Chapter 5
Fischer–Muszély Additivity: A Half Century Story

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Abstract This is an extended version of my talk presented at *the 30th International Summer Conference on Real Functions Theory* that was held in Stará Lesná (Slovakia) from September 4 to 9, 2016.

Keywords Fischer–Muszély equation (additivity) • Strictly convex spaces • General solution • The hierarchy of (non)commutativity • Pexiderization • Fischer–Muszély type inequalities • Stability

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5.1 Background

In the beginning was the word (of Fischer and Muszély in Hungarian and English:

A Cauchy-féle függvényegyenletek bizonyos típusú általánosí-tásai (see [11]) and *On some new generalizations of the functional equation of Cauchy* (see [12]):

Examining certain problems in physics M. Hosszu (*Észrevételek a relativitáselméleti időfogalom Reichenbach-féle értelmezéséhez,* NME magyarnyelvű Kőzleményi Miskolc (1964), 223–233) *obtained the functional equation*

$$f(x+y)^2 = [f(x) + f(y)]^2, \tag{$*$}$$

where x, y, f *are real.*

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In another paper of M. Hosszu (*Egy alternativ fűggvényegyenletrő,* Mat. Lapok 14 (1963), 98–102) *proved that Equation* ($*$) *is equivalent to the functional equation of Cauchy, i.e. to the equation*

$$f(x+y) = f(x) + f(y) \tag{$**$}$$

H. Światak examined in (*On the equation* $\varphi(x+y)^2 = [\varphi(x)g(y) + \varphi(y)g(x)]^2$, Zeszyty Naukowe Uniwersytetu Jagiellońskiego, Nr II. Prace Matematyczne, Zeszyt 10 (1965), 97–104) *a generalization of Equation* ($*$) *in the class of continuous functions.*

A similar alternative functional equation is considered in a paper of J. Aczél, K. Fladt and M. Hosszu (*Lösungen einer mit dem Doppelverhältnis zusamenhängender Funktionalgleichung,* MTA Mat. Kut. Int. Közl 7A (1962), 335–352).

At the end of his paper M. Hosszu puts the question: what is the general solution of Equation ($*$)*?*

E. Vincze was the first to give an answer to this question in his papers

- *Alternativ fűggvényegyenletek megoldásairól,* Mat Lapok 14 (1963), 179–195*;*
- *Beitrag zur Theorie der Cauchyschen Funktionalgleichungen,* Arch. Mat. 15 (1964), 132–135*;*
- *Über eine Verallgemeinerung der Cauchyschen Funktionalgleichung*, *Funkcialaj Ekvacioj 6 (1964), 55–62.*

He proved that the functional equation

$$f(x+y)^n = [f(x) + f(y)]^n$$

is equivalent to the functional equation of Cauchy, where x, y *are in an additive Abelian semigroup,* f *is an arbitrary complex-valued function and* n *is a natural number.*

5.2 Fischer–Muszély Equation

Plainly, Equation ($*$) may equivalently be written in the form

$$|f(x+y)| = |f(x) + f(y)|$$

and, if so, why not to replace the absolute value sign by the norm?

Throughout the years the functional equation

$$\|f(x+y)\| = \|f(x) + f(y)\| \tag{FM}$$

has extensively been studied by many authors, see, e.g., Fischer and Muszély [11, 12], Dhombres [9], Aczél and Dhombres [1], Berruti and Skof [4], Skof [28], Ger [16–22], Schöpf [27], Ger and Koclęga [23], Száz [29]. The reason why this functional equation was attracting so much attention is, on the one side, the facts established in the papers spoken of in the Background and, on the other side, because of its links with the theory of isometries; moreover, it leads to some characterizations of strictly convex normed linear spaces as well as to some of their generalizations. The main result from [18] states that any map f from a (not necessarily commutative) group into a strictly convex space has to be additive, i.e. to satisfy the Cauchy equation

$$f(x+y) = f(x) + f(y). \tag{C}$$

On the other hand, already in 1979 Dhombres [9] exhibited an example of a continuous solution $f : \mathbb{R} \to X$ of Equation (∗) that fails to satisfy (C).

In the case where the domain $\mathbb{R}$ is replaced by the halfline $[0, \infty)$ one may "produce" a rich family of C^∞-nonadditive solutions of Equation (FM).

This inspired Schöpf [27] to look for a description of all continuous (resp. differentiable) solutions of (FM) mapping the real line $\mathbb{R}$ into a not necessarily strictly convex normed linear space $(X, \|\cdot\|)$. Looking for some alternative representations Ger and Koclęga [23] have shown that any function f of that kind fulfilling merely very mild regularity assumptions has to be proportional to an odd isometry mapping $\mathbb{R}$ into X.

Last but not least, in 2003, Tabor [31] has obtained the additivity of surjective solutions to (FM).

Theorem 5.1 (Fischer and Muszély [11]) *Let $(X, +)$ be a semigroup and $(Y, (\cdot|\cdot))$ be a unitary space. Let further $f : X \to Y$ be a solution to functional equation* (FM). *Then f is additive.*

Problem Is it possible to replace the unitary target space by a strictly convex one?

Numerous characterizations of strictly convex spaces are known (see, e.g., the monograph of Day [6]). Among them the following one was given by Dhombres in [9]

A normed space (real or complex) $(X, \|\cdot\|)$ is strictly convex if and only if each function $f : \mathbb{R} \to X$ belonging to the class

$$\mathscr{F} := \{g : \mathbb{R} \to X : \ g \text{ has a measurable majorant on a set of positive measure}\}$$

and satisfying the functional equation (FM) *has to be additive.*

Moreover, Dhombres writes (p. 2.28 in [9]): *The problem of determining those normed spaces characterized by the equivalence of Equation* (FM) *and the equation of additivity, even in the case of the domain being some group like the additive $\mathbb{R}$, remains open.*

Actually, to show that the space considered is strictly convex it suffices to consider only continuous solutions of Equation (FM) (see Aczél and Dhombres [1] and Theorem 5.4 below). But while studying logical connections between (FM) and additivity it seems desirable indeed to get rid of the class $\mathscr{F}$. This is actually possible; namely, we have the following:

Theorem 5.2 *Let* $(G,+)$ *be a group (not necessarily commutative) and let* $(X, \|\cdot\|)$ *be a strictly convex space. Then every function* $f : G \to X$ *satisfying Equation* (FM) *for all* $x, y \in G$ *is additive.*

Proof Without the use of strict convexity one may show [see Dhombres (p. 2.23 in [9])] that the equality

$$\|f(2x)+f(x)\| = \|f(2x)\| + \|f(x)\|$$

holds true for all $x \in G$. Then strict convexity implies that for every $x \in G$ such that $f(x) \neq 0 \neq f(2x)$ there exists a positive number $\lambda(x)$ such that $f(2x) = \lambda(x)f(x)$. Since we obviously have

$$\|f(2x)\| = 2\|f(x)\|, \qquad x \in G, \tag{5.1}$$

we infer that $\lambda(x) = 2$ whenever $f(x) \neq 0 \neq f(2x)$. However, in view of (5.1), if one of the values $f(x)$ or $f(2x)$ vanishes, then so does the other; consequently, the equality $f(2x) = 2f(x)$ is fulfilled for all elements x from G.

Putting $y = -x$ in (FM) and taking into account that (5.1) implies the equality $f(0) = 0$, we derive the oddness of f. Now, observe that for all $x, y \in G$ one has

$$\| f(x+y) - \frac{1}{2} f(x) \| = \| \frac{1}{2}f(x) + f(y) \| = \| \frac{f(x+y)+f(y)}{2} \| . \tag{5.2}$$

In fact,

$$\begin{aligned} 2 \| f(x+y) - \frac{1}{2} f(x) \| &= \| 2f(x+y) - f(x) \| = \| f(x+y+x+y) + f(-x) \| \\ &= \| f(y+x+y) \| = \| f(x+y) + f(y) \|, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} 2 \| \frac{1}{2} f(x) + f(y) \| &= \| f(x) + 2f(y) \| = \| f(x) + f(2y) \| \\ &= \| f(x+2y) \| = \| f(x+y) + f(y) \|, \end{aligned}$$

which ends the proof of (5.2). Fix arbitrarily x and y from G and put $u := f(x+y) - \frac{1}{2} f(x)$ and $v := \frac{1}{2}f(x) + f(y)$; then (5.2) states that

$$\| u \| = \| v \| = \| \frac{u+v}{2} \|,$$

which, in view of the strict convexity of X, gives $u = v$. Thus

$$f(x+y) = f(x) + f(y),$$

which was to be proved. □

Remark 5.1 Under the assumption that the group considered is uniquely 2-divisible this result was presented by the author at the 26-th International Symposium on Functional Equations (Catalonia, 1988); see [16]. A year later, during the 27-th ISFE, the present version as well as its detailed proof was presented; see [17]. Assuming that the domain of the function in question yields a real linear space, in 1991 Berruti and Skof (Lemma fondamentale in [4]) proved the analogous assertion. Their proof relies essentially on Baker's lemma from [3].

Below we derive Baker's main result of [3] from ours.

Theorem 5.3 (Baker) *Let* $(E, \|\cdot\|)$ *and* $(X, \|\cdot\|)$ *be two real normed linear spaces and let* $f : E \to X$ *be an isometry. If the target space is strictly convex, then* f *has to be an affine function, i.e. there exists a constant* $c \in X$ *and a linear map* $L : E \to X$ *such that* $f(x) = L(x) + c$ *for all* $x \in E$.

Proof Put $c := f(0)$ and $L := f - c$. Then L is an isometry as well and $L(0) = 0$. Consequently,

$$\| L(x) - L(y) \| = \| x - y \| = \| L(x-y) \| \tag{5.3}$$

for all $x, y \in E$. Putting here $y = -x$, one gets

$$\| L(x) - L(-x) \| = 2 \| x \| = \| L(x) \| + \| -L(-x) \|$$

which, by means of the strict convexity of X, implies the oddness of L. This, jointly with (5.3), implies that the equality

$$\| L(x+y) \| = \| L(x) + L(y) \|$$

holds true for all $x, y \in E$. An appeal to Theorem 5.2 gives now the additivity of L which, being continuous, has to be linear. This ends the proof. □

The following characterization of strictly convex spaces in terms of the equivalence of Equation (FM) and the Cauchy functional equation yields a slight refinement of a result given by Aczél and Dhombres (p. 138 in [1]).

