Chapter 12
On the Construction of the Field of Reals by Means of Functional Equations and Their Stability and Related Topics

Jens Schwaiger

Abstract There are certain approaches to the construction of the field of real numbers which do not refer to the field of rationals. Two of these ideas are closely related to stability investigations for the Cauchy equation and for some homogeneity equation. The a priory different subgroups of $\mathbb{Z}^{\mathbb{Z}}$ used are shown to be more or less identical. Extension of these investigations shows that given a commutative semigroup G and a normed space X with completion X_c the group $\mathrm{Hom}(G, X_c)$ is isomorphic to $\mathscr{A}(G, X)/\mathscr{B}(G, X)$ where $\mathscr{B}(G, X)$ is the subgroup of X^G of all bounded functions and $\mathscr{A}(G, X)$ the subgroup of those $f: G \to X$ for which the Cauchy difference $(x, y) \mapsto f(x + y) - f(x) - f(y)$ is bounded.

The space $\mathrm{Hom}(\mathbb{N}, X_c)$ may be identified with X_c itself. With this in mind, we are able to show directly that $\mathscr{A}(\mathbb{N}, X)/\mathscr{B}(\mathbb{N}, X)$ is a completion of the normed space X.

Keywords Stability of the Cauchy equation • Completion of normed spaces

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12.1 Introduction

Stability of functional equations is a very active and topical field of research. The example par excellence is the famous result found in Hyers [6].

Theorem 12.1 *Let G be an abelian semigroup and X a complete normed space. Given $f: G \to X$, assume that $\gamma_f: G \times G \to X$, $\gamma_f(x, y) := f(x + y) - f(x) - f(y)$, is bounded. Then, there is a unique $g \in Hom(G, X)$ such that $f - g$ is bounded.*

J. Schwaiger (✉)
Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstraße 36, 8010 Graz, Austria
e-mail: jens.schwaiger@uni-graz.at

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Moreover, $\left\|\gamma_f\right\|_\infty := \sup\{\left\|\gamma_f(x,y)\right\| \mid x,y \in G\} \le \varepsilon$ *implies that* $\|f(x)-g(x)\| \le \varepsilon$ *for all* $x \in G$.

Similar results hold true if the abelian group $(G,\cdot)$ operates on some set M when considering the inequality

$$\|f(g\cdot x)-\varphi(g)f(x)\| \le c\psi(g), \quad x \in M,\ g \in G, \tag{12.1}$$

where $\varphi,\psi \in \mathrm{Hom}(G,\mathbb{R}\setminus\{0\})$ and $c \ge 0$. See Jabłoński and Schwaiger [7] for even more general results.

There are many construction methods for the field $\mathbb{R}$ of real numbers. They use many different principles and ideas. Contrasting most of these ideas some of them do not require the field of rationals as a tool. The starting point there is the ring of integers. In Faltin et al. [4], a subring of the ring of formal Laurent series over $\mathbb{Z}$ factored by some maximal ideal, namely the principal ideal generated by a certain *carry string*, is used.

Two other ones are closely related to functional equations and their stability. Schönhage [11] uses the subgroup $\mathscr{S} := \bigcup_{c\in\mathbb{N}} \mathscr{S}_c$ of $\mathbb{Z}^{\mathbb{N}}$, where

$$\mathscr{S}_c := \{g \in \mathbb{Z}^{\mathbb{N}} \mid |g(kn)-kg(n)| \le ck \text{ for all } n,k \in \mathbb{N}\}. \tag{12.2}$$

In A'Campo [1], the starting point for the construction is the subgroup $\mathscr{A} := \bigcup_{c\in\mathbb{N}} \mathscr{A}_c$ of $\mathbb{Z}^{\mathbb{Z}}$, where

$$\mathscr{A}_c := \{f \in \mathbb{Z}^{\mathbb{Z}} \mid |f(n+m)-f(n)-f(m)| \le c \text{ for all } n,m \in \mathbb{Z}\}. \tag{12.3}$$

Some basic tools are the following ones.

Theorem 12.2 *Let G be an abelian semigroup and X a normed vector space over the field $\mathbb{Q}$ of rationals. Given $c \ge 0$, let $f \in \mathscr{A}_c(G,X) := \{h \in X^G \mid \|\gamma_h\|_\infty \le c\}$, where $\gamma_h(x,y) := h(x+y)-h(x)-h(y)$ is the* Cauchy difference *of h and*

$$\|\gamma_h\|_\infty := \sup_{x,y\in G}\|\gamma_h(x,y)\|.$$

Then

$$\|f(nx)-nf(x)\| \le (n-1)c, \qquad x \in G, n \in \mathbb{N}, \tag{12.4}$$

and the sequence $(f(nx)/n)_{n\in\mathbb{N}}$ is a Cauchy sequence, since it satisfies

$$\left\|\frac{f(nx)}{n}-\frac{f(mx)}{m}\right\| \le \left(\frac{1}{n}+\frac{1}{m}\right)c, \qquad x \in G, n,m \in \mathbb{N}.$$

Moreover, if this sequence converges for all $x \in G$, the limit function a,

$$a(x) := \lim_{n\to\infty}\frac{f(nx)}{n},$$

is an element of Hom(G, X)*, the set of homomorphisms from G to X. This a satisfies* $\|f - a\|_\infty \le c$ *and it is the only homomorphism b, such that* $f - b$ *is bounded.*

Proof The first assertion is clear for $n = 1$. If it is true for n, we get

$$\begin{aligned}\|f((n+1)x) - (n+1)f(x)\| &= \|f(nx + x) - f(nx) - f(x) + f(nx) - nf(x)\| \\ &\le \|f(nx + x) - f(nx) - f(x)\| + \|f(nx) - nf(x)\| \\ &\le c + (n-1)c = nc.\end{aligned}$$

By the first part, $\|f(nmx) - nf(mx)\| \le (n-1)c$ and $\|f(mnx) - mf(nx)\| \le (m-1)c$. Thus,

$$\begin{aligned}\|nf(mx) - mf(nx)\| &\le \|nf(mx) - f(nmx)\| + \|f(nmx) - mf(nx)\| \\ &\le (n - 1 + m - 1)c.\end{aligned}$$

Dividing by nm gives the desired result. So, the second assertion also is proved.

