Chapter 3
On the Indicator Plurality Function

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Abstract This survey paper is dedicated to certain mathematization method of social choice, given by Roberts, and its generalizations.

Keywords Social choice • Plurality function • Indicator plurality function • Consistency condition • Cone • Additive function

Mathematics Subject Classification (2010) Primary 39B22; Secondary 39B52, 39B72, 39B82

3.1 Introduction

There is a long history in the theory of social choice of finding axioms that characterize a particular group consensus function and of finding all group consensus functions that satisfy certain axioms. Much of the early history of this theory has been concerned with impossible theorems, which show that under certain reasonable axioms there is no social choice function that merges individual judgements into a consensus judgement (see [8]). From 1974 there have been a variety of positive results. The first concerned an axiomatization of Borda's rule [21], next outcomes, among others, of social choice scoring functions [22], of the plurality rule [15], and of the plurality function [16].

Much of the literature of social choice functions falls into the following setting. Let A be a set of alternatives, for instance, alternative strategies, alternative new technologies, alternative diagnoses, or alternative candidates and let B be a set of individuals (voters or experts), who are expressing opinions about the alternatives in the set A. A social choice function is a function which, based on the opinions

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J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer Optimization and Its Applications 124,
DOI 10.1007/978-3-319-61732-9_3

of the individuals in B, picks a "consensus." In many contexts, the opinions of the individuals in B are given as rankings or linear orders of the elements of A. In other contexts, the opinions are simply a choice of the best or most preferred alternative in A. The "consensus" can be either a single element of A, a subset of alternatives in A, a ranking of elements of A, or a set of such rankings. In the situation where the opinions are rankings and the consensus is a subset of A, a well-known social choice function is the plurality rule, which chooses for the consensus all those alternatives which receive the greatest number of first place rankings. In [15], Richelson was able to characterize the plurality rule by giving five simple axioms, which are based on some earlier axioms of Young (see [22]).

In the situation where each of individual gives only the first choice from A (the opinions are elements of the set A) and the consensus is a subset of A a social choice function is a consensus function. The plurality function is that consensus function which chooses as consensus all alternatives which receive the largest number of first choices. The axioms that characterize the plurality function were introduced by Roberts [16]. Mathematical theory of this approach was developed by Bahyrycz [2–7], Forti and Paganoni [9, 10], Moszner [7, 11–14], Roberts [17, 18], and Rosenbaum [20]. In this paper we present some of these results.

In the second section we present the definitions and characterizations of the plurality function and the indicator plurality function and we give their election interpretation. In the Section 3.3 we determine all functions which are consistent. In the next section we describe a way of construction of all m-elements consistent system related to the indicator plurality function. In the last section we consider the systems of equations with unknown multifunctions related to the indicator plurality function.

3.2 The Plurality Function and the Indicator Plurality Function

In this section we start by recalling the results of Roberts [16, 17], which are based on some earlier axioms of Richelson [15].

Suppose A is a set of alternatives and each voter provides us with a first choice from A. *The plurality function* is the function F from $\cup_{n=1}^{\infty} A^n$ into 2^A, where $F(x_1, \ldots, x_n)$ is the set of all those y in A so that no z in A appears more often in $(x_1, \ldots, x_n)$ than y.

To state the characterization of the plurality function, we introduce the following definitions.

Anonymity For all permutations π of $\{1, \ldots, n\}$

$$F(x_{\pi(1)}, \ldots, x_{\pi(n)}) = F(x_1, \ldots, x_n)$$

for all sequences of alternatives $(x_1, \ldots, x_n)$.

The election interpretation of this property is the following. The election result does not depend on the order of inserting the votes into the ballot box.

Neutrality For all permutations σ of A

$$F(\sigma(x_1), \dots, \sigma(x_n)) = \sigma[F(x_1, \dots, x_n)]$$

where $\sigma(X) = \{\sigma(x) : x \in X\}$.

The election result does not depend on the order of placing the candidates on the list.

Consistency If $F(x_1, \dots, x_n) \cap F(y_1, \dots, y_m) \neq \emptyset$, then

$$F(x_1, \dots, x_n, y_1, \dots, y_m) = F(x_1, \dots, x_n) \cap F(y_1, \dots, y_m). \tag{3.1}$$

It expresses the following. If $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ are two vectors representing the votes of two different groups of voters among the same set of candidates and some candidate is chosen by both groups, then a candidate x is chosen by the combined group if and only if this candidate x is chosen by both groups, separately. The combined group is represented by the vector $(x_1, \dots, x_n, y_1, \dots, y_m)$.

The assumption that at least one candidate won the election in both groups is important, otherwise the equality (3.1) could not take place, the set on the right side would be empty and the set on the left side could never be empty.

Faithfulness $F(x) = \{x\}$ for all $x \in A$.

If we have one voter and this voter gives his or her first choice on a candidate x, then this candidate is chosen.

In [16] was given a following characterization of the plurality function.

Theorem 3.1 *Suppose* $F : \cup_{n=1}^{\infty} A^n \to 2^A$ *and* $F(x) \neq \emptyset$ *for any* $x \in A$. *Then the following are equivalent:*

(1) F is the plurality function.
(2) F is anonymous, neutral, consistent, and faithful.

For more characterization of the plurality function see [16].