Theorem 5.4 *A normed linear space* $(X, \|\cdot\|)$ *is strictly convex if and only if for every its two-dimensional subspace* $Y \subset X$ *the functions*

$$f_c(x) = x \cdot c, \qquad x \in \mathbb{R},$$

where c *stands for an arbitrarily fixed element of* Y, *are the only continuous solutions* $f : \mathbb{R} \to Y$ *of Equation* (FM).

Proof Necessity. Fix a two-dimensional subspace Y of X and a continuous solution $f : \mathbb{R} \to Y$ of Equation (FM). Obviously, $(Y, \|\cdot\|)$ is strictly convex; therefore, by means of Theorem 5.2, f is additive and being continuous has to have the form f_c for some $c \in Y$.

Sufficiency. Assume, for the indirect proof, that $(X, \|\cdot\|)$ is not strictly convex. Then there exist elements $a, b \in X$, $a \neq b$ such that

$$\|a\| = \|b\| = \| \frac{a+b}{2} \| = 1.$$

Such vectors are linearly independent; in fact, if we had $b = \lambda a$ for some scalar λ (real or complex) we would get $|\lambda| = 1$ and $|1+\lambda| = 2$ implying the equality $\lambda = 1$, which is impossible. Consequently, the space $Y := \mathrm{Lin}\{a, b\}$ is two-dimensional. A continuous function

$$f(x) := \begin{cases} x \cdot a & \text{for } x \in [-1, 1] \\ a + (x-1) \cdot b & \text{for } x \in (1, \infty) \\ -a + (x+1) \cdot b & \text{for } x \in (-\infty, -1), \end{cases}$$

mapping $\mathbb{R}$ into Y yields a solution to (FM) (see Dhombres [9] or Aczél and Dhombres [1]) which obviously fails to be an f_c function. This contradiction completes the proof. □

Now, we are going to show that our Theorem 5.2 carries over to the case of linear topological spaces, topologized through families of suitable seminorms. To this aim, we shall first recall the definition introduced by Diminnie and White Jr. in [7]. Let X be a linear space and let $\mathscr{P}$ be a nonempty family of nonzero seminorms on X. For $p \in \mathscr{P}$ we put $N_p := \{x \in X : p(x) = 0\}$. The pair $(X, \mathscr{P})$ is said to be strictly convex if and only if for every $p \in \mathscr{P}$ and every $a, b \in X$ the conditions

$$p(a) = p(b) = p\left(\frac{a+b}{2}\right) = 1 \qquad \text{and} \qquad N_p \cap \mathrm{Lin}\{a, b\} = \{0\}$$

imply that $a = b$. Without loss of generality, in what follows, we shall be assuming that the family $\mathscr{P}$ consists of just a single seminorm: $\mathscr{P} = \{p\}$.

Theorem 5.5 *Let $(G, +)$ be a group (not necessarily commutative) and let X be a linear space endowed with a nonzero seminorm p such that the pair $(X, \{p\})$ is strictly convex. Suppose that $f : G \to X$ satisfies the functional equation*

$$p(f(x+y)) = p(f(x) + f(y)), \qquad x, y \in G. \tag{5.4}$$

Then there exists exactly one additive function $a : G \to X$ and exactly one function $n : G \to N_p$ such that

$$f(x) = a(x) + n(x), \qquad x \in G;$$

in particular,

$$p(f(x+y)-f(x)-f(y)) = 0, \qquad x, y \in G.$$

Proof One of the four equivalent conditions for a pair $(X, \{p\})$ to be strictly convex given by Diminnie and White Jr. in [8] states that there exists a strictly convex normed space $(Y, \|\cdot\|)$ and a linear mapping $F : X \to Y$ such that $p(x) = \|F(x)\|$ for all $x \in X$. Consequently, Equation (5.4) says that

$$\|F(f(x+y))\| = \|F(f(x)+f(y))\| = \|F(f(x)) + F(f(y))\|, \quad x, y \in G.$$

Putting $g := F \circ f$ we obtain

$$\|g(x+y)\| = \|g(x) + g(y)\|$$

for all $x, y \in G$ and, by the strict convexity of the space $(Y, \|\cdot\|)$, Theorem 5.2 implies the additivity of the map g; in other words

$$F(f(x+y)) = F(f(x)) + F(f(y)), \qquad x, y \in G.$$

Now, the additivity of F gives

$$C_f(x, y) := f(x+y) - f(x) - f(y) \in \ker F,$$

whence

$$p(C_f(x, y)) = \| F(C_f(x, y)) \| = 0,$$

i.e. $C_f(x, y) \in N_p$ for all $x, y \in G$.

Let N_p^c denote the complementary space to the linear subspace N_p of the space X. Then, for every $x \in G$, the value $f(x)$ can uniquely be factorized as $a(x) + n(x)$, where $a(x) \in N_p^c$ and $n(x) \in N_p$. Since, for any $x, y \in G$, one has

$$N_p^c \ni a(x+y) - a(x) - a(y) = C_f(x, y) - n(x+y) + n(x) + n(y) \in N_p,$$

the function a is additive, which finishes the proof. □

Now we are going to present an example illustrating the utility of Theorem 5.2 while solving some functional equations.

Assume that we are given a (not necessarily commutative) group $(G, +)$ and real numbers α, β, γ such that

$$\alpha > 0 \qquad \text{and} \qquad \beta^2 - 4\alpha\gamma < 0. \tag{5.5}$$

We will find the general solution of the functional equation

$$\alpha\left[\varphi(x+y)^2-(\varphi(x)+\varphi(y))^2\right]+\beta\left[\varphi(x+y)\psi(x+y)\right.$$
$$\left.-(\varphi(x)+\varphi(y))(\psi(x)+\psi(y))\right]+\gamma\left[\psi(x+y)^2-(\psi(x)+\psi(y))^2\right]=0 \tag{e}$$

in the class of all functions $\varphi, \psi : G \to \mathbb{R}$. An easy calculation shows that Equation (e) may equivalently be written in the form

$$\left(\begin{bmatrix}\alpha, & \frac{1}{2}\beta\\ \frac{1}{2}\beta, & \gamma\end{bmatrix}\cdot\begin{bmatrix}\varphi(x+y)\\ \psi(x+y)\end{bmatrix}\,\Big|\,\begin{bmatrix}\varphi(x+y)\\ \psi(x+y)\end{bmatrix}\right)$$

$$=\left(\begin{bmatrix}\alpha, & \frac{1}{2}\beta\\ \frac{1}{2}\beta, & \gamma\end{bmatrix}\cdot\begin{bmatrix}\varphi(x)+\varphi(y)\\ \psi(x)+\psi(y)\end{bmatrix}\,\Big|\,\begin{bmatrix}\varphi(x)+\varphi(y)\\ \psi(x)+\psi(y)\end{bmatrix}\right)$$

for all $x, y \in G$; here $(\cdot|\cdot)$ stands for the usual inner product in $\mathbb{R}^2$. Let us put

$$A := \begin{bmatrix}\alpha, & \frac{1}{2}\beta\\ \frac{1}{2}\beta, & \gamma\end{bmatrix} \qquad \text{and} \qquad f(x) := (\varphi(x), \psi(x)), \qquad x \in G.$$

Then the latter equation states that

$$(A\cdot f(x+y)\,|\,f(x+y)) = (A\cdot(f(x)+f(y))\,|\,f(x)+f(y))$$

for all $x, y \in G$. Since conditions (5.5) guarantee that the matrix A is positive definite the formula

$$\langle u|v\rangle := (A\cdot u\,|\,v), \qquad u, v \in \mathbb{R}^2,$$

produces a new inner product in $\mathbb{R}^2$ and the equation considered assumes the form

$$\| f(x+y) \|^2 = \| f(x)+f(y) \|^2, \qquad x, y \in G,$$

where $\|u\|^2 = \langle u|u\rangle, u \in \mathbb{R}^2$. Since any inner product space is obviously strictly convex, Theorem 5.2 establishes the additivity of f and hence that of the component functions φ and ψ. Conversely, every pair of additive functions $\varphi, \psi : G \to \mathbb{R}$ yields a solution to Equation (e).

5.3 General Solution

In what follows, we are presenting a factorization of the general solution of Equation (FM) for functions mapping a commutative group into a real normed linear space (with no regularity assumptions whatsoever), into isometric and additive mappings. We believe that, in this way, we have finally achieved a clear explanation of seemingly divergent earlier approaches focused on different endeavours either to show that (FM) implies additivity or to express the solutions of (FM) in terms of isometries.

5.3.1 Preliminary Results

Given an Abelian group $(X, +)$ we call a function p mapping X into the set $\mathbb{R}$ of all real numbers *sublinear* provided that p is *subadditive*, i.e.

$$p(x + y) \leq p(x) + p(y), \qquad x, y \in X,$$

and satisfies a homogeneity condition

$$p(nx) = np(x),$$

for all $x \in X$ and all $n \in \mathbb{N}_0$ (nonnegative integers).

The following Hahn–Banach type theorem is a special case of Kranz's result (Theorem 2 in [24]).

Lemma 5.1 *Let* $(X, +)$ *be an Abelian group and let* $(X_0, +)$ *stand for a subgroup of* $(X, +)$. *Assume that we are given a sublinear functional* $p : X \to \mathbb{R}$ *and an additive functional* $a_0 : X_0 \to \mathbb{R}$ *such that*

$$a_0(x) \leq p(x), \qquad x \in X_0.$$

Then there exists an additive extension $a : X \to \mathbb{R}$ *of* a_0 *such that*

$$a(x) \leq p(x), \qquad x \in X.$$

As a matter of fact, the sublinearity assumption on the functional p above might simply be replaced by subadditivity alone but, in the sequel, we will need the following corollary in which sublinearity is actually essential.

Corollary 5.1 *Let* $(X, +)$ *be an Abelian group and let* $x_0 \in X$. *Given an even sublinear functional* $p : X \to \mathbb{R}$ *there exists an additive functional* $a : X \to \mathbb{R}$ *such that* $a \leq p$ *and* $a(x_0) = p(x_0)$.

Proof Denote by $\mathbb{Z}$ the set of all integers and put $X_0 := \{nx_0 : n \in \mathbb{Z}\}$. Obviously, a functional $a_0 : X_0 \to \mathbb{R}$ is unambiguously defined by the formula

$$a_0(nx_0) := np(x_0), \qquad n \in \mathbb{Z};$$

moreover, a_0 is additive and majorized by p on X_0 since p, being even, has to be nonnegative. Now, it suffices to apply Lemma 5.1 to complete the proof. □

In what follows, we are going to show that the validity of Fischer's conjecture (see [13] and Kuczma [25]) stating that an (even!) sublinear functional p admits a representation of the form $p = \|\cdot\| \circ A$ where $A : X \to Z$ stands for an additive map with values in a suitable real normed linear space $(Z, \|\cdot\|)$, carries over to groups. The idea of the proof is based on the paper of Berz [5]; we have only to ensure that the passage to commutative group domains is possible. To proceed we need yet another lemma.

