, $\circ = \circ_0$ *and* $\circ_\sigma = \circ_s$ *for some* $s > 0$*; then, for some* $\gamma \in \mathbb{R}$,

$$K(t) \equiv \kappa(t) \equiv (e^{\gamma t} - 1)/s\,, \qquad \psi(t) \equiv e^{\gamma t}\,;$$

(ii) $\rho = 0$, $\circ_\sigma = \circ_0$ *and* $\circ = +_b$ *for some* $b \in \mathbb{R}$; *then*

$$K(t) \equiv \kappa(t + b) = \kappa(t) + \kappa(b), \qquad \psi(t) \equiv 1 \qquad (t \in \mathbb{R})\,,$$

*and* $\kappa : \mathbb{G}_0 \to \mathbb{G}_0$ *is linear;*

(iii) $\rho > 0$, $\circ = \circ_\rho$ *and* $\circ_\sigma = \circ_s$ *for some* $s \geq 0$*; then, for some* $\gamma \in \mathbb{R}$,

$$K(t) \equiv \kappa(t) \equiv [(1 + \rho t)^\gamma - 1]/s, \quad (s > 0), \quad \text{or} \quad \gamma \log(1 + rt) \quad (s = 0)\,,$$
$$\psi(t) \equiv (1 + \rho t)^\gamma \quad (s > 0), \quad \text{or} \quad \psi(t) \equiv 1 \quad (s = 0)\,.$$

This recovers results in [33].

### 9.2.5 Beck Sequences, Integration, and Flows

Assuming continuity, we show in this section how to use integration to find the non-trivial solutions of the following variant of (*GFE*):

$$K(x + y\eta(x)) - K(y) = \psi(y)K(x)\,.$$

A key tool here, and also in later sections, is an appropriate partitioning of any interval (range of integration); for this we refer to what we term the *Beck* $\varphi$*-sequence* $t_m = t_m(u)$, defined recursively for $u > 0$ and $\varphi$ a solution of (*GS*) by

$$t_{m+1} = t_m \circ_\varphi u = t_m + u\varphi(t_m) \text{ with } t_0 = 0\,.$$

Albeit present in [43], the systematic use of such iterations seems to stem from Beck's oeuvre on continuous flows in the plane—[5, L. 1.6.4]. The Popa notation inserted above clarifies that this is the sequence of *Popa powers* of $u$ under $\circ_\varphi$ and so may also be written $u_\varphi^m$. So, from the group perspective, this is the natural *discretization* with 'mesh' size $u$ for the purposes of integration. As $\varphi$ is a homomorphism,

$$\varphi(u_\varphi^{m+1}) = \varphi(u)\varphi(u_\varphi^m) = \varphi(u)^{m+1}\varphi(0)\,.$$

So ([17, Theorem 5], or Theorem 8 below) the sequence $t_m$ is divergent, since either $\varphi(u) = 1$ and $t_m = mu$ (directly, from the inductive definition), or else $\varphi(u) \neq 1$

and

$$t_m = u\frac{\varphi(u)^m - 1}{\varphi(u) - 1} = (\varphi(u)^m - 1) \Big/ \frac{\varphi(u) - 1}{u} \qquad (**)$$

—see, e.g., by Ostaszewski [71, L. 4] (cf. a lemma of Bloom: BGT Lemma 2.11.2). In either case, for $u, t > 0$ there exists a unique integer $m = m_t(u)$, the *jump index of* $t$, satisfying

$$t_m \leq t < t_{m+1}\,.$$

**Application: Solutions by Integration** To solve the equation above for $K, \psi$ (and $\eta$) continuous, note that if $K$ is non-trivial with $K(0) = 0$, then for all small enough $u > 0$ we have $K(u)$ non-zero; otherwise the $\circ_\eta$-subgroup[2]

$$\{u : K(u) = 0\}$$

accumulates at the origin, and so is dense in $\mathbb{R}_+$ (forcing $K$ into triviality). Now proceed as follows. Fix $x_0,\ x_1 > 0$, and denote the corresponding jump indices $i_0 = i_0(u)$ and $i_1 = i_1(u)$: so for $j \in \{0, 1\}$

$$t_{i_j} \leq x_j < t_{i_j+1}\,.$$

Now, for the Beck $\eta$-sequence $t_m = u_\eta^m$,

$$K(t_{m+1}) - K(t_m) = K(u)\psi(t_m)\,.$$

Summing, and setting

$$h(t) := \psi(t)/\eta(t) \geq 0 \qquad (t \in \mathbb{R}_+)$$

(valid as $\eta$ is positive),

$$K(t_m) = K(t_m) - K(t_0) = K(u)\sum_{n=0}^{m-1}\psi(t_n) = \frac{K(u)}{u}\sum_{n=0}^{m-1}u\eta(t_n)h(t_n)\,,$$

since $t_0 = 0$.

As above, $K(u) > 0$ for small enough $u > 0$, so we may write with the obvious notation

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2

$$0 = K(0) = K(1_\eta) = K(u \circ_\eta u_\eta^{-1}) = \psi(u_\eta^{-1})K(u) + K(u_\eta^{-1}) = K(u_\eta^{-1}).$$

$$\frac{K(t_{i_1})}{K(t_{i_0})} = \frac{K(u)\sum_{n=0}^{i_1-1}\psi(t_n)}{K(u)\sum_{n=0}^{i_0-1}\psi(t_n)} = \frac{\sum_{n=0}^{i_1-1}u\eta(t_n)h(t_n)}{\sum_{n=0}^{i_0-1}u\eta(t_n)h(t_n)}$$
$$= \frac{\sum_{n=0}^{i_1-1}(t_{n+1}-t_n)h(t_n)}{\sum_{n=0}^{i_0-1}(t_{n+1}-t_n)h(t_n)} \to \frac{\int_0^{x_1}h(t)dt}{\int_0^{x_0}h(t)dt} = \frac{H(x_1)}{H(x_0)}\,.$$

Here passage to the limit in the rightmost terms is as $u \downarrow 0$. Above we assume without loss of generality that $H(x_0) > 0$. (Otherwise $\psi \equiv 0$ on $[0, \infty)$, implying that $K$ is constant and yielding the trivial case $K \equiv 0$.) Passing to the limit as $u \downarrow 0$ in the leftmost term above, by continuity of $K$, as $t_{i_j} \to x_j$

$$K(x_1)/K(x_0) = H(x_1)/H(x_0)\,.$$

Put

$$c := K(x_0)/H(x_0)\,;$$

then, with $x$ for $x_1$,

$$K(x) = cH(x) := c\int_0^x h(t)\,\mathrm{d}t\,,$$

valid for $x \geq 0$, as $K(0) = 0$.

*Remark* When

$$\eta(t) \equiv 1\,, \qquad \psi(t) \equiv e^{\gamma t}\,, \qquad h(t) \equiv e^{\gamma t}\,,$$

the analysis above lends new clarification, via the language of homomorphisms, to the ‘classical relation’ in RV that

$$K = c(\psi - 1)\,,$$

connecting $K$ and the auxiliary function $\psi$, as in Theorem 1.

**Flows (‘Translation Equation’)** Subject to $K(0) = 0$, assuming positivity of $K$ (i.e. to the right of 0), and continuity and positivity of $\psi$, we have just seen that the solution $K$ satisfies, for some $c \geq 0$,

$$K(x) = c \cdot \tau_f(x)\,,$$

for

$$\tau_f(x) := \int_0^x \mathrm{d}u/f(u), \text{ with } f := \eta/\psi\,.$$

Inspired by Beck [5, 5.25], we may interpret $\tau_f$ as the *occupation time measure* (of $[0, x]$) of the continuous $f$-flow: $dx/dt = f(x)$, where $f$ as above measures the relative velocity of $\eta$ and $\psi$. Furthermore, interpreting $\circ_\eta$ as a *flow* or *group action* (yielding the *translation equation*, cf. [69], [77]) it emerges surprisingly that the underlying homomorphy is now expressed not by $K$, but by the *relative flow-velocity* $f$: under mild regularity assumptions, if $K$ solves (*GBE-P*), then $f$ satisfies

$$f(x \circ_\eta y) = f(x)f(y) \qquad (x, y \in \mathbb{R}_+) .$$

There is a converse for $\psi := \eta/f$: see [17, 72].