Now, suppose that A is a finite set $\{v_1, v_2, \dots, v_m\}$ and F is a plurality function. We may rewrite any vector $(x_1, \dots, x_n)$ from A^n, after possibly permuting the subscripts, in the form

$$\underbrace{v_1, \dots, v_1}_{c_1},\ \underbrace{v_2, \dots, v_2}_{c_2},\ \dots,\ \underbrace{v_m, \dots, v_m}_{c_m},$$

where v_i occurs c_i times. If v_j doesn't occur in the vector $(x_1, \dots, x_n)$, then $c_j = 0$ and all of the c_i are non-negative integers and at least one of them is positive.

The vector $(x_1, \dots, x_n)$ we can write in the following way: $(c_1 v_1, \dots, c_m v_m)$. Since the function F is anonymous we have

$$F(x_1, \dots, x_n) = F(c_1 v_1, \dots, c_m v_m).$$

We can define a new function $f = (f_1, \ldots, f_m) : \mathbb{Z}(m) \to 0(m)$, where $\mathbb{Z}(m)$ is the set of all m-vectors of non-negative integer numbers, except the vector $\underline{0} := (0, \ldots, 0)$ and $0(m)$ is a subset of $\mathbb{Z}(m)$ in which each component is 0 or 1 and

$$f_k(c_1, \ldots, c_m) = 1 \Leftrightarrow v_k \in F(c_1 v_1, \ldots, c_m v_m) \quad \text{for } k \in \{1, \ldots, m\}.$$

We will think of f as the indicator function corresponding to the plurality function F.

From the above considerations follows that we may define the function f independently of the plurality function in the following way:

$$f_k(c_1, \ldots, c_m) = 1 \Leftrightarrow c_k \geq c_j \quad \text{for } j \in \{1, \ldots, m\}. \tag{3.2}$$

It expresses the following. If we vote for m candidates and $c_1, \ldots, c_m$ is a description of this vote (c_i is the number of votes which received the ith candidate on the list), then 1 in the kth position in $f(c_1, \ldots, c_m)$ means that the kth candidate on the list received at least as many votes as the other and he won the election maybe simultaneously with other candidates.

More generally, if we allow fractional votes or vote splitting, the domain of f would consist of the set of all m-vectors of non-negative rational numbers, except $\underline{0}$ $\big(\mathbb{Q}(m)\big)$ or even the set of all m-vectors of non-negative real numbers, except $\underline{0}$ $\big(\mathbb{R}(m)\big)$.

A function $f = (f_1, \ldots, f_m) : U \to 0(m)$, where $U \subset \mathbb{R}(m)$ is called *the indicator plurality function on U* if f satisfies (3.2) for all $(c_1, \ldots, c_m) \in U$.

From now on, we assume that $U \in \{\mathbb{Z}(m), \mathbb{Q}(m), \mathbb{R}(m)\}$.

The indicator plurality function on U has analogous properties to those defined above for the plurality function F. The anonymity was used to define the indicator plurality function and other properties have the following form.

Neutrality For all $(c_1, \ldots, c_m) \in U$ and all permutations π of $\{1, \ldots, m\}$

$$f_k(c_{\pi(1)}, \ldots, c_{\pi(m)}) = f_{\pi(k)}(c_1, \ldots, c_m) \quad \text{for } k \in \{1, \ldots, m\}.$$

Consistency For all $c, d \in U$

$$f(c) \cdot f(d) \neq \underline{0} \Rightarrow f(c + d) = f(c) \cdot f(d), \tag{3.3}$$

where $x + y := (x_1 + y_1, \ldots, x_m + y_m)$ and $x \cdot y := (x_1 \cdot y_1, \ldots, x_m \cdot y_m)$ for $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in U$.

Faithfulness For all $i \in \{1, \ldots, m\}$

$$f(e_i) = e_i$$

where e_i denotes the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the ith position.

We have the following characterization of the indicator plurality function on $\mathbb{Z}(m)$ and $\mathbb{Q}(m)$ (see [17]).

Theorem 3.2 *Suppose that* $V \in \{\mathbb{Z}(m), \mathbb{Q}(m)\}$ *and* $f : V \to 0(m)$. *Then the following are equivalent:*

(i) f *is the indicator plurality function on* V.
(ii) f *is neutral, consistent, and faithful.*

The original indicator plurality function is homogeneous, because the election result does not depend on the number of votes, but on the proportion of the votes for individual candidates. One can consider the following properties:

Weak Homogeneity For all $c \in U$

$$f(2c) = f(c).$$

Homogeneity For all positive real number r, all $c \in \mathbb{R}(m)$ and $U = \mathbb{R}(m)$

$$f(rc) = f(c),$$

where $rc := (rc_1, \ldots, rc_m)$ for $c = (c_1, \ldots, c_m) \in \mathbb{R}(m)$.

Homogeneity Faithful For all positive real number r, all $j \in \{1, \ldots, m\}$ and $U = \mathbb{R}(m)$

$$f(re_j) = f(e_j).$$

In the case, when we weaken the assumption in Theorem 3.2 that the range of f is contained in $O(m)$ we have the following (see [17]):

Theorem 3.3 *Suppose that* $V = \mathbb{Z}(m)$ *or* $V = \mathbb{Q}(m)$ *and* $f : V \to \mathbb{R}(m)$. *Then the following are equivalent:*

(i) f *is the indicator plurality function on* V.
(ii) f *is neutral, consistent, faithful, and weakly homogeneous.*

If we consider the function $f : \mathbb{R}(m) \to 0(m)$, then there exist the functions which are neutral, consistent, and faithful but different from the indicator plurality function which shows the following example ([20], see also [11]).

