Chapter 15
Stability of Systems of General Functional Equations in the Compact-Open Topology

Pavol Zlatoš

Abstract We introduce a fairly general concept of functional equation for k-tuples of functions $f_1, \ldots, f_k : X \to Y$ between arbitrary sets. The homomorphy equations for mappings between groups and other algebraic systems, as well as various types of functional equations and recursion formulas occurring in mathematical analysis or combinatorics, respectively, become special cases (of systems) of such equations. Assuming that X is a locally compact and Y is a completely regular topological space, we show that systems of such functional equations, with parameters satisfying rather a modest continuity condition, are stable in the following intuitive sense: Every k-tuple of "sufficiently continuous," "reasonably bounded" functions $X \to Y$ satisfying the given system with a "sufficient precision" on a "big enough" compact set is already "arbitrarily close" on an "arbitrarily big" compact set to a k-tuple of continuous functions solving the system. The result is derived as a consequence of certain intuitively appealing "almost-near" principle using the relation of infinitesimal nearness formulated in terms of nonstandard analysis.

Keywords System of functional equations • Continuous solution • Stability • Locally compact • Completely regular • Uniformity • Nonstandard analysis

Mathematics Subject Classification (2010) Primary 39B82; Secondary 39B72, 54D45, 54E15, 54J05

P. Zlatoš (✉)
Faculty of Mathematics, Physics and Informatics, Comenius University,
Mlynská dolina, 842 48 Bratislava, Slovakia
e-mail: zlatos@fmph.uniba.sk

J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer Optimization and Its Applications 124,
DOI 10.1007/978-3-319-61732-9_15

15.1 Introduction

The study of stability of functional equations in the spirit of Ulam started with examining the stability of additive functions and more generally of homomorphisms between metrizable topological groups, cf. [3, 13, 14, 19, 20, 27, 28]. Since that time it has developed to an established topic in mathematical and functional analysis and extended to a variety of (systems of) functional equations—see, e.g., [6, 9, 15, 21, 22, 26]. However, in most cases the stability issue was considered (explicitly or implicitly) either within the topology of uniform convergence or within the (strong) topology given by a norm on some functional space. On the other hand, especially when dealing with spaces of continuous functions defined on a locally compact space, the compact-open topology (i.e., the topology of uniform convergence on compact sets) is the most natural one. The systematic study of such local stability on compacts and its relation to the "usual" global or uniform stability was commenced by the author for homomorphisms between topological groups in [30] and extended to homomorphisms between topological universal algebras in [31]; cf. also [18, 24].

In the present paper we introduce a fairly general concept of functional equation for k-tuples of functions $f_1, \ldots, f_k : X \to Y$ between arbitrary sets. Then the homomorphy equations for mappings between groups and other algebraic systems, as well as various types of functional equations occurring in mathematical analysis (like, e.g., the sine and cosine addition formulas) or various recursion formulas occurring in combinatorics become just special cases (of systems) of such equations. Assuming that X is a locally compact and Y is a completely regular (i.e., uniformizable) topological space, we will show that systems of such functional equations, with functional parameters satisfying rather a modest continuity condition, are stable in the following intuitive sense, which will be made precise in the final Section 15.4 (Theorems 15.2, 15.3): Every k-tuple of "sufficiently continuous," "reasonably bounded" functions $X \to Y$ satisfying the given system with a "sufficient precision" on a "big enough" compact set is already "arbitrarily close" on an "arbitrarily big" compact set to a k-tuple of continuous functions solving the system. The result is a generalization comprising several former results by the author and his collaborators [24, 25, 29–31], as special cases. It is derived as a consequence of certain intuitively appealing stability or "almost-near" principle (in the sense of [2, 5]) using the relation of infinitesimal nearness formulated in terms of nonstandard analysis in Section 15.3 (Theorem 15.1, Corollary 15.2), generalizing a more specific principle of this kind from [24].

15.2 General Form of Functional Equations

Let X, Y be arbitrary nonempty sets and $k, m, n \geq 1$, $p \geq 0$ be integers. A k-tuple of functions $\boldsymbol{f} = (f_1, \ldots, f_k)$, $f_i : X \to Y$, is viewed as a single function $\boldsymbol{f} : X \to Y^k$. Further, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an m-tuple of p-ary operations $\alpha_j : X^p \to X$

(if $p = 0$, a nullary operation α on X is simply a constant $\alpha \in X$). We use the tensor product notation to denote the function $\boldsymbol{f} \otimes \boldsymbol{\alpha}: X^p \to Y^{k\times m}$ assigning to every p-tuple $\boldsymbol{x} = (x_1, \ldots, x_p) \in X^p$ the $k \times m$ matrix

$$(\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}) = \big((f_i \circ \alpha_j)(\boldsymbol{x})\big) = \begin{pmatrix} f_1\big(\alpha_1(\boldsymbol{x})\big) & \ldots & f_1\big(\alpha_m(\boldsymbol{x})\big) \\ \vdots & \ddots & \vdots \\ f_k\big(\alpha_1(\boldsymbol{x})\big) & \ldots & f_k\big(\alpha_m(\boldsymbol{x})\big) \end{pmatrix}.$$

In the trivial case when $k = m = 1$ we can identify $\boldsymbol{f} = f$, $\boldsymbol{\alpha} = \alpha$; then $\boldsymbol{f} \otimes \boldsymbol{\alpha}$ is just the composition of functions $f \circ \alpha: X^p \to Y$. If $m = 1$ and $\alpha(x) = x$ is the identity Id_X on X, then $\boldsymbol{f} \otimes \alpha = (f_1, \ldots, f_k) = \boldsymbol{f}$. If $m = p$ and $\boldsymbol{\alpha} = \boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$, where $\pi_j: X^m \to X$ is the jth projection, i.e., $\pi_j(x_1, \ldots, x_m) = x_j$, then $(\boldsymbol{f} \otimes \boldsymbol{\pi})(\boldsymbol{x}) = \big(f_i(x_j)\big) \in Y^{k\times m}$. In general, the function $\boldsymbol{f} \otimes \boldsymbol{\alpha}$ can be identified with the matrix of composed functions $f_i \circ \alpha_j: X^p \to Y$ ($i \le k, j \le m$).

Additionally, if $F: Y^{k\times m} \to Y$ is a $(k \times m)$-ary operation on Y, then $F(\boldsymbol{f} \otimes \boldsymbol{\alpha}) = F \circ (\boldsymbol{f} \otimes \boldsymbol{\alpha}): X^p \to Y$ denotes the function given by

$$F(f \otimes \boldsymbol{\alpha})(\boldsymbol{x}) = F\big((\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x})\big),$$

for $\boldsymbol{x} \in X^p$. More generally, for any mapping $F: Y^{k\times m} \times X^p \to Y$ we denote by $\widetilde{F}(\boldsymbol{f} \otimes \boldsymbol{\alpha}): X^p \to Y$ the function given by

$$\widetilde{F}(\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}) = F\big((\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big),$$

for $\boldsymbol{x} \in X^p$. Further on (except for some Examples) we will study exclusively the latter more general case which includes the former one, when the mapping F does not depend on $\boldsymbol{x}$, i.e., when $F(\boldsymbol{A}, \boldsymbol{x}) = F(\boldsymbol{A}, \boldsymbol{x'})$ for any matrix $\boldsymbol{A} \in Y^{k\times m}$ and all $\boldsymbol{x}, \boldsymbol{x'} \in X^p$.

A *general functional equation*, briefly a *GFE*, *of type* (k, m, n, p), with $k, m, n \ge 1, p \ge 0$, is a functional equation of the form

$$\widetilde{F}(\boldsymbol{f} \otimes \boldsymbol{\alpha}) = \widetilde{G}(\boldsymbol{f} \otimes \boldsymbol{\beta}), \tag{15.1}$$

where $\boldsymbol{f} = (f_1, \ldots, f_k)$ is a k-tuple of *functional variables* or "unknown" functions $f_i: X \to Y$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ is an m-tuple and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ is an n-tuple of p-ary operations on the set X, and, finally, $F: Y^{k\times m} \times X^p \to Y$ and $G: Y^{k\times n} \times X^p \to Y$ are any mappings. The operations (mappings) α_i, β_j, F, and G are called the *functional coefficients* or *parameters* of the equation. A k-tuple of functions $\boldsymbol{f} = (f_1, \ldots, f_k): X \to Y^k$ *satisfies* the GFE (15.1), or it is a *solution* of it, if the functions $\widetilde{F}(\boldsymbol{f} \otimes \boldsymbol{\alpha})$, $\widetilde{G}(\boldsymbol{f} \otimes \boldsymbol{\beta})$ coincide, i.e., if

$$F\big((\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big) = G\big((\boldsymbol{f} \otimes \boldsymbol{\beta})(\boldsymbol{x}), \boldsymbol{x}\big),$$

for all $\boldsymbol{x} \in X^p$. More generally, $\boldsymbol{f}$ *satisfies* the GFE (15.1) *on a set* $S \subseteq X^p$ if the above equation holds for each $\boldsymbol{x} \in S$; we say that $\boldsymbol{f}$ *satisfies* the GFE (15.1) *on a set* $A \subseteq X$ if it satisfies (15.1) on the set $A^p \subseteq X^p$.