### *9.2.6 Beurling's Tauberian Theorem*

For $\varphi : \mathbb{R} \to \mathbb{R}_+$ introduce the following 'Beurling convolution':

$$\begin{aligned} F *_\varphi H(x) &:= \int F\Big(\frac{x-u}{\varphi(x)}\Big)H(u)\frac{\mathrm{d}u}{\varphi(x)} \\ &= \int F(-t)H(x + t\varphi(x))\,\mathrm{d}t , \end{aligned}$$

reducing for $\varphi \equiv 1$ to the classical counterpart

$$F * H(x) = \int F(x-t)H(t)\,\mathrm{d}t .$$

See [18] for background. Substitution of $t = (u-x)/\varphi(x)$ yields

$$u = u_x(t) := x + t\varphi(x),$$

so that $t \mapsto u_x(t)$ is a 'speeded-up' version of the $x$-shift $t \mapsto x + t$. This includes for

$$H = (1/a)\mathbf{1}_{[0,a]}$$

and

$$G(x) := \sum_{n<x} g_n$$

the *moving average* 'speeded up' by $\varphi$, introducing alternative summability methods:

$$MA_a^\varphi(x) = G *_\varphi H(x) = \frac{1}{a}\int_x^{x+a\varphi(x)} G(u)du = \frac{1}{a}\sum_x^{x+a\varphi(x)} g_n .$$

**Theorem BT (Beurling's Tauberian Theorem)** *For $K \in L_1(\mathbb{R})$ with $\hat{K}$ non-zero on $\mathbb{R}$, and $\varphi$ Beurling slowly varying, i.e. with*

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1, \qquad (x \to \infty) \qquad (t \ge 0) : \qquad (BSV)$$

*if $H$ is bounded, and the following holds for some $c \in \mathbb{R}$*

$$K *_\varphi H(x) \to c \int K(y)\,\mathrm{d}y\,, \qquad (K *_\varphi H)$$

*then for all* $F \in L_1(\mathbb{R})$

$$F *_\varphi H(x) \to c \int F(y)\,\mathrm{d}y \qquad (x \to \infty)\,.$$

As a sample, we note that the Popa algebraicization enables the following generalization:

**Theorem 3 (Extension to Beurling's Tauberian Theorem, [18, Theorem 2])** *Suppose that:*

(i) $\varphi \in SE$, *i.e. locally uniformly in* $t$

$$\varphi(x + t\varphi(x))/\varphi(x) \to \eta(t) \in GS \qquad (x \to \infty) \qquad (t \geq 0)\,,$$

(ii) $K \in L_1(\mathbb{R})$ *with* $\hat{K}$ *non-zero on* $\mathbb{R}$,
(iii) $H$ *is bounded, and*
(iv) $(K *_\varphi H)$ *holds*
*then for all* $G \in L_1(\mathbb{R})$

$$G *_\varphi H(x) \to c \int G(y)\mathrm{d}y \qquad (x \to \infty)\,.$$

## 9.3 Beurling Kernels

We begin by describing the context in which Beurling kernels arise.

### *9.3.1 Asymptotics*

We refer below again to the self-equivarying functions defined by ($SE$) of Section 9.1. We adopt the additive formulation here. At its simplest, a functional equation such as (*GFE*) arises when taking limits

$$K_F(t) := \lim\nolimits_{x\to\infty}[F(x + t\varphi(x)) - F(x)] = \text{briefly, } \lim \Delta_t^\varphi F(x)\,, \qquad (BK)$$

for $\varphi \in SE$; then, with $\eta$ the associated limit as in ($SE$) above, for $s, t$ ranging over the set $\mathbb{A}$ on which the limit function $K_F$, the *Beurling kernel* of $F$, exists as a locally uniform limit:

$$K_F(s + t) = K_F(s/\eta(t)) + K_F(t) : \qquad K_F(t + s\eta(t)) = K_F(s) + K_F(t)\,.$$

So with $\circ_\eta$ in mind, both $\mathbb{A}$ and $K_F(\mathbb{A})$ carry group structures under which $K_F$ is a homomorphism. Thus, even in the classical context, (*GS*) plays a significant rôle albeit disguised and previously unnoticed, despite its finger-print: namely, the terms $+1$ or $-1$, appearing in the formulas for $K_F$ (as in Theorem 2).

The more general functional equation, arising in Beurling RV, is the *generalized Goldie–Beurling equation* on $\mathbb{R}_+ := [0, \infty)$, noted in Section 9.2.3:

$$K(x + y\eta(x)) - K(y) = \psi(y)K(x) \qquad (x, y \in \mathbb{R}_+) \qquad (GBE_\psi)$$

(in the two unknowns $K$ and $\psi$), where $\eta(x) = \eta_\rho(x)$ for some $\rho \in \mathbb{R}_+$. This arises quite similarly to (*BK*) in the context

$$K(t) = \lim \Delta_t^\varphi F(x) \big/ \Phi(x) \text{ with } \psi(t) := \lim \Phi(x + t\varphi(x))/\Phi(x)\,,$$

assuming these limits exist.

The classical Karamata case is $\rho = 0$ with $\mathbb{A} = \mathbb{R}$, and the general Beurling case $\rho > 0$ with $\mathbb{A} = \mathbb{G}_\rho^+$ (in which case $\Phi(x)$ is *Beurling $\varphi$-regularly varying*). In the RV literature this equation appears in [11], in work inspired by Bojanić and Karamata [26], and is due principally to Goldie. In both these cases the solution $K$ to (*GBE*) describes a function derived from the limiting behaviour of some *regularly varying* function $F$ for a suitable auxiliary $\Phi$.

*Example ([18, Corollary 2])* For $\varphi \in SE$, if $U$ satisfies

$$\frac{U(x + t\varphi(x)) - U(x)}{\varphi(x)} \to c_U t \text{ as } x \to \infty, \text{ for all } t \geq 0\,, \qquad (BMA_\varphi)$$

and

$$K_V(u) := \lim_{x\to\infty} \Delta_u^\varphi F(x)/\varphi(x)\,,$$

for $V(\cdot) := U(\tau_\varphi^{-1}(\cdot))$ with $\tau_\varphi$ as in Section 9.2.5, then for $\rho = \rho_\varphi$

$$K_V(s + t) = K_V(s)e^{\rho t} + K_V(t)\,,$$

and so with the notation $H_\rho$ of Section 9.2.1 above, for some $c$,

$$K_V(s) = cH_\rho(s)\,.$$

### 9.3.2 Some "Advanced" Popa Theory: Quantifier Weakening

We illustrate the usefulness of the Popa group structure by surveying some further results from the recent [18]. These culminate in a theorem on quantifier weakening (Theorem 5 below) in the demanding context of local uniformity; it in turn relies on the 'subgroup property' of the domain of definition of certain limit operations (the sets $\mathbb{A}^{\varphi}$ and $\mathbb{A}_u$ below). For additional motivation see Proposition 10 in Section 9.7(5).

The definition of *SE* in Section 9.1 demands locally uniform convergence: this motivates the introduction of the following weak notion of uniformity, which is key to Theorem 4 below. Say that $f_n \to f$ *uniformly near* $t$ if for every $\varepsilon > 0$ there is $\delta > 0$ and $m \in \mathbb{N}$ such that

$$f(t) - \varepsilon < f_n(s) < f(t) + \varepsilon \text{ for } n > m \text{ and } s \in (t - \delta, t + \delta)\,.$$

For instance, for $\varphi \in SN$, $x_n$ divergent, and $f(s) \equiv 1$, if $f_n(s) := \varphi(x_n + s\varphi(x_n))/\varphi(x_n)$, then '$f_n \to f$ uniformly near $t$ for all $t > 0$.'

The notion above is easier to satisfy than Hobson's 'uniform convergence *at t*' which replaces $f(t)$ above by $f(s)$ twice, [52, p. 110]; suffice it to refer to $f_n \equiv 0$, and $f$ with $f(0) = 0$ and $f \equiv 1$ elsewhere. (See also Klippert and Williams [62], where though Hobson's condition is satisfied at all points of a set, the choice of $\delta$ cannot itself be uniform in $t$.)

The above notion of uniformity may be equivalently stated in limsup language, which presently (in Proposition 6) brings to the fore the underlying *uniform upper and lower semicontinuity*. We refer to [18, Section 5] for details.

For $\varphi \in SE$ we now introduce a further binary operation, one in which a point $x$ appears as a parameter (we think of this as a *circle operation localized* to $x$):

$$s \circ_{\varphi x} t := s + t\eta_x^{\varphi}(s),$$

where

$$\eta_x^{\varphi}(s) := \varphi(x + s\varphi(x))/\varphi(x)\,.$$

This notation neatly summarizes two frequently used facts in (Karamata/Beurling) regular variation: firstly,

$$x \circ_{\varphi} (b \circ_{\varphi x} a) = y \circ_{\varphi} a, \text{ for } y := x \circ_{\varphi} b = x + b\varphi(x)$$

(so an 'absorption' property), and secondly, as $x \to \infty$, locally uniformly in $s, t$ :

$$s \circ_{\varphi x} t \to s \circ_{\eta} t, \text{ for } \eta := \lim_x \eta_x^{\varphi} \in GS\,.$$

Here $\eta$ satisfies (*GS*), by Ostaszewski [71], so the localized operation $\circ_{\varphi x}$ is asymptotic to a Popa operation $\circ_{\eta}$. This is used in Proposition 8.

