Chapter 10
Recent Developments in the Translation Equation and Its Stability

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Abstract The aim of this chapter is to present some of the recent results concerning the theory of the translation equation and its stability.

Keywords Translation equation • Stability of functional equations • Iterative roots • Embeddability in iteration groups

Mathematics Subject Classification (2010) Primary 39B12; Secondary 39B82, 26A18, 37E10, 37C15, 37E05, 39B22

10.1 Introduction

The functional equation

$$F(s, F(t, x)) = F(s \cdot t, x), \quad s, t \in G, x \in X,$$

where $F: G \times X \to X$, G is a set with binary operation $\cdot$, and X is an arbitrary set, is called *the translation equation*. Here, we gather only a personal choice of results concerning the translation equation and its stability published in recent years. We focus in a more detailed way only on these results, which are not discussed in the previous survey papers.[1] We refer the reader to the earlier survey papers on this topic:

[1]Especially the recent ones: [6, 25, 26] are published as open access papers, and [13] is also free available.

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J. Brzdęk et al. (eds.), *Developments in Functional Equations and Related Topics*, Springer Optimization and Its Applications 124,
DOI 10.1007/978-3-319-61732-9_10

1. In paper [11], Moszner listed several mathematical domains in which the translation equation appears. It includes among others abstract geometric and algebraic objects, groups of transformations, iterations, and dynamical systems. The author then presents many results concerning the solutions of the translation equation, including his own construction for the general solution on some domain. Continuity problems are also discussed.
2. In [13], Moszner continues the presentation of achievements in the theory. Further results on structure of solutions are listed. This paper also contains survey on regular (continuous, differentiable, analytic, and monotonic) solutions, problem of extendability of solutions to bigger domain, and papers of Smajdor on set-valued iteration semigroups.
3. The survey [14] covers among others the results on stability of the translation equation obtained by Mach and Moszner.
4. Zdun and Solarz [26] is an extensive survey on iteration theory. Here, we consider G an additive subgroup or subsemigroup of $\mathbb{R}$ or $\mathbb{C}$; in case $G = \mathbb{R}$ or $G = \mathbb{R}_+$, we say about iteration group (flow) or semigroup (semiflow), respectively. We usually write $F^t(x)$ instead of $F(t, x)$, hence, the translation equation takes the form

$$F^t \circ F^s(x) = F^{s+t}(x).$$

The origin of the notion of iteration group is extending the iterates F^n, $n \in \mathbb{N}$, of a given $F: X \to X$, to "real" iterates F^t, $t \in \mathbb{R}$. We often interpret $F^t(x)$ as the state of a point (object) x at the time t.

Topics covered in this paper (quite in detail)[2]:

- Measurable iteration semigroups: results of Baron, Chojnacki, Jarczyk, and Zdun on the problem *under what condition the measurability of iteration group/semigroup implies its continuity*;
- Embeddability of f into iteration groups or semigroups: *when for a given f there exists $\{F^t\}$ such that $F^1 = f$*, moreover, we can demand that iteration group or semigroup is of suitable regularity. This issue was examined for diffeomorphisms in $\mathbb{R}^N$, Brouwer homeomorphisms on the plane (mainly Leśniak's results), and interval homeomorphisms (mainly the result obtained by Zdun, Krassowska, and Zhang);
- When two commuting functions (f and g defined on an open interval, without fixed points) can be embeddable in the same iteration group (i.e. $f, g \in \{F^t\}$) (mainly the results of Zdun, Krassowska, and Ciepliński);
- Problem of existence of iterative roots (φ is an iterative root of order n of a given f, if $\varphi^n = f$, where φ^n denotes n-th iterate of φ) of piecewise monotonic functions, homeomorphism of the circle, and homeomorphisms of the plane (Zhang, Liu, Li, Yang, Jarczyk, Jarczyk, Zdun, and Solarz);

[2] Here, we signal them only, and mention some main authors; for detailed references, we refer the reader to [26].

- The structure of iteration groups of homeomorphisms of an interval, and of homeomorphisms of the circle (Zdun and Ciepliński);
- Different notions of "near" embeddability into iteration semigroup and characterization of such functions (Jarczyk and Przebieracz);
- A few problems concerning set-valued iteration semigroups (existence of iteration semigroup of single valued functions which is a selection of a given set-valued iteration group, and existence of majorizing iteration semigroups (Smajdor, Olko, Piszczek, and Łydzińska);
- Theorems of Matkowski and Jarczyk on iterates of mean-type mappings; and
- Stability of the translation equation (Moszner, Mach, Chudziak, Przebieracz, Reich, and Jabłoński).

