Comparison of Inference Relations Defined over Different Sets of Ranking Functions

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Abstract. Skeptical inference in the context of a conditional knowledge base $\mathcal{R}$ can be defined with respect to a set of models of $\mathcal{R}$. For the semantics of ranking functions that assign a degree of surprise to each possible world, we develop a method for comparing the inference relations induced by different sets of ranking functions. Using this method, we address the problem of ensuring the correctness of approximating c-inference for $\mathcal{R}$ by constraint satisfaction problems (CSPs) over finite domains. While in general, determining a sufficient upper bound for these CSPs is an open problem, for a sequence of simple knowledge bases investigated only experimentally before, we prove that using the number of conditionals in $\mathcal{R}$ as an upper bound correctly captures skeptical c-inference.

1 Introduction

For a knowledge base $\mathcal{R}$ containing conditionals of the form *If A then usually B*, various semantics have been proposed, e.g. [4,9]. Here, we will consider the approach of ranking functions (or *Ordinal Conditional Functions (OCF)* [10]), assigning a degree of surprise to each possible world. The models of $\mathcal{R}$ are then OCFs accepting all conditionals in $\mathcal{R}$, and every OCF model of $\mathcal{R}$ induces a nonmonotonic inference relation (e.g. [4,9,10]). For any set O of models of $\mathcal{R}$, skeptical inference with respect to O takes all elements of O into account. C-representations are particular ranking functions exibiting desirable inference properties [7], and c-inference is skeptical inference with respect to all c-representations of $\mathcal{R}$ [1].

The two main objectives of this paper are (1) to develop an approach for comparing the inference relations with respect to two different sets of OCFs O and O', and (2) to illustrate how this approach can be used for proving that in the context of c-representations [7], particular upper bounds in a finite domain constraint system are sufficient for correctly modeling skeptical c-inference [1] so that only a subset of all c-representations have to be taken into account.

For checking that the inference relations with respect to O and O' are identical, we introduce the notion of *merged order compatibility* and show that it suffices to check that their *inference cores* coincide if O and O' are merged order compatible. We demonstrate that there are knowledge bases $\mathcal{R}$ such that the set of all ranking modes of $\mathcal{R}$ is not merged order compatible, while at the

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same time the set of all c-representations of $\mathcal{R}$ is merged order compatible. We then investigate how this approach can be employed for c-representations [7] and skeptical c-inference [1]. For the sequence of knowledge bases $\mathcal{R}_n$ considered in [2] that contain only conditional facts of the form $(a|\top)$ we formally prove upper bounds that are sufficient for skeptical c-inference. This indicates that the concepts developed here may be helpful for addressing the open problem of determining upper bounds for general knowledge bases $\mathcal{R}$ that are sufficient for modelling skeptical c-inference for $\mathcal{R}$.

2 Background: Conditional Logic and OCFs

Let $\Sigma = \{v_1, ..., v_m\}$ be a propositional alphabet. A *literal* is the positive (v_i) or negated ($\overline{v_i}$) form of a propositional variable, $\dot{v}_i$ stands for either v_i or $\overline{v_i}$. From these we obtain the propositional language $\mathcal{L}$ as the set of formulas of Σ closed under negation $\neg$, conjunction $\wedge$, and disjunction $\vee$. For shorter formulas, we abbreviate conjunction by juxtaposition (i.e., AB stands for $A \wedge B$), and negation by overlining (i.e., $\overline{A}$ is equivalent to $\neg A$). Let Ω_Σ denote the set of possible worlds over $\mathcal{L}$; Ω_Σ will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$ and can be identified with the set of all complete conjunctions over Σ; we will often just write Ω instead of Ω_Σ. For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world ω. For any propositional formula A let $\Omega_A = \{\omega \in \Omega \mid \omega \models A\}$ be the set of all possible worlds satisfying A.

A *conditional* $(B|A)$ with $A, B \in \mathcal{L}$ encodes the defeasible rule "if A then usually B" and is a trivalent logical entity with the evaluation [5,7]

$$[\![(B|A)]\!]_\omega = \begin{cases} \textit{true} & \text{iff} \quad \omega \models AB \quad \text{(verification)} \\ \textit{false} & \text{iff} \quad \omega \models A\overline{B} \quad \text{(falsification)} \\ \textit{undefined} & \text{iff} \quad \omega \models \overline{A} \quad \text{(not applicable)} \end{cases}$$

An *Ordinal Conditional Function* (OCF, ranking function) [10] is a function $\kappa : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ that assigns to each world $\omega \in \Omega$ an implausibility rank $\kappa(\omega)$: the higher $\kappa(\omega)$, the more surprising ω is. OCFs have to satisfy the normalization condition that there has to be a world that is maximally plausible, i.e., $\kappa^{-1}(0) \neq \emptyset$. The rank of a formula A is defined by $\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$. An OCF κ *accepts* a conditional $(B|A)$, denoted by $\kappa \models (B|A)$, iff the verification of the conditional is less surprising than its falsification, i.e., iff $\kappa(AB) < \kappa(A\overline{B})$. This can also be understood as a nonmonotonic inference relation between the premise A and the conclusion B: We say that A *κ-entails* B, written $A \mathrel{|\!\sim}^\kappa B$, iff κ accepts the conditional $(B|A)$: $\kappa \models (B|A)$ iff $\kappa(AB) < \kappa(A\overline{B})$ iff $A \mathrel{|\!\sim}^\kappa B$.

