Incoherence Correction and Decision Making Based on Generalized Credal Sets

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Abstract. While making decisions we meet different types of uncertainty. Recently the concept of generalized credal set has been proposed for modeling conflict, imprecision and contradiction in information. This concept allows us to generalize the theory of imprecise probabilities giving us possibilities to process information presented by contradictory (incoherent) lower previsions. In this paper we propose a new way of introducing generalized credal sets: we show that any contradictory lower prevision can be represented as a convex sum of non-contradictory and fully contradictory lower previsions. In this way we can introduce generalized credal sets and apply them to decision problems. Decision making is based on decision rules in the theory of imprecise probabilities and the contradiction-imprecision transformation that looks like incoherence correction.

Keywords: Contradictory (incoherent) lower previsions · Decision making · Generalized credal sets · Incoherence correction

1 Introduction

Recently the extension of imprecise probabilities based on generalized credal sets has been proposed [3,4]. By the classical theory of imprecise probabilities [1,2,6,8] we can model two types of uncertainty: conflict associated with probability measures and imprecision (non-specificity) linked with the choice of a probability measure among possible alternatives. Generalized credal sets allow us also to model contradiction when the avoiding sure loss condition is not fulfilled. Each upper generalized credal set consists of special plausibility functions, conceived as lower probabilities, whose bodies of evidence consist of singletons and certain event. The part consisting of singletons models conflict in information and the part described by a certain event models contradiction.

In our previous research [3,4] we have shown how we can work with contradictory lower and upper previsions based on generalized credal sets, we introduce

A. Antonucci et al. (Eds.): ECSQARU 2017, LNAI 10369, pp. 271–281, 2017.
DOI: 10.1007/978-3-319-61581-3_25

the construction like natural extension in the classical theory of imprecise probabilities, we describe conditions when generalized credal sets generate models based on usual imprecise probabilities.

In the paper we show how generalized credal sets can be used for correcting incoherent information and how they can be applied to decision problems.

2 Monotone Measures: Basic Definitions and Notations

Let $X = \{x_1, ..., x_n\}$ be a finite set of elementary events, and let 2^X be the algebra of all subsets of X. A set function $\mu : 2^X \to [0, 1]$ is called a *monotone measure* if $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $\mu(A) \leqslant \mu(B)$ for any $A, B \in 2^X$ such that $A \subseteq B$. A monotone measure μ is

- a *probability measure* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any $A, B \in 2^X$ such that $A \cap B = \emptyset$;
- a *belief function* if there is a set function $m : 2^X \to [0, 1]$ called the *basic belief assignment* (bba) with $m(\emptyset) = 0$ and $\sum_{B \in 2^X} m(B) = 1$ such that $\mu(A) = \sum_{B \subseteq A} m(B)$.

In the sequel M_{mon} denotes the set of all monotone measures on 2^X; M_{pr} denotes the set of all probability measures on 2^X; and M_{bel} denotes the set of all belief functions on 2^X.

We define on M_{mon} the following operations and relations:

- $\mu = a\mu_1 + (1 - a)\mu_2$ for $\mu_1, \mu_2 \in M_{mon}$ and $a \in [0, 1]$ if $\mu(A) = a\mu_1(A) + (1 - a)\mu_2(A)$ for all $A \in 2^X$;
- $\mu_1 \leqslant \mu_2$ for $\mu_1, \mu_2 \in M_{mon}$ if $\mu_1(A) \leqslant \mu_2(A)$ for all $A \in 2^X$;
- μ^d is the dual of μ if $\mu^d(A) = 1 - \mu(A^c)$ for all $A \in 2^X$, where A^c is the complement of A.

Let $Bel \in M_{bel}$ with bba m, then

- Bel^d is called a *plausibility function*;
- a set $B \in 2^X$ is called a *focal element* if $m(B) > 0$;
- the set of all focal elements is called the *body of evidence*;
- a belief function is called *categorical* if its body of evidence contains one focal element $B \in 2^X$. This set function is denoted by $\eta_{\langle B \rangle}$ and can be computed as $\eta_{\langle B \rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & B \not\subseteq A. \end{cases}$;
- Any $Bel \in M_{bel}$ with bba m can represented as a convex sum of categorical belief functions $Bel = \sum\limits_{B \in 2^X} m(B)\eta_{\langle B \rangle}$.