5. The readers interested in the topic of iterative roots should read [6], where many results (recent and older) are presented in detail, also some open problems are listed. Here (in Section 10.3), we develop only the topic of conjugacy between F and its iterative root, for piecewise monotonic F.
6. In [25], Zdun discussed the existence of embeddings of given mappings in real iteration groups with suitable regularity, the conditions which imply the uniqueness of embeddings, and the formulas expressing the above embeddings or their general constructions. Here, in the next section, we refine some new approach to this subject proposed in [7].

10.2 Recent Advances in the Problem of Embeddability in Iteration Groups: Embeddability of Homeomorphisms of the Circle in Set-Valued Iteration Groups

Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with positive orientation, $cc[\mathbb{S}^1]$ be the family of all non-empty convex and compact subsets of $\mathbb{S}^1$ (that is, the family of closed arcs and points of $\mathbb{S}^1$). Let $F\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism without periodic points (its rotation number ρ is irrational). Let L_F be the set of all limit points of orbits of F (it is known that L_F is either equal to $\mathbb{S}^1$ or is a nowhere dense perfect set [4]). Moreover, F is embeddable in continuous iteration group[3] if and only if $L_F = \mathbb{S}^1$; in such a case, the continuous embedding is unique up to a constant [24]. Necessary and sufficient conditions for embeddability in the discontinuous iteration groups were given in [2] (in this case, F has infinitely many nonmeasurable embeddings).

In the paper [7], authors proposed a new approach to the problem of embeddability. They constructed some substitution of an iteration group in which F can be embedded.

[3] That is, there exists an iteration group $\{f^t\colon \mathbb{S}^1 \to \mathbb{S}^1;\ t \in \mathbb{R}\}$, such that $F = f^1$ and for every z the orbits $t \mapsto f^t(z)$ are continuous.

Before formulating main theorems from that paper, let us fix some notation. We assume that $L_F \neq \mathbb{S}^1$. In this case, the set $\mathbb{S}^1 \setminus L_F$ is a countable sum of pairwise disjoint open arcs, let $\mathscr{A}$ be a family of these arcs, $\alpha(I)$ be the middle point of the arc I, $M := \{\alpha(I) : I \in \mathscr{A}\}$, and $I_p := \alpha^{-1}(p)$ for $p \in M$. Hence, $\bigcup_{p\in M} I_p$ is a decomposition of $\mathbb{S}^1 \setminus L_F$ into open pairwise disjoint arcs. Let $L^* := \mathbb{S}^1 \setminus \bigcup_{p\in M} \operatorname{cl} I_p$. There exists exactly one continuous solution Φ of equation

$$\Phi(F(z)) = e^{2\pi i\rho}\Phi(z), \quad z \in \mathbb{S}^1,$$

such that $\Phi(1) = 1$. This solution is surjective and increasing (see [3, 23]). Define $F^t(z)$ as preimages of singletons:

$$F^t(z) := \Phi^{-1}[\{e^{2\pi it\rho}\Phi(z)\}], \quad t \in \mathbb{R}, z \in \mathbb{S}^1.$$

The family $\{F^t : \mathbb{S}^1 \to cc[\mathbb{S}^1];\ t \in \mathbb{R}\}$ is an iteration group such that $F(z) \in F^1(z)$ for $z \in \mathbb{S}^1$. It will be called *the main set-valued embedding* of F. It has the following properties:

(A1) $\forall_{t\in\mathbb{R},\, z\in\mathbb{S}^1}$ $F^t(z)$ is either a closed arc $\operatorname{cl} I_p$ for some $p \in M$ or a singleton belonging to L^*;
(A2) $\forall_{t\in\mathbb{R}}$ the function $z \mapsto F^t(z)$ is increasing and constant on the arcs $\operatorname{cl} I_p, p \in M$;
(A3) $\forall_{z\in\mathbb{S}^1}$ the function $t \mapsto F^t(z)$ is periodic with the period $\frac{1}{\rho}$ and strictly increasing on the arcs $\operatorname{cl} I_p, p \in M$;
(A4) if $F^u(z) \cap F^v(z) \neq \emptyset$, then $u = v + \frac{k}{\rho}$ for a $k \in \mathbb{Z}$;
(A5) $\forall_{p\in M}$ F^0 is constant on $\operatorname{cl} I_p$, $F^0[\operatorname{cl} I_p] = \operatorname{cl} I_p$; $F^0(z) = z$ for $z \in L^*$;
(A6) $\forall_{z\in\mathbb{S}^1}$ $\bigcup_{t\in\mathbb{R}} F^t(z) = \mathbb{S}^1$; and
(A7) $\forall_{z\in\mathbb{S}^1}$ $\exists_{t_1,t_2\in\mathbb{R}}$ $F^{t_1}(z)$ is an arc, $F^{t_2}(z)$ is a singleton.

