Chapter 4
The Translation Equation in the Ring of Formal Power Series Over $\mathbb{C}$ and Formal Functional Equations

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Abstract In this survey we describe the construction of one-parameter subgroups (iteration groups) of Γ, the group of all (with respect to substitution) invertible power series in one indeterminate x over $\mathbb{C}$. In other words, we describe all solutions of the translation equation in $\mathbb{C}[\![x]\!]$, the ring of formal power series in x with complex coefficients. For doing this the method of formal functional equations will be applied. The coefficient functions of solutions of the translation equation are polynomials in additive and generalized exponential functions. Replacing these functions by indeterminates we obtain formal functional equations. Applying formal differentiation operators to these formal translation equations we obtain three types of formal differential equations. They can be solved in order to get explicit representations of the coefficient functions. For solving the formal differential equations we apply Briot–Bouquet differential equations in a systematic way.

Keywords Translation equation • Formal functional equations • Formal partial differential equations • Aczél–Jabotinsky type equations • Briot–Bouquet equations • Formal iteration groups of type I • Formal iteration groups of type (II, k) • Ring of formal power series over $\mathbb{C}$ • Lie–Gröbner series

Mathematics Subject Classification (2010) Primary 39B12; Secondary 39B52, 13F25, 30D05

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J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer Optimization and Its Applications 124,
DOI 10.1007/978-3-319-61732-9_4

4.1 Introduction

As a motivation we mention the embedding problem from analytic mechanics [30] or geometric complex analysis [24].

4.1.1 *The Embedding Problem*

Consider a domain $U \subseteq \mathbb{C}^n$, $n \geq 1$, $\mathbf{0} = (0, \ldots, 0) \in U$, and a biholomorphic function $\tilde{F}\colon U \to U$ so that $\tilde{F}(\mathbf{0}) = \mathbf{0}$. We try to find a family $(F_t)_{t\in\mathbb{C}}$ of biholomorphic functions $F_t\colon U \to U$ so that $F_t(\mathbf{0}) = \mathbf{0}$, $t \in \mathbb{C}$, and

$$F_1 = \tilde{F}$$
$$F_s \circ F_t = F_{s+t} \qquad s, t \in \mathbb{C}. \tag{T}$$

The mapping $\mathbb{C} \times U \ni (t, x) \mapsto F_t(x) \in U$ is supposed to be holomorphic. The family $(F_t)_{t\in\mathbb{C}}$ is called a *flow*, a *one-parameter group*, an *iteration group*, or an *embedding* of $\tilde{F}$. Formula (T) is called the *translation equation*. If we represent the mappings F_t by their Taylor expansions in x and if we neglect the convergence of these series, then we obtain a solution of (T) in the ring of formal power series.

4.1.2 *The Ring of Formal Power Series with Complex Coefficients*

Now we want to study (T) in $\mathbb{C}[\![x]\!]$, the ring of all formal power series $F(x) = c_0 + c_1 x + \ldots$ in the indeterminate x over $\mathbb{C}$. For a detailed introduction to formal power series we refer the reader to [1] and [13]. Together with addition $+$ and multiplication $\cdot$ the set $\mathbb{C}[\![x]\!]$ forms a commutative ring. If $F \neq 0$, then the order of $F(x) = c_0 + c_1 x + \ldots$ is defined as $\operatorname{ord}(F) = \min\{n \geq 0 \mid c_n \neq 0\}$. Moreover, $\operatorname{ord}(0) = \infty$. The *composition* $\circ$ of formal series is defined as follows: Let $F, G \in \mathbb{C}[\![x]\!]$, $\operatorname{ord}(G) \geq 1$, then $(F \circ G)(x)$ is $F(G(x)) = \sum_{n\geq 0} c_\nu [G(x)]^\nu$. (This converges in the order topology.) Consider

$$\Gamma = \{F \in \mathbb{C}[\![x]\!] \mid F(x) = c_1 x + \ldots,\ c_1 \neq 0\} = \{F \in \mathbb{C}[\![x]\!] \mid \operatorname{ord}(F) = 1\}$$

and

$$\Gamma_1 = \{F \in \Gamma \mid c_1 = 1\}.$$

Then $(\Gamma, \circ)$ is the group of all invertible formal power series (with respect to $\circ$), and $(\Gamma_1, \circ)$ is a subgroup of $(\Gamma, \circ)$. It will be necessary to consider rings of formal power series in more than one variable, e.g., $\mathbb{C}[\![x, y]\!] = (\mathbb{C}[\![x]\!])[\![y]\!]$, $\mathbb{C}[\![x, y, z]\!]$, etc., and also rings of the form $(\mathbb{C}[y])[\![x]\!]$, where $\mathbb{C}[y]$ is the polynomial ring in y over $\mathbb{C}$, which are subrings of $\mathbb{C}[\![x, y]\!]$.

The derivation of $F \in \mathbb{C}[\![x]\!]$, $F(x) = \sum_{n\geq 0} c_n x^n$ is

$$F'(x) = \frac{dF}{dx}(x) = \sum_{n\geq 0}(n+1)c_{n+1}x^n.$$

In $\mathbb{C}[\![x, y]\!]$ or $\mathbb{C}[\![x, y, z]\!]$ we have derivations with respect to x, y, z. The chain rule is valid which means that for $F, G \in \mathbb{C}[\![x]\!]$, $\operatorname{ord}(G) \geq 1$, the derivation of $F \circ G \in \mathbb{C}[\![x]\!]$ is of the form $(F \circ G)'(x) = F'(G(x))G'(x)$. In rings of the form $\mathbb{C}[\![x, y]\!]$ or $\mathbb{C}[\![x, y, z]\!]$ the mixed chain rule holds true.

4.1.3 Iteration Groups

Iteration groups or *one-parameter groups* in $\mathbb{C}[\![x]\!]$ are families $(F_t)_{t\in\mathbb{C}}$, $F_t \in \Gamma$, $t \in \mathbb{C}$, satisfying (T). If we write $F_t(x)$ as

$$F(t, x) = \sum_{n\geq 1} c_n(t)x^n, \qquad t \in \mathbb{C},$$

then (T) is equivalent to

$$F(s + t, x) = F(s, F(t, x)), \qquad s, t \in \mathbb{C}.$$

Therefore $F_0(x) = x$ and $F_{-t}(x) = F_t^{-1}(x)$.

An iteration group in Γ can be seen as a homomorphism

$$\theta\colon (\mathbb{C}, +) \to (\Gamma, \circ), \qquad \theta(t) = F_t.$$

Moreover, in [17–19] and [16], Jabłoński and Reich were studying homomorphisms $\theta\colon (G, +) \to (\Gamma, \circ)$, where $(G, +)$ is a commutative group. In general the situation $G \neq \mathbb{C}$ is even more involved. In the present paper we will only deal with $G = \mathbb{C}$.

The problem to describe the one-parameter groups in the group of invertible formal power series in one indeterminate with complex coefficients and, more generally, to describe one-parameter groups of invertible formal power series transformations ("formally biholomorphic mappings") was studied by several authors, mainly in connection with the embedding problem, that is, whether a given formal power series (a formally biholomorphic mapping) can be embedded in such an iteration group. We mention Lewis [21], Sternberg [30], Chen [2], Peschl and Reich [24], Reich and Schwaiger [28], Mehring [23], and Praagman [25].

If $(F_t)_{t\in\mathbb{C}}$ is an iteration group in Γ and $S \in \Gamma$, then $(S^{-1} \circ F_t \circ S)_{t\in\mathbb{C}}$ is an iteration group as well. Two iteration groups $(F_t)_{t\in\mathbb{C}}$ and $(G_t)_{t\in\mathbb{C}}$ are called *conjugate* if there is some $S \in \Gamma$ so that $G_t = S^{-1} \circ F_t \circ S$ for all $t \in \mathbb{C}$.

4.1.4 The Main Problems

Motivated by the question of embeddability the problem arises to find the structure and the explicit form of iteration groups in detail, not necessarily as a part of the embedding problem. In the sequel we will study the following topics:

1. Construction of all iteration groups in Γ.
2. Find the detailed structure and explicit form of the coefficient functions $c_n: \mathbb{C} \to \mathbb{C}$ $(n \geq 1)$ of the solutions $F_t(x) = \sum_{n \geq 1} c_n(t)x^n$, $t \in \mathbb{C}$, of (T).
3. Describe the structure of all iteration groups and their normal forms with respect to conjugation.

The construction of iteration groups is strongly connected with the *maximal abelian subgroups of* $(\Gamma, \circ)$ (cf. [26]).

In the present paper we apply the method of formal functional equations which differs in many aspects from the approach by Jabłoński and Reich [17, 18]. This approach combines a detailed investigation of the systems (FE,I) and (FE, (II, k)) (see Section 4.2) for the coefficient functions of iteration groups with the a priori construction of the so-called *analytic* iteration groups, which have by definition entire coefficient functions, and with the application of certain polynomial relations associated with the coefficient functions. In our paper, however, we do not use any knowledge in analytic iteration groups.

We hardly ever present complete proofs, in some places we indicate some sketch of the proof. For details the reader is referred to the publications [4] in connection with iteration groups of type I and [5] for iteration groups of type (II, k).

