Chapter 7
Remarks on Analogies Between Haar Meager Sets and Haar Null Sets

Eliza Jabłońska

Abstract In the paper some analogies between Haar meager sets and Haar null sets in abelian Polish groups are presented.

Keywords Abelian Polish group • Haar meager set • Haar null set • Meager set • Set of Haar measure zero

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7.1 Introduction

It is well known [3] that a subset A of an abelian Polish group X is called *Haar null* if there are a universally measurable set $B \subset X$ with $A \subset B$ and a Borel probability measure μ on X such that

$$\mu(x + B) = 0$$

for all $x \in X$. In [5] Darji introduced another family of "small" sets in an abelian Polish group X; he called a set $A \subset X$ *Haar meager* if there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f : K \to X$ such that

$$f^{-1}(B + x) \text{ is meager in } K \text{ for every } x \in X.$$

In a locally compact group these two definitions are equivalent to definitions of Haar measure zero sets and meager sets, respectively. That is why we can say that

E. Jabłońska (✉)
Department of Discrete Mathematics, Rzeszów University of Technology,
Powstańców Warszawy 12, 35-959 Rzeszów, Poland
e-mail: elizapie@prz.edu.pl

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the notion of a Haar meager set is a topological analog to the notion of a Haar null set. Since lots of similarities between meager sets and sets of Haar measure zero are well known in locally compact abelian Polish groups (see, e.g., [24]), we would like to find as many analogies between Haar meager sets and Haar null sets as possible.

For each abelian Polish group Y we introduce the following notations:

$$\mathscr{HN}_Y := \{A \subset Y : A \text{ is Haar null}\},$$
$$\mathscr{HM}_Y := \{A \subset Y : A \text{ is Haar meager}\},$$
$$\mathscr{M}_Y := \{A \subset Y : A \text{ is meager}\};$$

and, if additionally Y is locally compact,

$$\mathscr{N}_Y := \{A \subset Y : A \text{ has Haar measure zero}\}.$$

Moreover, in the whole paper X is an abelian Polish group.

7.2 Basic Similarities

Let us start with the fact that both families, $\mathscr{HM}_X$ and $\mathscr{HN}_X$, are "small".

Theorem 7.1 ([3, Theorem 1]) *The family $\mathscr{HN}_X$ is a σ-ideal and, if X is locally compact,*

$$\mathscr{HN}_X = \mathscr{N}_X.$$

Theorem 7.2 ([5, Theorems 2.4, 2.9]) *The family $\mathscr{HM}_X$ is a σ-ideal and, if X is locally compact,*

$$\mathscr{HM}_X = \mathscr{M}_X.$$

Moreover, Darji (see [5, Theorem 2.2]) proved that in the case, where X is not locally compact,

$$\mathscr{HM}_X \subsetneqq \mathscr{M}_X.$$

Clearly an analogous inclusion for Haar null sets is impossible.

An important result obtained by Christensen [3] is a theorem of Steinhaus' type.

Theorem 7.3 ([3, Theorem 2]) *For every universally measurable subset A of X, with $A \notin \mathscr{HN}_X$, the set*

$$\{x \in X : (A + x) \cap A \notin \mathscr{HN}_X\}$$

is a neighbourhood of 0 *in X; consequently* $0 \in \operatorname{int}(A - A)$.

A topological analogue of the above theorem also holds.

Theorem 7.4 ([16, Theorem 2]) *For every Borel subset A of X, $A \notin \mathscr{HM}_X$, the set*

$$\{x \in X : (A + x) \cap A \notin \mathscr{HM}_X\}$$

is a neighbourhood of 0 *in X; i.e.,* $0 \in \operatorname{int}(A - A)$.

The following generalization of Theorem 7.3 has been proved by Gajda [13].

Theorem 7.5 ([13, Theorem 1]) *For every $n \in \mathbb{N}$ and every universally measurable set $A \notin \mathscr{HN}_X$ the set*

$$\{x \in X : \bigcap_{k \in \{-n,,\dots,n\}} (A + kx) \notin \mathscr{HN}_X\}$$

is a neighbourhood of 0 *in X.*

The above theorem is a very useful tool in functional equations. An analogous result has been proved in [17].