A system of GFEs

$$\widetilde{F}_\lambda(\boldsymbol{f} \otimes \boldsymbol{\alpha}_\lambda) = \widetilde{G}_\lambda(\boldsymbol{f} \otimes \boldsymbol{\beta}_\lambda) \qquad (\lambda \in \Lambda)\,, \tag{15.2}$$

with (finite or infinite) index set $\Lambda \neq \emptyset$, consist of GFEs of particular types $(k, m_\lambda, n_\lambda, p_\lambda)$ (with k fixed and m_λ, n_λ, p_λ depending on $\lambda \in \Lambda$). Then $f = (f_1, \ldots, f_k)$ is a *solution of the system* if $\boldsymbol{f}$ satisfies all the equations in it. Satisfaction of the system on some set $A \subseteq X$ is defined in the obvious way.

We do not maintain that the (systems of) GFEs of the form just defined cover all the (systems of) functional equations one can meet, as such a claim would be too ambitious and, obviously, not founded well enough. In particular, functional equations dealing with compositions of functional variables $f_i \circ f_j$ or with iterated compositions like $f, f^2 = f \circ f, f^3 = f \circ f \circ f$, etc., do not fall under this scheme. On the other hand, as indicated by the examples below, they still comprise a large and representative variety of (systems of) functional equations studied so far.

Let us start with three closely related examples of algebraic nature.

Example 15.1 Let $(X, *)$, $(Y, \star)$ be two groupoids, i.e., algebraic structures with arbitrary binary operations $*$, $\star$ on the sets X and Y, respectively. Let $\alpha\colon X^2 \to X$ be the operation $\alpha(x_1, x_2) = x_1 * x_2$ on X, π_1, $\pi_2\colon X^2 \to X$ be the projections on the first and the second variable, respectively, $F = \mathrm{Id}_Y\colon Y \to Y$ be the identity mapping and $G\colon Y^2 \to Y$ be the operation $G(y_1, y_2) = y_1 \star y_2$ on Y. Then the GFE

$$F(\boldsymbol{f} \otimes \boldsymbol{\alpha}) = G(\boldsymbol{f} \otimes \boldsymbol{\pi})$$

of type $(1, 1, 2, 2)$, with $\boldsymbol{f} = f\colon X \to Y$, $\boldsymbol{\alpha} = \alpha$ and $\boldsymbol{\pi} = (\pi_1, \pi_2)$, which rewrites as

$$f \circ \alpha = G\big(f \otimes (\pi_1, \pi_2)\big)\,,$$

simply means that

$$f(x_1 * x_2) = f(x_1) \star f(x_2)$$

for all $x_2, x_2 \in X$. In other words, a function f satisfies the above GFE if and only if it is a homomorphism $f\colon (X, *) \to (Y, \star)$.

If both $(X, *)$, $(Y, \star)$ coincide with the additive group $(\mathbb{R}, +)$ of reals, we get the Cauchy functional equation

$$f(x + y) = f(x) + f(y)\,.$$

If $(X, *) = (\mathbb{R}, +)$ and $(Y, \star)$ is the multiplicative group $(\mathbb{R}^+, \cdot)$ of positive reals, we obtain the equation

$$f(x + y) = f(x)f(y)\,,$$

characterizing exponential functions. If both $(X, *)$, $(Y, \star)$ denote the set $\mathbb{R}$ with the arithmetical mean $x * y = x \star y = (x + y)/2$, we have Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \,.$$

And the list could be continued indefinitely.

Example 15.2 More generally, let Λ be a set of operation symbols with finite arities p_λ $(\lambda \in \Lambda)$, and $\mathfrak{X} = (X, \alpha_\lambda)_{\lambda\in\Lambda}$, $\mathfrak{Y} = (Y, G_\lambda)_{\lambda\in\Lambda}$ be two universal algebras of signature $(p_\lambda)_{\lambda\in\Lambda}$, i.e., $\alpha_\lambda = \lambda^{\mathfrak{X}}: X^{p_\lambda} \to X$, $G_\lambda = \lambda^{\mathfrak{Y}}: Y^{p_\lambda} \to Y$ are p_λ-ary operations on the sets X, Y, respectively, corresponding to the symbol $\lambda \in \Lambda$, cf. [10]. A function $f: X \to Y$ is called a *homomorphism* from $\mathfrak{X}$ to $\mathfrak{Y}$, briefly $f: \mathfrak{X} \to \mathfrak{Y}$, if for each $\lambda \in \Lambda$ and any p_λ-tuple $\boldsymbol{x} = (x_1, \dots, x_{p_\lambda}) \in X^{p_\lambda}$ we have

$$f\big(\alpha_\lambda(x_1, \dots, x_{p_\lambda})\big) = G_\lambda\big(f(x_1), \dots, f(x_{p_\lambda})\big) \,,$$

(for nullary operation symbols $\lambda \in \Lambda$ this simply means that $f(\alpha_\lambda) = G_\lambda$). Similarly as in the previous Example 15.1, we see immediately that this is the case if and only if f satisfies the system of GFEs

$$f \circ \alpha_\lambda = G_\lambda\big(f \otimes (\pi_1, \dots, \pi_{p_\lambda})\big) \qquad (\lambda \in \Lambda) \,,$$

of types $(1, 1, p_\lambda, p_\lambda)$, where $\pi_j: X^{p_\lambda} \to X$, $\pi_j(\boldsymbol{x}) = x_j$, is the jth projection for $j \le p_\lambda$.

Example 15.3 Let $(\Lambda, +, \cdot, 0, 1)$, be a ring with unit $1 \neq 0$. A (left) Λ-module X is an abelian group $(X, +)$ with scalar multiplication $\Lambda \times X \to X$, sending each pair $(\lambda, x) \in \Lambda \times X$ to the scalar multiple $\lambda x \in X$, satisfying the usual axioms. Then each scalar $\lambda \in \Lambda$ can be regarded as an endomorphism $\lambda^X: X \to X$ of the abelian group $(X, +)$, and the assignment $\lambda \mapsto \lambda^X$ becomes a homomorphism of rings $(\Lambda, +, \cdot, 0, 1) \to \big(\mathrm{End}(X, +), +, \circ, 0, \mathrm{Id}_X\big)$, cf. [12]. In particular, if Λ is a field, then a Λ-module is just a vector space over Λ.

A homomorphism of Λ-modules X, Y is a mapping $f: X \to Y$, preserving the addition and scalar multiplication, i.e., satisfying

$$f(x + y) = f(x) + f(y) \,,$$
$$f(\lambda x) = \lambda f(x)$$

for any $x, y \in X$, $\lambda \in \Lambda$. If Λ is a field, then this is the usual definition of a linear mapping between the vector spaces X, Y.

Regarding $\Lambda^+ = \{+\} \cup \Lambda$ as a set of operation symbols (+ binary, and each $\lambda \in \Lambda$ unary), every Λ-module is simply a universal algebra $\mathfrak{X} = (X, +, \lambda)_{\lambda\in\Lambda}$, satisfying the Λ-module axioms, and a Λ-module homomorphism is a homomorphism of such algebras. Now, the previous Example 15.2 applies, i.e., $f: X \to Y$ is a Λ-module homomorphism if and only if it satisfies the system of GFEs consisting of

$$f \circ \alpha = G\big(f \otimes (\pi_1, \pi_2)\big)\,,$$

where α is the addition in X and G is the addition in Y, and

$$f \circ \lambda^X = \lambda^Y \circ f \qquad (\lambda \in \Lambda)\,.$$

We continue with two examples of more analytic character.