Finally, let

$$a_n(x) := \frac{f(nx)}{n}.$$

The properties of f imply

$$\|a_n(x + y) - a_n(x) - a_n(y)\| \le \frac{c}{n}.$$

Thus a,

$$a(x) := \lim_{n\to\infty} a_n(x),$$

lies in $\mathrm{Hom}(G, X)$. Moreover, the second part with $n = 1$ gives

$$\|f(x) - a_m(x)\| \le (1 + \frac{1}{m})c.$$

Taking the limit for m to ∞ shows that $\|f - a\|_\infty \le c$.

Finally, assume that for $b \in \mathrm{Hom}(G, X)$ the difference $f - b$ is bounded. Then the homomorphism $b - a$ is bounded as well. This implies $b - a = 0$. □

Let now G be merely a set on which $(\mathbb{N}, \cdot)$ operates via $(n, x) \mapsto nx$ such that $n(mx) = (nm)x$ and $1x = 1$. Furthermore, let

$$\mathscr{S}_c(G, X) := \{f \in X^G \mid \|f(nx) - nf(x)\| \le nc,\ x \in G, n \in \mathbb{N}\}.$$

Then we have the following result.

Theorem 12.3 *For any $f \in \mathscr{S}_c(G,X)$ and any $x \in G$, the sequence of the values $a_n(x) := f(nx)/n$ satisfies*

$$\|a_n(x) - a_m(x)\| \leq \left(\frac{1}{n} + \frac{1}{m}\right) c.$$

Therefore, it is a Cauchy sequence. If it converges for all x, the limit function a,

$$a(x) := \lim_{n\to\infty} a_n(x),$$

satisfies $a(nx) = na(x)$ for all n and all x. Moreover,

$$\|f - a\|_\infty \leq c$$

and a is the only homogeneous function b (i. e., $b(nx) = nb(x)$ for all $x \in G$ and all $n \in \mathbb{N}$), such that $f - b$ is bounded.

Proof $f \in \mathscr{S}_c(X)$ implies $\|f(nmx) - nf(mx)\| \leq nc$ and $\|f(mnx) - mf(nx)\| \leq mc$. As in the proof above, this means

$$\|a_n(x) - a_m(x)\| \leq \left(\frac{1}{n} + \frac{1}{m}\right) c.$$

This with $n = 1$ implies $\|f(x) - a(x)\| \leq c$. Using $\|f(mnx) - mf(nx)\| \leq mc$ or

$$\left\|\frac{f(nmx)}{n} - m\frac{f(nx)}{n}\right\| \leq \frac{m}{n}c$$

in the limit case $n \to \infty$ shows that $a(mx) - ma(x) = 0$ for all m and x. If $f - b$ is bounded and b homogeneous, then $a - b$ is also bounded and homogeneous. Thus, $a - b = 0$. □

Remark 12.1 Rational normed vector spaces are considered by Bourbaki, where in [3, TVS I.6] it is shown that the completion of such a space exists and that it is a real Banach space. Normed vector spaces over the rationals are special cases of normed abelian groups introduced in [13]: The homogeneity condition in normed abelian groups X reads as $\|nx\| = |n|\,\|x\|$ for all $n \in \mathbb{Z}$ and all $x \in X$.

Remark 12.2 For abelian semigroups G and normed abelian groups X, the set

$$\mathscr{A}(G,X) := \bigcup_{c\geq 0} \mathscr{A}_c(G,X)$$

is a subgroup of the abelian group X^G containing $\mathscr{B}(G,X)$ the subgroup of bounded functions in X^G.

If G is a set on which $\mathbb{N}$ operates in the above sense, the set

$$\mathscr{S}(G,X) := \bigcup_{c \geq 0} \mathscr{S}_c(G,X)$$

is also a subgroup of X^G containing $\mathscr{B}(G,X)$.

If additionally X is a module over some subring R of $\mathbb{C}$, the above subgroups are also modules over that ring. In particular, this applies to rational vector spaces X.

Proof $\mathscr{A}_c(G,X) \subseteq \mathscr{A}_d(G,X)$ for all $0 \leq c \leq d$ implies $0 \in \mathscr{A}_0(G,X) \subseteq \mathscr{A}(G,X)$. Moreover, by the triangle inequality

$$\mathscr{A}_c(G,X) + \mathscr{A}_d(G,X) \subseteq \mathscr{A}_{c+d}(G,X).$$

Finally,

$$r\mathscr{A}_c(G,X) \subseteq \mathscr{A}_{|r|c(G,X)} \text{ for all } r \in R.$$

Any $f \in \mathscr{B}(G,X)$ with $\|f(x)\| \leq c$ for all x is contained in $\mathscr{A}_{3c}(G,X)$.

The arguments are similar for $\mathscr{S}(G,X)$. In this case, $\|f(x)\| \leq c$ implies that $f \in \mathscr{S}_{2c}(G,X)$. □

Remark 12.3 (Least Absolute and Least Nonnegative Remainder) For further use, we also note that given $m \in \mathbb{N}$ any integer n may be written uniquely as $n = \alpha m + \rho$ with $\alpha, \rho \in \mathbb{Z}$ provided that ρ satisfies $-m \leq 2\rho < m$. The uniquely determined α will be denoted by $\langle n\colon m\rangle$. Thus,

$$-m \leq 2(n - \langle n\colon m\rangle m) < m$$

(and therefore a fortiori $|n - \langle n\colon m\rangle m| < m$).

n may also be written uniquely in the form $n = \beta m + \sigma$ with integers β, σ such that $0 \leq \sigma < m$. β will be denoted by $[n : m]$ and satisfies

$$0 \leq n - [n : m]m < m.$$

12.2 Two Constructions of the Reals and the Interplay Between Them

12.2.1 Schönhage

The base of construction in [11] is the set

$$\mathscr{S} := \bigcup_{c \in \mathbb{N}} \mathscr{S}_c$$

with

$$\mathscr{S}_c := \mathscr{S}_c(\mathbb{N}, \mathbb{Z}).$$

It is shown that $\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$ is an ordered field in which every subset bounded from above admits a supremum. Thus, this quotient group is a model of the set $\mathbb{R}$ of reals. All constructions are done completely in $\mathbb{Z}$, no noninteger rational numbers have to be used.