We finish the introduction by giving an outline of the results and adding several comments. In Section 4.2 we describe the basic distinction between iteration groups of type I and iteration groups of type (II, k), $k \geq 2$. After studying the infinite systems of functional equations characterizing the coefficient functions of iteration groups, namely (FE,I) for iteration groups of type I and (FE, (II, k)) for iteration groups of type (II, k) (see Lemmas 4.1 and 4.2), we reduce the construction to the investigation of the so-called formal iteration groups of type I and formal iteration groups of type (II, k) (Theorem 4.1). These objects are elements in $(\mathbb{C}[y])[\![x]\!]$ which are solutions of certain relations in $(\mathbb{C}[y,z])[\![x]\!]$, namely the formal translation equations (Tform, I) and (Tform, (II, k)), together with appropriate boundary conditions. The basic idea of this reduction is the possibility to replace in the case of iteration groups of type I, say $F_t(x) = c_1(t)x + \ldots$, $t \in \mathbb{C}$, the generalized exponential function $c_1 \neq 1$ by an indeterminate y and similarly in the case of iteration groups of type (II, k), $F_t(x) = x + c_k(t)x^k + \ldots$, $t \in \mathbb{C}$, the additive function $c_k \neq 0$ by an indeterminate y, in the systems (FE,I) and (FE, (II, k)), respectively. Furthermore, we deduce from (Tform, I) and (Tform, (II, k)) by formal differentiation two formal differential equations, namely (Dform, I), (PDform, I) and by combining these two (AJform, I) for formal iteration groups of type I, and (Dform, (II, k)), (PDform, (II, k)), and (AJform, (II, k)) for formal iteration groups of type (II, k). The partial differential equations (PDform, I) and (PDform, (II, k)) may be considered as the simplest since they do not require a substitution of the unknown series $G(y,x)$. The Aczél–Jabotinsky equations (AJform, I) and (AJform, (II, k)) are weaker than the other differential equations just mentioned, and we add a remark (Theorem 4.2) how these Aczél–Jabotinsky differential equations can be used to construct and describe maximal abelian subgroups of Γ. All these differential equations contain the generator $H(x)$ where $H(x) = \frac{\partial}{\partial y} G(y,x)|_{y=1} = x + h_2x^2 + \ldots$ for formal iteration groups of type I and $H(x) = \frac{\partial}{\partial y} G(y,x)|_{y=1} = x^k + h_{k+1}x^{k+1} + \ldots$, $k \geq 2$, for formal iteration groups of type (II, k). The coefficients h_ν of the generators play an important role as natural parameters in the representations we are going to obtain in the following sections. In Section 4.2.5 we draw attention to the reordering of a formal iteration group $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ as $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$ (for type I) or $G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n$ (for type (II, k)) which allows in several situations a simpler and more elegant integration of the differential system.

In Section 4.3 we present the main results about the explicit form of formal iteration groups. Theorems 4.3 and 4.4 give the form of the coefficient functions P_n as derived from (PDform, I) for formal iteration groups $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n$ of type I. The coefficient functions $P_n(y)$ are not only polynomials in y, but also universal polynomials in y and the coefficients $h_2, \ldots, h_n$ of the generator H, where $H(x) = x + \ldots$ can be chosen arbitrarily. We obtain rather explicit formulas for the P_n, including recursive relations describing the dependence on the parameters $(h_n)_{n\geq 2}$, as well as estimates of the degree of P_n. Using the reordering $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$ of the formal iteration group of type I in (PDform, I) leads to Briot–Bouquet differential equations for the coefficients ϕ_n. The result is Theorem 4.5 which gives the unique representation $G(y,x) = S^{-1}(yS(x))$ with $S \in \Gamma_1$, sometimes called standard form. This means that each formal iteration group of type I is conjugate to yx which has generator x.

Theorems 4.6 and 4.7 show another representation of the coefficient functions $P_n(y)$ of formal iteration groups of type I, as deduced from (Dform, I). Theorem 4.8 contains one more description of $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$, a formal iteration group of type I, which follows from (Dform, I), where $\phi_n(x)$ is expressed as $\varphi_n(\phi_1(x))$, $n \geq 1$, and a recurrence for $(\varphi_n)_{n\geq 1}$ without differentiation is deduced.

Theorem 4.9 refers to the solutions of (AJform, I). Here again Briot–Bouquet differential equations may be applied. The condition $G(y,x) = yx + \ldots$ leads exactly to the solutions of (Tform, I) (see Theorems 4.9 and 4.10). Theorem 4.11 is also based on (AJform, I), reordering of $G(y,x)$, and using Briot–Bouquet differential equations. It gives again the standard form and the recurrence of Theorem 4.8.

In Section 4.3.4 we sketch two further approaches to obtain the standard form, here directly without formal functional equations. In connection with the first approach we discuss the important connection (4.1) of the generators of two conjugate formal iteration groups of type I. We formulate this connection as a differential equation for the conjugating series $S \in \Gamma$, involving the generators H and $\tilde{H}$ of the conjugate formal iteration groups. Formula (4.1), also valid for formal iteration groups of type (II, k), will also appear later in the paper. The second approach to the standard form is a calculation in the field $\mathbb{C}\langle\langle x\rangle\rangle$ of formal Laurent series with finite principal part.

The results for formal iteration groups of type (II, k) follow in the next section. The situation is not only much more complicated from a technical point of view, but also offers new "aspects." Theorems 4.12 and 4.13 refer to explicit formulas for the coefficient functions of the formal iteration groups of type (II, k), $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$, derived from (PDform, (II, k)) or (Dform, (II, k)). Here P_n is a universal polynomial in y and the coefficients $h_{k+1}, \dots, h_{n-k+1}, h_n$ of the generator $H(x) = x^k + h_{k+1}x^{k+1} + \dots$. As a matter of fact, P_n does not depend on $h_{n-k+2}, \dots, h_{n-1}$. Estimates of the degree of the P_n are given.

A similar result follows from (AJform, (II, k)) (see Theorems 4.14 and 4.15).

Writing $G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n$ and substituting it into (PDform, (II, k)) we find a simple recurrence formula (PDR$_n$, (II, k)) for ϕ_n, this time with differentiation. Its solution under the boundary condition (BR, (II, k)) is contained in Theorems 4.16–4.18. The explicit formula for ϕ_{n+1} in Theorem 4.17 has as parameters certain coefficients h_ν of the generator H and certain coefficients of the P_n which are the coefficients of $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$, whereas the explicit formula for ϕ_n in Theorem 4.18 has as parameters the coefficients of the generator only.

Formal iteration groups of type I and those of type (II, k) have very different properties with respect to conjugation. We start Section 4.3.6 by claiming that each formal iteration group of type (II, k) is conjugate to a formal iteration group of type (II, k) with generator $\tilde{H}(x) = x^k + hx^{2k-1}$, a so-called normal form. This is unique if we restrict the conjugating series S to be an element of Γ_1. To see this, we have to solve (4.1) for $S \in \Gamma_1$. Theorem 4.19 describes in detail, using (PDform, (II, k)) and (B, (II, k)), the explicit form of formal iteration groups of type (II, k) with generators $x^k + hx^{2k-1}$. These normal forms have the simplified structure $G(y,x) = \sum_{n\geq 0} P_{n(k-1)+1}(y)x^{n(k-1)+1}$ which is, however, much more complicated than the standard form $S^{-1}(yS(x))$ of formal iteration groups of type I.

It follows that the normal form $G(y,x)$ determined by the generator $x^k + hx^{2k-1}$ has an expansion $G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r$ as a power series in h with coefficients $G_r(y,x) \in \mathbb{C}[\![y,x]\!]$, since h can be considered as a new indeterminate. The series $G_r(y,x)$ are determined from the recursive system (4.4) and (4.5). Their form is presented in Theorem 4.20. The differential equation (Dform, (II, k)) leads to a more compact description of $G_r(y,x)$, given in Theorem 4.21, involving a series of binomial type and a polynomial in $\ln(1 - (k-1)yx^{k-1})$. The series $G_0(y,x) = x(1 - (k-1)yx^{k-1})^{-1/(k-1)}$ plays a role in the theory of reversible power series (cf. [12]). Eventually Theorem 4.22 builds a bridge to Lie–Gröbner series.

We finish the paper by collecting some open problems. The most interesting one is the construction of iteration groups in higher dimensions by means of formal functional equations. So far only partial results are known.

4.2 First Classification of Iteration Groups

Let $(F_t)_{t\in\mathbb{C}}$ be an iteration group, $F_t(x) = \sum_{n\geq 1} c_n(t)x^n$, $t \in \mathbb{C}$. We consider three different types of iteration groups:

1. $F_t(x) = x$ for all $t \in \mathbb{C}$ is the *trivial iteration group*.
2. If $c_1 \neq 1$, then

$$c_1(s+t) = c_1(s)c_1(t), \qquad s,t \in \mathbb{C},$$

   thus c_1 is a non-trivial generalized exponential function. We call $(F_t)_{t\in\mathbb{C}}$ an iteration group of *type* I.
3. If $c_1 = 1$, then there exists some $k \geq 2$, so that $c_2 = \cdots = c_{k-1} = 0$, $c_k \neq 0$, and

$$c_k(s+t) = c_k(s) + c_k(t), \qquad s,t \in \mathbb{C},$$

   thus c_k is a non-trivial additive function. We say that $(F_t)_{t\in\mathbb{C}}$ is an iteration group of *type* (II, k).