Some of the above properties characterize the main set-valued embeddings of F, namely, if a set-valued group $\{F^t : \mathbb{S}^1 \to cc[\mathbb{S}^1];\ t \in \mathbb{R}\}$ fulfills conditions (A1), (A3), and (A6) (only for one point $z_0 \in \mathbb{S}^1$, not necessarily for all $z \in \mathbb{S}^1$) and $F(z) \in F^1(z)$, then it is the main set-valued embedding of F.

Moreover, the set

$$T := \{t \in \mathbb{R};\ \Phi[\mathbb{S}^1 \setminus L_F] = e^{2\pi it\rho}\Phi[\mathbb{S}^1 \setminus L_F]\},$$

is an additive, countable, and dense subgroup of $\mathbb{R}$ and $1 \in T$. It will be called *the supporting group of* F.

Let $\mathscr{F} := \{F^t : \mathbb{S}^1 \to cc[\mathbb{S}^1];\ t \in \mathbb{R}\}$ be the main set-valued embedding of F. It turns out that for every $t \in \mathbb{R}$ and every $p \in M$ the function F^t is constant on $\operatorname{cl} I_p$, whence, for every $z \in I_p$, $F^t(z) = F^t[\operatorname{cl} I_p]$. The set $F^t[\operatorname{cl} I_p]$ is either an arc or a point. Similarly, if $z \in L^*$, then $F^t(z)$ is either an arc or a point. Group T characterizes these indices for which F^t maps arcs $\operatorname{cl} I_p$ onto arcs and points from L^* onto points from L^*.

The subgroup $\{F^t: \mathbb{S}^1 \to cc[\mathbb{S}^1];\ t \in T\}$ of $\mathscr{F}$ is said to be the *refinement set-valued embedding of* F. It possesses a piecewise linear selection $\{v^t: \mathbb{S}^1 \to \mathbb{S}^1;\ t \in T\}$ of homeomorphisms. Moreover, $H \in \mathscr{F}$ has a continuous and injective selection if and only if H belongs to the refinement set-valued embedding of F.

10.3 Recent Advances in the Subject of Iterative Roots: Conjugacy Between Piecewise Monotonic Functions and Their Iterative Roots

First, we set some notations in order to formulate theorems in this section in a more concise way.

Let $I := [a, b]$ for $a < b < \infty$ and $F: I \to \mathbb{R}$ be a continuous function. A point $c \in (a, b)$ is called a *fort* of F if F is not strictly monotonic in any neighbourhood of c. We say that F is piecewise monotonic ($F \in \mathscr{P}\mathscr{M}[I]$) if the number $N(F)$ of forts of F is finite.

We put $S(F) := \{c_1, c_2, \ldots, c_{N(F)}\}$ for the set of all forts of piecewise monotonic F. Additionally, put $c_0 = a$ and $c_{N(F)+1} = b$ and define $I_i := [c_i, c_{i+1}]$ for $i = 0, 1, \ldots, N(F)$. It is known that [27, 28] either there exists an integer $r \in \mathbb{N} \cup \{0\}$ such that

$$0 = N(F^0) < N(F) < N(F^2) < \ldots < N(F^r) = N(F^{r+1}) = N(F^{r+2}) = \ldots,$$

or for every $k \in \mathbb{N} \cup \{0\}$ we have $N(F^k) < N(F^{k+1})$. In the first case, we put $H(F) := r$, and in the second $H(F) := \infty$, where $H(F)$ is called the *non-monotonicity height* of piecewise monotonic F.

For $F \in \mathscr{P}\mathscr{M}[I]$ with $H(F) = 1$, the maximal interval $K(F)$, containing $F[I]$ and such that F is monotonic on it, is called the *characteristic interval* of F [27, 28].

If f is a continuous iterative root of F of order n, then for every $i \in \{0, \ldots, N(F)\}$ there exists a positive integer $k \le \min\{n, N(F)\}$ and $i_1, \ldots i_{k-1} \in \{0, \ldots, N(F)\}$ such that

$$I_i \to I_{i_1} \to \ldots \to I_{i_{k-1}} \to K(F),$$

where by $I_{l_1} \to I_{l_2}$ we mean $f(I_{l_1}) \subset I_{l_2}$. Let $k_f(i)$ denote the number k described above. The *pace* ℓ, of iterative root f, is defined as $\max\{k_f(i);\ i \in \{0, 1, \ldots, N(F)\}\}$.

Every iterative root f of F can be extended from the characteristic interval $K(F)$ [9].