Note that κ-entailment is based on the total preorder on possible worlds induced by a ranking function κ as $A \mathrel{|\!\sim}^\kappa B$ iff for all $\omega' \in \Omega_{A\overline{B}}$, there is a $\omega \in \Omega_{AB}$ such that $\kappa(\omega) < \kappa(\omega')$.

The acceptance relation is extended as usual to a set $\mathcal{R}$ of conditionals, called a *knowledge base*, by defining $\kappa \models \mathcal{R}$ iff $\kappa \models (B|A)$ for all $(B|A) \in \mathcal{R}$. This is synonymous to saying that κ is *admissible* with respect to $\mathcal{R}$ [6], or that κ is a *ranking model* of $\mathcal{R}$; the set of all ranking models of $\mathcal{R}$ is denoted by $Mod(\mathcal{R})$. $\mathcal{R}$ is *consistent* iff it has a ranking model.

3 Skeptical Inference and Merged Order Inference

While each OCF κ accepting $\mathcal{R}$ induces a nonmonotonic inference relation, also each set O of such ranking functions induces an inference relation determined by taking all elements of O into account.

Definition 1 (skeptical inference). *Let $\mathcal{R}$ be a knowledge base, $O \subseteq Mod(\mathcal{R})$, and $A, B \in \mathcal{L}$.* Skeptical Inference *over O in the context of $\mathcal{R}$, denoted by $\mid\!\sim^{O}_{\mathcal{R}}$, is defined by $A \mid\!\sim^{O}_{\mathcal{R}} B$ iff $A \mid\!\sim^{\kappa} B$ for all $\kappa \in O$.*

Thus, $A \mid\!\sim^{O}_{\mathcal{R}} B$ holds if every $\kappa \in O$ accepts $(B|A)$. The skeptical inference relations defined over two different sets of OCFs may be identical. Instead of having to check the acceptance of all possible conditionals $(B|A)$ with respect to both sets of OCFs, we will investigate conditions under which it suffices to check only so-called base conditionals.

Definition 2 (base conditional). *A* base conditional *over the signature Σ is a conditional of the form $(\omega_1|\omega_1 \vee \omega_2)$ with ω_1, $\omega_2 \in \Omega_\Sigma$ and $\omega_1 \neq \omega_2$.*

Note that a base conditional $(\omega_1|\omega_1 \vee \omega_2)$ is accepted by a ranking model κ, iff $\kappa(\omega_1) < \kappa(\omega_2)$. To characterize the behavior of an inference relation $\mid\!\sim$ for these base conditionals, we define the *inference core* of an inference relation as the reduction of $\mid\!\sim$ from pairs of formulas to pairs of possible worlds.

Definition 3 (inference core, $\lfloor \mid\!\sim \rfloor$). *Let $\mid\!\sim$ be an inference relation. The* inference core *of $\mid\!\sim$, denoted by $\lfloor \mid\!\sim \rfloor$, is the set of all pairs $(\omega_1, \omega_2) \in \Omega \times \Omega$ with $\omega_1 \neq \omega_2$, such that $\omega_1 \vee \omega_2 \mid\!\sim \omega_1$, i.e., $\lfloor \mid\!\sim \rfloor = \{(\omega_1, \omega_2) \mid \omega_1 \vee \omega_2 \mid\!\sim \omega_1\}$.*

The notion of the inference core is based on an inference relation. The corresponding concept of a *merged order* is based solely on a set of ranking models:

Definition 4 (merged order, $<_O$). *Let O be a set of OCFs. The merged order $<_O$ is given by $<_O = \{(\omega_1, \omega_2) \mid \omega_1 \neq \omega_2, \kappa(\omega_1) < \kappa(\omega_2)$ for all $\kappa \in O\}$.*

Note that in general $<_O$ is a strict weak ordering, i.e. it is irreflexive, asymmetric, and transitive. The inference core of skeptical inference over a set of OCFs O coincides with the merged order induced by O:

Proposition 1 (inference core and merged order). *For any knowledge base $\mathcal{R}$ and any set $O \subseteq Mod(\mathcal{R})$ it holds that $\lfloor \mid\!\sim^{O}_{\mathcal{R}} \rfloor = <_O$.*

Proof

$$\begin{aligned}
<_O &= \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid \omega_1 \neq \omega_2, \kappa(\omega_1) < \kappa(\omega_2) \text{ for all } \kappa \in O\} \\
&= \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid \omega_1 \neq \omega_2, \kappa \models (\omega_1|\omega_1 \vee \omega_2) \text{ for all } \kappa \in O\} \\
&= \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid \omega_1 \neq \omega_2, \omega_1 \vee \omega_2 \mid\!\sim^{\kappa} \omega_1 \text{ for all } \kappa \in O\} \\
&= \lfloor \mid\!\sim^{O}_{\mathcal{R}} \rfloor
\end{aligned}$$

□

We now define an inference relation with respect to $<_O$ in a similar way to inference with respect to the total pre-order on worlds induced by an OCF.