Addition is the one inherited from $\mathscr{S}$. A quasiorder on $\mathscr{S}$ is defined by

$$f \leq g :\Longleftrightarrow \text{there is some integer } c \text{ such that } f(n) \leq g(n) + c \text{ for all } n \in \mathbb{N}.$$

This quasiorder is compatible with the addition. Both $f \leq g$ and $g \leq f$ are satisfied if and only if $f - g \in \mathscr{B}(\mathbb{N}, \mathbb{Z})$. This quasiorder is a total order, too. Accordingly, the relation

$$f + \mathscr{B}(\mathbb{N}, \mathbb{Z}) \leq g + \mathscr{B}(\mathbb{N}, \mathbb{Z}) :\Longleftrightarrow f \leq g$$

on $\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$ is well defined and, by the properties of the quasiorder on $\mathscr{S}$, it is a total order compatible with the addition of equivalence classes. A convenient fact,

$$\text{for any } f \in \mathscr{S}_c \text{ there is some } f' \in \mathscr{S}_2 \text{ such that } f - f' \in \mathscr{B}(\mathbb{N}, \mathbb{Z}),$$

is used several times, for instance, in the proof that any non-empty subset A of $\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$, which is bounded from above, admits a supremum. To this aim, A is written as

$$A = \{f + \mathscr{B}(\mathbb{N}, \mathbb{Z}) \mid f \in A'\}$$

with $A' \subseteq \mathscr{S}_2$. In the same manner, the set B of upper bounds of A is written as

$$B = \{f + \mathscr{B}(\mathbb{N}, \mathbb{Z}) \mid f \in B'\}$$

with $B' \subseteq \mathscr{S}_2$. Then, $f(n) \leq g(n) + 8$ for all $n \in \mathbb{N}, f \in A', g \in B'$. Accordingly, we may define $h \in \mathbb{Z}^{\mathbb{N}}$ by

$$h(n) := \max_{f \in A'} \{f(n)\}.$$

Then, $h(n) \leq g(n) + 8$ for all $n \in \mathbb{N}$, $g \in B'$. Moreover, it is shown that $h \in \mathscr{S}$. This and the definition of h implies that

$$h + \mathscr{B}(\mathbb{N}, \mathbb{Z})$$

is a supremum of A.

Remark 12.4 Given $f \in \mathscr{S}_c$, Schönhage uses the function

$$n \mapsto [f(kn) : k] =: g(n)$$

and shows that $g \in f + \mathscr{B}(\mathbb{N}, \mathbb{Z})$ and that $g \in \mathscr{S}_2$ for sufficiently large k. Using h, $h(n) := \langle f(kn) \colon k \rangle$, instead of g, it turns out that also $f - h$ is bounded and that even $h \in \mathscr{S}_1$ is true provided that k is suitably large.

In fact:

$$kh(n) = f(kn) + u(kn) \text{ with } 2\,|u(kn)| \le k \text{ and}$$

$$2k\,|h(n) - f(n)| = |2(f(kn) - kf(n)) - 2u(kn)| \le 2kc + k < 2(c+1)k.$$

Thus, $h - f$ is bounded. Moreover,

$$2k\,(h(mn) - mh(n)) = 2f(kmn) + 2u(kmn) - 2mf(kn) - 2mu(kn)$$

and

$$\begin{aligned} 2k\,|h(mn) - mh(n)| &\le 2\,|f(kmn) - mf(kn)| + |2u(kmn) - 2mu(kn)| \\ &\le 2cm + (m+1)k. \end{aligned}$$

For $k \ge 2c$, this implies $2k\,|h(mn) - mh(n)| \le (2m+1)k$ or $2\,|h(mn) - mh(n)| \le 2m + 1$. Since only integers are involved, this finally shows that

$$|h(mn) - mh(n)| \le 1 \cdot m.$$

Multiplication for $f, g \in \mathscr{S}$ can be defined by

$$n \mapsto \langle f(n)g(n) \colon n \rangle n.$$

(Schönhage used $n \mapsto [f(n)g(n) : n]$ instead.) Denoting this by $f * g$, it is verified that $f * g \in \mathscr{S}$ in the following way:

Without loss of generality, we may assume that there is some c common to f and g such that $f, g \in \mathscr{S}_c$. Thus, $|f(n) - nf(1)|$, $|g(n) - ng(1)| \le cn$ imply

$$|f(n)|,\ |g(n)| \le c'n$$

for $c' := c + \max\{|f(1|,\ |g(1)|)\}$. According to

$$\begin{aligned} nk(\langle f(nk)g(nk) \colon nk \rangle\ &- k\langle f(n)g(n) \colon n \rangle) \\ &= (nk\langle f(nk)g(nk) \colon nk \rangle - f(nk)g(nk)) \\ &\quad + f(nk)\,(g(nk) - kg(n)) + kg(n)\,(f(nk) - kf(n)) \\ &\quad + k^2\,(f(n)g(n) - n\langle f(n)g(n) \colon n \rangle) \end{aligned}$$

we get for all n, k that

$$|nk\,(\langle f(nk)g(nk)\colon nk\rangle - k\langle f(n)g(n)\colon n\rangle)| \leq nk + c'nkck + kc'nkc + k^2n,$$

implying that

$$|(\langle f(nk)g(nk)\colon nk\rangle - k\langle f(n)g(n)\colon n\rangle)| \leq 1 + 2cc'k + k \leq 2(1+cc')k.$$

So,

$$f * g \in \mathscr{S}_{2(1+cc')}.$$

For bounded h_1, h_2, it is seen easily that

$$(f + h_1) * (g + h_2) = f * g + h_3$$

for some bounded h_3. Thus, a product on

$$\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$$

may be defined by

$$(f + \mathscr{B}(\mathbb{N}, \mathbb{Z})) \cdot (g + \mathscr{B}(\mathbb{N}, \mathbb{Z})) := f * g + \mathscr{B}(\mathbb{N}, \mathbb{Z}).$$