Assume that M is an arbitrary subset of M_{mon}, then $M^d = \{\mu^d | \mu \in M\}$. In such a way M^d_{bel} denotes the set of all plausibility functions on 2^X.

3 Credal Sets, Lower and Upper Previsions

In the following any $P \in M_{pr}$ can be represented as a point $(P(\{x_1\}), ..., P(\{x_n\}))$ in $\mathbb{R}^n$. By definition [1,6], a *credal set* $\mathbf{P}$ is a non-empty subset of M_{pr}, which is convex and closed. Convexity of $\mathbf{P}$ means that $P_1, P_2 \in \mathbf{P}$ and $a \in [0,1]$ implies that $aP_1 + (1-a)P_2 \in \mathbf{P}$, and $\mathbf{P}$ is closed as a subset of $\mathbb{R}^n$. A model based on credal sets is one of the most general models of imprecise probabilities. We can describe credal sets using lower and upper previsions. Let K be a set of all real-valued functions $f : X \to \mathbb{R}$ on X. Then any $f \in K$ can be viewed as a random variable for a fixed $P \in M_{pr}$ and we can compute its expectation defined by $E_P(f) = \sum\limits_{x \in X} f(x)P(\{x\})$. Let K' be an arbitrary subset of K, then any functional $\underline{E} : K' \to \mathbb{R}$ is called a *lower prevision* if each value $\underline{E}(f)$, $f \in K'$, is viewed as a lower bound of expectation of the random variable f. This lower prevision is called non-contradictory (or it avoids sure loss) iff it defines the credal set

$$\mathbf{P}(\underline{E}) = \{P \in M_{pr} | \forall f \in K' : E_P(f) \geqslant \underline{E}(f)\} \tag{1}$$

Otherwise, when the set $\mathbf{P}(\underline{E})$ is empty, the lower prevision is called *contradictory* (or *incoherent*). Analogously, upper previsions are defined. Any functional $\bar{E} : K' \to \mathbb{R}$ is called an *upper prevision* if its values are viewed as upper bounds of expectations. It is non-contradictory (or it avoids sure loss) iff it defines the credal set

$$\mathbf{P}(\bar{E}) = \left\{P \in M_{pr} | \forall f \in K' : E_P(f) \leqslant \bar{E}(f)\right\},$$

and it incurs sure loss otherwise. Models of uncertainty based on upper and lower previsions are equivalent. It follows from the fact that every lower prevision $\underline{E} : K' \to \mathbb{R}$ and the corresponding upper prevision

$$\bar{E}(f) = -\underline{E}(-f), \quad -f \in K',$$

define the same credal set. The central role in reasoning based on imprecise probabilities plays the natural extension. Let $\underline{E} : K' \to \mathbb{R}$ be an non-contradictory lower prevision and P(E) be the credal set defined by formula (1), then the *natural extension* of $\underline{E}$ is a functional

$$\underline{E}'(f) = \inf_{P \in \mathbf{P}(\underline{E})} E_P(f), \quad f \in K'.$$

A lower prevision $\underline{E}$ is called *coherent* if $\underline{E}(f) = \underline{E}'(f)$ for all $f \in K'$. Analogously the natural extension of non-contradictory upper previsions is defined and coherent upper previsions are introduced. Monotone measures can be considered as special models of lower and upper previsions. In this case $K' = \{1_A\}_{A \in 2^X}$, where 1_A is the characteristic function of the set A, i.e. $\mu(A) = \underline{E}(1_A)$, $A \in 2^X$, can be viewed as a set function. A monotone measure μ is called a lower probability if its values give us lower bounds of probabilities. It is non-contradictory if it defines the credal set $\mathbf{P}(\mu) = \{P \in M_{pr} | \mu \leqslant P\}$. We can define analogously the natural extension of non-contradictory lower previsions and the family of coherent lower probabilities. In the same way we define upper probabilities that give us upper bounds of probabilities, the natural extension of non-contradictory upper probabilities and coherent upper probabilities.