Definition 5 (inference relation induced by merged order, $\mid\!\sim_{\mathcal{R}}^{<_O}$). *Let $\mathcal{R}$ be a knowledge base, $O \subseteq Mod(\mathcal{R})$, and $A, B \in \mathcal{L}$. Then*

$$A \mid\!\sim_{\mathcal{R}}^{<_O} B \quad \textit{iff} \quad \textit{for all } \omega' \in \Omega_{A\overline{B}} \textit{ there is a } \omega \in \Omega_{AB} \textit{ such that } \omega <_O \omega'.$$

Proposition 2. *For any two sets of ranking models O and O' of $\mathcal{R}$ it holds that if $<_O = <_{O'}$ then $\mid\!\sim_{\mathcal{R}}^{<_O} = \mid\!\sim_{\mathcal{R}}^{<_{O'}}$.*

The inference relation induced by the merged order of a set of OCFs O approximates skeptical inference over O.

Proposition 3. *For any knowledge base $\mathcal{R}$ and $O \subseteq Mod(\mathcal{R})$ it holds that*

$$\mid\!\sim_{\mathcal{R}}^{<_O} \subseteq \mid\!\sim_{\mathcal{R}}^{O} \tag{1}$$

Proof.

$$\begin{aligned}
A \mid\!\sim_{\mathcal{R}}^{<_O} B &\Leftrightarrow \forall \omega' \in \Omega_{A\overline{B}} \;\; \exists \omega \in \Omega_{AB} : \omega <_O \omega' \\
&\Leftrightarrow \forall \omega' \in \Omega_{A\overline{B}} \;\; \exists \omega \in \Omega_{AB} \;\; \forall \kappa \in O : \kappa(\omega) < \kappa(\omega') \\
&\Rightarrow \forall \kappa \in O : \min\{\kappa(\omega) \mid \omega \models AB\} < \min\{\kappa(\omega) \mid \omega \models A\overline{B}\} \\
&\Leftrightarrow \forall \kappa \in O : A \mid\!\sim^{\kappa} B \\
&\Leftrightarrow A \mid\!\sim_{\mathcal{R}}^{O} B
\end{aligned}$$ □

While it is always the case that an inference over the merged order of a set O is also a skeptical inference over that set, the other direction of (1) does not hold in general.

Proposition 4. *There is a knowledge base $\mathcal{R}$ and a set $O \subseteq Mod(\mathcal{R})$ with*

$$\mid\!\sim_{\mathcal{R}}^{O} \not\subseteq \mid\!\sim_{\mathcal{R}}^{<_O}. \tag{2}$$

Proof. Consider $\mathcal{R} = \{(a|\top)\}$ over $\Sigma = \{a, b\}$. Let κ_1 and κ_2 be defined as:

$$\kappa_1(\omega) = \begin{cases} 0 & \text{if } \omega = ab \\ 1 & \text{otherwise} \end{cases} \qquad \kappa_2(\omega) = \begin{cases} 0 & \text{if } \omega = a\overline{b} \\ 1 & \text{otherwise} \end{cases}$$

Both κ_1 and κ_2 accept $\mathcal{R}$, but for $O = \{\kappa_1, \kappa_2\}$ it holds that $<_O = \emptyset$. Thus, since both OCFs accept $\mathcal{R}$ it holds that $\top \mid\!\sim_{\mathcal{R}}^{O} a$, but since $<_O$ is empty, $\top \not\mid\!\sim_{\mathcal{R}}^{<_O} a$. □

Since (1) holds for all sets of OCF models, but the reverse direction does not hold in general, we introduce the notion of *merged order compatibility*, classifying the sets of OCFs for which the other direction of (1) holds.

Definition 6 (merged order compatible). *Let $\mathcal{R}$ be a knowledge base and $O \subseteq Mod(\mathcal{R})$. O is called* merged order compatible *iff $\mid\!\sim_{\mathcal{R}}^{O} \subseteq \mid\!\sim_{\mathcal{R}}^{<_O}$.*

Thus, for merged order comaptible O we immediately get:

Proposition 5. *If $O \subseteq Mod(\mathcal{R})$ is merged order compatible, then $\mid\!\sim_{\mathcal{R}}^{<_O} = \mid\!\sim_{\mathcal{R}}^{O}$.*

Since the merged order of a set of ranking models is equal to the inference core of the skeptical inference over that set of models, merged order compatibility ensures that equivalence of skeptical inference relations coincides with equivalence of inference cores.