Let b_0 be a positive irrational, b_1 be a non-zero rational, and H be a Hamel base of the space $\mathbb{R}$ over a field $\mathbb{Q}$ such that $b_0, b_1 \in H$. Every $x \in \mathbb{R}$ has a representation, unique up to terms with coefficients zero

$$x = \sum_{l=0}^{n} q_l b_l,$$

where $q_l \in \mathbb{Q}$, $b_l \in H$ for $l \in \{0, \dots, n\}$. We put $\gamma := q_0$ and $\delta := \sum_{l=1}^{n} q_l b_l$, then $x = \gamma b_0 + \delta$. We define a function $g = (g_1, \dots, g_m) : \mathbb{R}^m \to 0(m)$ by

$$g_j(x_1, \dots, x_m) = \begin{cases} 1 & \text{if } x_j \geq x_k \text{ for } k = 1, \dots, m, \\ 0 & \text{otherwise .} \end{cases}$$

Now, we define a function $f : \mathbb{R}(m) \to 0(m)$ in the following way:

$$f(x_1, \dots, x_m) = g(\delta_1 - \gamma_1, \dots, \delta_m - \gamma_m),$$

where $x_j := \gamma_j b_0 + \delta_j$ for $j \in \{1, \dots, m\}$.

The function f is neutral, consistent, and faithful but different from the indicator plurality function because

$$f(b_0, 0, \dots, 0) = g(-1, 0, \dots, 0) = (0, 1, \dots, 1).$$

On the other hand, we have the following theorems (see [17]):

Theorem 3.4 *Let $f : \mathbb{R}(m) \to 0(m)$ be an arbitrary function. Then the following are equivalent:*

(i) f is the indicator plurality function on $\mathbb{R}(m)$.
(ii) f is neutral, consistent, and homogeneous faithful.

Theorem 3.5 *Let $f : \mathbb{R}(m) \to \mathbb{R}(m)$ be an arbitrary function. Then the following are equivalent:*

(i) f is the indicator plurality function on $\mathbb{R}(m)$.
(ii) f is neutral, consistent, faithful, and homogeneous.

If we consider the function $f : \mathbb{R}(m) \to \mathbb{R}(m)$, then there exist the functions which are neutral, consistent, and homogeneous faithful but different from the indicator plurality function on $\mathbb{R}(m)$, which shows the following example (see [2, 12]).

We define a function $f = (f_1, \dots, f_m) : \mathbb{R}(m) \to \mathbb{R}(m)$ as follows:

$$f_i(x) = \begin{cases} \exp(x_1 + \cdots + x_m - x_i) & \text{for } x \in Z_i, \\ 0 & \text{for } x \in \mathbb{R}(m) \setminus Z_i, \end{cases}$$

where $Z_i := \{x \in \mathbb{R}(m) : x_i \geq x_j \text{ for } j \in \{1, \dots, m\}\}$ and $i \in \{1, \dots, m\}$.

The function f is neutral, consistent, and homogeneity faithful but f is not the indicator plurality function on $\mathbb{R}(m)$ because the range of f is not contained in $0(m)$, for example,

$$f(1, 1, 0 \dots, 0) = (e, e, 0 \dots, 0).$$

In the original problem of the social choice the function $f(x_1, \dots, x_m)$ is defined on the set $\mathbb{Z}(m)$ and $x_1 + \cdots + x_m$ is the sum of the votes cast. In practice this sum is limited, for example, by the number $c > 0$. This begs the idea of replacing the property consistency by the condition

$$x_1 + \cdots + x_m + y_1 + \cdots + y_m \leq c \ \wedge \ f(x) \cdot f(y) \neq \underline{0} \Rightarrow f(x+y) = f(x) \cdot f(y) \tag{3.4}$$

and a problem arises if each function f satisfying Equation (3.4) can be uniquely extended to the solution of Equation (3.3). We have the following (see [12, 13]):

Theorem 3.6 *Every function* $f : E := \{(x_1, \ldots, x_m) \in \mathbb{R}(m) : x_1 + \cdots + x_m \leq c\} \to \mathbb{R}(m)$ *which is the solution of Equation* (3.4) *can be uniquely extended to the solution of Equation* (3.3).

Every solution of Equation (3.4) *can be obtained by restricting the solution of Equation* (3.3) *to the set* E.

This result shows that the generalization of the considerations about the election of the case of natural numbers to the case of real numbers is not good for the description of the election, because the outcome of the election on a small population determines the result for the whole population. Note that this anomaly does not take place if the real numbers replace integers, because in this case an analogue of Theorem 3.6 is not true. Indeed, we consider the function f defined as follows (see [13]):

$$f(1,0) = f(2,0) = f(1,1) = (1,0) \quad \text{and} \quad f(0,1) = f(0,2) = (0,1).$$

This function f satisfies the condition (3.4) with $c = 2$ and $m = 2$ and can be extended onto $\mathbb{Z}(2)$ at least two different ways

$$f_1(x_1, x_2) = \begin{cases} (1,0) & \text{for } x_1 \geq x_2, \\ (0,1) & \text{for } x_1 < x_2 \end{cases} \tag{3.5}$$

and

$$f_2(x_1, x_2) = \begin{cases} (1,0) & \text{for } x_1 \neq 0, \\ (0,1) & \text{for } x_1 = 0. \end{cases}$$

These and other characterizations of the indicator plurality function one may be found in [2, 11–13, 17, 18].