An important rôle is played by the corresponding *localized Beck* $\eta_x^{\varphi}$*-sequence* (or *iteration*):

$$a_{\varphi x}^{n+1} = a_{\varphi x}^{n} \circ_{\varphi x} a, \qquad a_{\varphi x}^{1} = a\,. \tag{$\eta_x^{\varphi}$}$$

Its properties are listed below; here to avoid excessive bracketing, the usual arithmetic operations bind more strongly than Popa operations.

**Proposition 4 (Arithmetic of Popa Operations, [18, Proposition 2])**

| | | |
|---|---|---|
| *(i)* | $a^0_{\varphi x} = 1_{\varphi x} = 0\,, \qquad a \circ_{\varphi x} a^{-1}_{\varphi x} = 0\,,$ | *for* $a^{-1}_{\varphi x} := (-a)/\eta^{\varphi}_x(a)\,;$ |
| *(ii)* | $x \circ_{\varphi} (b \circ_{\varphi x} a) = y \circ_{\varphi} a\,,$ | *for* $y := x \circ_{\varphi} b\,;$ |
| *(iii)* | $x \circ_{\varphi} (b \circ_{\eta} a) = y \circ_{\varphi} a\eta(b)/\eta^{\varphi}_x(b)\,,$ | *for* $y := x \circ_{\varphi} b\,;$ |
| *(iv)* | $x = y \circ_{\varphi} b^{-1}_{\varphi x}\,,$ | *for* $y := x \circ_{\varphi} b\,;$ |
| *(v)* | $\eta^{\varphi}_x(a^m_{\varphi x}) = \prod_{k=0}^{m-1} \eta^{\varphi}_{y_k}(a)\,,$ | *for* $y_k = x \circ_{\varphi} a^k_{\varphi x}\,.$ |

**Definitions** Recalling from Section 9.3.1 that

$$\Delta^{\varphi}_t h(x) := h(x + t\varphi(x)) - h(x),$$

and, taking limits here and below as $x \to \infty$, as before (rather than sequentially as $n \to \infty$), put for $\varphi \in SE$ and with $\rho = \rho_{\varphi}$ and $\rho^* = -\rho^{-1}$

$$\mathbb{A}^{\varphi} := \{t > \rho^* : \Delta^{\varphi}_t h \text{ converges to a finite limit}\}\,,$$

$$\mathbb{A}_{\mathrm{u}} := \{t > \rho^* : \Delta^{\varphi}_t h \text{ converges to a finite limit locally uniformly near } t\}\,.$$

So

$$0 \in \mathbb{A}^{\varphi},$$

but we cannot yet assume either that $\mathbb{A}^{\varphi}$ is a subgroup, or that $0 \in \mathbb{A}_{\mathrm{u}}$, a critical point in Proposition 5 below. In the Karamata case $\varphi \equiv 1$, $\mathbb{A}^{\varphi} = \mathbb{A}^1$ is indeed a subgroup (see [20, Proposition 1] and Section 9.7(5) below).

For $t \in \mathbb{A}^{\varphi}$ put

$$K(t) := \lim_{x\to\infty} \Delta^{\varphi}_t h\,. \tag{K}$$

So $K(0) = 0$.

**Proposition 5 ([18, Proposition 6])** *For $\varphi \in SE$, $\mathbb{A}_{\mathrm{u}}$ is a subgroup of $\mathbb{G}^{\rho}_{+}$ for $\rho = \rho_{\varphi}$ iff $0 \in \mathbb{A}_u$; then $K : (\mathbb{A}_u, \circ) \to (\mathbb{R}, +)$, defined by (K) above, is a homomorphism.*

**Theorem 4 ([18, Theorem 4])** *If the pointwise convergence (K) holds on a co-meagre set in $\mathbb{G}^{\rho}_{+}$ with the limit function K upper semicontinuous also on a co-meagre set, and, furthermore, the one-sided condition*

$$K(t) = \lim_{\delta\downarrow 0} \limsup_{x\to\infty} \sup\{h(x + s\varphi(x)) - h(x) : s \in [t, t+\delta)\} \tag{UNIF$^+$}$$

*holds at the origin, then two-sided limsup convergence holds everywhere:*

$$\mathbb{A}^{\varphi} = \mathbb{A}_{\mathrm{u}} = \mathbb{G}_{+}^{\rho}\,.$$

This last result is based on the following monotone convergence theorem, akin to those of Dini and of Pólya-Szegő; the proof relies on the Baire category theorem.

**Proposition 6 (Uniform Upper Semicontinuity, [18, Proposition 4])** *If quasi everywhere $f_n$ converges pointwise to $f$, an upper semicontinuous limit satisfying quasi everywhere in its domain the one-sided condition*

$$f(t) = \lim_{\delta\downarrow 0} \limsup_{n} \sup\{f_n(s) : s \in [t, t+\delta)\},$$

*then quasi everywhere $f$ is uniformly upper semicontinuous:*

$$f(t) = \lim_{\delta\downarrow 0} \limsup_{n} \sup\{f_n(s) : s \in (t-\delta, t+\delta)\}\,.$$

**Definitions** For $\varphi \in SE$ and $\rho = \rho_{\varphi}$, put

$$H^{\dagger}(t) := \lim_{\delta\downarrow 0}\ \limsup_{x\to\infty}\ \sup\left\{h(x \circ_{\varphi} s) - h(x) : s \in [t, t+\delta)\right\} \qquad (t > \rho^{*}),$$

$$\mathbb{A}_{\mathrm{u}}^{\dagger} := \{t > \rho^{*} : H^{\dagger}(t) < \infty\}\,.$$

So $\mathbb{A}_{\mathrm{u}} \subseteq \mathbb{A}_{\mathrm{u}}^{\dagger}$, as $H^{\dagger}(t) = K(t)$ on $\mathbb{A}_{\mathrm{u}}$.

The following result clarifies the rôle of uniformity in classical 'Heiberg–Seneta boundedness' terms (for background see BGT (3.2.4) and [17, Section 1,2]).

**Proposition 7 ([18, Proposition 9])** *For $\varphi \in SE$, the following are equivalent:*

(i) $0 \in \mathbb{A}_u$ *(i.e. $\mathbb{A}_u \neq \emptyset$ and so a subgroup);*
(ii) $\lim_{x\to\infty}[h(x + u\varphi(x)) - h(x)] = 0$ *uniformly near $u = 0$;*
(iii) $H^{\dagger}(t)$ *satisfies the two-sided Heiberg–Seneta condition:*

$$\limsup_{u\to 0} H^{\dagger}(u) \leq 0\,. \qquad (HS_{\pm}(H^{\dagger}))$$

**Theorem 5 (Quantifier Weakening from Uniformity, [18, Theorem 6])** *If $\mathbb{A}_u$ is dense in $\mathbb{G}_{+}^{\rho}$ and $H^{\dagger}(t) = K(t)$ on $\mathbb{A}_u$—i.e. $H^{\dagger} : (\mathbb{A}_u, \circ_{\rho}) \to (\mathbb{R}, +)$ is a homomorphism, then $\mathbb{A}_u = \mathbb{G}_{+}^{\rho}$ and for some $c \in \mathbb{R}$:*

$$H^{\dagger}(t) = c\log(1 + \rho t) \quad (t > \rho^{*})\,.$$

This uses Proposition 2. Below, again working additively, we put for $\varphi \in SE$

$$H^{*}(t) := \limsup_{x\to\infty} h(x \circ_{\varphi} t) - h(x) \qquad (t > \rho_{\varphi}^{*}),$$

$$H_*(t) := \liminf_{x\to\infty} h(x \circ_\varphi t) - h(x) \qquad (t > \rho_\varphi^*).$$

**Theorem 6 ([18, Theorem 10])** *In the setting of Theorem 5, for $\varphi \in SE$, if the set $S$ on which $H^*(t)$ and $H_*(t)$ are both finite contains a half-interval $[a, \infty)$ for some $a > 0$, then there is a constant $K > 0$ such that for all large enough $x$ and $u$*

$$h(u\varphi(x) + x) - h(x) \le K \log u\,.$$

The proof parallels a classical result (that of BGT Theorem 2.0.1), but with the usual powers $a^n$ replaced by the (localized) Beck $\eta_x^\varphi$-iterates, as in Equation ($\eta_x^\varphi$) above. But there is heavy reliance on the estimation results below for $a_{\varphi x}^m$ that are uniform in $m$ (this only needs $\eta_x^\varphi \to \eta_\rho$ pointwise):

**Proposition 8 ([18, Proposition 11])** *If $\varphi \in SE$ with $\rho = \rho_\varphi > 0$, then for any $a > 1$ and $0 < \varepsilon < 1$ :*