Example 15.4 Let $\alpha: \mathbb{R}^2 \to \mathbb{R}$ be the addition on $\mathbb{R}$, $F_1 = \pi_1$, $F_2 = \pi_2: \mathbb{R}^2 \to \mathbb{R}$ denote the projections, and the functions $G_1, G_2: \mathbb{R}^{2\times 2} \to \mathbb{R}$ be given by

$$G_1\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \operatorname{Per}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + a_{21}a_{12}\,,$$

$$G_2\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \operatorname{Det}\begin{pmatrix} a_{21} & a_{12} \\ a_{11} & a_{22} \end{pmatrix} = a_{21}a_{22} - a_{11}a_{12}\,,$$

(notice the reversed order of elements in the first column of the determinant). Then the system of the following two GFEs, both of type $(2, 1, 2, 2)$, in the couple of functional variables $\boldsymbol{f} = (f_1, f_2)$, standing for the sine and cosine, respectively,

$$\pi_1(\boldsymbol{f} \otimes \alpha) = G_1\big(\boldsymbol{f} \otimes (\pi_1, \pi_2)\big)\,,$$

$$\pi_2(\boldsymbol{f} \otimes \alpha) = G_2\big(\boldsymbol{f} \otimes (\pi_1, \pi_2)\big)\,,$$

is nothing else but the well-known sine and cosine addition formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y\,,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y\,.$$

Example 15.5 Let $\sigma: \mathbb{C} \to \mathbb{C}$ denote the shift $\sigma(x) = x + 1$ and $G: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the multiplication $G(y, x) = yx$ on $\mathbb{C}$. Then the GFE

$$f \circ \sigma = \widetilde{G}(f)$$

of type $(1, 1, 1, 1)$ is the functional equation

$$f(x + 1) = f(x)\,x\,,$$

satisfied by the Euler function Γ on the open complex half plane $\{x \in \mathbb{C} \mid \operatorname{Re} x > 0\}$.

We conclude with two examples dealing with recursion in one and two variables.

Example 15.6 Let $\big(f(x)\big)_{x\in\mathbb{N}}$ be a sequence of elements of a set A, i.e., a function $f: \mathbb{N} \to A$, satisfying the recursion

$$f(x + n) = G\big(f(x), \ldots, f(x + n - 1)\big)$$

for a fixed $n \geq 1$ and an n-ary operation $G: A^n \to A$ given in advance. For each $j \in \mathbb{N}$ we denote by $\sigma^j: \mathbb{N} \to \mathbb{N}$ the shift $\sigma^j(x) = x + j$. Then the above recursion formula takes the form of the GFE

$$f \circ \sigma^n = G\big(f \otimes (\sigma^0, \dots, \sigma^{n-1})\big)$$

of type $(1, 1, n, 1)$. The more general recursion formula

$$f(x+n) = G\big(f(x), \dots, f(x+n-1), x\big),$$

where $G: A^n \times \mathbb{N} \to A$, takes the form of the GFE of type $(1, 1, n, 1)$

$$f \circ \sigma^n = \widetilde{G}\big(f \otimes (\sigma^0, \dots, \sigma^{n-1})\big).$$

Example 15.7 Let A be a set and $G: A^3 \times \mathbb{N}^2 \to A$ be an arbitrary mapping. Consider the following recursion formula:

$$f(x+1, y+1) = G\big(f(x,y), f(x+1,y), f(x,y+1), x, y\big),$$

expressing the value of a function $f: \mathbb{N}^2 \to A$ at $(x+1, y+1)$ in terms of its values at the preceding neighbors (x,y), $(x+1,y)$, $(x,y+1)$, and the position (x,y) itself. The notorious recursion formulas

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$
$$c(k+1, l+1) = c(k+1, l) + c(k, l+1),$$
$$s(n+1, k+1) = s(n,k) - n\, s(n, k+1),$$
$$S(n+1, k+1) = S(n,k) + (k+1) S(n, k+1),$$

for binomial coefficients (both in the usual form or for $c(k,l) = \binom{k+l}{k}$), as well as for Stirling numbers of the first and the second kind, respectively, are just some special cases of such functional equations for functions $f: \mathbb{N}^2 \to \mathbb{Z}$.

Let $\sigma_1, \sigma_2: \mathbb{N}^2 \to \mathbb{N}^2$ denote the shifts in the first and the second variable, respectively, i.e., $\sigma_1(x,y) = (x+1, y)$, $\sigma_2(x,y) = (x, y+1)$, and $\sigma_{12} = \sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1: \mathbb{N}^2 \to \mathbb{N}^2$ be the double shift, i.e., $\sigma_{12}(x,y) = (x+1, y+1)$. Then the original recursion formula can be written as the GFE

$$f \circ \sigma_{12} = \widetilde{G}\big(f \otimes (\sigma_0, \sigma_1, \sigma_2)\big),$$

with $\sigma_0 = \mathrm{Id}_{\mathbb{N}^2}: \mathbb{N}^2 \to \mathbb{N}^2$ denoting the identity. The generalization to recursion formulas for functions $f: \mathbb{N}^n \to A$ with $n \geq 2$ variables is straightforward.

15.3 Infinitesimal Nearness and S-Continuity

In this section we modify the short introduction to the nonstandard approach to continuity of mappings between topological groups from [24] to the more general situation of mappings between completely regular topological spaces. We use [8] as a reference source for general topology. In order to simplify our terminology, we assume that all (standard) topological or uniform spaces dealt with are Hausdorff.

The reader is assumed to have some basic acquaintance with nonstandard analysis in an extent covered either by the original Robinson's monograph [23] or, e.g., by Albeverio et al. [1], or Davis [7], or Arkeryd et al. [4], mainly in the parts [11] and [17]. In particular, some knowledge of the nonstandard approach to topology, based on the equivalence relation of infinitesimal nearness, is desirable.

Our exposition takes place in a nonstandard universe which is an elementary extension $^*\mathbf{V}$ of a superstructure $\mathbf{V}$ over some set of individuals containing at least all (classical) complex numbers and the elements of the universal algebras or topological spaces dealt with. In particular, this means that every standard universal algebra $\mathfrak{A} = (A, F_\lambda)_{\lambda\in\Lambda}$ is embedded into its nonstandard extension $^*\mathfrak{A} = (^*A, {}^*F_\lambda)_{\lambda\in\Lambda}$ via the canonic elementary embedding $a \mapsto {}^*a$, and identified with its image under *, in such a way that for any formula $\Phi(v_1, \dots, v_n)$ of the first-order language built upon the operation symbols $\lambda \in \Lambda$ and any $a_1, \dots, a_n \in A$ we have

$$\Phi(a_1, \dots, a_n) \text{ holds in } \mathfrak{A} \quad \text{if and only if} \quad {}^*\Phi(a_1, \dots, a_n) \text{ holds in } {}^*\mathfrak{A},$$

where $^*\Phi$ is the formula obtained from Φ by replacing each operation $F_\lambda\colon A^{p_\lambda} \to A$ by its extension $^*F_\lambda\colon {}^*A^{p_\lambda} \to {}^*A$. This rule is referred to as the *transfer principle*. However, this principle applies to any tuples of functions $\boldsymbol{f} = (f_1, \dots, f_k)\colon X \to Y^k$ and their nonstandard extensions $^*\boldsymbol{f} = (^*f_1, \dots, {}^*f_k)\colon {}^*X \to {}^*Y^k$, as well.

Objects belonging to the original universe are called *standard* and objects belonging to its nonstandard extension are called *internal*. Taking the advantage of the relation between the universes of standard and internal objects, we cannot avoid the so-called *external sets*, i.e., sets of internal objects, which themselves are not necessarily internal.

We assume that our nonstandard universe is κ*-saturated* for some sufficiently big uncountable cardinal κ, which will be specified later on. This is to say that any system of less than κ internal sets with the finite intersection property has itself nonempty intersection. Informally, we refer to this situation by the phrase that our nonstandard universe is *sufficiently saturated*. In a similar vein, a set of *admissible size* means a set of cardinality $< \kappa$.

If $(X, \mathscr{T})$ is a topological space, then the topology $\mathscr{T}$ (i.e., the system of open sets in X) gives rise to two different topologies on its nonstandard extension $^*X$.

The Q*-topology* is given by the base $^*\mathscr{T}$; it is Hausdorff if and only if the original topology $\mathscr{T}$ on X is Hausdorff. This topology plays rather an auxiliary role in our accounts.

The *S-topology* is given by the base $\{{}^*A \mid A \in \mathcal{T}\}$. Obviously, the S-topology is coarser than the Q-topology and it is not Hausdorff, unless $(X, \mathcal{T})$ is discrete.

We will systematically take advantage of the fact that if $(X, \mathcal{T})$ is a (Hausdorff) completely regular space, whose topology is induced by a uniformity $\mathcal{U}$ on X, then, in a sufficiently saturated nonstandard universe, the S-topology is fully determined by a single external equivalence relation

$$x \approx y \;\Leftrightarrow\; \forall\, U \in \mathcal{U} : (x, y) \in {}^*U\,,$$

called the *relation of infinitesimal nearness* on ${}^*X$. At the same time the system $\{{}^*U \mid U \in \mathcal{U}\}$ is a base of the *S-uniformity* on ${}^*X$. Uniform continuity with respect to it is referred to as the *uniform S-continuity*.