3.3 On the Functions Which Are Consistent

In [19] Roberts stated that it is of interest in the theory of social choice to determine all functions $f : \mathbb{R}(m) \to \mathbb{R}(m)$ which are consistent, i.e., satisfy the conditional functional equation

$$f(x) \cdot f(y) \neq \underline{0} \Rightarrow f(x+y) = f(x) \cdot f(y).$$

As a generalization, one may consider functions $f : \mathbb{R}(n) \to \mathbb{R}(m)$ (where n, m are arbitrary natural numbers, independent of each other) satisfying the condition

$$\forall_{x,y\in\mathbb{R}(n)} : \ f(x)\cdot f(y) \neq \underline{0} \Rightarrow f(x+y) = f(x)\cdot f(y). \tag{3.6}$$

It may be shown that in such a case the description of all the solutions $f = (f_1, \ldots, f_m)$ of Equation (3.6) takes the following form (see [4] and [11] for $n = m$):

$$f_\nu(x) = \begin{cases} \exp a_\nu(x) & \text{for } x \in Z_\nu, \\ 0 & \text{for } x \in \mathbb{R}(n) \setminus Z_\nu, \end{cases} \tag{3.7}$$

where $a_\nu : \mathbb{R}^n \to \mathbb{R}$ are additive functions for $\nu = 1, \ldots, m$, whereas the sets Z_ν satisfy the conditions

$$Z_1 \cup \cdots \cup Z_m = \mathbb{R}(n), \tag{3.8}$$

$$ij \neq 0_m \Rightarrow Z_1^{i_1} \cap \cdots \cap Z_m^{i_m} + Z_1^{j_1} \cap \cdots \cap Z_m^{j_m} \subset Z_1^{i_1j_1} \cap \cdots \cap Z_m^{i_mj_m}, \tag{3.9}$$

for every $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_m) \in 0(m)$, $E_1 + E_2 := \{x + y : x \in E_1, \ y \in E_2\}$ for $E_1, E_2 \subset \mathbb{R}^n$, $E^1 := E$, $E^0 := \mathbb{R}(n) \setminus E$ for $E \subset \mathbb{R}(n)$.

Let us observe that if sets $Z_1, \ldots, Z_m$ satisfy condition (3.8), then for every $a \in \mathbb{R}(n)$ and every $k \in \{1, \ldots, m\}$ there exists a unique $i_k \in \{0, 1\}$ such that $a \in Z_k^{i_k}$.

The parameters determining the solutions of Equation (3.6) are systems of sets $Z_1, \ldots, Z_m$ satisfying conditions (3.8) and (3.9), as well as additive functions $a_\nu : \mathbb{R}^n \to \mathbb{R}$. Additionally condition (3.9) has a complicated form. For this reason, it is interesting to find conditions equivalent to condition (3.9) under the assumption of condition (3.8) which are of simpler form than the ones obtained from (3.9). We have the following theorem (see [4]).

Theorem 3.7 *Assume that sets $Z_1, \ldots, Z_m$ satisfy condition* (3.8)*. The following conditions are equivalent:*

(i) *condition* (3.9)*;*

(ii) *the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$ for which*

$$Z_l^1 + Z_l^1 \cap Z_k^0 \subset Z_l^1 \cap Z_k^0 \tag{3.10}$$

for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$;

(iii) *the sets $Z_1, \ldots, Z_m$ satisfy the conditions*

$$Z_k^1 + Z_k^1 \subset Z_k^1 \tag{3.11}$$

for every $k \in \{1, \ldots, m\}$ and condition (3.10) *for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$;*

(iv) for all $x, y \in \mathbb{R}(n)$ if there exists $\nu \in \{1, \ldots, m\}$ such that $x \in Z_\nu$ and $y \in Z_\nu$, then

$$\forall k \in \{1, \ldots, m\} : \quad x + y \in Z_k \Leftrightarrow x \in Z_k \text{ and } y \in Z_k;$$

(v) for all $k, l \in \{1, \ldots, m\}$ the following implication holds:

$$ij \neq \underline{0} \Rightarrow Z_k^{i_k} \cap Z_l^{i_l} + Z_k^{j_k} \cap Z_l^{j_l} \subset Z_k^{i_k j_k} \cap Z_l^{i_l j_l},$$

where $i = (i_k, i_l)$, $j = (j_k, j_l) \in 0(2)$.

We observe (see [12]) that the function f satisfying the condition (3.6) is continous if and only if the sets $Z_1, \ldots, Z_m$ fulfilling the condition (3.8) are such that $Z_\nu = \emptyset$ or $Z_\nu = \mathbb{R}(n)$ and the additive functions a_ν for $\nu \in \{1, \ldots, m\}$ are continous.

We notice also that the function f satisfying the condition (3.6) can be measurable without being continuous. For example, for $m = n = 2$ it is enough to consider the function f_1 given by the formula (3.5).

Let us make the following definitions.

Definition 3.1 Let $C \subset \mathbb{R}(n)$ be a cone over $\mathbb{Q}$ ($x + y \in C$ and $qx \in C$ for all $x, y \in C$, $q \in \mathbb{Q}_+$). Denote:

$< C > -$ the linear subspace of $\mathbb{R}^n$ over the field $\mathbb{R}$ generated by C;
C^* $-$ the interior of the set C in $< C >$.