Remark 1. Obviously, $\min_{x\in X} f(x) \leqslant E_P(f) \leqslant \max_{x\in X} f(x)$ for any $P \in M_{pr}$ and $f \in K$. Thus, without decreasing generality we can assume that values $\underline{E}(f)$ of any lower prevision $\underline{E} : K' \to \mathbb{R}$ should be not larger than $\max_{x\in X} f(x)$, i.e. $\underline{E}(f) \leqslant \max_{x\in X} f(x)$ for any $f \in K'$. Analogously, we will assume that $\bar{E}(f) \geqslant \min_{x\in X} f(x)$ for any upper prevision $\bar{E} : K' \to \mathbb{R}$ and $f \in K'$, This assumption will be used later without mentioning about it.

4 Generalized Credal Sets for Describing Contradictory Lower Previsions

Assume that we have estimates $\hat{p}(x_i)$, $i = 1, ..., n$, of probabilities, but unfortunately $\sum_{i=1}^{n} \hat{p}(x_i) \neq 1$. What should we do? One can say that the available information is defective and it is not possible to use it. But if the value $\varepsilon = |\sum_{i=1}^{n} \hat{p}(x_i) - 1|$ is small, then this conclusion seems to be not useful. Otherwise we should correct $\hat{p}(x_i)$. Assume that $\sum_{i=1}^{n} \hat{p}(x_i) < 1$, then the correction can be done by adding to each $\hat{p}(x_i)$ a value $\alpha_i \geqslant 0$ such that $\sum_{i=1}^{n} (\hat{p}(x_i) + \alpha_i) = 1$. Thus, uncertainty can be modeled by the set of probability distributions

$$\left\{ (p(x_1), ..., p(x_n)) \, | p(x_i) \geqslant \hat{p}(x_i), i = 1, ..., n, \sum\nolimits_{i=1}^{n} p(x_i) = 1 \right\}.$$

Observe that in this case values $\hat{p}(x_i)$ looks like lower bounds of probabilities, but this does not follow from the problem statement. To avoid ambiguity we should decide whether $\hat{p}(x_i)$ give us lower or upper bounds of probabilities. Lower probabilities have been intensively investigated in the theory of imprecise probabilities and they describe two types of uncertainty: conflict associated with probability measures and non-specificity linked with the choice of a probability measure among possible alternatives. If values $\hat{p}(x_i)$ are viewed as upper probabilities then we say that the available information incurs sure loss or it is contradictory.

Let us analyze the above model in detail. If $\hat{p}(x_i) = 0$, $i = 1, ..., n$, and $\hat{p}(x_i)$ are viewed as lower bounds of probabilities, then the set

$$\left\{ (p(x_1), ..., p(x_n)) \, | p(x_i) \geqslant 0, i = 1, ..., n, \sum\nolimits_{i=1}^{n} p(x_i) = 1 \right\}$$

contains all possible probability distributions or probability measures on 2^X. Thus, in such a case, values $\hat{p}(x_i) = 0$, $i = 1, ..., n$, describe the situation of complete ignorance. We will describe this situation by a vacuous belief function $\eta_{\langle X \rangle}$ viewed as lower probability. Analogously, if $\hat{p}(x_i) = 0$, $i = 1, ..., n$, are viewed as upper bounds of probabilities, then we can describe contradiction by the set of all probability measures M_{pr}, or by $\eta_{\langle X\rangle}$ viewed as an upper probability. This situation can be understood as the case of full contradiction.

Although we can describe contradiction and non-specificity by the set of probability measures there is a principal difference between these two types of uncertainty. Non-specificity means that we don't know exactly what kind of probability model should be chosen among possible alternatives, but contradiction means that we have some deficiency in estimating probabilities. The last problem appears when we try to use simultaneously different probabilistic models for analyzing statistical data or to aggregate pieces of evidence from separate sources of information.

Let us remind the notion of contradiction from usual logic. Let we have a set of axioms A_1,...,A_m, and if we use the set-theoretical model, then any A_i can be represented as a subset of a finite set X. Then this system of axioms is contradictory iff $A_1 \cap ... \cap A_m = \emptyset$. In logic we can infer from the contradictory system of axiom that any conclusion is true. This situation can be described by the contradictory lower probability

$$\eta_X^d(A) = \begin{cases} 1, \; A \neq \emptyset, \\ 0, \; A = \emptyset. \end{cases}$$

Thus, the case of full contradiction can be described by any lower probability $\mu \in M_{mon}$ such that $\mu(A_1) = ... = \mu(A_m) = 1$ and $A_1 \cap ... \cap A_m = \emptyset$. In general the case of full contradiction can be described by the following definition.