Lemma 5.2 *Let $(X, +)$ be an Abelian group and let $p : X \to \mathbb{R}$ be an even sublinear functional. Then the equality*

$$p(x) = \sup\{a(x) : a : X \to \mathbb{R} \text{ is additive and } a \le p\}$$

holds true for all $x \in X$.

Proof By virtue of Corollary 5.1, the family T of all additive real functionals a on X majorized by p is nonvoid.

Therefore, the formula

$$\tilde{p}(x) = \sup\{a(x) : a \in T\}, \quad x \in X,$$

correctly defines a functional $\tilde{p} : X \to \mathbb{R}$. Plainly, we have $\tilde{p} \le p$. On the other hand, by means of Corollary 5.1 again, for an arbitrarily fixed $x_0 \in X$ there exists an $a \in T$ such that $p(x_0) = a(x_0) \le \tilde{p}(x_0)$. Thus, $p \le \tilde{p}$, which finishes the proof. □

In the sequel, as usual, given a nonempty set T by $B(T, \mathbb{R})$ we denote a Banach space of all bounded real functions on T, equipped with the uniform convergence norm $\|\cdot\|_\infty$.

Theorem 5.6 *Let $(X, +)$ be an Abelian group and let $p : X \to \mathbb{R}$ be an even sublinear functional. Then there exists a nonempty set $T \subset \mathbb{R}^X$ and an additive operator $A : X \to B(T, \mathbb{R})$ such that*

$$p(x) = \|A(x)\|_\infty, \qquad x \in X.$$

Proof Let $T \subset \mathbb{R}^X$ stand for the family of all additive real functionals a on X majorized by p. According to Lemma 5.2, we have

$$p(x) = \sup\{a(x) : a \in T\}, \qquad x \in X.$$

In view of the evenness of p as well as the oddness of the members of T we obtain the estimation $|a(x)| \le p(x)$ valid for every $x \in X$ and every $a \in T$. Therefore, the formula

$$A(x)(a) := a(x), \qquad a \in T, x \in X,$$

correctly defines a map $A : X \to B(T, \mathbb{R})$. Clearly, A yields an additive operator and, moreover, the equality

$$p(x) = \sup\{|A(x)(a)| : a \in T\} = \|A(x)\|_\infty,$$

is satisfied for all $x \in X$. Thus the proof has been completed. □

5.3.2 *Main Result*

Now, we are in a position to prove a factorization theorem announced at the beginning of the present section.

Theorem 5.7 *Let $(X, +)$ be an Abelian group and let $(Y, \|\cdot\|)$ be a real normed linear space. Let further $f : X \to Y$ be a solution to functional equation (FM). Then there exist: a nonempty set $T \subset \mathbb{R}^X$, an additive operator $A : X \to B(T, \mathbb{R})$ and an odd isometry $I : A(X) \to Y$ such that*

$$f(x) = I(A(x)), \qquad x \in X.$$

Conversely, for an arbitrary real normed linear space $(Z, \|\cdot\|_Z)$, any additive operator $A : X \to Z$ and any odd isometry $I : A(X) \to Y$ the superposition $f := I \circ A$ yields a solution of Equation (FM).

Proof Let f be a solution of Equation (FM) and let a functional $p : X \to \mathbb{R}$ be given by the formula

$$p(x) := \|f(x)\|, \qquad x \in X.$$

Equation (FM) implies easily the subadditivity of p as well as the relationship

$$p(2x) = 2p(x), \qquad x \in X.$$

A simple induction shows that then $p(nx) = np(x)$ holds true for every $x \in X$ and every positive integer n. In other words, the functional p is sublinear. Observe that $f(0) = 0$ [by putting $x = y = 0$ in (FM)] whence the oddness of f results by setting $y = -x$ in (FM). Consequently the sublinear functional p is even. Therefore, by virtue of Theorem 5.6, there exist: a nonempty set $T \subset \mathbb{R}^X$ and an additive operator $A : X \to B(T, \mathbb{R})$ such that

$$p(x) = \|A(x)\|_\infty, \qquad x \in X.$$

Denote by $\hat{X}$ the quotient space $X/\ker A$ and define an operator $\hat{A} : \hat{X} \to B(T, \mathbb{R})$ by the formula

$$\hat{A}(x + \ker A) := A(x), \qquad x \in X.$$

Obviously, the operator $\hat{A}$ is both additive and injective. Now, observe that the formula

$$\hat{f}(x + \ker A) := f(x), \qquad x \in X,$$

correctly defines a map $\hat{f} : \hat{X} \to Y$. Indeed, once we have $x + \ker A = y + \ker A$ for some x, y from X, then $x - y \in \ker A$ whence by means of (FM) and the oddness of f we get

$$0 = \|A(x-y)\|_\infty = p(x-y) = \|f(x-y)\| = \|f(x) - f(y)\|$$

and, a fortiori, $f(x) = f(y)$.

Clearly, the image $G := \hat{A}(\hat{X}) = A(X)$ yields a subgroup of the additive group $(B(T, \mathbb{R}), +)$ and the formula

$$I(u) := \hat{f}\left(\hat{A}^{-1}(u)\right), \qquad u \in G,$$

establishes a map from the group $(G, +)$ into the normed space $(Y, \|\cdot\|)$. We are going to show that

(i) $\|I(u) + I(v)\| = \|I(u+v)\|, \qquad u, v \in G,$
(ii) $\|I(u)\| = \|u\|_\infty, \qquad u \in G.$

In fact, to see that (i) holds true, fix arbitrarily u, v from G. Then there exist x, y in X such that $u = \hat{A}(x + \ker A)$ and $v = \hat{A}(y + \ker A)$. Thus $u + v = \hat{A}(x + y + \ker A)$ whence

$$\begin{aligned}\|I(u) + I(v)\| &= \|\hat{f}\left(\hat{A}^{-1}(u)\right) + \hat{f}\left(\hat{A}^{-1}(v)\right)\| \\ &= \|\hat{f}(x + \ker A) + \hat{f}(y + \ker A)\| = \|f(x) + f(y)\| = \|f(x+y)\| \\ &= \|\hat{f}(x + y + \ker A)\| = \|\hat{f}\left(\hat{A}^{-1}(u+v)\right)\| = \|I(u+v)\|.\end{aligned}$$

To check (ii), observe that for every $u \in G$ one has

$$\begin{aligned}\|I(u)\| &= \|\hat{f}\left(\hat{A}^{-1}(u)\right)\| = \|\hat{f}(x + \ker A)\| \\ &= \|f(x)\| = p(x) = \|A(x)\|_\infty = \|\hat{A}(x + \ker A)\|_\infty = \|u\|_\infty.\end{aligned}$$

Since, as we have seen already, (i) implies the oddness of I, we infer that for every $u, v \in G$ one has

$$\|I(u) - I(v)\| = \|I(u) + I(-v)\| = \|I(u-v)\| = \|u - v\|$$

because of (i) and (ii). Thus the map I yields an odd isometry mapping G into Y.

Finally, for any $x \in X$ we have

$$I(A(x)) = I\left(\hat{A}(x + \ker A)\right) = (I \circ \hat{A})(x + \ker A) = \hat{f}(x + \ker A) = f(x),$$

which completes the necessity part of the proof.

Conversely, given a real normed linear space $(Z, \|\cdot\|_Z)$, an additive operator $A : X \to Z$ and an odd (hence also norm preserving) isometry $I : A(X) \to Y$, we see that the superposition $f := I \circ A$ satisfies Equation (FM) because for all $x, y \in X$ one gets

$$\begin{aligned} \|f(x) - f(y)\| &= \|I(A(x)) - I(A(y))\| \\ &= \|A(x) - A(y)\|_Z = \|A(x - y)\|_Z = \|I(A(x - y))\| = \|f(x - y)\|; \end{aligned}$$

now, since f itself is odd as a superposition of an odd and additive mapping, it remains to replace here y by $-y$ to get (FM). This finishes the proof. □

In the case where the domain group $(X, +)$ is uniquely 2-divisible, it is worthwhile to note that actually the functional $p = \|\cdot\| \circ f$ discussed above is not merely sublinear but also *Jensen-convex*, i.e. it satisfies the functional inequality

$$p\left(\frac{x + y}{2}\right) \leq \frac{p(x) + p(y)}{2}$$

for all points x, y from X. In particular, assuming that $(X, +)$ is simply the additive group of a normed real linear space $(X, \|\cdot\|_X)$ we see that very mild regularity assumption imposed upon p (for instance, continuity at a single point, Baire measurability, boundedness on a second category Baire subset of X, etc.; see Kuczma's monograph [25] for numerous further much more delicate instances) implies its continuity. Consequently, we get easily the following:

Theorem 5.8 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real normed linear spaces. Let further $f : X \to Y$ be a solution to the functional equation* (FM) *such that the functional $p : X \to \mathbb{R}$ defined by the formula*

$$p(x) := \|f(x)\|_Y, \qquad x \in X,$$

satisfies any regularity condition that forces a Jensen-convex functional to be continuous. Then there exist: a nonempty set $T \subset \mathbb{R}^X$, a continuous linear operator $L : X \to B(T, \mathbb{R})$ and an odd isometry $I : L(X) \to Y$ such that

$$f(x) = I(L(x)), \qquad x \in X.$$

Conversely, for an arbitrary real normed linear space $(Z, \|\cdot\|_Z)$, any continuous linear operator $L : X \to Z$ and any odd isometry $I : L(X) \to Y$ the superposition $f := I \circ L$ yields a solution of Equation (FM) *and the corresponding functional p is continuous.*

Proof As we have already observed the functional p being Jensen-convex has to be continuous. Therefore the additive operator $A : X \to B(T, \mathbb{R})$ such that

$$p(x) = \|A(x)\|_\infty, \qquad x \in X,$$

is continuous as well. Therefore, since it is well known that additivity implies rational homogeneity, jointly with continuity it forces A to be linear (recall that we deal with real normed linear spaces).

Since the latter assertion is obvious, this ends the proof. □

5.3.3 Derivation of Earlier Results

We shall first derive the main result of [18] (cf. Theorem 5.2 above) from Theorem 5.9. To this end, we shall prove two propositions which, I believe, may present an interest of their own.