The external set of all elements indiscernible from $x \in {}^*X$ is called the *monad* of x, i.e.,

$$\mathrm{Mon}(x) = \{y \in {}^*X \mid y \approx x\}\,.$$

An element $x \in {}^*X$ is called *nearstandard* if $x \approx x_0$ for some $x_0 \in X$. The (external) set of all nearstandard elements in ${}^*X$ is denoted by $\mathrm{Ns}({}^*X)$, i.e.,

$$\mathrm{Ns}({}^*X) = \bigcup_{x \in X} \mathrm{Mon}(x)\,.$$

For $x \in \mathrm{Ns}({}^*X)$ we denote by ${}^\circ x$ the unique element $x_0 \in X$ infinitesimally close to x, called the *standard part* or *shadow* of x.

For the rest of this section $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ denote some completely regular topological spaces, whose topologies are induced by some uniformities $\mathcal{U}_X$, $\mathcal{U}_Y$, respectively, and ${}^*X$, ${}^*Y$ are their canonical extensions in a sufficiently saturated nonstandard universe; more precisely, we assume that our nonstandard universe is κ-saturated for some cardinal κ bigger than the cardinalities of some bases of the uniformities $\mathcal{U}_X$, $\mathcal{U}_Y$.

While the Q-continuity of internal functions $f: {}^*X \to {}^*Y$ is just the *continuity, their S-continuity can be characterized in the following intuitively appealing way in the spirit of the original infinitesimal calculus (below, we denote the relations of infinitesimal nearness on ${}^*X$, ${}^*Y$ by $\approx_X$, $\approx_Y$, respectively):

Proposition 15.1 *Let $f: {}^*X \to {}^*Y$ be an internal function. Then*

(a) *f is S-continuous in a point $x_0 \in {}^*X$ if and only if*

$$\forall\, x \in {}^*X : x \approx_X x_0 \;\Rightarrow\; f(x) \approx_Y f(x_0)\,;$$

(b) *f is S-continuous on a set $A \subseteq {}^*X$ (i.e., f is S-continuous in every point $a \in A$) if and only if*

$$\forall\, a \in A \;\; \forall\, x \in {}^*X : x \approx_X y \;\Rightarrow\; f(x) \approx_Y f(y)\,;$$

(c) *if $A \subseteq {}^*X$ is an intersection of admissibly many internal sets, then f is S-continuous on A if and only if f is uniformly S-continuous on A.*

In view of (a) and (b), S-continuity of an internal function $f\colon {}^*X \to {}^*Y$ can be alternatively *defined* as preservation of the relation of infinitesimal nearness by f. In particular, for the canonic extension ${}^*f\colon {}^*X \to {}^*Y$ of a standard function $f\colon X \to Y$ we have the following criteria (notice the subtle difference between (b) and (c)).

Corollary 15.1 *Let $f\colon X \to Y$ be a function. Then*

(a) *f is continuous in a point $x_0 \in X$ if and only if*

$$\forall\, x \in {}^*X : x \approx_X x_0 \;\Rightarrow\; {}^*f(x) \approx_Y f(x_0)\,;$$

(b) *f is continuous on a set $A \subseteq X$ (i.e., f is continuous in every point $a \in A$) if and only if*

$$\forall\, a \in A \;\; \forall\, x \in {}^*X : x \approx_X a \;\Rightarrow\; {}^*f(x) \approx_Y f(a)\,;$$

(c) *f is uniformly continuous on a set $A \subseteq X$ if and only if*

$$\forall\, x, y \in {}^*A : x \approx_X y \;\Rightarrow\; {}^*f(x) \approx_Y {}^*f(y)\,.$$

Notice that under the assumption of (b), ${}^*f$ is Q-continuous on ${}^*A$, as well.

An internal function $f\colon {}^*X \to {}^*Y$ is called *nearstandard* if $f(x) \in \mathrm{Ns}({}^*Y)$ for each $x \in X$. Let us remark that this is indeed equivalent to f be a nearstandard point in the nonstandard extension ${}^*\big(Y^X\big)$ of the Tikhonov product $Y^X = \{f \mid f\colon X \to Y\}$. Any nearstandard function $f\colon {}^*X \to {}^*Y$ gives rise to a function ${}^\circ f\colon X \to Y$ given by

$$({}^\circ f)(x) = {}^\circ(f(x))\,,$$

for $x \in X$, called the *standard part* of f. If f is additionally S-continuous on $\mathrm{Ns}({}^*X)$, then its standard part can be extended to a map ${}^\circ f\colon \mathrm{Ns}({}^*X) \to Y$ (denoted in the same way), such that

$${}^\circ f(x) = {}^\circ f({}^\circ x) = {}^\circ(f(x))$$

for any $x \in \mathrm{Ns}({}^*X)$. The situation can be depicted by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ns}({}^*X) & \xrightarrow{\;f\;} & \mathrm{Ns}({}^*Y) \\ {\scriptstyle\circ}\big\downarrow & & \big\downarrow{\scriptstyle\circ} \\ X & \xrightarrow[{}^\circ f]{} & Y \end{array}$$

A function $f: {}^*X \to {}^*Y$ is called *NS-continuous* if it is S-continuous on $\mathrm{Ns}({}^*X)$. Now we have the following supplement to Proposition 15.1 and its Corollary 15.1.

Proposition 15.2 *Let $f: {}^*X \to {}^*Y$ be a nearstandard internal function. Then the following implications hold:*

(a) *if f is NS-continuous, then its standard part ${}^\circ f: X \to Y$ is continuous and ${}^*({}^\circ f)(x) \approx_Y f(x)$ for $x \in \mathrm{Ns}({}^*X)$;*
(b) *if f is S-continuous on some internal set $A \supseteq \mathrm{Ns}({}^*X)$, then its standard part ${}^\circ f: X \to Y$ is uniformly continuous.*

Notice that the function ${}^*({}^\circ f)$ is also Q-continuous. However, even if f were S-continuous on the whole of ${}^*X$, the second conclusion in (a) still cannot be strengthened to ${}^*({}^\circ f)(x) \approx_Y f(x)$ for all $x \in {}^*X$.

Proof We will prove just the first statement in (a); then the second statement easily follows and (b) can be proved in a similar way.

Assume that f is *NS*-continuous and denote $g = {}^\circ f: X \to Y$ its standard part. In order to prove the continuity of g, pick an arbitrary $x_0 \in X$ and $V \in \mathscr{U}_Y$. Let $W \in \mathscr{U}_Y$ be symmetric, such that $W^3 = W \circ W \circ W \subseteq V$. As f is internal and *NS*-continuous, it is also continuous in x_0 with respect to the S-topology on ${}^*X$, hence there is a $U \in \mathscr{U}_X$ such that $(x, x_0) \in {}^*U$ implies $\big(f(x), f(x_0)\big) \in {}^*W$ for any $x \in {}^*X$. In particular, for $x \in X$ such that $(x, x_0) \in U$, we have $g(x) \approx_Y f(x)$, $\big(f(x), f(x_0)\big) \in {}^*U$, as well as $f(x_0) \approx_Y g(x_0)$, hence $\big(g(x), g(x_0)\big) \in {}^*W^3 \subseteq {}^*V$. Since $g(x), g(x_0) \in Y$, by transfer principle $\big(g(x), g(x_0)\big) \in V$.

Let us conclude this section with a remark that the introduced continuity notions can be easily generalized to tuples of functions $\boldsymbol{f} = (f_1, \ldots, f_k): X \to Y^k$. As well known, $\boldsymbol{f}$ has whatever standard continuity property if and only if all the functions f_i have this property. The relation of infinitesimal nearness $\approx_Y$ can be extended to ${}^*Y^k$ by

$$y \approx_Y z \;\Leftrightarrow\; y_1 \approx_Y z_1 \;\&\; \ldots \;\&\; y_k \approx_Y z_k\,,$$

(similarly, $\approx_X$ can be extended to ${}^*X^p$). Then an internal function $\boldsymbol{f}: {}^*X \to {}^*Y^k$ is nearstandard if and only if all the functions f_i are nearstandard; $\boldsymbol{f}$ has anyone of the S-continuity properties if and only if all the functions f_i have the corresponding property. If $\boldsymbol{f}$ is nearstandard, then the k-tuple ${}^\circ\boldsymbol{f} = \big({}^\circ f_1, \ldots, {}^\circ f_k\big)$ of functions ${}^\circ f_i: X \to Y$ is called the *standard part* of $\boldsymbol{f}$.