Proposition 6. *For any two merged order compatible sets of ranking models O and O' of a knowledge base $\mathcal{R}$ it holds that:*

$$\lfloor \mid\!\sim_{\mathcal{R}}^{O} \rfloor = \lfloor \mid\!\sim_{\mathcal{R}}^{O'} \rfloor \text{ iff } \mid\!\sim_{\mathcal{R}}^{O} = \mid\!\sim_{\mathcal{R}}^{O'} \tag{3}$$

Proof. The direction from right to left trivially holds since base conditionals are a subset of all conditionals. For the other direction we have:

$$\begin{aligned} \lfloor \mid\!\sim_{\mathcal{R}}^{O} \rfloor = \lfloor \mid\!\sim_{\mathcal{R}}^{O'} \rfloor &\Rightarrow <_O = <_{O'} && \text{(Proposition 1)} \\ &\Rightarrow \mid\!\sim_{\mathcal{R}}^{<_O} = \mid\!\sim_{\mathcal{R}}^{<_{O'}} && \text{(Proposition 2)} \\ &\Rightarrow \mid\!\sim_{\mathcal{R}}^{O} = \mid\!\sim_{\mathcal{R}}^{O'} && \text{(Proposition 5)} \end{aligned}$$

□

Note that according to Proposition 6, merged order compatibility provides a sufficient condition for reducing the question of skeptical inference equivalence to the equality of the inference cores.

4 C-Inference and Merged Order Compatibility

We will now illustrate merged order compatibility for a special kind of ranking models. C-Representations are special ranking models of a knowledge base $\mathcal{R}$, obtained by assigning individual impacts to the conditionals in $\mathcal{R}$. The rank of a possible world is then defined as the sum of impacts of falsified conditionals.

Definition 7 (c-representation [7,8]). *A* c-representation *of a knowledge base $\mathcal{R}$ is a ranking function κ constructed from integer impacts $\eta_i \in \mathbb{N}_0$ assigned to each conditional $(B_i|A_i)$ such that κ accepts $\mathcal{R}$ and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{4}$$

Every c-representation exibits desirable inference properties, and two c-representations induce the same inference relation if they induce the same total preorder on worlds. In [1], a modeling of c-representations as solutions of a constraint satisfaction problem $CR(\mathcal{R})$ is given and shown to be correct and complete with respect to the set of all c-representations of $\mathcal{R}$. Recently, it has been suggested to take inferential equivalence of c-representations into account and to sharpen $CR(\mathcal{R})$ by introducing an upper bound for the impact values η_i.

Definition 8 ($\boldsymbol{CR^u(\mathcal{R})}$ [3]). *Let* $\mathcal{R} = \{(B_1|A_1), \dots, (B_n|A_n)\}$ *and* $u \in \mathbb{N}$. *The finite domain constraint satisfaction problem* $CR^u(\mathcal{R})$ *on the constraint variables* $\{\eta_1, \dots, \eta_n\}$ *ranging over* $\mathbb{N}$ *is given by the conjunction of the constraints, for all* $i \in \{1, \dots, n\}$*:*

$$\eta_i \geqslant 0 \tag{5}$$

$$\eta_i > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j \tag{6}$$

$$\eta_i \leqslant u \tag{7}$$

A solution of $CR^u(\mathcal{R})$ is an n-tuple $(\eta_1, \dots, \eta_n)$ of natural numbers, its set of solutions is denoted by $Sol(CR^u(\mathcal{R}))$. For $\vec{\eta} \in Sol(CR^u(\mathcal{R}))$ and κ as in Eq. (4), κ is the *OCF induced by* $\vec{\eta}$, denoted by $\kappa_{\vec{\eta}}$, and the set of all induced OCFs is denoted by $\mathcal{O}(CR^u(\mathcal{R})) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR^u(\mathcal{R}))\}$. The constraint satisfaction problem $CR(\mathcal{R})$, given in [1], is obtained by removing the constraints (7) from $CR^u(\mathcal{R})$.

C-inference is skeptical inference over the set of all c-representations.

Definition 9 (c-inference, $\boldsymbol{\mid\!\sim^c_{\mathcal{R}}}$ [1]). *Let* $\mathcal{R}$ *be a knowledge base and let* A, B *be formulas.* B *is* a (skeptical) c-inference from A in the context of $\mathcal{R}$, *denoted by* $A \mid\!\sim^c_{\mathcal{R}} B$, *iff* $A \mid\!\sim^{\kappa} B$ *holds for all c-representations* κ *for* $\mathcal{R}$.

We will now illustrate c-representations, c-inference, and how the inference over the merged order of the set of all c-representations accepting a knowledge base can coincide with c-inference in the context of that knowledge base.