This classification is compatible with the conjugation of iteration groups, i.e., if $(F_t)_{t\in\mathbb{C}}$ and $(G_t)_{t\in\mathbb{C}}$ are conjugate, then they have the same type.

4.2.1 Systems of Functional Equations for the Coefficient Functions

Consider a family $(F_t)_{t\in\mathbb{C}}$, $F_t(x) = \sum_{n\geq 1} c_n(t)x^n$, $t \in \mathbb{C}$, where $c_1 \neq 1$. Then $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type I, if and only if the system

$$\begin{aligned} c_1(s+t) &= c_1(s)c_1(t) \\ c_2(s+t) &= c_1(s)c_2(t) + c_2(s)c_1(t)^2 \qquad \text{(FE,I)} \\ c_n(s+t) &= c_1(s)c_n(t) + c_n(s)c_1(t)^n + \tilde{P}_n\big(c_2(s),\dots,c_{n-1}(s),c_2(t),\dots,c_{n-1}(t)\big), \\ & \qquad n \geq 2 \end{aligned}$$

is satisfied for all $s,t \in \mathbb{C}$. The $\tilde{P}_n$ are universal polynomials which are linear in $c_2(s),\dots,c_{n-1}(s)$.

Lemma 4.1 ([4, Lemma 2]) *If $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type* I *of the form $F_t(x) = \sum_{n\geq 1} c_n(t)x^n$, $t \in \mathbb{C}$, then c_1 is a non-trivial generalized exponential function and there exists a sequence of polynomials $(P_n)_{n\geq 2}$ so that*

$$c_n(s) = P_n(c_1(s)) \quad \forall s \in \mathbb{C}, \text{ and } P_n(0) = 0, \qquad n \geq 2.$$

Since $c_1 \neq 1$, for $n \geq 2$ there exists some $t_n \in \mathbb{C}$ so that $c_1(t_n)^n - c_1(t_n) \neq 0$. From $c_2(s+t) = c_2(t+s)$, for all $s, t \in \mathbb{C}$, we obtain

$$c_2(s) = \frac{c_2(t_2)(c_1(s)^2 - c_1(s))}{c_1(t_2) - c_1(t_2)^2} = P_2(c_1(s)), \qquad s \in \mathbb{C}.$$

Using induction on n and $c_n(s+t) = c_n(t+s)$, $\forall s, t \in \mathbb{C}$, we obtain the assertion from (FE,I).

Hence we obtain from (FE,I)

$$\begin{aligned} P_n(c_1(s)c_1(t)) &= P_n(c_1(s+t)) = c_n(s+t) \\ &= c_1(s)P_n(c_1(t)) + P_n(c_1(s))c_1(t)^n \qquad (\hat{\mathrm{P}}, \mathrm{I}) \\ &\quad + \tilde{P}_n\big(P_2(c_1(s)), \ldots, P_{n-1}(c_1(s)), P_2(c_1(t)), \ldots, P_{n-1}(c_1(t))\big) \end{aligned}$$

for all $s, t \in \mathbb{C}$ and all $n \geq 2$.

Consider a family $(F_t)_{t\in\mathbb{C}}$, $F_t(x) = x + \sum_{n\geq k} c_n(t)x^n$, $t \in \mathbb{C}$, where $c_k \neq 0$. Then $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type (II, k), if and only if the system

$$\begin{aligned} c_n(s+t) &= c_n(s) + c_n(s), \qquad k \leq n \leq 2k-2, \\ c_{2k-1}(s+t) &= c_{2k-1}(s) + c_{2k-1}(t) + kc_k(s)c_k(t) \\ c_{2k}(s+t) &= c_{2k}(s) + c_{2k}(t) + kc_k(s)c_{k+1}(t) + (k+1)c_{k+1}(s)c_k(t) \\ c_n(s+t) &= c_n(s) + c_n(t) + kc_k(s)c_{n-(k-1)}(t) \qquad (\mathrm{FE}, (\mathrm{II}, k)) \\ &\quad + \big(n-(k-1)\big)c_{n-(k-1)}(s)c_k(t) \\ &\quad + \tilde{P}_n\big(c_k(s), \ldots, c_{n-k}(s), c_k(t), \ldots, c_{n-k}(t)\big), \qquad n > 2k, \end{aligned}$$

for all $s, t \in \mathbb{C}$, where $\tilde{P}_n$ are universal polynomials which are linear in $c_k(s), \ldots, c_{n-k}(s)$.

Lemma 4.2 ([5, Lemma 1]) *Consider some integer $k \geq 2$. If $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type (II, k), $F_t(x) = x + \sum_{n\geq k} c_n(t)x^n$, $t \in \mathbb{C}$, then c_k is a non-trivial additive function and there exists a sequence of polynomials $(P_n)_{n\geq k}$ so that*

$$c_n(s) = P_n(c_k(s)), \qquad s \in \mathbb{C},\ n \geq k.$$

The reader should remember that these polynomials P_n differ from the polynomials P_n of Lemma 4.1. Since $c_k \neq 0$ there exists some $t_0 \in \mathbb{C}$ so that $c_k(t_0) \neq 0$. Using (FE, (II, k)) for $n = 2k$ we obtain from $c_{2k}(s+t) = c_{2k}(t+s)$,

$$c_{k+1}(s) = \frac{c_{k+1}(t_0)}{c_k(t_0)}\, c_k(s) = P_{k+1}(c_k(s)), \qquad s \in \mathbb{C}.$$

By induction on n and $c_{n+k-1}(s+t) = c_{n+k-1}(t+s)$, for all $s, t \in \mathbb{C}$, we obtain the assertion from (FE, (II, k)).

Hence we obtain from (FE, (II, k))

$$\begin{aligned} P_n(c_k(s) + c_k(t)) &= P_n(c_k(s+t)) = c_n(s+t) \\ &= P_n(c_k(s)) + P_n(c_k(t)) + kc_k(s)P_{n-(k-1)}(c_k(t)) \qquad (\hat{\mathrm{P}}, (\mathrm{II}, k)) \\ &\quad + \big(n-(k-1)\big)P_{n-(k-1)}(c_k(s))c_k(t) \\ &\quad + \tilde{P}_n\big(c_k(s), \ldots, P_{n-k}(c_k(s)), c_k(t), \ldots, P_{n-k}(c_k(t))\big), \end{aligned}$$

for all $s, t \in \mathbb{C}$ and $n \geq k$, where $P_j = 0$ for $j < k$ and $\tilde{P}_j = 0$ for $j \leq 2k$.

4.2.2 Formal Functional Equations

Formal functional equations in connection with the translation equation were studied by Gronau [10, 11], and the present authors [4, 5]. Similar methods were also applied for the study of cocycle equations which occur in connection with covariant embeddings of the linear functional equation (cf. [3, 6, 7]). Assume that $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type I, $F_t(X) = \sum_{n\geq 1} c_n(t)x^n$, $t \in \mathbb{C}$, where $c_1(s+t) = c_1(s)c_1(t)$, $s, t \in \mathbb{C}$, $c_1 \neq 1$ and $c_1 \neq 0$. Since the image of c_1 contains infinitely many elements we can prove for any polynomial $Q(x, y) \in \mathbb{C}[x, y]$ that $Q(c_1(s), c_1(t)) = 0$ for all $s, t \in \mathbb{C}$ implies $Q = 0$. From ($\hat{\mathrm{P}}$, I) we obtain by replacing $c_1(s)$ and $c_1(t)$ by independent variables y, z, that

$$P_n(yz) = yP_n(z) + P_n(y)z^n + \tilde{P}_n\big(P_2(y), \ldots, P_{n-1}(y), P_2(z), \ldots, P_{n-1}(z)\big) \qquad (\mathrm{P}, \mathrm{I})$$

in $\mathbb{C}[y, z]$ for $n \geq 2$. Writing $G(y, x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$ we deduce from (P, I) that G satisfies the formal translation equation of type I

$$G(yz, x) = G(y, G(z, x)) \qquad (\mathrm{Tform}, \mathrm{I})$$

in $(\mathbb{C}[y, z])[\![x]\!]$. We call $G(y, x)$ a *formal iteration group of type* I. It also satisfies the condition

$$G(1, x) = x. \qquad (\mathrm{B}, \mathrm{I})$$

Assume that $(F_t)_{t\in\mathbb{C}}$ is an iteration group of type (II, k) for some $k \geq 2$, $F_t(x) = x + \sum_{n\geq k} c_n(t)x^n$, $t \in \mathbb{C}$, where $c_k(s+t) = c_k(s) + c_k(t)$, $s, t \in \mathbb{C}$, $c_k \neq 0$. Since the image of c_k contains infinitely many elements we can prove for any polynomial $Q(x, y) \in \mathbb{C}[x, y]$ that $Q(c_k(s), c_k(t)) = 0$ for all $s, t \in \mathbb{C}$ implies $Q = 0$.

From ($\hat{\mathrm{P}}$, (II, k)) we obtain by replacing $c_k(s)$ and $c_k(t)$ by independent variables y, z, that

$$P_n(y+z) = P_n(y) + P_n(z) + kyP_{n-(k-1)}(z) + \big(n-(k-1)\big)P_{n-(k-1)}(y)z$$
$$+ \tilde{P}_n\big(y, \dots, P_{n-k}(y), z, \dots, P_{n-k}(z)\big) \qquad \text{(P, (II, } k\text{))}$$

for all $n \geq k$.