15.4 An Infinitesimal "Almost-Near" Principle for Systems of General Functional Equations

Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$ be two completely regular topological spaces with topologies induced by some uniformities $\mathscr{U}_X$, $\mathscr{U}_Y$, respectively. If there is no danger of confusion, we omit the subscripts X, Y in the notation of the relations of infinitesimal nearness $\approx_X$, $\approx_Y$ on ${}^*X$, ${}^*Y$, respectively.

Consider the GFE (15.1) for a k-tuple of functional variables $\boldsymbol{f} = (f_1, \ldots, f_k)$. Embedding the situation into some nonstandard universe we say that an internal function $\boldsymbol{f} = (f_1, \ldots, f_k)\colon {}^*X \to {}^*Y^k$, *almost satisfies* Equation (15.1) *on* $\mathrm{Ns}({}^*X)$ if

$$^*\widetilde{F}(\boldsymbol{f} \otimes {}^*\boldsymbol{\alpha})(\boldsymbol{x}) \approx {}^*\widetilde{G}(\boldsymbol{f} \otimes {}^*\boldsymbol{\beta})(\boldsymbol{x})$$

for all $\boldsymbol{x} = (x_1 \ldots, x_p) \in \mathrm{Ns}({}^*X^p)$. Similarly, $\boldsymbol{f}$ *almost satisfies* the system of GFEs (15.2) *on* $\mathrm{Ns}({}^*X)$ if it almost satisfies on $\mathrm{Ns}({}^*X)$ every equation in it. (Notice that, due to the transfer principle, ${}^*\big(\widetilde{F}\big) = \widetilde{{}^*F}$, and similarly for G, hence the notation ${}^*\widetilde{F}$, ${}^*\widetilde{G}$ is unambiguous.)

Theorem 15.1 *Let the mappings* $F\colon Y^{k\times m} \times X^p \to Y$, $G\colon Y^{k\times n} \times X^p \to Y$ *be continuous in the "matrix" variables* $y_{ij} \in Y$ *for all* $i \le k$ *and* $j \le m, n$, *respectively. If a nearstandard internal function* $\boldsymbol{f} = (f_1, \ldots, f_k)\colon {}^*X \to {}^*Y^k$ *almost satisfies the GFE* (15.1) *on* $\mathrm{Ns}({}^*X)$, *then its standard part* ${}^\circ f = \big({}^\circ f_1, \ldots, {}^\circ f_k\big)$ *is a solution of the GFE* (15.1).

Proof Take an arbitrary $\boldsymbol{x} = (x_1, \ldots, x_p) \in X^p$. We have

$$^*\widetilde{F}(\boldsymbol{f} \otimes {}^*\boldsymbol{\alpha})(\boldsymbol{x}) \approx {}^*\widetilde{G}(\boldsymbol{f} \otimes {}^*\boldsymbol{\beta})(\boldsymbol{x})\,.$$

As $\boldsymbol{x}$ is standard, ${}^*\boldsymbol{\alpha}(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{x})$ is standard, as well, hence $f_i\big(\alpha_j(\boldsymbol{x})\big) \approx {}^\circ f_i\big(\alpha_j(\boldsymbol{x})\big)$ for any $i \le k, j \le m$, and, as ${}^*F$ is *NS*-continuous in the matrix variables y_{ij},

$$\begin{aligned} {}^*\widetilde{F}(\boldsymbol{f} \otimes {}^*\boldsymbol{\alpha})(\boldsymbol{x}) &= {}^*F\big((\boldsymbol{f} \otimes {}^*\boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big) = {}^*F\big((\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big) \\ &\approx {}^*F\big(({}^\circ f \otimes \boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big) = F\big(({}^\circ f \otimes \boldsymbol{\alpha})(\boldsymbol{x}), \boldsymbol{x}\big) = \widetilde{F}({}^\circ f \otimes \boldsymbol{\alpha})(\boldsymbol{x})\,. \end{aligned}$$

Similarly we can get

$$^*\widetilde{G}(\boldsymbol{f} \otimes {}^*\boldsymbol{\beta})(\boldsymbol{x}) \approx \widetilde{G}({}^\circ f \otimes \boldsymbol{\beta})(\boldsymbol{x})\,.$$

Therefore,

$$\widetilde{F}({}^\circ f \otimes \boldsymbol{\alpha})(\boldsymbol{x}) \approx \widetilde{G}({}^\circ f \otimes \boldsymbol{\beta})(\boldsymbol{x})\,,$$

and, as both the expressions are standard,

$$\widetilde{F}({}^\circ f \otimes \boldsymbol{\alpha})(\boldsymbol{x}) = \widetilde{G}({}^\circ f \otimes \boldsymbol{\beta})(\boldsymbol{x})\,,$$

i.e., ${}^\circ f$ is a solution of the GFE (15.1).

From Theorem 15.1 and Proposition 15.2 (b) we readily obtain the following consequence generalizing Theorem 2.2 from [24], dealing just with the homomorphy equation in topological groups.

Corollary 15.2 *Assume that F, G are continuous in the matrix variables y_{ij}. Then for every nearstandard NS-continuous internal function $\boldsymbol{f} = (f_1, \dots, f_k) \colon {}^*X \to {}^*Y^k$ which almost satisfies the system of GFEs* (15.2) *on* $\mathrm{Ns}({}^*X)$*, there is a continuous solution $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_k)$ of the system, such that $\boldsymbol{\varphi}(x) \approx \boldsymbol{f}(x)$ for each $x \in X$.*

15.5 Stability of Systems of General Functional Equations

In order to formulate a standard version of the just established nonstandard stability principle, we need to introduce some notions—cf. [24, 30, 31].

Definition 15.1 Let $(X, \mathscr{T}_X)$ be a topological space and $(Y, \mathscr{U}_Y)$ be a uniform space.

(a) A $(\mathscr{T}_X, \mathscr{U}_Y)$ *continuity scale* is a mapping $\Gamma \colon X \times \mathscr{B} \to \mathscr{T}_X$, such that $\mathscr{B}$ is a base of the uniformity $\mathscr{U}_Y$ and $\Gamma(x, V)$ is a neighborhood of x in $(X, \mathscr{T}_X)$, satisfying

$$V \subseteq W \;\Rightarrow\; \Gamma(x, V) \subseteq \Gamma(x, W)$$

for any $x \in X$, and $V, W \in \mathscr{B}$.

(b) Given a continuity scale $\Gamma \colon X \times \mathscr{B} \to \mathscr{T}_X$, a function $f \colon X \to Y$ is called *Γ-continuous in a point $x_0 \in X$*, or *continuous in x_0 with respect to Γ*, if

$$x \in \Gamma(x_0, V) \;\Rightarrow\; \big(f(x), f(x_0)\big) \in V$$

for each $x \in X$; f is *Γ-continuous on a set $A \subseteq X$* if it is Γ-continuous in each point $a \in A$; it is *Γ-continuous* if it is Γ-continuous on X.

(c) Given a continuity scale $\Gamma \colon X \times \mathscr{B} \to \mathscr{T}_X$ and an entourage $U \in \mathscr{B}$, a function $f \colon X \to Y$ is *(Γ, U)-precontinuous in a point $x_0 \in X$* if

$$x \in \Gamma(x_0, V) \;\Rightarrow\; \big(f(x), f(x_0)\big) \in V$$

for any $V \in \mathscr{B}$, such that $U \subseteq V$, and each $x \in X$; (Γ, U)-precontinuity on a set $A \subseteq X$ and on X are defined in the obvious way.

(d) If $(X, \mathscr{U}_X)$ is a uniform space, too, then a $(\mathscr{U}_X, \mathscr{U}_Y)$ *uniform continuity scale* is a mapping $\Gamma \colon \mathscr{B} \to \mathscr{U}_X$ such that $\mathscr{B}$ is some base of the uniformity $\mathscr{U}_Y$ and

$$V \subseteq W \;\Rightarrow\; \Gamma(V) \subseteq \Gamma(W)$$

for any $V, W \in \mathscr{B}$.

(e) Given a uniform continuity scale $\Gamma \colon \mathscr{B} \to \mathscr{U}_X$, a function $f \colon X \to Y$ is *uniformly Γ-continuous on a set $A \subseteq X$* if

$$(x, y) \in \Gamma(V) \;\Rightarrow\; \big(f(x), f(y)\big) \in V$$

for any $x, y \in A$; f is *uniformly Γ-continuous* if it is uniformly Γ-continuous on X.