(i) *($a_{\varphi x}^m$-estimates under $\eta_x^\varphi$) for all large enough $x$,*

$$(1-\varepsilon) \le \eta_x^\varphi (a_{\varphi x}^m)^{1/m}/\eta_\rho(a) \le (1+\varepsilon) \qquad (m \in \mathbb{N})\,,$$

(ii) *($a_{\varphi x}^m$-*estimates under *$\eta_\rho$) for all large enough $x$,*

$$\frac{\eta_\rho(a(1-\varepsilon))^m}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon} \le \eta_\rho(a_{\varphi x}^m) \le \frac{\eta_\rho(a(1+\varepsilon))^m}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} \qquad (m \in \mathbb{N})\,,$$

(iii) *$a_{\varphi x}^m \to \infty$, and*

(iv) *there are $C_\pm = C_\pm(\rho, a, \varepsilon) > 0$ such that, for all large enough $x$ and $u$,*

$$a_{\varphi x}^m \le u < a_{\varphi x}^{m+1} \Longrightarrow mC_- \le \log u \le (m+1)C_+\,.$$

### *9.3.3 Random Walks with Stable Laws: GFE Again*

A random variable $X$ has a *stable law* if the probability law (measure) $\mu$ of the random walk $S_n := X_1 + \ldots + X_n$, in which the steps are executed on the group of additive reals $\mathbb{R}$ independently and with identical law, is again of the same *type*. The latter means that the distribution function

$$F(x) = \mathrm{Prob}^\mu[X \le x]$$

of $X$ and that of each $S_n$ should be equal up to a change of 'scale and location':

$$S_n \overset{D}{=} a_n X + b_n\,, \tag{D}$$

for some (real) norming constants $a_n, b_n$ with $a_n > 0$. Here $\overset{D}{=}$ denotes equality of distributions. Such a law may be exactly characterized by its *characteristic functional equation,* Equation (*ChFE*) below, obtained from (D) on taking its Fourier transform (using the linearity and multiplicativity features of the transform). Since the characteristic function here is

$$\varphi(t) = \mathbb{E}[\exp(\mathrm{i}tX)] = \int_{\mathbb{R}} \exp(\mathrm{i}tx)\,\mathrm{d}F(x)$$

(identifying the characters as

$$\chi_t(x) = e^{\mathrm{i}tx}$$

—cf. [84, Example 3.7]), (D) above yields

$$\varphi(t)^n = \varphi(a_n t)\exp(\mathrm{i}b_n t) \qquad (n \in \mathbb{N})\,. \tag{ChFE}$$

In what follows we restrict attention to $t \geq 0$, without loss of generality (as $\varphi(-t)$ may be reconstructed via complex conjugation). The standard way of solving (*ChFE*) is to derive from it the equations satisfied by the functions $a : n \mapsto a_n$ and $b : n \mapsto a_n$. A direct approach to the characterization of the laws was recently demonstrated in Pitman and Pitman [75], who proceed by proving the map *a injective,* extending both of the maps $a$ and $b$ to $\mathbb{R}_+$, and exploiting the classical Cauchy functional equation (*CFE*) in both cases. For a background textbook account see [58] and for subsequent developments, based on the Choquet–Deny Theorem [45]; the stable laws are given a sketchy account in [78, Chapter 3], and more recent studies include [46] and [47].

Here, however, we indicate why (*ChFE*) can be re-configured to (*GFE*), so that (*GFE*) may be used just once, thereby simplifying the Pitman approach and yielding an even more direct approach. Though we adopt a somewhat cavalier fashion here, the procedure is made entirely rigorous in [73], and we comment below on the underlying justification. Take logarithms (trickery!—see below) and, adjusting notation, pass first to the form

$$f(g(n)t) = nf(t) + h(n)t \qquad (n \in \mathbb{N}, t \in \mathbb{R}_+)\,,$$

where now $\mathbb{R}_+ := (0, \infty)$. Suppose both that $g$ is injective and that one may pass to continuous arguments, in the manner of Kendall's Theorem, for which see Section 9.7(4) (for the double trickery involved here—again see below); then, taking $s = g(n)$, this is

$$f(st) = g^{-1}(s)f(t) + h(g^{-1}(s))t \qquad (s, t \in \mathbb{R}_+)\,,$$

or with $F(t) := f(t)/t$, $G(s) := g^{-1}(s)/s$, $H(s) := hg^{-1}(s)/s$, by symmetry:

$$F(st) = F(t)G(s) + H(s) = F(s)G(t) + H(t).$$

There are now two cases to consider, both leading to the multiplicative form of (*GFE*):

*Case* (i). If $F(1) = 0$, then taking $s = 1$ yields $F(t) = H(t)$, and so

$$F(st) - F(s) = F(t)G(s)\,.$$

So $\kappa = F$ indeed satisfies the *multiplicative* form of the Goldie equation.

*Case* (ii). On the other hand, if $F(1) \neq 0$, then passing from $F$ to $F/F(1)$ and from $H$ to $H/F(1)$ we may assume without loss of generality that $F(1) = 1$ (i.e. $f(1) = 1$); then, taking $t = 1$,

$$F(s) = G(s) + H(s).$$

Eliminating $H$ gives

$$F(st) - F(s) = (F(st) - 1) - (F(s) - 1) = (F(t) - 1)G(s)\,,$$

so $\kappa = F - 1$ now satisfies the multiplicative form of the Goldie equation.

Either way, putting $s = e^u$ and $t = e^v$, and $K(u) = \kappa(e^u)$ and $\psi(u) = G(e^u)$, we obtain the additive form:

$$K(u + v) - K(u) = K(v)\psi(u)\,.$$

So (*ChFE*) is (*GFE*), again in disguise!

As to the trickery above: application of the logarithm and the passage from discrete to continuous in the transformation of (*ChFE*) into (*GFE*) is justified in [73] from knowledge of the norming constants, that $a_n = n^k$ for some $k \neq 0$ (as then $a$ extends to an injective function $g$, and the values $a_m/a_n$ form a dense set). That is an acceptable way to proceed for probabilists, by virtue of an elementary probabilistic proof identifying the norming constants (cf. [42, VI.1, Theorem 1], [75, Lemma 5.3]); the next section (Section 9.4) rids us of this dependence on 'outside material'.

The first trick above (taking logarithms) is justified by Lemma 1 below; the subsequent trick relies on continuity of $K$ and on reference to a dense subset of $\mathbb{R}$, via the simple Corollary below, the routine proof of which we omit: it is similar in spirit to the proof of Lemma 1. (Unlike for the constants $a_n$, an explicit form for the $b_n$ is not needed.)

**Lemma 1 ([73, L. 1])** *For continuous $\varphi \not\equiv 0$ satisfying (ChFE) with $a_n = n^k$ ($k \neq 0$), $\varphi$ has no zeros on $\mathbb{R}_+$.*

*Proof* If $\varphi(\tau) = 0$ for some $\tau > 0$, then $\varphi(a_m\tau) = 0$ for all $m$, by (*ChFE*). Again by (*ChFE*),

$$|\varphi(\tau a_m/a_n)|^n = |\varphi(a_m\tau)| = 0,$$

so $\varphi$ is zero on the dense subset of points $\tau a_m/a_n$; then, by continuity, $\varphi \equiv 0$ on $\mathbb{R}_+$, a contradiction. □

***Corollary** ([73, C. 1]) Equation (ChFE) with $a_n = n^k$ ($k \neq 0$) holds on the dense subgroup*

$$\mathbb{A}_{\mathbb{Q}} := \{a_m/a_n : m, n \in \mathbb{N}\} :$$

*there are constants $\{b_{m/n}\}_{m,n\in\mathbb{N}}$ with*

$$\varphi(t)^{m/n} = \varphi(ta_m/a_n)\exp(ib_{m/n}t) \qquad (t \geq 0)\,.$$

Reference to case (ii) in the reduction to (*GFE*) above and to the known continuous solutions of (*GFE*) yields the form of the (non-degenerate) stable law: for some $\gamma \in \mathbb{R}$, $\kappa \in \mathbb{C}$ and with $A := \kappa/\gamma$ and $B := 1 - A$ (for $\gamma \neq 0$),

$$f(t) = \log\varphi(t) = \begin{cases} f(1)(At^{\gamma+1} + Bt), & \text{for } \gamma \neq 0, \\ f(1)(t + \kappa t\log t), & \text{for } \gamma = 0, \end{cases} \qquad (t > 0)\,. \tag{‡}$$

Here $\alpha := \gamma + 1$ is called the *characteristic exponent*.