Definition 1. The information described by a lower prevision $\underline{E} : K' \to \mathbb{R}$ is *fully contradictory* iff $\underline{E}$ can not be represented as a convex sum $\underline{E}(f) = a\underline{E}^{(1)}(f) + (1-a)\underline{E}^{(2)}(f)$ of a non-contradictory lower prevision $\underline{E}^{(1)} : K' \to \mathbb{R}$, and a (contradictory) lower prevision $\underline{E}^{(2)} : K' \to \mathbb{R}$ for some $a \in (0, 1]$.

Lemma 1. *A lower prevision $\underline{E} : K' \to \mathbb{R}$ is fully contradictory iff for any $a \in (0,1]$ the lower prevision $\underline{E}'(f) = \frac{1}{a}\left(\underline{E}(f) - (1-a)\max_{x\in X} f(x)\right)$, $f \in K'$, is contradictory.*

Lemma 2. *If the set of contradictory previsions on K' is not empty, then the lower prevision $\hat{\underline{E}}(f) = \max_{x\in X} f(x)$, $f \in K'$, is fully contradictory.*

Remark 2. It is possible to choose K' such that every lower prevision is non-contradictory. In this case $\hat{\underline{E}}$ is also a non-contradictory lower prevision. Because the aim of the paper is to deal with contradictory information, in the next we will assume that K' is chosen providing the lower prevision $\hat{\underline{E}}$ to be fully contradictory.

Let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision. Then by Lemma 1 and Lemma 2 (see also Remark 2) it can be always represented as a convex sum

$$\underline{E}(f) = a\underline{E}^{(1)}(f) + (1-a)\underline{E}^{(2)}(f), \tag{2}$$

where $\underline{E}^{(1)}$ is a non-contradictory lower prevision and a lower prevision $\underline{E}^{(2)}$ is fully contradictory. If $a \in (0, 1]$, then by Lemma 1 $\underline{E}^{(2)}$ can be chosen to be equal to $\hat{\underline{E}}$. If the lower prevision $\underline{E}$ is fully contradictory, then $\underline{E}^{(2)} = \underline{E}$, $a = 0$, and we can take a non-contradictory lower prevision $\underline{E}^{(1)}$ arbitrarily. We see that the largest value of a characterizes the amount of contradiction in $\underline{E}$. Thus, we can introduce the following definition.

Definition 2. Let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision and let A be the set of all possible values $a \in [0, 1]$, for which the representation (2) exists for some non-contradictory lower prevision $\underline{E}^{(1)}$ and a fully contradictory lower prevision $\underline{E}^{(2)}$. Then the *amount of contradiction* is defined by $Con(\underline{E}) = 1 - \sup\{a|a \in A\}$.

Obviously, by Definition 2 $Con(\underline{E}) = 0$ iff $\underline{E}$ is a non-contradictory lower prevision, and $Con(\underline{E}) = 1$ iff $\underline{E}$ is fully contradictory. Let us introduce new concepts, which will help us to simplify the computation of $Con(\underline{E})$. Consider monotone measures on 2^X of the type

$$P = a_0\eta^d_{\langle X\rangle} + \sum_{i=1}^{n} a_i\eta_{\langle\{x_i\}\rangle}, \tag{3}$$

where $\sum\limits_{i=0}^{n} a_i = 1$, $a_i \geqslant 0$, $i = 0, ..., n$, and P is viewed as a lower probability. Such a P can be represented also as $P = a_0\eta^d_{\langle X\rangle} + (1 - a_0)P'$, where $\eta^d_{\langle X\rangle}$ is a fully contradictory lower probability and P' is a probability measure defined by $P' = \frac{1}{1-a_0}\sum\limits_{i=1}^{n} a_i\eta_{\langle\{x_i\}\rangle}$ for $a_0 \neq 1$. We can extend P to the lower prevision on the set of all functions in K by

$$\underline{E}_P(f) = a_0 \max_{x\in X} f(x) + \sum_{i=1}^{n} a_i f(x_i).$$

Again $\underline{E}_P$ can be represented as a convex sum of fully contradictory lower prevision $\hat{\underline{E}}$ and linear prevision $E_{P'}$, i.e. $\underline{E}_P(f) = a_0\hat{\underline{E}}(f) + (1 - a_0)E_{P'}(f)$ for all $f \in K$. We will denote by M_{cpr} the set of all monotone measures defined by (3).

