A Hasse Diagram for Weighted Sceptical Semantics with a Unique-Status Grounded Semantics

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Abstract. We provide an initial study on the Hasse diagram that represents the partial order -w.r.t. set inclusion- among weighted sceptical semantics in Argumentation: grounded, ideal, and eager. Being our framework based on a parametric structure of weights, we can directly compare weighted and classical approaches. We define a unique-status weighted grounded semantics, and we prove that the lattice of strongly-admissible extensions becomes a semi-lattice.

1 Introduction

An *Abstract Argumentation Framework* (*AAF*) [9] is essentially a pair $\langle \mathscr{A}_{rgs}, R \rangle$ consisting of a set of arguments and a binary oriented relation of attack defined among them (*e.g.*, $a, b \in \mathscr{A}_{rgs}$, and $R(a, b)$). The key idea behind *extension-based* semantics is to identify some subsets of arguments (called *extensions*) that survive the conflict "together". For example, the arguments in an *admissible* extension [9] $\mathscr{B}$ are not in conflict and they counter-attack attacked arguments in $\mathscr{B}$, i.e., arguments in $\mathscr{B}$ are *defended*.

Several notions of weighted defence have been defined in the literature [4,5,8,11,12]. Attacks are associated with a weight indicating a "strength" value, thus generalising the notion of AAF into *Weighted AAF* (*WAAF*) [4,5,11]. In [3] we provide a new definition of defence for WAAFs, called w-defence, and we use this to redefine classical semantics [9] to their weighted counterpart, that is w-semantics [3] (e.g., *w-admissible*).

In formal Abstract Argumentation, as well as in non-monotonic inference in general, it is possible for a semantics to yield more than one extension: in a framework with two arguments a and b, if $R(a, b)$ and $R(b, a)$ then both $\{a\}$ and $\{b\}$ are admissible. Often, this is dealt with by using a sceptical approach: hence, it is desirable to also have sceptical semantics that always yields exactly one extension. The most well-known example of such unique-status semantics is the *grounded* semantics [9]; however, in the literature it is sometimes supposed to be too sceptical [7]. The *ideal* [10] and *eager* [7] semantics try to be less sceptical, i.e., grounded $\subseteq$ ideal $\subseteq$ eager [7].[1]

[1] The eager is a unique-status semantics only for finite AAFs [1], which we study in this paper.

M. Balduccini and T. Janhunen (Eds.): LPNMR 2017, LNAI 10377, pp. 49–56, 2017.
DOI: 10.1007/978-3-319-61660-5_6

In the paper we provide an initial study on the Hasse diagram that represents the partially order set (or *poset*) -w.r.t. set inclusion- among w-extensions; in particular, we curb to sceptical semantics. In a Hasse diagram, each vertex corresponds to an extension $\mathscr{B}$, and there is an edge between extensions $\mathscr{B}$ and $\mathscr{C}$ whenever $\mathscr{B} \subseteq \mathscr{C}$ and there is no extension $\mathscr{D}$ such that $\mathscr{B} \subseteq \mathscr{D} \subseteq \mathscr{C}$. We will build this study on a parametric framework based on an algebraic structure, which can represent [3] the frameworks in [8,9,12]. Differently from [8,11,12], our solution always provides a single grounded extension.

2 Background

C-semirings are commutative semirings where $\otimes$ is used to compose values, while an idempotent $\oplus$ is used to represent a partial order among them.

Definition 1 (C-semirings [2]). *A c-semiring is a five-tuple* $\mathbb{S} = \langle S, \oplus, \otimes, \bot, \top \rangle$ *such that* S *is a set,* $\top, \bot \in S$*, and* $\oplus, \otimes : S \times S \rightarrow S$ *are binary operators making the triples* $\langle S, \oplus, \bot \rangle$ *and* $\langle S, \otimes, \top \rangle$ *commutative monoids, satisfying,* (i) *distributivity* $\forall a, b, c \in S. a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, (ii) *annihilator* $\forall a \in A. a \otimes \bot = \bot$*, and* (iii) *absorptivity* $\forall a, b \in S. a \oplus (a \otimes b) = a$.