This product makes $\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$ to a commutative ring with unit element

$$\mathrm{id}_{\mathbb{N}} + \mathscr{B}(\mathbb{N}, \mathbb{Z}) =: 1.$$

Since $f * g \geq 0$ for $f, g \geq 0$, this is also an ordered ring. Finally, one verifies that this ring is even a field. If $f \in \mathscr{S}$ and $f + \mathscr{B}(\mathbb{N}, \mathbb{Z}) > 0$, we may additionally assume that $mf(n) \geq n, f(n) \geq 1$, and $f(n) \leq dn$ for some m, d, and all n. Then, $f'\colon \mathbb{N} \to \mathbb{Z}$,

$$f'(n) := \langle n^2 \colon f(n)\rangle,$$

is contained in $\mathscr{S}$:

$$\begin{aligned} &kn^2 \left|\langle k^2n^2 \colon f(kn)\rangle - k\langle n^2 \colon f(n)\rangle\right| \\ &\quad \leq m^2 f(kn)f(n) \left|\langle k^2n^2 \colon f(kn)\rangle - k\langle n^2 \colon f(n)\rangle\right| \\ &\quad = m^2 \left|f(n)\left(k^2n^2 - r_1\right) - kf(nk)\left(n^2 - r_2\right)\right| \end{aligned}$$

with $|r_1| < f(kn)$, $|r_2| < f(n)$. Therefore,

$$\begin{aligned} &kn^2 \left|\langle k^2n^2 \colon f(kn)\rangle - k\langle n^2 \colon f(n)\rangle\right| \\ &\quad \leq m^2 \left|f(n)r_1 + kf(nk)r_2\right| + kn^2 \left|kf(n) - f(nk)\right| \\ &\quad \leq m^2(dnnk + kdnkn + kn^2kc) = m^2n^2k(d + dk + kc) \end{aligned}$$

when we assume that $f \in \mathscr{S}_c$. This implies

$$\left|\langle k^2n^2{:}f(kn)\rangle - k\langle n^2{:}f(n)\rangle\right| \le m^2n^2k(2d+c)k$$

and

$$\left|\langle k^2n^2{:}f(kn)\rangle - k\langle n^2{:}f(n)\rangle\right| \le m^2(2d+c)k.$$

An (easier) calculation shows that

$$f * f' - \mathrm{id}_{\mathbb{N}} \in \mathscr{B}(\mathbb{N}, \mathbb{Z}).$$

This implies

$$(f + \mathscr{B}(\mathbb{N}, \mathbb{Z})) \cdot (f' + \mathscr{B}(\mathbb{N}, \mathbb{Z})) = 1.$$

For $g < 0$, $g \notin \mathscr{B}(\mathbb{N}, \mathbb{Z})$, choose $f \in g + \mathscr{B}(\mathbb{N}, \mathbb{Z})$ such that

$$(-f) * (-f)' - \mathrm{id}_{\mathbb{N}} \in \mathscr{B}(\mathbb{N}, \mathbb{Z}).$$

Then, $-(-f)' + \mathscr{B}(\mathbb{N}, \mathbb{Z})$ is the multiplicative inverse of $g + \mathscr{B}(\mathbb{N}, \mathbb{Z})$.

12.2.2 A'Campo et al.

Street [14] gave some hints on how to construct the reals using $\mathscr{A} := \mathscr{A}(\mathbb{Z}, \mathbb{Z})$. Street [15] contains a report of what happened since then. Ross Street refers to several papers, in particular to A'Campo [1]. Nadine Manschek, one of my students, gave full worked out proofs of all important steps in [10].

From Remark 12.2, it immediately follows that $\mathscr{A}$ is an abelian group with subgroup $\mathscr{B}(\mathbb{Z}, \mathbb{Z})$. Multiplication is defined by $(f, g) \mapsto f \circ g$. The proof that $f \circ g \in \mathscr{A}$ strongly depends on the fact that for $f \in \mathbb{Z}^{\mathbb{Z}}$ the boundedness of γ_f as defined in Theorem 12.1 implies that $\gamma_f(\mathbb{Z} \times \mathbb{Z})$ is finite.

Remark 12.5 This is not true for $\mathscr{A}(X, X)$ in general. In particular, there is some $f \in \mathscr{A}(\mathbb{R}, \mathbb{R})$ such that $f \circ f \notin \mathscr{A}(\mathbb{R}, \mathbb{R})$.

An example is given by $f = a + r$ with a additive and r bounded such that $a(\pi^{-n}) = 2^n$ for all n and $r(n) = \pi^{-n}$. (There is some additive a with this property, since $\{\pi^{-n} \mid n \in \mathbb{N}\}$ is linearly independent in the $\mathbb{Q}$-vector space $\mathbb{R}$.) Note that

$$f(f(x)) = a(a(x)) + a(r(x)) + r(f(x)).$$

Assuming $f \circ f \in \mathscr{A}(\mathbb{R}, \mathbb{R})$ would imply the existence of some additive b such that $f \circ f - b$ were bounded. By Theorem 12.2,

$$b(x) = \lim_{n\to\infty} \frac{(f \circ f)(nx)}{n}.$$

But for $x = 1$

$$f(f(n \cdot 1))/n = a(a(1)) + 2^n/n + r(f(n))/n$$

produces a divergent sequence.

Then, it is shown that $f \circ g - g \circ f$ is bounded and that $f \circ g - f' \circ g$ is bounded provided that $f - f'$ is. Thus, $\mathscr{A}(\mathbb{Z}, \mathbb{Z})/\mathscr{B}(\mathbb{Z}, \mathbb{Z})$ becomes a commutative ring with unit $\mathrm{id}_{\mathbb{Z}} + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ and

$$(f + \mathscr{B}(\mathbb{Z}, \mathbb{Z})) \cdot (g + \mathscr{B}(\mathbb{Z}, \mathbb{Z})) := (f \circ g + \mathscr{B}(\mathbb{Z}, \mathbb{Z})).$$

$f \in \mathscr{A}_c$ implies $|f(nm) - nf(m)| \leq nc$ for all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$. In particular, the restriction of f to $\mathbb{N}$, $f|_{\mathbb{N}}$, is contained in $\mathscr{S}_c$ for $f \in \mathscr{A}_c$. Accordingly, the quasiorder defined in $\mathscr{S}$ is meaningful in $\mathscr{A}$ and defines a total order in $\mathscr{A}(\mathbb{Z}, \mathbb{Z})/\mathscr{B}(\mathbb{Z}, \mathbb{Z})$.

To make this ring a field, several additional steps have to be considered:

1. If $f > 0$ and $f \notin \mathscr{B}(\mathbb{Z}, \mathbb{Z})$, there is some $f' \in f + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ which is also > 0 and additionally odd.
2. For this f' and all $m \in \mathbb{Z}$, the set $M_m := f'^{-1}(\{k \in \mathbb{Z} \mid k \leq m\})$ is non-empty and bounded from above.
3. $g \in \mathbb{Z}^{\mathbb{Z}}$, $g(m) := \max M_m$, is contained in $\mathscr{A}(\mathbb{Z}, \mathbb{Z})$.
4. $f' \circ g - \mathrm{id}_{\mathbb{Z}} \in \mathscr{B}(\mathbb{Z}, \mathbb{Z})$.