Theorem 7.6 ([17, Theorem 4]) *For every $n \in \mathbb{N}$ and Borel set $A \notin \mathscr{HM}_X$ the set*

$$\{x \in X : \bigcap_{k \in \{-n,,\dots,n\}} (A + kx) \notin \mathscr{HM}_X\}$$

is a neighbourhood of 0 *in X.*

Christensen and Fischer [4] generalized Theorem 7.5 as follows.

Theorem 7.7 ([4, Theorem 2]) *For every $N \in \mathbb{N}$ and every universally measurable set $A \notin \mathscr{HN}_X$ the set*

$$\{(x_1, \dots, x_N) \in X^N : A \cap \bigcap_{i=1}^{N} (A + x_i) \notin \mathscr{HN}_X\}$$

is a neighbourhood of 0 *in X^N.*

It turns out that an analogy to Theorem 7.7 also exists.

Theorem 7.8 ([18, Theorem 2.2]) *For every $N \in \mathbb{N}$ and Borel set $A \notin \mathscr{HM}_X$ the set*

$$\{(x_1, \dots, x_N) \in X^N : A \cap \bigcap_{i=1}^{N} (A + x_i) \notin \mathscr{HM}_X\}$$

is a neighbourhood of 0 *in X^N.*

From Theorems 7.3 and 7.4 we obtain that σ-compact sets in non-locally compact groups are “small” in both senses. More precisely we have the following.

Corollary 7.1 ([3], [16, Corollary 1]) *If X is not locally compact, then each σ-compact set is Haar null as well as Haar meager.*

One of the well-known results is the decomposition theorem stating that the real line can be decomposed into two disjoint "small" sets: a meager one and a Lebesgue measure zero one. Doležal, Rmoutil, Vejnar and Vlasák proved that some special spaces also can be decomposed into two disjoint "small" sets.

Theorem 7.9 ([10, Theorems 22 and 25]) *Each Banach space, or $\mathbb{R}^\omega$, can be decomposed into two disjoint sets: a Haar meager one and a Haar null one.*

Let us pay attention yet that the Kuratowski–Ulam Theorem and the Fubini Theorem, which are analogues of each other in the locally compact groups, fail in non-locally compact groups.

Example 7.1 ([10, Example 20]) The set

$$C := \{(s,t) \in \mathbb{Z}^\omega \times \mathbb{Z}^\omega : t_n \leq s_n \leq 0 \text{ for } n \in \omega\}$$

is neither Haar null nor Haar meager. But the set

$$C[t] := \{s \in \mathbb{Z}^\omega : (s,t) \in C\}$$

is Haar meager as well as Haar null for each $t \in \mathbb{Z}^\omega$ (because it is compact). On the other hand, the set

$$A := \{s \in \mathbb{Z}^\omega : s_n \leq 0 \text{ for } n \in \omega\}$$

is non-Haar meager and non-Haar null and, for each $s \in A$, the set

$$C[s] := \{t \in \mathbb{Z}^\omega : (s,t) \in C\}$$

is neither Haar meager nor Haar null.
From this example we see that there exists a non-Haar meager and non-Haar null set in $\mathbb{Z}^\omega \times \mathbb{Z}^\omega$ such that in one direction all its section are Haar meager, and in the other direction there are non-Haar meager many sections which are non-Haar meager.

It is rather obvious that every set containing a translation of each compact set is "large" in both senses; i.e., the following proposition is valid.

Proposition 7.1 *Every set containing a translation of each compact set is neither Haar null nor Haar meager.*

This proposition is very useful, because allows to observe some further similarities between Haar meager sets and Haar null sets.

In the paper [22] Matoušková and Zelený constructed closed sets A, B in a non-locally compact abelian Polish group X such that A, as well as B, includes a translation of each compact set and the set $(A + x) \cap B$ is compact for each $x \in X$. Consequently we obtain two analogies characterizations of locally compact groups.

Proposition 7.2 *An abelian Polish group X is locally compact if and only if*

$$int\,(A + B) \neq \emptyset$$

for each universally measurable non-Haar null sets $A, B \subset X$

Proposition 7.3 *An abelian Polish group X is locally compact if and only if*

$$int\,(A + B) \neq \emptyset$$

for each Borel non-Haar meager sets $A, B \subset X$.

Dodos [6] has used Matoušková's and Zelený's result from [22] to show that the invariance under bigger subgroups is not sufficient to establish a dichotomy. More precisely, he proved the following fact.