Example 1 ($\mathcal{R}_{lw}$). Consider $\Sigma_{lw} = \{l, w\}$ and $\mathcal{R}_{lw} = \{r_1, r_2, r_3\}$ with

$r_1 = (\overline{w}\mid l)$	*"land vehicles are usually not watercrafts"*
$r_2 = (l \vee \overline{w}\mid\top)$	*"usually, something is a land vehicle or not a watercraft"*
$r_3 = (w\mid\overline{l})$	*"things that are not land vehicles, are usually watercrafts"*

representing some default knowledge about vehicles in a country like Germany.

Using the verification and falsification behavior of the four possible worlds reveals that $\vec{\eta}_1, \dots, \vec{\eta}_5$ as given in Table 1 are solutions to $CR(\mathcal{R}_{lw})$. Furthermore, there are no other solutions of $CR(\mathcal{R}_{lw})$ inducing an ordering on worlds that is different from every of the orderings induced by $\kappa_{\vec{\eta}_1}, \dots, \kappa_{\vec{\eta}_5}$; for example, the solution $\vec{\eta}_6 = (3, 2, 3)$ induces the same ordering on worlds as $\kappa_{\vec{\eta}_5}$ and thus allows for exactly the same inferences.

Therefore, the merged order for $O = \{\kappa_{\vec{\eta}_1}, \dots, \kappa_{\vec{\eta}_5}\}$, given in the lower right corner of Table 1, coincides with the merged order over all c-representations of $\mathcal{R}_{lw}$. Checking all pairs of formulas over Σ_{lw} shows that for $\mathcal{R}_{lw}$ there is no difference between merged order inference over O and skeptical c-inference.

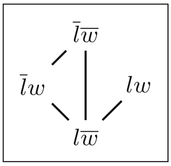
The following example illustrates an interesting difference between the set of all ranking models of a knowledge base and the set of its c-representations and shows that there are knowledge bases $\mathcal{R}$ such that the former set is not merged order compatible while the latter set is merged order compatible.

Table 1. Verification (v), falsification (f), impacts (η_i), solution vectors $\vec{\eta}_i$, induced OCFs $\kappa_{\vec{\eta}_i}$, and merged order of $\{\kappa_{\vec{\eta}_1}, \ldots, \kappa_{\vec{\eta}_5}\}$ for $CR(\mathcal{R}_{lw})$ in Example 1.

ω	r_1: $(\overline{w}\|l)$	r_2: $(l \vee \overline{w}\|\top)$	r_3: $(w\|\bar{l})$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$	$\kappa_{\vec{\eta}_3}(\omega)$	$\kappa_{\vec{\eta}_4}(\omega)$	$\kappa_{\vec{\eta}_5}(\omega)$
$l\,w$	f	v	–	η_1	1	3	1	2	2
$l\,\overline{w}$	v	v	–	0	0	0	0	0	0
$\bar{l}\,w$	–	f	v	η_2	2	1	1	1	1
$\bar{l}\,\overline{w}$	–	v	f	η_3	3	2	2	3	2

impacts:	η_1	η_2	η_3
$\vec{\eta}_1$	1	2	3
$\vec{\eta}_2$	3	1	2
$\vec{\eta}_3$	1	1	2
$\vec{\eta}_4$	2	1	3
$\vec{\eta}_5$	2	1	2

merged order:



Example 2. Consider the knowledge base $\mathcal{R}$ and Σ from the proof of Proposition 4, and let $P = Mod(\mathcal{R})$ and let O be the set of all c-representations accepting $\mathcal{R}$. For both P and O, a can be inferred skeptically from $\top$ in the context of $\mathcal{R}$, i.e. $\top \mid\!\sim^{P}_{\mathcal{R}} a$ and $\top \mid\!\sim^{O}_{\mathcal{R}} a$. The two ranking functions κ_1 and κ_2 used in the proof of Proposition 4 both accept $\mathcal{R}$ and are thus elements of P. Since there are no two distinct worlds ω and ω' with $\kappa_1(\omega) < \kappa_1(\omega')$ and $\kappa_2(\omega) < \kappa_2(\omega')$, the merged order $<_P$ is empty, and therefore $\top \not\mid\!\sim^{<_P}_{\mathcal{R}} a$. On the other hand, for every c-representation κ accepting $\mathcal{R}$ it holds that $\kappa(ab) = \kappa(a\bar{b})$ and $\kappa(\bar{a}b) = \kappa(\bar{a}\bar{b})$ and $\kappa(a\dot{b}) < \kappa(\bar{a}\dot{b})$. Thus, $<_O = \{(ab, \bar{a}b), (ab, \bar{a}\bar{b}), (a\bar{b}, \bar{a}b), (a\bar{b}, \bar{a}\bar{b})\}$ and hence $\top \mid\!\sim^{<_O}_{\mathcal{R}} a$. In fact, the set O of all c-representations accepting $\mathcal{R}$ is merged order compatible, while $\top \mid\!\sim^{P}_{\mathcal{R}} a$ and $\top \not\mid\!\sim^{<_P}_{\mathcal{R}} a$ shows that the set P of all ranking models of $\mathcal{R}$ is not merged order compatible.