Definition 3.2 For every subset $\{l_1, \ldots, l_k\} \subset \{1, \ldots, n\}$ we define the set

$$B_{l_1,\ldots,l_k} := \{(x_1, \ldots, x_n) \in \mathbb{R}(n) : x_{l_1} = \cdots = x_{l_k} = 0\},$$

and then we define the set

$$\mathbb{B} := \{B_{l_1,\ldots,l_k} : \{l_1, \ldots, l_k\} \subset \{1, \ldots, n\}\}.$$

Theorem 3.7 leads to the following.

Corollary 3.1 *If the sets $Z_1, \ldots, Z_m$ are pairwise disjoint and satisfy condition (3.8), then condition (3.9) is equivalent to the following condition: $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$.*

Corollary 3.2 *If a system of sets $Z_1, \ldots, Z_m$ satisfies conditions (3.8) and (3.9), then for every non-empty subset $\{l_1, \ldots, l_p\}$ of the set $\{1, \ldots, m\}$ and for every $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$ the set $Z_{l_1}^{i_{l_1}} \cap \cdots \cap Z_{l_p}^{i_{l_p}}$ is a cone over $\mathbb{Q}$.*
Let us observe that if the sets $Z_1, \ldots, Z_m$ satisfy conditions (3.8) and (3.9), then for $m \in \{1, 2\}$ the set Z_1^0 in the case of $m = 1$ and the sets Z_1^0, Z_2^0, $Z_1^0 \cap Z_2^0$ for $m = 2$

are also cones over $\mathbb{Q}$. If $m = 1$, then $Z_1 = \mathbb{R}(n)$, so $Z_1^0 = \emptyset$. If $m = 2$, then $Z_1^0 = Z_1^0 \cap Z_2^1$ and $Z_2^0 = Z_2^0 \cap Z_1^1$ are cones over $\mathbb{Q}$ and the set $Z_1^0 \cap Z_2^0$ is empty.

If the sets $Z_1, \ldots, Z_m$ satisfy conditions (3.8) and (3.9), then for $m > 2$ not every set $Z_{l_1}^0 \cap \cdots \cap Z_{l_p}^0$, where $\emptyset \neq \{l_1, \ldots, l_p\} \subset \{1, \ldots, m\}$, is necessarily a cone over $\mathbb{Q}$. Here is a suitable example for $n = 2$ and $m = 3$. Define

$$\begin{aligned} Z_1 &:= \{(x, y) \in \mathbb{R}(2) : y \leq \tfrac{1}{2}x\}, \\ Z_2 &:= \{(x, y) \in \mathbb{R}(2) : \tfrac{1}{2}x < y \leq 2x\}, \\ Z_3 &:= \{(x, y) \in \mathbb{R}(2) : y > 2x\}. \end{aligned}$$

The sets Z_1, Z_2, Z_3 satisfy conditions (3.8) and (3.9) but Z_2^0 is not a cone over $\mathbb{Q}$.

Corollary 3.3 *Let the sets $Z_1, \ldots, Z_m$ satisfy the conditions (3.8) and (3.9). If there exist $k, l \in \{1, \ldots, m\}$ such that $k \neq l$ and $(Z_k \cap Z_l)^* \neq \emptyset$, then*

$$Z_k \cap < Z_k \cap Z_l > = Z_l \cap < Z_k \cap Z_l >.$$

Corollary 3.4 *If the sets $Z_1, \ldots, Z_m$ satisfy the conditions (3.8) and (3.9), then for all $k, l \in \{1, \ldots, m\}$ $Z_k = Z_l$ or $Z_k \cap Z_l$ is a set with empty interior in $\mathbb{R}^n$.*

Corollary 3.5 *If a system $Z_1, \ldots, Z_m$ satisfies the conditions (3.8) and (3.9) and if there exists such $k \in \{1, \ldots, m\}$ that $Z_k = \mathbb{R}(n)$, then $Z_i \in \mathbb{B}$ for every $i \in \{1, \ldots, m\}$.*

From the above Corollary we obtain, for example, that if the sets Z_1, Z_2 satisfy the conditions (3.8) and (3.9) with $n = m = 2$ and $Z_1 = \mathbb{R}(2)$, then Z_2 must be equal to one of the sets $B_\emptyset = \mathbb{R}(2), B_1, B_2, B_{1,2} = \emptyset$.

It may be proved (see [12] for $n = m$) that every function $f : \mathbb{R}(n) \to \mathbb{R}(m)$ satisfying (3.6) and the condition

$$\exists_{r>0} : [r \neq 1 \wedge \forall_{x \in \mathbb{R}(n)} : \ f(rx) = f(x)] \tag{3.12}$$

with some r being an algebraic number must have values in the set $0(m)$. It is known (see [3] for $n = m$) that this property holds also with a transcendental number r if $m \leq 2$ and in the case when $m > 2$ there exists a solution of Equation (3.6) satisfying (3.12) with some transcedental number r which range is not contained in $0(m)$. In a very long construction of such function the Axiom of Choice is used. Moreover in [7] was shown that one cannot give this construction without using non-measurable set.