Proposition 7.4 ([6, Proposition 12]) *If X is not locally compact and G is a* σ*-compact subgroup of X, then there exists a G-invariant* F_σ *subset F of X such that neither F nor* $X \setminus F$ *is Haar null.*

In view of Proposition 7.1, in the same way as Dodos, we can prove that an another type of dichotomy also does not hold.

Proposition 7.5 ([18, Proposition 3.2]) *If X is not locally compact and G is a* σ*-compact subgroup of X, then there exists a G-invariant* F_σ *subset F of X such that neither F nor* $X \setminus F$ *is Haar meager.*

Let us also recall that each meager set is contained in an F_σ meager set, as well as each set of Lebesgue measure zero is contained in a G_δ set of Lebesgue measure zero. It turns out that both theorems cannot be generalized on the case of Haar null sets and Haar meager sets. More precisely, Elekes and Vindyánszky [11] proved the following.

Theorem 7.10 ([11, Theorem 4.1]) *Let* $1 \leq \xi < \omega_1$. *If X is non-locally compact, then there exists a Borel Haar null set that is not contained in any Haar null set from* $\Pi^0_\xi(X)$ *(i.e., the* ξ*th multiplicative Borel class in X).*

The same type result for a Haar meager set has been proved by Doležal and Vlásak in [9].

Theorem 7.11 ([9, Theorem 10]) *Let* $1 \leq \xi < \omega_1$. *If X is non-locally compact, then there exists a Borel Haar meager set that is not contained in any Haar meager set from* $\Sigma^0_\xi(X)$ *(i.e., the* ξ*th additive Borel class in X).*

Clearly, for $\xi = 2$, we obtain the existence of a Borel Haar null set without any G_δ Haar null hull, as well as the existence of a Borel Haar meager set without any F_σ Haar meager hull.

In the same papers we can also find the following theorems analogies each other.

Theorem 7.12 ([11, Theorem 4.1]) *If X is non-locally compact, then there exists a coanalytic Haar null set without any Borel Haar null hull.*

Theorem 7.13 ([9, Theorem 10]) *If X is non-locally compact, then there exists a coanalytic Haar meager set without any Borel Haar meager hull.*

Matoušková and Stegall [21] proved that a separable Banach space X is non-reflexive if and only if there exists a closed convex subset of X with empty interior, which contains a translation of any compact subset of X. Consequently, by Proposition 7.1, we obtain the following result.

Theorem 7.14 *Every separable nonreflexive Banach space contains a closed convex set with empty interior, which is neither Haar null nor Haar meager.*

Moreover, Matoušková [20, Theorem 4] has showed that this is unlike the situation in superreflexive spaces, where closed, convex, nowhere dense sets are Haar null. In turn Banakh [1, Proposition 5.7] has proved that each closed Haar null set in a Polish group is Haar meager. Hence we have the next theorem.

Theorem 7.15 *In separable superreflexive Banach spaces closed, convex, nowhere dense sets are Haar null as well as Haar meager.*

7.3 Generically Haar Meager Sets and Generically Haar Null Sets

Let us recall once again definitions of Haar meager sets and Haar null sets.

Definition 7.1 A set $A \subset X$ is *Haar null* if there is a universally measurable set $B \supset A$ and a Borel probability measure μ on X such that

$$\mu(x + B) = 0 \text{ for all } x \in X.$$

Definition 7.2 A set $A \subset X$ is *Haar meager* if there is a Borel set $B \supset A$, a compact metric space K and a continuous function $f : K \to X$ such that

$$f^{-1}(B + x) \in \mathscr{M}_K \text{ for all } x \in X.$$

It means that:

- each Haar null set has the only one witness parameter—*a test measure*;
- each Haar meager set has two witness parameters—*a witness metric space* and *a witness function*.

The following result has been proved in [2].

Proposition 7.6 *A Borel set $B \subset X$ is Haar meager if and only if there is a continuous function $f : 2^\omega \to X$ such that $f^{-1}(B + x)$ is meager in 2^ω for all $x \in X$.*
It means that a Haar meager set and a Haar null set have both the only one witness parameter—*a witness function* and *a test measure*, respectively.

Now, let $P(X)$ be the space of all Borel probability measures on X; this is a Polish space with Lévy metric.