Proposition 5.1 (A Modified Version of Baker's Theorem; See [3]) *Let $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ be two real normed linear spaces and let $(Y, \|\cdot\|_Y)$ be strictly convex. Let further $(G, +)$ be a subgroup of the additive group $(Z, +)$ such that $G = 2G$. If $I : G \to Y$ is an isometry vanishing at zero, then I is additive.*

Proof Fix arbitrarily elements $u, v \in G$. Then

$$\left\|I\left(\frac{u+v}{2}\right) - I(u)\right\|_Y = \left\|\frac{u+v}{2} - u\right\|_Z = \frac{1}{2}\|u-v\|_Z = \frac{1}{2}\,\|I(u) - I(v)\|_Y$$

as well as

$$\left\|I\left(\frac{u+v}{2}\right) - I(v)\right\|_Y = \left\|\frac{u+v}{2} - v\right\|_Z = \frac{1}{2}\|u-v\|_Z = \frac{1}{2}\,\|I(u) - I(v)\|_Y,$$

whence, in view of the uniqueness of the midpoint of a metric segment in a strictly convex space, implies the equality

$$I\left(\frac{u+v}{2}\right) = \frac{I(u)+I(v)}{2}.$$

Hence, on account of the assumption that $I(0) = 0$, we obtain the additivity of I. This ends the proof. □

It turns out that the assumption $G = 2G$ is superfluous whenever the isometry in question is odd. Namely, we have the following:

Proposition 5.2 *Let $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ be two real normed linear spaces and let $(Y, \|\cdot\|_Y)$ be strictly convex. Let further $(G, +)$ be a subgroup of the additive group $(Z, +)$. If $I : G \to Y$ is an odd isometry, then I is additive.*

Proof Put

$$\tilde{G} := \bigcup \{2^{-n}G : n \in \mathbb{N}_0\}.$$

It is easily seen that the structure $(\tilde{G}, +)$ yields a subgroup of the group $(Z, +)$ and that $G \subset \tilde{G}$. Moreover, we have $\tilde{G} = 2\tilde{G}$. Therefore, by means of Proposition 5.1, to finish the proof, it suffices to show that I admits an isometric extension onto $\tilde{G}$. This is actually the case, because I being an odd isometry satisfies Equation (FM) whence, in particular, $I(2u) = 2I(u)$ (see, e.g., [1, p. 139]). Consequently, the formula

$$\tilde{I}(2^{-n}u) := 2^{-n}I(u), \qquad u \in G, n \in \mathbb{N}_0,$$

unambiguously defines a map $\tilde{I} : \tilde{G} \to Y$ which, obviously, yields an extension of I. To see that $\tilde{I}$ itself is an isometry, fix arbitrarily $u, v \in G$ and $n, m \in \mathbb{N}_0$. Then

$$\begin{aligned} \|\tilde{I}(2^{-n}u) - \tilde{I}(2^{-m}v)\|_Y &= \|2^{-n}I(u) - 2^{-m}I(v)\|_Y \\ &= 2^{-n-m}\|2^{m}I(u) - 2^{n}I(v)\|_Y = 2^{-n-m}\|I(2^{m}u) - I(2^{n}v)\|_Y \\ &= 2^{-n-m}\|2^{m}u - 2^{n}v\|_Z = \|2^{-n}u - 2^{-m}v\|_Z, \end{aligned}$$

which completes the proof. □

Corollary 5.2 *Any solution of Equation* (FM) *mapping an Abelian group into a strictly convex real normed linear space* $(Y, \|\cdot\|_Y)$ *satisfies the Cauchy functional equation* (C).

Proof An appeal to Theorem 5.7 shows that $f = I \circ A$ where $A : X \to B(T, \mathbb{R})$ is an additive operator and $I : A(X) \to Y$ is an odd isometry. Plainly, $A(X)$ is a subgroup of the additive structure $(B(T, \mathbb{R}), +)$ whence, on account of Proposition 5.2, I is additive; therefore so is also the composition $f = I \circ A$. □

Remark 5.2 The main result of [12] (i.e. Theorem of Fischer and Muszély here) cannot, however, be derived from Corollary 5.2 (even with semigroups replaced by groups) because the commutativity of the domain was not assumed there. On the other hand, the only place in the proof of our Theorem 5.7, requiring commutativity of the domain was an indirect appeal to Corollary 5.1 via Lemma 5.2 and Theorem 5.6. Therefore, the following question arises in a natural way.

Problem Does Lemma 5.1 carry over to non-Abelian groups? An essential step towards a positive answer to that question will be discussed in Section 5.4.

Corollary 5.3 *Any solution* $f : \mathbb{R} \to Y$ *of Equation* (FM), *where* $(Y, \|\cdot\|_Y)$ *stands for a real normed linear space, such that the function* $p : \mathbb{R} \to \mathbb{R}$ *defined by the formula*

$$p(x) := \|f(x)\|_Y, \qquad x \in \mathbb{R},$$

satisfies any regularity condition that forces a Jensen-convex function to be continuous, has to be proportional to an odd isometry mapping $\mathbb{R}$ into Y.

Proof An appeal to Theorem 5.8 shows that $f = I \circ L$ where $L : \mathbb{R} \to B(T, \mathbb{R})$ is a continuous linear operator and $I : L(\mathbb{R}) \to Y$ yields an odd isometry. Clearly, we simply have

$$L(x) = x \cdot c, \qquad x \in \mathbb{R},$$

where c is a fixed element of $B(T, \mathbb{R})$. Without loss of generality we may assume that $c \neq 0$. Setting

$$\tilde{I}(x) := \frac{1}{\|c\|_\infty} I(x \cdot c), \qquad x \in \mathbb{R},$$

we infer that

$$\|\tilde{I}(x) - \tilde{I}(y)\|_Y = \frac{1}{\|c\|_\infty} \|I(x \cdot c) - I(y \cdot c)\|_Y = \frac{1}{\|c\|_\infty} \|x \cdot c - y \cdot c\|_\infty = |x - y|$$

for all $x, y \in \mathbb{R}$ stating that $\tilde{I}$ yields an isometry. The oddness of $\tilde{I}$ results from that of I. Finally,

$$f(x) = I(x \cdot c) = \|c\|_\infty \tilde{I}(x), \qquad x \in \mathbb{R},$$

i.e. f is proportional to the odd isometry $\tilde{I}$, as claimed. □

The subsequent corollary (the main result in Schöpf's paper [27]) does not follow directly from our Theorem 5.9. The derivation of condition (iii) below is possible via a structural result of Jacek Tabor describing the form of odd isometries on the real line (see Ja. Tabor, Isometries from $\mathbb{R}$ to a Banach space, oral communication). We omit the details here.

Corollary 5.4 *Any continuous solution $f : \mathbb{R} \to Y$ of Equation (FM), where $(Y, \|\cdot\|_Y)$ stands for a real normed linear space, satisfies the following conditions:*

(i) *f is odd,*
(ii) $\|f(xy)\| = |x|\|f(y)\|$ *for all* $x, y \in \mathbb{R}$,
(iii) $\operatorname{conv}\left\{\frac{f(y)-f(x)}{\|f(y)-f(x)\|} : x, y \in \mathbb{R},\ x < y\right\}$ *is contained in the unit sphere $S \subset X$.*

Conversely, any function $f : \mathbb{R} \to Y$ that enjoys properties (i), (ii) *and*

(*iii′*) *for every quadruple x, y, u, v of real numbers such that $x < y$ and $u < v$ the segment joining the points $\frac{f(y)-f(x)}{\|f(y)-f(x)\|}$ and $\frac{f(v)-f(u)}{\|f(v)-f(u)\|}$ is contained in S,*

is necessarily continuous and satisfies Equation (FM).

Corollary 5.5 *Let* $(X,+)$ *be an Abelian group with uniquely performable division by* 2 *and* 3 *and let* $(Y, \|\cdot\|_Y)$ *be a real Banach space. Then any surjective solution* $f: X \to Y$ *of Equation* (FM) *is additive.*

Proof An appeal to Theorem 5.7 shows that $f = I \circ A$ where $A : X \to B(T,\mathbb{R})$ is an additive map and I stand for an odd isometry mapping the set $G := A(X)$ into Y. Clearly, the subgroup $(G,+)$ of the additive group $(B(T,\mathbb{R}),+)$ enjoys the following property:

$$G = \lambda G, \qquad \lambda \in D := \{\pm 2^n 3^m : n, m \in \mathbb{Z}\}.$$

On the other hand, the surjectivity of f implies that I yields a surjective isometry of G onto the Banach space Y. Therefore G is a closed subset of the space $B(T,\mathbb{R})$ and, a fortiori,

$$G = \lambda G, \qquad \lambda \in \mathbb{R},$$

because of the density of the set D in $\mathbb{R}$. Hence, the isometry I yields a surjection of the Banach space $(G,+)$ onto Y and being odd has to be linear by means of the well-known Mazur–Ulam theorem. Consequently, f is additive as a composition of two additive maps. □

Remark 5.3 Corollary 5.5 is, however, a considerably weaker version of Tabor's result from [27] where neither commutativity nor divisibility assumptions were imposed upon the domain group.

Two further questions might be asked:

- what about the uniqueness of the factorization spoken of in Theorem 5.7?
- does the result carry over to the case of Abelian semigroups?

The first question has a negative answer; actually we are pretty far from any kind of uniqueness. This is visible already from the last part of the statement of Theorem 5.7 the platform space $(Z, \|\cdot\|)$ occurring in the "only if" part, whichever it could be, may always be replaced by the space $B(T,\mathbb{R})$ considered in the "if" part of the theorem.

The other question remained open for many years and finally has been partially answered by Badora who has shown in [2] that commutativity may be replaced by the requirement that the group in question is a so-called $\mathscr{G}$-group. We shall discuss this problem in the next section.

5.4 The Hierarchy of (Non)Commutativity

Recall that the essential part of the proof of Lemma 5.2 was to show that

the family of all additive real functionals a on X majorized by p is nonvoid.

In that connection Badora [2] decided to introduce the notion of $\mathscr{G}$-groups, as those enjoying this property. More exactly:

Definition 5.1 We say that a group $(G, +)$ belongs to the class $\mathcal{G}$ if and only if for each subadditive functional $p : G \to \mathbb{R}$ there exists an additive functional $a : G \to \mathbb{R}$ such that $a \leq p$.

It turns out that that notion is closely connected with the validity of Hahn–Banach extension theorem for groups. Namely, the following characterization of the class of $\mathcal{G}$-groups holds true.