The idempotency of $\oplus$, which derives from absorptivity, leads to the definition of a partial order $\leq_{\mathbb{S}}$ over S: $a \leq_{\mathbb{S}} b$ iff $a \oplus b = b$, which means that b is "better" than a. $\oplus$ is the *least upper bound* of the lattice $\langle S, \leq_{\mathbb{S}} \rangle$. Some c-semiring instances are: *Boolean* $\langle \{F, T\}, \vee, \wedge, F, T \rangle$, *Fuzzy* $\langle [0, 1], \max, \min, 0, 1 \rangle$, and *Weighted* $\langle \mathbb{R}^+ \cup \{+\infty\}, \min, +, +\infty, 0 \rangle$. Thus, the definition of WAAFs can represent different problems.

Definition 2 (c-semiring-based WAAF [5]). *A semiring-based Argumentation Framework (WAAF$_{\mathbb{S}}$) is a quadruple* $\langle \mathscr{A}_{rgs}, R, W, \mathbb{S} \rangle$*, where* $\mathbb{S}$ *is a semiring* $\langle S, \oplus, \otimes, \bot, \top \rangle$*,* $\mathscr{A}_{rgs}$ *is a set of arguments,* R *the attack binary-relation on* $\mathscr{A}_{rgs}$*, and* $W : \mathscr{A}_{rgs} \times \mathscr{A}_{rgs} \rightarrow S$ *is a binary function. Given* $a, b \in \mathscr{A}_{rgs}$*,* $\forall (a, b) \in R$*,* $W(a, b) = s$ *means that* a *attacks* b *with a weight* $s \in S$*. Moreover, we require that* $R(a, b)$ *iff* $W(a, b) <_{\mathbb{S}} \top$*.*

In [3] we define w-defence: a set $\mathscr{B}$ defends an argument b from a if the set-wise $\otimes$ of the attacks from all $c \in \mathscr{B}$ that defend b, i.e., $W(\mathscr{B}, a) = \bigotimes_{c \in \mathscr{B}} W(c, a)$, is worse than (i.e., stronger) or equal to the attacks to b and all the arguments in $\mathscr{B}$, i.e., $W(a, \mathscr{B} \cup b)$.

Definition 3 (w-defence [3]). *Given* $WF = \langle \mathscr{A}_{rgs}, R, W, S \rangle$*,* $\mathscr{B} \subseteq \mathscr{A}_{rgs}$ *w-defends* $b \in \mathscr{A}_{rgs}$ *from* $a \in \mathscr{A}_{rgs}$ *s.t.* $R(a, b)$*, iff* $W(a, \mathscr{B} \cup \{b\}) \geq_{\mathbb{S}} W(\mathscr{B}, a)$*;* $\mathscr{B}$ *w-defends* b *iff it defends* b *from any* a *s.t.* $R(a, b)$*.*

Note that (the weights of) counter-attacks of b can be exploited in the defence offered by $\mathscr{B}$, as in [9] happens for self-defence: this is why we consider all the attacks from $\mathscr{B} \cup \{b\}$ to a, and vice-versa. From [3] we report the definitions of w-semantics.

Definition 4 (w-semantics [3]). *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, $\mathscr{B}$ is a conflict-free set [9] iff $W(\mathscr{B}, \mathscr{B}) = \top$ (where $W(\mathscr{B}, \mathscr{D}) = \bigotimes_{b\in\mathscr{B}, d\in\mathscr{D}} W(b, d)$). $\mathscr{B}$ can be:*

- *a w-admissible (*wadm*) extension iff all the arguments in $\mathscr{B}$ are w-defended by $\mathscr{B}$;*
- *a w-complete (*wcom*) extension iff each argument $b \in \mathscr{A}_{rgs}$ s.t. $\mathscr{B} \cup \{b\}$ is w-admissible belongs to $\mathscr{B}$;*
- *a w-preferred (*wprf*) extension iff it is a maximal (w.r.t. set inclusion) w-admissible subset of $\mathscr{A}_{rgs}$;*
- *w-semi-stable (*wsst*) iff, given the* range *of $\mathscr{B}$ defined as $\mathscr{B} \cup \mathscr{B}^+$, where $\mathscr{B}^+ = \{a \in \mathscr{A}_{rgs} : W(\mathscr{B}, a) <_{\mathbb{S}} \top\}$, $\mathscr{B}$ is a w-complete extension with maximal (w.r.t. set inclusion) range.*[2]
- *a w-stable extension (*wstb*) iff $\forall a \notin \mathscr{B}, \exists b \in \mathscr{B}. W(b, a) <_{\mathbb{S}} \top$.*

If we use the *Boolean* c-semiring and consider $W(a, b) = false$ whenever $R(a, b)$, for each semantics in Definition 4 we exactly obtain the corresponding original Dung's one [3]: i.e., respectively admissible (adm), complete (com), preferred (prf), semi-stable (sst), and stable (stb) [9,13]. In this case, the notion of w-defence collapses to classical defence [9]: $\mathscr{B}$ w-defends a iff $\mathscr{B}$ defends a.[3]

We conclude the background by recalling in Definition 5 the definitions of the classical sceptical semantics, which we will later weigh in Sect. 3.