Thus, any unbounded $f + \mathscr{B}(\mathbb{Z}, \mathbb{Z}) > 0$ is invertible. Inverses for $f + \mathscr{B}(\mathbb{Z}, \mathbb{Z}) < 0$ are constructed as the additive inverse of the multiplicative inverse of $(-f) + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$.

Finally, it is shown that any non-empty subset A of $\mathscr{A}(\mathbb{Z}, \mathbb{Z})/\mathscr{B}(\mathbb{Z}, \mathbb{Z})$ has a supremum. Writing

$$A = \{f + \mathscr{B}(\mathbb{Z}, \mathbb{Z}) \mid f \in A'\}$$

with $A' \subseteq \mathscr{A}(\mathbb{Z}, \mathbb{Z})$, one may assume that all $f' \in A'$ are odd and elements of $\mathscr{A}_1(\mathbb{Z}, \mathbb{Z})$. For any odd $g \in \mathscr{A}(\mathbb{Z}, \mathbb{Z})$ such that $g + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ is an upper bound of A, it is seen that $f(n) \leq g(n) + 2$ for all $n \in \mathbb{N}_0$. Thus, $h \in \mathbb{Z}^{\mathbb{Z}}$ with

$$h(n) := \max\{f(n) \mid f \in A'\}$$

for $n \in \mathbb{N}_0$ and $h(n) := -h(-n)$ for $n \in \mathbb{Z}, n < 0$ is well defined. Then, some tedious calculation shows that $h \in \mathscr{A}_5$. Finally, it is shown that $h + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ is a least upper bound of A. (The definition of h is similar to the corresponding definition in Schönhage's approach.)

Remark 12.6 The abovementioned fact that f' may be chosen in $\mathscr{A}_1$ is shown by defining for given $f \in \mathscr{A}_c$ the function f' by

$$f'(n) := \langle f(nk) \colon k \rangle.$$

Then,

$$f - f' \in \mathscr{B}(\mathbb{Z}, \mathbb{Z})$$

and $f' \in \mathscr{A}_1$ for sufficiently large k. The proof for this is similar to that contained in Remark 12.4.

12.2.3 Synthesis

In Blatter [2], you may find the opinion that Schönhage's setting is a kind of predecessor of A'Campo's setting. In fact, in some sense, the settings are identical.

Theorem 12.4

$$\mathscr{S} = \mathscr{A}|_{\mathbb{N}} := \{f|_{\mathbb{N}} \mid f \in \mathscr{A}\} = \mathscr{A}(\mathbb{N}, \mathbb{Z})$$

and

$$\mathscr{A}/\mathscr{B}(\mathbb{Z}, \mathbb{Z}) \cong \mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z}).$$

Proof Certainly, $\mathscr{A}|_{\mathbb{N}} \subseteq \mathscr{S}$ by the inequality (12.4) of Theorem 12.2, which also applies when X is an abelian normed group only. Now, take any $f \in \mathscr{S}_c$. Then,

$$|f(k) - \langle kf(m) : m \rangle| \leq 2c + 1$$

provided that $k \leq m$:

$$2m\,|f(k) - \langle kf(m) : m \rangle| \leq |2mf(k) - 2kf(m) + 2r|$$

for some r such that

$$2\,|r| \leq m.$$

Thus,

$$2m\,|f(k) - \langle kf(m) : m \rangle| \leq 2c(m + k) + m \leq (4c + 1)m < 2(2c + 1)m$$

since $|f(km) - kf(m)| \leq kc$ and $|f(km) - mf(k)| \leq mc$ imply

$$|mf(k) - kf(m)| \leq (k + m)c.$$

Using this and the easy to verify inequality

$$2m\,|\langle a + b : m \rangle - \langle a : m \rangle - \langle b : m \rangle| \leq 3m$$

we may estimate $f(k+l) - f(k) - f(l)$ as follows:

$$\begin{aligned} |f(k+l) - f(k) - f(l)| &\le |f(k+l) - \langle (k+l)f(m){:}\, m\rangle| \\ &\quad + |f(k) - \langle kf(m){:}\, m\rangle| + |f(l) - \langle lf(m){:}\, m\rangle| \\ &\quad + |\langle (kf(m) + lf(m)){:}\, m\rangle - \langle (kf(m){:}\, m\rangle - \langle (lf(m){:}\, m\rangle| \\ &\le 3(2c+1) + 3 = 6(c+1) \text{ if } k, l \in \mathbb{N},\ k+l \le m. \end{aligned}$$

(Observe that the inequality $2m\,|\langle a+b{:}\, m\rangle - \langle a{:}\, m\rangle - \langle b{:}\, m\rangle| \le 3m$ implies

$$|\langle a+b{:}\, m\rangle - \langle a{:}\, m\rangle - \langle b{:}\, m\rangle| \le 1.)$$

Therefore,

$$f \in \mathscr{A}_{6(c+1)}(\mathbb{N}, \mathbb{Z}) \subseteq \mathscr{A}(\mathbb{N}, \mathbb{Z}).$$

For $g \in \mathscr{A}(\mathbb{N}, \mathbb{Z})$, we define (the odd extension) $g^* \in \mathbb{Z}^{\mathbb{Z}}$ by

$$g^*|_{\mathbb{N}} := g,$$

$g^*(0) := 0$, and $g^*(n) := -g(-n)$ for $n \in \mathbb{Z}, n < 0$. By considering the cases

(a) $n, m > 0$,
(b) $m = 0$ or $n = 0$,
(c) $n, m < 0$,
(d) $n < 0,\ m > 0,\ n+m = 0$,
(e) $n < 0,\ m > 0,\ n+m > 0$, and
(f) $n < 0,\ m > 0,\ n+m < 0$

and by observing that $g^*(n+m) - g^*(n) - g^*(m)$ is symmetric with respect to n and m, it can be verified that $f^* \in \mathscr{A}_{6(c+1)}(\mathbb{Z}, \mathbb{Z})$.

Altogether this shows that $\mathscr{S} = \mathscr{A}|_{\mathbb{N}}$ and $\mathscr{A}|_{\mathbb{N}} = \mathscr{A}(\mathbb{N}, \mathbb{Z})$.

Now, we prove the second assertion. Let $\varphi{:}\, \mathscr{A}(\mathbb{N}, \mathbb{Z}) \to \mathscr{A}(\mathbb{Z}, \mathbb{Z})/\mathscr{B}(\mathbb{Z}, \mathbb{Z})$ be defined by

$$\varphi(f) := f^* + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$$

with f^* as above. Then, φ is a homomorphism of abelian groups. Since $f^* \in \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ is equivalent to

$$f \in \mathscr{B}(\mathbb{N}, \mathbb{Z})$$

the kernel of φ equals $\mathscr{B}(\mathbb{N}, \mathbb{Z})$.