Writing $G(y,x) = x + yx^k + \sum_{n\geq k+1} P_n(y)x^n \in (\mathbb{C}[y])[\![y]\!]$ we deduce from (P, (II, k)) that G satisfies the formal translation equation of type (II, k)

$$G(y+z,x) = G(y, G(z,x)) \qquad \text{(Tform, (II, } k\text{))}$$

in $(\mathbb{C}[y,z])[\![x]\!]$. We call $G(y,x)$ a *formal iteration group of type* (II, k). It also satisfies the condition

$$G(0,x) = x. \qquad \text{(B, (II, } k\text{))}$$

Conversely, from each formal iteration group we can construct iteration groups in the following way (cf. [4, Theorem 3] and [5, Theorem 3]):

Theorem 4.1 1. *If $G(y,x)$ is a formal iteration group of type* I, *c_1 a generalized exponential function, $c_1 \neq 1$, then $(G(c_1(t),x))_{t\in\mathbb{C}}$ is an iteration group of type* I.
2. *If $G(y,x)$ is a formal iteration group of type* (II, k), *$k \geq 2$, c_k an additive function, $c_k \neq 0$, then $(G(c_k(t),x))_{t\in\mathbb{C}}$ is an iteration group of type* (II, k).

4.2.3 Differential Equations Obtained from the Translation Equation

Let $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ be a formal iteration group of type I. Then the infinitesimal generator of G is defined as

$$H(x) = \frac{\partial}{\partial y} G(y,x)\Big|_{y=1}.$$

It is of the form $H(x) = x + \sum_{n\geq 2} h_n x^n$. Differentiation of (Tform, I) with respect to y yields

$$z\frac{\partial}{\partial t} G(t,x)|_{t=yz} = \frac{\partial}{\partial y} G\big(y, G(z,x)\big).$$

For $y = 1$ we get

$$z\frac{\partial}{\partial z} G(z,x) = H(G(z,x)). \qquad \text{(Dform, I)}$$

Differentiation of (Tform, I) with respect to z and application of the mixed chain rule yields

$$y\frac{\partial}{\partial t}G(t,x)|_{t=yz}=\frac{\partial}{\partial t}G(y,t)|_{t=G(z,x)}\frac{\partial}{\partial z}G(z,x).$$

For $z=1$ we get

$$y\frac{\partial}{\partial y}G(y,x)=H(x)\frac{\partial}{\partial x}G(y,x). \qquad \text{(PDform, I)}$$

The advantage of this equation lies in the circumstance that no substitution of the unknown series $G(y,x)$ is needed and that (PDform, I) is a linear equation.

Combining (Dform, I) and (PDform, I), we obtain an Aczél–Jabotinsky differential equation of the form

$$H(x)\frac{\partial}{\partial x}G(y,x)=H(G(y,x)). \qquad \text{(AJform, I)}$$

In this equation the variable y is an internal parameter since it does not appear explicitly in (AJform, I).

Let $G(y,x)\in(\mathbb{C}[y])[\![x]\!]$ be a formal iteration group of type (II,k) for some $k\geq 2$. Then the infinitesimal generator of G is defined as

$$H(x)=\frac{\partial}{\partial y}G(y,x)\Big|_{y=0}.$$

It is of the form $H(x)=x^k+\sum_{n\geq k+1}h_nx^n$. Differentiation of (Tform, (II,k)) with respect to y yields

$$\frac{\partial}{\partial t}G(t,x)|_{t=y+z}=\frac{\partial}{\partial y}G\big(y,G(z,x)\big).$$

For $y=0$ we get

$$\frac{\partial}{\partial z}G(z,x)=H(G(z,x)). \qquad \text{(Dform, (II, }k\text{))}$$

Differentiation of (Tform, (II,k)) with respect to z and application of the mixed chain rule yields

$$\frac{\partial}{\partial t}G(t,x)|_{t=y+z}=\frac{\partial}{\partial t}G(y,t)|_{t=G(z,x)}\frac{\partial}{\partial z}G(z,x).$$

For $z=0$ we get

$$\frac{\partial}{\partial y}G(y,x)=H(x)\frac{\partial}{\partial x}G(y,x). \qquad \text{(PDform, (II, }k\text{))}$$

The advantage of this equation lies in the circumstance that no substitution of the unknown series $G(y,x)$ is needed and that (PDform, (II, k)) is a linear equation.

Combining (Dform, (II, k)) and (PDform, (II, k)), we obtain an Aczél–Jabotinsky differential equation of the form

$$H(x)\frac{\partial}{\partial x}G(y,x) = H(G(y,x)). \qquad \text{(AJform, (II, } k))$$

In this equation the variable y is an internal parameter since it does not appear explicitly in (AJform, (II, k)).

4.2.4 The Relevance of Aczél–Jabotinsky Differential Equations

The Aczél–Jabotinsky differential equations can be used to characterize maximal abelian subgroups of Γ (cf. [26]). The main result reads as follows:

Theorem 4.2 *A set* $\mathscr{F} \subset \Gamma$ *is a maximal abelian subgroup of* Γ *if and only if there exists some* $H \in \mathbb{C}[\![x]\!]$, $H \neq 0$, $\operatorname{ord}(H) \geq 1$, *so that*

$$\phi \in \mathscr{F} \iff H(x)\phi'(x) = H(\phi(x)).$$

It can be shown that either $\mathscr{F}$ is isomorphic to $\mathbb{C}^*$, or $\mathscr{F}$ is isomorphic to

$$\left\{ \begin{pmatrix} \rho & t \\ 0 & \rho \end{pmatrix} \;\middle|\; \rho^m = 1,\ t \in \mathbb{C} \right\},$$

where m is uniquely determined by $\mathscr{F}$.

4.2.5 Reordering the Summands

Let $G(y,x) = \sum_{n\geq 1} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!] \subset \mathbb{C}[\![y,x]\!]$ be a formal iteration group of type I, then it is possible to write $G(y,x)$ in the form

$$G(y,x) = \sum_{n\geq 1} \phi_n(x) y^n \in (\mathbb{C}[\![x]\!])[\![y]\!] \qquad \text{(R, I)}$$

where $\phi_1 \in \Gamma_1$, and $(\phi_n(x))_{n\geq 1}$ is a summable family in $\mathbb{C}[\![x]\!]$. Therefore the boundary condition (B, I)

$$G(1,x) = \sum_{n\geq 1} \phi_n(x) = x \qquad \text{(BR, I)}$$

makes sense. It is possible to use this representation of G in the differential equations (Dform, I), (PDform, I), and (AJform, I).

Let

$$G(y,x) = x + yx^k + \sum_{n\geq k+1} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!] \subset \mathbb{C}[\![y,x]\!],$$

$k \geq 2$, be a formal iteration group of type (II, k), then it is possible to write $G(y,x)$ in the form

$$G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n \in (\mathbb{C}[\![\, x \,]\!])[\![y]\!] \qquad (\mathrm{R}, (\mathrm{II}, k))$$

where $(\phi_n(x))_{n\geq 0}$ is a summable family in $\mathbb{C}[\![\, x \,]\!]$. The boundary condition (B, (II, k)) reads as

$$G(0,x) = \phi_0(x) = x. \qquad (\mathrm{BR}, (\mathrm{II}, k))$$

It is possible to use this representation of G in the differential equations (Dform, (II, k)), (PDform, (II, k)), and (AJform, (II, k)).

4.3 Solving the Translation Equation by a Purely Algebraic Differentiation Process

Here we present the construction of formal iteration groups by solving the differential equations (Dform, I), (PDform, I), or (AJform, I) for formal iteration groups of type I and (Dform, (II, k)), (PDform, (II, k)), or (AJform, (II, k)) for formal iteration groups of type (II, k) under the appropriate boundary conditions.

4.3.1 Formal Iteration Groups of Type I *Obtained from* (PDform, I) *and* (B, I)

Using the partial differential equation (PDform, I) we describe how the polynomials P_n, $n \geq 2$, depend on the coefficients h_j, $j \geq 2$, of the infinitesimal generator H of the formal iteration group G of type I. We determine all solutions of (PDform, I) and (B, I) and we show that each of them is a solution of (Tform, I).

Theorem 4.3 ([4, Theorem 4]) *For each generator* $H(x) = x + h_2x^2 + \dots$ *the partial differential equation* (PDform, I) *together with* (B, I) *has exactly one solution. It is given by*

$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

The polynomials P_n, $n \geq 2$, are of formal degree n (that is an upper bound for the degree), they satisfy $P_n(0) = 0$, and they are of the form

$$P_n(y) = \frac{h_n}{n-1}(y^n - y) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2, \ldots, h_{n-1})}{n-j}(y^n - y^j)$$

where the polynomials $\Phi_j^{(n)}$, $1 \leq j \leq n-1$, are (recursively) determined by

$$\sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2, \ldots, h_{n-1})y^j.$$

Theorem 4.4 ([4, Theorem 5]) *For each generator $H(x) = x + h_2x^2 + \ldots$ the solution $G(y,x)$ of* (PDform, I) *and* (B, I) *is a solution of the formal translation equation* (Tform, I).