Theorem 3.8 *If a function $f : \mathbb{R}(n) \to \mathbb{R}(m)$ fulfils the conditions (3.6), (3.12) and additionally for every $x \in \mathbb{R}(n)$ the set*

$$M_i(x) = \{tx \in \mathbb{R}(n) : f_i(tc) \neq 0\} \quad \textit{for } \ i \in \{1, \ldots, m\}$$

are Lebesgue linearly measurable, then f must have its values in the set $0(m)$.

From the description of the solution of Equation (3.6) follows that the function satisfying the conditions (3.6) and (3.12) has values in the set $0(m)$ if and only if all additive functions a_ν are identically equal to zero. The condition (3.12) imposes on the functions a_ν and the sets Z_ν ($\nu = 1, \ldots, m$) the conditions

$$rZ_\nu = Z_\nu \tag{3.13}$$

and

$$a_\nu(rx) = a_\nu(x) \quad \text{for } x \in Z_\nu, \tag{3.14}$$

and we have the following

Theorem 3.9 *The function $f : \mathbb{R}(n) \to \mathbb{R}(m)$ satisfying the conditions (3.6) and (3.12) has values only in the set $0(m)$ if and only if the sets Z_ν fulfilling the conditions (3.8), (3.9), and (3.13) satisfy the condition*

$$Z_\nu \subset (r-1)lin_{\mathbb{Q}} Z_\nu \quad \textit{for } \nu = 1, \ldots, m \tag{3.15}$$

with r occuring in (3.12).

The above Theorem was proved in [7] for $n = m$, but from Lemma 1 from the same paper we can obtain this fact for the arbitrary $n, m \in \mathbb{N}$.

We notice that for $m = 1$ we have $Z_1 = \mathbb{R}(n)$ and the condition (3.15) is obviously fulfilled. This condition is also satisfied for $m = 2$, because then the sets Z_1^0 and Z_2^0 are cones over $\mathbb{Q}$. In the paper [3] such a cone is constructed for which the condition (3.15) is not satisfied.

In [9, 10] was given a description of the construction of the solutions of a system of functional equations: (3.6) (with $n = m$) and equation

$$\forall_{r>0} \forall_{x \in \mathbb{R}(m)} : \; f(rx) = f(x). \tag{3.16}$$

To each function $f : \mathbb{R}(m) \to 0(m)$ a partition of $\mathbb{R}(m)$ is associated, given by the family of the non-empty level sets of f, i.e., the family $\{A_i, \; i \in \mathscr{I} \subset 0(m)\}$ where $A_i = \{x \in \mathbb{R}(m) : f(x) = i\}$ and $i \in \mathscr{I}$ if and only if $A_i \neq \emptyset$. The following theorem characterizes the solutions of the system of the functional equations (3.6) and (3.16) through the properties of the corresponding families of the non-empty level sets.

Theorem 3.10 *Let $\{A_i, \; i \in \mathscr{I} \subset 0(m)\}$ be the family of the non-empty level sets of a function $f : \mathbb{R}(m) \to 0(m)$. Then f is a solution of Equation (3.6) satisfying (3.16) if and only if*

(i) A_i is a cone over $\mathbb{R}$ for all $i \in \mathscr{I}$;
(ii) $ij \neq \underline{0} \Rightarrow A_i + A_j \subset A_{ij}$ for all $i, j \in \mathscr{I}$.

In [9] were described explicitly all solutions of that system in the case of dimension less or equal to three. The authors wrote that for higher dimension the task of giving an analogous description seemed hopeless. We present these results only for $m \in$

$\{1, 2\}$ because the construction of all solutions of that system for $m = 3$ in [9] occupies more than 12 pages.

For $m = 1$ the only solution is given by $f(x) = 1 \quad$ for $x \in \mathbb{R}(1)$.

In the case $m = 2$ we have the following possibilities:

(a) $\quad f(x) = i \quad$ for $\; x \in \mathbb{R}(2)$,

where $\quad i \in \{(1, 0), (0, 1), (1, 1)\}$;

(b) $$f(x) = \begin{cases} i & \text{for } x \in \mathbb{R}(2) \setminus U, \\ (1,1) \;\text{ or }\; (1,1) - i & \text{for } x \in U, \end{cases}$$

where U is one of the semiaxes of $\mathbb{R}(2)$ and $i \in \{(1, 0), (0, 1)\}$;

(c) $$f(x) = \begin{cases} i & \text{for } x \in Z, \\ (1,1) - i & \text{for } x \in Z', \\ i \;\text{ or }\; (1,1) - i \;\text{ or }\; (1,1) & \text{for } x \in L, \end{cases}$$

where L is a half-line in $\mathbb{R}(2)$ from the origin, Z, Z' are two non-empty and disjoint cones over $\mathbb{R}$ whose union is $\mathbb{R}(2) \setminus L$ and $i \in \{(1, 0), (0, 1)\}$.

In [10] the above problem was studied in a completely different way: first the authors have proved some lemmas of geometric-combinatorial type which highlight some properties that were the guidelines for developing the procedure for the construction of the solutions, then they have described an operative procedure to construct all solutions.

3.4 Construction of All *m*-Elements Consistent System

In this section motivated by problem of Aczel [1] and Roberts [19] we provide a way of construction of all families of the sets $Z_1, \ldots, Z_m$ satisfying the conditions (3.8) and (3.9) from the paper [6].