Theorem 5.9 (Badora [28]) *Let $(G, +)$ be a group. Then $(G, +) \in \mathcal{G}$ if and only if for each subgroup $(G_0, +)$ of the group $(G, +)$ and for every subadditive functional $p : G \to \mathbb{R}$ such that*

$$M(x) := \sup\{p(-a + x + a) - p(x) : a \in G_0\} < \infty$$

and

$$\liminf_{n\to\infty} \frac{1}{n} M(nx) = 0,$$

for all $x \in G$, and for every additive functional $a_0 : G_0 \to \mathbb{R}$ with $a_0 \leq p|_{G_0}$, there exists an additive functional $a : G \to \mathbb{R}$ such that $a|_{G_0} = a_0$ and $a \leq p$.

Corollary 5.6 *Let $(G, +)$ be a group from the class $\mathcal{G}$ and let $p : G \to \mathbb{R}$ be a subadditive functional such that*

$$p(2x) = 2p(x), \qquad x \in G.$$

Then for every subgroup $(G_0, +)$ of the group $(G, +)$ and for every additive functional $a_0 : G_0 \to \mathbb{R}$ enjoying the property $a_0 \leq p|_{G_0}$, there exists a functional $a : G \to \mathbb{R}$ such that $a|_{G_0} = a_0$ and $a \leq p$.

Moreover, Badora has shown in [2] that the following classes of groups $(G, +)$ are contained in class $\mathcal{G}$:

- Abelian groups
- *amenable groups*, i.e. those admitting a positive, translation invariant linear functional $M : B(G, \mathbb{R}) \to \mathbb{R}$ with $M(1) = 1$;
- *weakly commutative groups*, i.e. those enjoying the following property: for each $x, y \in G$ there exists a positive integer n such that $2^n(x + y) = 2^n x + 2^n y$.

By *Hyers groups* we comprehend those enjoying the following property: for each functional $f : G \to \mathbb{R}$ with bounded Cauchy difference $G \times G \ni (x, y) \mapsto f(x + y) - f(x) - f(y) \in \mathbb{R}$, there exists a homomorphism $a : G \to \mathbb{R}$ such that $f - a$ is bounded.

The following chain of inclusions holds true:

$$\begin{array}{c} \textit{Abel} \subset \textit{Amen} \subset \mathcal{G} \subset \textit{Hyers} \\ \qquad\qquad \bigcup \\ \textit{weak commutativity} \end{array}$$

It is known that free groups with two free generators fail to be Hyers ones (see Forti's remark [14]); consequently, such groups stay off the class $\mathscr{G}$. Till now it is not known whether anyone of the inclusions

$$\mathit{Amen} \subset \mathscr{G} \subset \mathit{Hyers}$$

is strict.

Undoubtedly, Badora's idea of introducing the class $\mathscr{G}$ proved to be extremely useful. In particular, in all corresponding results concerning Equation (FM), the commutativity assumption of the group considered may now be replaced by the requirement that this group belongs to the class $\mathscr{G}$. Above all, it holds true in the case of the factorization Theorem 5.7 which now, without any changes in the proof, may be improved as follows:

Theorem 5.10 *Let a group* $(X, +)$ *be a member of class* $\mathscr{G}$ *and let* $(Y, \|\cdot\|)$ *be a real normed linear space. Let further* $f : X \to Y$ *be a solution to functional equation* (FM). *Then there exist: a nonempty set* $T \subset \mathbb{R}^X$, *an additive operator* $A : X \to B(T, \mathbb{R})$ *and an odd isometry* $I : A(X) \to Y$ *such that*

$$f(x) = I(A(x)), \qquad x \in X.$$

Conversely, for an arbitrary real normed linear space $(Z, \|\cdot\|_Z)$, *any additive operator* $A : X \to Z$ *and any odd isometry* $I : A(X) \to Y$ *the superposition* $f := I \circ A$ *yields a solution of Equation* (FM).

5.5 Pexiderization

The results presented in the present section are published with detailed proofs in paper [19] of mine in which an answer to a question posed by Ludwig Reich during my stay at the Karl-Franzens Universität (Graz, Austria, Autumn 1995) gives a description of solutions to the functional equation

$$\|f(x+y)\| = \|g(x) + h(y)\|. \tag{PFM}$$

Surprisingly, in contrast to the preceding results, even in the case of strictly convex ranges, the pexiderized Equation (FM), i.e. Equation (PFM) fails to be equivalent to the Pexider functional equation

$$f(x+y) = g(x) + h(y). \tag{P}$$

Indeed, let $(X, +)$ be a groupoid and let $(Y, \|\cdot\|)$ be a normed linear space with $\dim Y \geq 2$. Fix arbitrarily a positive real number ϱ and a $d \in Y$. Denoting by $S(a, \varrho)$ the sphere $\{u \in Y : \|u - a\| = \varrho\}$, $a \in Y$, one can easily check that the triple

(f, g, d) yields a solution to (PFM) for quite arbitrary mappings $f : X \to S(0, \varrho)$ and $g : X \to S(-d, \varrho)$. Therefore, in general, Equation (PFM) enjoys an abundance of solutions being far away from translations of an additive map which are the only ones satisfying the Pexider equation (cf. Aczél and Dhombres [1] or Kuczma [25], for instance). As we shall see later on such a phenomenon is caused by the lack of zeros of the map f. If f vanishes at at least one point of its domain, then all the triples (f, g, h) fulfilling (PFM) may be expressed in terms of mappings G fulfilling the equation

$$\|G(x-y)\| = \|G(x) - G(y)\|. \tag{5.6}$$

5.5.1 Solutions Admitting Zeros

Assuming that either f or, equivalently, the two-place function $(x, y) \mapsto g(x) + h(y)$ vanishes at some point we shall reduce Equation (PFM) to (5.6). Namely we have the following:

Theorem 5.11 *Let $(X, +)$ be a group (not necessarily commutative) and let $(Y, \|\cdot\|)$ be a (real or complex) normed linear space. Assume that functions $f, g, h : X \to Y$ satisfy the functional equation (PFM) for all $x, y \in X$ and $f(x_0) = 0$ for some $x_0 \in X$. Then there exists a solution $G : X \to Y$ of Equation (5.6) and a vector $a \in Y$ such that*

$$g(x) = G(x) + a, \qquad x \in X, \tag{5.7}$$

$$h(x) = -G(x_0 - x) - a, \qquad x \in X, \tag{5.8}$$

and f is a selection of the multifunction

$$X \ni x \mapsto S(0, \|G(x) - G(x_0)\|) \subset Y. \tag{5.9}$$

Conversely, for every solution $G : X \to Y$ of Equation (5.6), for every vector $a \in Y$, for every point $x_0 \in X$ and for every selection f of the multifunction (5.9), the triple (f, g, h) with g and h given by (5.7) and (5.8), respectively, yields a solution to (PFM) with $f(x_0) = 0$.

Remark 5.4 The assumption on f to possess a zero in X may equivalently be replaced by the requirement

$$h^{-1}(-g(X)) \neq \emptyset \qquad \text{or} \qquad g^{-1}(-h(X)) \neq \emptyset.$$

In particular, this is the case provided that at least one of the maps g and h is surjective.

Theorem 5.12 *Let $(X, +)$ be a group (not necessarily commutative) and let $(Y, \|\cdot\|)$ be a (real or complex) strictly convex normed linear space. Assume that functions $f, g, h : X \to Y$ satisfy the functional equation (PFM) for all $x, y \in X$ and $f(x_0) = 0$ for some $x_0 \in X$. If either the even part of g is constant or the function $X \ni x \mapsto h(x + x_0) \in Y$ has constant even part, then there exists an additive map $G : X \to Y$ and constants $a, b \in Y$ such that*

$$g(x) = G(x) + a, \qquad x \in X,$$
$$h(x) = G(x) + b, \qquad x \in X,$$

and f is a selection of the multifunction

$$X \ni x \mapsto S(0, \|G(x) + a + b\|) \subset Y.$$

Conversely, for every additive function $G : X \to Y$, for every vectors $a, b \in Y$ and for every selection f of the above multifunction, the triple (f, g, h) with g and h given by the above formulae yields a solution to (PFM).

Remark 5.5 A particular selection

$$f(x) := G(x) + a + b, \qquad x \in X,$$

of the multifunction considered in Theorem 5.12 leads to a solution (f, g, h) of the Pexider equation (P). However, in general, Theorem 5.12 shows that even in the case of strictly convex ranges, a solution (f, g, h) of (PFM) may still be far from any triple solving (P) because of multitude of possible selections f. Nevertheless, remarkable is the fact that functions g and h in any such triple are exactly those occurring in solutions of the Pexider equation (translations of an additive function).

5.5.2 *Basic Equation and Additivity*

As we have seen, Equation (5.6) happened to be basic while studying (PFM). Obviously, each odd solution of (5.6) satisfies (FM) and every solution of (FM) is easily checked to be odd. Therefore

Remark 5.6 Equations (5.6) and (FM) are equivalent in the class of odd functions mapping a group into a normed linear space.

Replacing x by $x + y$ in (5.6) we arrive at

$$\|G(x)\| = \|G(x + y) - G(y)\|,$$

which, in case of Abelian domains, is equivalent to

$$\|G(x + y) - G(x)\| = \|G(y)\|. \tag{S}$$

Equally simple is the way back whence

Remark 5.7 Equations (5.6) and (S) are equivalent in the class of functions mapping a commutative group into a normed linear space.

Equation (S) was examined by Skof [28] in the case where the unknown function G is defined on a real linear space. Her principal goal was to give sufficient conditions for a solution of (S) to be additive. As we shall see later on, the main results (Theorems 1 and 2 in [28]) are special cases of our Theorem 5.13 (ii) and Corollary 5.8, respectively.

We proceed with the following:

Theorem 5.13 *Let $(X, +)$ be an Abelian group and let $(Y, \|\cdot\|)$ be a strictly convex normed linear space. If $G : X \to Y$ is a solution to the equation*

$$\|G(x-y)\| = \|G(x) - G(y)\|, \qquad x, y \in X,$$

then the following conditions are pairwise equivalent:

(i) *G is additive;*
(ii) *$G(X) = -G(X)$;*
(iii) *G is odd;*
(iv) *$\|G(2x)\| = 2\|G(x)\|$ for all $x \in X$.*

Remark 5.8 The commutativity of $(X, +)$ was used exclusively to show that (ii) $\Rightarrow$ (iii). Even in this case the relationship

$$\|G(x+y)\| = \|G(y+x)\|, \qquad x, y \in X, \tag{5.10}$$

is sufficient to conduct that part of the proof of Theorem 5.13. Indeed, having (5.10) we replace y by $y - x$ to get

$$\|G(y)\| = \|G(x+y-x)\| = \|G(x+y) - G(x)\|$$

and that is what was really needed. The question whether or not Equation (5.6) implies (5.10) in non-Abelian groups remains open.