It turns out that all continuous monotonic functions are conjugate to their iterative roots [29] (we say that f is conjugate to g if there exists a homeomorphism Φ such that $\Phi \circ f = g \circ \Phi$). It enables us to understand the topological dynamics properties of iterative root f (explicit formulas can be complicated) having given $F = f^n$. In [8], authors gave examples of continuous piecewise monotonic but not monotonic functions, in order to prove that such functions:

- May have no iterative roots conjugate to them;
- May have some iterative roots not conjugate to them; and
- May have some iterative roots ($n \neq 1$) conjugate to them.

Moreover, they give necessary and sufficient conditions under which piecewise monotonic F is conjugate to its iterative root f.

Theorem 10.1 *Suppose that the mapping $F \in \mathscr{PM}[I]$ with $N(F) \geq 1$ and its continuous iterative root F having pace 1 are conjugate. Suppose that F is strictly increasing on its characteristic interval $K(F)$. Moreover, assume that $K(F) = \mathrm{Fix}(f) \cup J_1 \cup J_2 \cup \ldots \cup J_d$, where $\mathrm{Fix}(f)$ is the set of all fixed points of f and J_m's ($m = 1, 2, \ldots d$) are pairwise different intervals with endpoints being fixed points of f and interiors without fixed points. Then, f is strictly increasing on $K(F)$ and for each interval J_m, $m = 1, \ldots, d$, either*

(H1) $\{f(c_i);\ i = 1, 2 \ldots, N(F)\} \cap \mathrm{int}\, J_m = \emptyset$, *or*
(H2) *There is a point $c^* \in \mathrm{int}\, J_m$ such that*

$$\begin{cases} f(c_j) \in (f^2(c^*), f(c^*)], & \text{if } f(c^*) < c^* \text{ or} \\ f(c_j) \in (c^*, f(c^*)], & \text{if } f(c^*) > c^* \end{cases}$$

for all c_j's ($j = 1, 2, \ldots, N(F)$) satisfying $f(c_j) \in \mathrm{int}\, J_m$.

Also,

(H1') $\{F(c_i);\ i = 1, 2 \ldots, N(F)\} \cap \mathrm{int}\, J_m = \emptyset$, *or*
(H2') *There is a point $c^* \in \mathrm{int}\, J_m$ such that*

$$\begin{cases} F(c_j) \in (F \circ (f|_{K(F)})(c^*), F(c^*)], & \text{if } F(c^*) < c^* \text{ or} \\ F(c_j) \in (F \circ (f|_{K(F)})^{-1}(c^*), F(c^*)], & \text{if } F(c^*) > c^* \end{cases}$$

for all c_j's ($j = 1, 2, \ldots, N(F)$) satisfying $F(c_j) \in \mathrm{int}\, J_m$.

Theorem 10.2 *Suppose that the mapping $F \in \mathscr{PM}[I]$ with $N(F) \geq 1$ is strictly increasing on its characteristic interval $K(F)$. Assume that $K(F) = \mathrm{Fix}(F) \cup J_1 \cup J_2 \cup \ldots \cup J_d$, where $\mathrm{Fix}(F)$ is the set of all fixed points of F and J_m's ($m = 1, 2, \ldots d$) are pairwise different intervals with endpoints being fixed points of F and interiors without fixed points. Suppose that a continuous iterative root f of F is strictly increasing on $K(F)$. Moreover, let F and f satisfy either*

(H3) $\{F(c_i);\ i = 0, 1, 2 \ldots, N(F) + 1\} \cap \mathrm{int}\, J_m = \emptyset$, *or*
(H4) *There is a point $c^* \in \mathrm{int}\, J_m$ such that*

$$\begin{cases} F(c_j) \in (F \circ (f|_{K(F)})(c^*), F(c^*)], & \text{if } F(c^*) < c^* \text{ or} \\ F(c_j) \in (F \circ (f|_{K(F)})^{-1}(c^*), F(c^*)], & \text{if } F(c^*) > c^* \end{cases}$$

for all c_j's ($j = 0, 1, 2, \ldots, N(F) + 1$) satisfying $F(c_j) \in \mathrm{int}\, J_m$.

Then, F is conjugate to f.

10.4 Different Definitions of Stability of the Translation Equation

The question of Ulam, concerning the stability of group homomorphisms, posed in 1940, and the partial affirmative answer of Hyers [5] is often considered as the origin of the theory of stability of functional equations. But, even in these papers: [5, 21, 22], the precise formulation of what to understand as stability differs. Moszner devoted a few papers to define different kind of stabilities and examined the relations between them. See [10, 12, 14–17]. In this section, we present some of the results concerning the different stabilities of the translation equation and, in the next section, of the systems of functional equations defining (equivalently) dynamical systems (see [17, 18] and [16]).