Let G be the solution of (PDform, I) and (B, I) for the generator H. In order to prove this theorem we show that both series

$$U(y,z,x) := G(yz,x)$$
$$V(y,z,x) := G(z, G(y,x))$$

satisfy the system

$$y\frac{\partial}{\partial y} f(y,z,x) = H(x)\frac{\partial}{\partial x} f(y,z,x)$$
$$f(1,z,x) = G(z,x)$$

which has a unique solution in $(\mathbb{C}[y,z])[\![x]\!]$.

Let G be the solution of (PDform, I) and (B, I) for the generator H. Reordering the summands of G we write $G(y,x)$ as $\sum_{n\geq 1} \phi_n(x)y^n$. Then from (PDform, I) and (B, I) we obtain

$$\sum_{n\geq 1} n\phi_n(x)y^n = H(x)\sum_{n\geq 1} \phi_n'(x)y^n \qquad \text{(PDR, I)}$$

and (BR, I). Equation (PDR, I) is equivalent to

$$n\phi_n(x) = H(x)\phi_n'(x) \qquad (\text{PDR}_n, \text{I})$$

for all $n \geq 1$. Each of these equations is equivalent to a Briot–Bouquet differential equation (in the non-generic case), thus it has solutions.

A *Briot–Bouquet differential equation* (cf. [20, Section 5.2], [14, Section 11.1], [15, Section 12.6]) is a complex differential equation

$$zw'(z) = az + bw(z) + \sum_{\alpha+\beta\geq 2} a_{\alpha,\beta} z^{\alpha} [w(z)]^{\beta},$$

where $w(z)$ is a power series in z with $w(0) = 0$, and the power series on the right-hand side is given. Cauchy's theorem on existence and uniqueness cannot be applied directly. In the case $b = n$, a positive integer, a formal solution $w(z)$ exists if, and only if, a certain polynomial $P(a, b, a_{\alpha,\beta} : \alpha + \beta \leq n)$ vanishes. If so, then the equation is called *solvable* or *non-generic of type n*, and all solutions take the shape

$$w_t(z) = c_1 z + \ldots + c_{n-1} z^{n-1} + tz^n + \sum_{\nu\geq n+1} Q_\nu(t) z^\nu, \qquad t \in \mathbb{C},$$

for polynomials $Q_\nu(t)$. The coefficients c_i, $1 \leq i \leq n-1$, are uniquely determined. The series $w_t(z)$ is convergent if the given right-hand side is convergent.

Let

$$H(x) = x(1 + \sum_{n\geq 1} h_n^* x^n) = xH^*(x),$$

then $h_n^* = h_{n+1}$, $n \geq 1$, and (PDR_n, I) is equivalent to

$$n\phi_n(x) = xH^*(x)\phi_n'(x)$$

or

$$x\phi_n'(x) = n\phi_n(x)[1 + h_1^* x + \ldots]^{-1}.$$

Finally for each $n \geq 1$ we end up with the system

$$x\phi_n'(x) = n\phi_n(x) + n \sum_{\alpha+\beta\geq 2} d_{\alpha,\beta} x^\alpha [\phi_n(x)]^\beta$$
$$\phi_n(0) = 0.$$

The set of solutions of (PDR_n, I) is then given by $\{\varphi_n^{(n)}[\phi_{1,0}(x)]^n \mid \varphi_n^{(n)} \in \mathbb{C}\}$, where $\phi_{1,0}(x)$ is the unique solution of (PDR_1, I) which belongs to Γ_1, i.e., which is of the form $\phi_{1,0}(x) = x + \ldots$. Denote this series by $S(x) = \phi_{1,0}(x)$ and let $\sum_{n\geq 1} \phi_n(x)$ be a solution of (PDR, I). From the boundary condition (BR, I) we obtain

$$x = \sum_{n\geq 1} \phi_n(x) = \sum_{n\geq 1} \varphi_n^{(n)} [S(x)]^n,$$

whence,

$$S^{-1}(x) = \sum_{n\geq 1} \varphi_n^{(n)} x^n,$$

from which it is possible to determine the values $\varphi_n^{(n)}$, $n \geq 1$.

The main result of this section is

Theorem 4.5 ([4, Theorem 7]) *If $G(y,x) = \sum_{n\geq 1} \phi_n(x) y^n$ is a solution of (Tform, I) and (B, I), then there exists exactly one $S \in \Gamma_1$ so that*

$$G(y,x) = S^{-1}\big(yS(x)\big).$$

Using the representation (R, I) we have $\phi_n(x) = \varphi_n^{(n)}[S(x)]^n$, where $\varphi_n^{(n)} \in \mathbb{C}$, $n \geq 1$.

Conversely, for every $S \in \Gamma_1$ the series

$$G(y,x) = S^{-1}\big(yS(x)\big)$$

is a solution of (Tform, I) and (B, I).

4.3.2 Formal Iteration Groups of Type I Obtained from (Dform, I) and (B, I)

For the differential equation (Dform, I) we obtain similar results as in the previous section (see also [4, Theorems 9, 10, 11]).

Theorem 4.6 *For each generator*

$$H(x) = x + h_2 x^2 + \dots$$

the differential equation (Dform, I) together with (B, I) has exactly one solution. It is given by

$$G(z,x) = zx + \sum_{n\geq 2} P_n(z) x^n \in (\mathbb{C}[z])[\![x]\!].$$

The polynomials P_n, $n \geq 2$, are of formal degree n, they satisfy $P_n(0) = 0$, and they are of the form

$$P_n(z) = \frac{h_n}{n-1}(z^n - z) + \sum_{j=2}^{n} \frac{\Psi_j^{(n)}(h_2, \dots, h_{n-1})}{j-1}(z^j - z)$$

where the polynomials $\Psi_j^{(n)}$, $2 \le j \le n$, *are (recursively) determined by*

$$\sum_{\nu=2}^{n-1} h_\nu \sum_{\substack{r_1+\ldots+r_\nu=n \\ r_j\ge 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(z)\right) = \sum_{j=2}^{n} \Psi_j^{(n)}(h_2,\ldots,h_{n-1}) z^j.$$

Theorem 4.7 *For each generator*

$$H(x) = x + h_2x^2 + \ldots$$

the solution $G(z,x)$ *of* (Dform, I) *and* (B, I) *is a solution of the formal translation equation* (Tform, I).

Using the representation (R, I) we obtain from (Dform, I)

$$\sum_{n\ge 1} n\phi_n(x)z^n = \sum_{\nu\ge 1} h_\nu \left[\sum_{n\ge 1}\phi_n(x)z^n\right]^\nu \qquad \text{(DR, I)}$$

which is equivalent to

$$n\phi_n(x) = \sum_{\nu=1}^{n} h_\nu \sum_{\substack{r_1+\ldots+r_\nu=n \\ r_j\ge 1}} \left(\prod_{j=1}^{\nu}\phi_{r_j}(x)\right) \qquad (\mathrm{DR}_n, \mathrm{I})$$

for all $n \ge 1$. This is a recursive formula for the ϕ_n without any differentiation process. The solutions of (DR, I) are given in

Theorem 4.8 *Consider* $H(x) = x + h_2x^2 + \ldots$.

1. *Every* $\phi_1(x) \in \mathbb{C}[\![x]\!]$ *satisfies* $(\mathrm{DR}_1, \mathrm{I})$.
2. *Let* $\phi_1 \in \mathbb{C}[\![x]\!] \setminus \{0\}$. *For each* $n \ge 2$ *there exists exactly one solution* ϕ_n *of* $(\mathrm{DR}_n, \mathrm{I})$, *depending on* ϕ_1. *It is given by* $\phi_n(x) := \varphi_n[\phi_1(x)]^n$, *where* $\varphi_1 = 1$ *and*

$$\varphi_n = \frac{1}{n-1}\sum_{\nu=2}^{n} h_\nu \sum_{r_1+\ldots+r_\nu=n} \prod_{j=1}^{\nu} \varphi_{r_j}, \qquad n \ge 2.$$

   Consequently, φ_n *does not depend on the choice of* ϕ_1.
3. *The system* (DR, I) *and* (BR, I) *has a unique solution. It is given by*

$$\sum_{n\ge 1} \varphi_n[\phi_1(x)]^n z^n$$

for $\varphi_1 = 1$, φ_n *for* $n \geq 2$ *given as above, and*

$$\phi_1(x) = (x + \sum_{n\geq 2} \varphi_n x^n)^{-1},$$

which is an element of Γ_1.

4.3.3 Formal Iteration Groups of Type I Obtained from (AJform, I) and (B, I)

Here we present some facts from [4, Section 2.3]. Writing the series H as $x(1 + h_2x + \ldots)$ and $\phi(x) := G(y, x)$ motivates that (AJform, I) is equivalent to

$$x\phi'(x) = [1 + h_2x + \ldots]^{-1} H(\phi(x))$$

or

$$x\phi'(x) = \phi(x) + \sum_{\substack{\alpha+\beta\geq 2\\ \beta\geq 1}} d_{\alpha,\beta}(h) x^\alpha [\phi(x)]^\beta$$

which is a Briot–Bouquet differential equation. It is well known that for each $\tilde{P}_1(y) \in \mathbb{C}[y]$ there exists exactly one solution

$$\tilde{G}(y, x) = \tilde{P}_1(y)x + \sum_{n\geq 2} \tilde{P}_n(y)x^n$$

of this Briot–Bouquet equation with coefficients $\tilde{P}_n(y)$ which are polynomials, $n \geq 2$.