Following Dodos [7, 8], given a universally measurable set $A \subset X$, by $T(A)$ we mean the set of all test measures for A, i.e.

$$T(A) := \{\mu \in P(X) : \ \mu(x + A) = 0 \ \text{ for every } \ x \in X\}.$$

Dodos [7] has proved the following.

Theorem 7.16 ([7, Proposition 5]) *If $A \subset X$ is a universally measurable Haar null set, then:*

- *$T(A)$ is dense in $P(X)$;*
- *if A is analytic, then either $T(A)$ is meager or $T(A)$ is comeager in $P(X)$;*
- *if A is σ-compact, then $T(A)$ is comeager in $P(X)$.*

Using Theorem 7.16, Dodos [8] has introduced the notion of a generically Haar null set and next he has proved a theorem of Steinhaus' type.

Definition 7.3 A set $A \subset X$ is *generically Haar null* if $T(A)$ is comeager in $P(X)$.

Theorem 7.17 ([8, Proposition 11]) *If $A \subset X$ is analytic, non-generically Haar null, then $A - A$ is non-meager.*

Now, let $C(2^\omega, X)$ be the space of all continuous functions $f : 2^\omega \to X$; this is a Polish space with the supremum metric (similarly as the space $P(X)$ with Lévy metric). For every Borel set $A \subset X$ we define

$$W(A) := \{f \in C(2^\omega, X) : \ f^{-1}(x + A) \in \mathscr{M}_{2^\omega} \ \text{ for every } \ x \in X\},$$

i.e., the set of all witness functions for A. Clearly, if $A \in \mathscr{HM}_X$, then $W(A) \neq \emptyset$, so this notation is analogous to $T(A)$.

In [1] and [2] an analogous result to Theorem 7.16 has been proved.

Theorem 7.18 ([2]) *Let $A \subset X$ be a Borel Haar meager set. Then:*

- *$W(A)$ is dense in $C(2^\omega, X)$;*
- *either $W(A)$ is meager, or $W(A)$ is comeager in $C(2^\omega, X)$;*
- *if A is σ-compact, then $W(A)$ is comeager in $C(2^\omega, X)$.*

Theorem 7.19 ([1], [2]) *If $A \subset X$ is analytic, non-generically Haar meager (i.e., $W(A)$ is not comeager in $C(2^\omega, X)$), then $A - A$ is non-meager.*

7.4 Analogies in Functional Equations

In this part (only) we assume that X is a Polish real linear space to present some further similarities between Haar meager sets and Haar null sets, which are very important in functional equations.

Lemma 7.1 ([23, Lemma 5]) *Let $A \notin \mathcal{HN}$ be a universally measurable set and $x \in X \setminus \{0\}$. Then there exists a Borel set $B \subset A$ such that the set $k_x^{-1}(B + z)$ has a positive Lebesgue measure in $\mathbb{R}$ for each $z \in X$, where $k_x : \mathbb{R} \to X$ is given by $k_x(\alpha) = \alpha x$.*

Lemma 7.2 ([19, Lemma 1]) *Let $A \notin \mathcal{HM}$ be a Borel set and $x \in X \setminus \{0\}$. Then there exists a Borel set $B \subset A$ such that the set $k_x^{-1}(B + z)$ is non-meager with the Baire property in $\mathbb{R}$ for each $z \in X$.*

Due to those two lemmas t-Wright convex functions, that are bounded on a "large" set, can be characterized.

Theorem 7.20 ([23, Theorem 8]) *Let $D \subset X$ be a nonempty convex open set and $t \in (0, 1)$. Each t-Wright convex function $f : D \to \mathbb{R}$ bounded on a non-Haar null universally measurable set $T \subset D$ is continuous.*

Theorem 7.21 ([19, Theorem 4]) *Let $D \subset X$ be a nonempty convex open set and $t \in (0, 1)$. Each t-Wright convex function $f : D \to \mathbb{R}$ bounded on a non-Haar meager Borel set $T \subset D$ is continuous.*

Now, using a weaker version of Lemma 7.1, the additive functions, that are bounded above on a "large" set, can be characterized. More precisely, the following theorem is true.

Theorem 7.22 ([14, Corollary 1]) *If $f : X \to \mathbb{R}$ is additive and bounded above on a universally measurable set $C \notin \mathcal{HN}$, then f is linear.*

Replacing [14, Lemma 1] by Lemma 7.2 in the proof of the above theorem, we obtain an analogous result.