(f) Given a uniform continuity scale $\Gamma: \mathscr{B} \to \mathscr{U}_X$ and an entourage $U \in \mathscr{B}$, a function $f: X \to Y$ is *uniformly (Γ, U)-precontinuous on a set $A \subseteq X$* if

$$(x, y) \in \Gamma(V) \;\Rightarrow\; \big(f(x), f(y)\big) \in V$$

for any $V \in \mathscr{B}$, such that $U \subseteq V$ and all $x, y \in A$; f is *uniformly (Γ, U)-precontinuous* if it is (Γ, U)-precontinuous on X.

Obviously, if a function $f: X \to Y$ is Γ-continuous with respect to some continuity scale Γ, then it is continuous. Conversely, if f is continuous, then, given any base $\mathscr{B}$ of $\mathscr{U}_Y$, the assignment

$$\Gamma(x_0, V) = \big\{x \in X \;\big|\; \big(f(x), f(x_0)\big) \in V\big\},$$

for $x_0 \in X$, $V \in \mathscr{B}$, defines a $(\mathscr{T}_X, \mathscr{U}_Y)$ continuity scale $\Gamma: X \times \mathscr{B} \to \mathscr{T}_X$, and, of course, f is continuous with respect to it.

The other way round, f is Γ-continuous if and only if it is (Γ, U)-precontinuous for all $U \in \mathscr{B}$. Thus each particular condition of (Γ, U)-precontinuity for an entourage $U \in \mathscr{U}_Y$ can be regarded as an approximate continuity property. Informally, f is "almost Γ-continuous" if it is (Γ, U)-precontinuous for a "sufficiently small" $U \in \mathscr{B}$. The relation between the uniform versions of these notions is similar.

If (X, d), (Y, e) are metric spaces, then a (d, e)-continuity scale is just a mapping $\gamma: X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(x, \epsilon) \le \gamma(x, \epsilon')$ for any $x \in X$, $\epsilon' \ge \epsilon > 0$. Then a function $f: X \to Y$ is γ-continuous in $x_0 \in X$ if

$$d(x, x_0) < \gamma(x_0, \epsilon) \;\Rightarrow\; e\big(f(x), f(x_0)\big) < \epsilon$$

for all $\epsilon > 0$ and $x \in X$. A uniform (d, e)-continuity scale is an isotone mapping $\gamma: \mathbb{R}^+ \to \mathbb{R}^+$. A function $f: X \to Y$ is uniformly γ-continuous if

$$d(x, y) < \gamma(\epsilon) \;\Rightarrow\; e\big(f(x), f(y)\big) < \epsilon$$

for all $\epsilon > 0$ and $x, y \in X$.

Definition 15.2 Let X, Y be arbitrary sets.

(a) A *bounding relation* from X to Y is any binary relation $R \subseteq X \times Y$ such that all its stalks $R[x] = \{y \in Y \mid (x, y) \in R\}$, for $x \in X$, are nonempty.

(b) Given a bounding relation $R \subseteq X \times Y$, a function $f: X \to Y$ is *R-bounded on a set $A \subseteq X$* if $f(a) \in R[a]$ for each $a \in A$; f is *R-bounded* if it is R-bounded on X, i.e., if $f \subseteq R$.

(c) A bounding relation $R \subseteq X \times Y$ is *stalkwise finite* if all its stalks $R[x]$ are finite. If, additionally, $(Y, \mathscr{T}_Y)$ is a topological space, then R is called *stalkwise compact* if all its stalks $R[x]$ are compact.

Definition 15.3 Let X be any set, $(Y, \mathscr{U}_Y)$ be a uniform space and $V \in \mathscr{U}_Y$.

(a) Two functions $f, g\colon X \to Y$ are *V-close on a set $A \subseteq X$* if $\big(f(a), g(a)\big) \in V$ for all $a \in A$.
(b) A k-tuple $\boldsymbol{f} = (f_1, \dots, f_k)$ of functions $f_i\colon X \to Y$ is a *V-solution* of the GFE (15.1) *on a set $S \subseteq X^p$* if

$$\big(\widetilde{F}(\boldsymbol{f} \otimes \boldsymbol{\alpha})(\boldsymbol{x}),\ \widetilde{G}(\boldsymbol{f} \otimes \boldsymbol{\beta})(\boldsymbol{x})\big) \in V$$

for all $\boldsymbol{x} \in S$; $\boldsymbol{f}$ is a *V-solution* the GFE (15.1) *on a set $A \subseteq X$* if it is its V-solution on A^p; $\boldsymbol{f}$ is a *V-solution* of the system of GFEs (15.2) on $A \subseteq X$ if it is a V-solution of every equation in the system on A.

For brevity's sake we say that a function $\boldsymbol{f} = (f_1, \dots, f_k)\colon X \to Y^k$ has any of the just introduced Γ-continuity properties if and only if each particular function f_i has the corresponding property. Similarly, $\boldsymbol{f}$ is R-bounded (on a set $A \subseteq X$) if and only each function f_i is R-bounded. We say that two such functions $\boldsymbol{f}, \boldsymbol{g}\colon X \to Y^k$ are V-close on $A \subseteq X$ if f_i, g_i are V-close on A for each $i \le k$.

The system of all nonempty compact sets of a topological space $(X, \mathscr{T}_X)$ is denoted by $\mathscr{K}(X)$.

Theorem 15.2 *Let $(X, \mathscr{T}_X)$ be a locally compact topological space, $(Y, \mathscr{U}_Y)$ be a uniform space, $\Gamma\colon X \times \mathscr{B} \to \mathscr{T}_X$ be a $(\mathscr{T}_X, \mathscr{U}_Y)$ continuity scale, and $R \subseteq X \times Y$ be a stalkwise compact bounding relation. Assume that all the functional coefficients $F_\lambda\colon Y^{k\times m_\lambda} \times X^{p_\lambda} \to Y$, $G_\lambda\colon Y^{k\times n_\lambda} \times X^{p_\lambda} \to Y$ in the system of GFEs (15.2) are continuous in the matrix variables y_{ij}. Then for each pair $D \in \mathscr{K}(X)$, $V \in \mathscr{U}_Y$ there exists a pair $C \in \mathscr{K}(X)$, $U \in \mathscr{U}_Y$ such that $D \subseteq C$ and the following implication holds true:*

If a U-solution $\boldsymbol{f} = (f_1, \dots, f_k)\colon X \to Y^k$ of the system (15.2) on C is both (Γ, U)-precontinuous and R-bounded on C, then there exists a continuous solution $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_k)$ of the system, such that $\boldsymbol{f}$, $\boldsymbol{\varphi}$ are V-close on D.

Proof Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{U}_Y)$, $\Gamma\colon X \times \mathscr{B} \to \mathscr{T}_X$, $R \subseteq X \times Y$, as well as the system of GFEs (15.2) satisfy the assumptions of the theorem. Then $(X, \mathscr{T}_X)$ is completely regular, as well, hence its topology is induced by some uniformity $\mathscr{U}_X$. Admit, in order to obtain a contradiction, that there is a pair $D \in \mathscr{K}(X)$, $V \in \mathscr{U}_Y$ for which the conclusion of the theorem fails. For each pair $C \in \mathscr{K}(X)$, $U \in \mathscr{B}$ such that $C \supseteq D$ we denote by $\mathscr{F}(C, U)$ the set of all U-solutions $\boldsymbol{f} = (f_1, \dots, f_k)\colon X \to Y^k$ of the system of GFEs (15.2) on C which are both (Γ, U)-precontinuous and R-bounded on C, nonetheless, $\boldsymbol{f}$ is not V-close on D to any continuous solution $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_k)$ of the system (15.2). According to our assumption, all the sets $\mathscr{F}(C, U)$ are nonempty, and, for all $C, C' \in \mathscr{K}(X)$, $U, U' \in \mathscr{B}$, we obviously have

$$D \subseteq C \subseteq C' \,\&\, U' \subseteq U \ \Rightarrow\ \mathscr{F}(C', U') \subseteq \mathscr{F}(C, U)\,.$$

Let us embed the situation into a sufficiently saturated nonstandard universe. More precisely, we assume that it is κ-saturated for some uncountable cardinal κ

such that card $\mathscr{B} < \kappa$, as well as card $\mathscr{C} < \kappa$ for some cofinal subset $\mathscr{C} \subseteq \mathscr{K}(X)$ such that $D \subseteq C$ for each $C \in \mathscr{C}$. Then

$$\bigcap_{C \in \mathscr{C},\, U \in \mathscr{B}} {}^*\mathscr{F}(C, U) \neq \emptyset\,.$$