Definition 5 (Sceptical semantics). *Given a framework $F = \langle \mathscr{A}_{rgs}, R\rangle$:*

*(a) $\mathscr{B} \subseteq \mathscr{A}_{rgs}$ is grounded (*grd*) iff $\mathscr{B}$ is complete and $\forall \mathscr{B}' \in com(F)$, $\mathscr{B} \subseteq \mathscr{B}'$. The grounded extension is the minimal (w.r.t. set inclusion) complete set [9].*
*(b) $\mathscr{B} \subseteq \mathscr{A}_{rgs}$ is ideal (*ide*) iff $\mathscr{B}$ is admissible and $\forall \mathscr{B}' \in prf(F)$, $\mathscr{B} \subseteq \mathscr{B}'$. The ideal extension is the maximal (w.r.t. set inclusion) ideal set [10].*
*(c) $\mathscr{B} \subseteq \mathscr{A}_{rgs}$ is eager (*eag*) iff $\mathscr{B}$ is admissible and $\forall \mathscr{B}' \in sst(F)$, $\mathscr{B} \subseteq \mathscr{B}'$. The eager extension is the maximal (w.r.t. set inclusion) eager set [7].*

In the following, we use $WF_{\Downarrow}$ to denote the classical framework $F = \langle \mathscr{A}_{rgs}, R\rangle$ of Dung [9] that can be obtained by just lifting W and $\mathbb{S}$ from $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$. In practice, $WF_{\Downarrow}$ drops the weighted system in WF.

3 Definitions and Formal Results

We focus on weighted sceptical semantics, starting from the grounded one. If we directly extend Definition 5a to WAAFs, there are frameworks where the set of complete extensions has more than one minimal element: hence, there is no unique least-set as it happens in [9], and as we also desire for WAAFs.

[2] Even if new and not in [3], we introduce this semantics in Definition 4 for the sake of presentation.

[3] Since Dung's definitions of semantics are directly encompassed by our framework (just by using the *Boolean* semiring), we do not introduce them in this paper for the sake of brevity.

For example, we consider a WAAF with arguments $\mathscr{A}_{rgs} = \{a, b, c, d\}$, and $R(a, b)$, $R(b, c)$, $R(b, d)$, all with a weight of 1 (using the *Weighted* semiring): $W(a, b) = 1, W(b, c) = 1, W(b, d) = 1$. The set of w-complete extensions is $\{\{a, c\}, \{a, d\}\}$, and then there is no single least element. According to [9] instead, the grounded extension in $WF_{\Downarrow}$ is $\{a, c, d\}$: the least element exists.

Since our goal is to preserve its uniqueness in WAAFs, we improve Definition 5 in order to always have one single solution. We follow the same approach used in [1] to define the ideal and eager semantics.

Definition 6 (w-grounded). *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, an extension $\mathscr{B} \in wgrd(F)$, iff $\mathscr{B} \in wadm(WF)$, and $\mathscr{B} \subseteq \bigcap wcom(WF)$, and $\nexists \mathscr{B}' \in wadm(WF)$ satisfying $\mathscr{B}' \subseteq \bigcap wcom(WF)$ s.t. $\mathscr{B} \subsetneq \mathscr{B}'$.*

In words, the w-grounded extension is any maximal (w.r.t. set inclusion) w-admissible extension included in the intersection of w-complete extensions. We now relate the w-grounded semantics in Definition 6 to the classical one by Dung [9].

Proposition 1. *In $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, if $\mathbb{S}$ is* Boolean *then the w-grounded extension is equivalent to the classical grounded extension on $WF_{\Downarrow}$.*

Moreover, from Definition 6 we can derive some noticeable properties in the following.