The solutions of (AJform, I) with $\tilde{P}_1(y) = y$ are determined in the next theorem.

Theorem 4.9 *1. For each generator* $H(x) = x+h_2x^2+\ldots$ *the differential equation* (AJform, I) *has exactly one solution of the form*

$$G(y, x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

2. The polynomials P_n, $n \geq 2$, *(from the unique solution* $G(y, x) \equiv yx \bmod x^2$*) are of formal degree n, they satisfy* $P_n(0) = 0$, *and they are of the form*

$$P_n(y) = \frac{h_n}{n-1}(y^n - y) + \sum_{j=2}^{n} \frac{\Theta_j^{(n)}(h_2, \ldots, h_{n-1})}{n-1}(y^j - y)$$

where the polynomials $\Theta_j^{(n)}$, $2 \le j \le n$, *are (recursively) determined by*

$$\sum_{\nu=2}^{n-1} h_\nu \left(\sum_{\substack{r_1+\ldots+r_\nu=n \\ r_j\ge 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(y) \right) - (n-\nu+1)P_{n-\nu+1}(y) \right) = \sum_{j=2}^{n} \Theta_j^{(n)}(h_2,\ldots,h_{n-1})(y^j - y).$$

Applying the same method as in the proof of Theorem 4.4 we obtain

Theorem 4.10 *For each generator* $H(x) = x + h_2x^2 + \ldots$ *the solution* $G(y,x)$ *of the differential equation* (AJform, I) *with* $G(y,x) \equiv yx \bmod x^2$ *is a solution of* (Tform, I).

Using the representation (R, I) we obtain from (AJform, I)

$$H(x)\sum_{n\ge 1}\phi_n'(x)y^n = \sum_{\nu\ge 1} h_\nu \left[\sum_{n\ge 1}\phi_n(x)y^n \right]^\nu \qquad \text{(AJR, I)}$$

which is equivalent to

$$H(x)\phi_n'(x) = \phi_n(x) + \sum_{\nu=2}^{n} h_\nu \sum_{\substack{r_1+\ldots+r_\nu=n \\ r_j\ge 1}} \left(\prod_{j=1}^{\nu} \phi_{r_j}(x) \right) \qquad (\text{AJR}_n, \text{I})$$

for all $n \ge 1$. Again these equations are Briot–Bouquet differential equations since, for all $n \ge 1$,

$$x\phi_n'(x) = [1 + h_2x + \ldots]^{-1} \left(\phi_n(x) + \sum_{\nu=2}^{n} h_\nu \sum_{\substack{r_1+\ldots+r_\nu=n \\ r_j\ge 1}} \left(\prod_{j=1}^{\nu} \phi_{r_j}(x) \right) \right).$$

We are mainly interested in solutions where $\phi_1(x) = x + \ldots$ since they lead to iteration groups. The set of all solutions of (AJR, I) is described in

Theorem 4.11 *Consider* $H(x) = x + h_2x^2 + \ldots$.

1. *For every* $c \in \mathbb{C}$, *there is exactly one solution*

$$\phi_1(x) \equiv cx \bmod x^2$$

of (AJR_1, I).

2. *Assume that* $\phi_1 = cx + \dots$, $c \neq 0$, *is a solution of* $(\mathrm{AJR}_1, \mathrm{I})$. *Then for each* $n \geq 2$ *there exists exactly one solution* $\phi_n(x)$ *of* $(\mathrm{AJR}_n, \mathrm{I})$. *It is given by* $\phi_n(x) = \varphi_n[\phi_1(x)]^n$, *where* $\varphi_1 = 1$ *and*

$$\varphi_n = \sum_{\nu=2}^{n} \frac{h_\nu}{n-1} \sum_{r_1+\dots+r_\nu=n} \prod_{j=1}^{\nu} \varphi_{r_j}, \qquad n \geq 2.$$

Consequently, φ_n *does not depend on the choice of* ϕ_1.
3. *The unique solution* ϕ_1 *of system* $(\mathrm{AJR}_1, \mathrm{I})$ *which belongs to* Γ_1, *(i.e.,* $c = 1$*) leads to the solution*

$$\sum_{n\geq 1} \varphi_n[\phi_1(x)]^n y^n$$

of $(\mathrm{AJR}, \mathrm{I})$, *where* $\varphi_1 = 1$ *and* φ_n *for* $n \geq 2$ *given as above. Moreover* $\phi_1(x) = (x + \sum_{n\geq 2} \varphi_n x^n)^{-1}$.

Based on these results it is possible to give another simple proof of Theorem 4.5.

4.3.4 Normal Forms of Iteration Groups of Type I

From Theorem 4.5 we know that each formal iteration group $G(y, x)$ of type I is conjugate to yx. We call it the *normal form of formal iteration groups of type* I. Let $(F_t)_{t\in\mathbb{C}}$ be an iteration group of type I, $F_t(x) = \sum_{n\geq 1} c_n(t)x^n$ for all $t \in \mathbb{C}$. Then there exists some $S \in \Gamma_1$ so that $F_t(x) = S^{-1}(c_1(t)S(x))$, $t \in \mathbb{C}$.

We want to present two further methods for finding this normal form.

1. Consider the generator $H(x) = \frac{\partial}{\partial y} G(y, x)|_{y=1} = x + h_2x^2 + \dots$ of a formal iteration group of type I, and some $S \in \Gamma$. Then $\tilde{G}(y, x) = S^{-1}(G(y, S(x)))$ is a solution of (Tform, I). We calculate its generator

$$\tilde{H}(x) = \frac{\partial}{\partial y} S^{-1}(G(y, x))|_{y=1}$$

by an application of the chain rule:

$$\frac{\partial}{\partial y} S^{-1}(G(y, Sx)) = (S^{-1})'(G(y, Sx)) \frac{\partial}{\partial y} G(y, Sx).$$

Putting $y = 1$ we obtain $\tilde{H}(x) = (S^{-1})'(Sx)H(Sx)$. Since $(S^{-1})'(Sx)S'(x) = 1$ we get

$$\left(\frac{\partial}{\partial x} S(x)\right) \tilde{H}(x) = H(S(x)). \tag{4.1}$$

If we choose $\tilde{H}(x) = x$, then (4.1) yields the Briot–Bouquet differential equation

$$x\frac{\partial}{\partial x}S(x) = S(x) + h_2[S(x)]^2 + \dots \qquad (4.2)$$

(see [20, Section 5.2], [14, Section 11.1], [15, Section 12.6]). It is known that (4.2) has exactly one solution in $S \in \Gamma_1$. Using this S it follows that $\tilde{G}(y, x)$ has the generator $\tilde{H}(x) = x$, hence from (B, I) we get $yx = \tilde{G}(y, x) = S^{-1}(G(y, S(x)))$, or equivalently

$$G(y, x) = S(yS^{-1}(x)). \qquad (4.3)$$

2. Consider for some $H(x) = x + h_2x^2 + \dots \in \mathbb{C}[\![x]\!]$ the Aczél–Jabotinsky equation

$$H(x)\Phi'(x) = H(\Phi(x)), \text{ for } \Phi(x) = \rho x + \dots, \ \rho \neq 0. \qquad \text{(AJ)}$$

We compute the standard form of its set of solutions by computation in $\mathbb{C}\langle\langle x\rangle\rangle$, the ring of formal Laurent series with finite principal part. Again we write $H(x) = xH^*(x)$ and assume that $[H^*(x)]^{-1} = 1 + h_1^*x + h_2^*x^2 + \dots$. Then from (AJ) we get $xH^*(x)\Phi'(x) = \Phi(x)H^*(\Phi(x))$ thus

$$\frac{\Phi'(x)}{\Phi(x)}\left(1 + \sum_{n\geq 1} h_n^*[\Phi(x)]^n\right) = \frac{1}{x}\left(1 + \sum_{n\geq 1} h_n^* x^n\right)$$

and

$$\frac{\Phi'(x)}{\Phi(x)} - \frac{1}{x} = -\sum_{n\geq 1} h_n^*\Phi'(x)[\Phi(x)]^{n-1} + \sum_{n\geq 1} h_n^* x^{n-1}.$$

Using the differentiation operator this can be written as

$$\frac{\partial}{\partial x}\left(\ln\frac{\Phi(x)}{\rho x}\right) = -\frac{\partial}{\partial x}\left(\sum_{n\geq 1}\frac{h_n^*}{n}[\Phi(x)]^n\right) + \frac{\partial}{\partial x}\left(\sum_{n\geq 1}\frac{h_n^*}{n}x^n\right),$$

therefore

$$\ln\frac{\Phi(x)}{\rho x} = -T(\Phi(x)) + T(x) \text{ for } T(x) = \sum_{n\geq 1}\frac{h_n^*}{n}x^n.$$

Applying the exponential series we deduce

$$\frac{\Phi(x)}{\rho x} = \frac{\exp(T(x))}{\exp(T(\Phi(x)))}$$

or equivalently $\Phi(x)\exp(T(\Phi(x))) = \rho x \exp(T(x))$. The series S given by $S(x) = x\exp(T(x))$ is in Γ_1 and satisfies $S(\Phi(x)) = \rho S(x)$, whence $\Phi(x) = S^{-1}(\rho S(x))$. The coefficients of S are polynomials in the coefficients h_n.