In this section, let (S, d) be a metric space, $(G, \cdot)$ a groupoid. We start with reminding some definitions.

Definition 10.1 We say that the translation equation is *stable in the Hyers–Ulam sense* (shortly *stable*) if there exists a function $\Phi\colon (0,\infty) \to (0,\infty)$ (called *measure of stability*) such that for every $\varepsilon > 0$ and every function $H\colon G \times S \to S$, if

$$d(H(x, H(y,\alpha)), H(x \cdot y, \alpha)) \le \Phi(\varepsilon), \quad \alpha \in S,\ x, y \in G,$$

then there exists a solution $F\colon G \times S \to S$ of the translation equation

$$F(x, F(y,\alpha)) = F(x \cdot y, \alpha) \tag{10.1}$$

such that

$$d(G(x,\alpha), F(x,\alpha)) \le \varepsilon, \quad x \in G,\ \alpha \in S.$$

Moreover, if there exists such a function Φ which is unbounded, we say that Equation (10.1) is *normally stable*.

If there exists such Φ of the form $\Phi(\varepsilon) = K\varepsilon$, we say that Equation (10.1) is *strongly stable*.

Definition 10.2 We say that the translation equation is *uniformly b-stable* if there exists a function $\Psi\colon (0,\infty) \to (0,\infty)$ (called *measure of uniform b-stability*), such that for every $\delta > 0$ and every function $H\colon G \times S \to S$, if

$$d(H(x, H(y,\alpha)), H(x \cdot y, \alpha)) \le \delta, \quad \alpha \in S,\ x, y \in G,$$

then there exists a solution $F\colon G \times S \to S$ of the translation equation (10.1) such that

$$d(H(x,\alpha), F(x,\alpha)) \le \Psi(\delta), \quad \alpha \in S,\ x \in G.$$

Moreover, if there exists such a function Ψ which is unbounded, we say that the *uniform b-stability is normal*.

If there exists such Ψ of the form $\Psi(\delta) = k\delta$, we say that Equation (10.1) is *strongly b-stable*.

Definition 10.3 We say that the translation equation is *b-stable* if for every function $H: G \times S \to S$ such that

$$G \times G \times S \ni (x, y, \alpha) \mapsto d(H(x, H(y, \alpha)), H(x \cdot y, \alpha))$$

is bounded there exists a solution F of (10.1) such that

$$G \times S \ni (x, \alpha) \mapsto d(H(x, \alpha), F(x, \alpha))$$

is bounded.

Notice that uniform b-stability implies b-stability.

We have the following results concerning these notions.

Theorem 10.3 (1–4 in [17], 5 in [19] and [1])

1. *If the stability of (10.1) is normal, then this equation is uniformly b-stable.*
2. *Stable equation (10.1) does not need to be necessarily b-stable.*
3. *If the b-stability of (10.1) is uniform and normal, then this equation is normally stable.*
4. *Uniform b-stability of (10.1) does not necessarily imply stability.*
5. *The translation equation is normally stable with $\Phi(\varepsilon) = \varepsilon/10$ and normally uniformly b-stable with $\Psi(\delta) = 10\delta$, in the class of continuous functions with $(G, \cdot) = (\mathbb{R}, +)$ and S being a real interval.*

10.5 Stability of Dynamical Systems

In this section, we confine ourselves to continuous function $\mathbb{R} \times I \to I$, where $I \subset \mathbb{R}$ is nondegenerate interval. Such class of function is natural for consideration of dynamical systems.

Definition 10.4 The continuous function $F: \mathbb{R} \times I \to I$ is called *dynamical system* if F is a solution of the translation equation

$$F(s, F(t, x)) = F(s + t, x), \quad s, t \in \mathbb{R},\ x \in I, \tag{10.2}$$

and satisfies *one* or (equivalently, as it appears), *every*, of the following conditions:

1. $F(0, x) = x$, for $x \in I$,
2. $(F^0)'(x) = 1$, for $x \in I$, where $F^0 = F(0, \cdot)$,

3. $I \ni x \mapsto F(0, x)$ is strictly increasing,
4. $(F^0)'$ exists, and
5. F is a surjection.

Hence, we can consider stability problem for systems: translation equation and one of the equations appearing in the first two of the above conditions; and stability problem of the translation equation in the class of functions described by one of the last three of the above conditions. Full research on this topic can be found in [16–18]. Here, we present the selected results. The definitions from the previous section can be complemented by the notion of *restricted uniform b-stability* (definition almost the same as the definition of uniform b-stability, only the function Ψ is defined on some interval $(0, \delta_0)$ instead of on the whole positive halfline).