*Remark* The form (‡) here takes no account of a further probabilistic ingredient: restrictions on the two parameters $\gamma$ and $\kappa$ (equivalently $\alpha$ and $\kappa$). Such restrictions follow from the asymptotic analysis of the 'initial' behaviour of the characteristic function $\varphi$ (i.e. near the origin). This is equivalent to the 'final' or tail behaviour (i.e. at infinity) of the corresponding distribution function, and relates to its *skewness*, i.e. its 'tail balance' ratio—the asymptotic ratio of the distribution's tail difference to its tail sums; for the details see [75, Section 8].

## 9.4 The Stable Laws Equation on $\mathbb{R}$

Treating the stable laws equation (*ChFE*) *purely* as a functional equation for determining continuous solutions calls for the removal of spurious probabilistic assumptions. It emerges that knowledge of $a_n$ may be deduced from (*ChFE*) provided the continuous solution $\varphi$ is to be *non-trivial,* i.e. neither $|\varphi| \equiv 0$ nor $|\varphi| \equiv 1$ holds on $[0, \infty)$. That is: the explicit form of $a_n$ may be deduced without assuming that $\varphi$ is the characteristic function of a (non-degenerate) distribution, as we now show.

**Theorem 7** *If $\varphi$ is a non-trivial continuous function and satisfies (ChFE) for some sequence $a_n \geq 0$, then $a_n = n^k$ for some $k \neq 0$.*

We will first need to establish a further lemma and proposition.

**Lemma 2** *If (ChFE) is satisfied by a continuous and non-trivial function $\varphi$, then the sequence $a_n$ is either convergent to 0, or divergent ('convergent to $+\infty$').*

*Proof* Suppose otherwise. Assume first that, as $a_n \geq 0$, for some infinite $\mathbb{M} \subseteq \mathbb{N}$, and $a > 0$,

$$a_m \to a \text{ through } \mathbb{M}\,.$$

Without loss of generality $\mathbb{M} = \mathbb{N}$, otherwise interpret $m$ below as restricted to $\mathbb{M}$. For any fixed $t$, $a_m t \to at$, so

$$K_t := \sup_m \{|\varphi(a_m t)|\}$$

is finite by the continuity of $\varphi$. Then, for all $m$,

$$|\varphi(t)|^m = |\varphi(a_m t)| \leq K_t,$$

and so $|\varphi(t)| \leq 1$, for each $t$. Then, by continuity,

$$|\varphi(at)| = \lim_m |\varphi(a_m t)| = \lim_m |\varphi(t)|^m = 0 \text{ or } 1\,.$$

So, setting $N_k := \{t : |\varphi(at)| = k\}$,

$$\mathbb{R}_+ = N_0 \cup N_1\,.$$

By the connectedness of $\mathbb{R}_+$, one of $N_0, N_1$ is empty, as the disjoint sets $N_k$ are closed; so respectively $|\varphi| \equiv 0$ or $|\varphi| \equiv 1$, contradicting non-triviality.

To complete the proof, suppose there exist $\mathbb{M} \subseteq \mathbb{N}$ and $\mathbb{M}' \subseteq \mathbb{N}$ such that $\lim_{m\in\mathbb{M}} a_m = \infty$ and $\lim_{m\in\mathbb{M}'} a_m = 0$. The former implies that $|\varphi(0)| = 1$ : as $\varphi$ is non-trivial, we may choose $t$ with $|\varphi(t)| \neq 0$; then, by continuity at 0,

$$|\varphi(0)| = \lim_{m\in\mathbb{M}} |\varphi(t/a_m)| = \lim_{n\in\mathbb{M}} \exp\left(\frac{1}{m} \log|\varphi(t)|\right) = 1.$$

But, again by continuity at 0, for each $t$,

$$\lim_{m\in\mathbb{M}'} |\varphi(t)|^m = \lim_{m\in\mathbb{M}'} |\varphi(a_m t)| = |\varphi(0)| = 1,$$

and so $|\varphi(t)| = 1$ for all $t$, contradicting non-triviality. □

The next result essentially contains [75, Lemma 5.2]; the latter relies on $|\varphi(0)| = 1$, the continuity of $\varphi$, and the existence of some $t$ with $\varphi(t) < 1$ (guaranteed below by the non-triviality of $\varphi$). We assume less here, and so must also consider the possibility that $|\varphi(0)| = 0$ (automatically excluded if $\varphi$ is the characteristic function of a distribution [42, Chapter XV, Lemma 1]).

**Proposition 9** *If* (*ChFE*) *is satisfied by a continuous and non-trivial function* $\varphi$ *and for some* $c > 0$, $|\varphi(t)| = |\varphi(ct)|$ *for all* $t > 0$, *then* $c = 1$.

*Proof* Note first that $a_n > 0$ for all $n$; indeed, otherwise, $a_k = 0$ for some $k \geq 1$ and

$$|\varphi(t)|^k = |\varphi(0)| \qquad (t \geq 0)\,.$$

Assume first that $k > 1$; taking $t = 0$ yields $|\varphi(0)| = 0$ or $1$, which as in Lemma 2 implies $|\varphi| \equiv 0$ or $|\varphi| \equiv 1$. If $k = 1$, then $|\varphi(t)| = |\varphi(0)|$, and for all $n > 1$,

$$|\varphi(0)|^n = |\varphi(0)|;$$

so again $|\varphi(0)| = 0$ or $1$, which again implies $|\varphi| \equiv 0$ or $|\varphi| \equiv 1$.

Applying Lemma 2, the sequence $a_n$ converges either to 0 or to $\infty$.

We consider these two cases separately.

(i) Suppose that $a_n \to 0$. Then, as above (referring again to $K_t$), we obtain

$$|\varphi(t)| \leq 1,$$

for all $t$. Now, since

$$|\varphi(0)| = \lim_n |\varphi(a_n t)| = \lim_n |\varphi(t)|^n\,,$$

if $|\varphi(t)| = 1$ for *some* $t$, then $|\varphi(0)| = 1$, and that in turn yields, for the very same reason, that

$$|\varphi(t)| \equiv 1$$

for *all* $t$, a trivial solution, which is ruled out. So in fact $|\varphi(t)| < 1$ for *all* $t$, and so $|\varphi(0)| = 0$.

Now suppose that for some $c > 0$, $|\varphi(t)| = |\varphi(ct)|$ for all $t > 0$. We show that $c = 1$. If not, without loss of generality $c < 1$, (otherwise replace $c$ by $c^{-1}$ and so, by hypothesis, $|\varphi(t/c)| = |\varphi(ct/c)| = |\varphi(t)|$ ); then

$$0 = |\varphi(0)| = \lim_n |\varphi(c^n t)| = |\varphi(t)|, \text{ for } t > 0\,,$$

and also for $t = 0$; so $\varphi$ is trivial, a contradiction. So indeed $c = 1$ in this case.

(ii) Suppose now that $a_n \to \infty$. Choose $s$ with $\varphi(s) \neq 0$; then, by (*ChFE*),

$$|\varphi(0)| = \lim_n |\varphi(s/a_n)| = \lim_n \exp\left(\frac{1}{n}\,\log|\varphi(s)|\right) = 1\,,$$

i.e. $|\varphi(0)| = 1$. Again as in case (i) above, suppose that for some $c > 0$,

$$|\varphi(t)| = |\varphi(ct)|$$

for all $t > 0$. To show that $c = 1$, suppose again without loss of generality that $c < 1$; then

$$1 = |\varphi(0)| = \lim_n |\varphi(c^n t)| = |\varphi(t)| \text{ for } t > 0\,,$$

and so $|\varphi(t)| \equiv 1$, for $t \geq 0$, again a trivial solution. So again $c = 1$. □

*Proof of the Theorem 7* (*ChFE*) implies that

$$|\varphi(a_{mn}t)| = |\varphi(t)|^{mn} = |\varphi(a_m t)|^n = |\varphi(a_m a_n t)| \qquad (t \geq 0)\,.$$

By Proposition 9, $a_n$ satisfies the discrete version of the Cauchy equation

$$a_{mn} = a_m a_n \qquad (m, n \in \mathbb{N})\,,$$

whose solution is known to take the form $n^k$, since $a_n > 0$ (as at the start of the proof of Proposition 9). If $a_n = 1$ for some $n > 1$, then, for each $t > 0$, $|\varphi(t)| = 0$ or 1 (as $|\varphi(t)| = |\varphi(t)|^n$) and so again, by continuity as in Lemma 2, $\varphi$ is trivial. So $k \neq 0$. □

*Remark* Continuity is essential to the theorem: take $a_n \equiv 1$, then a Borel function $\varphi$ may take the values 0 and 1 arbitrarily.

## 9.5 Positive Solutions of *GS*

In this section we include various new arguments providing information on the positive solutions of (*GS*) by way of fairly direct links to the equation. Theorem BM, with a family resemblance to Theorem 1, is derived here more directly than if we were to specialize results from Brzdęk [29] and Brzdęk-Mureńko [31]. Theorem B, which follows it and in combination yields *the dichotomy*: $f$ is either never 1 or always 1 on $\mathbb{R}_+ := (0, \infty)$, is taken from these papers, but again the proof here is more direct, and shorter. The final result is Theorem 9, suggested by the recent [71, Theorem 6].