Remark 5.9 Unlike (FM) Equation (5.6) always admits nonadditive solutions (no matter whether or not the target space is strictly convex) provided that the domain constitutes a group possessing subgroups of index 2. If that is the case, $(K, +)$ is a subgroup of index 2 of the group $(X, +)$ and $c \neq 0$ is an arbitrarily fixed vector of the normed linear space $(Y, \|\cdot\|)$, then any function $G : X \to Y$ given by the formula

$$G(x) = \begin{cases} 0 \text{ if } x \in K \\ c \text{ if } x \in X \setminus K \end{cases} \tag{5.11}$$

yields a nonadditive solution of Equation (5.6). Indeed, G being even and nonzero cannot be additive since, otherwise, it would be odd. To check that it satisfies

Equation (5.2) fix arbitrarily a pair $(x, y) \in X^2$. The following three possibilities have to be distinguished:

(a) $x, y \in K$: then so does $x - y$ and both sides of (5.6) are equal to 0;
(b) $x, y \in X \setminus K$: then $x - y$ is in K and we have the equalities

$$G(x - y) = 0 = c - c = G(x) - G(y);$$

(c) exactly one of the arguments x, y is in K : then $x - y \in X \setminus K$ whence $G(x - y) = c$ and $G(x) - G(y) \in \{-c, c\}$; thus (5.6) is satisfied as well.

Remark 5.10 Functions of the form (5.11) are, jointly with the additive solutions, the only ones that satisfy Mikusiński's functional equation

$$G(x + y) \neq 0 \quad \text{implies} \quad G(x + y) = G(x) + G(y) \tag{M}$$

(cf. Dubikajtis et al. [10] or Kuczma [25]). Therefore, in the light of Remark 5.9, each solution of Equation (M) satisfies the basic equation (5.6). In the sequel we shall show, among others, that the converse is true in the case of real functionals on groups.

5.5.3 Solutions with Values in Inner Product Spaces

Except for Theorem 5.14 below, in the present section we deal with solutions to the basic equation (5.6) which map a given group into an inner product space. So, we replace the assumption of strict convexity upon the target space by a stronger requirement: the norm comes from an inner product structure.

Theorem 5.14 *Let $(X, +)$ be a group (not necessarily commutative) such that $X = 2X$ and let $(Y, (\| \cdot \|)$ be a normed linear space (real or complex). Then any even solution of Equation* (5.6)*, mapping X into Y vanishes identically on X.*

Proof Let $G : X \to Y$ be an even solution of (5.6). Replacing y by $-y$ in (5.6) leads to

$$\|G(x + y)\| = \|G(x) - G(y)\|, \qquad x, y \in X,$$

whence, by putting here $y = x$ we obtain the equality $G(2x) = 0$ valid for all $x \in X$. Since, by assumption, $2X = X$ this completes the proof. □

Remark 5.11 In view of Remark 5.10 the 2-divisibility assumption is essential because each function of the form (5.11) is even.

In what follows we wish to realize how far are the solutions of (5.6) from those of (FM). The following two results jointly with Corollary 5.7 provide some information in that direction.

Theorem 5.15 *Let* $(X, +)$ *be a group (not necessarily commutative) and let* $(Y, (\cdot|\cdot))$ *be an inner product space (real or complex). Then* $G : X \to Y$ *is a solution of Equation* (5.6) *if and only if*

$$\|G(x) + G(y)\|^2 = \|G(x + y)\|^2 + 4\Re(G(x)|G_e(y))$$

for all $x, y \in X$*, where* G_e *stands for the even part of* G*.*

Theorem 5.16 *Let* $(X, +)$ *be a commutative group and let* $(Y, (\cdot|\cdot))$ *be a real inner product space. Then Equation* (5.6) *is equivalent to the system*

$$\|G(x) + G(y)\|^2 = \|G(x + y)\|^2 + \|G(x) + G(y) - G(x + y)\|^2$$
$$\|G(x) + G(y) - G(x + y)\|^2 = 4\,(G(x)|G_e(y))$$

assumed for all $x, y \in X$*. In particular, any solution* $G : X \to Y$ *of* (5.6) *enjoys the property*

$$G(x + y) \perp G(x) + G(y) - G(x + y).$$

Observe that due to the commutativity of the group $(X, +)$ the assertion of Theorem 5.15 implies the equality

$$(G(x)|G_e(y)) = (G(y)|G_e(x))$$

valid for all $x, y \in X$. Plainly, we have also

$$(G(-x)|G_e(y)) = (G(y)|G_e(x)), \qquad x, y \in X,$$

which, by subtracting these two equalities side by side, we deduce the following:

Corollary 5.7 *Under the assumptions of Theorem 5.16 every solution* $G : X \to Y$ *of Equation* (5.6) *has the following property:*

$$G_o(x) \perp G_e(y), \qquad x, y \in X,$$

where G_o *and* G_e *stand for the odd and even part of* G*, respectively. In particular, if the set* $\{G_o(x) : x \in X\}$ *is total, then* G *is additive.*

Finally, we shall show that in the case of real functionals the basic equation (5.6) and Mikusiński's equation (M) are equivalent.

Theorem 5.17 *Let* $(X, +)$ *be a commutative group. Then a function* $G : X \to \mathbb{R}$ *satisfies the equation*

$$|G(x - y)| = |G(x) - G(y)|, \qquad x, y \in X, \tag{5.12}$$

if and only if G is a solution to Mikusiński's equation

$$G(x+y)\ [G(x+y)-G(x)-G(y)]=0, \qquad x,y\in X. \tag{5.13}$$

Proof Let $G : X \to \mathbb{R}$ be a solution of (5.12). An appeal to Theorem 5.16 shows that

$$G(x+y) \perp G(x+y)-G(x)-G(y)$$

for all $x, y \in X$ which, in the real case, states nothing else but (5.13).

As to the converse Remark 5.10 may directly be applied. This ends the proof. □

Remark 5.10 jointly with Theorem 5.17 immediately implies the following:

Corollary 5.8 *If $(X, +)$ is a commutative group with no subgroups of index 2, then a function $G : X \to \mathbb{R}$ satisfies Equation (5.12) if and only if G is additive.*

5.6 Inequality Case

Is there any chance to obtain nontrivial results for the case where the equality sign in Equation (FM) would be replaced by that of inequality? More precisely, there are two possibilities:

- to assume that for every x, y from the domain (semigroup, at least, written additively) of a function f whose codomain is a normed linear space, one has

$$\|f(x+y)\| \leq \|f(x)+f(y)\|;$$

- to assume that for every x, y from the domain (semigroup, at least, written additively) of a function f whose codomain is a normed linear space, one has

$$\|f(x+y)\| \geq \|f(x)+f(y)\|.$$

The first possibility seems to be pointless because of the abundance of solutions that might be expected. For instance, given any normed linear space $(E, \|\cdot\|)$ the function $f : E \to \mathbb{R}$ defined by $f(x) = \|x\|,\ x \in E$, is a solution. For any nonnegative increasing subadditive function $\varphi : [0, \infty) \to \mathbb{R}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = \varphi(|x|),\ x \in [0, \infty)$, is a solution as well.

What concerns the other possibility, the following very interesting result of Gyula Maksa and Peter Volkmann has been obtained in their paper [26]. In what follows, the details will be reported on.

Theorem 5.18 (Maksa and Volkmann [26]) *Let $(X, +)$ be a group and $(Y, (\cdot|\cdot))$ be a real or complex inner product space. Let further $f : X \to Y$ be a solution to the functional inequality*

$$\|f(x+y)\| \geq \|f(x)+f(y)\|, \qquad x,y\in X. \tag{MV}$$

Then f is additive.

Proof Putting $x = y = 0$ in (MV) we infer that $f(0) = 0$. Consequently, on setting $y = -x$ in (MV) we get $f(-x) = -f(x),\ x \in X$. Squaring both sides of (MV) we arrive at

$$\|f(x+y)\|^2 \geq \|f(x)\|^2 + 2\Re(f(x)|f(y)) + \|f(y)\|^2. \tag{5.14}$$

Replacing here x and y by $x + y$ and $-y$, respectively, and taking into account the oddness of f, we obtain the inequality

$$\|f(x)\|^2 \geq \|f(x+y)\|^2 - 2\Re(f(x+y)|f(y)) + \|f(y)\|^2$$

whence

$$-\|f(x+y)\|^2 \geq -\|f(x)\|^2 - 2\Re(f(x+y)|f(y)) + \|f(y)\|^2.$$

Now, adding the latter inequality to (5.14) side by side we infer that

$$2\Re(f(x)|f(y)) - 2\Re(f(x+y)|f(y)) + 2\|f(y)\|^2 \leq 0,$$

or, equivalently,

$$\Re(f(x) + f(y) - f(x+y)|f(y)) \leq 0. \tag{5.15}$$

Replacing in (5.14) x and y by $-x$ and $x + y$, respectively, and taking into account the oddness of f, we obtain the inequality

$$\|f(y)\|^2 \geq \|f(x)\|^2 - 2\Re(f(x)|f(x+y)) - \|f(x+y)\|^2$$

whence

$$-\|f(x+y)\|^2 \geq \|f(x)\|^2 - 2\Re(f(x)|f(x+y)) - \|f(y)\|^2.$$

Now, adding the latter inequality to (5.14) side by side we infer that

$$2\|f(x)\|^2 + 2\Re(f(x)|f(y) - f(x+y)) \leq 0,$$

or, equivalently,

$$\Re(f(x) + f(y) - f(x+y)|f(x)) \leq 0. \tag{5.16}$$

Replacing here x and y by $x + y$ and $-y$, respectively, and taking into account the oddness of f, we get

$$\Re(f(x+y) - f(y) - f(x)|f(x+y)) \leq 0,$$

or, equivalently,

$$\Re(f(x) + f(y) - f(x+y)| - f(x+y)) \leq 0. \tag{5.17}$$

Now, adding (5.15)–(5.17), side by side, we deduce finally that the inequality

$$\|f(x) + f(y) - f(x+y)\|^2 \leq 0,$$

holds true for all elements x, y from X. This implies the additivity of f and finishes the proof. □

That kind result fails to hold in the case where the group domain is replaced by a semigroup one. In fact, take $(X, +) = ([0, \infty), +)$, $(Y, (\cdot|\cdot)) = (\mathbb{R}, \cdot)$, and $f : [0, \infty) \to \mathbb{R}$ given by the formula $f(x) = x^2,\ x \in [0, \infty)$. Then

$$|f(x+y)| = (x+y)^2 = x^2 + y^2 + 2xy \geq x^2 + y^2 = |f(x) + f(y)|.$$

The authors of [26] have posed also the following:

Problem Is it possible to replace the unitary target space by a strictly convex one?