Theorem 10.4 *1. The translation equation is normally stable and normally uniformly b-stable in the class of surjective functions.*

2. *The translation equation is not stable in any of the classes: such F that F^0 is strictly increasing, and such F that the derivative of F^0 exists.*
3. *The translation equation is b-stable, uniformly b-stable, restrictedly uniformly b-stable, and normally uniformly b-stable only for I bounded , in both classes: such F that F^0 is strictly increasing, and such F that the derivative of F^0 exists.*
4. *The system of equations: "(10.2) & $(F^0)' \equiv 1$" is stable and restrictedly normally uniformly b-stable for every I; normally stable, normally uniformly b-stable, b-stable, and uniformly b-stable only for I bounded.*
5. *The system of equations: "(10.2) & $F^0 = \mathrm{id}$" is stable and normally stable only for $I = \mathbb{R}$; b-stable, uniformly b-stable, restrictedly uniformly b-stable, and normally uniformly b-stable only for I bounded and $I = \mathbb{R}$.*

10.6 Approximate Continuous Solutions of the Translation Equation

In this section, we concentrate only on a class of continuous function $\mathbb{R} \times I \to I$, where $I \subset \mathbb{R}$ is a nondegenerate interval.

In paper [20], there were listed some conditions which every approximate continuous solution of the translation equation, G, satisfies. These conditions show similarities between an exact solution and approximate solution of the translation equation. One of them is the existence of an exact solution of the translation equation in some neighbourhood of G. It is of interest that assuming only the existence of a solution of the translation equation in a neighbourhood of $G: \mathbb{R} \times I \to I$ does not suffice to obtain that G satisfies the translation equation approximately. More precisely, in paper [18] it was shown that

- The translation equation is not *inversely stable* (i.e. it is **not** true that

  for every $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every continuous function* $H: \mathbb{R} \times I \to I$ *if there exists a continuous solution* F *of the translation equation such that*

$$|F(t,x) - H(t,x)| \le \delta, \qquad t \in \mathbb{R},\ x \in I,$$

  then

$$|H(s, H(t,x)) - H(t+s,x)| \le \varepsilon, \qquad s,t \in \mathbb{R},\ x \in I);$$

- The translation equation is not *inversely b-stable* for unbounded intervals I (i.e. it is **not** true that

  for every continuous $F, H: \mathbb{R} \times I \to I$ *if* F *is a solution of the translation equation and*

$$\mathbb{R} \times I \ni (t,x) \mapsto |F(t,x) - H(t,x)|$$

  is bounded, then

$$\mathbb{R} \times \mathbb{R} \times I \ni (t,s,x) \mapsto |H(s, H(t,x)) - H(t+s,x)|$$

  is bounded); *and*

- The translation equation is not *inversely uniformly b-stable* for unbounded intervals I (i.e. it is **not** true that

  for every $\delta > 0$ *there exists a* $\varepsilon > 0$ *such that for every continuous function* $H: \mathbb{R} \times I \to I$ *if there exists a continuous solution* F *of the translation equation such that*

$$|F(t,x) - H(t,x)| \le \delta, \qquad t \in \mathbb{R},\ x \in I,$$

  then

$$|H(s, H(t,x)) - H(t+s,x)| \le \varepsilon, \qquad s,t \in \mathbb{R},\ x \in I).$$

Now, we are going to remind the characterization of a continuous solution of the translation equation (Theorem 10.5). Next, we present the necessary (Theorem 10.6) and sufficient condition (Theorem 10.7) for satisfying the translation equation approximately.

Theorem 10.5 *Let* $F: \mathbb{R} \times I \to I$ *be a solution of the translation equation, i.e.*

$$F(s, F(t,x)) = F(s+t,x), \quad s,t \in \mathbb{R}, x \in I.$$

Put $V = H(\mathbb{R} \times I)$. Then, there exist open, disjoint, intervals $U_n \subset V$ and homeomorphisms $h_n: \mathbb{R} \to U_n$ such that for every $x \in U_n$

$$F(t,x) = h_n(h_n^{-1}(x) + t), \quad t \in \mathbb{R},$$

and

$$F(t,x) = x, \quad x \in V \setminus \bigcup_n U_n,\ t \in \mathbb{R}.$$

Moreover, there exists a continuous function $f: I \to V$, such that $f(x) = x$ for $x \in V$ and

$$F(t,x) = F(t,f(x)), \quad t \in \mathbb{R}, x \in I \setminus V.$$

Conversely, for every continuous $f: I \to I$ such that $f \circ f = f$, a family of open, disjoint intervals $\{U_n;\ n \in N \subset \mathbb{N}\}$ such that $U_n \subset f(I)$, and a family of homeomorphisms $h_n: \mathbb{R} \to U_n$, $n \in N$, every function of the form

$$F(t,x) = \begin{cases} h_n(h_n^{-1}(f(x)) + t), & \text{if } f(x) \in U_n,\ t \in \mathbb{R}; \\ f(x), & \text{if } f(x) \notin \bigcup_{n \in N} U_n,\ t \in \mathbb{R} \end{cases}$$

is a continuous solution of the translation equation.