For completeness, as it is needed in Theorem B (and obliquely referred to in Section 9.2.5 above), we begin with the following, which we quote verbatim, as it is short.

**Theorem 8 (From [17, Theorem 5])** *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (GS), then $\varphi(x) \geq 1$ for all $x > 0$.*

*Proof* Suppose that $\varphi(u) < 1$ for some $u > 0$; then $v := u/(1 - \varphi(u)) > 0$ and so, since $v = u + v\varphi(u)$,

$$0 < \varphi(v) = \varphi\,(u + v\varphi(u)) = \varphi(u)\varphi(v)\,.$$

So, cancelling by $\varphi(v) > 0$, one has $\varphi(u) = 1$, a contradiction. □

In Theorems BM and B below we use $f$ rather than $\varphi$ for ease of comparison with [31].

**Theorem BM ([31, Lemma 7])** *For $f > 0$ on $\mathbb{R}_+$ a solution of (GS), if $f \neq 1$ at all points, then $f(x) = 1 + cx$ $(x > 0)$ for some $c > 0$.*

*Proof* By symmetry, for any $x, y > 0$

$$f(x + yf(x)) = f(x)f(y) = f(y + xf(y)).$$

Fix $x$ and $y$ and put $u := x + yf(x)$ and $v := y + xf(y)$. If these are unequal, without loss of generality suppose that $v > u$. Then $(v - u)/f(u) > 0$, so

$$0 < f(u) = f(v) = f(u + f(u)(u - v)/f(u)) = f(u)f((u - v)/f(u)).$$

Cancelling by $f(u) > 0$ gives $f((u-v)/f(u)) = 1$, contradicting the hypothesis that $f$ is never 1. So $u = v$: that is, for all $x, y > 0$

$$x + yf(x) = y + xf(y);$$

equivalently, for all $x, y > 0$

$$x/(1 - f(x)) = y/(1 - f(y)) = \text{ const. } = c,$$

say. Then $f(x) = 1 + cx$ for all $x > 0$. So $c > 0$. □

Below we suppose that $f(a) = 1$, for some fixed $a > 0$. Note that $t_n := na$ is a *Beck* sequence under $\circ_f$ with step size $a$; so $f(na) = 1$, since $f(t_n) = f(t_1)^n$ (see Section 9.2.5).

For $f$ a positive solution of (GS), we denote here the *positive range* of $f$ by

$$R_f := \{w : (\exists x > 0) w = f(x)\}.$$

If $f \equiv 1$, then $R_f = \{1\}$.

**Lemma B ([29, Corollary 1], cf. [31, Lemmas 1,2])** *If the value 1 is achieved at $a > 0$ by a solution $f > 0$ on $\mathbb{R}_+$ of (GS), then*

(i) *the range set $R_f$ is a multiplicative subgroup;*
(ii) *$f(x + a) = f(x)$ for all $x > 0$;*
(iii) *$f(wa) = 1$ for $w \in R_f$.*

*Proof* For (i), (GS) itself implies that $R_f$ is a semigroup. We only need to find the inverse of $w := f(x)$ with $x > 0$. Choose $n \in \mathbb{N}$ with $na > x$. Put $y = (na - x)/f(x)$; then $y > 0$ and

$$f(x)f(y) = f(x + yf(x)) = f(na) = 1.$$

So $f(y) \in R_f$. For (ii), note that, as $f(a) = 1$,

$$f(x) = f(x)f(a) = f(a + xf(a)) = f(x + a).$$

For (iii), since (i) holds, this time write $w = 1/f(x)$ for some $x > 0$; then by (ii)

$$f(x) = f(x + a) = f(x + f(x)a/f(x)) = f(x)f(aw)\,,$$

and cancelling by $f(x) > 0$ gives $f(aw) = 1$. □

**Theorem B ([29, Theorem 3])** *For $f$ a positive solution of (GS), if $1 \in R_f$, then $f \equiv 1$.*

*Proof* Suppose otherwise; then, by Theorem 8 above, $f(u) > 1$, for some $u > 0$. Choose $a > 0$ with $f(a) = 1$ and $n \in \mathbb{N}$ with

$$na > u/(f(u) - 1) > 0.$$

Put

$$v := na + u/(1 - f(u)) > 0\,; \quad v + naf(u) = u + vf(u) + na\,.$$

So, since $f(u) \in R_f$, applying Lemma B (first (iii) with $f(u)$ in place of $w$ giving $f(af(u)) = 1$, then (ii) repeatedly, but with $af(u)$ in place of $a$, and then again (ii) repeatedly, but this time with $a$)

$$\begin{aligned} 0 < f(v) &= f(v + naf(u)) = f(u + vf(u) + na) \\ &= f(u + vf(u)) = f(u)f(v)\,, \end{aligned}$$

yielding the contradiction $f(u) = 1$. Hence $f(x) = 1$ for all $x$. □

We now revert to the $\varphi$ notation. In Section 9.2.5 above, $K(u) > 0$ was posited for $u > 0$ near 0. Below a similar assumption, justified by Theorem 8 above, is made for $K := \varphi - 1$. For $\varphi : [0, \infty) \to \mathbb{R}$, denote its level set above unity by:

$$L_+(\varphi) := \{t \in \mathbb{R}_+ : \varphi(t) > 1\}\,.$$

**Theorem 9** *If the continuous solution $\varphi$ of (GS) with $\varphi(0) = 1$ has a nonempty level set $L_+(\varphi)$ containing an interval $(0, \delta)$ for some $\delta > 0$, then $\varphi$ is differentiable and for some $\rho > 0$*

$$\varphi(t) = 1 + \rho t\,.$$

*Proof* For $T \in L_+ := L_+(\varphi)$ and $u > 0$, write $m(u) = m_T(u)$ for the jump index of $T$ for the Beck sequence $t_m(u)$, as in Section 9.2.5 above; then

$$t_{m(u)}(u) \leq T < t_{m(u)+1}(u)\,.$$

By (**) of Section 9.2.5 (with $m = m(u)$) and continuity at 0 of $\varphi$,

$$\Delta_{m(u)}(u) := t_{m(u)+1}(u) - t_{m(u)}(u) = u\varphi(u)^{m(u)}$$
$$\leq T(\varphi(u) - 1) + u \to 0 \quad \text{as } u \to 0\,,$$

for $u \in L_+$ *uniformly* in $T > 0$ on compacts. Likewise for $u \notin L_+$, as then

$$\Delta_{m(u)}(u) = u.$$

Consider any null sequence $u_n \to 0$ with $u_n > 0$. We will show that

$$\{(\varphi(u_n) - 1)\,/u_n\}$$

is convergent, by showing that down every subsequence $\{(\varphi(u_n) - 1)\,/u_n\}_{n\in\mathbb{M}}$ there is a convergent sub-subsequence with limit independent of $\mathbb{M}$.

Without loss of generality we take $0 < u_n \in L_+$ for all $n$ (so $u_n < \delta$). Now consider an arbitrary $T \in L_+$. Passing, if necessary, to a subsequence (dependent on $T$) of $\{(\varphi(u_n) - 1)\,/u_n\}_{n\in\mathbb{M}}$, we may suppose, for $k(n) := m_T(u_n)$, that

$$\Delta_{k(n)}(u_n) \to 0\,;$$

then along $\mathbb{M}$

$$|T - t_{m(u_n)}(u_n)| \leq \Delta_{m(u_n)}(u_n)\,,$$

and so

$$t_{k(n)}(u_n) = t_{m(u_n)}(u_n) \to T.$$

Again by (**) and continuity at $T$ of $\varphi$, putting $\rho := (\varphi(T) - 1)/T > 0$,

$$\frac{\varphi(u_n) - 1}{u_n} = \frac{\varphi(u_n)^{m(u_n)} - 1}{t_{m(u_n)}(u_n)} = \frac{\varphi(t_{m(n)}(u_n)) - 1}{t_{m(u_n)}(u_n)} \to \frac{\varphi(T) - 1}{T} = \rho\,,$$

along $\mathbb{M}$ to a limit $\rho$ dependent only on $T$ (and not on $\mathbb{M}$). So $\{(\varphi(u_n) - 1)\,/u_n\}$ is itself convergent to $\rho$. But this holds for any null sequence $\{u_n\}$ in $\mathbb{R}_+$, so the function $\varphi$ is differentiable at 0, and so is right-differentiable everywhere in $L_+$ (see [71, Lemma 3]). It is also left-differentiable at any $x > 0$, as follows. For $y$ with $0 < y < x$, put

$$t := (x - y)/\varphi(y) > 0.$$

Then $x = y + t\varphi(y)$, so

$$\frac{\varphi(x)-\varphi(y)}{x-y}=\frac{\varphi(y+t\varphi(y))-\varphi(y)}{x-y}=\frac{[\varphi(t)-1]\varphi(y)}{x-y}=\frac{\varphi(t)-1}{t}.$$