Proposition 2. *The w-grounded extension always exists and is unique.*

Proposition 3. *The w-grounded extension corresponds to the set of sceptically accepted arguments in $wcom(WF)$: $grd(WF) = \{a \in \mathscr{A}_{rgs} \mid \forall \mathscr{B} \in wcom(WF), a \in \mathscr{B}\}$.*

According to Proposition 2, the w-grounded extension is a subset of $\bigcap wcom(WF)$, which always exists, is unique and w-admissible. This uniqueness is novel w.r.t. [8,11,12], where the described frameworks offer several grounded scenarios. Furthermore, when there is only one minimal w-complete extension, it corresponds to the w-grounded one.

Theorem 1. *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, and $\mathbb{S}$ any semiring, if $\forall \mathscr{B}' \in wcom(F)$, $\mathscr{B} \in wcom(F)$ s.t. $\mathscr{B} \subseteq \mathscr{B}'$, then $\mathscr{B} = wgrd(WF)$. Consequently, $\mathscr{B}$ is w-complete.*

Theorem 1 is always satisfied when the *Boolean* semiring is used. Moreover, we have that the w-grounded extension is a subset of each minimal w-complete extension.

Proposition 4. *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, $\mathbb{S}$ any semiring, and $wcom_{\subseteq}(WF) = \{\mathscr{B} \in wcom(WF) \mid \nexists \mathscr{B}' \in wcom(WF). \mathscr{B}' \subset \mathscr{B}'\}$, then $\forall \mathscr{B} \in wcom_{\subseteq}(WF). wgrd(WF) \subseteq \mathscr{B}$.*

In addition, each minimal w.r.t. set inclusion of the w-complete extensions is always a subset of the classical grounded extension in Definition 5a.

Proposition 5. *Given* $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S} \rangle$, $\mathbb{S}$ *any semiring, and* $wcom_{\subseteq}(WF) = \{\mathscr{B} \in wcom(WF) \mid \nexists \mathscr{B}' \in wcom(WF). \mathscr{B}' \subset \mathscr{B}'\}$, *then* $\forall \mathscr{B} \in wcom_{\subseteq}(WF). \mathscr{B} \subseteq grd(WF_{\Downarrow})$.

We now introduce the w-strongly-admissible semantics, from which Proposition 6 follows, relating it with the w-grounded one.

Definition 7 (w-strongly-admissible). *Given* $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S} \rangle$, $\mathscr{B} \subseteq \mathscr{A}_{rgs}$ *is* w*-strongly-admissible iff every* $b \in \mathscr{B}$ *is* w*-defended by some* $\mathscr{B}' \subseteq \mathscr{B} \setminus \{b\}$.

Proposition 6. *The* w*-grounded extension is* w*-strongly-admissible.*

In Fig. 1 we present a Hasse diagram summarising some of the formal results above. The reference WAAF is $\mathscr{A}_{rgs} = \{a, b, c, d\}$, and $R(a, b)$, $R(b, c)$, $R(b, d)$. The w-grounded extension is $\{a\}$, the minimal w-complete ones are $\{a, c\}$ and $\{a, d\}$, and the grounded extension (Definition 5) is $\{a, c, d\}$. Hence, $\{a\}$ corresponds to the set of sceptically accepted arguments in $wcom(WF)$ (Proposition 3), $\{a\} \subseteq \{a, c\}$ and $\{a\} \subseteq \{a, d\}$ (Proposition 4), and both $\{a, c\}$ and $\{a, d\}$ are a subset of $\{a, c, d\}$ (Proposition 5). In the following we extend the other two well-known sceptical semantics in order to make them consider weights:

Definition 8 (w-ideal). *Given* $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S} \rangle$, $\mathscr{B} = w$*-ideal(F) (*wide*), iff* $\mathscr{B}$ *is* w*-admissible and* $\forall \mathscr{B}' \in wprf(WF)$, $\mathscr{B} \subseteq \mathscr{B}'$. *The* w*-ideal extension is the maximal (w.r.t. set inclusion)* w*-ideal set.*

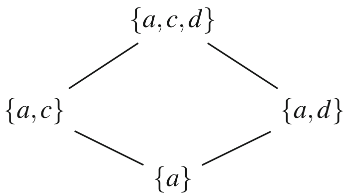

Fig. 1. Given the WAAF with $\mathscr{A}_{rgs} = \{a, b, c, d\}$, and $R(a, b)$, $R(b, c)$, $R(b, d)$, the w-grounded extension is $\{a\}$, the minimal w-complete (they are w-admissible and include all w-defended arguments) ones are $\{a, c\}$ and $\{a, d\}$, and the grounded extension [9] is $\{a, c, d\}$.