Theorem 10.6 *Suppose that $H: \mathbb{R} \times I \to I$ is a continuous solution of*[4]

$$|H(s, H(t,x)) - H(s+t,x)| \leq \delta, \quad x \in I,\ s,t \in \mathbb{R}.$$

Then,

(a) There exist open, disjoint intervals $U_n \subset I$, $n \in N$, of the length greater or equal to 6δ, homeomorphisms $h_n: \mathbb{R} \to U_n$, $n \in N$, and a continuous function $f: I \to I$, such that $f \circ f = f$, $U_n \subset f(I)$, $n \in N$,

$$|H(t,x) - f(x)| \leq 10\delta, \qquad t \in \mathbb{R}, f(x) \notin \bigcup_{n \in N} U_n,$$

$$|H(t,x) - h_n(h_n^{-1}(f(x)) + t)| \leq 10\delta, \qquad t \in \mathbb{R}, f(x) \in U_n,\ n \in N;$$

(b) $\forall_{(x \in I,\, n \in N)}\ (f(x) \in U_n \ \Rightarrow\ H(\mathbb{R}, x) = U_n)$;
(c) $\forall_{(x \in I,\, n \in N)}\ (x \in U_n \ \Rightarrow\ f(x) = x)$;
(d) $\forall_{(x \in I,\, t \in \mathbb{R})}\ (|f(H(t,x)) - H(t,x)| \leq 2\delta)$;

[4]The proof of this theorem can be found in [19] and [20], and the construction of homeomorphisms $h_n: \mathbb{R} \to U_n$ was done in [1].

(e) $\forall_{x\in I}\left(f(x)\notin\bigcup_{n\in N}U_n \;\Rightarrow\; \left(\forall_{t\in\mathbb{R}} f(H(t,x))\notin\bigcup_{n\in N}U_n\right)\right)$;

(f) $\forall_{x\in I}\left(f(x)\notin\bigcup_{n\in N}U_n \;\Rightarrow\; (\forall_{s_1,s_2\in\mathbb{R}}\,|H(s_1,x)-H(s_2,x)|\le 6\delta)\right)$;

(g) *The set of values of function* f, V_f, *is contained in the set of values of function* H, V_H, *i.e.* $V_f\subset V_H$;

(h) *Every interval* U_n *is "invariant", more precisely*

$$H(\mathbb{R},x)=U_n,\qquad x\in U_n,\ n\in N,$$

and

$$H(t,U_n)=U_n,\qquad t\in\mathbb{R},\ n\in N;$$

(i) *For every* $n\in N$, *put* $a_n:=\inf U_n$, $b_n:=\sup U_n$. *Either* h_n *is an increasing homeomorphism,*

$$\lim_{t\to\infty}H(t,x)=b_n,\quad \lim_{t\to-\infty}H(t,x)=a_n,\qquad x\in U_n,$$

and $H(\cdot,x)$ *"almost increases", i.e. for every* $t\in\mathbb{R}$ *we have* $H(s,x)>H(t,x)-2\delta$ *for* $s>t$; *or* h_n *is a decreasing homeomorphism,*

$$\lim_{t\to\infty}H(t,x)=a_n,\quad \lim_{t\to-\infty}H(t,x)=b_n,\qquad x\in U_n,$$

and $H(\cdot,x)$ *"almost decreases", i.e. for every* $t\in\mathbb{R}$ *we have* $H(s,x)<H(t,x)+2\delta$ *for* $s>t$;

(j) *For every* $n\in N$

$$H(t,a_n)=a_n,\ H(t,b_n)=b_n,\qquad t,\in\mathbb{R},$$

whenever a_n, b_n *are in* I;

(k) *For every* $x\in I$ *such that* $x\notin\bigcup_{n\in N}U_n$ *but there are* $n,m\in N$ *with* $b_n\le x\le a_m$, *we have*

$$|H(t,x)-x|\le 6\delta,\qquad t\in\mathbb{R};$$

(l)

$$|H(t,x)-H(t,f(x))|\le 10\delta,\qquad t\in\mathbb{R},\ x\in I;\textit{ and}$$

(m) Moreover, for every $n \in N$ there are two possibilities:

- *Either there exists $\eta_n > 0$ such that*

$$|t_1 - t_2| \leq \eta_n \;\Rightarrow |h_n(t_1) - h_n(t_2)| \leq 21\delta, \quad t_1, t_2 \in \mathbb{R}, \tag{10.3}$$

for $\eta_n^* := \sup\{\eta_n > 0 : (10.3)\text{ holds}\} \in (0, \infty]$ we have[5]

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \leq H(t,x) \leq h_n(t + \eta_n^* + h_n^{-1}(f(x))), \quad t \in \mathbb{R}, f(x) \in U_n,$$

if h_n is increasing,

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \geq H(t,x) \geq h_n(t + \eta_n^* + h_n^{-1}(f(x))), \quad t \in \mathbb{R}, f(x) \in U_n,$$

if h_n is decreasing,

- *or such η_n, for which (10.3) holds, does not exist and*

$$H(t,x) = h_n(t + h_n^{-1}(f(x)), \quad t \in \mathbb{R}, f(x) \in U_n.$$

Theorem 10.7 *Let I be a nondegenerate real interval, $\delta, A_1, A_2, B, C, D > 0$, suppose that $H\colon \mathbb{R} \times I \to I$ is a continuous function. If*

(a) There exist open, disjoint intervals $U_n \subset I$, $n \in N$, homeomorphisms $h_n\colon \mathbb{R} \to U_n$, $n \in N$, and a continuous function $f\colon I \to I$, such that $f \circ f = f$, $U_n \subset f(I)$, $n \in N$,

$$|H(t,x) - f(x)| \leq A_1\delta, \qquad t \in \mathbb{R}, f(x) \notin \bigcup_{n\in N} U_n,$$

$$|H(t,x) - h_n(h_n^{-1}(f(x)) + t)| \leq A_2\delta, \qquad t \in \mathbb{R}, f(x) \in U_n, n \in N;$$

(b) $\forall_{(x\in I, n\in N)}\ (f(x) \in U_n \;\Rightarrow\; H(\mathbb{R}, x) \subset U_n)$;
(c) $\forall_{(x\in I, n\in N)}\ (x \in U_n \;\Rightarrow\; f(x) = x)$;
(d) $\forall_{(x\in I, t\in\mathbb{R})}\ (|f(H(t,x)) - H(t,x)| \leq B\delta)$;
(e) $\forall_{x\in I}\left(f(x) \notin \bigcup_{n\in N} U_n \;\Rightarrow\; \left(\forall_{t\in\mathbb{R}} f(H(t,x)) \notin \bigcup_{n\in N} U_n\right)\right)$;
(f) $\forall_{x\in I}\left(f(x) \notin \bigcup_{n\in N} U_n \;\Rightarrow\; (\forall_{s_1,s_2\in\mathbb{R}}\, |H(s_1,x) - H(s_2,x)| \leq C\delta)\right)$; and
(g) Moreover, for every $n \in N$ there are two possibilities:

- *Either there exists $\eta_n > 0$ such that*

$$|t_1 - t_2| \leq \eta_n \;\Rightarrow |h_n(t_1) - h_n(t_2)| \leq D\delta, \qquad t_1, t_2 \in \mathbb{R}, \tag{10.4}$$

[5] If $\eta_n^* = \infty$, then by $h_n(\pm\infty)$ we understand $\lim_{t\to\pm\infty} h_n(t)$.

for $\eta_n^* := \sup\{\eta_n > 0 :$ *(10.4) holds*$\} \in (0, \infty]$ *we have*[6]

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \le H(t,x) \le h_n(t + \eta_n^* + h_n^{-1}(f(x))), \qquad t \in \mathbb{R}, f(x) \in U_n,$$

if h_n *is increasing,*

$$h_n(t - \eta_n^* + h_n^{-1}(f(x))) \ge H(t,x) \ge h_n(t + \eta_n^* + h_n^{-1}(f(x))), \qquad t \in \mathbb{R}, f(x) \in U_n,$$

if h_n *is decreasing,*

- *or such* η_n*, for which (10.4) holds, does not exist and*

$$H(t,x) = h_n(t + h_n^{-1}(f(x)), \qquad t \in \mathbb{R}, f(x) \in U_n,$$

then

$$|H(s, H(t,x)) - H(t+s,x)| \le E\delta, \qquad s,t \in \mathbb{R},\ x \in I,$$

where $E := \max\{(2A_2 + D), \min\{3A_1 + B, A_1 + B + C\}\}$.

Acknowledgements This research is a part of *Iterative functional equations and real analysis* program (Institute of Mathematics, University of Silesia, Katowice, Poland).

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