But $t \downarrow 0$ as $y \uparrow x$ (by continuity of $\varphi$ at $x$), and

$$(\varphi(t)-1)/t \to \varphi'(0).$$

So $\varphi$ is left-differentiable at $x$ and so differentiable; from here

$$\varphi'(x)=\varphi'(0).$$

Integration then yields the form of $\varphi(x)$; also, since $T$ above was arbitrary, for any $T \in L_+$ and with $u_n \to 0$ as above,

$$\rho = \lim_{n\in\mathbb{N}}\{(\varphi(u_n)-1)\,/u_n\} = \varphi'(0) = (\varphi(T)-1)/T:$$

$$\varphi(x) = 1+\rho x \quad (x \in \mathbb{R}_+)\,.$$

□

## 9.6 Two Random Walks in $\mathbb{R}^3$

We close by taking note of two higher-dimensional analogues of the random walk of Section 9.3.2, one unbounded, the other not. These are random walks involving independence both of the step size and of the direction, the latter with (directional) symmetry, i.e. its probability law is invariant under rotation; the object of study is the distribution of the distance from a designated starting point $o$. The unbounded, locally compact, case is a motion in space starting from the origin with spherical symmetry (which can thus be described by the distribution of its radial component), the other, compact, case a motion on the sphere with starting point $o$ at its north pole (yielding angular, or great circle, distance from $o$). The correspondingly radial or angular-wise characteristic function satisfies a functional equation involving an ‘averaging homomorphy’:

$$K(x)K(y) = \int_{-1}^{1} K(x \circ_\lambda y)\,\mathrm{d}\,\psi(\lambda)\,, \qquad (AH)$$

with the auxiliary function $\psi$ a direction-cosine distribution, and two corresponding commutative binary operations with real parameter $\lambda$:

$$x \circ_\lambda y = (x^2+y^2+2\lambda xy)^{1/2}\,,$$

$$x \circ_\lambda y = xy + \lambda\sqrt{1-x^2}\sqrt{1-y^2}\,.$$

These expressions arise from the cosine rules for Euclidean and spherical Triangles, respectively. The first of the two gives the radial distance generated by the two step lengths $x, y$ with $\lambda$ the direction-cosine of the angle between them (note the relation to the *Gauss functional equation* [2, Chapter 3, Example 6]); similarly, the second measures angular distance. As the action which generates motion is not associative in the usual sense, associativity has to be replaced by a probabilistic variant. Replacing the step-length realizations by random variables, the usual associativity property is re-interpreted modulo 'equality in distribution' (cf. $\stackrel{D}{=}$ in Section 9.3.2) for the corresponding random outcomes '$(X \circ_\psi Y) \circ_\psi Z$' and '$X \circ_\psi (Y \circ_\psi Z)$' (with $\psi$ denoting the law of $\lambda$). The two kinds of motion were studied, respectively, first by Kingman [61] and next by Bingham [8]. They were very much driven by the work of Bochner, especially [23–25]; indeed, on the basis of this link, one may regard Bochner as the forerunner to/founding father of hypergroups.

The Kingman non-degenerate case finds that probabilistic associativity holds iff the direction-determining *auxiliary function* is $\psi_\sigma$ with

$$d\psi_\sigma(\lambda) \propto (1-\lambda^2)^{\sigma-1/2} d\lambda$$

(for a parameter $\sigma > -1/2$), a matter earlier recognized by Haldane [48]; the (radial) characteristic function of the walk is then

$$K(u) = \int_{-1}^{1} e^{\mathrm{i}u\lambda} \mathrm{d}\,\psi_\sigma(\lambda) \equiv \Lambda_\sigma(u)\,,$$

where the *lambda Bessel function* is defined by

$$\Lambda_\sigma(t) := (t/2)^{-\sigma} J_\sigma(t) \Gamma(\sigma+1)\,.$$

The Bingham *non-degenerate* case finds that probabilistic associativity holds iff the auxiliary function $\psi$ again has the same $\psi_\sigma$ form and, up to normalization, the corresponding (angular) characteristic functions $K$ are the *Gegenbauer orthogonal polynomials* (ultraspherical polynomials): Gegenbauer's original analysis plays a rôle in both random walks.

The two *degenerate* cases of (*AH*) in the spherical case correspond to $\psi$ representing either $\delta_0$—a unit point-mass at 0, or $\frac{1}{2}(\delta_{-1} + \delta_{+1})$—two half-unit masses at $\pm 1$. The former yields the Cauchy multiplicative equation on $[-1, 1]$, as may be expected, the latter the *cosine functional equation*.

The general framework for non-deterministic binary operations is provided by the theory of *hypergroups,* as noted in the introduction. Thus the two examples above yield *Kingman's Bessel hypergroups* [22, 3.5.68] (cf. [86, Section 4.1], [87]), and *Bingham's Gegenbauer polynomial hypergroups* [22, 3.4.23] (cf. [86, Chapter 2]). A few words may help to provide some context.

The latter 'polynomial hypergroup' is the easier to describe. Its underlying topological space is discrete: $\mathbb{N}$. Convolution is defined using a family of orthogonal polynomials $\{C_n(t)\}$ acting as a base in the linear space of all polynomials; the binary operation on the pair $k, l \in \mathbb{N}$ is computed from the product $C_k C_l$ via its 'linearization'—its orthogonal expansion. The indices $n$ for $C_n$ with non-zero coefficients in the expansion (the direction cosines) are the possible locations in $\mathbb{N}$, with the cosines prescribing the probability of random selection. This calculation is also at the heart of [8, Proposition 3b], which uses classical orthogonal polynomials with weight function $\psi_\sigma$.

The other example is a hypergroup on $\mathbb{R}_+ := [0, \infty)$ with Euclidean topology. The connection with Bessel's differential equation makes Kingman's random walks a canonical example of hypergroups generated by a standard Sturm–Liouville (S-L) differential operator

$$\mathscr{L}_x := -\partial_x^2 - \frac{p'(x)}{p(x)}\,\partial_x\,,$$

where $p(x)$ denotes, as usual, the S-L coefficient function, so that the subscript $x$ signifies the variable of differentiation (cf. [22, 3.5]). The convolution of two unit point-masses at $x$ and $y$ is determined by their action on a $C^\infty(\mathbb{R}_+)$ function $f$, which action maps $f$ to the evaluation $u_f(x, y)$ at $(x, y)$ of the unique function $u = u_f(.,.)$ defined on $\mathbb{R}_+^2$ and satisfying the p.d.e.

$$\mathscr{L}_x u(x, y) = \mathscr{L}_y u(x, y)\,,$$

with boundary information along the axes $x = 0$ and $y = 0$ provided by $f$.

The upshot of this is to fulfil a like aim as in the earlier example: to define a binary operation $\star$. The continuous analogue, based on ($AH$) above, is

$$f(x \star y) = \int_{\mathbb{R}_+} f(t)(\delta_x \star \delta_y)(\mathrm{d}t) := \int_{-1}^{1} f(x \circ_\lambda y)\psi(\mathrm{d}\lambda)\,,$$

where $f(x \star y)$ stands for $f(\delta_x \star \delta_y)$, and so is the mean value of $f$ under the measure $\delta_x \star \delta_y$, and the function $u(x, y) := f(\delta_x \star \delta_y)$ is to satisfy the S-L p.d.e. as above. (This assumes $f$ is integrable with respect to such measures.)

The characteristic function $K$ now solves ($AH$) above iff it solves the functional equation

$$K(x \star y) = K(x)K(y)\,, \tag{$\star$}$$

and now this again expresses homomorphy. In the Sturm–Liouville case, by dint of the construction of the hypergroup relying on the operator $\mathscr{L}_x$, Equation ($\star$) reduces (via separation of variables) to solving a Sturm–Liouville eigenvalue problem:

$$\mathscr{L}_x K(x) = \text{const.}\,,$$

with

$$p(x)/p'(x) \equiv x\,,$$

which identifies that $K$ is a lambda Bessel function [22, 3.5.23]. In the polynomial case, Equation ($\star$) reduces to a polynomial recurrence equation, with solution yielding the Gegenbauer polynomials.

*Remarks* We note two significant underlying features, correlated with the homomorphy asserted by ($\star$).

Firstly, the $r$-th normalized coefficient (i.e. modulo division by the usual binomial coefficients) $\theta_r$ in any valid *finite* Taylor expansions of $\log K(t)$ is 'additive':

$$\theta_r(X \circ_\psi Y) = \theta_r(X) + \theta_r(Y)$$

(these are the Haldane 'cumulants')—see [61, Section 4]; here by a *valid* expansion is meant that the powers in the expansion corresponding to $r$ are taken only as far as the finiteness of the corresponding moments allows.