For studying the exact relationship between $CR(\mathcal{R})$ and $CR^u(\mathcal{R})$, the concept of a *sufficient* $CR^u(\mathcal{R})$ was introduced in [3] to capture the idea that only a finite number of c-representations is needed for modeling c-inference.

Definition 10 (sufficient). *Let $\mathcal{R}$ be a knowledge base and let $u \in \mathbb{N}$. Then $CR^u(\mathcal{R})$ is called* sufficient *(for skeptical inference) iff for all formulas A, B we have*

$$A \mid\!\sim^{c}_{\mathcal{R}} B \quad \textit{iff} \quad A \mid\!\sim^{\mathcal{O}(CR^u(\mathcal{R}))}_{\mathcal{R}} B.$$

If $CR^u(\mathcal{R})$ is sufficient, we will also call u sufficient for $\mathcal{R}$.

In terms of the classical skeptical inference relation over a set of ranking models given in Definition 1 this means that $CR^u(\mathcal{R})$ is sufficient iff

$$A \mid\!\sim^{\mathcal{O}(CR(\mathcal{R}))}_{\mathcal{R}} B \quad \text{iff} \quad A \mid\!\sim^{\mathcal{O}(CR^u(\mathcal{R}))}_{\mathcal{R}} B. \tag{8}$$

For various $\mathcal{R}$ and u, we will now use merged order compatibility for proving (8).

5 Proving Sufficient Upper Bounds

In this section, we continue the investigation from [2] and use the concepts from the previous section to formally prove an experimental result from [2].

Definition 11 **($\Sigma_n, \mathcal{R}_n$).** *For* $n \geqslant 1$ *and* $\Sigma_n = \{a_1, \dots, a_n\}$, $\mathcal{R}_n = \{(a_1|\top), \dots, (a_n|\top)\}$ *is called the* knowledge base of n conditional facts.

Note that from the constraints in $CR(\mathcal{R}_n)$ and $CR^u(\mathcal{R}_n)$ it follows that for all impacts in c-representations accepting $\mathcal{R}_n$ it holds that $\eta_i \geqslant 1$. In the rest of this section, we investigate how the concepts of merged order compatibility and inference cores can be used to prove that for $\mathcal{R}_n$ the CSP $\mathcal{O}(CR^{n-1}(\mathcal{R}_n))$ is indeed sufficient. In [2] this was solely illustrated by means of some examples.

Because the structure of knowledge bases $\mathcal{R}_n$ is very simple, the rank of a world ω over Σ_n assigned by a c-representation depends on the set of falsified atoms in ω in a very predictable way.

Definition 12 **($f(\omega), <_f$).** *For* $\omega \in \Omega_{\Sigma_n}$, $f(\omega) = \{\, i \mid \omega \models \overline{a_i},\, i \in \{\, 1, \dots, n \,\} \,\}$ *is the set of indices of the negated literals in* ω*. The ordering* $<_f$ *on* Ω_{Σ_n} *is defined such that for two worlds* $\omega, \omega' \in \Omega_{\Sigma_n}$, $\omega <_f \omega'$ *iff* $f(\omega) \subsetneqq f(\omega')$.

As the ordering $<_O$ on worlds, also $<_f$ induces an inference relation.

Definition 13 **($\mathrel{|\!\!\sim}^{<_f}_{\mathcal{R}_n}$).** *For* $n > 1$ *and formulas* $A, B \in \mathcal{L}_{\Sigma_n}$

$$A \mathrel{|\!\!\sim}^{<_f}_{\mathcal{R}_n} B \quad \textit{iff} \quad \textit{for every } \omega' \in \Omega_{A\overline{B}}, \textit{ there is a } \omega \in \Omega_{AB} \textit{ such that } \omega <_f \omega'.$$

The following proposition generalizes a proposition from [2] regarding the ranking of worlds ω and ω' incomparable in $<_f$.

Proposition 7. *Let* $n > 1$, $\omega' \in \Omega_{\Sigma_n}$ *and* $\Omega_V = \{\omega_1, \dots, \omega_m\} \subseteq \Omega_{\Sigma_n}$*. If for all* $i \in \{1, \dots, m\}$, $f(\omega') \not\subseteq f(\omega_i)$ *and* $f(\omega_i) \not\subseteq f(\omega')$*, then there exists a c-representation* κ *accepting* $\mathcal{R}_n$ *such that for all* $i \in \{1, \dots, m\}$, $\kappa(\omega') \leqslant \kappa(\omega_i)$.