The aforesaid result of Maksa and Volkmann has recently been generalized by Száz in [29]. The generalization consists in replacing the target inner product space by a group $(Y, +)$ endowed with an inner product $Q : Y \times Y \to \mathbb{C}$ subjected to satisfy the following conditions:

(a) $Q(x, x) \geq 0$ and $Q(x, x) = 0$ forces x to be 0;
(b) $Q(y, x) = \overline{Q(x, y)}$;
(c) $Q(x + y, z) = Q(x, z) + Q(y, z)$,

for all x, y, z from Y.

Theorem 5.19 (Száz [29, 30]) *Let $(X, +)$ be a group and $(Y+)$ be a group endowed with an inner product Q. Put*

$$q(u) := \sqrt{Q(u, u)}, \qquad u \in Y.$$

Then for every map $f : X \to Y$ the following conditions are pairwise equivalent:

- *f is additive;*
- *$q(f(x+y)) \geq q(f(x) + f(y))$ for all $x, y \in X$;*
- *f is odd and*

$$\Re Q(f(x), f(y)) \leq \frac{1}{2}\left(q(f(x+y))^2 - q(f(x))^2 - q(f(y))^2\right)$$

for all $x, y \in X$.

In a final Remark 3.4 of his paper spoken of, Száz emphasizes that his proof of the above theorem "does not requires particular tricks" (author's spelling) and therefore it is "more simple" than that presented by Maksa and Volkmann (see the proof of Theorem 5.18 above).

In a feature article of Száz *Remarks and Problems at the Conference on Inequalities and Application* [30], containing 228 references, item nr [207] is a self-citation and reads as follows:

[207] Á. Száz, *A generalization of a theorem of Maksa and Volkmann on additive functions*, Tech. Rep., Inst. Math., Univ. Debrecen 2016/5, 6 pp. (The publication of an improved and enlarged version of this work in the Anal. Math. was probably prevented by a close colleague of Ger.)

No comments.

5.7 Stability

We shall present two single results in two categories:

- Hyers–Ulam stability of the Fischer–Muszély equation;
- Fischer–Muszély equation postulated almost everywhere.

It turns out that Fischer–Muszély equation is stable in the sense of Hyers and Ulam. More precisely we have the following result established by Tabor in his paper [31] for the class of surjective mappings.

Theorem 5.20 (Tabor [31]) *Let* $(G, +)$ *be a group and let* $(X, \|\cdot\|)$ *be a Banach space. If a surjective map* $f : G \to X$ *satisfies the inequality*

$$\big|\|f(x+y)\| - \|f(x)+f(y)\|\big| \leq \varepsilon, \qquad x, y \in G,$$

with a given $\varepsilon \geq 0$*, then*

$$\|f(x+y) - f(x) - f(y)\| \leq 13\varepsilon, \qquad x, y \in G.$$

In particular $(\varepsilon = 0)$*, any surjective solution of Equation* (FM) *is additive.*

Corollary 5.9 *If* $(G, +)$ *is amenable, or more generally, if* $(G, +)$ *happens to be a* $\mathscr{G}$*-group, then there exists exactly one additive map* $a : G \to X$ *such that* $\|f(x) - a(x)\| \leq 13\varepsilon$ *for all* $x \in G$*. Consequently, in that case, the Fischer–Muszély functional equation is stable in the class of surjective mappings.*

Now we want to exhibit another stability property: we shall show that under suitable assumptions a function satisfying the Fischer–Muszély functional equation postulated almost everywhere has to coincide with an additive map almost everywhere.

In what follows the symbol $(G, +)$ will stand for an additively written group. Recall that a nonempty family $\mathscr{J} \subset 2^G \setminus \{G\}$ is called a *proper linearly invariant ideal* (briefly: p.l.i. ideal) in G provided that it satisfies the following conditions:

(i) if $A, B \in \mathscr{J}$, then $A \cup B \in \mathscr{J}$;
(ii) if $A \in \mathscr{J}$ and $B \subset A$, then $B \in \mathscr{J}$;
(iii) if $A \in \mathscr{J}$ and $x \in G$, then $x - G \in \mathscr{J}$.

We say that a property $\mathscr{P}(x)$ holds $\mathscr{J}$*-almost everywhere in G* whenever $\mathscr{P}(x)$ is valid for all $x \in G \setminus U$ for some set $U \in \mathscr{J}$.

For a subset $M \subset G^2$ and $x \in G$ we define a *section*

$$M[x] := \{y \in G : \ (x, y) \in M\}.$$

An ideal $\widehat{\mathscr{J}}$ in G^2 is said to be *conjugate* with an ideal $\mathscr{J}$ in G if and only if for every set $M \in \widehat{\mathscr{J}}$ the appurtenance $M[x] \in \mathscr{J}$ takes place $\mathscr{J}$-almost everywhere in G.

The family

$$\Omega(\mathscr{J}) := \{M \subset G^2 : \ M[x] \in \mathscr{J} \text{ for } \mathscr{J}\text{-almost all } x \in G\}$$

yields the largest (in sense of the set inclusion) p.l.i. ideal in G^2 being conjugate to $\mathscr{J}$ [see, e.g., Kuczma [25, Ch. XVII, §5]].

Our main result reads as follows.

Theorem 5.21 *Given a p.l.i. ideal $\mathscr{J}$ in a group $(G, +)$ and a real or complex inner product space $(H, (\cdot|\cdot))$, assume that a map $f : G \to H$ satisfies Equation* (FM) *for all pairs $(x, y) \in G^2$ off a set $M \in \Omega(\mathscr{J})$ such that $T_1(M)$ and $T_2(M)$ stay in $\Omega(\mathscr{J})$ for $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x - y)$, $(x, y) \in G^2$.*

If, moreover, for any set U from $\mathscr{J}$ the set $\frac{1}{2}\, U := \{x \in G : 2x \in U\}$ belongs to $\mathscr{J}$ and there exists a member E of $\mathscr{J}$ such that

$$M \cap \bigcup_{k=1}^{3} \{(x, kx) \in G^2 : x \notin E\} = \emptyset,$$

then there exists a unique additive map $a : G \to H$ such that

$$\{x \in G : f(x) \neq a(x)\} \in \mathscr{J}.$$

Proof To apply the technique used by Fischer and Muszély in [12] (see also p. 139 in Aczél and Dhombres [1], fix arbitrarily an $x \in G \setminus (E \cup \frac{1}{2}\, E)$; then all the pairs (x, x), $(x, 2x)$ and $(x, 3x)$ as well as $(2x, 2x)$ are off M and we have

$$\|f(2x)\| = 2\|f(x)\|, \ \ \|f(3x)\| = \|f(x) + f(2x)\|,$$
$$4\|f(x)\| = \|f(4x)\| = \|f(x) + f(3x)\|,$$

which like in [12], forces the equality

$$f(2x) = 2f(x), \qquad x \in G \setminus (E \cup \frac{1}{2}\, E). \tag{5.18}$$

Since M is supposed to be a member of $\Omega(\mathscr{J})$, there exists a set $U \in \mathscr{J}$ such that for every $x \in G \setminus U$ the section $M[x]$ falls into $\mathscr{J}$.

Let N stand for the set-theoretical union of the following seven sets: $M, (E \cup (\frac{1}{2}\, E)) \times G, G \times (E \cup (\frac{1}{2} E))$ and

$$M_1 := \{(x, y) \in G^2 : x \in \frac{1}{2}\, U \text{ or } y \in M[2x]\},\ \ M_2 := \{(x, y) \in G^2 : x \in U \text{ or } y \in \frac{1}{2} M[x]\},$$
$$M_3 := \{(x, y) \in G^2 : x \in U \text{ or } y \in -x + M[x]\},\ \ M_4 := (T_1 \circ T_2)(M).$$

Each one of these seven sets yields a member of the ideal $\Omega(\mathscr{J})$. Indeed, this is obvious for the first three sets as well as, by the invariance assumptions, for the set M_4. To check that $M_1 \in \Omega(\mathscr{J})$ note that for every $x \notin \frac{1}{2}\, U \in \mathscr{J}$ the section

$$M_1[x] = \{y \in G : (x, y) \in M_1\} = \{y \in G : y \in M[2x]\} = M[2x] \text{ belongs to } \mathscr{J}.$$

Similarly, since for every $x \notin U \in \mathscr{J}$ the section

$$M_2[x] = \{y \in G : (x, y) \in M_2\} = \{y \in G : y \in \frac{1}{2}\, M[2x]\} = \frac{1}{2} M[2x] \text{ belongs to } \mathscr{J},$$

we infer that $M_2 \in \Omega(\mathscr{J})$. Finally, for every $x \notin U \in \mathscr{J}$ the section

$$M_3[x] = \{y \in G : (x, y) \in M_3\} = \{y \in G : y \in -x + M[x]\} = -x + M[x] \text{ belongs to } \mathscr{J},$$

which shows that $M_3 \in \Omega(\mathscr{J})$.

Consequently, the union N of all the sets spoken of yields a member of the ideal $\Omega(\mathscr{J})$ as well. Now, fix arbitrarily a pair $(x, y) \in G^2 \setminus N$. Then:

1. $\|f(x + y)\| = \|f(x) + f(y)\|$ because $(x, y) \notin M$;
2. $f(2x) = 2f(x)$ and $f(2y) = 2f(y)$ because of (5.18) and the fact that $x, y \notin E \cup \frac{1}{2}\, E$;
3. $\|f(2x+y)\| = \|f(2x)+f(y)\|$ because $(x, y) \notin M_1$ which forces the pair $(2x, y)$ to stay off the set M;
4. $\|f(2x + y)\| = \|f(x) + f(x + y)\|$ because $(x, y) \notin M_3$ which forces the pair $(x, x + y)$ to stay off the set M;
5. $\|f(x+2y)\| = \|f(x)+f(2y)\|$ because $(x, y) \notin M_2$ which forces the pair $(x, 2y)$ to stay off the set M;
6. $\|f(x + 2y)\| = \|f(x + y) + f(y)\|$ because $(x, y) \notin M_4$ which forces the pair $(x + y, y)$ to stay off the set M.