Let $\boldsymbol{f} = (f_1, \dots, f_k)$ belong to this intersection. Then $\boldsymbol{f}\colon {}^*X \to {}^*Y^k$ is an internal function, for all $U \in \mathscr{U}_Y$, $C \in \mathscr{C}$, $\boldsymbol{f}$ is ${}^*(\Gamma, U)$-precontinuous and ${}^*R$-bounded on ${}^*C$ and it satisfies

$$\left({}^*\widetilde{F}_\lambda(\boldsymbol{f} \otimes {}^*\boldsymbol{\alpha}_\lambda)(\boldsymbol{x}),\, {}^*\widetilde{G}_\lambda(\boldsymbol{f} \otimes {}^*\boldsymbol{\beta}_\lambda)(\boldsymbol{x})\right) \in {}^*U$$

for any $\lambda \in \Lambda$ and $\boldsymbol{x} \in {}^*C^{p_\lambda}$. Since X is locally compact, $\mathrm{Ns}({}^*X) = \bigcup_{C \in \mathscr{C}} {}^*C$. It follows that $\boldsymbol{f}$ is *NS*-continuous and almost satisfies the system (15.2) on $\mathrm{Ns}({}^*X)$. Finally, $\boldsymbol{f}(x) \in \left({}^*R[x]\right)^k$ for any $C \in \mathscr{C}$ and $x \in {}^*C$. As $R[x]$ is compact for $x \in X$, in that case we have $\boldsymbol{f}(x) \in \left({}^*R[x]\right)^k \subseteq \mathrm{Ns}\left({}^*Y^k\right)$. Thus $\boldsymbol{f}$ is nearstandard. According to Corollary 15.2, there is a continuous solution $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_k)$ of the system (15.2), such that $\boldsymbol{f}(x) \approx \boldsymbol{\varphi}(x)$ each $x \in X$. On the other hand, ${}^*\boldsymbol{\varphi}$ is Q-continuous (i.e., *continuous), hence $\boldsymbol{f}$ and ${}^*\boldsymbol{\varphi}$ cannot be ${}^*V$-close on ${}^*D$. Thus there are an $i \le k$ and an $x \in {}^*D$ such that $\left(f_i(x), {}^*\varphi_i(x)\right) \notin {}^*V$. However, as D is compact, ${}^*D \subseteq \mathrm{Ns}({}^*X)$. Since both f_i and ${}^*\varphi_i$ are *NS*-continuous, taking an $x_0 \in X$ such that $x \approx x_0$, we obtain

$${}^*\varphi_i(x) \approx \varphi_i(x_0) \approx f_i(x_0) \approx f_i(x)\,,$$

i.e., a contradiction.

Like in Theorem 15.2, we assume in the next three Corollaries that all the mappings F_λ, G_λ in the system of GFEs (15.2) are continuous in the matrix variables y_{ij} (but, for brevity's sake, we do not mention that explicitly). In the fourth Corollary 15.6 this assumption is superfluous as it is satisfied automatically.

If $(Y, \mathscr{U}_Y)$ is compact, then $R = X \times Y$ is a stalkwise compact bounding relation such that every function $\boldsymbol{f}\colon X \to Y^k$ is R-bounded. This makes possible to avoid mentioning any bounding relation in the formulation of Theorem 15.2.

Corollary 15.3 *Let $(X, \mathscr{T}_X)$ be a locally compact topological space, $(Y, \mathscr{U}_Y)$ be a compact uniform space, and $\Gamma\colon X \times \mathscr{B} \to \mathscr{T}_X$ be a $(\mathscr{T}_X, \mathscr{U}_Y)$ continuity scale. Then for each pair $D \in \mathscr{K}(X)$, $V \in \mathscr{U}_Y$ there is a pair $C \in \mathscr{K}(X)$, $U \in \mathscr{U}_Y$ such that $D \subseteq C$ and the following implication holds true:*

If a U-solution $\boldsymbol{f} = (f_1, \dots, f_k)\colon X \to Y^k$ of the system of GFEs (15.2) *on C is (Γ, U)-precontinuous on C, then there exists a continuous solution $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_k)$ of the system, such that $\boldsymbol{f}$, $\boldsymbol{\varphi}$ are V-close on D.*

If $(X, \mathscr{T}_X)$ is compact, then its topology is induced by a unique uniformity $\mathscr{U}_X$ and, at the same time, it is enough to control the continuity of functions $\boldsymbol{f}\colon X \to Y^k$ by means of a uniform continuity scale. Choosing $D = X$ we get the following *global* version of Theorem 15.2.

Corollary 15.4 *Let $(X, \mathscr{U}_X)$ be a compact and $(Y, \mathscr{U}_Y)$ be an arbitrary uniform space, $\Gamma: \mathscr{B} \to \mathscr{U}_X$ be a $(\mathscr{U}_X, \mathscr{U}_Y)$ uniform continuity scale and $R \subseteq X \times Y$ be a stalkwise compact bounding relation. Then for each $V \in \mathscr{U}_Y$ there is a $U \in \mathscr{U}_Y$ such that the following implication holds true:*

If $\boldsymbol{f} = (f_1, \ldots, f_k): X \to Y^k$ is a uniformly (Γ, U)-precontinuous and R-bounded U-solution of the system of GFEs (15.2), then there exists a (uniformly) continuous solution $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$ of the system, such that $\boldsymbol{f}$, $\boldsymbol{\varphi}$ are V-close on X.

Under the assumptions of both Corollaries 15.3 and 15.4 we have

Corollary 15.5 *Let $(X, \mathscr{U}_X)$, $(Y, \mathscr{U}_Y)$ be compact uniform spaces and $\Gamma: \mathscr{B} \to \mathscr{U}_X$ be a $(\mathscr{U}_X, \mathscr{U}_Y)$ uniform continuity scale. Then for each $V \in \mathscr{U}_Y$ there is a $U \in \mathscr{U}_Y$ such that the following implication holds true:*

If $\boldsymbol{f} = (f_1, \ldots, f_k): X \to Y^k$ is a uniformly (Γ, U)-precontinuous U-solution of the system of GFEs (15.2), then there exists a (uniformly) continuous solution $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$ of the system, such that $\boldsymbol{f}$, $\boldsymbol{\varphi}$ are V-close on X.

The interested reader can easily formulate the metric versions of Theorem 15.2, as well as of Corollaries 15.3–15.5.

Endowing both the sets X, Y with discrete topologies (uniformities), all the functions $X \to Y$ become (uniformly) continuous. Then compact subsets of X are just the finite ones and, similarly, a stalkwise compact bounding relation $R \subseteq X \times Y$ is a stalkwise finite one. In that case, choosing $U = \mathrm{Id}_Y$ in Theorem 15.2, we obtain the following result on extendability of functions satisfying a system of GFEs (15.2) on some finite set to its (global) solutions.

Corollary 15.6 *Let X and Y be arbitrary sets and $R \subseteq X \times Y$ be a stalkwise finite bounding relation. Then for each finite set $D \subseteq X$ there is a finite set $C \subseteq X$ such that $D \subseteq C$ and for every R-bounded partial solution $\boldsymbol{f} = (f_1, \ldots, f_k): X \to Y^k$ of the system of GFEs (15.2) on C there exists a solution $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$ of the system, such that $\boldsymbol{\varphi}(x) = \boldsymbol{f}(x)$ for all $x \in D$.*

If the arity numbers p_λ in the system of GFEs (15.2) have a common upper bound p, then all the particular equations in the system can be considered as being of types $(k, m_\lambda, n_\lambda, p)$. In such a case, given a $U \in \mathscr{U}_Y$, we say that a function $\boldsymbol{f}: X \to Y^k$ is a *U-solution* of the system (15.2) *on a set $S \subseteq X^p$* if it is a U-solution of each its particular equation on S. Then we have the following variant of Theorem 15.2. Its proof can be obtained by slight modifications of the proof of Theorem 15.2 and is left to the reader.