Proof. Let I be $I = (\bigcup_{i=1}^m f(\omega_i)) \setminus f(\omega')$. Note that because of the precondition $f(\omega') \not\subseteq f(\omega_i)$ and $f(\omega_i) \not\subseteq f(\omega')$, it holds that $I \neq \emptyset$. Let $\vec{\eta} = (\eta_1, \dots, \eta_n)$ with

$$\eta_i = \begin{cases} 1 & i \notin I \\ n-1 & i \in I \end{cases}$$

Since for every $i \in f(\omega')$ the impact vector $\vec{\eta}$ assigns 1 to the corresponding conditional $(a_i|\top) \in \mathcal{R}_n$ and because we know that $\omega' \neq \overline{a_1} \dots \overline{a_n}$, we get $\kappa_{\vec{\eta}}(\omega') = |f(\omega')| \leqslant n-1$. Because I is not empty, for every $i \in \{1, \dots, m\}$, there is some $k \in f(\omega_i)$ such that $\eta_k = n-1$. Thus, we get $\kappa_{\vec{\eta}}(\omega_i) \geqslant n-1$. Therefore, it holds that $\kappa_{\vec{\eta}}(\omega') \leqslant \kappa_{\vec{\eta}}(\omega_i)$ for every $i \in \{1, \dots, m\}$. □

We now use Proposition 7 to show that the inference relation $\mathrel{|\!\!\sim}^{<_f}_{\mathcal{R}_n}$ defined over the ordering on worlds $<_f$ is equal to the skeptical inference over all c-representations accepting $\mathcal{R}_n$.

Proposition 8. *For $n > 1$ and $O = \mathcal{O}(CR(\mathcal{R}_n))$, $\mid\!\sim^{<_f}_{\mathcal{R}_n} = \mid\!\sim^{O}_{\mathcal{R}_n}$.*

Proof. Let A and B be arbitrary formulas from $\mathcal{L}_{\Sigma_n}$. If $A \mid\!\sim^{<_f}_{\mathcal{R}_n} B$ then for all $\omega' \in \Omega_{A\overline{B}}$ there is a $\omega \in \Omega_{AB}$ such that $\omega <_f \omega'$. Thus $f(\omega) \subsetneq f(\omega')$, and because $\kappa(\omega)$ for a c-representation κ is defined by the sum of all impacts of negative literals in ω, it also holds that $\kappa(\omega) < \kappa(\omega')$ for $\kappa \in O$. Thus $A \mid\!\sim^{\kappa} B$ holds for all $\kappa \in O$, implying that $\mid\!\sim^{<_f}_{\mathcal{R}_n} \subseteq \mid\!\sim^{O}_{\mathcal{R}_n}$.

To show the other direction, we assume that $A \not\mid\!\sim^{<_f}_{\mathcal{R}_n} B$ and show that $A \not\mid\!\sim^{O}_{\mathcal{R}_n} B$. If $A \not\mid\!\sim^{<_f}_{\mathcal{R}_n} B$, then there is a world $\omega' \in \Omega_{A\overline{B}}$, such that for all worlds $\omega \in \Omega_{AB}$ $\omega \not<_f \omega'$ holds. If $\omega' <_f \omega$, then $\kappa(\omega') \leqslant \kappa(\omega)$ for every c-representation κ with $\kappa \models \mathcal{R}_n$ and therefore $A \not\mid\!\sim^{O}_{\mathcal{R}_n} B$. If $\omega' \not<_f \omega$ then for all worlds $\omega \in \Omega_{AB}$ $f(\omega') \not\subseteq f(\omega)$ and $f(\omega) \not\subseteq f(\omega')$, and we use Proposition 7 by setting $\Omega_V = \Omega_{AB}$ and construct a c-representation κ such that $\kappa(\omega') \leqslant \kappa(\omega)$ for all $\omega \in \Omega_{AB}$. Thus, $min\left\{\kappa(\omega) \mid \omega \models A\overline{B}\right\} \leqslant \min\left\{\kappa(\omega) \mid \omega \models AB\right\}$, implying $A \not\mid\!\sim_{\kappa} B$ and therefore $A \not\mid\!\sim^{O}_{\mathcal{R}_n} B$. $\square$

Since both $<_f$ and $<_O$ are orderings of worlds and $\mid\!\sim^{<_f}_{\mathcal{R}_n}$ and $\mid\!\sim^{<_O}_{\mathcal{R}_n}$ are defined in the same way, it is now straightforward to show that $\mathcal{O}(CR(\mathcal{R}_n))$ is merged order compatible for any $\mathcal{R}_n$.

Proposition 9. *For $n > 1$, $\mathcal{O}(CR(\mathcal{R}_n))$ is merged order compatible for $\mathcal{R}_n$.*

Proof. To show that $O = \mathcal{O}(CR(\mathcal{R}_n))$ is merged order compatible for $\mathcal{R}_n$, we need to show that $\mid\!\sim^{O}_{\mathcal{R}_n} \subseteq \mid\!\sim^{<_O}_{\mathcal{R}_n}$. Since we already know $\mid\!\sim^{O}_{\mathcal{R}_n} = \mid\!\sim^{<_f}_{\mathcal{R}_n}$ (Proposition 8), due to Proposition 2 it suffices to show that $<_f = <_O$. If $\omega <_f \omega'$, then $f(\omega) \subsetneq f(\omega')$. As was already pointed out in the proof of Proposition 8, this means that for all c-representations κ we have $\kappa(\omega) < \kappa(\omega')$ and thus $\omega <_O \omega'$. We now have $\mid\!\sim^{O}_{\mathcal{R}_n} = \mid\!\sim^{<_f}_{\mathcal{R}_n} = \mid\!\sim^{<_O}_{\mathcal{R}_n}$ and $\mathcal{O}(CR(\mathcal{R}_n))$ is merged order compatible. $\square$