4.3.5 Formal Iteration Groups of Type (II, k) *Obtained from the Three Differential Equations*

The solutions of (PDform, (II, k)) [or (Dform, (II, k))] together with (B, (II, k)) and the polynomials $P_n(y)$, $n > k$, occurring as their coefficient functions are completely described in

Theorem 4.12 ([5, Theorems 4, 9]) *Consider some* $k \geq 2$.

1. *For each generator* $H(x) = x^k + \sum_{n>k} h_n x^n$ *the system of* (PDform, (II, k)) *and* (B, (II, k)) *(or* (Dform, (II, k)) *and* (B, (II, k))*] has exactly one solution. It is given by*

$$G(y, x) = x + yx^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

2. *The polynomials* P_n, $n \geq k$, *have a formal degree* $\lfloor (n-1)/(k-1) \rfloor$ *and they are of the form*

$$P_n(y) = \begin{cases} h_n y & k \leq n < 2k-1 \\ h_{2k-1}y + \frac{k}{2}y^2 & n = 2k-1 \\ h_n y + \frac{n+1}{2} h_{n-k+1}y^2 + \Phi_n(y, h_{k+1}, \dots, h_{n-k}) & n \geq 2k, \end{cases}$$

where Φ_n *are polynomials in* y *and in the coefficients* $h_{k+1}, \dots, h_{n-k}$. *They satisfy* $\Phi_n(0, h_{k+1}, \dots, h_{n-k}) = 0$. *For* $n > 2k$ *a formal degree of* Φ_n *as a polynomial in* y *is* $\lfloor (n-1)/(k-1) \rfloor$.

Theorem 4.13 ([5, Theorems 5, 10]) *For each generator* $H(x) = x^k + \sum_{n>k} h_n x^n$ *the solution* $G(y, x)$ *of the system* (PDform, (II, k)) *and* (B, (II, k)) *[or* (Dform, (II, k)) *and* (B, (II, k))*] is a solution of* (Tform, (II, k)).

For the Aczél–Jabotinsky equation we obtain

Theorem 4.14 ([5, Theorem 13]) *Consider some* $k \geq 2$.

1. *For each generator* $H(x) = x^k + \sum_{n>k} h_n x^n$ *and for any polynomial* $P_k(y) \in \mathbb{C}[y]$ *with* $P_k(0) = 0$ *the differential equation* (AJform, (II, k)) *together with* (B, (II, k)) *has exactly one solution of the form*

$$G(y, x) = x + P_k(y)x^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

The polynomials $P_n(y)$ for $n > k$ are given by

$$P_n(y) = \begin{cases} h_n P_k(y) & \text{if } n < 2k-1 \\ h_{2k-1}P_k(y) + \frac{k}{2}P_k(y)^2 & \text{if } n = 2k-1 \\ h_n P_k(y) + \frac{n+1}{2}\, h_{n-k+1}P_k(y)^2 & \\ \qquad + \Phi_n(P_k(y), h_{k+1}, \ldots, h_{n-k}) & \text{if } n \geq 2k, \end{cases}$$

with polynomials Φ_n, $n \geq 2k$, in $P_k(y)$ and $h_{k+1}, \ldots, h_{n-k}$.

2. *Assume that $P_k(y) = y$. The polynomials P_n, $n \geq k$, have a formal degree $\lfloor (n-1)/(k-1) \rfloor$ and their coefficients are given in Theorem 4.12.*

Applying the same method as in the proof of Theorem 4.4 we obtain

Theorem 4.15 ([5, Theorem 14]) *For each generator $H(x) = x^k + \sum_{n>k} h_n x^n$ the solution $G(y,x)$ of (AJform, (II, k)) with $G(y,x) \equiv x + yx^k \bmod x^{k+1}$ is a solution of (Tform, (II, k)).*

Let G be the solution of (PDform, (II, k)) and (B, (II, k)) for the generator $H(x) = x^k + \sum_{n>k} h_n x^n$. Reordering the summands of G we write $G(y,x)$ as $\sum_{n\geq 0} \phi_n(x) y^n$. Then (PDform, (II, k)) yields

$$\sum_{n\geq 1} n\phi_n(x)y^{n-1} = H(x)\sum_{n\geq 0} \phi_n'(x)y^n, \qquad \text{(PDR, (II, } k))$$

where $(\phi_n'(x)y^n)_{n\geq 0}$ is a summable family. We note that (PDR, (II, k)) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1} H(x)\phi_n'(x) \qquad (\text{PDR}_n, (\text{II}, k))$$

holds true for all $n \geq 0$.

The solutions of (PDR, (II, k)) and (BR, (II, k)) are thoroughly analyzed in the following theorems.

Theorem 4.16 ([5, Theorem 15]) *For each generator $H(x) = \sum_{n\geq k} h_n x^n$, $k \geq 2$, $h_k = 1$, the system (PDR$_n$, (II, k)) and (BR, (II, k)) has a unique solution. For $n \geq 0$ the order of $\phi_n(x)$ is equal to $n(k-1)+1$ and $\phi_n(0) = 0$.*

Theorem 4.17 ([5, Corollary 16, Theorem 18]) *Consider some $k \geq 2$ and assume that $\sum_{n\geq 0} \phi_n(x)y^n = \sum_{r\geq 1} P_r(y)x^r$ is the solution of (PDR$_n$, (II, k)) and (BR, (II, k)) for a given generator $H(x)$. Writing*

$$P_r(y) = \sum_{j\geq 0} P_{r,j}y^j, \quad r \geq 1, \text{ and } \phi_n(x) = \sum_{r\geq 1} P_{r,n}x^r, \quad n \geq 0,$$

we deduce that $P_r = 0$ for $2 \leq r < k$. Moreover for $r \geq k$ the series $P_r(y)$ is a polynomial which has a formal degree $\lfloor (r-1)/(k-1) \rfloor$ and which satisfies $P_r(0) = 0$. Consequently

$$\sum_{n\geq 0}\phi_n(x)y^n = x + \sum_{r\geq k} P_r(y)x^r \in (\mathbb{C}[y])[\![x]\!].$$

If $\phi_n(x) = \sum_{r\geq n(k-1)+1} P_{r,n}x^r$ *and* $H(x) = \sum_{r\geq k} h_r x^r$*, then*

$$\phi_{n+1}(x) = \frac{1}{n+1}\sum_{r\geq (n+1)(k-1)+1}\left(\sum_{\nu=n(k-1)+1}^{r+1-k} \nu h_{r+1-\nu}P_{\nu,n}\right)x^r, \qquad n\geq 0.$$

Theorem 4.18 ([5, Theorem 19]) *Let* $H(x) = \sum_{n\geq k} h_n x^n$*,* $k\geq 2$*,* $h_k = 1$*, be a generator and assume that* $\sum_{n\geq 0}\phi_n(x)y^n$ *is the solution of* (PDR$_n$, (II, k)) *and* (BR, (II, k))*. Then*

$$\phi_n(x) = \frac{1}{n!}\sum_{r\geq n(k-1)+1}\left(\sum_{(\nu_1,\dots,\nu_{n-1})}^{*r}\prod_{s=1}^{n-1} h_{\nu_s}\Big(r+s-\sum_{t=1}^{s}\nu_t\Big)h_{r+(n-1)-\sum_{t=1}^{n-1}\nu_t}\right)x^r$$

for $n\geq 1$*. In* $\sum_{(\nu_1,\dots,\nu_{n-1})}^{*r}$ *we are taking the sum over all* $(n-1)$*-tuples* $(\nu_1,\dots,\nu_{n-1})$ *of integers, such that* $k\leq \nu_s\leq r-(n-s)k+(n-1)-\sum_{t=1}^{s-1}\nu_t$*.*
This theorem shows that the coefficient $P_{r,n}$ of x^r in $\phi_n(x)$ depends only on the elements $h_k,\dots,h_{r-(n-1)(k-1)}$.

4.3.6 Normal Forms of Iteration Groups of Type (II, k)

Assume that $G(y,x)$ is a formal iteration group of type (II, k) for some $k\geq 2$, i.e., G is a solution of (Tform, (II, k)) and (B, (II, k)). For all $S\in\Gamma_1$ the series

$$\tilde{G}(y,x) := S^{-1}\big(G(y,S(x))\big)$$

is also a solution of (Tform, (II, k)) and (B, (II, k)). Assume that H is the infinitesimal generator of G, then according to (4.1) the infinitesimal generator of $\tilde{G}$ is

$$\tilde{H}(x) = \big[S'(x)\big]^{-1}H\big(S(x)\big).$$

This differential equation for S is not a Briot–Bouquet equation. However, it can be reduced to such an equation by putting $S(x) = x\exp(\theta(x))$, where $\theta(x)\in\mathbb{C}[\![x]\!]$, $\theta(0)=0$. For each $H(x) = \sum_{n\geq k} h_n x^n$, $k\geq 2$, $h_k = 1$, there exist some $S(x)\in\Gamma_1$ and exactly one $h\in\mathbb{C}$, so that

$$\tilde{H}(x) = x^k + hx^{2k-1}.$$

This is the *normal form* of the generator of a formal iteration group of type (II, k). (A direct proof not using the theory of Briot–Bouquet equations can be found in [5, Theorem 28].) We say that a (formal) iteration group of type (II, k) with generator

$$H(x) = x^k + hx^{2k-1}, \qquad h \in \mathbb{C},$$

is a *normal form* and we describe these normal forms in the next theorems.