Secondly, the radial characteristic function encodes homomorphy:

$$\mathbb{E}[K(tX)]\mathbb{E}[K(tY)] = \mathbb{E}[K(t(X \circ_\psi Y))]\,.$$

## 9.7 Complements

*1. Additive Versus Multiplicative, and Double Sweep* The definition of a regularly varying $f$ defined on $\mathbb{R}_+$ is usually given in multiplicative form, as that is generally found most useful in applications; the definition immediately suggests a connection with *scaling phenomena*, as in the *Fechner theorem* in physics—see [10]. One is tempted to interpret these phenomena as *functional equations of absent scaling*: to solve $f(x) = \varphi(g(x))$ in the absence of any natural scaling effect between $f$ and $g$. This is solved on the assumption of *asymptotic* scale independence of $f$ from $g$ :

$$f(\lambda x) \sim \psi(\lambda) f(x)$$

for some $\psi$, i.e. on the assumption that $f$ is regularly varying. [10] is a very illuminating survey of the applications of RV also in other fields.

The theoretical work in RV, on the other hand, prefers the equivalent additive form of regular variation (as in Section 9.3.1), with $f$ defined on $\mathbb{R}$ satisfying

$$f(x+t) - f(x) \to k(t)\,,$$

so that $k$ will satisfy the additive Cauchy equation. This limit function $k$ may be regarded as the first-order derivative of $f$ 'at infinity'. Of interest is then a second-order asymptotic form arising from the divided difference:

$$[f(x+t)-f(x)]\,/g(x)$$

(comparing growth rates) studied in the Bojanić-Karamata/de Haan theory, BGT Chapter 3. The general denominator yields the advantages of 'double sweep' (BGT 3.13.1) by capturing both first- and second-order at once (setting $g \equiv 1$ in the former case). Consequently, the Beurling divided difference story of BRV captures the best of both worlds and encompasses all the forms of RV see especially [20, §7].

*2. Automatic Continuity* In the presence of even the merest hint of additional good behaviour, an additive function is beautifully well-behaved—it is (continuous, and hence) linear. The general context for results like this is that of *automatic continuity*, studied, e.g., by us [12, 13, 15, 17] for real analysis, Hoffmann-Jørgensen in [80, Part 3, Section 2], [83] and [82] for groups, etc. For Banach algebras and Gelfand theory, see, e.g., Dales [37, 38], Helson [49, p. 51], [39, 40], and the recent [60, esp. Corollary 16.7]. The pathology of discontinuity in the absence of good behaviour here is tied to set-theoretic axioms (cf. the foundational discussion in [19, Appendix 1]).

For a study of these features and the *up-grade phenomenon* (as in Theorem 9), that continuity implies differentiability, see [44] and the textbook [56].

*3. Generalized Quantifiers* Relevant for us are weakenings of the universal quantifier, along such lines as 'for quasi all $x$', i.e. for $x$ off a negligible set (and elsewhere 'there exist an infinite subset of $\mathbb{N}$' [20]). Mostowski [68] was the first in modern times to begin a study of generalized quantifiers, followed by Lindström [66] (for a textbook treatment see [6, Chapter 13]), and most notably Barwise [4]—see [89] for an account of this important development, and e.g. [65] for some recent developments in this field. Van Lambalgen [88] traces connections here with the conditional expectation of probability theory.

*4. Sequential Limits* The quantifier weakening here has been concerned with thinning as much as possible the set of $\lambda$ occurring in $\lambda + x$ or $\lambda x$. Related, and equally important, is the question of thinning the set of $x$ here—that is, in letting $x \to \infty$ through not all the reals, but some thinned subset. The most familiar case is taking limits *sequentially*, as in *Kendall's theorem* (BGT, Theorem 1.9.2; cf. [10] and Section 9.3.3): for any sequence $\{x_n\}$ with $\limsup x_n = \infty$ and $\limsup x_{n+1}/x_n \to 1$ (for instance, $x_n = n$), if $f$ is smooth enough (e.g. continuous) and

$$a_n f(\lambda x_n) \to g(\lambda) \in (0,\infty) \qquad \forall \lambda \in I$$

for some finite interval $I \subseteq (0,\infty)$ and some sequence $a_n \to \infty$, then $f$ is regularly varying. (Here $a_n$ regularly varying follows from smoothness of $f$.) The question arises of simultaneous thinning of $\lambda$ and $x$ together. Another case here is regular variation—in many dimensions, or of measures:

$$n\mathbb{P}(a_n^{-1}\mathbf{x} \in .) \to \nu(.) \qquad (n \to \infty)\,,$$

(here regular variation of $a_n \to \infty$ is assumed) and the limit (spectral) measure $\nu$ is on the unit sphere $\mathbf{S}$; see, e.g., Hult et al. [53] or [79, Chapter 6] for background. Now thinning is to be done on subsets of $\mathbf{S}$ on which convergence is assumed. For *convergence-determining classes* here, see, e.g., Billingsley [7, Section 1.2], Landers [64], Rogge [81].

*5. Regular Variation Without Limits* In the absence of limit functions one studies the 'limsup' variants. As these are subadditive, one asks when does this subadditivity lead to additivity. The following identifies where naturally to apply quantifier weakening; Theorem 5 of Section 9.3.2 yields a sample answer: see also [18, 20].

**Proposition 10 (Additive Kernel, [20, Proposition 1])** *For $F : \mathbb{R} \to \mathbb{R}$ put*

$$\mathbb{A}_F := \{u : \lim_{x\to\infty} [F(u + x) - F(x)] \text{ exists and is finite}\},$$

*and, for $a \in \mathbb{A}_F$, put $G(a) := \lim_{x\to\infty}[F(a + x) - F(x)]$. For $u \in \mathbb{R}$ define*

$$F^*(u) := \limsup_{x\to\infty}[F(u + x) - F(x)] .$$

*Then:*

(i) *$\mathbb{A}_F$ is an additive subgroup;*
(ii) *$G$ is an additive function on $\mathbb{A}_F$;*
(iii) *$F^* : \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a subadditive extension of $G$;*
(iv) *$F^*$ is finite-valued and additive iff $\mathbb{A}_F = \mathbb{R}$ and $F^*(u) = G(u)$ for all $u$.*

This directly connects to Theorem 1 in Section 9.2, as the identity

$$uv - u - v + 1 \equiv (1 - u)(1 - v)$$

gives that $(1-e^{-\gamma x})/\gamma$ is *subadditive* on $\mathbb{R}_+ := (0, \infty)$ for $\gamma \geq 0$, and *superadditive* on $\mathbb{R}_+$ for $\gamma \leq 0$.

*6. Functional Equations of Associativity* The equivalence noticed by Javor of (*GS*) with the associativity of $\circ_\eta$ has further analogues in connecting functional equations with the associativity of binary operations. For example, one may consider the operations

$$x *_\lambda y := xy \pm \lambda^2 p(x)p(y)$$

with $p$ either involutary or skew-involutary. These are associative iff $g(x) := \lambda p(x)/x$ solves the equation

$$g(x *_\lambda y) = \frac{g(x) + g(y)}{1 \mp g(x)g(y)/\lambda^2} ;$$

converting $g$ into a homomorphism calls for the right-hand side to be interpreted as the combination of the elements $u = g(x)$ and $v = g(y)$ by means of a group operation on the interval $(-\lambda, \lambda)$, $\circ_\lambda$ say, given by

$$u \circ_\lambda v = \frac{u+v}{1 \pm uv/\lambda^2} \ .$$

Then $g$ is seen to satisfy the *functional equation of competition* introduced recently by Kahlig and Matkowski [59]; cf. the hyperbolic semi-group of [51, 8.3]. As there, the choice of sign '$-$' or '$+$' yields the familiar tangent or hyperbolic tangent addition formulas. In the skew case the operations $*_\lambda$ include both

$$xy \pm \lambda^2(1-x)(1-y)$$

and the 'cosine formula', similarly as in Section 9.6:

$$xy \pm \lambda^2 \sqrt{1-x^2}\sqrt{1-y^2}\,.$$

The operation

$$x * y = xy + p(x) + p(y)\,,$$

with $p(0) = 0$, is associative only for $p(x) \equiv 0$ and $p(x) \equiv x$.

*7. The Cocycle Equation* The cocycle functional equation

$$F(st, x) = F(s, tx)F(t, x)$$

for $F : G \times X \to G$ may be regarded as an entry-point into RV, using flow language, as in [70, Section 4] and [14]; indeed, if $F$ is to be a $h$-coboundary for some continuous $h$, then

$$h(tx) = F(t, x)h(x)\,.$$

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