It remains to show that φ is surjective. For given $g + \mathscr{B}(\mathbb{Z}, \mathbb{Z})$ with $g \in \mathscr{A}(\mathbb{Z}, \mathbb{Z})$, assume that $g \in \mathscr{A}_c(\mathbb{Z}, \mathbb{Z})$. Then,

$$|g(0) - g(n) - g(-n)| \le c.$$

Thus, $|g(0)| \leq c$ and

$$|g(n) - (-g(-n))| \leq c + |g(0)| \leq 2c$$

for all n which with

$$f := g|_{\mathbb{N}} \in \mathscr{A}(\mathbb{N}, \mathbb{Z})$$

implies $g - f^* \in \mathscr{B}(\mathbb{Z}, \mathbb{Z})$. Accordingly,

$$\varphi(f) = f^* + \mathscr{B}(\mathbb{Z}, \mathbb{Z}) = g + \mathscr{B}(\mathbb{Z}, \mathbb{Z}).$$

□

Remark 12.7 From the proof, it follows that an isomorphism $\widehat{\varphi}$ between $\mathscr{S}/\mathscr{B}(\mathbb{N}, \mathbb{Z})$ and $\mathscr{A}/\mathscr{B}(\mathbb{Z}, \mathbb{Z})$ is given by

$$f + \mathscr{B}(\mathbb{N}, \mathbb{Z}) \mapsto f^* + \mathscr{B}(\mathbb{Z}, \mathbb{Z}).$$

Till now, the usage of the set of real or even of the rational numbers has been avoided. Since at present we already have (two) models for the field $\mathbb{R}$ and since

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

we may conclude from Theorem 12.2 that for any $f \in \mathscr{A}(\mathbb{N}, \mathbb{Z})$ and any $g \in \mathscr{A}(\mathbb{Z}, \mathbb{Z})$ there are real numbers α, β such that the set of $|f(n) - \alpha n|$, $n \in \mathbb{N}$, is bounded and that the same holds true for the set of $|g(n) - n\beta|$, $n \in \mathbb{Z}$. (The Cauchy sequences appearing in that theorem converge, since order completeness implies sequentially completeness; see, for example, Lang [9, Chapter 2, Theorem 1.5].) Moreover, any homomorphism a defined on $\mathbb{N}$ or $\mathbb{Z}$ is of the form $n \mapsto \gamma n$ with $\gamma = a(1)$. This will be used to show the following result, where, given any real x also the Gaussian bracket

$$[x] := \max\{m \in \mathbb{Z} \mid m \leq x\}$$

is involved.

Theorem 12.5 *For any $f, g \in \mathscr{S} = \mathscr{A}(\mathbb{N}, \mathbb{Z})$, we have*

$$\widehat{\varphi}(f * g + \mathscr{B}(\mathbb{N}, \mathbb{Z})) = (f^* \circ g^*) + \mathscr{B}(\mathbb{Z}, \mathbb{Z}).$$

Thus, the multiplication in the sense of Schönhage and A'Campo coincides.

Proof Let $\alpha, \beta \in \mathbb{R}$ be such that

$$f(n) = \alpha n + u(n)$$

and

$$g(n) = \beta n + v(n)$$

for all $n \in \mathbb{N}$ with certain bounded functions u, v. Then,

$$f(n)g(n) = \alpha\beta n^2 + n(\alpha v(n) + \beta u(n)) + u(n)v(n).$$

Let

$$f(n)g(n) = n\langle f(n)g(n) : n\rangle + r(n)$$

with some function r satisfying $2\,|r(n)| \leq n$. Then,

$$n\,|\langle f(n)g(n) : n\rangle - \alpha\beta n| \leq cn$$

for all n with suitable c. Note that

$$\alpha\beta n = [\alpha\beta n] + s(n),$$

where $0 \leq s(n) < 1$. Thus,

$$n \mapsto ((f * g)(n) - [\alpha\beta n])$$

is bounded. Obviously,

$$\mathbb{Z} \ni n \mapsto [\alpha\beta n] \in \mathscr{A}(\mathbb{Z}, \mathbb{Z}).$$

Thus,

$$f * g - \mathrm{int}_{\alpha\beta}|_{\mathbb{N}} \in \mathscr{B}(\mathbb{N}, \mathbb{Z}),$$

where

$$\mathrm{int}_\gamma(n) := [\gamma n].$$

So,

$$\widehat{\varphi}(f * g + \mathscr{B}(\mathbb{N}, \mathbb{Z})) = \mathrm{int}_{\alpha\beta}|_{\mathbb{N}}^{*} + \mathscr{B}(\mathbb{N}, \mathbb{Z}).$$

If $f^*, g^* \in \mathscr{A}(\mathbb{Z}, \mathbb{Z})$ are the odd extensions of f, g, we may write

$$f^* = \alpha \mathrm{id}_{\mathbb{Z}} + u^*, g^* = \beta \mathrm{id}_{\mathbb{Z}} + v^*.$$

Then,

$$f^* \circ g^* = \alpha\beta \mathrm{id}_{\mathbb{Z}} + \alpha v^* + u^* \circ g^*$$

is implying that

$$f^* \circ g^* - \alpha\beta \mathrm{id}_{\mathbb{Z}}$$

is bounded. But, therefore also

$$f^* \circ g^* - \mathrm{int}_{\alpha\beta}$$

and

$$f^* \circ g^* - \mathrm{int}_{\alpha\beta}|^*_{\mathbb{N}}$$

are bounded. This finally implies the assertion. □

12.3 Stability and Completeness

Now, the interplay between the stability of the Cauchy equation and the completeness of the involved normed space will be investigated. In the following, Remark 12.2 should be taken into account.