Lemma 3. *Let* $P = a_0\eta^d_{\langle X\rangle} + \sum\limits_{i=1}^{n} a_i\eta_{\langle\{x_i\}\rangle}$ *be in* M_{cpr}*. Then* $Con(P) = a_0$*.*

We will identify each $P \in M_{cpr}$ from (3) with a point $(a_1, ..., a_n)$ in $\mathbb{R}^n$. Let $P_1, P_2 \in M_{cpr}$ and $P_i = (a_1^{(i)}, ..., a_n^{(i)})$, $i = 1, 2$, then $P_1 \leqslant P_2$ iff $a_k^{(1)} \geqslant a_k^{(2)}$, $k = 1, ..., n$. Clearly, such P_1 and P_2 can describe the same information, but P_2 is a lower probability with higher contradiction.

Definition 3. A subset $\mathbf{P}$ of M_{cpr} is called an *upper generalized credal set* (UG-credal set) if

(1) $P_1 \in \mathbf{P}$, $P_2 \in M_{cpr}$, and $P_1 \leqslant P_2$ implies $P_2 \in \mathbf{P}$;

(2) $P_1, P_2 \in \mathbf{P}$ implies $aP_1 + (1-a)P_2 \in \mathbf{P}$ for every $a \in [0,1]$;
(3) $\mathbf{P}$ is a closed set if we consider it as a subset of $\mathbb{R}^n$.

We will describe any lower prevision $\underline{E} : K' \to \mathbb{R}$ by a UG-credal set $\mathbf{P}$ defined by

$$\mathbf{P} = \{P \in M_{cpr} | \forall f \in K' : \underline{E}(f) \leqslant \underline{E}_P(f)\}. \tag{4}$$

Remark 3. Obviously, the set defined by (4) is not empty, because it always contains the measure $\eta^d_{\langle X \rangle}$.

Proposition 1. *Let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision, and let $\mathbf{P}$ be its corresponding UG-credal set defined by (4). Then*

$$Con(\underline{E}) = \inf \{Con(P) | P \in \mathbf{P}\}. \tag{5}$$

5 Decision Making Based on Contradictory Lower Previsions

Assume that $\underline{E} : K' \to \mathbb{R}$ is a lower prevision and $Con(\underline{E}) = b$. If $b = 1$ then $\underline{E}$ is fully contradictory and $\underline{E}$ does not contain useful information. Therefore, this case is identical to the case of complete ignorance. Let $b < 1$, then for any $a \in (0, 1-b]$ the lower prevision $\underline{E}$ can be represented as $\underline{E}(f) = a\underline{E}^{(1)}(f) + (1-a)\underline{E}^{(2)}(f)$, $f \in K'$, where $\underline{E}^{(1)}$ is a non-contradictory and $\underline{E}^{(2)}$ is a fully contradictory lower prevision. Obviously, decision making should be based on information in $\underline{E}^{(1)}$. Notice also that decreasing parameter a we get information in $\underline{E}^{(1)}$ more imprecise. Therefore, it makes a sense taking $a = 1-b$. It is also possible to choose $\underline{E}^{(2)} = \hat{\underline{E}}$. After this choice the above representation can be rewritten as $\underline{E}(f) = (1-b)\underline{E}^{(1)}(f) + b\hat{\underline{E}}(f)$, $f \in K'$.

Assume that a non-contradictory lower prevision $\underline{E}^{(1)}$ defines the credal set $\mathbf{P}' = \left\{P \in M_{pr} | \forall f \in K' : \underline{E}^{(1)}(f) \leqslant E_P(f)\right\}$. Then taking in account that $\hat{\underline{E}}$ describes the case of full contradiction, we can describe $\underline{E}$ by a credal set $\mathbf{P}''$ represented as a convex sum of two credal sets in which the first is $\mathbf{P}'$ and the second describes the case of complete ignorance, i.e.

$$\mathbf{P}'' = \{(1-b)P_1 + bP_2 | P_1 \in \mathbf{P}', P_2 \in M_{pr}\}. \tag{6}$$

The following proposition shows how the above set $\mathbf{P}''$ can be found based on UG-credal sets.