We start from the following:

Definition 3.3 A system of sets $(Z_1, \ldots, Z_m)$ is called an m-elements consistent system if it satisfies the conditions (3.8) and (3.9).

Definition 3.4 We call a system of sets $Z_1, \ldots, Z_p \subset \mathbb{R}(n)$ can be extended to an m-elements consistent system ($p < m$), if there exists a system of sets $Z_{p+1}, \ldots, Z_m \subset \mathbb{R}(n)$ such that $(Z_1, \ldots, Z_m)$ is m-elements consistent system.
From now on, we assume that $p < m$. We have the following:

Theorem 3.11 *A system of sets $Z_1, \ldots, Z_p \subset \mathbb{R}(n)$ can be extended to an m-elements consistent system if and only if the sets $Z_1, \ldots, Z_p$ are cones over $\mathbb{Q}$ satisfying the condition (3.10) for every $k, l \in \{1, \ldots, p\}$ such that $k \neq l$ and the set $\mathbb{R}(n) \setminus$*

$\bigcup_{i=1}^{p} Z_i$ has a representation as a sum of system $A_{p+1}, \ldots, A_m$, which elements are pairwise disjoint cones over $\mathbb{Q}$. Then the system $(Z_1, \ldots, Z_p, A_{p+1}, \ldots, A_m)$ is m-elements consistent system extending the system $Z_1, \ldots, Z_p$.

Now we describe a way of construction of the set of all m-elements consistent systems extending the system $Z_1, \ldots, Z_p$.

Let a system $Z_1, \ldots, Z_p \subset \mathbb{R}(n)$ be such that it can be extended to an m-elements consistent system. We denote

$$U_p := \{C_p = (Z_1, \ldots, Z_p, A_{p+1}, \ldots, A_m) : A_{p+1}, \ldots, A_m \text{ are pairwise disjoint cones over } \mathbb{Q} \text{ such that } A_{p+1} \cup \cdots \cup A_m = \mathbb{R}(n) \setminus \bigcup_{i=1}^{p} Z_i\}.$$

For every $C_p = (Z_1, \ldots, Z_p, A_{p+1}, \ldots, A_m) \in U_p$ we construct corresponding sets

$$\begin{aligned} Z_{p+1}^{C_p} := \{Z_{p+1} :\ & Z_{p+1} \text{ is a cone over } \mathbb{Q} \text{ satisfying the following conditions:} \\ & A_{p+1} \subset Z_{p+1} \subset A_{p+1} \cup Z_1 \cup \ldots \cup Z_p, \\ & Z_k^1 + Z_k^1 \cap Z_{p+1}^0 \subset Z_k^1 \cap Z_{p+1}^0 \text{ and } Z_{p+1}^1 + Z_{p+1}^1 \cap Z_k^0 \subset Z_{p+1}^1 \cap Z_k^0 \\ & \text{for every } k \in \{1, \ldots, p\}\}, \\ U_{p+1}^{C_p} := \{&(Z_1, \ldots, Z_p, Z_{p+1}, A_{p+2}, \ldots, A_m) : Z_{p+1} \in Z_{p+1}^{C_p}\}, \\ U_{p+1} := \bigcup&_{C_p \in U_p} U_{p+1}^{C_p}. \end{aligned}$$

Each element of the set U_{p+1} is an m-elements consistent system and if $m-p > 1$, then satisfies the condition $A_{p+2} \cup \cdots \cup A_m = \mathbb{R}(n) \setminus \bigcup_{i=1}^{p+1} Z_i$. Proceeding in the analogous way, after $(m-p)$ steps, we can construct the set U_m, which each element is an m-elements consistent system extending the system $Z_1, \ldots, Z_p$.

From the above consideration we have the following:

Theorem 3.12 *Constructing in the above way the set U_m is the set of all m-elements consistent systems extending the system $Z_1, \ldots, Z_p$.*

Let us make the following definition:

Definition 3.5 A system $(A_1, \ldots, A_m)$ is called an m-elements basis system of $\mathbb{R}(n)$ if the sets $A_1, \ldots, A_m$ are pairwise disjoint cones over $\mathbb{Q}$, such that $A_1 \cup \cdots \cup A_m = \mathbb{R}(n)$.

We denote

$$\alpha := \{B = (A_1, \ldots, A_m) : A_1, \ldots, A_m \text{ is an } m\text{-elements basis system of } \mathbb{R}(n)\}.$$

We define an equivalence relation $\sim$ on the set α in the following way:

$$\forall_{B=(A_1,\ldots,A_m),B^*=(A_1^*,\ldots,A_m^*)\in\alpha} (B \sim B^* \Leftrightarrow A_1 = A_1^*).$$

Each equivalence class $[B]_\sim$, where $B = (A_1, \ldots, A_m) \in \alpha$ is the set of those elements belonging to the set α which first element equals A_1. By β we denote the set of all equivalence classes given an equivalence relation $\sim$ on α.

We notice that the equivalence class $[B]_\sim \in \beta$, where $B = (A_1, \ldots, A_m)$, is equal to the set U_1 constructing according to above description for one-element set A_1, i.e., the set of all m-elements consistent systems extending the system A_1, such that $m-1$ remaining cones are pairwise disjoint and their sum is equal to $\mathbb{R}(n) \setminus A_1$. By Theorem 3.12, for each equivalence class $[B = (A_1, \ldots, A_m)]_\sim \in \beta$ we can construct corresponding set $U_m([B]_\sim)$, which is the set of all m-elements consistent systems extending the system A_1. Hence we derive the following result:

Theorem 3.13 *A set*

$$U = \bigcup_{[B]_\sim \in \beta} U_m([B]_\sim)$$

is the set of all m-elements consistent systems of $\mathbb{R}(n)$.
The way of construction of all elements of the set α is unknown.