Theorem 12.6 *Let G be an abelian semigroup, suppose X to be a normed vector space (over $\mathbb{Q}$) with completion X_c. Then, $\mathscr{A}(G,X)/\mathscr{B}(G,X) \cong Hom(G,X_c)$, the group of homomorphisms defined on G with values in X_c.*

Proof Since

$$\mathscr{A}(G,X) \subseteq \mathscr{A}(G,X_c)$$

Theorem 12.2 may be applied. Thus, given $f \in \mathscr{A}(G,X)$ the mapping a_f with

$$a_f(x) := \lim_{n\to\infty} \frac{f(nx)}{n}$$

is contained in $\mathrm{Hom}(G,X_c)$. Moreover, $f - a_f$ is bounded. Let

$$\varphi: \mathscr{A}(G,X) \to \mathrm{Hom}(G,X_c)$$

be defined by $\varphi(f) := a_f$. Then, obviously φ is a homomorphism. Since f is bounded iff a_f is bounded, the kernel of φ equals $\mathscr{B}(G,X)$. Since

$$\mathscr{A}(G,X)/\ker(\varphi) \cong \varphi(\mathscr{A}(G,X))$$

it remains to show that φ is surjective.

To this aim, let $a \in \mathrm{Hom}(G, X_c)$ be arbitrary. For any $x \in G$, we may choose some $f(x) \in X$ such that $\|a(x) - f(x)\| < 1$. Then, f is an element of $\mathscr{A}_3(G, X)$ because of

$$\|f(x+y) - f(x) - f(y)\| \leq \|f(x+y) - a(x+y)\| \\ + \|f(x) - a(x)\| + \|f(y) - a(y)\| < 3.$$

The definition of f implies $\varphi(f) = a$. □

Corollary 12.1 *The groups $\mathscr{A}(\mathbb{N}, X)/\mathscr{B}(\mathbb{N}, X)$, $\mathscr{A}(\mathbb{Z}, X)/\mathscr{B}(\mathbb{Z}, X)$ both are isomorphic to X_c. In particular, for $X = \mathbb{Q}$ these groups are isomorphic to $\mathbb{R}$.*

Proof In both cases for G, the mapping $\mathrm{Hom}(G, X_c) \ni a \mapsto a(1) \in X_c$ is an isomorphism. □

It was proved for $G = \mathbb{Z}$ in Schwaiger [12] and for arbitrary abelian groups G containing at least one element of infinite order in Forti and Schwaiger [5] that the following theorem holds true.

Theorem 12.7 (Hyers' Theorem and Completeness) *If G is an abelian group as above and X a normed space such that for any $f \in \mathscr{A}(G, X)$ there is some $a \in$ Hom(G, X) such that $f - a$ is bounded, then X necessarily must be complete.*

Proof (Alternative Method) In Forti and Schwaiger [5], it was shown that it is enough to prove the result for $G = \mathbb{Z}$. There the latter task was managed by constructing to any Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ a suitable $f \in \mathscr{A}(\mathbb{Z}, X)$ and to use the hypotheses of the theorem for this f. Here, it is done in the following way: Choose any $\alpha \in X_c$ and $f \colon \mathbb{Z} \to X$ such that

$$\|f(n) - \alpha n\| < 1$$

for all $n \in \mathbb{Z}$. Then, $f \in \mathscr{A}(\mathbb{Z}, X)$ and by assumption there is some $a \in \mathrm{Hom}(\mathbb{Z}, X)$ such that $f - a$ is bonded. Since a is a homomorphism defined on $\mathbb{Z}$, there is some $\beta \in X$ such that

$$a(n) = \beta n$$

for all $n \in \mathbb{Z}$. Thus, also $\alpha \mathrm{id}_{\mathbb{Z}} - a$ is bounded implying $\alpha = \beta \in X$. Therefore, $X_c = X$. □

12.4 A Construction Method for the Completion of a Normed Space

There are well-known methods to construct the completion of metric and normed spaces. The most common ones use the set of Cauchy sequences on the underlying space. A different one, probably first mentioned by Kunugui [8], is contained in the following remark.

Remark 12.8 Let (X, d) be any (non-empty) metric space. Then, there is an isometry j from X into the Banach space $\mathscr{B}(X, \mathbb{R})$ of real-valued bounded functions defined on X. Thus, the closure of $j(X)$ is a completion of X.

The embedding is constructed by fixing some x_0 in X and by defining $j(x)$ pointwise as

$$j(x)(y) := d(y, x) - d(y, x_0).$$

If X has some additional structure, say of a normed space, this can be carried over easily to the completion $\overline{j(X)}$.

In the context of the present considerations, it has already been shown that for any normed space the factor group $\mathscr{A}(\mathbb{N}, X)/\mathscr{B}(\mathbb{N}, X)$ is isomorphic to the completion X_c of X. But, it seems to be desirable and interesting to give a proof that $\mathscr{A}(\mathbb{N}, X)/(\mathbb{N}, X)$ is a completion of X not using the existence of a completion of X a priori.

Theorem 12.8 *Let X be a real normed space, let $\mathscr{A} := \mathscr{A}(\mathbb{N}, X)$ and $\mathscr{B} := \mathscr{B}(\mathbb{N}, X)$. Then, $\mathscr{A}/\mathscr{B}$ is a completion of X, if addition and multiplication by a real number are defined as usual and if $\|f + \mathscr{B}\|$ is given by the well-defined limit*

$$\lim_{n\to\infty} \left\| \frac{f(n)}{n} \right\|.$$

Proof If X is a normed space, the abelian groups $\mathscr{A}$ and $\mathscr{B}$ are not only abelian groups but vector spaces by Remark 12.2. Thus, $\mathscr{A}/\mathscr{B}$ is a real vector space. Given $f \in \mathscr{A}$, we know by Theorem 12.2 that the sequence $(f(n)/n)_{n\in\mathbb{N}}$ is Cauchy in X. In detail

$$\left\| \frac{f(n)}{n} - \frac{f(m)}{m} \right\| \leq c \left(\frac{1}{n} + \frac{1}{m} \right)$$

for $f \in \mathscr{A}_c := \mathscr{A}_c(\mathbb{N}, X)$. Thus, by the reversed triangle inequality

$$|\|a\| - \|b\|| \leq \|a - b\|$$

the sequence $(\|f(n)/n\|)_{n\in\mathbb{N}}$ is Cauchy in the complete normed space $\mathbb{R}$. Thus, we may define

$$\|f\| := \lim_{n\to\infty} \left\| \frac{f(n)}{n} \right\|.$$

Obviously, this is a seminorm on $\mathscr{A}$. Now, it is shown that

$$\{f \in \mathscr{A} \mid \|f\| = 0\} = \mathscr{B}.$$

If $f \in \mathscr{B}$, the sequence of $f(n)$ is bounded. Thus,

$$\|f\| = \lim_{n\to\infty} \left\| \frac{f(n)}{n} \right\| = 0.$$

On the other hand, let $f \in \mathscr{A}$ satisfy $\|f\| = 0$. Then, $\|f(n)/n\|$ tends to 0 for n tending to ∞. Therefore, the sequence of $f(n)/n$ converges to 0 in X. This implies that

$$\lim_{n\to\infty} \frac{f(nm)}{n} = m \lim_{n\to\infty} \frac{f(nm)}{nm} = 0$$

for all $m \in \mathbb{N}$. Theorem 12.2 thus implies that $f - 0 = f$ is bounded.