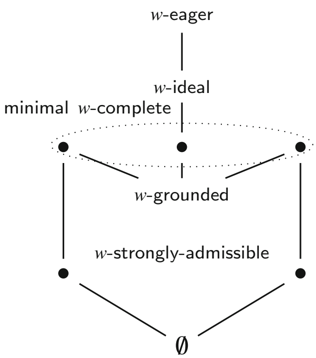

Fig. 2. The partial ordered for sceptical w-semantics.

Definition 9 (w-eager). *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, $\mathscr{B} = w\text{-}eager(F)$ (*weag*), iff $\mathscr{B}$ is admissible and $\forall \mathscr{B}' \in wsst(F)$, $\mathscr{B} \subseteq \mathscr{B}'$. The w-eager extension is the maximal (w.r.t. set inclusion) w-eager set.*

As done in Proposition 1 for the w-grounded semantics, we relate the w-ideal and w-eager semantics to their classical counterparts:

Proposition 7. *In $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, if $\mathbb{S}$ is* Boolean *then the w-ideal and w-eager extensions are equivalent to the classical ideal and eager extension, i.e., $wide(WF) = ide(WF_{\Downarrow})$ and $weag(WF) = eag(WF_{\Downarrow})$.*

We prove the semantics in Definitions 8 and 9 are satisfied by only one extension:

Proposition 8. *The w-ideal and the w-eager are unique-status semantics.*

From Definitions 6, 8 and 9 we obtain the same inclusion-result as for their corresponding unweighted semantics [7]: $grd(F) \subseteq ide(F) \subseteq eag(F)$.

Theorem 2. *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, then $wgrd(WF) \subseteq wide(WF) \subseteq weag(WF)$.*

Then, we can relate sceptical w-semantics to their unweighted counterpart given in Definition 5: each of such extensions results to be a subset of the corresponding original one:

Theorem 3. *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$ and $F = \langle \mathscr{A}_{rgs}, R\rangle$ (same $\mathscr{A}_{rgs}$ and R), then $wgrd(WF) \subseteq grd(WF_{\Downarrow})$, $wide(WF) \subseteq ide(WF_{\Downarrow})$, and $weag(WF) \subseteq eag(WF_{\Downarrow})$.*

Classical strongly-admissible extensions form a lattice w.r.t. $\subseteq$ [7], that is a partial order where all the subsets of elements have an infimum, in this case $\emptyset$, and a supremum, in this case the grounded extension. With w-strongly-admissible ones instead, only a semi-lattice can be obtained: $\emptyset$ is still the infimum, but no supremum in this case. The minimal w-complete extensions are the maximal elements of such a semi-lattice.

Theorem 4. *Given $WF = \langle \mathscr{A}_{rgs}, R, W, \mathbb{S}\rangle$, the set of w-strongly-admissible extensions forms a semi-lattice w.r.t. $\subseteq$, with $\emptyset$ as the infimum. With $WF = \langle \mathscr{A}_{rgs}, R, W, Boolean\rangle$, the lattice structure is preserved, with $\emptyset$ as infimum and $grd(WF_{\Downarrow})$ as supremum.*

Figure 2 presents the Hasse diagram (w.r.t. $\subseteq$) for sceptical semantics, considering a generic semiring: in case of a *Boolean* semiring, we still have a complete lattice. Note that w-ideal and w-eager are not w-strongly-admissible, as for classical frameworks [7].

4 Conclusion

We have extended the weighted framework in [3] by proposing a unique-status grounded semantics, and a Hasse diagram that represents the partial order - w.r.t. set inclusion- among sceptical extensions and w-strongly admissible ones. By having a general framework based on semirings, it is easier to check which relations among semantics change when the defence considers weights. According to its sceptical nature, it is desirable to provide a single grounded extension, differently from the frameworks in [8,11,12].

In the future we will study the framework in [6], which partitions the arguments into sets satisfying the same semantics. In addition, we would like to define the upper part of the Hasse diagram: for instance, what the relation is between w-preferred and w-stable extensions, or the conditions when the preferred or stable extensions are unique.

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