Denote by $\alpha^{\mathbb{R}} := \{(A_1, \ldots, A_m) \in \alpha$ such that $A_1, \ldots, A_m$ are cones over $\mathbb{R}\}$.

Theorem 3.14 *The set*

$$U^{\mathbb{R}} := \bigcup_{[B]_\sim \in \beta^{\mathbb{R}}} U_m([B]_\sim)$$

is the set of all m-elements consistent systems such that all their elements are cones over $\mathbb{R}$, *where* $\beta^{\mathbb{R}}$ *is the set of all equivalence classes given the equivalence relation* $\sim$ *on the set* $\alpha^{\mathbb{R}}$.
In [9, 10] was described the completely different way of construction of all m-elements consistent systems such that their elements are cones over $\mathbb{R}$ and $n = m$. Theorem 3.14 may be treated as a generalization of this construction for the case of n, m being arbitrarily chosen natural numbers, independent of each other.

3.5 On the Multifunctions Related to the Plurality Function

As a generalization, in [14] were considered the multifunctions $Z : T \to 2^G$, where T is an arbitrary non-empty set, $(G, +)$ is an arbitrary groupoid. The conditions (3.8) and (3.9) were replaced by

$$\bigcup_{t \in T} Z(t) = G, \tag{3.17}$$

and

$$(\exists_{t \in T} : i(t)j(t) \neq 0) \Rightarrow \bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} \subset \bigcap_{t \in T} Z(t)^{i(t)j(t)}, \tag{3.18}$$

respectively, where $Z(t)^1 := Z(t), Z(t)^0 := G \setminus Z(t)$, and $i(t), j(t) : T \to \{0, 1\}$ are arbitrary functions not identically equal to zero.

It is known that the multifunction $Z(t) : T \to 2^G$ fulfilling condition (3.17), satisfies condition (3.18) if and only if $Z(t)$ satisfies condition

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} \subset Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2},$$

for all $t_1, t_2 \in T$ and for all $k_1, k_2, l_1, l_2 \in \{0, 1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$.

Moreover, we have the following theorem (see [5]):

Theorem 3.15 *Let T be an arbitrary set with at least 2 elements and let the multifunction $Z(t) : T \to 2^{\mathbb{R}(n)}$ satisfy condition*

$$\bigcup_{t \in T} Z(t) = \mathbb{R}(n). \tag{3.19}$$

If the multifunction $Z(t)$ fulfils the system of conditional equations

$$(\exists_{t \in T} : i(t) j(t) \neq 0) \Rightarrow \bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} = \bigcap_{t \in T} Z(t)^{i(t) j(t)} \tag{3.20}$$

for the arbitrary functions $i(t), j(t) : T \to \{0, 1\}$ not identically equal to zero, then $Z(t)$ satisfies the system of equations

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} = Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2}, \tag{3.21}$$

for all $t_1, t_2 \in T$ and all $k_1, k_2, l_1, l_2 \in \{0, 1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$.

The converse of Theorem 3.15 for the set T with at least 2 elements is not true, and here is an example for $T = \{1, 2, 3\}$.

Let H be a Hamel base of the space $\mathbb{R}^n$, such that

$h_0 = (\sqrt{2}, 0, \dots, 0) \in \mathbb{R}(n)$,

$h_i = (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0) \in \mathbb{R}(n)$ for $i = 1, \dots, n$

belong to H.

Every $x \in \mathbb{R}^n$ has a representation, unique up to terms with coefficients zero

$$x = \sum_{l=0}^{k} q_l h_l,$$

where $q_l \in \mathbb{Q}$ and $h_l \in H$ for $l \in \{0, \dots, k\}$.

We define the multifunction $Z(t) : \{1, 2, 3\} \to 2^{\mathbb{R}(n)}$ in the following way:

$$Z(t) = \begin{cases} \{x \in \mathbb{R}(n) : q_o \geq 0\} & \text{for } t = 1, \\ \{x \in \mathbb{R}(n) : q_o = 0\} & \text{for } t = 2, \\ \{x \in \mathbb{R}(n) : q_o \leq 0\} & \text{for } t = 3. \end{cases}$$

It can be easily checked that the sets $Z(1)$, $Z(2)$, $Z(3)$ satisfy the conditions (3.19) and (3.21) for all $t_1, t_2 \in \{1, 2, 3\}$ and all $k_1, k_2, l_1, l_2 \in \{0, 1\}$ such that $k_1 l_1 + k_2 l_2 \neq 0$. The condition (3.20) is not satisfied because

$$\underset{\emptyset}{\underset{\|}{Z(1)^1 \cap Z(2)^1 \cap Z(3)^0}} + \underset{\emptyset}{\underset{\nparallel}{Z(1)^1 \cap Z(2)^0 \cap Z(3)^0}} \subsetneq \underset{\emptyset}{\underset{\nparallel}{Z(1)^1 \cap Z(2)^0 \cap Z(3)^0}}.$$

Acknowledgements The author would like to thank the referee for his helpful suggestions for improving the present paper.

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