Since we do not make use of impacts $\eta_i > n - 1$, the proofs of Propositions 8 and 9 also work for $O = \mathcal{O}(CR^{n-1}(\mathcal{R}_n))$, implying:

Proposition 10. *For $n > 1$, $\mathcal{O}(CR^{n-1}(\mathcal{R}_n))$ is merged order compatible for $\mathcal{R}_n$.*

These results now enable us to prove that $n - 1$ is sufficient for $\mathcal{R}_n$, implying that the inference relation induced by the solutions of $CR^{n-1}(\mathcal{R}_n)$ is equal to the skeptical inference over all c-representations for $\mathcal{R}_n$.

Proposition 11. *For $n > 1$, $CR^{n-1}(\mathcal{R}_n)$ is sufficient for $\mathcal{R}_n$.*

Proof. We need to show that $\mid\!\sim^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))}_{\mathcal{R}_n} = \mid\!\sim^{\mathcal{O}(CR(\mathcal{R}_n))}_{\mathcal{R}_n}$. Since both $\mathcal{O}(CR^{n-1}(\mathcal{R}_n))$ and $\mathcal{O}(CR(\mathcal{R}_n))$ are merged order compatible, it suffices to show that the inference cores are equal, i.e. $\lfloor \mid\!\sim^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))}_{\mathcal{R}_n} \rfloor = \lfloor \mid\!\sim^{\mathcal{O}(CR(\mathcal{R}_n))}_{\mathcal{R}_n} \rfloor$.

It is easy to see that if a pair of possible worlds (ω, ω') is in $\lfloor \mid\!\sim^{\mathcal{O}(CR(\mathcal{R}_n))}_{\mathcal{R}_n} \rfloor$, then it is also in $\lfloor \mid\!\sim^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))}_{\mathcal{R}_n} \rfloor$ since $\mid\!\sim^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))}_{\mathcal{R}_n}$ allows for possibly more inferences. To show the other direction, we assume that $(\omega, \omega') \notin \lfloor \mid\!\sim_{\mathcal{R}_n}^{\mathcal{O}(CR(\mathcal{R}_n))} \rfloor$ and show that $(\omega, \omega') \notin \lfloor \mid\!\sim_{\mathcal{R}_n}^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))} \rfloor$.

If (ω, ω') is not in the inference core of the unbounded skeptical c-inference, it means that there is a c-representation κ in which $\kappa(\omega) \geqslant \kappa(\omega')$. If $f(\omega') \subseteq f(\omega)$, then for $\vec{\eta} = (1, \ldots, 1)$ it holds that $\kappa_{\vec{\eta}}(\omega) \geqslant \kappa_{\vec{\eta}}(\omega')$. If $f(\omega) \subseteq f(\omega')$, then there is no c-representation κ such that $\kappa(\omega) \geqslant \kappa(\omega')$, contradicting the assumption. If neither $f(\omega') \subseteq f(\omega)$ nor $f(\omega) \subseteq f(\omega')$ holds, the precondition of Proposition 7 is met for ω' and $\Omega_V = \{\omega\}$, and we can construct a c-representation κ in $\mathcal{O}(CR^{n-1}(\mathcal{R}_n))$ such that $\kappa(\omega) \geqslant \kappa(\omega')$; hence $(\omega, \omega') \notin \lfloor \mid\!\sim_{\mathcal{R}_n}^{\mathcal{O}(CR^{n-1}(\mathcal{R}_n))} \rfloor$. □

6 Conclusions and Further Work

We introduced the notion of inference core of a nonmonotonic inference relation taking only so called base conditionals into account. By showing that a set of ranking models is merged order compatible, we can reduce the question of equality of inference relations to equivalence of inference cores. We illustrated arising differences between the set of all ranking models of a knowledge base $\mathcal{R}$ and the set of all c-representations of $\mathcal{R}$, and we applied our approach to skeptical c-inference for proving that for certain knowledge bases a maximal impact of $|\mathcal{R}| - 1$ is sufficient to fully capture the behavior of skeptical c-inference.

In our current work, we employ the concepts of inference cores and merged order compatibility for extending our investigations on sufficient upper bounds for $CR(\mathcal{R})$ to more general kinds of knowledge bases, and for addressing the open problems of characterizing knowledge bases whose set of c-representations is merged order compatible or whether this property holds for all knowledge bases. This goes along with finding a suitable characterization of merged order compatible sets of ranking models, and exploring relationships to approaches employing e.g. possibilistic or probabilistic semantics.

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