Theorem 4.19 ([5, Theorem 29]) *Consider some $k \geq 2$. The solution of* (PDform, (II, k)) *and* (B, (II, k)) *for $H(x) = x^k + hx^{2k-1}$ is given by*

$$G(y, x) = \sum_{n \geq 0} P_{n(k-1)+1}(y) x^{n(k-1)+1}$$

where

$$P_{n(k-1)+1}(y) = \begin{cases} 1 & \text{if } n = 0 \\ y & \text{if } n = 1 \\ \prod_{i=1}^{n-1} \big(i(k-1)+1\big) \dfrac{y^n}{n!} + hQ_n(y, h) & \text{if } n \geq 2, \end{cases}$$

and where $Q_n(y, h)$, $n \geq 2$, is a polynomial in y of degree $n-1$ and a polynomial in h of degree $\lfloor n/2 \rfloor - 1$.

Now we assume that h is an indeterminate over $(\mathbb{C}[y])[\![x]\!]$. It is interesting to note that the normal forms of iteration groups of type (II, k) have expansions in powers of the parameter h. Since for $n \geq 2$ the degree of $P_{n(k-1)+1}(y)$ as a polynomial in h is $\lfloor n/2 \rfloor$, we can write $G(y, x)$ as

$$G(y, x) = \sum_{r \geq 0} G_r(y, x) h^r \in (\mathbb{C}[\![x, y]\!])[\![h]\!].$$

From (B, (II, k)) we deduce that $G_0(0, x) = x$ and $G_r(0, x) = 0$ for $r \geq 1$. Instead of (PDform, (II, k)) we obtain

$$\begin{aligned} \sum_{r \geq 0} \frac{\partial}{\partial y} G_r(y, x) h^r &= \big(x^k + hx^{2k-1}\big) \left(\sum_{r \geq 0} \frac{\partial}{\partial x} G_r(y, x) h^r \right) \\ &= \sum_{r \geq 0} x^k \frac{\partial}{\partial x} G_r(y, x) h^r + \sum_{r \geq 0} x^{2k-1} \frac{\partial}{\partial x} G_r(y, x) h^{r+1} \end{aligned}$$

This is a system of equations for $G_r(y, x)$, $r \geq 0$, given by

$$\frac{\partial}{\partial y} G_0(y, x) = x^k \frac{\partial}{\partial x} G_0(y, x) \tag{4.4}$$

and

$$\frac{\partial}{\partial y} G_r(y,x) = x^k \frac{\partial}{\partial x} G_r(y,x) + x^{2k-1} \frac{\partial}{\partial x} G_{r-1}(y,x), \qquad r \geq 1. \tag{4.5}$$

Theorem 4.20 ([5, Theorem 30]) *Consider $H(x) = x^k + hx^{2k-1}$ where h is an indeterminate over $\mathbb{C}[\![x,y]\!]$. The solution of (4.4), (4.5), and (B, (II, k)) is given by*

$$\sum_{r \geq 0} G_r(y,x) h^r$$

where

$$G_r(y,x) = \sum_{n \geq r} \sum_{\substack{(j_1,\dots,j_r) \\ 1 \leq j_1 \\ j_s \geq j_{s-1}+2,\ s \geq 2 \\ j_r \leq n+r-1}} \frac{\prod_{i=1}^{n+r-1} [i]}{\prod_{s=1}^{r} [j_s]} \frac{x^{[n+r]}}{n!} y^n, \qquad r \geq 0,$$

where $[r] = r(k-1)+1$.

Concerning the differential equation (Dform, (II, k)) we have

Theorem 4.21 ([5, Theorem 33]) *Consider some $k \geq 2$. The solution of (Dform, (II, k)) and (B, (II, k)) for $H(x) = x^k + hx^{2k-1}$ is given by*

$$G(y,x) = \sum_{r \geq 0} G_r(y,x) h^r$$

with

$$G_r(y,x) = x^{[r]} (1-(k-1)yx^{k-1})^{-[r]/(k-1)} P_r(\ln(1-(k-1)yx^{k-1})), \qquad r \geq 0,$$

where $[r] = r(k-1)+1$ and P_r are polynomials of degree r. Moreover $P_0 = 1$ and

$$P_1(z) = -z/(k-1).$$

The binomial series is used in order to compute

$$(1-(k-1)yx^{k-1})^{-[r]/(k-1)}.$$

The particular situation $r = 0$ yields

$$G_0(y,x) = x(1-(k-1)yx^{k-1})^{-1/(k-1)}.$$

$G_0(y,x)$ together with its conjugates occur in the problem of reversible power series (c.f. [12, Section 0.3]).

There exists also an approach with *Lie–Gröbner-series* (cf. [8] or [9, Chapter 1]) to solve (PDR, (II, k)) and (BR, (II, k)). We note that Lie–Gröbner-series in the context of iteration groups have already been used by St. Scheinberg [29] and also by Reich and Schwaiger in [27]. Define an operator

$$D\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \qquad D(f(x)) := H(x)f'(x).$$

Lemma 4.3 ([5, Lemma 23]) *Let H be a generator of order $k \geq 2$. If $(\phi_n)_{n\geq 0}$ satisfies the system* (PDR, (II, k)) *and* (BR, (II, k))*, then*

$$\phi_n(x) = \frac{1}{n!} D^n(x), \qquad n \geq 0.$$

Theorem 4.22 ([5, Theorem 24]) *The series*

$$G(y, x) := \sum_{n\geq 0} \frac{1}{n!} D^n(x) y^n,$$

is a Lie–Gröbner-series. It satisfies (Tform, (II, k)) *and* (B, (II, k)).

4.4 Concluding Remarks and Open Problems

At the end of this paper we present some open problems concerning the construction of iteration groups.

1. It is an important problem to study iteration groups in higher dimension. This means in our situation to change to the ring $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ of formal power series in $n \geq 2$ indeterminates $\mathbf{x} = (x_1, \ldots, x_n)^T$ over $\mathbb{C}$ and to consider n-tuples

$$F(\mathbf{x}) = F\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1((x_1, \ldots, x_n)^T) \\ \vdots \\ F_n((x_1, \ldots, x_n)^T) \end{pmatrix} = \begin{pmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix},$$

i.e., elements of $(\mathbb{C}[\![\mathbf{x}]\!])^n$. By $\mathrm{ord}(F(\mathbf{x}))$ we understand $\min\{\mathrm{ord}(F_1), \ldots, \mathrm{ord}(F_n)\}$. We consider the substitution of $G(\mathbf{x}) \in (\mathbb{C}[\![\mathbf{x}]\!])^n$ into $F(\mathbf{x}) \in (\mathbb{C}[\![\mathbf{x}]\!])^n$ provided that $\mathrm{ord}(G) \geq 1$.

Each $F(\mathbf{x}) \in (\mathbb{C}[\![\mathbf{x}]\!])^n$ can be written as $F(\mathbf{x}) = A \cdot \mathbf{x} + R(\mathbf{x})$, where A is a complex $n \times n$-matrix and $R(\mathbf{x}) \in (\mathbb{C}[\![\mathbf{x}]\!])^n$ with $\mathrm{ord}(R) \geq 2$. If $\det(A) \neq 0$ we call F a *formally biholomorphic* mapping. The set of all formally biholomorphic mappings forms a group $\boldsymbol{\Gamma}$ with respect to substitution $\circ$, and a family $(F_t(\mathbf{x}))_{t\in\mathbb{C}}$, $F_t(\mathbf{x}) \in \boldsymbol{\Gamma}$, satisfying the translation equation

$$F_{s+t} = F_s \circ F_t, \qquad s, t \in \mathbb{C}, \tag{T}$$

is called an iteration group in n dimensions.

The construction of all iteration groups of dimension $n \geq 2$ is an open problem and very likely the method of formal functional equations and differential equations will lead to a solution.

Mehring has shown in [22, 23] that the coefficient functions of an iteration group are polynomials in a finite number of additive or generalized exponential functions, however, the detailed structure is not known.

2. Jabłoński and Reich studied in [19] the iteration groups of truncated formal power series. It is an open question how to construct these groups using the method of formal functional equations.
3. The method of formal functional equations should also be applied in the problem of constructing maximal abelian subgroups of Γ or $\boldsymbol{\Gamma}$, in particular in higher dimension.
4. The various representations of the coefficient functions of iteration groups presented in this paper and the representations obtained by Jabłoński and Reich have so far not been compared by direct computation. This could yield interesting polynomial identities.
5. We notice that from the representation $G(y,x) = S(yS^{-1}(x))$ given in (4.3) we can derive a representation

$$G(y,x) = yx + \sum_{\nu \geq 2} Q_\nu(y, s_2, \ldots, s_\nu)x^\nu$$

where each Q_ν is a polynomial in y and in the coefficients $s_2, \ldots, s_\nu$ of $S(x) = x + s_2x^2 + \ldots$. Formula (4.2) describes a connection between the generator $H(x) = x + h_2x^2 + \ldots$ and the conjugating series $S(x)$. This gives eventually another (maybe new) representation of the coefficients P_n of G from Theorem 4.3 as polynomials in y and $h_2, \ldots, h_n$.

Acknowledgements We would like to thank the anonymous referee for helpful comments.

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