Proposition 2. *Let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision, $Con(\underline{E}) = b$, and let $\mathbf{P}$ be its corresponding UG-credal set. Then*

$$\mathbf{P}'' = \{P' \in M_{pr} | \exists P \in \mathbf{P} : Con(P) = b, P' \leqslant P\}. \tag{7}$$

The above transformation $\underline{E}: K' \to \mathbb{R}$ of a contradictory lower prevision to the non-contradictory information presented by the credal set $\mathbf{P}''$ can be considered as incoherence correction in which full contradiction is transformed to complete ignorance. After this transformation we can use known models of decision making considered in imprecise probabilities. In our paper we will consider the decision rule justified in many works (e.g. [1,8].).

We will identify each decision with a function in K. Assume that available information is described by a credal set $\mathbf{P}'' \subseteq M_{pr}$. Then decision $f_2 \in K$ is at least preferable as decision $f_1 \in K$ ($f_1 \preccurlyeq f_2$) if $E_{P'}(f_1) \leqslant E_{P'}(f_2)$ for every $P' \in \mathbf{P}''$. This rule can be rewritten as $f_1 \preccurlyeq f_2$ if $\underline{E}_{\mathbf{P}''}(f_2 - f_1) \geqslant 0$, where $\underline{E}_{\mathbf{P}''}(f) = \inf_{P \in \mathbf{P}''} \underline{E}_P(f)$, $f \in K$.

Lemma 4. *Let we use notations as in formula (6). Then the expression for $\underline{E}_{\mathbf{P}''}(f)$ can be transformed to*

$$\underline{E}_{\mathbf{P}''}(f) = (1-b)\underline{E}_{\mathbf{P}'}(f) + b \min_{x \in X} f(x),$$

where $\underline{E}_{\mathbf{P}'}(f) = \inf_{P \in \mathbf{P}'} \underline{E}_P(f)$.

Let us consider the computational scheme by which this decision rule can be realized. A function $f \in K$ is called *normalized from above* if $\max_{x \in X} f(x) = 0$. The following lemma shows how we can normalize functions for a given lower prevision.

Lemma 5. *Let $\underline{E}: K' \to \mathbb{R}$ be a lower prevision. Consider the set $K'' = \left\{ \bar{f} = f - \max_{x \in X} f(x) | f \in K' \right\}$ of normalized from above functions. Then a lower prevision $\underline{E}': K'' \to \mathbb{R}$ defines the same UG-credal set as $\underline{E}$ if $\underline{E}'(\bar{f}) = \underline{E}(f) - \max_{x \in X} f(x)$ for all $f \in K'$.*

Clearly the above lemma allows us to assume that functions in K', on which a lower prevision $\underline{E}$ is defined, are normalized from above.

Proposition 3. *Let K' be a finite subset of normalized functions from above in K and let $\underline{E}: K' \to \mathbb{R}$ be a lower prevision. Then $Con(\underline{E}) = \max\{0, b\}$, where b is the solution of the following linear programming problem:*

$$b = 1 - \sum_{i=1}^{n} a_i \to \min,$$

$$\begin{cases} \sum\limits_{i=1}^{n} a_i f_k(x_i) \geqslant \underline{E}(f_k), & f_k \in K', \\ \quad a_i \geqslant 0, & i = 1, ..., n, \end{cases}$$

Proposition 4. *Let K' be a finite subset of normalized functions from above in K and let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision with $Con(\underline{E}) = b$. Then $c = (1-b)\underline{E}_{\mathbf{P}'}(f)$ for any $f \in K$ is the solution of the following linear programming problem:*

$$c = \sum_{i=1}^{n} a_i f(x_i) \to \min,$$

$$\begin{cases} \sum\limits_{i=1}^{n} a_i f_k(x_i) \geqslant \underline{E}\left(f_k\right), \quad f_k \in K', \\ \sum\limits_{i=1}^{n} a_i = 1 - b, \quad a_i \geqslant 0, \quad i = 1, ..., n. \end{cases}$$