Theorem 15.3 *Let $(X, \mathscr{T}_X)$ be any topological space, $(Y, \mathscr{U}_Y)$ be a uniform space, $\Gamma: X \times \mathscr{B} \to \mathscr{T}_X$ be a $(\mathscr{T}_X, \mathscr{U}_Y)$ continuity scale, and $R \subseteq X \times Y^k$ be a stalkwise compact bounding relation. Assume that all the equations in the system of GFEs (15.2) have the same arity $p_\lambda = p$, S is a locally compact subspace of X^p and each of the maps $F_\lambda: Y^{k \times m_\lambda} \times X^p \to Y$, $G_\lambda: Y^{k \times n_\lambda} \times X^p \to Y$ is continuous in the matrix variables y_{ij}. Then for each pair $D \in \mathscr{K}(X)$, $V \in \mathscr{U}_Y$, such that $D^p \subseteq S$, there is a triple $C \in \mathscr{K}(X)$, $K \in \mathscr{K}\left(X^p\right)$, $U \in \mathscr{U}_Y$, such that $D \subseteq C$, $D^p \subseteq K \subseteq S$ and the following implication holds true:*

If a U-solution $\boldsymbol{f} = (f_1, \ldots, f_k) \colon X \to Y^k$ *of the system* (15.2) *on K is both* (Γ, U)*-precontinuous and R-bounded on C, then there exists a continuous solution* $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k)$ *of the system on S, such that* $\boldsymbol{f}$, $\boldsymbol{\varphi}$ *are V-close on D.*

Formulation of the corresponding modified versions of Corollaries 15.3–15.6 is left to the reader, as well.

Comparing the "local" stability Theorems 15.2, 15.3 and Corollaries 15.3, 15.6 with "global" Corollaries 15.4, 15.5 and other global stability results we see that while global stability deals with approximation of functions $\boldsymbol{f} = (f_1, \ldots, f_k) \colon X \to Y^k$ by continuous solutions $\boldsymbol{\varphi} = (\varphi_1, \ldots \varphi_k)$ of the given (system of) functional equation(s) on the whole space X, local stability deals with approximate extension (and if Y is discrete, then right by extension) of restrictions $\boldsymbol{f} \restriction D = (f_1 \restriction D, \ldots, f_k \restriction D)$ of such functions to some (in the present setting compact) subset $D \subseteq X$ to continuous solutions of the (system of) functional equation(s).

The interested reader can find a brief discussion of the role of nonstandard analysis in establishing our results as well of the possibility to replace it by some standard methods in the final part of [24].

Final Remark The general form of functional equations introduced in Section 15.1 was designed with the aim to prove the stability Theorems 15.2, 15.3 for all of them in a uniform way. I expected that in order to achieve this goal it will be necessary to assume that all the functional coefficients $F_\lambda, G_\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\beta}_\lambda$ are continuous (in all their variables). Having succeeded just with the continuity of F_λ and G_λ in the "matrix" variables $y_{ij} \in Y$, only, without requiring their continuity in the remaining variables $x_i \in X$, and, at the same time, without any continuity assumption on the tuples of operations $\boldsymbol{\alpha}_\lambda, \boldsymbol{\beta}_\lambda$, was then a true surprise for me.

A revision of the results established in [24, 25, 29–31] from such a point of view reveals that in most of them some continuity assumptions can be omitted. For instance, Theorem 3 from [30] (as well as Theorem 2.6 from [24]) on stability of continuous homomorphisms from a locally compact topological group G into any topological group H remains true *without assuming that* G *is a topological group*. It suffices that G be both a group and a locally compact topological space. Similarly, Theorem 3.1 from [31] on stability of continuous homomorphisms from a locally compact topological algebra $\mathfrak{A}$ into a completely regular topological algebra $\mathfrak{B}$ remains true for any universal algebra $\mathfrak{A}$ endowed with a locally compact (Hausdorff) topology, without assuming continuity of the operations in $\mathfrak{A}$.

Theorems 15.2, 15.3 also show that both the above-mentioned results admit a generalization in yet another direction, for the former one stated already in Theorem 2.6 in [24]. Namely for a mapping $f \colon G \to H$ or $f \colon A \to B$ in order to be close to a continuous homomorphism it is *not* necessary to assume that it is Γ-continuous with respect to the given continuity scale Γ (as both the above-mentioned theorems in [30] and [31] do); it is enough that f be (Γ, U)-precontinuous for a sufficiently small entourage U.

On the other hand, as shown by several counterexamples in [25] and [30], even in those weaker results one cannot manage without the control of the examined functions by means of some continuity scale and a stalkwise compact bounding relation. The more interesting are then the stability results not requiring the continuity scale and/or the bounding relation in their formulation. This is, e.g., the case of the global stability result for homomorphisms from amenable groups into the group of unitary operators on a Hilbert space in [16] (covering many more specific results proved both before and afterwards), as well as of the local stability result for homomorphisms from amenable groups into the unit circle $\mathbb{T}$ in [30].

Acknowledgements Research supported by grants no. 1/0608/13 and 1/0333/17 of the Slovak grant agency VEGA.

References

1. Albeverio, S., Fenstad, J.E., Høegh-Krohn, R., Lindstrøm, T.: Nonstandard Methods in Stochastic Analysis and Mathematical Physics. Academic, London (1986)
2. Anderson, R.M.: "Almost" implies "near." Trans. Am. Math. Soc. **196**, 229–237 (1986)
3. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
4. Arkeryd, L.O., Cutland, N.J., Henson, C.W. (eds.): Nonstandard Analysis, Theory and Applications. Kluwer Academic Publishers, Dordrecht (1997)
5. Boualem, H., Brouzet, R.: On what is the almost-near principle. Am. Math. Mon. **119**(5), 381–393 (2012)
6. Czerwik, S.: Functional Equations and Inequalities in Several Variables. World Scientific, Singapore (2002)
7. Davis, M.: Applied Nonstandard Analysis. Wiley, New York (1977)
8. Engelking, R.: General Topology. PWN — Polish Scientific Publishers, Warszawa (1977)
9. Forti, G.L.: Hyers-Ulam stability of functional equations in several variables. Aequationes Math. **50**, 143–190 (1995)
10. Grätzer, G.: Universal Algebra. Van Nostrand, Princeton (1968)
11. Henson, C.W.: Foundations of nonstandard analysis: a gentle introduction to nonstandard extensions. In: Arkeryd, L.O., Cutland, N.J., Henson, C.W. (eds.) Nonstandard Analysis, Theory and Applications, pp. 1–49. Kluwer Academic Publishers, Dordrecht (1997)
12. Hilton, P.J., Stammbach, U.: A Course in Homological Algebra. Springer, New York (1970)
13. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
14. Hyers, T.H., Rassias, T.M.: Approximate homomorphisms. Aequationes Math. **44**, 125–153 (1992)
15. Hyers, D.H., Isac, G., Rassias, T.M.: Stability of Functional Equations in Several Variables. Birkhäuser Verlag, Basel (1998)
16. Kazhdan, D.: On ϵ-representations. Isr. J. Math. **43**, 315–323 (1982)
17. Loeb, P.A.: Nonstandard analysis and topology. In: Arkeryd, L.O., Cutland, N.J., Henson, C.W. (eds.) Nonstandard Analysis, Theory and Applications, pp. 77–89. Kluwer Academic Publishers, Dordrecht (1997)
18. Mačaj, M., Zlatoš, P.: Approximate extension of partial ϵ-characters of abelian groups to characters with application to integral point lattices. Indag. Math. **16**, 237–250 (2005)
19. Mauldin, R.D.: The Scottish Book. Birkhäuser Verlag, Boston (1981)
20. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
21. Rassias, T.M.: On the stability of functional equations and a problem of Ulam. Acta Appl. Math. **62**, 23–130 (2000)

22. Rassias, T.M. (ed.): Functional Equations and Inequalities. Mathematics and Its Applications, vol. 518. Kluwer Academic Publishers, Dordrecht (2000)
23. Robinson A.: Non-Standard Analysis (revised edn.). Princeton University Press, Princeton (1996)
24. Sládek, F., Zlatoš, P.: A local stability principle for continuous group homomorphisms in nonstandard setting. Aequationes Math. **89**, 991–1001 (2015)
25. Špakula, J., Zlatoš, P.: Almost homomorphisms of compact groups. Ill. J. Math. **48**, 1183–1189 (2004)
26. Székelyhidi, L.: Ulam's problem, Hyers's solution — and where they led. In: Rassias, T.M. (ed.) Functional Equations and Inequalities. Mathematics and Its Applications, vol. 518, pp. 259–285. Kluwer, Dordrecht (2000)
27. Ulam, S.M.: A Collection of Mathematical Problems. Interscience Publications, New York (1961)
28. Ulam, S.M.: Problems in Modern Mathematics. Wiley, New York (1964)
29. Zlatoš, P.: Stability of homomorphisms between compact algebras. Acta Univ. M. Belii. Ser. Math. **15**, 73–78 (2009)
30. Zlatoš, P.: Stability of group homomorphisms in the compact-open topology. J. Log. Anal. **2:3**, 1–15 (2010)
31. Zlatoš, P.: Stability of homomorphisms in the compact-open topology. Algebra Univers. **64**, 203–212 (2010)