Accordingly, we may define a norm on $\mathscr{A}/\mathscr{B}$ by

$$\|f + \mathscr{B}\| := \|f\| .$$

To show that $\mathscr{A}/\mathscr{B}$ equipped with this norm is complete, we need the following result.

Claim For any $f \in \mathscr{A}$ and any $\varepsilon > 0$, there is some $g \in (f + \mathscr{B}) \cap \mathscr{A}_\varepsilon$.

For the proof, let $f \in \mathscr{A}_c$, i. e.,

$$\|f(n+m) - f(n) - f(m)\| \le c$$

for all $n, m \in \mathbb{N}$. Taking any $m \in \mathbb{N}$ such that $c/m \le \varepsilon$, the function g is defined by

$$g(n) := \frac{f(mn)}{m},$$

a construction similar to some used earlier. Then,

$$\|g(k+l) - g(k) - g(l)\| = \frac{1}{m}\,\|f(mk+ml) - f(ml) - f(mk)\| \le \frac{c}{m} \le \varepsilon.$$

Accordingly, $g \in \mathscr{A}_\varepsilon$. The estimation

$$\|f(mn) - mf(n)\| \le cm$$

from Theorem 12.2 implies that $g \in f + \mathscr{B}$.

To show the completeness of the normed space $\mathscr{A}/\mathscr{B}$, it is enough to show that any Cauchy sequence admits a convergent subsequence. So, let the Cauchy sequence $(f_n + \mathscr{B})$ with $f_n \in \mathscr{A}$ be given. By eventually passing to a subsequence, we may assume that

$$\|f_{n+1} + \mathscr{B} - (f_n + \mathscr{B})\| \le \frac{1}{2^{n+1}}, \qquad n \in \mathbb{N}.$$

Additionally, by the claim above, we may also assume that

$$f_n \in \mathscr{A}_{\frac{1}{2^n}}, \qquad n \in \mathbb{N}.$$

Using

$$\|f_{n+1} + \mathscr{B} - (f_n + \mathscr{B})\| \leq \frac{1}{2^{n+1}}$$

the triangle inequality shows that

$$\|f_{n+m} + \mathscr{B} - (f_n + \mathscr{B})\| \leq \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+m}} < \frac{1}{2^n}, \qquad n, m \in \mathbb{N}.$$

Moreover,

$$f_{n+m} - f_n \in \mathscr{A}_{\frac{1}{2^{n+m}}} + \mathscr{A}_{\frac{1}{2^n}} \subseteq \mathscr{A}_{\frac{1}{2^{n+1}}+\frac{1}{2^n}} = \mathscr{A}_{\frac{3}{2^{n+1}}}.$$

Thus, by Theorem 12.2

$$\|(f_{n+m} - f_n)(k) - k(f_{n+m} - f_n)(1)\| \leq k\frac{3}{2^{n+1}}$$

and

$$\|(f_{n+m} - f_n)(1)\| \leq \frac{3}{2^{n+1}} + \left\| \frac{(f_{n+m} - f_n)(k)}{k} \right\|,$$

which for $k \to \infty$ results in

$$\|f_{n+m}(1) - f_n(1)\| \leq \frac{5}{2^{n+1}}.$$

Now, let $f \in X^{\mathbb{N}}$ be defined by $f(n) := nf_n(1)$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}(n+m)f_{n+m}(1) &- nf_n(1) - mf_m(1) \\ &= n(f_{n+m}(1) - f_n(1)) + m(f_{n+m}(1) - f_m(1)\end{aligned}$$

implies

$$\begin{aligned}\|f(n+m) - f(n) - f(m)\| &\leq \frac{5n}{2^{n+1}} + \frac{5m}{2^{m+1}} = \frac{5}{2}\left(\frac{n}{2^n} + \frac{m}{2^m}\right) \\ &\leq \frac{5}{2}2\frac{1}{2} = \frac{5}{2}\end{aligned}$$

and thus $f \in \mathscr{A}_{\frac{5}{2}}$.

Next, we consider $f - f_n$. The equality

$$\begin{aligned}(f - f_n)(n+m) = (n+m)f_{n+m}(1) &- (n+m)f_n(1) \\ &+ ((n+m)f_n(1) - f_n(n+m))\end{aligned}$$

implies

$$\|(f - f_n)(n+m)\| \leq (n+m)\,\|f_{n+m}(1) - f_n(1)\| + \frac{1}{2^{n+1}}(n+m)$$

and

$$\begin{aligned}\left\| \frac{(f - f_n)(n+m)}{n+m} \right\| &\leq \|f_{n+m}(1) - f_n(1)\| + \frac{1}{2^{n+1}} \\ &\leq \frac{5}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{3}{2^n}.\end{aligned}$$

Thus,

$$\|(f - f_n) + \mathscr{B}\| = \lim_{m\to\infty} \left\| \frac{(f - f_n)(n+m)}{n+m} \right\| \leq \frac{3}{2^n}$$

implying that $f_n + \mathscr{B}$ tends to $f + \mathscr{B}$ in $\mathscr{X}$.

Finally, we find an isometry $j: X \to \mathscr{A}/\mathscr{B}$ such that $j(X)$ is dense in $\mathscr{A}/\mathscr{B}$. Given $x \in X$, the function α_x,

$$\alpha_x(n) := nx,$$

is contained in $\mathscr{A}_0$. Let

$$j(x) := \alpha_x + \mathscr{B}.$$

Then, j is a vector space homomorphism from X onto $j(X)$. This is also an isometry since $\|\alpha_x\| = \|x\|$. Let $f + \mathscr{B} \in \mathscr{A}/\mathscr{B}$ and $\varepsilon > 0$. We may assume that $f \in \mathscr{A}_\varepsilon$. Then, with $x := f(1)$ we have

$$\|f(n) - nx\| \leq n\varepsilon,$$

which implies that

$$\|f + \mathscr{B} - \alpha_x\| \leq \varepsilon.$$

□

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