Example 1. Let we have two pieces of evidence. The first says that the probability that it will be sunny tomorrow is higher or equal than 0.3. The second says that the probability of rain is higher or equal than 0.8. We can describe this information by the states of the world: $x_1 :=$ *sunny*, $x_2 :=$ *rain*, and denote $X = \{x_1, x_2\}$. Then we have $\underline{E}\left(1_{\{x_1\}}\right) = 0.3$, $\underline{E}\left(1_{\{x_2\}}\right) = 0.8$. For using our computational scheme functions $1_{\{x_1\}}$ and $1_{\{x_2\}}$ should be normalized from above. Doing it we get functions $f_1 = 1_{\{x_1\}} - 1_X$ and $f_2 = 1_{\{x_2\}} - 1_X$ with $\underline{E}\left(f_1\right) = -0.7$ and $\underline{E}\left(f_2\right) = -0.2$. Then the amount of contradiction can be computed by solving the following linear programming problem:

$$b = 1 - a_1 - a_2 \to \min$$

$$\begin{cases} -a_2 \geqslant -0.7, \\ -a_1 \geqslant -0.2, \\ a_1, a_2 \geqslant 0. \end{cases}$$

Thus, $b = 0.1$. Assume that we need to compute $c = (1-b)\underline{E}_{\mathbf{P}'}(f)$ for some $f \in K$. Then c can be computed by solving the following linear programming problem:

$$c = a_1 f(x_1) + a_2 f(x_2) \to \min,$$

$$\begin{cases} -a_2 \geqslant -0.7, \\ -a_1 \geqslant -0.2, \\ a_1 + a_2 = 0.9, \;\; a_1, a_2 \geqslant 0. \end{cases}$$

Thus, $c = 0.2 f(x_1) + 0.7 f(x_2)$. In this case by Lemma 4 $\underline{E}_{\mathbf{P}''}(f) = 0.2 f(x_1) + 0.7 f(x_2) + 0.1 \min\limits_{x \in X} f(x)$. Assume, for example, that we have two decisions: $g_1 :=$ *go to the park*; $g_2 :=$ *go to the theater*; defined by $g_1(x_1) = 3$, $g_1(x_2) = -1$, $g_2(x_1) = 1$, $g_2(x_2) = 1$. Then

$$\underline{E}_{\mathbf{P}''}(g_2 - g_1) = 0.2 \cdot (-2) + 0.7 \cdot 2 + 0.1 \cdot (-2) = 0.8 > 0,$$

i.e. decision g_2 is more preferable than decision g_1.

6 The Comparison with Previous Works

Incoherence correction has been considered in the papers by A. Capotorti and others (see [5] and references therein), and in the work [7] by E. Quaeghebeur. The main idea described in [5] is to use distances between incoherent lower prevision and the set of all possible coherent previsions, i.e. the best approximation is to use the closest coherent lower prevision to the available assessments. Among possible distances (divergences) are L_1- and L_2-distances, the logarithmic Bregman divergence, the discrepancy measure. In [7] the correction is produced by the lower envelope of maximal coherent lower previsions, which are lower than a given incoherent lower prevision.

Let us compare the incoherence correction based on generalized credal sets and the mentioned above approaches. Assume that $\mu \in M_{mon}$ is an upper envelope of the set of probability measures $\mathbf{P}$, i.e.

$$\mu(A) = \sup_{P \in \mathbf{P}} P(A), \quad A \in 2^X,$$

but it is viewed as a lower probability. Obviously, μ is a contradictory lower probability if $\mathbf{P}$ contains at least two different probability measures. If we apply methods from [5], then we choose some optimal approximation $P \in \mathbf{P}$ of μ. Thus, using this correction we cannot take in account that information is contradictory - every two decisions are comparable. If we use the approach considered in [7], then obviously after correction we get the coherent lower probability

$$\mu^d(A) = \inf_{P \in \mathbf{P}} P(A), \quad A \in 2^X.$$

Although sometimes corrections based on our approach and this one give us the same result (this is fulfilled for Example 1), but in some cases they can give us sufficiently different results, when, for example, we choose $\mathbf{P}$ such that μ is a fully contradictory lower probability and $\mathbf{P} \neq M_{pr}$. In this case, by our approach μ does not give us useful information, but the Quaeghebeur's approach supposes that μ contains some useful information that seems to be not correct.

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