Theorem 7.23 *If $f : X \to \mathbb{R}$ is additive and bounded above on a Borel set $C \notin \mathcal{HM}$, then f is linear.*

Moreover, using a weaker version of Lemma 7.1 and Theorem 7.22, solutions of a generalized Gołąb–Schinzel equation, that are bounded on a "large" set, can be characterized.

Theorem 7.24 ([15, Theorem 1]) *Let $f : X \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$ and $|f(D)| \subset (0, a)$ for a positive number a and a universally measurable set $D \notin \mathcal{HN}$. Then functions f and M satisfy the equation*

$$f(x + M(f(x))y) = f(x)f(y) \tag{7.1}$$

if and only if one of the following three conditions holds:

(i) $f = 1$;

(ii) $M|_{(0,\infty)} = 1$ *and there exists a nontrivial linear functional* $h : X \to \mathbb{R}$ *such that*

$$f(x) = \exp h(x) \ \textit{for} \ x \in X;$$

(iii) there exists a nontrivial linear functional $h : X \to \mathbb{R}$ *and* $c \in \mathbb{R} \setminus \{0\}$ *such that either*

$$M(y) = |y|^{1/c} \ sgn\, y \ \textit{for} \ y \in \mathbb{R},$$

$$f(x) = \begin{cases} |h(x) + 1|^c \ sgn\,(h(x) + 1), & x \in X, \ h(x) \neq -1; \\ 0, & x \in X, \ h(x) = -1 \end{cases}$$

or

$$M(y) = y^{1/c} \ \textit{for} \ y \in [0, \infty),$$

$$f(x) = \begin{cases} (h(x) + 1)^c, & x \in X, \ h(x) > -1; \\ 0, & x \in X, \ h(x) \leq -1. \end{cases}$$

Observe that using the method from [15] we can prove a theorem which is analogous to Theorem 7.24; the most important change in the proof is to replace:

- [15, Lemma 6] by Theorem 7.4,
- [15, Lemma 7] by Lemma 7.2,
- [15, Lemma 8] by Theorem 7.23.

Then we obtain the following theorem.

Theorem 7.25 *Let* $f : X \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$ *and* $|f(D)| \subset (0, a)$ *for a positive number a and a Borel set* $D \notin \mathcal{HM}$. *Then functions f and M satisfy Equation* (7.1) *if and only if one of the conditions (i)–(iii) of Theorem 7.24 holds.*

7.5 Modified Darji's and Christensen's Definitions

Doležal, Rmoutil, Vejnar and Vlasák [10] modified Darji's notion of meagerness in the following way.

Definition 7.4 A set $A \subset X$ is *naively Haar meager* if there is a compact metric space K and a continuous function $f : K \to X$ such that

$$f^{-1}(x + A) \ \text{is meager in} \ K \ \text{for every} \ x \in X.$$

They also have proved the next theorem.

Theorem 7.26 ([10, Theorem 16]) *If X is uncountable, then there exists a naively Haar meager subset of X, which is not Haar meager.*

In a similar way Elekes and Vindyánszky [12] have defined naively Haar null sets and showed a result analogous to Theorem 7.26.

Definition 7.5 A set A is called *naively Haar null* if there is a Borel probability measure μ on X such that

$$\mu(x + A) = 0 \text{ for all } x \in X.$$

Theorem 7.27 ([12, Theorem 1.3]) *If X is uncountable, then there exists a naively Haar null subset of X which is not Haar null.*

Moreover, in non-abelian Polish groups definitions of Haar meager sets and Haar null sets have been modified in the following way.

Definition 7.6 A subset A of a Polish group X is *Haar null* if there are a universally measurable set $B \subset X$ with $A \subset B$ and a Borel probability measure μ on X such that

$$\mu(x + B + y) = 0 \text{ for all } x, y \in X.$$

Definition 7.7 A subset A of a Polish group X is *Haar meager* if there are a Borel set $B \subset X$ with $A \subset B$, a compact metric space K and a continuous function $f : K \to X$ such that

$$f^{-1}(x + B + y) \text{ is meager in } K \text{ for every } x, y \in X.$$

Then both families—of all Haar null sets and of all Haar meager sets in X—form σ-ideals (see [12] and [10, Theorem 3]).

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