Relations 3. and 4. jointly with 2. imply that

$$\|f(x) + (f(x) + f(y))\| = \|f(x) + f(x+y)\|, \tag{5.19}$$

whereas a similar conclusion

$$\|(f(x) + f(y)) + f(y)\| = \|f(x+y) + f(y)\|, \tag{5.20}$$

can be drawn from relations 5. and 6. jointly with 2. By means of 1., after squaring both sides of (5.19) and (5.20), by a simple calculation, we derive the equalities

$$\Re((f(x)|f(x+y) - f(x) - f(y))) = 0 = \Re((f(y)|f(x+y) - f(x) - f(y))),$$

respectively, which immediately imply that

$$\Re((f(x) + f(y)|f(x+y) - f(x) - f(y))) = 0. \tag{5.21}$$

Along the same lines as in the paper [12] of Fischer and Muszély, from the trivial equality

$$\|f(x+y)\|^2 = \|(f(x) + f(y)) + (f(x+y) - f(x) - f(y))\|^2$$

with the aid of 1. and (5.21) we derive the relationship

$$\|f(x+y) - f(x) - f(y)\|^2 = 0.$$

This clearly forces the additivity relation

$$f(x+y) = f(x) + f(y)$$

that remains valid for all pairs $(x, y) \in G^2 \setminus N$, i.e. $\Omega(\mathscr{J})$-almost everywhere in G^2. Now, it remains to apply a de Bruijn's type result from [15]: there exists a unique additive function $a : G \to H$ such that the equality $f(x) = a(x)$ holds for $\mathscr{J}$-almost all $x \in G$, i.e.

$$\{x \in G : f(x) \neq a(x)\} \in \mathscr{J}.$$

Thus the proof has been completed. □

Remark 5.12 The leading idea of the proof above was to run along the lines of the proof presented in [12] treating it as the obstacle race. However, the set of obstacles, although basically caused by the fact that the validity of the (FM) equation is postulated merely almost everywhere, was enlarged by another one; namely, close to the bottom of page 199 in [12] the authors write:

If we interchange the variables x and y in Equation (16) *we get*

$$[\Re(f(y), f(x+y) - (f(x) + f(y))] = 0], \tag{17}$$

which is wrong; actually, we get then

$$[\Re(f(y), f(y+x) - (f(x) + f(y))] = 0],$$

and not (17) because of the lack of the commutativity of the domain semigroup.

In what follows we shall present a few corollaries illustrating some consequences of the theorem just proved.

Corollary 5.10 *Let* $(X, \|\cdot\|)$ *stand for a normed linear space and let* $(H, (\cdot|\cdot))$ *be an inner product space. If a map* $f : X \to H$ *satisfies the Fischer–Muszély functional equation* (FM) *in a vicinity of infinity (outside an arbitrarily given ball centred at the origin), then there exist a unique additive map* $a : X \to H$ *and a bounded set* $B \subset X$ *such that* $f(x) = a(x)$ *for all* $x \in X \setminus B$.

Proof Let $\mathscr{J}$ stand for the p.l.i. ideal of all bounded subsets of the space X. Clearly, any bounded set and, in particular, any ball $M := B((0,0), r)$ in the product space X^2 yields a member of $\Omega(\mathscr{J})$. Assume that

$$\|f(x+y)\| = \|f(x) + f(y)\|, \qquad (x, y) \in X^2 \setminus M.$$

Put $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x-y)$, $(x, y) \in X^2$. The images $T_1(M)$ and $T_2(M)$ are contained in M and $\sqrt{5}M$, respectively, so that they stay in $\Omega(\mathscr{J})$. Moreover, $\frac{1}{2}\, U$ is bounded for any bounded set U. Finally, since the set $E := \{x \in X : \|x\| \le r\}$ belongs to $\mathscr{J}$ and for every $x \in X \setminus E$ one has

$$\|(x, kx)\| = \sqrt{1+k^2}\|x\| \ge \sqrt{2}r > r, \qquad k \in \{1, 2, 3\},$$

the condition

$$M \cap \bigcup_{k=1}^{3} \{(x, kx) \in G^2 : x \notin E\} = \emptyset$$

is satisfied. Thus all the assumptions of Theorem 5.21 are fulfilled which ends the proof. □

Corollary 5.11 *Let* $(G, +)$ *stand for a uniquely* 2*-divisible locally compact group and let* $(H, (\cdot|\cdot))$ *be an inner product space. Denote by* h_1 *and* h_2 *the left Haar measures in* G *and* G^2*, respectively, with* $h_1(G) = \infty$*; moreover, let* h_1^* *be the outer Haar measure associated with* h_1*. Assume that for every set* $U \subset G$ *one has* $h^*(\{x \in G : 2x \in U\}) < \infty$ *provided that* $h^*(U) < \infty$*. If a map* $f : G \to H$ *satisfies the Fischer–Muszély functional equation* (FM) *for all* $(x, y) \in G^2 \setminus M$ *where* $M \subset G^2$ *is a set of finite measure* h_2 *and such that*

$$M \cap \bigcup_{k=1}^{3} \{(x, kx) \in G^2 : x \in G\} = \emptyset,$$

then there exist a unique additive map $a : G \to H$ *and a set* $B \subset G$ *such that* $h_1^*(B) < \infty$ *and* $f(x) = a(x)$ *for all* $x \in G \setminus B$.

Proof Let $\mathscr{J}$ stand for the p.l.i. ideal of all subsets of G having finite outer measure h_1^*. Since, by Fubini's theorem, one has

$$\infty > h_2(M) = \int_G h_1(M[x])dh_1(x),$$

we infer that h_1-almost all sections $M[x]$ are of finite h_1 measure. This proves that M falls into the ideal $\Omega(\mathscr{J})$. Let T_1 and T_2 be defined as in the statement of Theorem 5.18. Directly from the definition of the product measure it follows that $h_2(T_1(M)) = h_2(M) < \infty$ and

$$h_2(T_2(M)) = \int_G h_1(T_2(M)[x])dh_1(x) = \int_G h_1(-x + T_1(M)[x])dh_1(x)$$
$$= \int_G h_1(T_1(M)[x])dh_1(x) = h_2(T_1(M)) = h_2(M) < \infty.$$

Therefore, h_1-almost all sections $T_2(M)[x]$ are of finite h_1 measure which forces the image $T_2(M)$ to fall into the ideal $\Omega(\mathscr{J})$. To finish the proof it suffices to apply Theorem 5.18. □

Corollary 5.12 *Let* $(G, +)$ *stand for a uniquely 2-divisible Polish topological group and let* $(H, (\cdot|\cdot))$ *be an inner product space. Assume that the map* $G \ni x \mapsto \frac{1}{2}x \in G$ *is a homeomorphism of* G *onto itself. If a map* $f : G \to H$ *satisfies the Fischer–Muszély functional equation* (FM) *for all* $(x, y) \in G^2 \setminus M$ *where* $M \subset G^2$ *is a first category (in the sense of Baire) subset of the group* G^2 *and such that*

$$M \cap \bigcup_{k=1}^{3} \{(x, kx) \in G^2 : x \in G\} = \emptyset,$$

then there exist a unique additive map $a : G \to H$ *and a first category set* $B \subset G$ *such and* $f(x) = a(x)$ *for all* $x \in G \setminus B$.

Proof Let $\mathscr{J}$ stand for the p.l.i. ideal of all first category sets in G. Then with the aid of the celebrated Kuratowski–Ulam theorem we establish the fact that M belongs to the ideal $\Omega(\mathscr{J})$. Since the maps $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x-y)$, $(x, y) \in G^2$ yield homeomorphic self-mappings of G^2 we infer that both the images $T_1(M)$ and $T_2(M)$ stay in $\Omega(\mathscr{J})$. Moreover since, by assumption, the map $G \ni x \mapsto \frac{1}{2} x \in G$ is a homeomorphism of G onto itself, the set $\frac{1}{2}U$ is of the first Baire category provided that so is U. To finish the proof it remains to apply Theorem 5.18. □

Corollary 5.13 *Let* $(\mathbb{Z}, +)$ *be the additive group of all integers and let* $(H, (\cdot|\cdot))$ *be an inner product space. If a sequence* $(a_n)_{n\in\mathbb{Z}}$ *of elements of the space* H *satisfies the Fischer–Muszély equation*

$$\|a_{n+m}\| = \|a_n + a_m\| \tag{5.22}$$

for all but finite set of pairs $(n, m) \in \mathbb{Z}^2$*, then there exists a unique vector* $c \in H$ *such that* $a_n = nc$ *for all but finite number of integers* n*.*

Proof Let $\mathscr{J}$ stand for the p.l.i ideal of all finite subsets of $\mathbb{Z}$. Assuming that relation (5.22) holds for all $n, m \in \mathbb{Z}$ off a set $M := \{(n, m) \in \mathbb{Z}^2 : |n|, |m| \leq n_0\}$ where n_0 is a positive integer, we see that M belongs to the ideal $\Omega(\mathscr{J})$. Plainly the maps $T_1(n, m) := (m, n)$ and $T_2(n, m) := (m, n - m),\ (n, m) \in \mathbb{Z}^2$ transform finite sets into finite sets, which forces the images $T_1(M)$ and $T_2(M)$ to stay in $\Omega(\mathscr{J})$. Moreover, for every finite set $U \subset \mathbb{Z}$ the set $\{n \in \mathbb{Z} : 2n \in U\}$ is finite as well. Finally, on setting $E := \{-n_0, \ldots, -1, 0, 1, \ldots, n_0\}$ we have $E \in \mathscr{J}$ and M is disjoint with the union

$$\bigcup_{k=1}^{3} \{(n, kn) \in \mathbb{Z}^2 : n \notin E\}$$

that is contained in $\mathbb{Z}^2 \setminus M$. Thus all the assumptions of Theorem 5.21 are fulfilled which implies the existence of a unique additive map $a : \mathbb{Z} \to H$ such that the set $\{n \in \mathbb{Z} : a(n) \neq a_n\}$ is finite. Since, obviously, $a(n) = na(1),\ n \in \mathbb{Z}$, we get the equality $a_n = nc$ for all but finite number of integers n, with a unique $c := a(1) \in H$, as claimed. □

Remark 5.13 As it states, the formulation of Theorem 5.21 leaves room for improvements. For instance, it would be desirable to have

- the group considered replaced by a semigroup;
- the inner product space replaced by a strictly convex one;
- the assumption

$$M \cap \bigcup_{k=1}^{3} \{(x, kx) \in G^2 : x \notin E\} = \emptyset,$$

  removed.

Unfortunately, at present none of these three wishes can be accomplished because of the proof technique applied.

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